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STRONG SOLUTIONS TO THE COMPRESSIBLE LIQUID CRYSTAL SYSTEM

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We prove the existence of local strong solutions of the compressible liquid crystal system.

1. Introduction

We consider the following simplified system of Ericksen–Leslie equations:

$$\rho_t + \operatorname{div}(\rho u) = 0, \tag{1.1}$$

$$\rho u_t + \rho u \cdot \nabla u + \nabla p - \mu \Delta u + \lambda \Big(\operatorname{div}(\nabla n \otimes \nabla n) - \nabla \frac{|\nabla n|^2}{2} \Big) = 0, \tag{1.2}$$

$$\frac{\partial n}{\partial t} + u \cdot \nabla n - \nu (\Delta n + |\nabla n|^2 n) = 0, \tag{1.3}$$

with the following initial and boundary conditions:

$$(\rho, u, n)|_{t=0} = (\rho_0, u_0, n_0), \quad x \in \Omega, \tag{1.4}$$

$$u(x,t) = u_0(x) = 0, \quad n(x,t) = n_0(x), \quad x \in \partial\Omega, \tag{1.5}$$

where u is the velocity field, n the macroscopic average of the nematic liquid crystal orientation field, $\rho_0 \ge 0$, $|n_0| = 1$, and pressure $p = a\rho^\gamma$ with $\gamma > 1$, where γ is the adiabatic constant (in the physically relevant case of a monoatomic gas, $\gamma = \frac{5}{3}$). This system is modeled after the theory of Oseen [1933] and Frank [1958]; see the articles [Ericksen 1962; Forster et al. 1971; Leslie 1966; 1968] or the books [Ericksen and Kinderlehrer 1987; Gennes and Prost 1993; Pasechnik et al. 2009; Stephen 1970; Xie 1988].

The system (1.1)–(1.3) is much more complicated than the compressible Navier–Stokes equations, because equation (1.3), like the situation with heat flow into a sphere, makes the strongly coupling term $\operatorname{div}(\nabla n \otimes \nabla n) - \nabla \frac{|\nabla n|^2}{2}$ have a weak convergence. So far, the existence of weak solutions to the system remains open, though there are celebrated contributions by Lions [1998]; see also [Feireisl 2004;

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Feireisl et al. 2001]. Liu and Qing [2011] proved the global existence of finite energy weak solutions to the case where the free energy is replaced by the Ginzburg–Landau approximation energy,

$$\min_{n\in H^1(\Omega;R^3)} \int_\Omega \tfrac{1}{2}|\nabla n|^2 + \frac{1}{4\sigma^2}(|n|^2-1)^2\,dx.$$

In the incompressible case, F. H. Lin and C. Liu, among others [Lin 1989; Lin and Liu 1995; Lin and Liu 2001; Lin and Liu 2000; Lin and Liu 1996; Calderer and Liu 2000], systematically studied the incompressible liquid crystal dynamics system based on the Ericksen–Leslie model (that is, the Ginzburg–Landau approximation case with ρ being a constant in system (1.1) makes the velocity field divergence free) and proved the global existence of weak solutions, classical solutions, and partial regularity. Liu and Zhang [2009] also studied the existence of weak solutions to the incompressible liquid crystal system with the Ginzburg–Landau approximation and ρ nonconstant.

It is well known that there exist no global solutions to the system (1.1)–(1.3) even in the incompressible case. Surprisingly, we can prove the local existence of a strong solution to the compressible liquid crystal system with initial density $\rho_0 \geq 0$. We gained enlightenment from the corresponding results of the compressible Navier–Stokes equations. There is a huge literature on the compressible Navier–Stokes equations, under the crucial assumption that the initial density ρ_0 is bounded below away from zero. The existence results were obtained by Nash, Itaya, Tani, Matsumura, and Nishida, among others. For general nonnegative initial density, Cho, Kim, and Choe [Choe and Kim 2003; Cho et al. 2004; Cho and Kim 2006] obtained the existence of a local strong solution to a compressible Navier–Stokes equation.

We first have the energy law

$$\frac{dE}{dt} + \int_\Omega \mu|\nabla u|^2 + \lambda\nu|\Delta n + |\nabla n|^2 n|^2 = 0$$

with

$$E(t) = \int_\Omega \Big(\tfrac{1}{2}\rho u^2 + \frac{\lambda}{2}|\nabla n|^2 + \frac{a}{\gamma-1}\rho^\gamma\Big).$$

From the definition of velocity,

$$\frac{dx(X,t)}{dt} = u(x(X,t),t), \tag{1.6}$$

$$x(X,0) = X. \tag{1.7}$$

The continuity equation can be rewritten as

$$\frac{d\rho(x(X,t),t)}{dt} + \rho\,\mathrm{div}\,u = 0,$$

that is,

$$\rho(x,t) = \rho_0 \exp\left(-\int_0^t \operatorname{div} u\right). \tag{1.8}$$

We need the following regularity for ρ_0, n_0, and u_0:

$$\rho_0 \in W^{1,6}(\Omega), \quad u_0 \in H_0^1(\Omega) \cap H^2(\Omega), \quad n_0 \in H^3(\Omega). \tag{1.9}$$

We also need some compatibility condition on the initial data: for some $g \in L^2$,

$$\mu \Delta u_0 - \lambda \operatorname{div}(\nabla n_0 \otimes \nabla n_0 - \tfrac{1}{2}|\nabla n_0|^2 I) - a\nabla \rho_0^\gamma = \rho_0^{\frac{1}{2}} g. \tag{1.10}$$

The following is our main result.

Theorem 1.1. *Assume Ω is a smooth bounded domain in $\mathbb{R}^3$ and (ρ_0, n_0, u_0) satisfies regularity condition (1.9) and compatibility condition (1.10). Then there exist a small time $T^* > 0$ and a unique strong solution (ρ, n, u) of the compressible liquid crystal system (1.1)–(1.3) in $(0, T^*) \times \Omega$, satisfying initial and boundary conditions (1.4) and (1.5), such that*

$$\begin{aligned}
&\rho \in C([0,T^*); W^{1,6}), && \rho_t \in C([0,T^*); L^6),\\
&u \in C([0,T^*); H_0^1 \cap H^2) \cap L^2(0,T^*; W^{2,6}), && u_t \in L^2(0,T^*; H_0^1),\\
&n \in C([0,T^*); H^2) \cap L^2(0,T^*; W^{2,6}), && n_t \in C([0,T^*); H_0^1),\\
&\sqrt{\rho} u_t \in C([0,T^*); L^2).
\end{aligned}$$

2. Approximation solutions

We now consider the linearized equations as follows: for fixed smooth functions $v, d : \Omega \times [0,T] \to \mathbb{R}^3$ with

$$\frac{dx(X,t)}{dt} = v(x(X,t),t)$$

and $x(X,0) = X$, and $v(x,0) = u_0(x)$, $d(x,0) = n_0(x)$,

$$\rho_t + \operatorname{div}(\rho v) = 0, \tag{2.1}$$

$$(\rho u)_t + \operatorname{div}(\rho v \otimes v) + a\nabla \rho^\gamma = \mu \Delta u - \lambda \operatorname{div}(\nabla n \otimes \nabla n - \tfrac{1}{2}|\nabla n|^2 I), \tag{2.2}$$

$$n_t - \gamma \Delta n = \lambda |\nabla d|^2 d - v \cdot \nabla d, \tag{2.3}$$

with initial and boundary conditions

$$(\rho, u, n)|_{t=0} = (\rho_0 + \delta, u_0, n_0), \quad x \in \Omega, \tag{2.4}$$

$$u(x,t) = u_0(x) = 0, \quad n(x,t) = n_0(x), \quad x \in \partial\Omega. \tag{2.5}$$

Here $\delta > 0$ is a constant, and $\rho_0 \geq 0$, $|n_0| = 1$.

We use the following notations: Suppose Banach spaces

$$\mathscr{A} = L^\infty(0, T;\, H^2(\Omega)) \cap L^2(0, T;\, W^{2,6}(\Omega)) \cap W_2^{1,1}((0, T) \times \Omega),$$
$$\mathscr{B} = L^\infty(0, T;\, W^{2,6}(\Omega)) \cap W_\infty^{1,1}((0, T) \times \Omega) \cap W_2^{2,1}((0, T) \times \Omega)$$

with norm respectively

$$\|v\|_{\mathscr{A}} = \|v\|_{L^\infty(0,T;H^2(\Omega))} + \|v\|_{L^2(0,T;W^{2,6}(\Omega))} + \|v_t\|_{L^2(0,T;H^1(\Omega))},$$
$$\|d\|_{\mathscr{B}} = \|d_t\|_{L^2(0,T;H^2(\Omega))} + \|d_t\|_{L^\infty(0,T;H^1(\Omega))} + \|d\|_{L^\infty(0,T;W^{2,6}(\Omega))}.$$

Lemma 2.1. *For given v with $\|v\|_{\mathscr{A}} \le A$, the unique solution ρ of (2.1) satisfies*

$$\|\rho\|_{L^\infty(0,T;W^{1,6}(\Omega))} \le cc_0(1 + T^{\frac{1}{2}} A) \exp(cT^{\frac{1}{2}} A), \tag{2.6}$$
$$\|\rho_t\|_{L^\infty(0,T;L^6(\Omega))} \le cc_0 A \exp(cT^{\frac{1}{2}} A). \tag{2.7}$$

In particular,

$$\|p\|_{L^\infty(0,T;W^{1,6}(\Omega))} \le cc_0(1 + T^{\frac{1}{2}} A) \exp(cT^{\frac{1}{2}} A), \tag{2.8}$$
$$\|p_t\|_{L^\infty(0,T;L^6(\Omega))} \le cc_0 A \exp(cT^{\frac{1}{2}} A), \tag{2.9}$$

where c is an absolute constant, perhaps dependent on Ω, λ, μ, γ, etc., and c_0 is a constant dependent on initial and boundary data.

Proof. Since

$$\nabla\rho = \nabla\rho_0 \exp\Big(-\int_0^t \operatorname{div} v\Big) - \rho_0 \int_0^t \nabla \operatorname{div} v \exp\Big(-\int_0^t \operatorname{div} v\Big),$$
$$\rho_t = -\rho_0 \operatorname{div} v \exp\Big(-\int_0^t \operatorname{div} v\Big),$$

we have, from the Minkowski inequality,

$$\begin{aligned}
\|\nabla\rho\|_{L^6(\Omega)} &\le c\|\rho_0\|_{W^{1,6}(\Omega)}\Big(1 + \Big\|\int_0^t \nabla^2 v\Big\|_{L^6(\Omega)}\Big) \exp\Big(\int_0^T \|\operatorname{div} v\|_{L^\infty(\Omega)}\Big)\\
&\le c\|\rho_0\|_{W^{1,6}(\Omega)}\Big(1 + \int_0^T \|\nabla^2 v\|_{L^6(\Omega)}\Big) \exp\Big(\int_0^T \|\operatorname{div} v\|_{L^\infty(\Omega)}\Big)\\
&\le c\|\rho_0\|_{W^{1,6}(\Omega)}(1 + T^{\frac{1}{2}}\|v\|_X) \exp(cT^{\frac{1}{2}}\|v\|_X)\\
&\le cc_0(1 + T^{\frac{1}{2}} A) \exp(cT^{\frac{1}{2}} A),\\
\|\rho_t\|_{L^6(\Omega)} &\le c\|\rho_0\|_{L^\infty(\Omega)}\|\nabla v\|_{L^6(\Omega)} \exp\Big(\int_0^T \|\operatorname{div} v\|_{L^\infty(\Omega)}\Big)\\
&\le cc_0 \exp(cT^{\frac{1}{2}} A)\|v\|_{H^2(\Omega)} \le cc_0 A \exp(cT^{\frac{1}{2}} A),
\end{aligned}$$

where $X = L^2(0, T;\, W^{2,6}(\Omega))$. □

Lemma 2.2. *Suppose* $\|v\|_{\mathcal{A}} \le A$, $\|d\|_{\mathcal{B}} \le B$. *Then* (2.3) *with initial condition* $n(x,0) = n_0(x)$ *has a unique solution* n *and a constant* K_1, *depending only on* n_0 *and* u_0, *such that, for* $T = T(A, B)$ *small enough,*

$$\|n\|_{\mathcal{B}} = \|n_t\|_{L^2(0,T;H^2(\Omega))} + \|n_t\|_{L^\infty(0,T;H^1(\Omega))} + \|n\|_{L^\infty(0,T;W^{2,6}(\Omega))} \le K_1. \tag{2.10}$$

Proof. The existence of a solution to (2.3) is standard. We just give the estimates as follows. Differentiating (2.3) with respect to time t,

$$n_{tt} - \nu\Delta n_t = \nu(|\nabla d|_t^2 d + |\nabla d|^2 d_t) + (v_t\cdot\nabla)d - (v\cdot\nabla)d_t.$$

Multiplying by Δn_t, integrating over Ω, and using the Cauchy inequality, we get

$$\begin{aligned}\tfrac12 \frac{d}{dt}\int_\Omega |\nabla n_t|^2 + \nu\int_\Omega |\Delta n_t|^2 &\\ = -\int_\Omega \nu(|\nabla d|_t^2 d + |\nabla d|^2 d_t)\cdot\Delta n_t + (v_t\cdot\nabla)d\cdot\Delta n_t - (v\cdot\nabla)d_t\cdot\Delta n_t &\\ \le \int_\Omega 2\nu|\nabla d||\nabla d_t||d||\Delta n_t| + \nu|\nabla d|^2|d_t||\Delta n_t| &\\ + \int_\Omega |\nabla v_t||\nabla d||\nabla n_t| + |v_t||\nabla^2 d||\nabla n_t| + |v||\nabla d_t||\Delta n_t| &\\ = \sum_{i=1}^5 I_i. &\end{aligned} \tag{2.11}$$

We have the following estimates for I_i:

$$\begin{aligned} I_1 &= \int_\Omega 2\nu|\nabla d||\nabla d_t||d||\Delta n_t| \le c\int_\Omega |\nabla d|^2|\nabla d_t|^2|d|^2 + \frac{\nu}{6}\|\Delta n_t\|^2_{L^2(\Omega)},\\ I_2 &= \int_\Omega \nu|\nabla d|^2|d_t||\Delta n_t| \le c\int_\Omega |\nabla d|^4|d_t|^2 + \frac{\nu}{6}\|\Delta n_t\|^2_{L^2},\\ I_3 &= \int_\Omega |\nabla v_t||\nabla d||\nabla n_t| \le A^{-2}B^{-2}\int_\Omega |\nabla v_t|^2|\nabla d|^2 + A^2B^2\int_\Omega |\nabla n_t|^2,\\ I_4 &= \int_\Omega |v_t||\nabla^2 d||\nabla n_t| \le A^{-2}B^{-2}\int_\Omega |v_t|^2|\nabla^2 d|^2 + A^2B^2\int_\Omega|\nabla n_t|^2\\ &\le cA^{-2}B^{-2}\|\nabla v_t\|^2_{L^2}\|\nabla^2 d\|_{L^2}\|\nabla^2 d\|_{L^6} + A^2B^2\int_\Omega |\nabla n_t|^2,\\ I_5 &= \int_\Omega |v||\nabla d_t||\Delta n_t| \le \frac{3}{\nu}\int_\Omega |v|^2|\nabla d_t|^2 + \frac{\nu}{6}\|\Delta n_t\|^2_{L^2}.\end{aligned}$$

Substituting all the estimates into (2.11), we get

$$\begin{aligned}\frac{d}{dt}\int_\Omega |\nabla n_t|^2 + \nu\int_\Omega |\Delta n_t|^2 \le c\int_\Omega |\nabla d|^2|\nabla d_t|^2||d|^2 + c\int_\Omega |\nabla d|^4|d_t|^2 &\\ + cA^{-2}B^{-2}\int_\Omega |\nabla v_t|^2|\nabla d|^2 + cA^2B^2\int_\Omega |\nabla n_t|^2 &\\ + c\int_\Omega |v|^2|\nabla d_t|^2 + cA^{-2}B^{-2}\|\nabla v_t\|^2_{L^2}\|\nabla^2 d\|_{L^2}\|\nabla^2 d\|_{L^6}, &\end{aligned}$$

that is,

$$\int_\Omega |\nabla n_t|^2 + \nu \int_0^T \int_\Omega |\Delta n_t|^2 \\ \leq cB^6T + cA^2B^2T + c + cA^2B^2 \int_0^T \int_\Omega |\nabla n_t|^2 + c(n_0, u_0),$$

where

$$c(n_0, u_0) \\ = c\int_\Omega |\Delta \nabla n_0|^2 + |\nabla n_0|^2 |\nabla^2 n_0|^2 + |\nabla n_0|^6 + c\int_\Omega |\nabla u_0|^2 |\nabla n_0|^2 + |u_0|^2 |\nabla^2 n_0|^2.$$

Using Gronwall's inequality, we obtain

$$\int_\Omega |\nabla n_t|^2 \leq (cB^6T + cA^2B^2T + c_0)\exp(cA^2B^2T)$$

and

$$\int_\Omega |\nabla n_t|^2 + \nu \int_0^T \int_\Omega |\Delta n_t|^2 \leq c(B^6T + A^2B^2T + c_0)(1 + \exp(cA^2B^2T)).$$

Taking $T = T(A, B)$ small, we get

$$\int_\Omega |\nabla n_t|^2 + \nu \int_0^T \int_\Omega |\Delta n_t|^2 \leq c.$$

The elliptic estimates can be deduced from (2.3):

$$\begin{aligned} \|n\|_{W^{2,6}(\Omega)} &\leq \|n_t\|_{L^6} + \|v \cdot \nabla d\|_{L^6} + \||\nabla d|^2 d\|_{L^6} + \|n_0\|_{W^{2,6}} \\ &\leq \|v \cdot \nabla d\|_{L^6} + \| |\nabla d|^2 d\|_{L^6} + c_0. \end{aligned}$$

We estimate each item:

$$\begin{aligned} &\|v \cdot \nabla d\|_{L^6} \\ &= \left(\int_\Omega |v|^6 |\nabla d|^6\right)^{\frac{1}{6}} \leq \left(\int_\Omega |v - u_0|^6 |\nabla d|^6\right)^{\frac{1}{6}} + \|u_0\|_{L^\infty}\left(\int_\Omega |\nabla d|^6\right)^{\frac{1}{6}} \\ &\leq cB\left(\int_\Omega |\nabla v - \nabla u_0|^2\right)^{\frac{1}{2}} + c\|u_0\|_{L^\infty}\left(\int_\Omega |\nabla d - \nabla n_0|^6\right)^{\frac{1}{6}} + c\|u_0\|_{L^\infty}\|\nabla n_0\|_{L^\infty} \\ &\leq cB\left(\int_\Omega \left|\int_0^t \nabla v_t\right|^2\right)^{\frac{1}{2}} + c_0 B^{\frac{2}{3}}\left(\int_\Omega \left|\int_0^t \nabla d_t\right|^2\right)^{\frac{1}{6}} + c_0 \\ &\leq cBT^{\frac{1}{2}}\|\nabla v_t\|_{L^2(Q_T)} + c_0 T^{\frac{1}{3}} B + c_0 \leq cABT^{\frac{1}{2}} + c_0 BT^{\frac{1}{3}} + c_0 \end{aligned}$$

and

$$\begin{aligned}\||\nabla d|^2 d\|_{L^6} &= \Big(\int_\Omega ||\nabla d|^2 d|^6\Big)^{\frac16} \le \Big(\int_\Omega |\nabla d|^{12}|d-n_0|^6\Big)^{\frac16} + c_0\Big(\int_\Omega |\nabla d|^{12}\Big)^{\frac16}\\ &\le cB^2\Big(\int_\Omega |d-n_0|^6\Big)^{\frac16} + c_0\Big(\int_\Omega |\nabla d-\nabla n_0|^{12}\Big)^{\frac16} + c_0\\ &\le cB^2\Big(\int_\Omega |\nabla d-\nabla n_0|^2\Big)^{\frac12} + c_0B\Big(\int_\Omega |\nabla d-\nabla n_0|^6\Big)^{\frac16} + c_0\\ &\le cAB^2T^{\frac12} + c_0B^2T^{\frac13} + c_0.\end{aligned}$$

Taking $T = T(A, B)$ small enough, we obtain the desired $\|n\|_{W^{2,6}} \le c_0$. ☐

For (2.2) we have following Lemma.

Lemma 2.3. *Under the conditions of Lemma 2.2, suppose n satisfies (2.3) and ρ (2.1). Then there exists a unique solution u satisfying (2.2), and there is a constant K_2, depending only on n_0 and u_0, such that, for $T = T(A, B)$ small enough,*

$$\|u\|_{\mathscr{A}} \equiv \|u\|_{L^\infty(0,T;H^2(\Omega))} + \|u\|_{L^2(0,T;W^{2,6}(\Omega))} + \|u_t\|_{L^2(0,T;H^1(\Omega))} \le K_2. \tag{2.12}$$

Proof. Since

$$\rho \ge \delta \exp\Big(-\int_0^T |\nabla v|_{L^\infty((0,T)\times\Omega)}\Big) > 0,$$

the standard theory of parabolic equations implies the existence of the solution to (2.2). Differentiating (2.2) with respect to time t, we get

$$\begin{aligned}&\rho u_{tt} - \mu\Delta u_t \\ &= -\lambda\,\mathrm{div}((\nabla d\otimes\nabla d)_t - \tfrac12|\nabla d|_t^2 I) - \nabla p_t - (\rho v\cdot\nabla)v_t - (\rho_t v\cdot\nabla)v - (\rho v_t\cdot\nabla)v - \rho_t u_t.\end{aligned} \tag{2.13}$$

Multiplying by u_t, integrating by parts, and using the continuity of (2.1), we get

$$\begin{aligned}&\frac12\frac{d}{dt}\int_\Omega \rho|u_t|^2 + \mu\int_\Omega |\nabla u_t|^2\\ &= \lambda\int_\Omega ((\nabla d\otimes\nabla d)_t - \tfrac12|\nabla d|_t^2 I)\cdot\nabla u_t\\ &\qquad - \int_\Omega \nabla p_t\cdot u_t - (\rho v\cdot\nabla)v_t\cdot u_t - (\rho_t v\cdot\nabla)v\cdot u_t - \int_\Omega (\rho v_t\cdot\nabla)v\cdot u_t + \rho_t|u_t|^2\\ &\le 3\lambda\int_\Omega |\nabla d||\nabla d_t||\nabla u_t| + \int_\Omega p_t\,\mathrm{div}(u_t) + \rho|v||\nabla v_t||u_t|\\ &\qquad + \int_\Omega \rho|v||\nabla v|^2|u_t| + \rho|v|^2|\nabla^2 v||u_t| + \rho|v||\nabla v||\nabla u_t|\\ &\qquad\qquad + \int_\Omega \rho|v_t||\nabla v||u_t| + 2\rho|v||\nabla u_t||u_t|\\ &= \sum_{i=1}^8 I_i.\end{aligned}$$

For each I_i we have

$$I_1=3\lambda\int_\Omega|\nabla d||\nabla d_t||\nabla u_t|\le c\int_\Omega|\nabla d|^2|\nabla d_t|^2+\frac{\mu}{12}\int_\Omega|\nabla u_t|^2\le cB^4+\frac{\mu}{12}\int_\Omega|\nabla u_t|^2,$$

$$\begin{aligned}I_2&=\int_\Omega p_t\,\mathrm{div}(u_t)\le c\int_\Omega|p_t|^2+\frac{\mu}{12}\int_\Omega|\nabla u_t|^2\\&\le c_0\exp\left(\int_0^T 2\|\nabla v\|_{L^\infty(\Omega)}\right)\int_\Omega|\nabla v|^2+\frac{\mu}{12}\int_\Omega|\nabla u_t|^2\\&\le c_0A^2\exp(cAT^{\frac12})+\frac{\mu}{12}\int_\Omega|\nabla u_t|^2,\end{aligned}$$

$$I_3=\int_\Omega|\rho||v||\nabla v_t||u_t|\le A^4\int_\Omega\rho|u_t|^2+c_0A^{-2}\exp(cAT^{\frac12})\int_\Omega|\nabla v_t|^2,$$

$$I_4=\int_\Omega|\rho||v||\nabla v|^2|u_t|\le A^6\int_\Omega\rho|u_t|^2+c_0\exp(cAT^{\frac12}),$$

$$I_5=\int_\Omega|\rho||v|^2|\nabla^2 v||u_t|\le A^6\int_\Omega\rho|u_t|^2+c_0\exp(cAT^{\frac12}),$$

$$\begin{aligned}I_6&=\int_\Omega\rho|v||\nabla v||\nabla u_t|\le c\int_\Omega\rho^2|v|^2|\nabla v|^2+\frac{\mu}{12}\int_\Omega|\nabla u_t|^2\\&\le c_0A^4\exp(cAT^{\frac12})+\frac{\mu}{12}\int_\Omega|\nabla u_t|^2,\end{aligned}$$

$$\begin{aligned}I_7&=\int_\Omega\rho|v_t||\nabla v||u_t|\le A^4\int_\Omega\rho|u_t|^2+A^{-4}\int_\Omega\rho|v_t|^2|\nabla v|^2\\&\le A^2\int_\Omega\rho|u_t|^2+c_0A^{-2}\exp(cAT^{\frac12})\int_\Omega|v_t|^2,\end{aligned}$$

$$\begin{aligned}I_8&=2\int_\Omega\rho|v||\nabla u_t||u_t|\le c\int_\Omega\rho|u_t|^2(\rho|v|^2)+\frac{\mu}{12}\int_\Omega|\nabla u_t|^2\\&\le c_0A^2\exp(cAT^{\frac12})\int_\Omega\rho|u_t|^2+\frac{\mu}{12}\int_\Omega|\nabla u_t|^2.\end{aligned}$$

From the above estimates, we get

$$\begin{aligned}&\int_\Omega\rho|u_t|^2+\int_0^T\int_\Omega|\nabla u_t|^2\\&\qquad\le cB^4T+c_0A^4T\exp(cAT^{\frac12})+c_0+c_0A^4\exp(cAT^{\frac12})\int_0^T\int_\Omega\rho|u_t|^2,\end{aligned}$$

which implies that

$$\int_\Omega\rho|u_t|^2+\int_0^T\int_\Omega|\nabla u_t|^2\le(cB^4T+c_0A^4T\exp(cAT^{\frac12}))c_0A^4T\exp(cAT^{\frac12}).$$

Taking $T = T(A, B)$ small enough, we deduce

$$\int_\Omega \rho|u_t|^2 + \int_0^T \int_\Omega |\nabla u_t|^2 \le C(c_0). \tag{2.14}$$

Finally, we estimate

$$\|u\|_{L^\infty(0,T;H^2(\Omega))} \quad \text{and} \quad \|u\|_{L^2(0,T;W^{2,6}(\Omega))}.$$

From (2.2), we get

$$\begin{aligned}&\|u\|_{H^2(\Omega)}\\ &\quad \le c(\|\nabla p\|_{L^2(\Omega)} + \|\rho u_t\|_{L^2(\Omega)} + \|\nabla^2 n \nabla n\|_{L^2(\Omega)}) + c(\|(\rho v\cdot\nabla)v\|_{L^2(\Omega)} + c_0).\end{aligned}$$

Now we have

$$\begin{aligned}\|\nabla p\|_{L^2(\Omega)} &\le c_0 \exp(cAT^{\frac12}) + c_0 A T^{\frac12}\exp(cAT^{\frac12}),\\ \|\rho u_t\|_{L^2(\Omega)} &\le c_0\exp(cAT^{\frac12})\|\sqrt{\rho}u_t\|_{L^2(\Omega)},\\ \|\nabla^2 n\nabla n\|_{L^2(\Omega)} &\le \|\nabla^2 n\|_{L^6(\Omega)}\|\nabla n\|^{\frac12}_{L^2(\Omega)}\|\nabla n\|^{\frac12}_{L^6(\Omega)} \le K_1^2,\end{aligned}$$

and

$$\begin{aligned}&\|\rho v\cdot\nabla v\|^2_{L^2(\Omega)}\\ &\qquad \le \|\rho\|^2_{L^\infty(\Omega)}\int_\Omega |v|^2|\nabla v|^2\\ &\qquad \le c_0\exp(cAT^{\frac12})\Big(\int_\Omega |v-u_0|^2|\nabla v|^2 + \|u_0\|^2_{L^\infty}\int_\Omega|\nabla v - \nabla u_0|^2 + c_0\Big)\\ &\qquad \le c_0\exp(cAT^{\frac12})\Big(\int_\Omega\Big|\int_0^t v_t\Big|^2|\nabla v|^2 + c_0\int_\Omega\Big|\int_0^t \nabla v_t\Big|^2 + c_0\Big)\\ &\qquad \le c_0\exp(cAT^{\frac12})(A^4T + c_0A^2T + c_0).\end{aligned}$$

Similarly, we have

$$\begin{aligned}\|\nabla p\|_{L^6(\Omega)} &\le c_0\exp(cAT^{\frac12}) + c_0AT^{\frac12}\exp(cAT^{\frac12}),\\ \|\rho u_t\|_{L^2(0,T;L^6(\Omega))} &\le c_0\exp(cAT^{\frac12})\|\nabla u_t\|_{L^2(0,T;L^2(\Omega))}\\ &\le c_0\exp(cAT^{\frac12})C(c_0),\\ \|\nabla^2 n\nabla n\|_{L^2(0,T;L^6(\Omega))} &\le \|\nabla^2 n\|_{L^2(0,T;L^6(\Omega))}\|\nabla n\|_{L^\infty(\Omega)} \le K_1^2,\end{aligned}$$

and

$$\begin{aligned}
&\|\rho v\cdot\nabla v\|^2_{L^2(0,T;L^6(\Omega))}\\
&\qquad\le \|\rho\|^2_{L^\infty(\Omega)}\int_0^T\Big(\int_\Omega |v|^6|\nabla v|^6\Big)^{\frac13}\\
&\qquad\le c_0\exp(cAT^{\frac12})\int_0^T \|v\|^2_{L^\infty(\Omega)}\|\nabla v\|^{\frac43}_{L^\infty(\Omega)}\times\Big(\int_\Omega|\nabla v-\nabla u_0|^2+1\Big)^{\frac13}\\
&\qquad\le c_0\exp(cAT^{\frac12})A^2\int_0^T\|\nabla v\|^{\frac43}_{L^\infty(\Omega)}\times\Big(\int_\Omega\Big|\int_0^t\nabla v_t\Big|^2+1\Big)^{\frac13}\\
&\qquad\le c_0\exp(cAT^{\frac12})\Big(T\int_0^T\int_\Omega|\nabla v_t|^2+1\Big)^{\frac13}\times\Big(\int_0^T\|v\|^2_{W^{2,6}(\Omega)}\Big)^{\frac23}T^{\frac13}\\
&\qquad\le c_0\exp(cAT^{\frac12})(TA^2+1)^{\frac13}A^{\frac43}T^{\frac13}.
\end{aligned}$$

Thus

$$\int_\Omega\rho|u_t|^2dx+\mu\int_0^T\int_\Omega|\nabla u_t|^2\,dx\,dt+\|u\|_{L^\infty(0,T;H^2(\Omega))}+\|u\|_{L^2(0,T;W^{2,6}(\Omega))}\le C(c_0).$$

This concludes the proof. □

If (n^δ, u^δ) denotes a unique solution of (2.2) and (2.3) with

$$\rho(x,0)=\rho_0+\delta$$

and initial and boundary conditions, then taking $\delta\to 0$, we obtain a unique solution (n, u) of the linearized system (2.1)–(2.3) with $\rho(x,0)=\rho_0$ and initial and boundary conditions such that $\|n\|_{\mathscr{B}}\le K_1$, $\|u\|_{\mathscr{A}}\le K_2$. So we can define a map

$$\mathscr{T}:\mathscr{W}\to\mathscr{W},\quad (d, v)\mapsto(n,u),$$

where Banach space

$$\mathscr{W}=(\mathscr{A}\otimes\mathscr{B})\cap\mathscr{C}=\mathscr{A}\otimes\mathscr{B}$$

with

$$\mathscr{C}=\{(n,u):\|(n,u)\|_{\mathscr{C}}=\|n\|_{L^2(0,T;H^2(\Omega))}+\|u\|_{L^2(0,T;H^1(\Omega))}<\infty\}.$$

The following lemma tells us that the map $\mathscr{T}$ is contracted in the sense of weaker norm for $(d, v)\in\mathscr{W}$.

Lemma 2.4. *There is a constant $0<\theta<1$ such that for any $(d^i, v^i)\in\mathscr{W}$, $i=1,2$,*

$$\|\mathscr{T}(d^1,v^1)-\mathscr{T}(d^2,v^2)\|_{\mathscr{C}}\le\theta\|(d^1-d^2,v^1-v^2)\|_{\mathscr{C}}$$

for some small $T>0$.

Proof. Suppose ρ_i, n^i, and u^i are the solutions to (2.1)–(2.3) corresponding to given $(d^i, v^i) \in \mathcal{W}$. Define $\rho = \rho_2 - \rho_1$, $d = d^2 - d^1$, $v = v^2 - v^1$, $n = n^2 - n^1$, $u = u^2 - u^1$, and

$$\rho_i = \rho_0 \exp\left(-\int_0^t \operatorname{div} v^i\right),$$

$i = 1, 2$. Then

$$\rho_t + \operatorname{div}(\rho v^2) = -\operatorname{div}(\rho_1 v), \tag{2.15}$$

$$n_t - \nu\Delta n = \nu|\nabla d^2|^2 d^2 - \nu|\nabla d^1|^2 d^1 - v^2\nabla d^2 + v^1\nabla d^1, \tag{2.16}$$

$$\begin{aligned}\rho_2 u_t - \mu\Delta u = (\rho_1 - \rho_2)u_t^1 + \rho_1 v^1\nabla v^1 - \rho_2 v^2\nabla v^2 + \nabla p_1 \\ - \nabla p_2 - \lambda\nabla\cdot(\nabla n^2 \otimes \nabla n^2 - \tfrac{1}{2}|\nabla n^2|^2 I) \\ + \lambda\nabla\cdot(\nabla n^1 \otimes \nabla n^1 - \tfrac{1}{2}|\nabla n^1|^2 I).\end{aligned} \tag{2.17}$$

Multiplying (2.16) by n and integrating over Ω, we get

$$\begin{aligned}\tfrac{1}{2}\frac{d}{dt}\int_\Omega |n|^2\,dx + \nu\int_\Omega |\nabla n|^2\,dx \\ \le \int_\Omega |\nabla d^2|^2 d^2\cdot n - |\nabla d^1|^2 d^1\cdot n - v\nabla d^2\cdot n - v^1\nabla d\cdot n \\ \le \eta\int_\Omega (|\nabla d|^2 + |\nabla v|^2) + c(\eta, A, B)\int_\Omega |n|^2,\end{aligned} \tag{2.18}$$

where $c(\eta, A, B)(s)$ satisfies

$$\int_0^T c(\eta, A, B)(s)ds \le K_3 \tag{2.19}$$

for small $T = T(A, B, \eta)$, where K_3 is a constant dependent on initial and boundary data c_0.

Differentiating (2.16) with respect to x_i, multiplying by ∇n, and integrating over Ω, we deduce

$$\begin{aligned}\tfrac{1}{2}\frac{d}{dt}\int_\Omega |\nabla n|^2\,dx + \frac{\nu}{2}\int_\Omega |\nabla^2 n|^2\,dx \\ \le \eta\int_\Omega (|\nabla v|^2 + |\nabla d|^2 + |\nabla^2 d|^2) + c(\eta, A, B)\int_\Omega |\nabla n|^2,\end{aligned} \tag{2.20}$$

where $c(\eta, A, B)$ satisfies (2.19), and we have used the following identities and estimates:

$$\nabla d^2\nabla^2 d^2 d^2 - \nabla d^1\nabla^2 d^1 d^1 = \nabla d\nabla^2 d^2 d^1 + \nabla d^1\nabla^2 d d^1 + \nabla d^1\nabla^2 d^1 d,$$
$$|\nabla d^2|^2\nabla d^2 - |\nabla d^1|^2\nabla d^1 = |\nabla d^2|^2\nabla d + (|\nabla d^2|^2 - |\nabla d^1|^2)\nabla d^1,$$

and

$$\int_\Omega |\nabla n|^2|\nabla^2 d^2|^2 \le \left(\int_\Omega |\nabla^2 d^2|^6\right)^{\frac{1}{3}}\left(\int_\Omega |\nabla n|^3\right)^{\frac{2}{3}}$$
$$\le cB^2\|\nabla n\|_{L^2(\Omega)}\|\nabla^2 n\|_{L^2(\Omega)} \le \frac{\nu}{2}\int_\Omega |\nabla^2 n|^2 + cB^4\int_\Omega |\nabla n|^2.$$

Multiplying (2.15) by ρ and using the Minkowski inequality, we have

$$\begin{aligned}\frac{d}{dt}\int_\Omega \tfrac{1}{2}|\rho|^2 &= \int_\Omega -\tfrac{1}{2}|\rho|^2 \operatorname{div} v^2 - \int_\Omega \rho(\nabla\rho_1 v + \rho_1 \operatorname{div} v)\\
&\le c\int_\Omega |\rho|^2|\nabla v^2| + c\|\rho\|_{L^2(\Omega)}\|\nabla\rho_1\|_{L^3(\Omega)}\|v\|_{L^6(\Omega)}\\
&\qquad + c\|\rho\|_{L^2(\Omega)}\|\rho_1\|_{L^\infty(\Omega)}\|\nabla v\|_{L^2(\Omega)}\\
&\le c\|v^2\|_{W^{2,6}(\Omega)}\|\rho\|^2_{L^2(\Omega)} + \eta\|\nabla v\|^2_{L^2(\Omega)}\\
&\qquad + c_0\eta^{-1}\exp(cAT^{\frac{1}{2}})\left(1 + \left\|\int_0^t \nabla^2 v^1\right\|^2_{L^3(\Omega)}\right)\|\rho\|^2_{L^2(\Omega)}\\
&\le \eta\|\nabla v\|^2_{L^2(\Omega)} + c\|v^2\|_{W^{2,6}(\Omega)}\|\rho\|^2_{L^2(\Omega)}\\
&\qquad + c_0\eta^{-1}\exp(cAT^{\frac{1}{2}})(1 + T\|\nabla^2 v^1\|^2_{L^2(0,T;L^6(\Omega))})\|\rho\|^2_{L^2(\Omega)}\\
&\le c_0\eta^{-1}\exp(cAT^{\frac{1}{2}})(1 + TA^2 + \|v^2\|_{W^{2,6}(\Omega)})\|\rho\|^2_{L^2(\Omega)} + \eta\|\nabla v\|^2_{L^2(\Omega)},\end{aligned}$$

that is,

$$\frac{d}{dt}\int_\Omega \tfrac{1}{2}|\rho|^2 \le \eta\|\nabla v\|^2_{L^2(\Omega)} + c(\eta, A, T)\|\rho\|^2_{L^2(\Omega)}, \tag{2.21}$$

where $c(\eta, A, T)$ satisfies (2.19).

Multiplying (2.17) by u and integrating over Ω, we deduce

$$\begin{aligned}&\tfrac{1}{2}\frac{d}{dt}\int_\Omega \rho_2|u|^2\,dx + \mu\int_\Omega |\nabla u|^2\,dx \qquad (2.22)\\
&= \int_\Omega -\rho_2 v^2 u\nabla u + (\rho_1-\rho_2)u^1_t\cdot u + \rho_1 v^1\nabla v^1\cdot u - \rho_2 v^2\nabla v^2\cdot u + (p_2-p_1)\operatorname{div} u\\
&\qquad + \lambda(\nabla n^2\otimes\nabla n^2 - \tfrac{1}{2}|\nabla n^2|^2 I)\nabla u - \lambda(\nabla n^1\otimes\nabla n^1 - \tfrac{1}{2}|\nabla n^1|^2 I)\nabla u\\
&= \int_\Omega -\rho_2 v^2 u\nabla u + (\rho_1-\rho_2)(u^1_t + v^1\nabla v^1)\cdot u\\
&\qquad - \rho_2(v\nabla v^2 + v^1\nabla v)\cdot u + (p_1-p_2)\operatorname{div} u\\
&\qquad + \lambda(\nabla n^2\otimes\nabla n^2 - \tfrac{1}{2}|\nabla n^2|^2 I)\nabla u - \lambda(\nabla n^1\otimes\nabla n^1 - \tfrac{1}{2}|\nabla n^1|^2 I)\nabla u\\
&\le \eta\int_\Omega |\nabla v|^2 + \frac{2\mu}{3}\int_\Omega |\nabla u|^2 + c(\eta, A, B)\int_\Omega \rho_2|u|^2 + |\rho|^2 + |\nabla n|^2,\end{aligned}$$

where $c(\eta, A, B)$ satisfying (2.19). Here we have used the key estimates

$$\begin{aligned}\int_\Omega \rho_2|v\nabla v^2+v^1\nabla v||u| &\le \|\nabla v^2\|_{L^6(\Omega)}\|\rho_2 u\|_{L^2(\Omega)}\|v\|_{L^6(\Omega)} \\ &\qquad + \|\nabla v\|_{L^2(\Omega)}\|\rho_2 u\|_{L^2(\Omega)}\|v^1\|_{L^\infty(\Omega)} \\ &\le c_0\exp(cAT^{\frac12})\|\sqrt{\rho_2}u\|_{L^2(\Omega)}\|\nabla v^2\|_{H^2(\Omega)}\|\nabla v\|_{L^2(\Omega)} \\ &\qquad + c_0\exp(cAT^{\frac12})\|\sqrt{\rho_2}u\|_{L^2(\Omega)}\|v^1\|_{H^2(\Omega)}\|\nabla v\|_{L^2(\Omega)} \\ &\le \eta\|\nabla v\|^2_{L^2(\Omega)} + c\eta^{-1}A^2\exp(cAT^{\frac12})\|\sqrt{\rho_2}u\|^2_{L^2(\Omega)}, \\ \int_\Omega |\nabla n||\nabla u||\nabla n^2| &\le \eta\int_\Omega |\nabla u|^2 + c\eta^{-1}|\nabla n^2|^2_{L^\infty(\Omega)}\int_\Omega |\nabla n|^2 \\ &\le \frac{\mu}{3}\int_\Omega |\nabla u|^2 + cB^2\int_\Omega |\nabla n|^2,\end{aligned}$$

and

$$\begin{aligned}\int_\Omega (\rho_1-\rho_2)(u_t^1+v^1\nabla v^1)\cdot u &\le \|\rho\|_{L^{\frac32}(\Omega)}\|u_t^1+v^1\nabla v^1\|_{L^6(\Omega)}\|u\|_{L^6(\Omega)} \\ &\le c\|\rho\|_{L^2(\Omega)}\|u_t^1+v^1\nabla v^1\|_{H^1(\Omega)}\|\nabla u\|_{L^2(\Omega)} \\ &\le \frac{\mu}{3}\|\nabla u\|^2_{L^2(\Omega)} + c(A,T)(t)\|\rho\|^2_{L^2(\Omega)},\end{aligned}$$

where $c(\eta, A, T)(t)$ satisfies (2.19).

Summing inequalities (2.18) and (2.20)–(2.22), we obtain

$$\begin{aligned}&\frac{d}{dt}\int_\Omega |\rho|^2+|n|^2+|\nabla n|^2+\rho_2|u|^2 + \int_\Omega |\nabla n|^2+|\nabla^2 n|^2+|\nabla u|^2 \\ &\le c\eta\int_\Omega |\nabla v|^2+|\nabla d|^2+|\nabla^2 d|^2 + c(\eta, A, B, T)\int_\Omega |\rho|^2+|n|^2+|\nabla n|^2+\rho_2|u|^2,\end{aligned}$$

which implies, by (2.19) and taking $T = T(\eta, A, B))$ small enough,

$$\begin{aligned}&\int_\Omega |\rho|^2+|n|^2+|\nabla n|^2+\rho_2|u|^2 \\ &\qquad \le \eta\exp\left(\int_0^T c(\eta, A, B)(s)ds\right)\int_0^T\int_\Omega |\rho|^2+|n|^2+|\nabla n|^2+\rho_2|u|^2 \\ &\qquad \le c\eta\int_0^T\int_\Omega |\rho|^2+|n|^2+|\nabla n|^2+\rho_2|u|^2.\end{aligned}$$

Thus, taking η small, we obtain

$$\|\rho\|_{L^\infty(0,T;L^2(\Omega))} + \|n\|_{L^\infty(0,T;H^1(\Omega))} + \|\sqrt{\rho_2}u\|_{L^\infty(0,T;L^2(\Omega))} \le c \tag{2.23}$$

and

$$\int_0^T \int_\Omega |\nabla n|^2 + |\nabla^2 n|^2 + |\nabla u|^2 \le \theta \int_0^T \int_\Omega |\nabla d|^2 + |\nabla^2 d|^2 + |\nabla v|^2$$

with $0 < \theta < 1$. Since n and u are zero on boundary, we finish the proof. □

3. Proof of Theorem 1.1

Proof. By the contractibility of $\mathcal{T}$, we can easily obtain a unique solution (n, u) of (1.3) and (1.2), and ρ is from u by formula (1.8), that is, ρ is a unique solution of (1.1). Lemmas 2.1–2.3 and the lower semicontinuity of norms imply that the solutions (ρ, n, u) satisfy the same estimates. Multiplying (1.3) by n, we get

$$|n|_t^2 + (u \cdot \nabla)|n|^2 = \nu \Delta |n|^2 + (|n|^2 - 1)|\nabla n|^2,$$

that is,

$$(|n|^2 - 1)_t + (u \cdot \nabla)(|n|^2 - 1) = \nu \Delta (|n|^2 - 1) + (|n|^2 - 1)|\nabla n|^2.$$

Define $D = (|n|^2 - 1)\exp(\|\nabla n\|^2_{L^\infty(Q_T)} t)$, where $Q_T = \Omega \times [0, T]$. Then

$$D_t + (u \cdot \nabla) D = \nu \Delta D + (|\nabla n|^2 - \|\nabla n\|^2_{L^\infty(Q_T)}) D$$

with $D|_{\partial\Omega} = 0$. So from the maximum principle of parabolic equations,we deduce

$$D \equiv 0 \quad \text{in} \quad ((0, T) \times \Omega).$$

Thus we complete the proof of the theorem. □

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