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BOUNDED AND UNBOUNDED CAPILLARY SURFACES IN A CUSP DOMAIN

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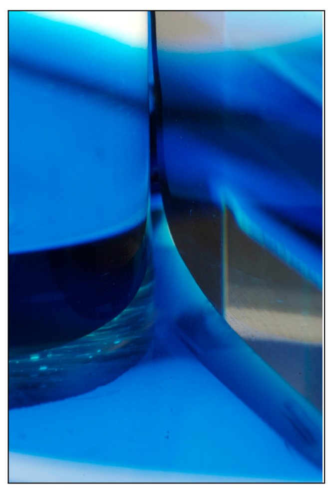
We study asymptotic behavior of the height of a static liquid surface in a cusp domain as modelled by the Laplace–Young capillary surface equation. We introduce a new form of an asymptotic expansion in terms of the functions defining the boundary curves forming a cusp. We are able to address the asymptotic behavior of the capillary surface in cusp domains not previously considered, such as an exponential cusp. In addition, we have shown that the capillary surface in a cusp domain is bounded if the contact angles of the boundary walls forming a cusp are supplementary angles, which implies the continuity of the capillary surface at the cusp.

1. Introduction

Background. In everyday life, it is often safe to assume that the surface of water at rest is almost flat; however, careful observation shows that the surface of water in a container can exhibit complicated geometry near the interface where the water meets the container. One of the most extreme examples is when the container has a sharp (cusped) boundary. As seen in the photo, the static liquid surface (capillary surface) rises very steeply near a cusp — formed in the case illustrated here by the tangency between a circular cylinder and a straight wall. This behavior can be understood through a singular solution of the Laplace–Young capillary surface equation.



As noted in [Finn 1986], the study of a singular capillary surface can be traced back to Brook Taylor in 1712. Later contributions to the study of singular capillary surfaces by Concus and Finn [1969] and Miersemann [1993] spurred considerable interest in the field; see, for example [King et al. 1999; Scholz 2001; 2004; Norbury et al. 2005; Aoki 2007]. In particular, Scholz's work on capillary surfaces in a

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domain containing a cusp where the boundaries can be approximated by power series (including fractional powers) led him to conclude that "[the capillary surface] rises with the same order [as] the order of contact of the two arcs, which form the cusp" [Scholz 2004]. Since this is a is very intuitive statement, our curiosity led us to ask whether this statement holds for cases that Scholz did not consider in his paper [2004].

In this paper we extend Scholz's results in two directions. We first consider cusp domains not limited to the power-law cusp. Instead of approximating the boundary by power series, we directly use the distance between two arcs forming a cusp in the asymptotic expansion. Although one may argue that most of the shapes used in real life applications can be approximated by power series, our main focus was to justify the above statement in a more direct and intuitive manner, by avoiding the extra approximation step. The second direction of extension is to include cases in which the contact angles of the boundary walls forming a cusp are supplementary angles. Although all the known results suggest that a capillary surface in a domain with a cusp is unbounded, we have shown that a capillary surface can be bounded, and hence continuous, if the contact angles are supplementary angles.

Statement of the problems. Here we state the problems we are going to consider in this paper. We first define a cusp domain. Without loss of generality, and for simplicity of writing, we consider the following domain (see Figure 1):

$$\Omega = \{(x, y) : x > 0,\ f_2(x) < y < f_1(x)\}, \tag{1-1}$$

where

$$\begin{gathered} f_1(x),\ f_2(x) \in C^3(0,\infty), \quad f_1(x) > f_2(x) \quad \text{for } x > 0, \\ \lim_{x\to 0^+} f_1(x) = \lim_{x\to 0^+} f_2(x) = 0, \quad \lim_{x\to 0^+} f_1'(x) = \lim_{x\to 0^+} f_2'(x) = 0. \end{gathered} \tag{1-2}$$

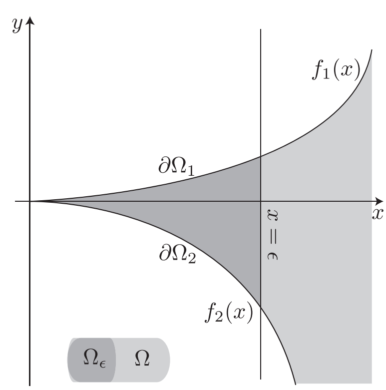

Figure 1. The cusped domain Ω and its boundary.

Also we denote the boundaries as follows:

$$\partial\Omega_1 = \{(x, y) : x > 0,\ y = f_1(x)\}, \quad \partial\Omega_2 = \{(x, y) : x > 0,\ y = f_2(x)\}.$$

Although we base our dicussion on this infinite domain, all of the results presented in this paper only depend locally on a domain sufficiently close to the cusp, so the results hold for any domain that coincides with Ω in a neighborhood of the origin.

We now state the partial differential equation that interests us, the Laplace–Young capillary surface equation. Let $u(x, y)$ be the height of a capillary surface in domain Ω. It satisfies the following boundary value problem (see [Finn 1986] for a derivation):

$$\nabla \cdot Tu = \kappa u \quad \text{in } \Omega, \tag{1-3}$$

$$\vec{\nu}_1 \cdot Tu = \cos\gamma_1 \quad \text{on } \partial\Omega_1, \tag{1-4}$$

$$\vec{\nu}_2 \cdot Tu = \cos\gamma_2 \quad \text{on } \partial\Omega_2, \tag{1-5}$$

where

$$Tu = \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}, \tag{1-6}$$

κ is the capillarity constant, $\vec{\nu}_1$ and $\vec{\nu}_2$ are exterior unit normal vectors on the boundaries $\partial\Omega_1$ and $\partial\Omega_2$, and γ_1, γ_2 are the contact angles. The capillarity constant κ can be normalized by rescaling x, y, and u. In the sequel we let $\kappa = 1$.

Here we introduce the big theta notation to replace the statement "is of the same order as", to make this expression more precise. If $f(x) = \Theta(g(x))$, there exist constants $k_1,\ k_2 > 0$ and $x_0 > 0$ such that

$$k_1|g(x)| < |f(x)| < k_2|g(x)| \quad \text{for all } x < x_0. \tag{1-7}$$

We note that Θ is a more strict order relation than that of O, i.e., if $f(x) = \Theta(g(x))$ then $f(x) = O(g(x))$; however the converse is not true.

We can now write our core research questions as follows:

- Suppose $\gamma_1 + \gamma_2 \neq \pi$. Does $u(x, y) = \Theta\left(\dfrac{1}{f_1(x) - f_2(x)}\right)$ hold for any $f_1(x)$ and $f_2(x)$ satisfying (1-2)?
- How does $u(x, y)$ behave asymptotically as $x \to 0^+$ when $\gamma_1 + \gamma_2 = \pi$?

Structure of the paper. As the title of this paper suggests, there are two main parts: unbounded and bounded cases.

In Section 2 we consider unbounded capillary surfaces in cusp domains. We first prove in Section 2A that capillary surfaces are unbounded if $\gamma_1 + \gamma_2 \neq \pi$. Then in Section 2B the formal asymptotic expansion is presented. Using the formal asymptotic expansion, in Section 2C we prove the asymptotic behavior of the solution. In Section 2D we give examples of power-law and non-power-law cusps with the intention of comparing our findings with the results in [Scholz 2004].

In Section 3 we consider bounded capillary surfaces in cusp domains. We first prove in Section 3A that capillary surfaces are bounded if $\gamma_1 + \gamma_2 = \pi$ and the curvature of the boundaries is finite. In Section 3B we show that if a capillary surface is bounded at the cusp, then it is continuous at the cusp. Section 4 contains concluding remarks summarizing our findings and suggesting some future extensions of our results. In addition, an Appendix we have included the Concus–Finn comparison principle and its Corollary used in Sections 2C and 3A.

2. Unbounded capillary surfaces

In this section, we assume $\gamma_1 + \gamma_2 \neq \pi$ and aim to prove that

$$u(x, y) = \Theta\left(\frac{1}{f_1(x) - f_2(x)}\right) \quad \text{as } x \to 0^+,$$

with as few restrictions on $f_1(x)$ and $f_2(x)$ as possible.

2A. ***Unboundedness of the capillary surface when $\gamma_1 + \gamma_2 \neq \pi$.*** We show that $u(x, y) \neq O(1)$. This is intuitively obvious from the remarkable result of Concus and Finn [1969], as a cusp can be considered as a corner with zero opening angle.

Lemma 2.1 (unboundedness of $u(x, y)$ when $\gamma_1 + \gamma_2 \neq \pi$). *Let $u(x, y)$ be the solution of the boundary value problem* (1-3)–(1-5).

If $\cos\gamma_1 + \cos\gamma_2 > 0$, *then $u(x, y)$ cannot be bounded from above.*

If $\cos\gamma_1 + \cos\gamma_2 < 0$, *then $u(x, y)$ cannot be bounded from below.*

Proof. Similar to the proof in [Concus and Finn 1969], we work by contradiction. First consider the case $\cos\gamma_1 + \cos\gamma_2 > 0$, and assume there exists a constant $M > 0$ such that $u(x, y) < M$ in Ω. Integrate the PDE (1-3) in a subdomain Ω_ϵ given by

$$\Omega_\epsilon = \{(x, y) : 0 < x < \epsilon,\ f_2(x) < y < f_1(x)\}.$$

By applying the divergence theorem and the boundary conditions (1-4) and (1-5), we obtain after some calculation the equation

$$\text{(2-1)} \quad \int_{x=0}^{\epsilon} \int_{y=f_2(x)}^{f_1(x)} u\, dy\, dx = \int_{x=0}^{\epsilon} \left(\cos\gamma_1 \sqrt{1+f_1'^2} + \cos\gamma_2 \sqrt{1+f_2'^2}\right) dx + \int_{y=f_2(\epsilon)}^{f_1(\epsilon)} \frac{u_x}{\sqrt{1+u_x^2+u_y^2}}\bigg|_{x=\epsilon} dx.$$

The trick is to realize that the last term of (2-1) can be bounded from below, i.e.,

$$\frac{u_x}{\sqrt{1+u_x^2+u_y^2}} > -1,$$

which implies

$$\int_{y=f_2(\epsilon)}^{f_1(\epsilon)} \frac{u_x}{\sqrt{1+u_x^2+u_y^2}}\bigg|_{x=\epsilon} dx > -(f_1(\epsilon)-f_2(\epsilon)).$$

We now apply the assumption $u(x, y) < M$ and the preceding inequality to (2-1) and obtain the inequality

$$\epsilon M \max_{0<x\le\epsilon} (f_1(x)-f_2(x)) + (f_1(\epsilon)-f_2(\epsilon)) \\ > \int_{x=0}^{\epsilon} \Big(\cos\gamma_1\sqrt{1+f_1'^2}+\cos\gamma_2\sqrt{1+f_2'^2}\Big)\,dx.$$

Dividing both sides by $\epsilon > 0$ and taking the limit as ϵ approaches 0 gives

$$\lim_{\epsilon\to 0^+} M \max_{0<x\le\epsilon} (f_1(x)-f_2(x)) + \lim_{\epsilon\to 0^+} \frac{f_1(\epsilon)-f_2(\epsilon)}{\epsilon} \\ \ge \lim_{\epsilon\to 0^+} \frac{\int_{x=0}^{\epsilon}\Big(\cos\gamma_1\sqrt{1+f_1'^2}+\cos\gamma_2\sqrt{1+f_2'^2}\Big)\,dx}{\epsilon}.$$

Applying the definition of the derivative together with (1-2) then gives

$$f_1'(0)-f_2'(0) \ge \Big(\cos\gamma_1\sqrt{1+f_1'(0)^2}+\cos\gamma_2\sqrt{1+f_2'(0)^2}\Big),$$

which implies $0 \ge \cos\gamma_1+\cos\gamma_2$. Hence we obtain a contradiction. The proof for the case where $\cos\gamma_1+\cos\gamma_2 < 0$ can be constructed similarly. □

Lemma 2.1 and Corollary A.1 together imply that $u(x, y)$ is unbounded at the cusp and bounded away from the cusp.

2B. ***Formal asymptotic expansion of the boundary value problem* (1-3)–(1-5).** The main idea is to consider an asymptotic expansion of the form

$$v(x, y) = \frac{A}{f_1(x)-f_2(x)} + g(x, y)\frac{f_1'(x)-f_2'(x)}{f_1(x)-f_2(x)} + h(x, y)\frac{(f_1'(x)-f_2'(x))^2}{f_1(x)-f_2(x)}, \tag{2-2}$$

where $g(x, y), h(x, y) \in O(1)$ as $x \to 0^+$. Recalling that $\lim_{x\to 0^+} f_1(x) = 0$ and $\lim_{x\to 0^+} f_2(x) = 0$, we have the first term significantly larger than the second term near the cusp. Also note that the leading order term is of the same order as the reciprocal of the distance between two boundaries measured in $\vec{y}$ direction.

The aim of this subsection is to find $g(x, y)$ and $h(x, y)$ such that (2-2) satisfies asymptotically the PDE (1-3) and the boundary conditions (1-4) and (1-5).

For simplicity of computation, we introduce coordinate variables s and t as follows:

$$s := x, \quad t := \frac{2y-(f_1(x)+f_2(x))}{f_1(x)-f_2(x)}.$$

We have chosen t so that $y = f_1(x)$ when $t = 1$, and $y = f_2(x)$ when $t = -1$.

Lemma 2.2 (first two terms of the formal asymptotic expansion). *In* (2-2), *let* $A = \cos\gamma_1 + \cos\gamma_2$, *and*

$$g(s,t) = -\sqrt{1 - \left(\frac{\cos\gamma_1(t+1) + \cos\gamma_2(t-1)}{2}\right)^2} + C_1$$

(*where* C_1 *is an arbitrary constant*), *and* $h(s,t) = 0$. *If* $f_1(s)$ *and* $f_2(s)$ *satisfy*

$$\begin{gathered} f_1(s) - f_2(s) = o\left(f_1'(s) - f_2'(s)\right), \quad \frac{f_1''(s) - f_2''(s)}{f_1(s) - f_2(s)} = o\left(\frac{f_1'(s) - f_2'(s)}{(f_1(s) - f_2(s))^2}\right), \\ \frac{f_1'''(s) - f_2'''(s)}{f_1'(s) - f_2'(s)} = o\left(\frac{1}{(f_1(s) - f_2(s))^2}\right), \end{gathered} \tag{2-3}$$

as $s \to 0^+$, *then*

$$\begin{gathered} \vec{\nu}_1 \cdot Tv|_{t=1} = \cos\gamma_1 + o(1), \quad \vec{\nu}_2 \cdot Tv|_{t=-1} = \cos\gamma_2 + o(1), \\ \nabla \cdot Tv - v = o\left(\frac{1}{f_1(s) - f_2(s)}\right) \end{gathered} \tag{2-4}$$

as $s \to 0^+$.

A tedious but straightforward calculation will verify this lemma. Instead of showing this calculation, we briefly explain here how the expressions for A, g, and h in the statement of the lemma were deduced. We first let

$$v(s,t) = \frac{A}{f_1(s) - f_2(s)} + g(t)\frac{f_1'(s) - f_2'(s)}{f_1(s) - f_2(s)}.$$

(It is desirable — and, as it turns out, possible — to make the function g depend only on t, so we will suppress the dependence of g on s; the same applies to the function h.) After some lengthy calculations with assumptions (2-3) we obtain

$$\begin{gathered} \vec{\nu}_1 \cdot Tv|_{t=1} = \frac{2g'(1)}{\sqrt{A^2 + 4g'^2(1)}} + o(1), \quad \vec{\nu}_2 \cdot Tv|_{t=-1} = -\frac{2g'(-1)}{\sqrt{A^2 + 4g'^2(-1)}} + o(1), \\ \nabla \cdot Tv - v = \left(\frac{4g''(t)A^2}{(A^2 + 4g'^2(t))^{3/2}} - A\right)\frac{1}{f_1(s) - f_2(s)} + o\left(\frac{1}{f_1(s) - f_2(s)}\right). \end{gathered}$$

We now impose the desired equalities (2-4) and obtain a nonlinear ordinary differential equation of the first order in $g'(t)$,

$$\frac{4g''(t)A^2}{(A^2 + 4g'^2(t))^{3/2}} = A \quad \text{for } -1 < t < 1, \tag{2-5}$$

with boundary conditions

$$\frac{2g'(1)}{\sqrt{A^2+4g'^2(1)}}=\cos\gamma_1, \qquad -\frac{2g'(-1)}{\sqrt{A^2+4g'^2(-1)}}=\cos\gamma_2. \tag{2-6}$$

Though there are two boundary conditions for this first-order ODE, note that A is an indeterminate constant. Both $g'(t)$ and A are determined by first integrating (2-5) udner the boundary conditions (2-6). One essential observation from this derivation is that the coefficient A of the leading-order term was found together with that of the second-order term, $g(t)$. In fact this pattern continues; the constant on the second-order term C_1 will be determined (it vanishes) at the same time as the third-order term of the formal asymptotic expansion is found.

Lemma 2.3 (first three terms of the formal asymptotic expansion). *In* (2-2), *let* $A=\cos\gamma_1+\cos\gamma_2$,

$$g(t)=-\sqrt{1-\left(\frac{\cos\gamma_1(t+1)+\cos\gamma_2(t-1)}{2}\right)^2},$$

and

$$h(t)=-\frac{A}{4}\left(\delta t+\frac{t^2}{2}\right)+\frac{1-\alpha}{2A}g(t)^2+C_2,$$

where C_2 is an arbitrary constant. If $f_1(s)$ and $f_2(s)$ satisfy the conditions

$$f_1'(s)>f_2'(s) \quad \textit{for } s>0, \tag{2-7}$$

$$f_1(s)-f_2(s)=o\left(f_1'(s)-f_2'(s)\right), \tag{2-8}$$

$$\frac{f_1''(s)-f_2''(s)}{f_1(s)-f_2(s)}=\alpha\frac{(f_1'(s)-f_2'(s))^2}{(f_1(s)-f_2(s))^2}+o\left(\frac{(f_1'(s)-f_2'(s))^2}{(f_1(s)-f_2(s))^2}\right), \tag{2-9}$$

$$\frac{f_1'''(s)-f_2'''(s)}{f_1'(s)-f_2'(s)}=O\left(\frac{(f_1'(s)-f_2'(s))^2}{(f_1(s)-f_2(s))^2}\right), \tag{2-10}$$

$$f_1'(s)+f_2'(s)=\delta(f_1'(s)-f_2'(s))+o(f_1'(s)-f_2'(s)), \tag{2-11}$$

$$f_1''(s)+f_2''(s)=O(f_1''(s)-f_2''(s)), \tag{2-12}$$

as $s\to 0^+$, where $\alpha,\ \delta\in\mathbb{R}$, then

$$\vec{\nu}_1\cdot Tv\,|_{t=1}=\cos\gamma_1+o(f_1'(s)-f_2'(s)), \qquad \vec{\nu}_2\cdot Tv\,|_{t=-1}=\cos\gamma_2+o(f_1'(s)-f_2'(s)),$$

$$\nabla\cdot Tv-v=o\left(\frac{f_1'(s)-f_2'(s)}{f_1(s)-f_2(s)}\right)$$

as $s\to 0^+$.

Again, a long tedious calculation will prove this lemma. We followed similar steps to determine $h(t)$, although solving the differential equation for $h(t)$ was not nearly as straightforward as for $g(t)$. The constant C_1 was determined to be 0 when $h(t)$ was determined and a new unknown constant C_2 appeared in the third-order term.

Comparing assumptions (2-3) with assumptions (2-8)–(2-12), we can see that the restrictions on f_1 and f_2 increase as the number of terms in the formal asymptotic expansion increases from two terms to three terms. Although these assumptions are not proven to be necessary conditions for these lemmas to hold, it is our suspicion that as the number of the terms in the asymptotic expansion increases, the restrictions on f_1 and f_2 do become more strict.

2C. ***Asymptotic behavior of the capillary surface.*** The main result of Section 2 is stated and proven in this subsection. We first show that the asymptotic growth order of the solution is the same order as the reciprocal of the distance between two arcs forming a cusp.

Theorem 2.1 (growth order of $u(x, y)$). *Let $u(x, y)$ be the solution of the boundary value problem* (1-3)–(1-5). *If $f_1(s)$ and $f_2(s)$ satisfy the conditions* (2-3) *and $|\cos\gamma_1| \neq 1$ and $|\cos\gamma_2| \neq 1$, then there exist positive constants s_0, k_1 and k_2 such that*

$$k_2\left(\frac{1}{f_1(s)-f_2(s)}\right) < |u(s,t)| < k_1\left(\frac{1}{f_1(s)-f_2(s)}\right), \quad \text{for } s < s_0. \tag{2-13}$$

Proof. The main idea of our proof is to construct a supersolution and a subsolution by modifying the formal asymptotic expansion given in Lemma 2.2. We prove these modified equations are in fact supersolution and subsolution by applying the Concus–Finn comparison principle (Theorem A.1). Let

$$v(s,t;K_1,K_2) = \frac{A(K_1)}{f_1(s)-f_2(s)} + g(t;K_1)\frac{f_1'(s)-f_2'(s)}{f_1(s)-f_2(s)} + K_2,$$

where

$$A(K_1) = \cos\gamma_1 + \cos\gamma_2 + K_1,$$

$$g(t;K_1) = -\frac{A}{A-\frac{1}{3}K_1}\sqrt{1-\left(\frac{\cos\gamma_1(t+1)+\cos\gamma_2(t-1)}{2} - \frac{K_1}{6}t\right)^2}; \tag{2-14}$$

here we choose K_1 and K_2 appropriately to construct the supersolution and the subsolution. The trick of this proof is to realize that A and $g(t)$, the first and second terms of the formal asymptotic expansion, need to be modified to obtain a supersolution and a subsolution. We first impose the following conditions on K_1

so that the quantities in (2-14) behave reasonably:

$$|K_1| < |\cos\gamma_1 + \cos\gamma_2|, \tag{2-15}$$

$$|K_1| < 6(1 - |\cos\gamma_1|), \tag{2-16}$$

$$|K_1| < 6(1 - |\cos\gamma_2|). \tag{2-17}$$

We restrict the choice of K_1 so that the sign of $A(K_1)$ only depends on the sign of $\cos\gamma_1 + \cos\gamma_2$. Also, if K_1 is chosen to satisfy (2-15)–(2-17), then $g(t, K_1)$ is real and bounded. After some calculations assuming (2-3), we obtain

$$\vec{\nu}_1 \cdot Tv|_{t=1} = \cos\gamma_1 + \tfrac{1}{3}K_1 + o(1), \quad \vec{\nu}_2 \cdot Tv|_{t=-1} = \cos\gamma_2 + \tfrac{1}{3}K_1 + o(1), \tag{2-18}$$

$$\nabla \cdot Tv - v = -\tfrac{1}{3}K_1 \frac{1}{f_1(s) - f_2(s)} - K_2 + o\left(\frac{f_1'(s) - f_2'(s)}{f_1(s) - f_2(s)}\right), \tag{2-19}$$

as $s \to 0^+$. The essential observation in this step of the proof is that the expressions in (2-18) do not depend on K_2 including the "small o" terms. Similarly, (2-19) has K_2 dependence only at the second term and not in the "small o" term.

We now construct a function v^+ that satisfies inequalities (A-1)–(A-4) in the Appendix, and is therefore a supersolution. We denote the associated constants by K_1^+ and K_2^+; i.e., $v^+ = v(s, t; K_1^+, K_2^+)$. Firstly, K_1^+ are chosen to be a small enough positive real number so as to satisfy (2-15)–(2-17). Then we choose a constant $s_0^+ > 0$ so that for all $s < s_0^+$ the inequalities

$$\vec{\nu}_1 \cdot Tv^+|_{t=1} - \cos\gamma_1 > 0, \quad \vec{\nu}_2 \cdot Tv^+|_{t=-1} - \cos\gamma_2 > 0, \tag{2-20}$$

$$\nabla \cdot Tv^+ - v^+ + K_2^+ < 0. \tag{2-21}$$

are satisfied. Based on our previous observation we note that the choice of s_0^+ is independent of K_2^+. Let Ω_0^+ be the subdomain of Ω such that $s < s_0^+$. By adding a restriction on K_2^+ to be a positive real number, it follows from (2-21) that

$$\nabla \cdot Tv^+ - v^+ < 0 \quad \text{in } \Omega_0^+.$$

Note that v^+ now satisfies conditions (A-1)–(A-3) of the Concus–Finn comparison principle (Theorem A.1). It remains to choose K_2^+ so as to satisfy condition (A-4). According to Corollary A.1, $u(s, t)$ is bounded at $s = s_0^+$. Hence there exists K_2^+ such that

$$v^+ > u \quad \text{on } s = s_0^+.$$

Thus by Theorem A.1 we have shown that there exists Ω_0^+, K_1^+, K_2^+ such that

$$v^+(s, t; K_1^+, K_2^+) > u(s, t) \quad \text{in } \Omega_0^+.$$

Similarly we can construct a subsolution $v^-(s,t;K_1^-,K_2^-)$ such that

$$v^-(s,t;K_1^-,k_2^-) < u(s,t) \quad \text{in } \Omega_0^-.$$

Hence in $\Omega_0^+ \cap \Omega_0^-$ we have $v^- < u < v^+$, i.e.,

$$\frac{A(K_1^-)}{f_1(s)-f_2(s)} + g(t;K_1^-)\frac{f_1'(s)-f_2'(s)}{f_1(s)-f_2(s)} + K_2^- < u$$

and

$$u < \frac{A(K_1^+)}{f_1(s)-f_2(s)} + g(t;K_1^+)\frac{f_1'(s)-f_2'(s)}{f_1(s)-f_2(s)} + K_2^+.$$

Since K_1^+ and K_1^- were chosen to satisfy (2-15), $A(K_1^+)$ and $A(K_1^-)$ have the same sign. Without loss of generality assume $A(K_1^+) > 0$. Let

$$m_1(s) = A(K_1^+) + \Big(\max_{-1<t<1} \big\{ g(t;K_1^+)(f_1'(s)-f_2'(s)) \big\} + K_2^+(f_1(s)-f_2(s)) \Big),$$
$$m_2(s) = A(K_1^-) + \Big(\min_{-1<t<1} \big\{ g(t;K_1^-)(f_1'(s)-f_2'(s)) \big\} + K_2^-(f_1(s)-f_2(s)) \Big).$$

Since $f_1'(s)-f_2'(s)$ and $f_1(s)-f_2(s)$ are $o(1)$ and continuous, there exists $s_0 > 0$ so that $m_1(s), m_2(s) > 0$ for $s < s_0$. By choosing

$$k_1 = \max_{0<s<s_0} m_1(s), \quad k_2 = \min_{0<s<s_0} m_2(s), \tag{2-22}$$

we obtain (2-13). □

Note that the proof holds for arbitrarily small $|K_1^\pm|$. Hence it is natural to guess that $(\cos\gamma_1+\cos\gamma_2)/(f_1(s)-f_2(s))$ is the correct leading-order term of the asymptotic expansion. We now show that the leading-order term of the formal asymptotic expansion is in fact the first-order term of the asymptotic expansion of $u(s,t)$.

Theorem 2.2 (leading-order behavior of $u(x,y)$). *Let $u(x,y)$ be the solution of the boundary value problem* (1-3)–(1-5). *Assume that $f_1(s)$ and $f_2(s)$ satisfy the conditions* (2-8)–(2-12). *Then*

$$u(s,t) = \frac{\cos\gamma_1+\cos\gamma_2}{f_1(s)-f_2(s)} + O\left(\frac{f_1'(s)-f_2'(s)}{f_1(s)-f_2(s)}\right) \quad \text{as } s \to 0^+. \tag{2-23}$$

Proof. We let

$$v(s,t;K_3,K_4,K_5) = \frac{A}{f_1(s)-f_2(s)} + g(t,K_3)\frac{f_1'(s)-f_2'(s)}{f_1(s)-f_2(s)} + h(t;K_4)\frac{(f_1'(s)-f_2'(s))^2}{f_1(s)-f_2(s)} + K_5,$$

where

$$A = \cos\gamma_1 + \cos\gamma_2,$$

$$g(t;\,K_3) = -\sqrt{1-\left(\frac{\cos\gamma_1(t+1)+\cos\gamma_2(t-1)}{2}\right)^2} + K_3,$$

$$h(t;\,K_4) = -\frac{A}{4}\left(\delta t+\frac{t^2}{2}\right)+\frac{1-\alpha}{2A}\left\{1-\left(\frac{\cos\gamma_1(t+1)+\cos\gamma_2(t-1)}{2}\right)^2\right\}+\frac{K_4}{2}t^2.$$

Unlike the proof of Theorem 2.1, we can choose K_3 and K_4 as any real numbers.

After some calculations assuming (2-8)–(2-12), we obtain

$$\vec{\nu}_1\cdot Tv|_{t=1} = \cos\gamma_1 + K_4\frac{(f_1'(s)-f_2'(s))}{\left(A^2+4(g'(t))^2\right)^{3/2}} + o(f_1'(s)-f_2'(s)), \tag{2-24}$$

$$\vec{\nu}_2\cdot Tv|_{t=-1} = \cos\gamma_2 + K_4\frac{(f_1'(s)-f_2'(s))}{\left(A^2+4(g'(t))^2\right)^{3/2}} + o(f_1'(s)-f_2'(s)), \tag{2-25}$$

$$\nabla\cdot Tv - v = \left\{\left(-\frac{12g'(t)t}{A^2+4(g'(t))^2}+\frac{4A^2}{\left(A^2+4(g'(t))^2\right)^{3/2}}\right)K_4 - K_3\right\}\frac{f_1'(s)-f_2'(s)}{f_1(s)-f_2(s)}$$
$$-K_5 + o\left(\frac{f_1'(s)-f_2'(s)}{f_1(s)-f_2(s)}\right), \tag{2-26}$$

as $s\to 0^+$.

We now construct a supersolution. Let v^+ denote the supersolution, with associate constants K_3^+, K_4^+, K_5^+; i.e., $v^+ = v(s,t;\,K_3^+,K_4^+,K_5^+)$. We first choose the positive constant K_4^+ arbitrarily. Then we choose K_3^+ big enough so that

$$\left\{\left(-\frac{12g'(t)t}{A^2+4(g'(t))^2}+\frac{4A^2}{\left(A^2+4(g'(t))^2\right)^{3/2}}\right)K_4^+ - K_3^+\right\} < 0 \quad \text{for } -1 < t < 1.$$

We now choose $s_2^+ > 0$ so that

$$\vec{\nu}_1\cdot Tv|_{t=1} - \cos\gamma_1 > 0, \quad \vec{\nu}_2\cdot Tv|_{t=-1} - \cos\gamma_2 > 0, \quad \nabla\cdot Tv - v + K_5^+ < 0$$

for $0 < s < s_2^+$. Let Ω_2^+ be the subdomain of Ω such that $s < s_2^+$. By Corollary A.1, we know that $u(s_2^+, t)$ is bounded; hence there exists a large enough positive constant K_5^+ so that

$$v^+ > u \quad \text{on } s = s_2^+.$$

Thus by the Concus–Finn comparison principle (Theorem A.1) we have

$$v^+ > u \quad \text{in } \Omega_2^+.$$

Similarly we can construct a subsolution v^- by choosing suitable K_3^-, K_4^-, K_5^- and s_2^-. Thus we can bound the solution $u(s,t)$ by v^- and v^+; i.e.,

$$v^- < u < v^+ \quad \text{in } \Omega_2^+ \cap \Omega_2^-,$$

and (2-23) holds. □

From this section, we conclude that the height of a capillary surface near a cusp is proportional to the reciprocal of the distance between the two arcs forming the cusp, assuming these arcs satisfy (2-3).

2D. ***Examples of cusp domains.*** In the previous subsection, we have shown the behavior of the capillary surface near a cusp under certain assumptions $f_1(x)$ and $f_2(x)$ giving the shape of the boundaries. Those assumptions, expressed by (2-3) or (2-8)–(2-12), are left in these forms in order to make the theorem as general as possible. On the other hand, it is hard to grasp what kind of cusps are allowed or not. In this subsection, we will show through examples when the theorem is applicable and when it is not.

It is easy to show that if the difference between f_1 and f_2 can be written in the following form, these functions satisfy (2-8)–(2-10):

$$f_1(x) - f_2(x) = c\, x^{a_0} \exp\left(\sum_{i=1}^{\infty} a_i x^{b_i}\right), \tag{2-27}$$

where $c > 0$, $a_1 < 0$, $b_1 < 0$, $b_{i+1} > b_i$. An alternative way to write this is

$$f_1(x) - f_2(x) = \exp\left(\int_c^x \frac{\sum_{i=0}^{\infty} \tilde{a}_i \zeta^{\tilde{b}_i}}{\sum_{i=0}^{\infty} a_i \zeta^{b_i}} d\zeta\right), \tag{2-28}$$

where $c > 0$, $b_0 - \tilde{b}_0 \geq 1$, $b_{i+1} > b_i$, $a_0 > 0$ and $\tilde{a}_0 > 0$. As (2-8)–(2-10) are stricter requirements for $f_1(x)$ and $f_2(x)$ than (2-3), if $f_1(x) - f_2(x)$ can be written as (2-27) or (2-28), then f_1 and f_2 satisfy (2-3).

Note that (2-11) and (2-12) can be interpreted as saying that some osculating cusps (cusps with boundaries tangent to second order) are not allowed, and Equation (2-7) can be interpreted as saying that infinitely oscillating cusp boundaries are not allowed.

Example 1 (fractional power cusp). We now consider a cusp that can be analyzed through the result of Scholz. Consider (2-28) and let $b_0 > 1$, $\tilde{a}_i = a_i\, b_i$ and $\tilde{b}_i = b_i - 1$. Then we have

$$f_1(x) - f_2(x) = \tilde{c} \sum_{i=0}^{\infty} a_i x^{b_i}. \tag{2-29}$$

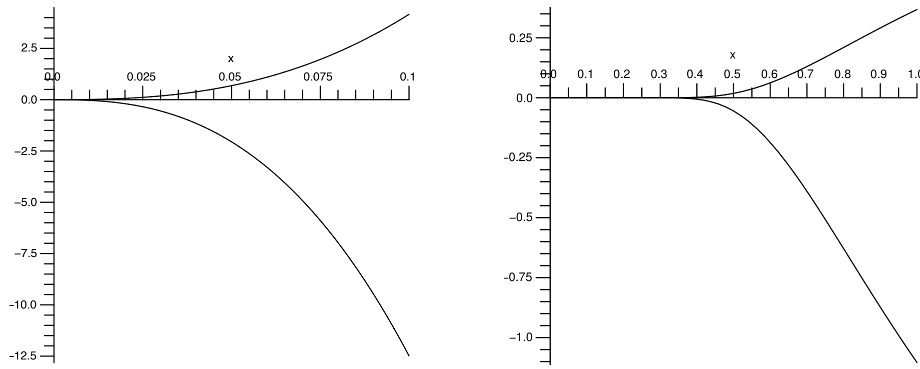

Figure 2. Left: fractional power cusp (Example 1). Right: exponential cusp (Example 2). In both cases, $p = 1$ and $q = -3$.

To be more specific, we consider the cusp boundaries

$$f_1(x) = p\,(x^{5/2} + x^3), \qquad f_2(x) = q\,(x^{5/2} + x^3), \tag{2-30}$$

with constants $p > q$ (see Figure 2, left). According to Theorem 2.2, we obtain the asymptotic expansion

$$\begin{aligned} u(x, y) &= \frac{\cos\gamma_1 + \cos\gamma_2}{(p-q)(x^{5/2} + x^3)} + O(x^{-1}) \\ &= \frac{\cos\gamma_1 + \cos\gamma_2}{p-q}\left(\frac{1}{x^{5/2}} - \frac{1}{x^2} + \frac{1}{x^{3/2}}\right) + O(x^{-1}) \end{aligned}$$

as $x \to 0^+$. We note that this result is consistent with that of Scholz. It is noteworthy that by finding the first order term of our asymptotic expansion we find the first three terms of the asymptotic series solution in power series.

Example 2 (exponential cusp). We now consider cusps to which the results of Scholz do not apply. Equation (2-27) implies that $f_1(x)$ and $f_2(x)$ can contain exponential terms. We now consider a very sharp cusp, an "exponential cusp", where

$$f_1(x) = p\,e^{-1/x^2}, \qquad f_2(x) = q\,e^{-1/x^2}.$$

with constants $p > q$ (see Figure 2, right). According to Theorem 2.2, we obtain the asymptotic expansion

$$u(x, y) = \frac{\cos\gamma_1 + \cos\gamma_2}{p-q} e^{1/x^2} + O(x^{-3}) \quad \text{as } x \to 0^+.$$

This example shows that our result has extended the result of Scholz on the leading order behavior of a capillary surface in a cusp domain.

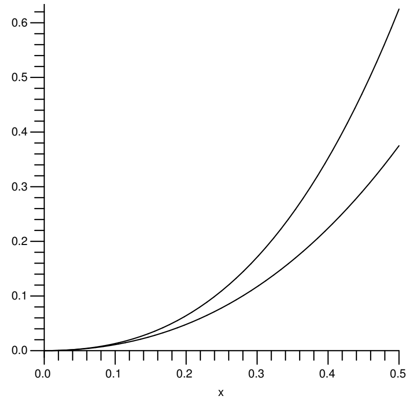

Figure 3. Osculatory cusp ($p = 3$, $q = 1$).

Example 3 (osculatory cusp). We now consider a case where Theorem 2.2 cannot be applied. Consider the cusp boundaries

$$f_1(x) = x^2 + px^3, \quad f_2(x) = x^2 + qx^3, \tag{2-31}$$

with constants $p > q$ (see Figure 3).

These functions do not satisfy (2-11)–(2-12); hence Theorem 2.2 does not apply. On the other hand, if $|\cos\gamma_1| \neq 1$ and $|\cos\gamma_2| \neq 1$, Theorem 2.1 applies, as this f_1 and f_2 satisfy (2-3). Hence even the case of the osculating cusp, we have shown that the height of the capillary surface rises as the same order as the reciprocal of the distance of two arcs forming a cusp, i.e.,

$$u(x, y) = \Theta\left(\frac{1}{x^3}\right). \tag{2-32}$$

As the two functions f_1 and f_2 forming a cusp only appear as $(f_1(x) - f_2(x))$ or $(f_1'(x) - f_2'(x))$ in the asymptotic expansion (2-2), it is not immediately obvious as to why we cannot conduct the asymptotic analysis of this problem similarly to the case where $f_1(x) = px^3$, $f_2(x) = qx^3$. However, the difference in asymptotic order between $f_1(x) - f_2(x)$ on the one hand and $f_1(x)$ or $f_2(x)$ on the other becomes crucial in calculating the asymptotic relations (2-24)–(2-26) of the boundary conditions and the PDE. For example, for the calculation of (2-24), since

$$\vec{\nu}_1 = \frac{(-f_1'(x), 1)}{\sqrt{1 + (f_1'(x))^2}},$$

the function $f_1(x)$ appears without subtracting $f_2(x)$. As a result, the asymptotic relation (2-24) does not hold for the case of osculatory cusp. Thus for the osculatory cusps, we cannot use the asymptotic expansion (2-2) to prove the leading order behavior.

3. Bounded capillary surfaces

In this section we assume $\gamma_1 + \gamma_2 = \pi$ and prove that $u(x, y)$ is bounded.

3A. *Proof of the boundedness of the capillary surface when* $\gamma_1 + \gamma_2 = \pi$.

Theorem 3.1 (boundedness of $u(x, y)$ when $\gamma_1 + \gamma_2 = \pi$). *Let* $u(x, y)$ *be the solution of the boundary value problem* (1-3)–(1-4) *with* $\gamma_1 = \gamma$ *and* $\gamma_2 = \pi - \gamma$. *If the boundaries* $\partial\Omega_1$ *and* $\partial\Omega_2$ *have finite curvatures in the neighborhood of the cusp*, *in other words*, *if there exists* ϵ_o *such that*

$$f_1(x),\ f_2(x) \in C^2([0, \epsilon_o]), \tag{3-1}$$

then $u(x, y)$ *is bounded.*

Proof. It follows immediately from Corollary A.1 that $u(x, y)$ is bounded in the domain away from the origin. Hence our problem reduces to show that $u(x, y)$ is bounded in the neighborhood of the origin.

First we show that $u(x, y)$ is bounded above at the origin by using the Concus–Finn comparison principle (Theorem A.1). In order to apply Theorem A.1, we need to construct a surface that satisfies (A-1)–(A-4). The most difficult part of this proof is to construct a surface that satisfies both (A-2) and (A-3). Our unique idea is to construct a surface that satisfies (1-4) exactly hence (A-2) and also satisfies (A-3). Such surface can be constructed by a surface with contour lines parallel to the boundary $\partial\Omega_1$. In other words by letting the height of the surface only depends on the distance from the boundary $\partial\Omega_1$, we can easily construct a surface with exact constant contact angle γ on this boundary. We choose a surface so that the height and the mean curvature is bounded so that Inequalities (A-1) and (A-4) can easily be satisfied by shifting this surface upwards.

We now translate the above statement to the precise language of mathematics. Without loss of generality we assume $0 \le \gamma \le \pi/2$. First we define a coordinate system such that the one family of the coordinate curves is parallel curves of the boundary $\partial\Omega_1$ and another family of the coordinate curves is lines perpendicular to the boundary $\partial\Omega_1$. Let s and t be new coordinate variables defined implicitly as the following (note that s here has different meaning from s used in Section 2):

$$(x, y) = (s, f_1(s)) - t\,\vec{\nu}_1(s), \tag{3-2}$$

where $\vec{\nu}_1(s)$ is the exterior unit normal vector of the boundary $\partial\Omega_1$ at $(s, f_1(s))$. More explicitly, the coordinate variables of Cartesian coordinate system x and y can be written using the new coordinate variables s and t as follows:

$$x = s + t\frac{f_1'(s)}{\sqrt{1 + (f_1'(s))^2}}, \qquad y = f_1(s) - t\frac{1}{\sqrt{1 + (f_1'(s))^2}}. \tag{3-3}$$

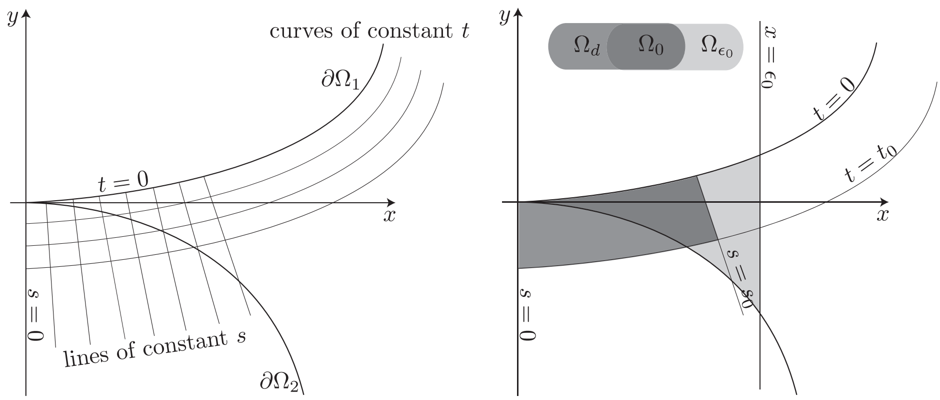

Figure 4. Left: coordinate lines of the s-t coordinate system. Right: the domain Ω_0.

The variable t can be interpreted as the distance of the point from the boundary $\partial\Omega_1$. The coordinate curves are sketched in Figure 4, left.

The Jacobian of (3-3) is calculated to be

$$\frac{\partial(x, y)}{\partial(s, t)} = \frac{f_1'(s)^2 - 1}{\sqrt{1 + (f_1'(s))^2}} \left(1 + t \frac{f_1''(s)}{(1 + (f_1'(s))^2)^{3/2}}\right).$$

This gives that the point (x, y) in the Cartesian coordinate system can be specified uniquely by the new coordinate variables (s, t) defined by (3-3) if both

$$f_1'(s)^2 - 1 \neq 0 \tag{3-4}$$

and

$$1 + t \frac{f_1''(s)}{(1 + (f_1'(s))^2)^{3/2}} \neq 0. \tag{3-5}$$

Since $f_1(s) \in C^2([0, \epsilon_o])$ and $\lim_{s \to 0^+} f_1(s) = 0$, there exists $0 < s_0 \le \epsilon_0$ so that (3-4) is satisfied for all $s \in [0, s_0]$. Also due to the smoothness of $f_1(s)$, we can find $t_0 > 0$ such that (3-5) holds for all $t \in [0, t_0]$ in $s \in [0, s_0]$. That is to say, the coordinate system defined in (3-3) is valid in the domain

$$\Omega_d := \{(s, f_1(s)) - t\,\vec{\nu}_1(s) \in \mathbb{R}^2 : 0 \le s \le s_0, 0 \le t \le t_0\}.$$

Then we choose the subdomain

$$\Omega_0 := \Omega_d \cap \Omega_{\epsilon_0},$$

where $\Omega_{\epsilon_0} := \{(x, y) \in \mathbb{R}^2 : 0 < x < \epsilon_0,\ f_2(x) < y < f_1(x)\}$, as depicted in Figure 4, right. Since $\bar{\Omega}_0$ contains the cusp at the origin, finding an upper bound for the surface u in domain Ω_0 by using Theorem A.1 would prove that the capillary surface

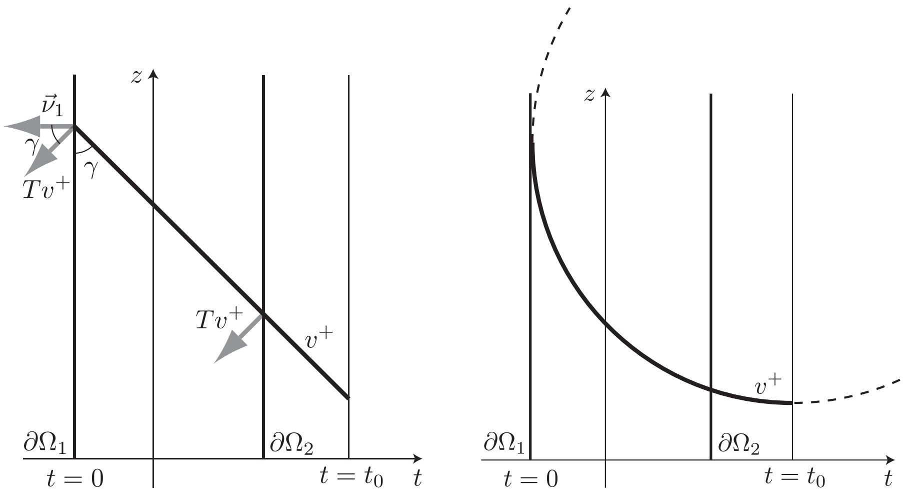

Figure 5. Cross section of a surface $v^+(s,t)$ on the line of constant s: Choice of function $g(t)$ for $\gamma \neq 0$ (left) and for $\gamma = 0$ (right).

is bounded above at the cusp. Using the parameters t and s, we now construct a surface $v^+(s,t)$ in Ω_0, with components (x, y, z), as follows:

(3-6)
$$x(s,t) = s + t\frac{f_1'(s)}{\sqrt{1+(f_1'(s))^2}}, \quad y(s,t) = f_1(s) - t\frac{1}{\sqrt{1+(f_1'(s))^2}}, \quad z(s,t) = g(t).$$

The choice of the height function $g(t)$ depends on the contact angle γ. In our opinion, the simplest choice such that the surface v^+ satisfies (1-4) exactly and also satisfies (A-3) is

$$g(t) = \begin{cases} -\cot\gamma\, t + K & \text{for } \gamma \neq 0, \\ -\sqrt{t_0^2 - (t-t_0)^2} + K & \text{for } \gamma = 0, \end{cases} \tag{3-7}$$

where K is a constant that we will specify later. The cross section of this surface on a line of constant s is depicted in Figure 5, left.

The surface $v^+(s,t)$ can be sketched as in Figure 6. For example, if the curve $\partial\Omega_1$ is a part of a circle, then the surface $v^+(s,t)$ for the case $\gamma \neq 0$ becomes a part of a cone, and for the case $\gamma = 0$ it becomes a part of a torus.

We now verify that the surface $v^+(s,t)$ satisfies (1-4) exactly and also satisfies (A-3). We first consider the case $\gamma \neq 0$, as the vector Tv^+ can be interpreted as a unit downwards vector of the surface v^+, it follows immediately from Figure 5 (left) that $Tv^+(s,t)$ can be written as

$$Tv^+ = \cos\gamma\, \vec{\nu}_1 - \sin\gamma\, \hat{z},$$

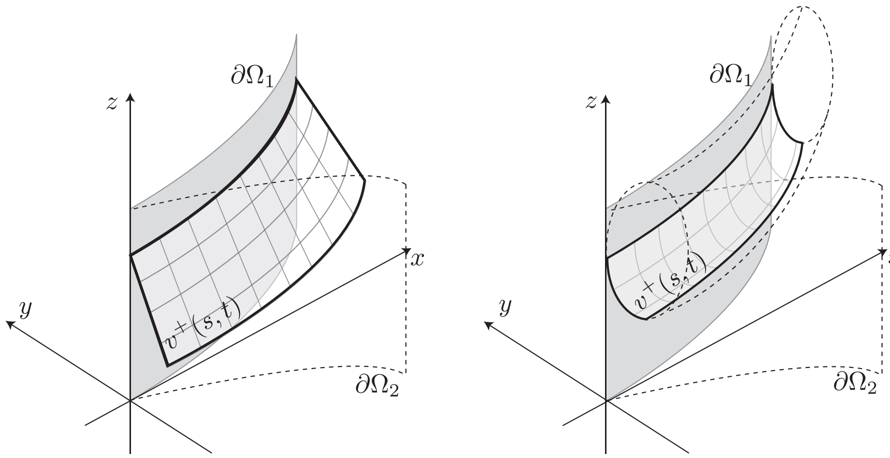

Figure 6. Sketch of the surface $v^+(s,t)$ for $\gamma \neq 0$ (left) and for $\gamma = 0$ (right).

where $\hat{z}$ is a unit vector in z direction. Noting that the vector $\vec{\nu}_1$ is orthogonal to $\hat{z}$, we obtain that (1-4) is satisfied exactly by the surface $v^+(s,t)$, i.e.,

$$\vec{\nu}_1 \cdot Tv^+ = \cos\gamma \quad \text{on } \partial\Omega_1 \cap \partial\Omega_0.$$

We now verify that the surface $v^+(s,t)$ satisfies Inequality (A-3). By noticing $\vec{\nu}_2$ and $\hat{z}$ are orthogonal and both $\vec{\nu}_1$ and $\vec{\nu}_2$ are unit vectors, we obtain the inequality

$$\vec{\nu}_2 \cdot Tv^+ = \cos\gamma\,\vec{\nu}_1 \cdot \vec{\nu}_2, > -\cos\gamma, = \cos(\pi - \gamma).$$

Although the case of $\gamma = 0$ may look complicated, it follows immediately from Figure 5 (right) that the angle between the unit downward normal vector of v^+ and $\vec{\nu}_1$ are parallel on the boundary, on $\partial\Omega_1 \cap \partial\Omega_0$,

$$\vec{\nu}_1 \cdot Tv^+ = 1 = \cos 0.$$

Also it follows immediately from the definition of the differential operator T that $|Tv^+| \leq 1$; see (1-6). By noting that $\vec{\nu}_2$ is a unit vector, i.e., $|\vec{\nu}_2| = 1$, we have

$$\nu_2 \cdot Tv^+ > -1 = \cos(\pi - 0).$$

Hence the surface $v^+(s,t)$ defined by (3-6)–(3-7) satisfies Inequalities (A-2) and (A-3). We now show that the surface $v^+(s,t)$ satisfies (A-1) by choosing large enough constant K.

Since $\nabla \cdot Tv^+$ is twice the mean curvature of the surface v^+, it is given by the well-known formula (see [Moon and Spencer 1970], for example)

$$\nabla \cdot Tv^+ \;=\; -2H(v^+) = -\frac{EN + GL - 2FM}{EG - F^2},$$

where

$$E = (x_s)^2 + (y_s)^2 + (z_s)^2, \quad F = x_s x_t + y_s y_t + z_s z_t, \quad G = (x_t)^2 + (y_t)^2 + (z_t)^2,$$

and

$$L = \frac{\begin{vmatrix} x_{ss} & y_{ss} & z_{ss} \\ x_s & y_s & z_s \\ x_t & y_t & z_t \end{vmatrix}}{\sqrt{EG - F^2}}, \qquad M = \frac{\begin{vmatrix} x_{st} & y_{st} & z_{st} \\ x_s & y_s & z_s \\ x_t & y_t & z_t \end{vmatrix}}{\sqrt{EG - F^2}}, \qquad N = \frac{\begin{vmatrix} x_{tt} & y_{tt} & z_{tt} \\ x_s & y_s & z_s \\ x_t & y_t & z_t \end{vmatrix}}{\sqrt{EG - F^2}}.$$

After some calculation we obtain

$$\nabla \cdot Tv^+ = \frac{g_1''(t)}{(1 + (g'(t))^2)^{3/2}} + \frac{f_1''(s)}{(1 + (f_1'(s))^2)^{3/2}\left(1 + t\dfrac{f_1''(s)}{(1 + (f_1'(s))^2)^{3/2}}\right)} \frac{g'(t)}{\sqrt{1 + (g'(t))^2}}.$$

Recalling that we have chosen the domain Ω_0 so that (3-5) holds in Ω_0 and that $f_1''(s) \in C^2([0, \epsilon_o])$, in order to show $\nabla \cdot Tv^+$ is bounded, all we need to show is that $g_1''(t)/(1 + (g'(t))^2)^{3/2}$ is bounded, that is to say, the curvature of the curve $g(t)$ is bounded. For the case of $\gamma \neq 0$, we have chosen $g(t)$ to be a linear function, so $g''(t)$ is zero. For the case of $\gamma = 0$, we have chosen $g(t)$ to be the part of a circle with radius t_0, so $g_1''(t)/(1 + (g'(t))^2)^{3/2} = 1/t_0$. In either case, it follows that $\nabla \cdot Tv^+$ is bounded. We now consider the quantity $\nabla \cdot Tv^+ - v^+$, which can be written as

$$\nabla \cdot Tv^+ - v^+ = \nabla \cdot Tv^+ - (g(t) + K).$$

It follows immediately from the choice of $g(t)$ that it is bounded in the domain $\bar{\Omega}_0$ and also we have shown that twice the mean curvature $\nabla \cdot Tv^+$ is bounded and does not depend on K. Hence there exists a constant K_0 such that

$$\nabla \cdot Tv^+ - v^+ = \nabla \cdot Tv^+ - (g(t) + K) \leq 0 \quad \text{for all } K \geq K_0.$$

Thus we have shown that the surface v^+ satisfies the (A-1) when $K > K_0$.

We now put the last piece of the puzzle in place by showing v^+ satisfies (A-4) for an appropriate choice of the constant K. Corollary A.1 implies that the capillary surface u is bounded away from the cusp, hence it is bounded on

$$\partial\Omega_0 \backslash (\partial\Omega_1 \cup \partial\Omega_2 \cup \{(0, 0)\}).$$

Since $g(t)$ is bounded in the domain $\bar{\Omega}_0$, there exists a constant $K_1 \geq K_0$ such that $g(t) + K_1 > u$ on $\partial\Omega_0 \backslash (\partial\Omega_1 \cup \partial\Omega_2 \cup \{(0, 0)\})$. Thus the surface v^+ satisfies (A-4) when $K = K_1$.

We have shown that the surface $v^+(s,t)$ defined in (3-6)–(3-7) satisfies inequalities (A-1)–(A-4), so by the Concus–Finn comparison principle we have

$$v^+(s,t) \geq u(x,y) \quad \text{in } \Omega_0.$$

Therefore the capillary surface at the cusp is bounded above when $\gamma_1+\gamma_2=\pi$ and each boundary $(\partial\Omega_1, \partial\Omega_2)$ has finite curvature near the cusp.

We can follow the similar steps for constructing the subsurface to show that this capillary surface is bounded below. We first construct a coordinate system such that one of the families of the coordinate curves is parallel curves of the boundary $\partial\Omega_2$ and another is perpendicular lines of the boundary $\partial\Omega_2$. Then choose a surface v^- so that the heigh only depends on the distance from $\partial\Omega_2$ which satisfies the contact angle condition exactly on $\partial\Omega_2$ and also it satisfies $\vec{\nu}_1 \cdot Tv^- - \cos\gamma \leq 0$. By choosing v^- to have the bounded height and the finite mean curvature, we can shift this surface downwards enough to satisfy $\nabla \cdot Tv^- - v^- \geq 0$ in Ω_0 and $v^- \leq u$ on $\partial\Omega_0 \backslash (\partial\Omega_1 \cup \partial\Omega_2 \cup \{(0,0)\})$. Then using the Concus–Finn comparison principle, we can prove that $u(x,y)$ is bounded below.

Thus by showing that there exist bounded sub- and supersolutions of the Laplace–Young capillary surface equation, we have proven that the capillary surface is bounded if the contact angles of the boundaries are supplementary angles and boundaries have finite curvatures near the cusp. □

3B. *Proof of the continuity of the capillary surface when $\gamma_1+\gamma_2=\pi$.*

Theorem 3.2. *If the capillary surface satisfies the conditions in Theorem 3.1, it is continuous at the cusp.*

Proof. Having established the boundedness of the solution, we can use the methods of [Lancaster and Siegel 1996] to establish a parametric description of the surface, with parameter domain at first the unit disk. The above comparison surface is needed in proving Case 5 (page 173) in that reference. Assuming the surface is discontinuous at the corner implies that an arc of the unit circle corresponds to the points on the surface above the corner point. A change of coordinates allows us to use the half-unit disk as the parameter domain, where the boundary line segment corresponds to the points on the surface above the corner point. Following the proof of Step 3 (page 175) of [Lancaster and Siegel 1996], for two different heights, there are level curves going through the corner point, and this leads to a contradiction (last paragraph of page 175 of the same reference). □

4. Concluding remarks

We have shown that the validity of the statement "[the capillary surface] rises with the same order like the order of contact of the two arcs, which form the cusp"

[Scholz 2004] is not restricted to power-law cusps; it can be extended further. Our proof directly uses the the functions $f_1(x)$ and $f_2(x)$ without approximating them by series. This idea has given us an advantage in the sense that our leading order term expression gives clearer intuitive understanding of the relationship between the shape of the domain and the shape of the singular capillary surface. Also as shown in an Example in Subsection 2.4.1, our leading order term gives first three terms of the power series asymptotic expansion, owing to the fact we have avoided approximating the boundary by the power series.

Even though we have extended the results beyond power-series cusps, our results still suffer from certain restrictions, including (2-8)–(2-12). Also a complete asymptotic series solution maybe desirable in order to claim a complete understanding of the asymptotic behavior; however, this will require further assumptions to the boundary functions f_1 and f_2. The authors suspect that functions f_1 and f_2 of a form similar to the right-hand side of (2-27) can be potential candidates for a type of cusp for which a complete asymptotic series can be determined.

Also we have shown the previously unknown phenomenon of a bounded capillary surface in a cusp domain is possible when the contact angles of the two walls are supplementary (i.e., $\gamma_1 + \gamma_2 = \pi$). Although our proof covers most of the cases when the boundaries are smooth except at the cusp, the behavior of the capillary surface is unknown when the curvature of the boundary is not finite at the cusp. For example, it is unknown whether or not the capillary surface is bounded in a cusp domain bounded by $f_1 = x^{3/2}$ and $f_2 = -x^{3/2}$ when the contact angles of the two walls are supplementary.

The phenomenon that the capillary surface can be bounded or unbounded in a cusp domain depending on the contact angle can be interesting physically, as it indicates that a gradual change in the contact angle (e.g., by changing the temperature of the liquid) can cause a dramatic change in the liquid surface from unbounded to bounded. However, as the bounded capillary surface in a cusp domain only appears when the contact angles are exactly supplementary, it is not unknown to the authors how easily this phenomena can be observed through an experiment.

Thus we end this paper by remarking that the further exploration of singular capillary surfaces through theoretical, experimental and possibly numerical analyses is desired.

Appendix: The Concus–Finn comparison principle

In Sections 2C and 3A we have used the Concus–Finn comparison principle. We present it here for readers unfamiliar with it; see [Finn 1986, pages 110–113; 1989] for detailed discussions and proofs. We use the following formulation of the comparison principle:

Theorem A.1 (supersolution). *Let $u(x, y)$ be a solution of the boundary value problem* (1-3)–(1-5) *and let Ω_0 be a subdomain of Ω, with boundary $\partial\Omega_0$. Suppose a function $v^+(x, y)$ satisfies the inequalities*

$$\nabla \cdot T v^+ - v^+ \le 0 \quad \text{in } \Omega_0, \tag{A-1}$$

$$\vec{\nu}_1 \cdot T v^+ - \cos\gamma_1 \ge 0 \quad \text{on } \partial\Omega_1 \cap \partial\Omega_0, \tag{A-2}$$

$$\vec{\nu}_2 \cdot T v^+ - \cos\gamma_2 \ge 0 \quad \text{on } \partial\Omega_2 \cap \partial\Omega_0, \tag{A-3}$$

$$v^+(x, y) \ge u(x, y) \quad \text{on } \partial\Omega_0 \backslash (\partial\Omega_1 \cup \partial\Omega_2 \cup \{(0,0)\}). \tag{A-4}$$

Then $v^+(x, y)$ is a supersolution of the boundary value problem (1-3)–(1-5), *i.e.*,

$$v^+(x, y) \ge u(x, y) \quad \text{in } \Omega_0.$$

A similar statement holds for subsolutions.

Also we make use of one of the corollaries of the comparison principle to construct an upper bound for the solution; see [Concus and Finn 1974] or pages 113–114 of [Finn 1986].

Corollary A.1 (bound by hemispheres). *Let $u(x, y)$ be a solution of the boundary value problem* (1-3)–(1-5) *and $B_{r_0}(x_0, y_0)$ a disk of radius $r_0 > 0$ centered at (x_0, y_0). If $B_{r_0}(x_0, y_0) \subseteq \Omega$, then*

$$-\left(\frac{1}{r_0} + r_0\right) \le u(x, y) \le \frac{1}{r_0} + r_0 \quad \text{in } B_{r_0}(x_0, y_0). \tag{A-5}$$

Recalling from (1-2) that the boundary is assumed to be of class C^3 away from the origin, it follows immediately from Corollary A.1 that $u(x, y)$ can only be unbounded at the origin (cusp).

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