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A full characterization is given of ordinary and restricted enveloping algebras which are normal with respect to the principal involution.

1. Introduction

Let A be an algebra with involution $*$ over a field $\mathbb{F}$. We recall that A is said to be normal if $xx^* = x^*x$ for every $x \in A$. Over the decades, normal algebras with involutions have been extensively investigated on their own; see, for example, [Beidar et al. 1981; Bovdi et al. 1985; Bovdi 1990; 1997; Bovdi and Siciliano 2007; Brešar and Vukman 1989; Herstein 1976; Knus et al. 1998; Lim 1977; 1979; Maxwell 1972]. Moreover, they have several applications in linear algebra and functional analysis; see, for example, [Berberian 1959; Fuglede 1950; Maxwell 1972; Mosić and Djordjević 2009; Putnam 1951; Yood 1974]. It is well-known that any normal algebra with involution satisfies the standard polynomial identity of degree 4 [Herstein 1976, Section 5]. Moreover, Maxwell [1972] determined the structure of a normal simple algebra of matrices with entries in a field with involution. He also proved that a division algebra D with involution is normal if and only if D is either a field or a generalized quaternion algebra over its center. Furthermore, a characterization of group algebras which are normal under the standard involution was established by Bovdi, Gudivok, and Semirot [Bovdi et al. 1985]. Subsequently, such a result has been extended to twisted group algebras [Bovdi 1990; 1997] and to group algebras under a Novikov involution [Bovdi and Siciliano 2007].

On the other hand, it seems that the rather natural problems of characterizing ordinary and restricted enveloping algebras which are normal under their canonical involutions have not been settled yet. The present paper is just devoted to answering these questions.

For an arbitrary Lie algebra L we denote by $U(L)$ the universal enveloping algebra of L. Moreover, if L is restricted with a p-map $[p]$ over a field $\mathbb{F}$ of

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characteristic $p > 0$, then we denote by $u(L)$ the restricted enveloping algebra of L. We consider $U(L)$ and $u(L)$ with the *principal involution* $*$, namely, the unique $\mathbb{F}$-antiautomorphism such that $x^* = -x$ for every x in L; see [Bourbaki 2007, Section 2] or [Dixmier 1974, Section 2]. Note that $*$ is just the antipode of the $\mathbb{F}$-Hopf algebras $U(L)$ or $u(L)$.

We use the symbols $Z(L)$ and L' for the center of L and the derived subalgebra of L, respectively. If $S \subseteq L$, we denote by $\langle S\rangle_{\mathbb{F}}$ the $\mathbb{F}$-vector space generated by S. Also, if L is restricted, $\langle S\rangle_p$ denotes the restricted subalgebra generated by S, and we put $S^{[p]} = \{x^{[p]} \mid x \in S\}$. In our first main result we completely settle the restricted case:

Theorem 1.1. *Let L be a restricted Lie algebra over a field $\mathbb{F}$ of characteristic $p > 0$. Then $u(L)$ is normal if and only if either L is abelian or $p = 2$, L is nilpotent of class* 2, *and one of the following conditions holds*:

(i) *L contains an abelian restricted ideal I of codimension* 1.

(ii) $\dim_{\mathbb{F}} L/Z(L) = 3$.

(iii) $\dim_{\mathbb{F}} L' = 1$ *and* $\left(L'\right)^{[2]} = 0$.

(iv) $L = \langle x, x_1, x_2, x_3\rangle_p + Z(L)$ *with*

$$[x_1, x_2] = \xi[x, x_3],$$
$$[x_1, x_3] = \mu[x, x_2],$$
$$[x_2, x_3] = \lambda[x, x_1],$$

and

$$\lambda[x, x_1]^{[2]} + \mu[x, x_2]^{[2]} + \xi[x, x_3]^{[2]} = 0$$

for some $\lambda, \mu, \xi \in \mathbb{F}$.

Afterwards we apply Theorem 1.1 in order to solve the ordinary case:

Theorem 1.2. *Let L be a Lie algebra over an arbitrary field $\mathbb{F}$. Then $U(L)$ is normal if and only if either L is abelian or $p = 2$, L is nilpotent of class* 2, *and one of the following conditions holds*:

(i) *L contains an abelian ideal of codimension* 1.

(ii) $\dim_{\mathbb{F}} L/Z(L) = 3$.

2. Proofs

For any associative algebra A, we shall consider the Lie bracket on A defined by $[a, b] := ab - ba \in A$, $a, b \in A$. The symbol $Z(A)$ will denote the center of A. Moreover, for a subset S of a Lie algebra L we shall denote by $C_L(S)$ the centralizer of S in L.

It is easy to verify that a normal algebra with involution satisfies the $*$-polynomial identity $[x, y] = [x^*, y^*]$. The converse is also true in characteristic different from 2, but in general it fails without such an assumption [Lim 1977]. However, for restricted Lie algebras we have the following:

Lemma 2.1. *Let L be a restricted Lie algebra over a field $\mathbb{F}$ of characteristic 2 such that $[x, y] = [x^*, y^*]$ for every $x, y \in u(L)$. Then L is nilpotent of class at most 2 and $u(L)$ is normal.*

Proof. For every $a, b, c \in L$, we have

$$0 = [ab, c] + [(ab)^*, c^*] = [[a, b], c].$$

Hence L is nilpotent of class at most 2.

Let $(e_i)_{i \in I}$ be an ordered $\mathbb{F}$-basis of L. Then every element u of $u(L)$ is an $\mathbb{F}$-linear combination of elements $e_{i_1} \cdots e_{i_m}$, where $m \geq 0$ and the indices $i_1 < \cdots < i_m$ are in I. As L is nilpotent of class at most 2, for every $z \in L$ we have $z^{[2]} \in Z(L)$, and then

$$[e_{i_1} \cdots e_{i_m}, (e_{i_1} \cdots e_{i_m})^*] = 0.$$

Moreover, by hypothesis we clearly have $[x, y^*] = [x^*, y]$ for every $x, y \in u(L)$. We conclude that $[u, u^*] = 0$, so that $u(L)$ is normal. □

Lemma 2.2. *Let L be a restricted Lie algebra over a field $\mathbb{F}$ of characteristic $p > 0$ such that $u(L)$ is normal. Then either L is abelian, or $p = 2$ and L is nilpotent of class 2.*

Proof. As $u(L)$ satisfies the $*$-polynomial identity $[x, y] = [x^*, y^*]$, if $p = 2$, Lemma 2.1 assures that L is nilpotent of class at most 2. Now suppose $p > 2$. For every $x, y \in L$, we have

$$0 = [x^2 + y, (x^2 + y)^*] = -4x[x, y] + 2[x, [x, y]].$$

Since $p > 2$, in view of the Poincaré–Birkhoff–Witt (PBW) theorem for restricted Lie algebras [Strade and Farnsteiner 1988, Section 2, Theorem 5.1], the previous relation is possible only when $[x, y] = 0$, so that L is abelian. This yields the claim. □

Let L be a restricted Lie algebra over a field of characteristic 2. For every $a, b, c, d \in L$, we put

$$\Theta(a, b, c, d) := [a, b][c, d] + [a, c][b, d] + [a, d][b, c] \in u(L).$$

The following result will be extremely useful in the sequel.

Lemma 2.3. *Let L be a restricted Lie algebra over a field $\mathbb{F}$ of characteristic* 2, *and suppose L to be nilpotent of class* 2. *Then $u(L)$ is normal if and only if $\Theta(a, b, c, d) = 0$ for all $a, b, c, d \in L$.*

Proof. If $u(L)$ is normal, for all $a, b, c, d \in L$ we have

$$\Theta(a, b, c, d) = [a, bcd] + [a, dcb] = [a, bcd] + [a, (bcd)^*] = 0.$$

Conversely, assume that $\Theta(a, b, c, d) = 0$ for all $a, b, c, d \in L$. Let $(e_j)_{j \in J}$ be an ordered $\mathbb{F}$-basis of L containing an $\mathbb{F}$-basis of $Z(L)$. Since $u(L)$ is a free $u(Z(L))$-module, there exists a unique homomorphism of $u(Z(L))$-modules

$$\phi : u(L) \to u(L),$$

which vanishes on 1 and L, and such that for every $n > 1$ and $j_1 < \ldots < j_n$, one has

$$\phi(e_{j_1} \cdots e_{j_n}) = \sum_{1 \leq h < k \leq n} e_{j_1} \cdots \hat{e}_{j_h} \cdots \hat{e}_{j_k} \cdots e_{j_n} [e_{j_h}, e_{j_k}],$$

where the symbol $\hat{e}_{i_h}$ indicates that e_{i_h} is to be omitted.

We claim that

$$\mathrm{Im}(\phi) \subseteq Z(u(L)).$$

For this purpose it is enough to prove that $[x, \phi(e_{j_1} \cdots e_{j_n})] = 0$ for every $x \in L$, $n > 1$, and $j_1, \ldots, j_n \in J$ with $j_1 < \ldots < j_n$. Indeed, by the hypothesis we have

$$\begin{aligned}
[x, \phi(e_{j_1} \cdots e_{j_n})] &= \left[x, \sum_{1 \leq h < k \leq n} e_{j_1} \cdots \hat{e}_{j_h} \cdots \hat{e}_{j_k} \cdots e_{j_n} [e_{j_h}, e_{j_k}] \right] \\
&= \sum_{1 \leq h < k \leq n} \sum_{\substack{1 \leq s \leq n \\ s \neq h,k}} e_{j_1} \cdots \hat{e}_{j_h} \cdots \hat{e}_{j_s} \cdots \hat{e}_{j_k} \cdots e_{j_n} [e_{j_h}, e_{j_k}][x, e_{j_s}] \\
&= \sum_{1 \leq h < k < s \leq n} e_{j_1} \cdots \hat{e}_{j_h} \cdots \hat{e}_{j_k} \cdots \hat{e}_{j_s} \cdots e_{j_n} \big([e_{j_h}, e_{i_k}][x, e_{j_s}] \\
&\qquad + [e_{j_h}, e_{i_s}][x, e_{j_k}] + [e_{j_k}, e_{i_s}][x, e_{j_h}] \big) = 0,
\end{aligned}$$

yielding the claim.

Now we shall prove that

$$a = a^* + \phi(a)$$

for every $a \in u(L)$. For this purpose it is enough to show that for all $n \geq 0$ and $j_1, \ldots, j_n \in J$ with $j_1 < \ldots < j_n$, one has

$$e_{j_1} \cdots e_{j_n} = e_{j_n} \cdots e_{j_1} + \phi(e_{j_1} \cdots e_{j_n}).$$

Let us proceed by induction on n. By the proved claim and the inductive assumption, we have, for $n > 0$,

$$\begin{aligned}
&e_{j_1} \cdots e_{j_n} \\
&\quad = (e_{j_{n-1}} \cdots e_{j_1})e_{j_n} + \phi(e_{j_1} \cdots e_{j_{n-1}})e_{j_n} \\
&\quad = e_{j_n}e_{j_{n-1}} \cdots e_{j_1} + [e_{j_{n-1}} \cdots e_{j_1}, e_{j_n}] + \phi(e_{j_1} \cdots e_{j_{n-1}})e_{j_n} \\
&\quad = e_{j_n}e_{j_{n-1}} \cdots e_{j_1} + [e_{j_1} \cdots e_{j_{n-1}}, e_{j_n}] + [\phi(e_{j_1} \cdots e_{j_{n-1}}), e_{j_n}] + \phi(e_{j_1} \cdots e_{j_{n-1}})e_{j_n} \\
&\quad = e_{j_n} \cdots e_{j_1} + \phi(e_{j_1} \cdots e_{j_n}),
\end{aligned}$$

completing the inductive step.

Finally, by applying the properties proved above, for all $a, b \in u(L)$, we have

$$[a, b] = [a^* + \phi(a), b^* + \phi(b)] = [a^*, b^*].$$

Hence $u(L)$ is normal by Lemma 2.1, as required. □

Remark 2.4. Since Θ is an alternating $\mathbb{F}$-multilinear function, by Lemma 2.3 it is clear that in order to conclude that $u(L)$ is normal, it suffices to check that $\Theta(a, b, c, d) = 0$ for all pairwise distinct noncentral elements a, b, c, d in a fixed $\mathbb{F}$-basis of L.

We are now in position to prove Theorem 1.1:

Proof of Theorem 1.1. Assume that $u(L)$ is normal and L is not abelian. Then, by Lemma 2.3, we know that $\mathbb{F}$ has characteristic 2 and L is nilpotent of class 2. Let us proceed with a case-by-case analysis.

Case 1. $\max\{\dim_{\mathbb{F}}[L, x] \mid x \in L\} = 1$. Let x_1 and y_1 be two noncommuting element of L and put $z_1 := [x_1, y_1]$. By assumption we have $[L, x_1] = [L, y_1] = \mathbb{F} z_1$ and $L = \mathbb{F} y_1 \oplus C_L(x_1)$. Now, if $C_L(x_1)$ is abelian, L satisfies alternative (i) of the statement. Suppose then that there exist $x_2, y_2 \in C_L(x_1)$ such that $[x_2, y_2] := z_2 \neq 0$. From Lemma 2.3 it follows that

$$z_1 z_2 = \Theta(x_1, y_1, x_2, y_2) = 0. \tag{1}$$

Therefore the PBW theorem for restricted Lie algebras entails that $z_1 = \lambda z_2$ for some $\lambda \in \mathbb{F}$, which shows that $L' = \mathbb{F} z_1$. Also, as $\lambda \neq 0$, by (1), we have $z_1^{[2]} = 0$. Thus $\left(L'\right)^{[2]} = 0$, and alternative (iii) of the statement holds.

Case 2. $\max\{\dim_{\mathbb{F}}[L, x] \mid x \in L\} = 2$. Let $x, x_1, x_2 \in L$ such that $z_1 := [x, x_1]$ and $z_2 := [x, x_2]$ are $\mathbb{F}$-linearly independent. We clearly have $L = \langle x_1, x_2\rangle_{\mathbb{F}} \oplus C_L(x)$. Furthermore, by Lemma 2.3, we have, for all $y_1, y_2 \in C_L(x)$,

$$0 = \Theta(x, x_1, y_1, y_2) = z_1[y_1, y_2] \quad \text{and} \quad 0 = \Theta(x, x_2, y_1, y_2) = z_2[y_1, y_2].$$

Since z_1 and z_2 are $\mathbb{F}$-linearly independent, the PBW theorem forces $[y_1, y_2] = 0$. Hence $C_L(x)$ is abelian. Again by Lemma 2.3, for every $y \in C_L(x)$, we have

$$0 = \Theta(x, x_1, x_2, y) = z_1[x_2, y] + z_2[x_1, y]. \tag{2}$$

At this stage, a straightforward application of the PBW theorem yields

$$[x_1, y] = \lambda_{11}(y)z_1 + \lambda_{12}(y)z_2 \quad \text{and} \quad [x_2, y] = \lambda_{21}(y)z_1 + \lambda_{22}(y)z_2$$

for some $\lambda_{11}(y), \lambda_{12}(y), \lambda_{21}(y), \lambda_{22}(y) \in \mathbb{F}$. From (2) it follows that

$$(\lambda_{11}(y) + \lambda_{22}(y))z_1z_2 = \lambda_{21}(y)z_1^2 + \lambda_{12}(y)z_2^2 \in L,$$

and, again by the PBW theorem, the preceding relation is possible only when $\lambda_{11}(y) = \lambda_{22}(y) := \lambda(y)$. With the notation just introduced, we consider the following subcases.

Subcase 2.1. *For every $u \in C_L(x)$, one has $\lambda_{12}(u) = \lambda_{21}(u) = 0$.* Let $y \in C_L(x)$ and put $\bar{y} := \lambda(y)x + y$. Then we have $[\bar{y}, x] = [\bar{y}, x_1] = [\bar{y}, x_2] = 0$. As $C_L(x)$ is abelian, it follows that $\bar{y} \in Z(L)$ and then $C_L(x) = \mathbb{F}x \oplus Z(L)$. Thus $\dim_{\mathbb{F}} L/Z(L) = 3$, and alternative (ii) of the statement holds.

Subcase 2.2. *There exists $u \in C_L(x)$ such that $\lambda_{12}(u) \neq 0$ and $\lambda_{21}(u) = 0$.* By replacing u by $\lambda_{12}^{-1}(u)u$, we can suppose that $\lambda_{12}(u) = 1$. Put $y := \lambda(u)x + u$. Then we have

$$[x_1, y] = z_2 \quad \text{and} \quad [x_2, y] = 0.$$

Let $y_1 \in C_L(x)$. Since $C_L(x)$ is abelian, by Lemma 2.3 we have

$$0 = \Theta(x_1, x_2, y, y_1) = z_2[x_2, y_1] = z_2(\lambda_{21}(y_1)z_1 + \lambda(y_1)z_2). \tag{3}$$

Consequently, as z_1 and z_2 are $\mathbb{F}$-linearly independent, the PBW theorem forces $\lambda_{21}(y_1) = 0$. Also, from relation (3) (applied for $y_1 = x$), we infer that $z_2^{[2]} = 0$. Now put $\bar{y}_1 := \lambda(y_1)x + \lambda_{12}(y_1)y + y_1$. Then $\bar{y}_1 \in Z(L)$, and $C_L(x) = \mathbb{F}x \oplus \mathbb{F}y \oplus Z(L)$. We conclude that $L = \langle x, x_1, x_2, y\rangle_p + Z(L)$, and it is clear that L is a restricted Lie algebra satisfying alternative (iv) of the statement.

Subcase 2.3. *There exists $u \in C_L(x)$ such that $\lambda_{12}(u) = 0$ and $\lambda_{21}(u) \neq 0$.* This is analogous to Subcase 2.2.

Subcase 2.4. *There exists $u \in C_L(x)$ such that $\lambda_{12}(u) \neq 0$ and $\lambda_{21}(u) \neq 0$.* By replacing u by $\lambda_{12}^{-1}(u)u$, we can suppose that $\lambda_{12}(u) = 1$. Put $y := \lambda(u)x + u$. Then we have

$$[x_1, y] = z_2 \quad \text{and} \quad [x_2, y] = \lambda_{21}(u)z_1.$$

Moreover, Lemma 2.3 yields

$$0 = \Theta(x, x_1, x_2, y) = \lambda_{21}(u)z_1^2 + z_2^2.$$

Let $y_1 \in C_L(x)$ and put $\bar{y}_1 := \lambda(y_1)x + y_1$. As $C_L(x)$ is abelian, Lemma 2.3 yields

$$0 = \Theta(x_1, x_2, y, \bar{y}_1) = z_2[x_2, \bar{y}_1] + \lambda_{21}(u)z_1[x_1, \bar{y}_1] = (\lambda_{21}(\bar{y}_1) + \lambda_{21}(u)\lambda_{12}(\bar{y}_1))z_1z_2,$$

so that $\lambda_{21}(\bar{y}_1) = \lambda_{21}(u)\lambda_{12}(\bar{y}_1)$. Put $\hat{y}_1 := \bar{y}_1 + \lambda_{12}(\bar{y}_1)y$. Then we have $[x_1, \hat{y}_1] = 0$. Now, if for some $y_1 \in C_L(x)$ one has $[x_2, \hat{y}_1] = \lambda_{21}(\hat{y}_1)z_1 \neq 0$ then we can replace y by $\hat{y}_1$ and conclude by Subcase 2.3 that alternative (iv) holds. On the other hand, if $[x_2, \hat{y}_1] = 0$ for every $y_1 \in C_L(x)$ then $L = \langle x, x_1, x_2, y\rangle_p + Z(L)$, and it is clear that, also in this case, L is a restricted Lie algebra satisfying alternative (iv).

Case 3. $\max\{\dim_{\mathbb{F}}[L, x] \mid x \in L\} = 3$. Let $x, u_1, u_2, u_3 \in L$ such that $z_1 := [x, u_1]$, $z_2 := [x, u_2]$, and $z_3 := [x, u_3]$ are $\mathbb{F}$-linearly independent. We clearly have $L = \langle u_1, u_2, u_3\rangle_{\mathbb{F}} \oplus C_L(x)$, and one can show that $C_L(x)$ is abelian in the same way as in Case 2. Moreover, in view of Lemma 2.3, we have

$$0 = \Theta(x, u_1, u_2, u_3) = z_1[u_2, u_3] + z_2[u_1, u_3] + z_3[u_1, u_2]. \tag{4}$$

Thus, for every $1 \leq i < j \leq 3$, by the PBW theorem, we see that

$$[u_i, u_j] = \sum_{k=1}^{3} \alpha_{ij}^{(k)} z_k, \tag{5}$$

where $\alpha_{ij}^{(k)} \in \mathbb{F}$, $k = 1, 2, 3$. By (4) and (5), another application of the PBW theorem yields

$$\alpha_{12}^{(1)} = \alpha_{23}^{(3)}, \quad \alpha_{12}^{(2)} = \alpha_{13}^{(3)}, \quad \alpha_{13}^{(1)} = \alpha_{23}^{(2)}.$$

Put

$$x_1 := u_1 + \alpha_{12}^{(2)}x, \quad x_2 := u_2 + \alpha_{12}^{(1)}x, \quad x_3 := u_3 + \alpha_{13}^{(1)}x,$$

and, moreover, $\alpha_{23}^{(1)} := \lambda$, $\alpha_{13}^{(2)} := \mu$, and $\alpha_{12}^{(3)} := \xi$. Then we have

$$[x_1, x_2] = \xi z_3, \quad [x_1, x_3] = \mu z_2, \quad [x_2, x_3] = \lambda z_1.$$

From Lemma 2.3 it follows that

$$\lambda z_1^{[2]} + \mu z_2^{[2]} + \xi z_3^{[2]} = \Theta(x, x_1, x_2, x_3) = 0.$$

Now, let $y \in C_L(x)$. By Lemma 2.3 we obtain

$$\Theta(x, x_1, x_2, y) = z_1[x_2, y] + z_2[x_1, y] = 0,$$
$$\Theta(x, x_1, x_3, y) = z_1[x_3, y] + z_3[x_1, y] = 0,$$
$$\Theta(x, x_2, x_3, y) = z_2[x_3, y] + z_3[x_2, y] = 0.$$

Consequently, by the PBW theorem there exists $\beta \in \mathbb{F}$ such that $[x_i, y] = \beta z_i$ for every $i = 1, 2, 3$. Put $\bar{y} := y + \beta x$. Then $\bar{y} \in Z(L)$ and $C_L(x) = \mathbb{F}x \oplus Z(L)$. We conclude that $L = \langle x, x_1, x_2, x_3\rangle_p + Z(L)$, and alternative (iv) is satisfied.

Case 4. $\max\{\dim_{\mathbb{F}}[L, x] \mid x \in L\} > 3$. Let $S := (u_i)_{i\in I}$ be a subset of L such that the elements $z_i := [x, u_i]$, $i \in I$, are $\mathbb{F}$-linearly independent, and $[S, x] = [L, x]$. We clearly have $L = \langle S\rangle_{\mathbb{F}} \oplus C_L(x)$, and one can show that $C_L(x)$ is abelian by proceeding in a similar way as in Case 2. Let $i, j \in I$, $i \neq j$. In view of Lemma 2.3, for every $k \in I\backslash\{i, j\}$, we have

$$0 = \Theta(x, u_i, u_j, u_k) = z_i[u_j, u_k] + z_j[u_i, u_k] + z_k[u_i, u_j].$$

At this stage, by arguing as in the first case of Case 3, we have that $[u_i, u_j] \in \mathbb{F} z_k$. As $|I| > 3$, we conclude that $[u_i, u_j] = 0$. Finally, let $y \in C_L(x)$. By Lemma 2.3, for all pairwise distinct elements i, j, k of I, we have

$$\Theta(x, u_i, u_j, y) = z_i[u_j, y] + z_j[u_i, y] = 0,$$
$$\Theta(x, u_i, u_k, y) = z_i[x_k, y] + z_k[u_i, y] = 0.$$

Therefore, an application of the PBW theorem shows that there exists $\beta \in \mathbb{F}$ such that $[u_i, y] = \beta z_i$ for every $i \in I$. Put $\bar{y} := y + \beta x$. Then $\bar{y} \in Z(L)$, so that $C_L(x) = \mathbb{F} x \oplus Z(L)$. Therefore, as $L^{[2]} \subseteq Z(L)$, we conclude that $Z(L) + \langle S\rangle_{\mathbb{F}}$ is an abelian restricted ideal of codimension 1 in L, and the proof of the necessity part is finished.

Now let us prove sufficiency. The claim is trivial if L is abelian. Then assume that the ground field has characteristic 2 and L is nilpotent of class 2. If L has an abelian restricted ideal of codimension 1, it is clear that $\Theta(a, b, c, d) = 0$ for any $a, b, c, d \in L$, and so, by Lemma 2.3, $u(L)$ is normal. Also, if $\dim_{\mathbb{F}} L/Z(L) = 3$ then $u(L)$ is normal by Lemma 2.3 and Remark 2.4. Furthermore, the claim is clear whenever $L' = \mathbb{F} z$ for some $0 \neq z \in L$ with $z^{[2]} = 0$. Finally suppose that alternative (iv) holds. We can assume that x, x_1, x_2, and x_3 are $\mathbb{F}$-linearly independent (otherwise alternative (i) or (ii) holds). Extend the set $\{x, x_1, x_2, x_3\}$ by central elements in order to form an $\mathbb{F}$-basis of L. We have

$$\begin{aligned}\Theta(x, x_1, x_2, x_3) &= [x, x_1][x_2, x_3] + [x, x_2][x_1, x_3] + [x, x_3][x_1, x_2]\\ &= \lambda[x, x_1]^{[2]} + \mu[x, x_2]^{[2]} + \xi[x, x_3]^{[2]} = 0.\end{aligned}$$

From Lemma 2.3 and Remark 2.4 it follows that $u(L)$ is normal. □

Finally, we deal with ordinary universal enveloping algebras of arbitrary Lie algebras. Indeed, we shall prove Theorem 1.2 as a consequence of Theorem 1.1.

Proof of Theorem 1.2. Suppose first that ground field $\mathbb{F}$ has characteristic zero. If L is abelian then $U(L)$ is obviously normal. On the other hand, if $U(L)$ is normal then it satisfies the standard polynomial identity of degree 4 [Herstein 1976, Section 5]. Therefore, in view of a theorem of Latyšev [Bahturin 1987, Section 6.7, Theorem 25], L is necessarily abelian. Now suppose $p > 0$. Put

$$\hat{L} := \sum_{k\geq 0} L^{p^k} \subseteq U(L),$$

where L^{p^k} is the $\mathbb{F}$-vector space spanned by the set $\{l^{p^k} \mid l \in L\}$. Then $\hat{L}$ is a restricted Lie algebra with $h^{[p]} = h^p$ for all $h \in \hat{L}$. Moreover, by [Strade 2004, Section 1, Corollary 1.1.4], we have $U(L) = u(\hat{L})$, and then Theorem 1.1 applies. Suppose first that $U(L)$ is normal. If $p > 2$, Theorem 1.1 forces $\hat{L}$ (and so L) to be abelian. Now assume that $p = 2$ and L is not abelian. Then $\hat{L}$ satisfies one of the alternatives (i)–(iv) in the statement of Theorem 1.1. If $\hat{L}$ contains an abelian restricted ideal of codimension 1 then L contains an abelian ideal of codimension 1. Likewise, if $\dim_{\mathbb{F}} \hat{L}/Z(\hat{L}) = 3$, $\dim_{\mathbb{F}} L/Z(L) = 3$. Observe that, as $u(\hat{L}) = U(L)$ is a domain, alternative (iii) in the statement of Theorem 1.1 cannot occur. Finally, suppose that $\hat{L} = \langle x, x_1, x_2, x_3\rangle_p + Z(\hat{L})$, where x, x_1, x_2, and x_3 are elements of L with $[x_1, x_2] = \xi[x, x_3]$, $[x_1, x_3] = \mu[x, x_2]$, $[x_2, x_3] = \lambda[x, x_1]$, and

$$\lambda[x, x_1]^{[2]} + \mu[x, x_2]^{[2]} + \xi[x, x_3]^{[2]} = 0$$

for some $\lambda, \mu, \xi \in \mathbb{F}$. Now, if $\dim_{\mathbb{F}} L' = 3$, the PBW theorem for ordinary enveloping algebras forces $\lambda = \mu = \xi = 0$. Hence L contains an abelian ideal of codimension 1. If $\dim_{\mathbb{F}} L' = 2$, we can suppose without loss of generality that $[x, x_1]$ and $[x, x_2]$ are $\mathbb{F}$-linearly independent and $[x, x_3] = \alpha[x, x_1] + \beta[x, x_2]$ for suitable $\alpha, \beta \in \mathbb{F}$. Consequently, we have

$$\alpha^2\xi[x, x_1]^2 + \beta^2\xi[x, x_2]^2 = \xi[x, x_3]^2 = \lambda[x, x_1]^2 + \mu[x, x_2]^2,$$

and the PBW theorem gets $\lambda = \alpha^2\xi$ and $\mu = \beta^2\xi$. Put

$$y := \alpha\beta\xi x + \alpha x_1 + \beta x_2 + x_3.$$

Then $y \in Z(\hat{L})$ and $\hat{L} = \langle x, x_1, x_2, y\rangle_p + Z(\hat{L})$. It follows that $\dim_{\mathbb{F}} \hat{L}/Z(\hat{L}) = 3$ and then $\dim_{\mathbb{F}} L/Z(L) = 3$ as well. Finally, if $\dim_{\mathbb{F}} L' = 1$ then it is easy to see that L contains an abelian ideal of codimension 1, and the necessity part is proved. Sufficiency easily follows from Theorem 1.1 and the fact that $U(L) = u(\hat{L})$. □

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