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The Lempert function for a set of poles in a domain of $\mathbb{C}^n$ at a point z is obtained by taking a certain infimum over all analytic disks going through the poles and the point z; it majorizes the corresponding multipole pluricomplex Green function. Coman proved that both coincide in the case of sets of two poles in the unit ball. We give an example of a set of three poles in the unit ball where this equality fails.

1. Introduction

Let Ω be a domain in $\mathbb{C}^n$, and $a_j \in \Omega$, $j = 1, \ldots, N$. The pluricomplex Green function with logarithmic singularities at $S := \{a_1, \ldots, a_N\}$ is defined by

$$G_S(z) := \sup\{u \in \mathrm{PSH}(\Omega, \mathbb{R}_-) : u(z) \le \log|z - a_j| + C_j,\ j = 1, \ldots, N\},$$

where $\mathrm{PSH}(\Omega, \mathbb{R}_-)$ stands for the set of all negative plurisubharmonic functions in Ω. When Ω is hyperconvex, this solves the Monge–Ampère equation with right hand side equal to $\sum_{i=1}^{N} \delta_{a_j}$.

Pluricomplex Green functions have been studied by many authors at different levels of generality. See [Demailly 1987; Zahariuta 1984; Lempert 1981; Lelong 1989; Lárusson and Sigurdsson 1998].

A deep result due to Poletsky [1993], and see also [Lárusson and Sigurdsson 1998; Edigarian 1997], is that the Green function may be computed from analytic disks:

$$G_S(z) = \inf\Big\{ \sum_{\alpha:\varphi(\alpha)\in S} \log|\alpha| : \text{such that there exists } \varphi \in \mathbb{O}(\mathbb{D}, \Omega) \text{ with } \varphi(0) = z \Big\}. \tag{1-1}$$

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However, it is tempting to pick only one $\alpha_j \in \varphi^{-1}(a_j)$ in the range $1 \leq j \leq N$, which motivated the Coman's definition of the Lempert function [2000]:

$$\ell_S(z) := \inf \left\{ \sum_{j=1}^{N} \log|\zeta_j| : \varphi(0) = z,\ \varphi(\zeta_j) = a_j,\ j = 1, \dots, N \text{ for some } \varphi \in \mathbb{O}(\mathbb{D}, \Omega) \right\}, \tag{1-2}$$

where $\mathbb{D}$ is the unit disc in $\mathbb{C}$.

One easily sees that $\ell_S(z) \geq G_S(z)$ without recourse to (1-1); the fact that equality holds when $N = 1$ and Ω is convex is part of Lempert's celebrated theorem [1981], which was, in fact, the starting point for many of the notions defined above; see also [Edigarian 1995]. Coman [2000] proved that equality holds when $N = 2$ and $\Omega = \mathbb{B}^2$, the unit ball of $\mathbb{C}^2$. The goal of this note is to present an example that shows that this is as far as it can go.

Theorem 1.1. *There exists a set of* 3 *points* $S \subset \mathbb{B}^2$ *such that* $\ell_S(z) > G_S(z)$ *for some* $z \in \mathbb{B}^2$.

Other examples in the same vein have been found in [Carlehed and Wiegerinck 2003; Thomas and Trao 2003; Nikolov and Zwonek 2005]. The interesting features of this one are that it involves no multiplicities and is minimal in the ball. Examples with an arbitrary number of points can be deduced from it. Let $z_0 \in \mathbb{B}^2$ satisfy $\ell_S(z_0) - G_S(z_0) =: \varepsilon_0 > 0$. Consider $S' := S \cup \{a_4, \dots, a_N\}$ with all the a_j close enough to the boundary so that $\ell_{S'}(z_0) \geq \ell_S(z_0) - \varepsilon_0/2$ (the Schwarz lemma shows that $|\zeta_j| \to 1$ when $\varphi(\zeta_j) = a_j$ and $|a_j| \to 1$). Then $\ell_{S'}(z_0) > G_S(z_0) \geq G_{S'}(z_0)$, as was to be shown. (I thank Nikolai Nikolov for sharing this observation with me).

Moreover, the corresponding Green function can be recovered, up to a bounded error, by using an analytic disk with just one more preimage than the number of points: One of the points has exactly two preimages and each of the other two points, only one; see [Magnússon et al. 2012, §6.8.2, Lemma 6.16].

More specifically, the theorem will follow from a precise calculation in the bidisk $\mathbb{D}^2$. Let $S_\varepsilon = \{(0, 0), (\rho(\varepsilon), 0), (0, \varepsilon)\} \subset \mathbb{D}^2$, where $\lim_{\varepsilon \to 0} \rho(\varepsilon)/\varepsilon = 0$.

Proposition 1.2. *There exists* $C_1 > 0$ *such that for any* $\delta \in (0, 1/4)$ *there exists* $\varepsilon_0 = \varepsilon_0(z, \delta) > 0$ *and* $r_0 = r_0(\delta) > 0$ *such that*

$$G_{S_\varepsilon}(z) \leq 2 \log|z_2| + C_1, \tag{1-3}$$

$$\ell_{S_\varepsilon}(z) \geq (2 - \delta) \log|z_2|. \tag{1-4}$$

for any ε *with* $|\varepsilon| < \varepsilon_0$ *and any* $z = (z_1, z_2) \in \mathbb{D}^2$ *such that*

$$\tfrac{1}{2}|z_2|^{3/2} \leq |z_1| \leq |z_2|^{3/2} \quad \text{and} \quad \|z\| < r_0. \tag{1-5}$$

Proof of Theorem 1.1. If U and V are domains, and $S \subset U \subset V$, then the definitions of the Green and Lempert functions imply that $G_S^U(z) \geq G_S^V(z)$ and $\ell_S^U(z) \geq \ell_S^V(z)$.

For $|\varepsilon|$ small enough, $S_\varepsilon \subset \mathbb{B}^2$. When $|z_1| = |z_2|^{3/2}$, so that z verifies (1-5), the inclusion $\mathbb{B}^2 \subset \mathbb{D}^2$ implies

$$\ell_{S_\varepsilon}^{\mathbb{B}^2}(z) \geq \ell_{S_\varepsilon}^{\mathbb{D}^2}(z) \geq (2-\delta)\log|z_2|.$$

Using the fact that $\mathbb{D}^2/\sqrt{2} \subset \mathbb{B}^2$ and the invariance of the Green function under biholomorphic mappings, we have

$$G_{S_\varepsilon}^{\mathbb{B}^2}(z) \leq G_{S_\varepsilon}^{\mathbb{D}^2/\sqrt{2}}(z) = G_{\sqrt{2}S_\varepsilon}^{\mathbb{D}^2}(\sqrt{2}z) \leq 2\log|z_2| + \log 2 + C_1.$$

The last inequality follows from the fact that $\sqrt{2}z$ still verifies (1-5), and $\sqrt{2}S_\varepsilon$ has the same form as S_ε, so we can apply (1-3).

Comparing the last two estimates, we see that $G_{S_\varepsilon}^{\mathbb{B}^2}(z) < \ell_{S_\varepsilon}^{\mathbb{B}^2}(z)$ for $|z_2|$ small enough and $|\varepsilon| < \varepsilon_0$. □

Open questions

This example is minimal in the ball, in terms of number of poles; what is the situation for the bidisk? Are the Green and Lempert functions equal when one takes two poles, not lying on a line parallel to the coordinate axes? Do they at least have the same order of singularity as one pole tends to the other?

What is the precise order of the singularity of the limit as $\varepsilon \to 0$ of the Lempert function in this case? Looking at the available analytic disks that give the correct order of the singularity of the limit of the Green function, one finds $\frac{3}{2}\log|z_2|$, so one would hope that the proposition can still be proved at least for $\delta < 1/2$.

Do the analytic disks from [Magnússon et al. 2012] yield the Green function itself, without any bounded error term?

More generally, when one is given a finite number of points in a given bounded (hyperconvex) domain, is there a bound on the number of preimages required to attain the Green function in the Poletsky formula? For instance, is 4 the largest possible number of preimages required when looking at 3 points in the ball?

2. Upper estimate for the Green function

Proof of (1-3) *of Proposition 1.2.* The upper bound (1-3) follows from [Magnússon et al. 2012, §6.8.2, Lemma 6.16]. For the reader's convenience, and since that paper is not generally available, we repeat the proof here in the case that concerns us.

We now construct an analytic disk passing twice through one of the poles. Our disk will be a perturbation of the Neil parabola $\zeta \mapsto (\zeta^3, \zeta^2)$.

We write $s(\varepsilon) = \rho(\varepsilon)/\varepsilon = o(1)$.

Choose complex numbers λ and μ such that

$$\lambda^2 := \frac{z_1}{z_2(z_2-\varepsilon)}\left(\frac{z_1}{z_2-\varepsilon}+s(\varepsilon)\right) \quad \text{and} \quad \mu^2 := \varepsilon + \left(\frac{s(\varepsilon)}{2\lambda}\right)^2.$$

Let

$$\Psi_{\lambda,\mu}(\zeta) := \left(\left(\lambda\zeta - \tfrac{1}{2}s(\varepsilon)\right)(\zeta^2-\mu^2), \zeta^2 - \left(\frac{s(\varepsilon)}{2\lambda}\right)^2\right).$$

Then by construction $\Psi_{\lambda,\mu}(\mu) = \Psi_{\lambda,\mu}(-\mu) = (0, \varepsilon)$,

$$\Psi_{\lambda,\mu}\left(\frac{s(\varepsilon)}{2\lambda}\right) = (0,0) \quad \text{and} \quad \Psi_{\lambda,\mu}\left(-\frac{s(\varepsilon)}{2\lambda}\right) = (\varepsilon s(\varepsilon), 0),$$

so we have a disk passing through all three poles of G_ε. Furthermore, choosing

$$\zeta_z := \frac{1}{\lambda}\left(\frac{z_1}{z_2-\varepsilon} + \frac{s(\varepsilon)}{2}\right),$$

we have $\Psi_{\lambda,\mu}(\zeta_z) = z$. Notice that

$$\zeta_z^2 = \frac{z_2(z_2-\varepsilon)}{z_1}\left(\frac{z_1}{z_2-\varepsilon}+\frac{s(\varepsilon)}{2}\right)^2\left(\frac{z_1}{z_2-\varepsilon}+s(\varepsilon)\right)^{-1},$$

so for any $\eta > 0$ there exists $\varepsilon_0(\delta,\eta) > 0$ such that for $|\varepsilon| < \varepsilon_0(\delta,\eta)$

$$\big||\zeta_z| - |z_2|^{1/2}\big| \le \eta \tag{2-1}$$

for any z such that $\delta \le \frac{1}{2}|z_2|^{3/2} \le |z_1| \le |z_2|^{3/2} \le 1$. In particular, by choosing η small enough we ensure that $\zeta_z \in \mathbb{D}$. We need a more general fact.

Claim. *Let $\eta > 0$, and $\delta > 0$. Then there exists $\varepsilon_1 = \varepsilon_1(\delta,\eta) > 0$ such that for any ε with $|\varepsilon| \le \varepsilon_1$, we have $\Psi_{\lambda,\mu}(D(0, 1-\eta)) \subset \mathbb{D}^2$ for any z such that $\delta \le \frac{1}{2}|z_2|^{3/2} \le |z_1| \le |z_2|^{3/2} \le 1$.*

Proof. For $|\varepsilon| \le \delta^{2/3}/2$, we have $|z_2|/2 \le |z_2-\varepsilon| \le 2|z_2|$, so

$$|\lambda|^2 \ge \left|\frac{z_1}{2z_2^2}\right|\left(\left|\frac{z_1}{2z_2}\right| - |s(\varepsilon)|\right) \ge \left|\frac{z_1^2}{8z_2^3}\right| \ge \frac{1}{32}$$

for ε small enough. So when $|\zeta| \le 1-\eta$,

$$|\Psi_{\lambda,\mu,2}(\zeta)| \le (1-\eta)^2 + 256|s(\varepsilon)|^2 < 1$$

for ε small enough.

In a similar way, given η', for ε small enough depending on δ and η', we have $|z_2| \le (1+\eta')|z_2-\varepsilon|$, so

$$|\lambda|^2 \le (1+\eta')^2\left|\frac{z_1}{z_2^2}\right|\left(\left|\frac{z_1}{z_2}\right| + \frac{|s(\varepsilon)|}{(1+\eta')}\right) \le (1+\eta')^3\left|\frac{z_1^2}{z_2^3}\right| \le (1+\eta')^3$$

for ε small enough. Choose η' so that $(1+\eta')^3 = (1+\eta)$. When $|\zeta| \le 1-\eta$,

$$|\Psi_{\lambda,\mu,1}(\zeta)| \le \big((1+\eta)(1-\eta) + \tfrac{1}{2}|s(\varepsilon)|\big)\big((1-\eta)^2 + |\varepsilon| + 64^2|s(\varepsilon)|^2\big) < 1$$

for ε small enough. □

So now the function $v(\zeta) := G_\varepsilon(\Psi_{\lambda,\mu}((1-\eta)\zeta))$ is negative and subharmonic on $\mathbb{D}$. Furthermore, it has logarithmic poles at the points

$$\pm\frac{\mu}{1-\eta} \quad \text{and} \quad \pm\frac{s(\varepsilon)}{2\lambda(1-\eta)};$$

in the cases when $\mu = 0$ or $s(\varepsilon) = 0$, we get a double logarithmic pole at the corresponding point.

Denote by $d_G(\zeta, \xi) := |(\zeta - \xi)/(1 - \zeta\bar{\xi})|$ the invariant (pseudohyperbolic) distance between points of the unit disk. Then

$$\begin{aligned} G_\varepsilon(z) = v(\zeta_z) \le \log d_G\Big(\zeta_z, \frac{\mu}{1-\eta}\Big) &+ \log d_G\Big(\zeta_z, -\frac{\mu}{1-\eta}\Big) \\ &+ \log d_G\Big(\zeta_z, \frac{s(\varepsilon)}{2\lambda(1-\eta)}\Big) + \log d_G\Big(\zeta_z, -\frac{s(\varepsilon)}{2\lambda(1-\eta)}\Big). \end{aligned}$$

By (2-1), choosing $m(\delta, \eta)$ accordingly, we have $G_\varepsilon(z) \le 4\log|z_2|^{1/2} + O(\eta)$ for $|\varepsilon| \le m$. □

3. Lower estimate for the Lempert function

Proof of (1-4) *of Proposition 1.2.* The proof will follow the methods and notations of [Thomas 2007]. We will make repeated use of the involutive automorphisms of the unit disk given by $\phi_a(\zeta) := (a-\zeta)/(1-\bar{a}\zeta)$ for $a \in \mathbb{D}$, which exchange 0 and a. Notice that the invariant (pseudohyperbolic) distance verifies

$$d_G(a, b) := |\phi_a(b)| = |\phi_b(a)|.$$

Write $\rho(\varepsilon) = \varepsilon s(\varepsilon)$ with $\lim_{\varepsilon\to 0} s(\varepsilon) = 0$.

We will assume that the conclusion fails. That is, for any $\delta \in (0, 1/4)$, there exist arbitrarily small values of $|z_2| = \max(|z_1|, |z_2|)$, and $|\varepsilon|$ such that

$$\ell_{S_\varepsilon}(z) < (2-\delta)\log|z_2|. \tag{3-1}$$

After applying, for each analytic disk, an automorphism of the disk that exchanges the preimage of (0, 0) and 0, the assumption implies that there exists a holomorphic map φ from $\mathbb{D}$ to $\mathbb{D}^2$ and points $\zeta_j \in \mathbb{D}$, depending on z and ε, satisfying the conditions

$$\begin{aligned} \varphi(0) &= (0,0), & \varphi(\zeta_1) &= (\varepsilon s(\varepsilon), 0), \\ \varphi(\zeta_0) &= (z_1, z_2), & \varphi(\zeta_2) &= (0, \varepsilon), \end{aligned} \tag{3-2}$$

with

$$\log|\zeta_0| + \log|\phi_{\zeta_0}(\zeta_1)| + \log|\phi_{\zeta_0}(\zeta_2)| \le (2-\delta)\log|z_2|. \tag{3-3}$$

The interpolation conditions in (3-2) are equivalent to the existence of holomorphic functions h_1 and h_2 from $\mathbb{D}$ to itself such that

$$\varphi(\zeta) = (\zeta\phi_{\zeta_2}(\zeta)h_1(\zeta), \zeta\phi_{\zeta_1}(\zeta)h_2(\zeta)),$$

such that furthermore

$$h_1(\zeta_1) = \frac{\varepsilon s(\varepsilon)}{\zeta_1\phi_{\zeta_2}(\zeta_1)} =: w_1, \tag{3-4}$$

$$h_1(\zeta_0) = \frac{z_1}{\zeta_0\phi_{\zeta_2}(\zeta_0)} =: w_2, \tag{3-5}$$

$$h_2(\zeta_2) = \frac{\varepsilon}{\zeta_2\phi_{\zeta_1}(\zeta_2)} =: w_4, \tag{3-6}$$

$$h_2(\zeta_0) = \frac{z_2}{\zeta_0\phi_{\zeta_1}(\zeta_0)} =: w_3. \tag{3-7}$$

By the invariant Schwarz lemma, the existence of a holomorphic function h_1 mapping $\mathbb{D}$ to itself and satisfying (3-4) and (3-5) is equivalent to

$$|w_1| < 1, \quad |w_2| < 1 \quad \text{and} \quad d_G(w_1, w_2) < d_G(\zeta_1, \zeta_0) = |\phi_{\zeta_1}(\zeta_0)|. \tag{3-8}$$

In the same way, the existence of h_2 is equivalent to

$$|w_3| < 1, \quad |w_4| < 1 \quad \text{and} \quad d_G(w_3, w_4) < d_G(\zeta_2, \zeta_0) = |\phi_{\zeta_2}(\zeta_0)|. \tag{3-9}$$

As in [Thomas 2007], we start by remarking that (3-3) can be rewritten as

$$\begin{aligned} -\log|w_2| - \log|w_3| &= \log\left|\frac{\zeta_0\phi_{\zeta_1}(\zeta_0)}{z_2}\right| + \log\left|\frac{\zeta_0\phi_{\zeta_0}(\zeta_2)}{z_1}\right| \\ &\le \log|\zeta_0| + (2-\delta)\log|z_2| - \log|z_1| - \log|z_2| \\ &\le \log|\zeta_0| - \left(\tfrac{1}{2}+\delta\right)\log|z_2| + \log 2, \end{aligned} \tag{3-10}$$

by (1-5). We can rewrite this in a more symmetric fashion:

$$\log\frac{1}{|w_2|} + \log\frac{1}{|w_3|} + \log\frac{1}{|\zeta_0|} \le \left(\tfrac{1}{2}+\delta\right)\log\frac{1}{|z_2|} + \log 2. \tag{3-11}$$

Since all terms are positive by (3-8) and (3-9), each of the terms on the left hand side is bounded by the right hand side.

We will proceed as follows: We have used the contradiction hypothesis (3-3) to prove that $|\zeta_0|$ and $|w_3|$ are relatively big. We will prove that $|\phi_{\zeta_2}(\zeta_0)|$ has to be relatively small, which by (3-9) forces $|w_4|$ to be roughly as large as $|w_3|$. This then allows us to bound $|\phi_{\zeta_1}(\zeta_2)|$ by a quantity that becomes as small as desired when ε can be made small, and hence allows us to bound $|\phi_{\zeta_1}(\zeta_0)|$ by the triangle inequality.

The final contradiction will concern $w_2 = z_1/(\zeta_0\phi_{\zeta_2}(\zeta_0))$. On the one hand, (3-11) guarantees that it is not too small; but an explicit computation of the quotient w_1/w_4 shows that w_1 must be small, and by (3-8) and the estimate on $|\phi_{\zeta_1}(\zeta_0)|$, $|w_2|$ is small as well.

We provide the details. From (3-11),

$$\log|w_3| \ge \left(\tfrac{1}{2}+\delta\right)\log|z_2| - \log 2. \tag{3-12}$$

From (3-5) and (3-10),

$$\begin{aligned} \log|\phi_{\zeta_2}(\zeta_0)| &= \log|z_1/\zeta_0| - \log|w_2| \\ &\le \log|z_1/\zeta_0| + \log|\zeta_0| - \left(\tfrac{1}{2}+\delta\right)\log|z_2| + \log 2 \\ &\le (1-\delta)\log|z_2| + \log 2. \end{aligned} \tag{3-13}$$

Since $\delta < 1/4$, (3-13) and (3-12) imply that $|\phi_{\zeta_2}(\zeta_0)| < \frac{1}{2}|w_3|$ for $|z_2| \le r_1(\delta)$, so by (3-9) and the triangle inequality for d_G,

$$|w_4| \ge \tfrac{1}{2}|w_3|. \tag{3-14}$$

We now prove that both ζ_1 and ζ_2 must be close to ζ_0 and even closer to each other. First, since (3-11) implies that $\log|\zeta_0| \ge (\frac{1}{2}+\delta)\log|z_2| - \log 2$, by (3-13), $|\phi_{\zeta_2}(\zeta_0)| \le \frac{1}{2}|\zeta_0|$ for $|z_2| \le r_2(\delta)$. By the triangle inequality for d_G,

$$\tfrac{1}{2}|\zeta_0| \le |\zeta_2| \le \tfrac{3}{2}|\zeta_0|. \tag{3-15}$$

On the other hand, from (3-11),

$$\log|w_3| + \log|\zeta_0| \ge (\tfrac{1}{2}+\delta)\log|z_2| - \log 2, \quad \text{that is, } |w_3\zeta_0| \ge \tfrac{1}{2}|z_2|^{\delta+1/2}.$$

Therefore, applying (3-14) and (3-15),

$$|\phi_{\zeta_1}(\zeta_2)| = \left|\frac{\varepsilon}{\zeta_2 w_4}\right| \le 4\left|\frac{\varepsilon}{\zeta_0 w_3}\right| \le 8|\varepsilon||z_2|^{-\delta-1/2}. \tag{3-16}$$

In particular, for

$$|\varepsilon| < \tfrac{1}{8}|z_2|^{3/2}, \tag{3-17}$$

this implies $|\phi_{\zeta_1}(\zeta_2)| < |z_2|^{1-\delta}$, and by the triangle inequality,

$$|\phi_{\zeta_1}(\zeta_0)| < |\phi_{\zeta_2}(\zeta_0)| + |\phi_{\zeta_1}(\zeta_2)| < 3|z_2|^{1-\delta}. \tag{3-18}$$

We now establish the two (contradictory) estimates for w_2. On the one hand, (3-11) implies that

$$\log|w_2| \ge \left(\tfrac{1}{2}+\delta\right)\log|z_2| - \log 2, \quad \text{that is, } |w_2| \ge \tfrac{1}{2}|z_2|^{\delta+1/2}. \tag{3-19}$$

On the other hand,

$$\left|\frac{w_1}{w_4}\right| = \left|\frac{\varepsilon s(\varepsilon)}{\zeta_1\phi_{\zeta_2}(\zeta_1)}\frac{\zeta_2\phi_{\zeta_1}(\zeta_2)}{\varepsilon}\right| = \left|s(\varepsilon)\frac{\zeta_2}{\zeta_1}\right|.$$

By the triangle inequality for d_G, when (3-17) holds, the lower bound in (3-15) and the corollary to (3-16) imply

$$|\zeta_1| \geq |\zeta_2| - |\phi_{\zeta_1}(\zeta_2)| \geq \tfrac{1}{2}|\zeta_0| - |z_2|^{1-\delta} \geq \tfrac{1}{4}|\zeta_0|$$

for $|z_2|$ small enough, because of (3-11) again. So finally, using the upper bound in (3-15), $|w_1/w_4| \leq 6|s(\varepsilon)|$. We choose $\varepsilon_0 < \frac{1}{8}|z_2|^{3/2}$ so that for any ε with $|\varepsilon| \leq \varepsilon_0$,

$$|s(\varepsilon)| < |z_2|^{1-\delta}. \tag{3-20}$$

The triangle inequality for d_G and (3-18) imply that when $|\varepsilon| \leq \varepsilon_0$,

$$|w_2| \leq |w_1| + |\phi_{\zeta_1}(\zeta_0)| \leq 6|s(\varepsilon)| + 3|z_2|^{1-\delta} \leq 9|z_2|^{1-\delta}.$$

Finally, if we choose $|z_2| \leq r_0(\delta)$, with

$$r_0(\delta) \leq \min(r_1(\delta), r_2(\delta)) \quad \text{and} \quad 9r_0(\delta)^{1-\delta} < \tfrac{1}{2}r_0(\delta)^{1/2+\delta},$$

we see that for any ε with $|\varepsilon| \leq \varepsilon_0$, this last bound contradicts (3-19). □

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