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PRESENTATIONS FOR THE HIGHER-DIMENSIONAL THOMPSON GROUPS nV

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M. G. Brin has introduced the higher-dimensional Thompson groups nV that are generalizations to the Thompson group V of self-homeomorphisms of the Cantor set and found a finite set of generators and relations in the case $n = 2$. We show how to generalize his construction to obtain a finite presentation for every positive integer n. As a corollary, we obtain another proof that the groups nV are simple (first proved by Brin).

1. Introduction

The higher-dimensional groups nV were introduced by Brin in [2004; 2005] and generalize Thompson's group V. The group V is a group of self-homeomorphisms of the Cantor set $\mathfrak{C}$ that is simple and finitely presented — the standard introduction to V is the paper by Cannon, Floyd and Parry [1996]. The groups nV generalize the group V and act on powers of the Cantor set $\mathfrak{C}^n$. Brin shows in [2004] that the groups V and $2V$ are not isomorphic and shows in [2005] that the group $2V$ is finitely presented. Bleak and Lanoue [2010] have recently shown that two groups mV and nV are isomorphic if and only if $m = n$.

In this paper we give a finite presentation for each of the higher-dimensional Thompson groups nV. The argument extends to the ascending union ωV of the groups nV and returns an infinite presentation of the same flavor. As a corollary, we obtain another proof that the groups nV and ωV are simple. Our arguments follow closely and generalize those of Brin in [2004; 2005] for the group $2V$.

This work arose during a Research Experience for Undergraduates program at Cornell University. The motivation for the project sprang from a commonly held opinion that the bookkeeping required to generalize Brin's presentations to the groups nV would be overwhelming. One would expect from the similarity of the groups' constructions that all arguments for $2V$ would carry over to nV for all n. Standing in the way of this are the cross relations. Thus our paper has two kinds

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of arguments: those that verify the parts of [Brin 2005] that carry over with no change to nV and those involving the cross relations that have to be modified to hold in nV (see Lemmas 6 and 20 and Remark 13 below).

Following a suggestion of Collin Bleak the authors have also explored an alternative generating set (see Section 8). An interesting project would be to find a set of relators for this alternative generating set in order to use a known procedure that significantly reduces the number of relations, and which has been successfully implemented in a number of papers by Guralnick, Kantor, Kassabov and Lubotzky; see for example [Guralnick et al. 2011].

After a careful reading of Brin's original paper [2005], it became clear what was needed to generalize his proof, and the current paper borrows heavily from Brin's. Brin was already aware that many of his arguments would probably extend (and he points out in several places in [2004; 2005] where it is evident that they do). We show how to deal with generators in higher dimensions and what steps are needed to obtain the same type of normalized words that are built for $2V$ in [Brin 2005].

We also mention that Brin asks in [2005] whether or not the group $2V$ has type F_∞ (that is, it has a classifying space that is finite in each dimension). This has recently been answered by Kochloukova, Martinez-Perez and Nucinkis [2010], who have shown that the groups $2V$ and $3V$ have type F_∞, therefore obtaining a new proof that these groups are finitely presented.

2. The main ingredient and structure of this paper

Many arguments of Brin [2004; 2005] generalize verbatim from $2V$ to nV. The key observation that allows us to restate many results without proof (or with little additional effort) is the following: Many statements of Brin do not depend on dimension 2, except those that need to make use of the "cross relation" (relation (18) in Section 4 below) to rewrite a cut in dimension d followed by a cut in dimension d' as one in dimension d' followed by one in dimension d.

As a result, proofs that need to make use of this new relation require a slight generalization (for example, the normalization of words in the monoid across fully divided dimensions) while those that do not can be obtained directly using Brin's original proof. In any case, since statements need to be adapted to our context we sketch certain proofs to make it clear that they generalize directly. For example, we will show why Brin's proof that $2V$ is simple does not use the new relation (18) and therefore it lifts immediately to higher dimensions.

3. The monoid Π_n

In [2004, Section 4.5], Brin defines the monoid Π and $\widehat{2V}$ and observes that one can extend the definition for all n. Elements of Π_n are given by numbered patterns

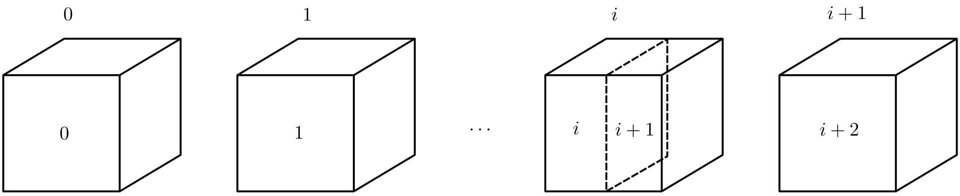

Figure 1. The generator $s_{i,d}$.

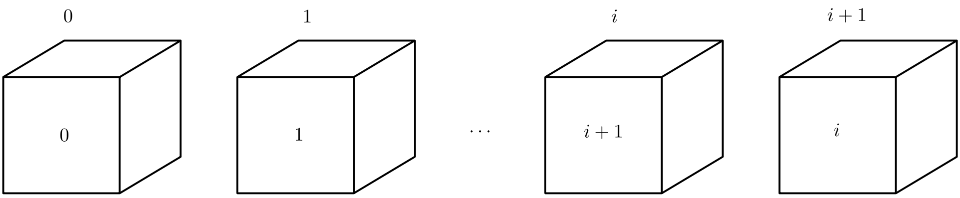

Figure 2. The generator σ_i.

in X, where X is the union of the set $\{S_0, S_1, \dots\}$ of unit n-cubes. Fix $n \in \mathbb{N}$ and fix an ordering on the dimensions d for $1 \leq d \leq n$. The monoid Π_n is generated by the elements $s_{i,d}$ and σ_i, where $s_{i,d}$ denotes the element that cuts the rectangle S_i in half across the d-th dimension (see Figure 1) and σ_i is the transposition that switches the rectangle labeled i with that labeled $i+1$, as defined for $2V$ (see Figure 2).

After each cut, the numbering shifts as before. The following relations hold in Π_n.

(M1) $$s_{j,d'} s_{i,d} = s_{i,d} s_{j+1,d'} \qquad \text{for } i < j,\ 1 \leq d, d' \leq n,$$

(M2) $$\sigma_i^2 = 1 \qquad \text{for } i \geq 0,$$

(M3) $$\sigma_i \sigma_j = \sigma_j \sigma_i \qquad \text{for } |i-j| \geq 2,$$

(M4) $$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \qquad \text{for } i \geq 0,$$

(M5a) $$\sigma_j s_{i,d} = s_{i,d} \sigma_{j+1} \qquad \text{for } i < j,$$

(M5b) $$\sigma_j s_{i,d} = s_{j+1,d} \sigma_j \sigma_{j+1} \qquad \text{for } i = j,$$

(M5c) $$\sigma_j s_{i,d} = s_{j,d} \sigma_{j+1} \sigma_j \qquad \text{for } i = j+1,$$

(M5d) $$\sigma_j s_{i,d} = s_{i,d} \sigma_j \qquad \text{for } i > j+1,$$

(M6) $$s_{i,d} s_{i+1,d'} s_{i,d'} = s_{i,d'} s_{i+1,d} s_{i,d} \sigma_{i+1} \qquad \text{for } i \geq 0,\ d \neq d'.$$

Relations (M5b) and (M5c) are actually equivalent, because σ_i is its own inverse.

Remark 1. The proofs of [Brin 2005, Section 2] that use relations (M1)–(M5d) do not depend on the dimension being 2. For this reason, they generalize immediately to the case of the monoid Π_n and we do not prove them again. This includes every result up to and including [Brin 2005, Lemma 2.9].

On the other hand, Proposition 2.11 in [Brin 2005] uses the cross relation (M6) and it requires us to choose how we write elements to get some underlying pattern. Brin achieves this type of normalization by writing elements so that vertical cuts appear first, whenever possible. We generalize his argument by describing how to order nodes in forests (which represent cuts in some dimension).

The following definition is given inductively on the subtrees.

Definition 2. Given a forest F, we say that a subtree T of some tree of F is *fully divided* across some dimension d if the root of T is labeled d or if both its left and right subtrees are fully divided across dimension d. We say a forest F is *normalized* if every subtree T is such that if T is fully divided across different the dimensions $d_1 < d_2 < \cdots < d_u$, then the root of T is labeled with d_1, the lowest among all possible dimensions over which T is fully divided.

Given that a word w is a word in the generators $\{s_{i,d}, \sigma_i\}$, we define the *length* $\ell(w)$ of w to be the number of times an element of $\{s_{i,d}\}$ appears in w. It can easily be seen that the length of a word is preserved by relations (M1)–(M6).

We restate some results adapted to our case.

Lemma 3 [Brin 2005, Lemma 2.7]. *If the numbered, labeled forest F comes from a word in $\{s_{i,d} \mid d, i \in \mathbb{N}\}$, then the leaves of F are numbered so that the leaves in F_i have numbers lower than those in F_j whenever $i < j$ and the leaves in each tree of F are numbered in increasing order under the natural left-right ordering of the leaves.*

Lemma 4 [Brin 2005, Lemma 2.8]. *If two words in the generators*

$$\{s_{i,d}, \sigma_i \mid i \in \mathbb{N}, 1 \le d \le n\}$$

lead to the same numbered, labeled forest, then they are related by (M1)–(M5d).

Lemma 5 [Brin 2005, Lemma 2.9]. *If F is a numbered, labeled forest with the numbering as in Lemma 3, and if a linear order is given on the interior vertices (and thus of the carets) of F that respects the ancestor relation, then there is a unique word w in $\{s_{i,d} \mid d, i \in \mathbb{N}\}$ leading to F such that the order on the interior vertices of F derived from the order on the entries in w is identical to the given linear order on the interior vertices.*

The next lemma and corollary are used to prove results analogous to [Brin 2005, Lemma 2.10 and Proposition 2.11].

Lemma 6. *Let w be a word in the set $\{s_{i,d}, \sigma_i\}$ and suppose that the underlying pattern P has a fully divided hypercube S_i across dimension d. Then $w \sim w' = s_{i,d}a$ for some word $a \in \langle s_{i,d}, \sigma_i \rangle$.*

Proof. We use induction on $g := \ell(w)$. By using relations (M5a)–(M5d) as in [Brin 2005, Lemma 2.3] we can assume that $w = pq$, where $p \in \langle s_{i,d} \rangle$ and $q \in \langle \sigma_i \rangle$. This does not alter the length of w. If $g = 3$, then $p = p_1 p_2 p_3$. If $p_1 = s_{i,d}$, we are done; otherwise we have two cases: either $p_2 = s_{i+1,d}$ and $p_3 = s_{i,d}$ or $p_2 = s_{i,d}$ and $p_3 = s_{i+2,d}$. Up to using relation (M1), we can assume that $p_2 = s_{i+1,d}$ and $p_3 = s_{i,d}$ which is what we want to apply relation (M6) to p to get $w \sim w' = s_{i,d} s_{i+1,k} s_{i,k} q$.

Now assume the thesis true for all words of length less than g. We consider the word p and look at the labeled unnumbered tree F_i corresponding to S_i with root vertex u and children u_0 and u_1. Let T_r be the subtree of F_i with root vertex u_r for $r = 0, 1$. We choose an ordering of the vertices of F_i that respects the ancestor relation and such that u corresponds to 1, u_0 corresponds to 2, the other interior nodes of T_0 correspond to the numbers from 3 to $j = \#(\text{interior nodes of } T_0)$ and u_2 corresponds to $j+1$.

By Lemma 5, the word p is equivalent to

$$p \sim s_{i,k}(s_{i,m} p_0)(s_{f,l} p_1),$$

where $s_{i,m} p_0$ is the subword corresponding to the subtree T_0 and $s_{f,l} p_1$ is the subword corresponding to the subtree T_1 and with $p_0, p_1 \in \langle s_{i,d} \rangle$. We observe that

$$\ell(s_{i,m} p_0) < \ell(p) = g \quad \text{and} \quad \ell(s_{f,l}\, p_1) < \ell(p) = g$$

and that the underlying squares S_i for $s_{i,m} p_0$ and S_{i+1} for $s_{f,l}\, p_1$ are fully divided across dimension d. We can thus apply the induction hypothesis and rewrite

$$s_{i,m} p_0 \sim s_{i,d} \tilde{p}_0 \tilde{q}_0 \quad \text{and} \quad s_{f,l}\, p_2 \sim s_{f,d} \tilde{p}_1 \tilde{q}_1.$$

We restrict our attention to the subword $s_{i,d} \tilde{p}_0 \tilde{q}_0 s_{f,d}$. Using the relations (M5a)–(M5d), we can move $\tilde{q}_0$ to the right of $s_{f,d}$ and obtain

$$s_{i,d} \tilde{p}_0 \tilde{q}_0 s_{f,d} \sim s_{i,d} \tilde{p}_0 s_{g,d} \tilde{q}$$

for some permutation word $\tilde{q}$. Since the word $\tilde{p}_0$ acts on the rectangle S_i and $s_{g,d}$ acts on the rectangle S_{i+1}, we can apply Lemma 4 and 5 and put a new order on the nodes so that the node corresponding to $s_{i,d}$ is 1 and $s_{g,d}$ is 2. Thus we have

$$s_{i,d} \tilde{p}_0 s_{g,d} \tilde{q} \sim s_{i,d} s_{i+2,d} \tilde{p} \tilde{q}$$

for some $\tilde{p}$ word in the set $\{s_{i,d}\}$. Thus we have $w \sim w'' = s_{i,k} s_{i,d} s_{i+2,d} \tilde{p}\, \tilde{q}$ and so, by applying the cross relation (M6) to the first three letters of w'', we get

$$w \sim w'' \sim w' = s_{i,d} s_{i,k} s_{i+2,k} \tilde{p} \tilde{q} = s_{i,d} a. \qquad \square$$

We have now proved [Brin 2005, Lemma 2.10], since in order for a tree in a forest to be nonnormalized, one of the rectangles in the pattern corresponding to that tree must be fully divided across two different dimensions.

Lemma 7 [Brin 2005]. *If two different forests correspond to the same pattern in X, then at least one of the two forests is not normalized.*

Remark 8. Lemma 6 is used in our extension of [Brin 2005, Proposition 2.11] so that we can push dimension d under the root. This is explained better in the following corollary.

Corollary 9. *Let w be a word in the generators $\{s_{i,d}, \sigma_i\}$ such that its underlying square S_i is fully divided across dimensions d and ℓ. Then*

$$w \sim w' = s_{i,d} s_{i,\ell} s_{i+2,\ell} a \sim w'' = s_{i,\ell} s_{i,d} s_{i+2,d} b$$

for some suitable words a and b in the generators $\{s_{i,d}, \sigma_i\}$.

Proof. This is achieved by a repeated application of Lemma 6. We apply it to w and obtain $w \sim s_{i,d} a_1$. By construction, the underlying squares S_i and S_{i+1} of a_1 are fully divided across dimension ℓ, so we can apply the previous lemma to a_1 to get $a_1 \sim s_{i,\ell} a_2$ and finally we apply it again to $a_2 \sim s_{i+2,\ell} a$. Hence $w \sim w' = s_{i,\ell} s_{i+2,\ell} a$. To get w'' we apply the cross relation (M6) to the subword $s_{i,\ell} s_{i,d} s_{i+2,d}$. □

Proposition 10. *A word w is related by (M1)–(M6) to a word corresponding to a normalized, labeled forest.*

Proof. We proceed by induction on the length of w. Let g be the length of w and assume the result holds for all words of length less than g. As before, write $w = pq$, where $p = s_{i_0} s_{i_1} \cdots s_{i_{n-1}}$ (here, the i_j refers to the cube that is being cut; we omit the second index indicating dimension as it is unimportant for now). Write $w = s_{i_0} w'$; since the order of the interior vertices of the forest for p given by the order of the letters in p must respect the ancestor relation, we know that the interior vertex corresponding to s_{i_0} must be a root of some tree T. As w' is a word of length less than g, we may apply our inductive hypothesis and assume that w' can be rewritten via relations (M1)–(M6) to obtain a corresponding normalized forest. The pattern P for w is obtained from the pattern P' for w' by applying the pattern of P' in unit square S_i to the rectangle numbered i in the pattern for s_{i_0}. The forest F for w is obtained from the forest F' for w' by attaching the i-th tree of F' to the i-th leaf of the forest for s_{i_0}. Since F' is normalized, it is seen that F has all interior vertices normalized except possibly for the root vertex of one tree, T.

Let u be the root vertex of T with label k and with children u_1 and u_2. Let T_1 and T_2 be the subtrees of T whose roots are u_1 and u_2, respectively. By hypothesis, T_1 and T_2 are already normalized. If T is not normalized already, then T must be fully divided across the dimension k that u is labeled with, and some other dimension less than k. Let d be the minimal dimension across which T is fully divided. Since T_1 and T_2 are also fully divided across d, by Lemma 6, we may apply relations (M1)–(M6) to the subwords of w corresponding to T_1 and T_2 until u_1 and u_2 are each labeled d. Now by [Brin 2005, Lemma 2.9], we may assume $w = s_{i_0,k}s_{i_0,d}s_{i_0+2,d}w''$, where w'' is the remainder of w. We apply relation (M6) to obtain

$$w = s_{i_0,d}s_{i_0,k}s_{i_0+2,k}\sigma_{i_0}w''.$$

Now, we have normalized the vertex u, and we may now use the inductive hypothesis to renormalize the trees T_1 and T_2. The result is a normalized forest. □

The proof of the next result follows the argument of [Brin 2005, Theorem 1], using [Lemma 2.10] and Proposition 10 (to extend [Proposition 2.11]).

Theorem 11. *The monoid Π_n is presented by using the generators $\{s_{i,d}, \sigma_i\}$ and relations (M1)–(M6).*

4. Relations in nV

4.1. ***Generators for nV.*** The following generators are defined as in [Brin 2004] and analogous arguments show why they are a generating set for nV.

$$\begin{aligned}
X_{i,d} &= (s_{0,1}^{i+1}s_{1,d}, s_{0,1}^{i+2}) && \text{for } i \geq 0, 1 \leq d \leq n, \\
C_{i,d} &= (s_{0,1}^{i}s_{0,d}, s_{0,1}^{i+1}) && \text{for } i \geq 0, 2 \leq d \leq n, && \text{(baker's maps)}, \\
\pi_i &= (s_{0,1}^{i+2}\sigma_1, s_{0,1}^{i+2}) && \text{for } i \geq 0 && (\sigma_i \text{ defined as above}), \\
\bar{\pi}_i &= (s_{0,1}^{i+1}\sigma_0, s_{0,1}^{i+1}) && \text{for } i \geq 0
\end{aligned}$$

4.2. ***Relations involving cuts and permutations.*** In the following relations (1)–(7), the reader can assume that $1 \leq d, d' \leq n$ unless otherwise stated.

$$X_{q,d}X_{m,d'} = X_{m,d'}X_{q+1,d} \quad \text{for } m < q, \tag{1}$$

$$\pi_q X_{m,d} = X_{m,d}\pi_{q+1} \quad \text{for } m < q, \tag{2}$$

$$\pi_q X_{q,d} = X_{q+1,d}\pi_q\pi_{q+1} \quad \text{for } q \geq 0, \tag{3}$$

$$\pi_q X_{m,d} = X_{m,d}\pi_q \quad \text{for } m > q+1, \tag{4}$$

$$\bar{\pi}_q X_{m,d} = X_{m.d}\bar{\pi}_{q+1} \quad \text{for } m < q, \tag{5}$$

$$\bar{\pi}_m X_{m,1} = \pi_m\bar{\pi}_{m+1} \quad \text{for } m \geq 0, \tag{6}$$

$$X_{m,d}X_{m+1,d'}X_{m,d'} = X_{m,d'}X_{m+1,d}X_{m,d}\pi_{m+1} \quad \text{for } m \geq 0, d \neq d'. \tag{7}$$

4.3. ***Relations involving permutations only.*** We have

$$\pi_q\pi_m = \pi_m\pi_q \qquad \text{for } |m-q|>2, \tag{8}$$
$$\pi_m\pi_{m+1}\pi_m = \pi_{m+1}\pi_m\pi_{m+1} \qquad \text{for } m\ge 0, \tag{9}$$
$$\bar\pi_q\pi_m = \pi_m\bar\pi_q \qquad \text{for } q\ge m+2, \tag{10}$$
$$\pi_m\bar\pi_{m+1}\pi_m = \bar\pi_{m+1}\pi_m\bar\pi_{m+1} \qquad \text{for } m\ge 0, \tag{11}$$
$$\pi_m^2 = 1 \qquad \text{for } m\ge 0, \tag{12}$$
$$\bar\pi_m^2 = 1 \qquad \text{for } m\ge 0. \tag{13}$$

4.4. ***Relations involving baker's maps.*** In the relations (14)–(18) the reader can assume that $2\le d\le n$ and $1\le d'\le n$ unless otherwise stated.

$$\bar\pi_m X_{m,d} = C_{m+1,d}\pi_m\bar\pi_{m+1} \qquad \text{for } m\ge 0, \tag{14}$$
$$C_{q,d}X_{m,d'} = X_{m,d'}C_{q+1,d} \qquad \text{for } m<q, \tag{15}$$
$$C_{m,d}X_{m,1} = X_{m,d}C_{m+2,d}\pi_{m+1} \qquad \text{for } m\ge 0, \tag{16}$$
$$\pi_q C_{m,d} = C_{m,d}\pi_q \qquad \text{for } m>q+1, \tag{17}$$
$$C_{m,d}X_{m,d'}C_{m+2,d'} = C_{m,d'}X_{m,d}C_{m+2,d}\pi_{m+1} \qquad \text{for } m\ge 0,\ 1<d'<d\le n. \tag{18}$$

Relations (1)–(17) are generalizations of those given in [Brin 2004] and their proofs are completely analogous. The only new family of relations is (18), which we prove using relation (M6) from the monoid:

Proof. We have

$$\begin{aligned}
C_{m,d}X_{m,d'}C_{m+2,d'} &= (s_{0,1}^m s_{0,d},\, s_{0,1}^{m+1})(s_{0,1}^{m+1}s_{1,d'},\, s_{0,1}^{m+2})(s_{0,1}^{m+2}s_{0,d'},\, s_{0,1}^{m+3})\\
&= (s_{0,1}^m s_{0,d}s_{1,d'}s_{0,d'},\, s_{0,1}^{m+3})\\
&= (s_{01}^m s_{0,d'}s_{1,d}s_{0,d}\sigma_1,\, s_{0,1}^{m+3})\\
&= (s_{0,1}^m s_{0,d'},\, s_{0,1}^{m+1})(s_{0,1}^{m+1}s_{1,d},\, s_{0,1}^{m+2})(s_{0,1}^{m+2}s_{0,d},\, s_{0,1}^{m+3})(s_{0,1}^{m+3}\sigma_1,\, s_{0,1}^{m+3})\\
&= C_{m,d'}X_{m,d}C_{m+2,d}\pi_{m+1}.
\end{aligned}$$
□

Lemma 12 (subscript raising formulas). *We have*

$$C_{r,d}\sim C_{r+1,d}X_{r,d}\pi_{r+1}X_{r,1}^{-1} \quad \textit{and} \quad \bar\pi_r\sim\pi_r\bar\pi_{r+1}X_{r,1}^{-1}\sim X_{r,1}\bar\pi_{r+1}\pi_r.$$

The first formula of Lemma 12 follows from relations (15) and (16), while the second is a generalization of the one found in [Brin 2005].

4.5. ***Secondary relations for nV.*** These are as follows.

$$X_{q,d}^{-1}X_{r,d} \sim \begin{cases} X_d X_d^{-1} & \text{if } r \neq q, \\ 1 & \text{if } r = q \end{cases} \quad \text{for } 1 \leq d \leq n,$$

$$X_{q,d}^{-1}X_{r,d'} \sim \begin{cases} X_{d'} X_d^{-1} & \text{if } r \neq q, \\ w(X_{d'})\pi w(X_d^{-1}) & \text{if } r = q \end{cases} \quad \text{for } 1 \leq d, d' \leq n, d \neq d',$$

$$C_{q,d}^{-1}X_{r,d'} \sim \begin{cases} X_{d'} C_d^{-1} & \text{if } r < q, \\ w(X_1, \pi, X_d^{-1})X_{d'}C_d^{-1} & \text{if } r \geq q \end{cases} \quad \text{for } 2 \leq d \leq n, 1 \leq d' \leq n,$$

$$X_{r,d'}^{-1}C_{q,d} \sim \begin{cases} C_d X_{d'}^{-1} & \text{if } r < q, \\ C_d X_{d'}^{-1} w(X_d, \pi, X_1^{-1}) & \text{if } r \geq q \end{cases} \quad \text{for } 2 \leq d \leq n, 1 \leq d' \leq n,$$

$$\pi_q X_{r,d} \sim X_d w(\pi) \quad \text{for } 1 \leq d \leq n,$$

$$\bar{\pi}_q X_{r,1} \sim \begin{cases} X_1 \bar{\pi} & \text{if } r < q, \\ \pi \bar{\pi} & \text{if } r = q, \\ w(X_1)\bar{\pi} w(\pi) & \text{if } r > q, \end{cases}$$

$$\bar{\pi}_q X_{r,d} \sim \begin{cases} X_d \bar{\pi} & \text{if } r < q, \\ C_d \pi \bar{\pi} & \text{if } r = q, \\ w(X_1)X_d \bar{\pi} w(\pi) & \text{if } r > q \end{cases} \quad \text{for } 2 \leq d \leq n,$$

$$\pi_q C_{r,d} \sim \begin{cases} C_d \pi & \text{if } r > q+1, \\ C_d w(X_1^{-1}, \pi, X_d) & \text{if } r \leq q+1 \end{cases} \quad \text{for } 2 \leq d \leq n,$$

$$\bar{\pi}_q C_{r,d} \sim \begin{cases} X_d \bar{\pi} \pi & \text{if } r = q+1, \\ w(X_1)X_d \bar{\pi} w(\pi) & \text{if } r > q+1, \\ w(X_d)C_d \pi \bar{\pi} w(\pi, X_1^{-1}) & \text{if } r < q+1 \end{cases} \quad \text{for } 2 \leq d \leq n,$$

$$C_{q,d}^{-1}C_{r,d} \sim \begin{cases} w(X_1^{-1}, \pi, X_d) & \text{if } q < r, \\ 1 & \text{if } q = r, \\ w(X_1, \pi, X_d^{-1}) & \text{if } q > r \end{cases} \quad \text{for } 2 \leq d \leq n,$$

$$C_{q,d}^{-1}C_{r,d'} \sim \begin{cases} X_{d'} C_{d'} \pi C_d^{-1} X_d^{-1} w(X_{d'}, \pi, X_1^{-1}) & \text{if } q > r, \\ X_{d'} C_{d'} \pi C_d^{-1} X_d^{-1} & \text{if } q = r, \\ w(X_1, \pi, X_{d'}^{-1}) X_d C_d \pi C_{d'}^{-1} X_{d'}^{-1} & \text{if } q < r \end{cases} \quad \text{for } 1 \leq d' < d \leq n.$$

Proof. We only prove the last set of secondary relations as it is the only one that does not immediately descend from the computations in [Brin 2005]. If $q > r$ we can apply the subscript raising formulas repeatedly for j times until $r + j = q$ and rewrite the product as

$$C_{q,d}^{-1}C_{r,d'} \sim C_{q,d}^{-1}C_{r+1,d'}X_{r,d'}\pi_{r+1}X_{r,1}^{-1} \sim \cdots \sim C_{q,d'}^{-1}C_{r+j,d'}w(X_{d'}, \pi, X_1^{-1}).$$

We argue similarly if $q < r$. We now have to study the product $C_{q,d}^{-1}C_{q,d'}$. Without loss of generality we assume $d' < d$ and apply relation (18):

$$C_{q,d}^{-1}C_{q,d'} = X_{q,d'}C_{q+2,d'}\pi_{q+1}C_{q+2,d}^{-1}X_{q,d}^{-1},$$

which is what was claimed. Similar relations can be derived if $d' > d$. □

Remark 13. The last two secondary relations allow us to rewrite a word of type $w(X, C, \pi, C^{-1}, X^{-1})$ in LMR form without increasing the number of times C appears, and thereby to generalize the proof of [Brin 2005, Lemma 4.6]; see Lemma 15 below. This observation also lets us generalize [Brin 2005, Lemma 4.7]; see Lemma 16 below. In fact, all our secondary relations are immediate generalizations of those in [Brin 2005]; the last one does not introduce appearances of $\bar{\pi}$ and therefore all the letters in the last secondary relations can be migrated to their needed position by means of the previous secondary relations, without altering the original argument of [Brin 2005, Lemma 4.7]. Therefore even in the case of nV one is able to do the bookkeeping without risk of creating extra letters that cannot be passed safely without recreating them, and hence we obtain an argument that terminates.

5. Presentations for nV

We now show how the relations above enable us to put our group elements into a normal form, starting with words in the generators of nV corresponding to elements from $\widehat{nV}$.

Lemma 14. *Let w be a word in $\{X_{i,d}, \pi_i, X_{i,d}^{-1} \mid 1 \le d \le n, i \in \mathbb{N}\}$. Then $w \sim LMR$, where L and R^{-1} are words in $\{X_{i,d}\}$ and M is a word in $\{\pi_i\}$.*

Proof. There is a homomorphism from $\widehat{nV}$ to nV given by $s_{i,d} \mapsto X_{i,d}$ and $\sigma_i \mapsto \pi_i$. This follows from the correspondence between the relations for $\widehat{nV}$ and nV as given below:

(M1) $\to$ (1),	(M5a) $\to$ (2),
(M2) $\to$ (12),	(M5b), (M5c) $\to$ (3),
(M3) $\to$ (8),	(M5d) $\to$ (4),
(M4) $\to$ (9),	(M6) $\to$ (7).

Hence, any word w as given above is the image under this homomorphism of a word w' in $\widehat{nV}$. Since $\widehat{nV}$ is the group of right fractions of the monoid Π_n, we can represent w' as pq^{-1}, where p and q are words in $\{s_{i,d}, \sigma_i \mid 1 \le d \le n, i \in \mathbb{N}\}$.

Now, as noted before in the proof of Lemma 6, we can assume p and q are of the form ab, where $a \in \langle s_{i,d} \rangle$ and $b \in \langle \sigma_i \rangle$. Hence, we have written w' as lmr for $l, r^{-1} \in \langle s_{i,d} \rangle$ and $m \in \langle \sigma_i \rangle$ since elements of $\langle \sigma_i \rangle$ are their own inverse. Applying the homomorphism to w' puts w in the desired form. □

The next two results follow the original proofs of [Brin 2005, Lemmas 4.6 and 4.7] via Remark 13.

Lemma 15. *Let w be of the form $w(X, C, \pi, X^{-1}, C^{-1})$. Then $w \sim LMR$, where L and R^{-1} are words of the form $w(X, C)$ and M is of the form $w(\pi)$. Further the number of appearances of C in L will be no larger than the number of appearances of C in w and the number of appearances of C^{-1} in R will be no larger than the number of appearances of C^{-1} in w.*

Lemma 16. *Let w be a word in the generating set*

$$\{X_{i,d}, C_{i,d'}, \pi_i, \overline{\pi}_i, X_{i,d}^{-1}, C_{i,d'}^{-1} \mid 1 \leq d \leq n, 2 \leq d' \leq n, i \in \mathbb{N}\}.$$

Then $w \sim LMR$, where L and R^{-1} are words of the form $w(X, C)$ and M is of the form $w(\pi, \overline{\pi})$.

Lemma 17. *Let w be a word in the generating set*

$$\{X_{i,d}, C_{i,d'}, \pi_i, \overline{\pi}_i, X_{i,d}^{-1}, C_{i,d'}^{-1} \mid 1 \leq d \leq n, 2 \leq d' \leq n, i \in \mathbb{N}\}.$$

Then $w \sim LMR$, where

- *$L = C_{i_0,d_0} C_{i_1,d_1} \ldots C_{i_g,d_g} q$ with $i_0 < i_1 < \cdots < i_g$ for $g \geq -1$ and q is a word in the set $\{X_{i,d} \mid 1 \leq d \leq n, i \in \mathbb{N}\}$*
- *$R^{-1} = C_{j_0,d'_0} C_{j_1,d'_1} \ldots C_{j_m,d'_m} q'$ with $j_0 < j_1 < \cdots < j_m$ for $m \geq -1$ and q' is a word in the set $\{X_{i,d} \mid 1 \leq d \leq n, i \in \mathbb{N}\}$*
- *M is a word in the set $\{\pi_i, \overline{\pi}_i \mid i \in \mathbb{N}\}$*

Proof. By using the secondary relations, we can assume that $w \sim LMR$, where L and R^{-1} are words in $\{X_{i,d}, C_{i,d}\}$ and M is a word in $\{\pi_i, \overline{\pi}_i\}$ by analogous arguments used in [Brin 2005, Lemmas 4.6 and 4.7]. We then improve L using the subscript raising formula for the $C_{i,d}$ and relation (15) as in the proof of [ibid., Lemma 4.8]. To adapt the quoted lemmas from [Brin 2005] we need to use Remark 13 to make sure that appearances of C and $\overline{\pi}$ do not increase. □

We define the notions of *primary* and *secondary tree* and of *trunk* exactly the same way that Brin does [2005]. The primary tree is the tree corresponding to the word t in Lemma 18 and any extension to the left is a secondary tree for L. The following extends [Brin 2005, Lemma 4.15] adapted to our case. The proof is completely analogous.

Lemma 18. *Let*

$$L = C_{i_0,d_0} C_{i_1,d_1} \cdots C_{i_g,d_g} X_{i_{n+1},d_{n+1}} \cdots X_{i_{l-1},d_{l-1}},$$

where $i_0 < i_1 < \cdots < i_g$, where $2 \le d_k \le n$ for $k \in \{0, \ldots, g\}$ and $1 \le d_k \le n$ for $k \in \{g+1, \ldots, l-1\}$. Let m equal the maximum of

$$\{i_j + g + 2 - j \mid g+1 \le j \le l-1\} \cup \{i_g + 1\}.$$

Then L can be represented as $L = (t, s_{0,1}^k)$, where t is a word in $\{s_{i,d}\}$ and k is the length of t, so that $k = m + l - g$, and so that the tree T for t is the primary tree for L and is described as follows. The tree T consists of a trunk Λ with a finite forest F attached. The trunk Λ has m carets and $m+1$ leaves numbered 0 through m in the right-left order. If the carets in Λ are numbered from 0 starting at the top, then the label of the i-th caret is d_k if $i = i_k$ for k in $\{0, 1, \ldots g\}$ and 1 otherwise.

The following two lemmas are used in proving Remark 13, which allows us to assume the trees corresponding to our group elements are in normal form.

Lemma 19. *Let*

$$L = C_{i_0,d_0} C_{i_1,d_1} \cdots C_{i_g,d_g} u \quad \textit{and} \quad L' = C_{k_0,d'_0} C_{k_1,d'_1} \cdots C_{k_g,d'_g} u',$$

where $i_0 < i_1 < \cdots < i_g$, where $k_0 < k_1 < \cdots < k_g$, where u is a word in the set $\{X_{i,d} \mid 1 \le d \le n, i \in \mathbb{N}\}$, and where u' is a word in the set $\{X_{i,d}, \pi_i \mid 1 \le d \le n, i \in \mathbb{N}\}$. Assume that L is expressible as $(t, s_{0,1}^p)$ as an element of $\widehat{nV}$ with t a word in $\{s_{i,d}\}$ and p the length of t. Let m be the number of carets of the trunk of the tree T corresponding to t and assume that $m \ge k_g + 1$.

If $L \sim L'$, then there is a word u'' in $\{X_{i,d}\}$, and there is a word z in $\{\pi_i \mid i \le p-2\}$ such that setting $L_1 = C_{k_0,d'_0} C_{k_1,d'_1} \cdots C_{k_g,d'_g} u''$ and $L_2 = L_1 z$ gives that $L \sim L_2$ and L_1 is expressible as $(t', s_{0,1}^p)$ with t' a word in $\{s_{i,d}\}$ of length p, so that the tree T' for t' is normalized except possibly at interior vertices in the trunk of the tree, and so that the trunk of T' has m carets.

Proof. The homomorphism $\widehat{nV} \to nV$ given by $s_{i,d} \mapsto X_{i,d}$ and $\sigma_i \mapsto \pi_i$ allows us to write $u' \sim u'' z'$ with u'' a word in $\{X_{i,d}\}$ and z' a word in $\{\pi_i \mid i \in \mathbb{N}\}$ such that the forest F for u'' is normalized. The rest of the proof goes through as before, but we describe the slight modifications needed for our case. We write $L = (ts_{0,1}^k, s_{0,1}^{p+k}) = (\hat{t}s_{1,0}^r x, s_{1,0}^{q+r}) = L_2$ as elements in $\widehat{nV}$, where x is a word in $\{\sigma_i\}$ and $p + k = q + r$. As before, we can conclude that the unnumbered patterns for $ts_{0,1}^k$ and $\hat{t}s_{1,0}^r$ are identical.

In the tree for $ts_{0,1}^k$, let the left edge vertices be $a_0, a_1, \ldots, a_b$ reading from the top, so that a_0 is the root of the tree. Since we assume the trunk of the tree has m carets, we know $b = m + k$ and for $m \le i < b$, the label for a_i is 1. Similarly, in the tree for $\hat{t}s_{1,0}^r$, let the left edge vertices be $a'_0, a'_1, \ldots, a'_b$ reading from the top. Note that remark $(*)$ in the proof of [Brin 2005, Theorem 4.21] (which we are about to restate) remains true in our general case, by giving a new definition: For each left edge vertex a_i, define the n-tuple $(x_1^i, \ldots, x_n^i)$, where x_k^i equals the number of left edge vertices above a_i with label k. (Note we are using i to denote an index, not an exponent). It follows that $x_1^i + x_2^i + \cdots + x_n^i$ is the total number of left edge vertices above a_i. Then we can say,

$(*)$ The rectangle corresponding to a left edge vertex a_i depends only on the n-tuple $(x_1^i, \ldots, x_n^i)$.

In other words, for the rectangle labeled 0 in any pattern, the order of the different cuts does not matter. This is because the rectangle labeled 0 must contain the origin and its size in each dimension k will be $2^{-x_k^i}$. Hence, the analogous statement for our case follows, and we conclude that the n-rectangle R corresponding to a_m is identical to the n-rectangle R' corresponding to a'_m Since R is divided k times across dimension 1, so is R', and hence the tree below a'_m must consist of an extension to the left by k carets all labeled 1, and we can conclude that $r \geq k$. The rest of the proof follows exactly as before. □

Here, we define a notion of *complexity* to measure progress in the following lemma and proposition towards normalizing trees. If T is a labeled tree, we let $a_0, a_1, \ldots, a_m$ be the interior, left edge vertices of T reading from top to bottom so that a_0 is the root. Let $b_0 b_1 \ldots b_m$ be a word in $\{1, 2, \ldots, n\}$ where $b_i = k$ if a_i is labeled k for $0 \leq i \leq m$. We say $b_0 b_1 \ldots b_m$ is the complexity of T. We impose the length-lex ordering on such words, that is, if w_1 and w_2 are two such words, then we say $w_1 < w_2$ if w_1 is shorter than w_2 or if $w_1 = b_0^1 \ldots b_m^1$ and $w_2 = b_0^2 \ldots b_m^2$ are two such words of the same length, then $w_1 < w_2$ if when we take $j \in \{0, \ldots, m\}$ minimal where $b_j^1 \neq b_j^2$, we have $b_j^1 < b_j^2$.

Lemma 20. *Let $L = C_{i_0,d_0} C_{i_1,d_1} \cdots C_{i_g,d_g} u$, where $i_0 < i_1 < \cdots < i_g$ and u is a word in the set $\{X_{i,d}\}$. Assume that the primary tree T for L is normalized except at one or more vertices in the trunk of T. Let m be the number of carets in the trunk of T. Then $L \sim L' = C_{k_0,c_0} C_{k_1,c_1} \cdots C_{k_g,c_g} u'$, where $k_0 < k_1 < \cdots < k_g$ and u' is a word in the set $\{X_{i,d}, \pi_s\}$, so that $m \geq k_g + 1$, and so that the complexity of the primary tree T' of L' is strictly less than the complexity of T.*

Proof. We want to use the relations to push a suitable instance of an $X_{u,v}$ in the word L as far as possible to the left to be able to apply a cross relation. This operation normalizes a suitable vertex and decreases the complexity of the primary tree T.

Let Λ be the trunk of T. The interior vertices of Λ are the interior, left edge vertices of T and let these be $a_0, a_1, \ldots, a_{m-1}$. Let r be the highest value with $0 \leq r < m$ for which a_r is not normalized. This is the lowest nonnormalized interior vertex of Λ, and since a_r is not normalized it is labeled $\ell \neq 1$ and must correspond to some $C_{i_j,\ell}$. From Lemma 18, we have $i_j = r$.

Since it is not normalized, a_r must correspond to some hypercube S_{i_j} that is fully divided across dimension ℓ and some other dimension d, with $1 \le d < \ell$.

By rewriting L as $(t, s^k_{0,1})$ (which we can do by Lemma 18) and applying Corollary 9 to t, we can assume that the children of a_r, v_1 and v_2, are both labeled d. We divide our work in two cases, $d = 1$ and $d > 1$. We observe that the case $d = 1$ is entirely analogous to the proof of [Brin 2005, Theorem 4.22] while the case $d > 1$ is slightly different.

Case 1: $d = 1$. In this case, the left child v_1, which is in the trunk Λ, is labeled 1. In the case that $j < n$ we observe that $i_{j+1} > r + 1 = i_j + 1$, since the interior vertex of the trunk corresponding to $C_{i_{j+1},d_{j+1}}$ is not labeled 1 (otherwise, $a_r = a_{i_j}$ would not be the lowest nonnormalized interior vertex). Since the right child v_2 is an interior vertex not on the trunk, there must be a letter $X_{q,1}$ corresponding to it. By Lemma 5 we can assume that $X_{q,1}$ occurs as the first letter of u, that is, $u = X_{q,1}u''$. Hence

$$L = C_{i_0} \cdots C_{i_{j-1}} \underline{C_{i_j,\ell}} C_{i_{j+1}} \cdots C_{i_g} \underline{X_{q,1}} u'',$$

where we have omitted all the dimension subscripts of the baker's maps $C_{i,d}$ (except for one map) since they are not important for the argument. The subword $C_{i_0} \cdots C_{i_j,\ell} \cdots C_{i_g} X_{q,1}$ is a trunk with a single caret labeled 1 attached at the caret i_j of the trunk on its right child. By a careful observation of the right-left ordering it is evident that $q = i_j$. By using relation (15) repeatedly on L we can move $X_{q,1} = X_{i_j,1}$ to the left and rewrite the word L as

$$C_{i_0} \cdots C_{i_{j-1}} \underline{C_{i_j,\ell} X_{i_j,1}} C_{i_{j+1}+1} \cdots C_{i_g+1} u'',$$

since $i_0 < i_1 < \cdots < i_g$ and $i_{j+1} > i_j + 1$. Combining relations (15) and (16) on the product $C_{i_j,\ell} X_{i_j,1}$, we rewrite L as

$$C_{i_0} \cdots C_{i_{j-1}} \underline{C_{i_j+1,\ell} X_{i_j,\ell} \pi_{i_j+1}} C_{i_{j+1}+1} \cdots C_{i_g+1} u''.$$

Now we apply (17) to commute π_{i_j+1} back to the right without affecting the indices of the baker's maps. This is possible since $i_{j+1} > i_j + 1$ and therefore $i_{j+1} + 1 > i_j + 2$. Now we apply (15) repeatedly to the word

$$C_{i_0} \cdots C_{i_{j-1}} \underline{C_{i_j+1,\ell} X_{i_j,\ell}} C_{i_{j+1}+1} \cdots C_{i_g+1} \underline{\pi_{i_j+1}} u''$$

to bring $X_{i_j,\ell}$ back to the right, decreasing the indices of the baker's maps by 1

$$C_{i_0} \cdots C_{i_{j-1}} \underline{C_{i_j+1,\ell}} C_{i_{j+1}} \cdots C_{i_g} \underline{X_{i_j,\ell} \pi_{i_j+1}} u''.$$

By setting $u' = X_{i_j,\ell}\pi_{i_j+1}u''$ in the previous equation and relabeling the indices with the k_i, we obtain the word $L' = C_{k_0,c_0}C_{k_1,c_1}\cdots C_{k_g,c_g}u'$ whose primary tree T' is the same as T up until the vertex a_r, which is now labeled $d = 1$ instead of ℓ. Thus, $L \sim L' = C_{k_0,c_0}C_{k_1,c_1}\cdots C_{k_g,c_g}u'$ and the complexity of the primary tree T' of L' is strictly less than the complexity of T.

The only thing we still need to prove in this case is that $m \geq k_g + 1$. However, it has been observed above that $i_j = r < m - 1$ so $i_j + 2 \leq m$. This gives the result in the case that $j = n$. If $j < n$, then $k_g = i_g$ and $m \geq i_g + 1$ by Lemma 18.

Case 2: $1 < d < \ell$. We observe that a_r corresponds to $C_{i_j,\ell}$ and that v_1 corresponds to $C_{i_k,d}$. By Lemma 18, we have $r + 1 = i_k$, which implies $i_k = i_j + 1 = i_{j+1}$. In fact, if $i_j + 1 < i_{j+1}$, there would be a vertex labeled 1 on the trunk between the vertices i_j and i_{j+1} (and this is impossible since $d > 1$). Let $X_{i_j,d}$ correspond to the right child v_2. Arguing as in the case $d = 1$ we have

$$L = C_{i_0}\cdots C_{i_{j-1}}\underline{C_{i_j,\ell}C_{i_j+1,d}}C_{i_{j+2}}\cdots C_{i_g}\underline{X_{q,d}}u''.$$

We apply relation (15) as before to move $X_{q,d} = X_{i_j,d}$ to the left while increasing the subscript of each baker's map by 1:

$$C_{i_0}\cdots C_{i_{j-1}}\underline{C_{i_j,\ell}X_{i_j,d}C_{i_j+2,d}}C_{i_{j+2}+1}\cdots C_{i_g+1}u''.$$

By using the cross relation (18) on the underlined portion, we read it as

$$C_{i_0}\cdots C_{i_{j-1}}\underline{C_{i_j,d}X_{i_j,\ell}C_{i_j+2,\ell}\pi_{i_j+1}}C_{i_{j+2}+1}\cdots C_{i_g+1}u''.$$

Since $i_{j+2} > i_{j+1}$, then $i_{j+2} + 1 > i_{j+1} + 1$; hence π_{i_j+1} and the baker's maps to its right commute, so the word becomes

$$C_{i_0}\cdots \underline{C_{i_j,d}X_{i_j,\ell}C_{i_j+2,\ell}}C_{i_{j+2}+1}\cdots C_{i_g+1}\underline{\pi_{i_j+1}}u''.$$

We apply (15) repeatedly and move $X_{i_j,\ell}$ back to the right to obtain

$$L \sim C_{i_0}\cdots \underline{C_{i_j,d}C_{i_j+1,\ell}}C_{i_{j+2}}\cdots C_{i_g}X_{i_j,\ell}\pi_{i_j+1}u'',$$

where the product $C_{i_j,d}C_{i_j+2,\ell}$ has been underlined to stress that the new trunk has the vertices labeled d and ℓ, which are now switched. Thus the complexity of the tree has been lowered. In this second case, the new sequence $k_0 < \cdots < k_g$ is exactly equal to the initial one $i_0 < \cdots < i_g$. By the definition of m (given in Lemma 18) applied on the initial word L, we have $m \geq i_g + 1$ and so, since $k_g = i_g$, we are done. □

Remark 21. As observed in the proof above, the case $d = 1$ is equivalent to [Brin 2005, Theorem 4.22], though the proof therein leads to a condition that is equivalent to lowering the complexity. When the index in some $C_{i_j,d}$ goes up by 1, this corresponds to switching the vertices with labels d and 1 in the primary tree and thus lowering the complexity by making more vertices normalized.

Proposition 22. *Let w be a word in the generating set*

$$\{X_{i,d}, C_{i,d'}, \pi_i, \bar{\pi}_i, X_{i,d}^{-1}, C_{i,d'}^{-1} \mid 1 \le d \le n, 2 \le d' \le n, i \in \mathbb{N}\}.$$

Then $w \sim LMR$ as in Lemma 17 and when expressed as elements of $\widehat{nV}$ we have

$$L = ts_{0,1}^{-p}, \quad R^{-1} = ys_{0,1}^{-p}, \quad M = s_{0,1}^{p} u s_{0,1}^{-p},$$

where t, y are words in $\{s_{i,d} \mid 1 \le d \le n, i \in \mathbb{N}\}$, u is a word in $\{\sigma_j \mid 0 \le j \le p-1\}$, and the lengths of t and y are both p. Further, we may assume the trees for t and y are normalized, and if u can be reduced to the trivial word using relations (2)–(4), then M can be reduced to the trivial word using relations (13)–(17).

Proof. The proof of the first conclusion is exactly the same as that of [Brin 2010, Lemma 4.19]. In order to assume the trees for t and y are normalized, we alternate applying Lemmas 19 and 20. We have L expressed as $(t, s_{0,1}^{p})$, where p is the length of t and the number of carets in the trunk of the tree T for t is m. Setting $L = L'$ certainly gives that $L \sim L'$ and $m \ge k_g + 1$ by Lemma 18, so we have satisfied the hypotheses of Lemma 19. Therefore, $L \sim L_1 z$ where L_1 expressed as $(t', s_{0,1}^{p})$, where the trunk of the tree T' for t' has m carets. Since we set $L = L'$, we see that the trunks of T and T' are identical and the only way in which the two trees differ is that T' is normalized off the trunk. Since z is a word in $\{\pi_i\}$, z can be absorbed into M without disrupting the assumptions on M, namely, M can still be written in the form $M = s_{0,1}^{p} u s_{0,1}^{-p}$ as above. We now replace L with L_1 and proceed to use Lemma 20.

Since the tree for L is now normalized off the trunk, we satisfy the hypotheses of Lemma 20 and write $L \sim L'$, where the tree for L' has complexity lower than the tree for L and $m \ge k_g + 1$. Hence, we can now apply Lemma 19 again and obtain $L \sim L_1 z$ and let z be absorbed into M. We apply this process over and over, decreasing the complexity of the tree associated to L each time. Since there are only finitely many linearly ordered complexities, eventually this process will terminate, at which point the tree for L will be normalized. We can apply the same procedure to the inverse of LMR to normalize the tree for R. The last statement regarding M follows immediately from [Brin 2005, Lemma 4.18]. ☐

Theorem 23. *Let w be a word in the generating set*

$$\{X_{i,d}, C_{i,d'}, \pi_i, \bar{\pi}_i, X_{i,d}^{-1}, C_{i,d'}^{-1} \mid 1 \le d \le n, 2 \le d' \le n, i \in \mathbb{N}\}$$

that represents the trivial element of nV. Then $w \sim 1$ using the relations in (1)–(18). Hence, we have a presentation for nV.

Proof. Using Proposition 22, we can assume

$$w \sim LMR = (ts_{0,1}^{-p})(s_{0,1}^{p}us_{0,1}^{-p})(s_{0,1}^{p}y^{-1}), = tuy^{-1}$$

where t and y are words in $\{s_{i,d} \mid 1 \le d \le n,\ i \in \mathbb{N}\}$, u is a word in $\{\sigma_j \mid 0 \le j \le p-1\}$, and the trees associated to t and y are normalized. By assumption, $tuy^{-1} = (tu, y)$ is the trivial element of $\widehat{nV}$ and so tu and y represent the same numbered patterns in Π_n. Furthermore, t and y must give the same unnumbered pattern, while u enacts a permutation on the numbering. Since the forests for t and y are normalized and give the same pattern, the forests are identical with the same labeling by Lemma 7. The numbering on the leaves for both forests follows the left-right ordering; hence t and y give the same numbered patterns, which implies that u enacts the trivial permutation and $M \sim 1$ by Proposition 22.

We now wish to show that $L \sim R^{-1}$. By Lemma 17, we have

$$L = C_{i_0,d_0}C_{i_1,d_1}\cdots C_{i_g,d_g}q \quad \text{and} \quad R^{-1} = C_{j_0,d'_0}C_{j_1,d'_1}\cdots C_{j_m,d'_m}q'.$$

Since we know that the trunks of the trees corresponding to L and R^{-1} are identical with the same labeling, the sequences $(i_0, i_1, \dots, i_g)$ and $(j_0, j_1, \dots, j_m)$ are identical and $d_k = d'_k$ for each $k \in \{0, 1, \dots, n = m\}$. Hence, the subwords $C_{i_0,d_0}C_{i_1,d_1}\cdots C_{i_g,d_g}$ and $C_{j_0,d'_0}C_{j_1,d'_1}\cdots C_{j_m,d'_m}$ are the same and it remains to show that $q \sim q'$. This follows from Lemma 4 and the homomorphism from $\widehat{nV}$ to nV as before. □

6. Finite presentations

6.1. ***Finite presentation for $\widehat{nV}$.*** We now give a finite presentation for $\widehat{nV}$, using arguments analogous to those found in [Brin 2005] to show that the full set of relations is the result of only finitely many of them.

Theorem 24. *The group $\widehat{nV}$ is presented by the $2n+2$ generators $\{s_{i,d}, \sigma_i \mid i \in \{0, 1\}, 1 \le d \le n\}$ and the $5n^2+7n+6$ relations given below*:

(M1)	$s_{1,1}^{-1}s_{1+k,d'}s_{1,1} = s_{2+k,d'}$	*for* $k = 1, 2,$
	$s_{i,d}^{-1}s_{i+k,d'}s_{i,d} = s_{i+k+1,d'}$	*for* $i = 0, 1,\ k = 1, 2,\ 2 \le d \le n,$
(M2)	$\sigma_i^2 = 1$	*for* $i = 0, 1,$
(M3)	$\sigma_i\sigma_{i+k} = \sigma_{i+k}\sigma_i$	*for* $i = 0, 1,\ k = 2, 3,$
(M4)	$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$	*for* $i = 0, 1,$
(M5a)	$\sigma_{k+1}s_{1,1} = s_{1,1}\sigma_{k+2}$	*for* $k = 1, 2,$
	$\sigma_{i+k}s_{i,d} = s_{i,d}\sigma_{i+k+1}$	*for* $i = 0, 1,\ k = 1, 2,\ 2 \le d \le n,$
(M5b)/(M5c)	$\sigma_i s_{i,d} = s_{i+1,d}\sigma_i\sigma_{i+1}$	*for* $i = 0, 1,$

(M5d) $\quad \sigma_i s_{i+k,d} = s_{i+k,d}\sigma_i \qquad$ *for* $i = 0, 1,\ k = 2, 3,$

(M6) $\quad s_{i,d}s_{i+1,d'}s_{i,d'} = s_{i,d'}s_{i+1,d}s_{i,d}\sigma_{i+1} \qquad$ *for* $i = 0, 1,\ d \neq d'.$

Proof. First, recall our generating set is $\{s_{i,d}, \sigma_i \mid i \in \mathbb{N}, 1 \leq d \leq n\}$. When $i < j$, relations (M1) and (M5a) give $s_{i,1}^{-1} x_j s_{i,1} = x_{j+1}$, where $x_j = s_{j,d}$ (for some d) or σ_j. Hence, we can use

$$s_{i,d} = s_{0,1}^{1-i} s_{1,d} s_{0,1}^{i-1} \quad \text{and} \quad \sigma_i = s_{0,1}^{1-i}\sigma_1 s_{0,1}^{i-1}$$

as definitions for $i \geq 2$. Therefore, $\widehat{nV}$ is generated by

$$\{s_{i,d}, \sigma_i \mid i \in \{0, 1\}, 1 \leq d \leq n\},$$

which gives a generating set of size $2n + 2$ for each n.

We treat relations (M1)–(M6) as they are treated in [Brin 2005]. Relations involving only one parameter, such as (M2), (M4), and (M6), are obtained for $i \geq 2$ by setting $i = 1$ and conjugating by powers of $s_{0,1}$; therefore the only necessary relations to include are those having $i = 0$ and $i = 1$. As before, (M2) and (M4) follow from $\sigma_0^2 = 1$, $\sigma_1^2 = 1$, $\sigma_0\sigma_1\sigma_0 = \sigma_1\sigma_0\sigma_1$, and $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$, or 4 relations for each n. Relation (M6) follows from 2 relations for each pair of distinct dimensions, giving $2\binom{n}{2} = n(n-1)$ relations for each n.

Relation (M3) is treated the same way as in [Brin 2005] for each n. Hence, for all i and j, (M3) follows from the 4 relations $\sigma_0\sigma_2 = \sigma_2\sigma_0$, $\sigma_0\sigma_3 = \sigma_3\sigma_0$, $\sigma_1\sigma_3 = \sigma_3\sigma_1$, $\sigma_1\sigma_4 = \sigma_4\sigma_1$.

For relation (M1), which can be rewritten as $s_{i,d}^{-1} s_{i+k,d'} s_{i,d} = s_{i+k+1,d'}$ for $k > 0$, we have two cases: the case where $d = 1$ and the case where $d \neq 1$. If $d = 1$, then the case $i = 0$ follows by definition, and by the same induction argument used in [Brin 2005] implies that the relation for all i and k follows from the cases where $i = 1$ and $k = 1, 2$; hence we need only 2 relations per dimension. If $d \neq 1$, we do not get the case $i = 0$ by definition and we must include $i = 0, 1$ and $k = 1, 2$, that is, 4 relations per each pair of dimensions. There are $n - 1$ choices for d, as $d \neq 1$, and n choices for d', so this case yields $4n(n-1)$ relations. Hence, in total (M1) can be obtained for all i and k by $2n + 4n(n-1) = 4n^2 - 2n$ relations.

For relation (M5b), $\sigma_i s_{i,d} = s_{i+1,d}\sigma_i\sigma_{i+1}$, there is only a single parameter to deal with; hence the relation for $i \geq 2$ can be obtained from the cases where $i = 0, 1$ by conjugating by $s_{0,1}$ as before. Relation (M5c) is actually equivalent to (M5b); hence for each n we only need $2n$ relations for (M5b) and (M5c). We treat (M5a) $\sigma_{i+k}s_{i,d} = s_{i,d}\sigma_{i+k+1}$ for $k > 0$ the same way as for (M1), hence 2 relations are required for $d = 1$ and 4 for $d \neq 1$ for a total of $4n - 2$ relations. And lastly, (M5d) $\sigma_i s_{i+k,d} = s_{i+k,d}\sigma_i$ can be obtained in the same way as the second case of (M1) where the relation for all i, k is obtained by $i = 0, 1, k = 2, 3$, that is, $4n$ relations. □

6.2. *Finite presentation for nV.*

Theorem 25. *The group nV is presented by the $2n+4$ generators*

$$\{X_{i,d}, \pi_i, \overline{\pi}_i \mid i \in \{0, 1\}, 1 \le d \le n\},$$

the $5n^2+7n+6$ relations obtained from the homomorphism $\widehat{nV} \to nV$, and the additional $5n^2+3n+4$ relations given below, for a total of $10n^2+10n+10$ relations.

(5)	$\overline{\pi}_{k+1} X_{1,1} = X_{1,1} \overline{\pi}_{k+2}$	*for* $k = 1, 2,$
	$\overline{\pi}_{m+k} X_{m,d} = X_{m,d} \overline{\pi}_{m+k+1}$	*for* $m = 0, 1, k = 1, 2,$ $2 \le d \le n,$
(10)	$\overline{\pi}_{m+k} \pi_m = \pi_m \overline{\pi}_{m+k}$	*for* $m = 0, 1, k = 2, 3,$
(11)	$\pi_m \overline{\pi}_{m+1} \pi_m = \overline{\pi}_{m+1} \pi_m \overline{\pi}_{m+1}$	*for* $m = 0, 1$
(13)	$\overline{\pi}_m^2 = 1$	*for* $m = 0, 1,$
(6)	$\overline{\pi}_m X_{m,1} = \pi_m \overline{\pi}_{m+1}$	*for* $m = 0, 1,$
(14)	$\overline{\pi}_m X_{m,d} = C_{m+1,d} \pi_m \overline{\pi}_{m+1}$	*for* $m = 0, 1, d \ne 1,$
(15)	$C_{k+1,d} X_{1,1} = X_{1,1} C_{k+2,d}$	*for* $k = 1, 2,$
	$C_{m+k,d} X_{m,d'} = X_{m,d'} C_{m+k+1,d}$	*for* $m = 0, 1,\ k = 1, 2,$ $2 \le d, d' \le n,$
(16)	$C_{m,d} X_{m,1} = X_{m,d} C_{m+2,d} \pi_{m+1}$	*for* $m = 0, 1,\ 2 \le d \le n,$
(17)	$\pi_m C_{m+k,d} = C_{m+k,d} \pi_m$	*for* $m = 0, 1,\ k = 2, 3,$
(18)	$C_{m,d} X_{m,d'} C_{m+2,d'} = C_{m,d'} X_{m,d} C_{m+2,d} \pi_{m+1}$	*for* $m = 0, 1,$ $1 < d' < d \le n,$

Proof. We can use the relations in nV to write, for $i \ge 2$ and $1 \le d \le n$,

$$X_{i,d} = X_{0,1}^{1-i} X_{1,d} X_{0,1}^{i-1}, \quad \pi_i = X_{0,1}^{1-i} \pi_1 X_{0,1}^{i-1}, \quad \overline{\pi}_i = X_{0,1}^{1-i} \overline{\pi}_1 X_{0,1}^{i-1}.$$

We can also use the relations for nV as in [Brin 2004, Proposition 6.2] to write

$$C_{m,d} = (\overline{\pi}_m X_{m,d} \overline{\pi}_{m+1} \pi_m)(X_{m,d} \pi_{m+1} X_{m,1}^{-1})$$

for $m \ge 0$ and $2 \le d \le n$, which we use as a definition. Hence, the $C_{m,d}$ are not needed to generate nV.

The homomorphism $\widehat{nV} \to nV$ given by $s_{i,d} \mapsto X_{i,d}$ and $\sigma_i \mapsto \pi_i$ implies that the work done for the relations for $\widehat{nV}$ carries over to relations (1)–(4), (7)–(9), and (12) (see Lemma 14). Relations (10), (11), (13) and (6) are exactly the same as those from $2V$ and can be treated as in [Brin 2005], contributing a total of 10 relations to our finite set.

Relation (5) can be treated in a manner similar to (M1) from $\widehat{nV}$, where 2 relations are needed for dimension 1 and 4 for all others, contributing a total of $4(n-1)+2$ relations. Relations (14) and (16) include only one parameter and hence can be obtained from the cases where $i=0, 1$ as before, contributing $2(n-1)$ relations apiece. And (17) requires 4 relations for each $d \neq 1$, hence adding an additional $4(n-1)$ relations.

For relation (15), we have two cases: For $d'=1$, all cases follow from when $i=0, 1$, giving us $2(n-1)$ relations since $2 \leq d \leq n$. For $d' \neq 1$, four relations are required for each pair $d, d' \in \{2, \ldots, n\}$, contributing $4(n-1)(n-1)$ relations. Lastly, since (18) involves only one parameter in the first component, we only need 2 relations for each $1 < d' < d \leq n$, the number of pairs being $(n-1)(n-2)/2$. □

Remark 26. Since ωV is an ascending union of the nV, a word

$$w \in \{X_{i,d}, \pi_i, \bar{\pi}_i \mid i \in \{0, 1\}, d \in \mathbb{N}\}$$

such that $w =_{\omega V} 1$ must be contained in some nV (for some $n \in \mathbb{N}$) and so we can use the same ideas and the relations inside nV to transform w into the empty word. Therefore, the following result is an immediate consequence of Theorem 25.

Corollary 27. *The group ωV is generated by the set $\{X_{i,d}, \pi_i, \bar{\pi}_i \mid i \in \{0, 1\}, d \in \mathbb{N}\}$ and satisfies the family of relations in Theorem 25 with the only exception that the parameters $d, d' \in \mathbb{N}$.*

7. Simplicity of nV and ωV

Brin [2010] proved that the groups nV and ωV are simple by showing that the baker's map is a product of transpositions and following the outline of an existing proof that V is simple.

We prove again Brin's simplicity result verify that Brin's original proof that $2V$ is simple [2004, Theorem 7.2] generalizes using the generators and the relations that have been found.

Theorem 28. *The groups nV equal their commutator subgroups for $n \leq \omega$.*

Proof. The goal is to show that the generators $X_{m,i}$, π_m and $\bar{\pi}_m$ are products of commutators. We write $f \simeq g$ to mean that $f = g$ modulo the commutator subgroup. The arguments below are independent of the dimension i.

From relation (1) we see that $X_{q,i}^{-1} X_{0,1}^{-1} X_{q,i} X_{0,1} = X_{q,i}^{-1} X_{q+1,i}$ for $q \geq 1$ and so $X_{q+1,i} \simeq X_{q,i}$. Therefore $X_{q,i} \simeq X_{1,i}$, for $q \geq 1$. Using relation (2) and arguing similarly, we see that $\pi_q \simeq \pi_1$ for $q \geq 1$.

From relation (3) we see that $\pi_0 X_{0,i} \pi_0^{-1} X_{0,i}^{-1} = X_{1,i} \pi_1 X_{0,i}^{-1}$ so that $X_{0,i} \simeq X_{1,i} \pi_1$. Also, by relation (3), $X_{2,i} \simeq X_{1,i}$, and the fact that $\pi_2 \simeq \pi_1$, we see $\pi_1 X_{1,i} = X_{2,i} \pi_1 \pi_2 \simeq X_{1,i} \pi_1 \pi_1 = X_{1,i}$. Therefore $\pi_1 \simeq 1$ and so $X_{0,i} \simeq X_{1,i}$.

Relation (9) and $\pi_1 \simeq 1$ give $\pi_0^2 \simeq \pi_0\pi_1\pi_0 = \pi_1\pi_0\pi_1 \simeq \pi_0$, which implies $\pi_0 \simeq 1$.

By relation (6) and the fact that $\pi_1 \simeq 1$ and $\bar{\pi}_1 \simeq \bar{\pi}_0$, we get $\bar{\pi}_1 X_{1,1} = \pi_1\bar{\pi}_2 \simeq \bar{\pi}_1$. Hence $X_{0,1} \simeq X_{1,1} \simeq 1$.

Now, relation (6) and $X_{0,1} \simeq 1$ give that $\bar{\pi}_0 \simeq \bar{\pi}_0 X_{0,1} = \bar{\pi}_1$. Relation (11) and $\pi_0 \simeq 1$ lead to $\bar{\pi}_1 \simeq \pi_0\bar{\pi}_1\pi_0 = \bar{\pi}_1\pi_0\bar{\pi}_1 \simeq \bar{\pi}_1^2$. Therefore $\bar{\pi}_0 \simeq \bar{\pi}_1 \simeq 1$.

Finally, by relation (7) and $X_{0,1} \simeq X_{1,1} \simeq 1 \simeq \pi_1$ we get

$$X_{1,i}X_{0,i} \simeq X_{0,1}X_{1,i}X_{0,i} = X_{0,i}X_{1,1}X_{0,1}\pi_1 \simeq X_{0,i},$$

which implies $X_{0,i} \simeq X_{1,i} \simeq 1$. We have thus proved that all the generators of nV are in the commutator subgroup. The case of ωV is identical: Each generator lies in some nV and can be written as a product of commutators within that subgroup. □

From [Brin 2004, Section 3.1] (which generalizes to nV and ωV as observed by Brin [2005; 2010]) the commutator subgroup of nV and ωV are simple; therefore Theorem 28 implies the following result.

Theorem 29. *The groups nV are simple for $n \leq \omega$.*

8. An alternative generating set

For any $n \in \mathbb{N}$, we have $(n-1)V \times V \leq nV$. It can be shown that another generating set for nV is given by taking a generating set for $(n-1)V \times V$ and adding an involution that swaps two disjoint subcubes of $[0, 1]^n$, one of which has the origin as one of its vertices and the other of which contains the vertex $(1, \dots, 1)$. This second generating set has the advantage of taking the generators of $(n-1)V$ and adding only the generators of V plus another one. This leads to a smaller generating set, which was suggested to us by Collin Bleak. It seems feasible that a good set of relations exist for this alternative generating set.

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