Pacific Journal of Mathematics

DIFFERENTIAL HARNACK INEQUALITIES FOR NONLINEAR HEAT EQUATIONS WITH POTENTIALS UNDER THE RICCI FLOW

JIA-YONG WU

Volume 257 No. 1 May 2012

PACIFIC JOURNAL OF MATHEMATICS
Vol. 257, No. 1, 2012

DIFFERENTIAL HARNACK INEQUALITIES FOR NONLINEAR HEAT EQUATIONS WITH POTENTIALS UNDER THE RICCI FLOW

JIA-YONG WU

We prove several differential Harnack inequalities for positive solutions to nonlinear backward heat equations with different potentials coupled with the Ricci flow. We also derive an interpolated Harnack inequality for the nonlinear heat equation under the ε-Ricci flow on a closed surface. These new Harnack inequalities extend the previous differential Harnack inequalities for linear heat equations with potentials under the Ricci flow.

1. Introduction and main results

Background. The study of differential Harnack estimates for parabolic equations originated with the work of P. Li and S.-T. Yau [1986], who first proved a gradient estimate for the heat equation via the maximum principle (though a precursory form of their estimate appeared in [Aronson and Bénilan 1979]). Using their gradient estimate, the same authors derived a classical Harnack inequality by integrating the gradient estimate along space-time paths. This result was generalized to Harnack inequalities for some nonlinear heat-type equations in [Yau 1994] and for some non-self-adjoint evolution equations in [Yau 1995]. Recently, J. Li and X. Xu [2011] gave sharper local estimates than previous results for the heat equation on Riemannian manifolds with Ricci curvature bounded below. Surprisingly, R. Hamilton employed similar techniques to obtain Harnack inequalities for the Ricci flow [Hamilton 1993a], and the mean curvature flow [Hamilton 1995]. In dimension two, a differential Harnack estimate for the positive scalar curvature was proved in [Hamilton 1988], and then extended by B. Chow [1991a] when the scalar curvature changes sign. Similar techniques were used to obtain the Harnack inequalities for the Gauss curvature flow [Chow 1991b] and the Yamabe flow [Chow 1992]. H.-D. Cao [1992] proved a Harnack inequality for the Kähler–Ricci

This work is partially supported by the NSFC (No.11101267) and the Science and Technology Program of Shanghai Maritime University (No. 20120061).
MSC2010: 53C44.
Keywords: Harnack inequality, interpolated Harnack inequality, nonlinear heat equation, nonlinear backward heat equation, Ricci flow.

flow. B. Andrews [1994] derived several Harnack inequalities for general curvature flows of hypersurfaces. Chow and Hamilton [1997] gave extensions of the Li–Yau Harnack inequality, which they called constrained and linear Harnack inequalities. For more detailed discussion, we refer the interested reader to [Chow et al. 2006, Chapter 10].

Hamilton [1993b] also generalized the Li–Yau Harnack inequality to a matrix Harnack form on a class of Riemannian manifolds with nonnegative sectional curvature. This result was extended to the constrained matrix Harnack inequalities in [Chow and Hamilton 1997]. H.-D. Cao and L. Ni [2005] proved a matrix Harnack estimate for the heat equation on Kähler manifolds. Chow and Ni [2007] proved a matrix Harnack estimate for Kähler–Ricci flow using interpolation techniques from [Chow 1998].

In another direction, differential Harnack inequalities for (backward) heat-type equations coupled with the Ricci flow have become an important object, which can be traced back to [Hamilton 1988]. This subject was further explored by Chow [1998], Chow and Hamilton [1997], Chow and D. Knopf [2002], and H.-B. Cheng [2006], among others. Perhaps the most spectacular result is G. Perelman's [2002] differential Harnack inequality for the fundamental solution to the backward heat equation coupled with the Ricci flow without any curvature assumption. Perelman's Harnack inequality has many important applications (it is essential in proving pseudolocality theorems), and it has been extended by X. Cao [2008] and independently by S.-L. Kuang and Qi S. Zhang [2008]. Those authors proved a differential Harnack inequality for all positive solutions to the backward heat equation under the Ricci flow on closed manifolds with nonnegative scalar curvature. X. Cao and Qi S. Zhang [2011a] have established Gaussian upper and lower bounds for the fundamental solution to the backward heat equation under the Ricci flow.

On the subject of differential Harnack inequalities for the linear heat equation coupled with the Ricci flow, there have been many important contributions; see, for example, [Bailesteanu et al. 2010; Cao and Hamilton 2009; Chau et al. 2011; Chow et al. 2010; Guenther 2002; Liu 2009; Wu and Zheng 2010; Zhang 2006].

In recent years there has been increasing interest in the study of the nonlinear heat-type equations coupled with the Ricci flow. A nice example of a nonlinear heat equation, introduced by L. Ma [2006], is

$$\frac{\partial}{\partial t} f = \Delta f - af \ln f - bf, \tag{1-1}$$

where a and b are real constants. Ma first proved a local gradient estimate for positive solutions to the corresponding elliptic equation

$$\Delta f - af \ln f - bf = 0 \tag{1-2}$$

on a complete manifold with a fixed metric. Indeed, F. R. K. Chung and S.-T. Yau [1996] observed that equation (1-2) is linked with the gross logarithmic Sobolev inequality. They also established a logarithmic Harnack inequality for this equation when $a < 0$. Y. Yang [2008] derived local gradient estimates for positive solutions to (1-1) on a complete manifold with a fixed metric; see also [Chen and Chen 2009; Huang and Ma 2010; Wu 2010a; 2010b]. Yang's result has been generalized by L. Ma [2010a; 2010b], who obtained Hamilton and new Li–Yau type gradient estimates for the nonlinear heat equation (1-1), and also by S.-Y. Hsu [2011], who proved local gradient estimates for the nonlinear heat equation (1-1) under the Ricci flow, similar to the gradient estimates of [Yang 2008] for the fixed metric case.

We remind the reader that equations (1-1) and (1-2) often appear in geometric evolution equations, and are also closely related to the gradient Ricci solitons. See, for example, [Cao and Zhang 2011b; Ma 2006] for nice explanations on this subject.

Very recently, X. Cao and Z. Zhang [2011b] used the argument from [Cao and Hamilton 2009] to prove an interesting differential Harnack inequality for positive solutions to the forward nonlinear heat equation

$$\frac{\partial}{\partial t} f = \Delta f - f \ln f + Rf \tag{1-3}$$

coupled with the Ricci flow equation

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij} \tag{1-4}$$

on a closed manifold. Here Δ, R and R_{ij} are the Laplacian, scalar curvature and Ricci curvature of the metric $g(t)$ moving under the Ricci flow.

Main results. In this paper, we will be concerned with general time-dependent nonlinear backward heat equations of the type (1-1) with different potentials on closed manifolds under the Ricci flow.

Before studying nonlinear backward heat equations, we first study the nonlinear forward heat equation (1-3) with the metric evolving under the Ricci flow. Suppose $(M, g(t))$, $t \in [0, T)$, is a solution to the ε-Ricci flow ($\varepsilon \geq 0$)

$$\frac{\partial}{\partial t} g_{ij} = -\varepsilon R g_{ij} \tag{1-5}$$

on a closed surface. Let f be a positive solution to the nonlinear forward heat equation with potential εR, that is,

$$\frac{\partial}{\partial t} f = \Delta f - f \ln f + \varepsilon R f. \tag{1-6}$$

In this case, we can derive a new differential interpolated Harnack inequality, which is originated with B. Chow [1998].

Theorem 1.1. *Let $(M, g(t))$, $t \in [0, T)$, be a solution to the ε-Ricci flow* (1-5) *on a closed surface with $R > 0$. Let f be a positive solution to the nonlinear heat equation* (1-6), *$u = -\ln f$ and $H_\varepsilon = \Delta u - \varepsilon R$. Then, for all time $t \in (0, T)$,*

$$H_\varepsilon \le \frac{1}{t},$$

that is,

$$\frac{\partial}{\partial t} \ln f - |\nabla \ln f|^2 + \ln f + \frac{1}{t} = \Delta \ln f + \varepsilon R + \frac{1}{t} \ge 0.$$

In Theorem 1.1, if we take $\varepsilon = 0$, we can get the following differential Harnack inequality for the nonlinear heat equation on closed surfaces with a fixed metric:

Corollary 1.2. *If $f : M \times [0, T) \to \mathbb{R}$, is a positive solution to the nonlinear heat equation*

$$\frac{\partial}{\partial t} f = \Delta f - f \ln f$$

on a closed surface (M, g) with $R > 0$, then, for all time $t \in (0, T)$,

$$\frac{\partial}{\partial t} \ln f - |\nabla \ln f|^2 + \ln f + \frac{1}{t} = \Delta \ln f + \frac{1}{t} \ge 0.$$

If we take $\varepsilon = 1$ in Theorem 1.1, we get:

Corollary 1.3. *Let $(M, g(t))$, $t \in [0, T)$, be a solution to the Ricci flow on a closed surface with $R > 0$. If f is a positive solution to the nonlinear heat equation* (1-3), *then for all time $t \in (0, T)$,*

$$\frac{\partial}{\partial t} \ln f - |\nabla \ln f|^2 + \ln f + \frac{1}{t} = \Delta \ln f + R + \frac{1}{t} \ge 0.$$

Remark 1.4. X. Cao and Z. Zhang [2011b] have proved a differential Harnack inequality for Equation (1-3) under the Ricci flow on manifolds of any dimension. However, on a closed surface, the result of Corollary 1.3 is better than theirs.

Remark 1.5. Interestingly, Theorem 1.1 is a nonlinear interpolated Harnack inequality which links Corollary 1.2 to Corollary 1.3.

Secondly, we now consider differential Harnack inequalities for positive solutions to the nonlinear backward heat equation with potential $2R$, that is,

$$\frac{\partial}{\partial t} f = -\Delta f + f \ln f + 2Rf \tag{1-7}$$

under the Ricci flow. X. Cao and Z. Zhang [2011b] made nice explanations that the nonlinear forward heat equation (1-3) is closely related to expanding gradient

Ricci solitons. Analogously to the argument of Cao and Zhang, our consideration of the Equation (1-7) is motivated by *shrinking* gradient Ricci solitons proposed in [Hamilton 1993a]. Recall that a shrinking gradient Ricci soliton (M, g) is defined by the form (see [Chow et al. 2006])

$$R_{ij} + \nabla_i \nabla_j w = c g_{ij}, \tag{1-8}$$

where w is some Ricci soliton potential and c is a positive constant. Taking the trace of both sides of (1-8) yields

$$R + \Delta w = \text{const.} \tag{1-9}$$

Using the contracted Bianchi identity, we can easily deduce that

$$R - 2cw + |\nabla w|^2 = -\text{const.} \tag{1-10}$$

From (1-9) and (1-10), we get

$$2|\nabla w|^2 = -\Delta w + |\nabla w|^2 + 2cw - 2R. \tag{1-11}$$

Recall that the Ricci flow solution for a complete gradient Ricci soliton [Chow et al. 2006, Theorem 4.1] is the pullback of g under $\varphi(t)$, up to a scale factor $c(t)$:

$$g(t) = c(t) \cdot \varphi(t)^* g,$$

where $c(t) := -2ct + 1 > 0$ and $\varphi(t)$ is the 1-parameter family of diffeomorphisms generated by

$$\frac{1}{c(t)} \nabla_g w.$$

Then the corresponding Ricci soliton potential $\varphi(t)^* w$ satisfies

$$\frac{\partial}{\partial t} \varphi(t)^* w = \left|\nabla \varphi(t)^* w\right|^2.$$

Note that along the Ricci flow, (1-11) becomes

$$2|\nabla \varphi(t)^* w|^2 = -\Delta \varphi(t)^* w + |\nabla \varphi(t)^* w|^2 + \frac{2c}{c(t)} \cdot \varphi(t)^* w - 2R.$$

Hence the evolution equation for the Ricci soliton potential $\varphi(t)^* w$ is

$$2\frac{\partial \varphi(t)^* w}{\partial t} = -\Delta \varphi(t)^* w + |\nabla \varphi(t)^* w|^2 + \frac{2c}{c(t)} \cdot \varphi(t)^* w - 2R.$$

If we let $\varphi(t)^* w = -\ln \tilde{f}$, this equation becomes

$$2\frac{\partial \tilde{f}}{\partial t} = -\Delta \tilde{f} + 2R\tilde{f} + \frac{2c}{c(t)} \cdot \tilde{f} \ln \tilde{f}. \tag{1-12}$$

Notice that (1-7) and (1-12) are closely related and only differ by the time scaling and their last terms.

For the nonlinear backward heat equation (1-7) under the Ricci flow, we have:

Theorem 1.6. *Let* $(M, g(t))$, $t \in [0, T]$, *be a solution to the Ricci flow on a closed manifold of dimension* n. *Let* f *be a positive solution to the nonlinear backward heat equation* (1-7), $u = -\ln f$, $\tau = T - t$ *and*

$$H = 2\Delta u - |\nabla u|^2 + 2R - 2\frac{n}{\tau}. \tag{1-13}$$

Then, for all time $t \in [0, T)$,

$$H \le \frac{n}{2}.$$

Remark 1.7. We can easily see that $H \le n/2$ is equivalent to

$$\frac{|\nabla f|^2}{f^2} - 2\left(\frac{f_\tau}{f} + \ln f + R\right) \le 2\frac{n}{\tau} + \frac{n}{2}.$$

In [Yang 2008] (see also [Wu 2010b]), the classical Li–Yau gradient estimate for positive solutions to the nonlinear heat equation (1-1) is

$$\frac{|\nabla f|^2}{f^2} - 2\left(\frac{f_t}{f} + a\ln f + b\right) \le 2\frac{n}{t} + na$$

on manifolds with a fixed metric satisfying nonnegative Ricci curvature. Hence our Harnack inequality is similar to the classical Li–Yau gradient estimate for the nonlinear heat equation (1-1).

If we assume instead that our solution to the Ricci flow is defined for $t \in [0, T)$ (where $T < \infty$ is the blow-up time) and is of type I, meaning that

$$|\mathrm{Rm}| \le \frac{d_0}{T - t} \tag{1-14}$$

for some constant d_0, then we can show this:

Theorem 1.8. *Let* $(M, g(t))$, $t \in [0, T)$ (*where* $T < \infty$ *is the blow-up time*) *be a solution to the Ricci flow on a closed manifold of dimension* n, *and assume that* g *is of type I*, *that is*, *it satisfies* (1-14), *for some constant* d_0. *Let* f *be a positive solution to the nonlinear backward heat equation* (1-7), $u = -\ln f$, $\tau = T - t$ *and*

$$H = 2\Delta u - |\nabla u|^2 + 2R - d\frac{n}{\tau},$$

where $d = d(d_0, n) \ge 2$ *is some constant such that* $H(\tau) < 0$ *for small* τ. *Then*, *for all time* $t \in [0, T)$,

$$H \le \frac{n}{2}.$$

Thirdly, we consider the nonlinear backward heat equation

$$\frac{\partial}{\partial t} f = -\Delta f + f \ln f + Rf \tag{1-15}$$

under the Ricci flow. This equation is very similar to (1-7) and only differs by the last potential. We also find that (1-15) can be regarded as the extension of the linear backward heat equation considered in [Cao 2008, Theorem 1.3] and [Kuang and Zhang 2008, Theorem 2.1]. In fact, we only have the additional term $f \ln f$ in the linear backward heat equation. For this system, we prove:

Theorem 1.9. *Let $(M, g(t))$, $t \in [0, T]$, be a solution to the Ricci flow on a closed manifold of dimension n with nonnegative scalar curvature. Let f be a positive solution to the nonlinear backward heat equation* (1-15), *$u = -\ln f$, $\tau = T - t$ and*

$$H = 2\Delta u - |\nabla u|^2 + R - 2\frac{n}{\tau}. \tag{1-16}$$

Then, for all time $t \in [0, T)$,

$$H \le \frac{n}{4}.$$

By modifying the Harnack quantity of Theorem 1.9, we can deduce the following differential Harnack inequality *without* assuming the nonnegativity of R:

Theorem 1.10. *Let $(M, g(t))$, $t \in [0, T]$, be a solution to the Ricci flow on a closed manifold of dimension n. Let f be a positive solution to the nonlinear backward heat equation* (1-15), *$v = -\ln f - \frac{1}{2} n \ln(4\pi\tau)$, $\tau = T - t$, and*

$$P = 2\Delta v - |\nabla v|^2 + R - 3\frac{n}{\tau}.$$

Then, for all time $t \in [T/2, T)$,

$$P \le \frac{n}{4}.$$

Remark 1.11. Theorems 1.6–1.10 extend to the nonlinear case Theorems 1.1–1.3 and 3.6 of [Cao 2008] and Theorem 2.1 of [Kuang and Zhang 2008].

The proof of all our theorems nearly follows from the arguments of X. Cao [2008], X. Cao and R. Hamilton [2009], X. Cao and Z. Zhang [2011b], and S.-L. Kuang and Qi S. Zhang [Kuang and Zhang 2008], where computations of evolution equations and the maximum principle for parabolic equations are employed. The major differences are that one of our results gives an interpolation Harnack inequality for a nonlinear forward heat equation along the ε-Ricci flow on a closed surface, and the others provide differential Harnack estimates for various *nonlinear backward* heat equations under the Ricci flow.

One interesting feature of this paper is that our differential Harnack inequalities are not only like the Perelman's Harnack inequalities, but also similar to the classical Li–Yau Harnack inequalities for the corresponding nonlinear heat equation (see Remark 1.7 above). Another feature is that our Harnack quantities of nonlinear backward heat equations are nearly the same as those of linear backward heat equations considered by X. Cao [2008], and S.-L. Kuang and Qi S. Zhang [2008]. Due to the fact that Ricci soliton potentials are linked with some nonlinear backward heat equations, we expect that our differential Harnack inequalities will be useful in understanding the Ricci solitons.

The rest of this paper is organized as follows: In Section 2, we will prove a new differential interpolated Harnack inequality on a surface, that is, Theorem 1.1. In Section 3, we firstly derive differential Harnack inequalities for positive solutions to the nonlinear backward heat equation with potential $2R$ under the Ricci flow (Theorems 1.6 and 1.8). Then a classical integral version of the Harnack inequality will be proved (Theorem 3.2). In the latter part of this section, we will establish Harnack inequalities for another nonlinear backward heat equation with potential R under the Ricci flow (Theorem 1.9) as well as its classical Harnack version (Theorem 3.4). By modifying the Harnack quantity of Theorem 1.9, we can prove another differential Harnack inequalities without the nonnegative assumption of scalar curvature (Theorem 1.10). Finally, in Section 4, we will prove gradient estimates for positive and bounded solutions to the nonlinear (including backward) heat equation without potentials under the Ricci flow, that is, Theorems 4.1 and 4.3.

2. Nonlinear heat equation with potentials

In this section, we will prove a differential interpolated Harnack inequality for positive solutions to nonlinear forward heat equations with potentials coupled with the ε-Ricci flow on a closed surface.

Let f be a positive solution to the nonlinear forward heat equation (1-6). By the maximum principle, we conclude that the solution will remain positive along the Ricci flow when scalar curvature is positive. If we let

$$u = -\ln f,$$

then u satisfies the equation

$$\frac{\partial}{\partial t} u = \Delta u - |\nabla u|^2 - \varepsilon R - u.$$

Proof of Theorem 1.1. The proof involves a direct computation and the parabolic maximum principle. Let f and u be defined as above. Under the ε-Ricci flow (1-5) on a closed surface, we have that

$$\frac{\partial R}{\partial t} = \varepsilon(\Delta R + R^2) \quad \text{and} \quad \frac{\partial}{\partial t}(\Delta) = \varepsilon R \Delta,$$

where the Laplacian Δ is acting on functions. Define the Harnack quantity

$$H_\varepsilon = \Delta u - \varepsilon R. \tag{2-1}$$

Using the evolution equations above, we first compute that

$$\begin{aligned}
\frac{\partial}{\partial t} H_\varepsilon &= \Delta\left(\frac{\partial}{\partial t} u\right) + \left(\frac{\partial}{\partial t}\Delta\right) u - \varepsilon \frac{\partial R}{\partial t} \\
&= \Delta(\Delta u - |\nabla u|^2 - \varepsilon R - u) + \varepsilon R \Delta u - \varepsilon \frac{\partial R}{\partial t} \\
&= \Delta H_\varepsilon - \Delta |\nabla u|^2 - \Delta u + \varepsilon R H_\varepsilon + \varepsilon^2 R^2 - \varepsilon \frac{\partial R}{\partial t}
\end{aligned}$$

Since

$$\Delta |\nabla u|^2 = 2|\nabla\nabla u|^2 + 2\nabla \Delta u \cdot \nabla u + R|\nabla u|^2$$

on a two-dimensional surface, we then have

$$\begin{aligned}
\frac{\partial}{\partial t} H_\varepsilon &= \Delta H_\varepsilon - 2|\nabla\nabla u|^2 - 2\nabla\Delta u \cdot \nabla u - R|\nabla u|^2 + \varepsilon R H_\varepsilon + \varepsilon^2 R^2 - \varepsilon \frac{\partial R}{\partial t} - \Delta u \\
&= \Delta H_\varepsilon - 2|\nabla\nabla u|^2 - 2\nabla H_\varepsilon \cdot \nabla u \\
&\qquad - 2\varepsilon \nabla R \cdot \nabla u - R|\nabla u|^2 + \varepsilon R H_\varepsilon + \varepsilon^2 R^2 - \varepsilon \frac{\partial R}{\partial t} - \Delta u \\
&= \Delta H_\varepsilon - 2\left|\nabla_i \nabla_j u - \frac{\varepsilon}{2} R g_{ij}\right|^2 - 2\varepsilon R \Delta u - 2\nabla H_\varepsilon \cdot \nabla u \\
&\qquad - 2\varepsilon \nabla R \cdot \nabla u - R|\nabla u|^2 + \varepsilon R H_\varepsilon + 2\varepsilon^2 R^2 - \varepsilon \frac{\partial R}{\partial t} - \Delta u.
\end{aligned}$$

Since $\Delta u = H_\varepsilon + \varepsilon R$ by (2-1), these equalities become

$$\begin{aligned}
\frac{\partial}{\partial t} H_\varepsilon &= \Delta H_\varepsilon - 2\left|\nabla_i \nabla_j u - \frac{\varepsilon}{2} R g_{ij}\right|^2 - \varepsilon R H_\varepsilon - 2\nabla H_\varepsilon \cdot \nabla u \\
&\qquad - 2\varepsilon \nabla R \cdot \nabla u - R|\nabla u|^2 - \varepsilon \frac{\partial R}{\partial t} - \Delta u.
\end{aligned}$$

Rearranging terms yields

$$\begin{aligned}
\frac{\partial}{\partial t} H_\varepsilon &= \Delta H_\varepsilon - 2\left|\nabla_i \nabla_j u - \frac{\varepsilon}{2} R g_{ij}\right|^2 - 2\nabla H_\varepsilon \cdot \nabla u - \varepsilon R H_\varepsilon \\
&\qquad - R\,|\nabla u + \varepsilon \nabla \ln R|^2 - \varepsilon R\left(\frac{\partial \ln R}{\partial t} - \varepsilon |\nabla \ln R|^2\right) - \Delta u \\
&\le \Delta H_\varepsilon - H_\varepsilon^2 - 2\nabla H_\varepsilon \cdot \nabla u - (\varepsilon R + 1) H_\varepsilon + \frac{\varepsilon}{t} R - \varepsilon R.
\end{aligned} \tag{2-2}$$

The reason for this last inequality is that the trace Harnack inequality for the ε-Ricci flow on a closed surface proved in [Chow 1998] (see also [Wu and Zheng

2010, Lemma 2.1]) states that

$$\frac{\partial \ln R}{\partial t} - \varepsilon|\nabla \ln R|^2 = \varepsilon(\Delta \ln R + R) \geq -\frac{1}{t},$$

since $g(t)$ has positive scalar curvature. Besides this, we also used (2-1) and the elementary inequality

$$\left|\nabla_i \nabla_j u - \frac{\varepsilon}{2} R g_{ij}\right|^2 \geq \frac{1}{2}(\Delta u - \varepsilon R)^2 = \frac{1}{2} H_\varepsilon^2.$$

Adding $-1/t$ to H_ε in (2-2) yields

$$\begin{aligned}\text{(2-3)} \quad \frac{\partial}{\partial t}\left(H_\varepsilon - \frac{1}{t}\right) \leq \Delta\left(H_\varepsilon - \frac{1}{t}\right) - 2\nabla\left(H_\varepsilon - \frac{1}{t}\right) \cdot \nabla u \\ - \left(H_\varepsilon + \frac{1}{t}\right)\left(H_\varepsilon - \frac{1}{t}\right) - (\varepsilon R + 1)\left(H_\varepsilon - \frac{1}{t}\right) - \frac{1}{t} - \varepsilon R.\end{aligned}$$

Clearly, for t small enough we have $H_\varepsilon - 1/t < 0$. Since $R > 0$, applying the maximum principle to the evolution formula (2-3) we conclude that $H_\varepsilon - 1/t \leq 0$ for all time t, and the proof of this theorem is completed. □

We remark that Theorem 1.1 can be regarded as a nonlinear version of an interpolated Harnack inequality proved by B. Chow:

Theorem 2.1 [Chow 1998]. *Let $(M, g(t))$ be a solution to the ε-Ricci flow* (1-5) *on a closed surface with $R > 0$. If f is a positive solution to*

$$\frac{\partial}{\partial t} f = \Delta f + \varepsilon R f,$$

then

$$\frac{\partial}{\partial t} \ln f - |\nabla \ln f|^2 + \frac{1}{t} = \Delta \ln f + \varepsilon R + \frac{1}{t} \geq 0.$$

3. Nonlinear backward heat equation with potentials

We next study several differential Harnack inequalities for positive solutions to the nonlinear backward heat equation under the Ricci flow, proving Theorems 1.6, 1.8, 1.9, and 1.10 from the Introduction. The first two of these theorems deal with the case where the potential equals $2R$, and the last two with the potential R. The proofs are largely based on the maximum principle.

Potential $2R$. Theorems 1.6 and 1.8 deal with differential Harnack inequalities for positive solutions to the equation

$$\frac{\partial}{\partial t} f = -\Delta f + f \ln f + 2Rf$$

under the Ricci flow. We follow the trick used to prove Theorem 1.1 in [Cao and Zhang 2011b] to simplify a tedious calculation of the evolution equations. Also, the evolution equation of u in this case is very similar to what is considered in [Cao 2008]. So we can borrow Cao's computation for the very general setting there to simplify our calculation. The only difference is that we have extra terms coming from the time derivative $\partial u/\partial\tau$.

Proof of Theorem 1.6. As before, it is easy to compute that u satisfies

$$\frac{\partial}{\partial\tau}u = \Delta u - |\nabla u|^2 + 2R - u. \tag{3-1}$$

Recall from (1-13) that $H = 2\Delta u - |\nabla u|^2 + 2R - 2n/\tau$. Adapting [Cao 2008, (2.4)] and using (3-1) as well as the elementary inequality

$$\left|\nabla_i\nabla_j u - R_{ij} - \frac{1}{\tau}g_{ij}\right|^2 \geq \frac{1}{n}\left(\Delta u - R - \frac{n}{\tau}\right)^2,$$

we can write

$$\begin{aligned}\frac{\partial}{\partial\tau}H &= \Delta H - 2\nabla H\cdot\nabla u - \frac{2}{\tau}H - \frac{2}{\tau}|\nabla u|^2 - 2|\mathrm{Rc}|^2 - 2\left|\nabla_i\nabla_j u + R_{ij} - \frac{1}{\tau}g_{ij}\right|^2 \\ &\qquad - 2(\Delta u - |\nabla u|^2) \\ &\leq \Delta H - 2\nabla H\cdot\nabla u - \frac{2}{\tau}H - \frac{2}{\tau}|\nabla u|^2 - \frac{2}{n}R^2 - \frac{2}{n}\left(\Delta u + R - \frac{n}{\tau}\right)^2 \\ &\qquad - 2(\Delta u - |\nabla u|^2),\end{aligned}$$

By the definition of H, we have

$$-2(\Delta u - |\nabla u|^2) = -2H + 2\left(\Delta u + R - \frac{n}{\tau}\right) + 2R - \frac{2n}{\tau}.$$

Plugging this into the preceding inequality yields

$$\begin{aligned}\frac{\partial}{\partial\tau}H &\leq \Delta H - 2\nabla H\cdot\nabla u - \left(\frac{2}{\tau}+2\right)H - \frac{2}{\tau}|\nabla u|^2 - \frac{2}{n}R^2 \\ &\qquad - \frac{2}{n}\left(\Delta u + R - \frac{n}{\tau} - \frac{n}{2}\right)^2 + \frac{n}{2} + 2R - \frac{2n}{\tau} \\ &= \Delta H - 2\nabla H\cdot\nabla u - \left(\frac{2}{\tau}+2\right)H - \frac{2}{\tau}|\nabla u|^2 \\ &\qquad - \frac{2}{n}\left(\Delta u + R - \frac{n}{\tau} - \frac{n}{2}\right)^2 - \frac{2}{n}\left(R - \frac{n}{2}\right)^2 - \frac{2n}{\tau} + n.\end{aligned}$$

Adding $-n/2$ to H, we then get

$$\begin{aligned}\frac{\partial}{\partial\tau}\left(H - \frac{n}{2}\right) &\leq \Delta\left(H - \frac{n}{2}\right) - 2\nabla\left(H - \frac{n}{2}\right)\cdot\nabla u - \left(\frac{2}{\tau}+2\right)\left(H - \frac{n}{2}\right) \\ &\qquad - \frac{2}{\tau}|\nabla u|^2 - \frac{2}{n}\left(\Delta u + R - \frac{n}{\tau} - \frac{n}{2}\right)^2 - \frac{2}{n}\left(R - \frac{n}{2}\right)^2 - \frac{3n}{\tau}.\end{aligned} \tag{3-2}$$

If τ is small enough, $H - n/2 < 0$. Then applying the maximum principle to the evolution equation (3-2) yields $H - n/2 \leq 0$ for all τ, hence for all $t \in [0, T)$. □

An easy modification of the preceding proof, using (1-14) to ensure that we can apply the maximum principle as $\tau \to 0$, verifies Theorem 1.8. We omit the details.

Remark 3.1. Theorem 1.6 is also true on a complete noncompact Riemannian manifolds, as long as we can apply the maximum principle.

From Theorem 1.6, we can derive a classical Harnack inequality by integrating along a space-time path.

Theorem 3.2. *Let $(M, g(t))$, $t \in [0, T]$, be a solution to the Ricci flow on a closed manifold of dimension n. Let f be a positive solution to the nonlinear backward heat equation* (1-7). *Assume that (x_1, t_1) and (x_2, t_2), $0 \le t_1 < t_2 < T$, are two points in $M \times [0, T)$. Then we have*

$$e^{t_2} \ln f(x_2, t_2) - e^{t_1} \ln f(x_1, t_1) \le \frac{1}{2} \int_{t_1}^{t_2} e^{T-t} \left(|\dot{\gamma}|^2 + 2R + \frac{n}{2} + \frac{2n}{T-t} \right) dt,$$

where γ is any space-time path joining (x_1, t_1) and (x_2, t_2).

Proof. This is similar to Theorem 2.3 in [Cao 2008]; we include the proof for completeness. Consider the solutions to

$$\frac{\partial}{\partial \tau} u = \Delta u - |\nabla u|^2 + 2R - u.$$

Combining this with

$$H - \frac{n}{2} = 2\Delta u - |\nabla u|^2 + 2R - 2\frac{n}{\tau} - \frac{n}{2} \le 0,$$

we have

$$2\frac{\partial}{\partial \tau} u + |\nabla u|^2 - 2R - 2\frac{n}{\tau} + 2u - \frac{n}{2} \le 0.$$

If $\gamma(x, t)$ is a space-time path joining (x_2, τ_2) and (x_1, τ_1), with $\tau_1 > \tau_2 > 0$, we have along γ

$$\begin{aligned} \frac{du}{d\tau} = \frac{\partial u}{\partial \tau} + \nabla u \cdot \gamma &\le -\frac{1}{2}|\nabla u|^2 + R + \frac{n}{\tau} - u + \frac{n}{4} + \nabla u \cdot \gamma \\ &\le \frac{1}{2}\left(|\dot{\gamma}|^2 + 2R + \frac{n}{2} \right) + \frac{n}{\tau} - u, \end{aligned}$$

where in the last step we used the inequality $-\frac{1}{2}|\nabla u|^2 + \nabla u \cdot \gamma - \frac{1}{2}|\dot{\gamma}|^2 \le 0$. Rearranging terms yields

$$\frac{d}{d\tau}(e^{\tau} \cdot u) \le \frac{e^{\tau}}{2} \left(|\dot{\gamma}|^2 + 2R + \frac{n}{2} + \frac{2n}{\tau} \right).$$

Integrating this inequality we obtain

$$e^{\tau_1}\cdot u(x_1,\tau_1)-e^{\tau_2}\cdot u(x_2,\tau_2)\le \frac{1}{2}\int_{\tau_2}^{\tau_1} e^{\tau}\left(|\dot{\gamma}|^2+2R+\frac{n}{2}+\frac{2n}{\tau}\right)d\tau,$$

which can be rewritten as

$$e^{t_1}\cdot u(x_1,t_1)-e^{t_2}\cdot u(x_2,t_2)\le \frac{1}{2}\int_{t_1}^{t_2} e^{T-t}\left(|\dot{\gamma}|^2+2R+\frac{n}{2}+\frac{2n}{T-t}\right)dt.$$

Note that $u=-\ln f$. Hence the desired classical Harnack inequality follows. □

Potential R. We now turn to the equation with potential R:

$$\frac{\partial}{\partial t}f=-\Delta f+f\ln f+Rf.$$

Here we need to assume that the initial metric $g(0)$ has nonnegative scalar curvature. It is well known that this property is preserved by the Ricci flow.

Proof of Theorem 1.9. This time u satisfies

$$\frac{\partial}{\partial \tau}u=\Delta u-|\nabla u|^2+R-u.$$

Adapting [Cao 2008, (3.2)], we can write

$$\frac{\partial}{\partial \tau}H=\Delta H-2\nabla H\cdot\nabla u-\frac{2}{\tau}H-\frac{2}{\tau}|\nabla u|^2-2\frac{R}{\tau} -2\left|\nabla_i\nabla_j u+R_{ij}-\frac{1}{\tau}g_{ij}\right|^2-2(\Delta u-|\nabla u|^2). \tag{3-3}$$

Since H is now given by (1-16), we have

$$-2(\Delta u-|\nabla u|^2)=-2H+2\left(\Delta u+R-\frac{n}{\tau}\right)-\frac{2n}{\tau}.$$

Plugging this into (3-3), we obtain

$$\begin{aligned}\frac{\partial}{\partial \tau}H&\le \Delta H-2\nabla H\cdot\nabla u-\left(\frac{2}{\tau}+2\right)H-\frac{2}{\tau}|\nabla u|^2-2\frac{R}{\tau}\\&\qquad -\frac{2}{n}\left(\Delta u+R-\frac{n}{\tau}\right)^2+2\left(\Delta u+R-\frac{n}{\tau}\right)-\frac{2n}{\tau}\\&=\Delta H-2\nabla H\cdot\nabla u-\left(\frac{2}{\tau}+2\right)H-\frac{2}{\tau}|\nabla u|^2-2\frac{R}{\tau}\\&\qquad -\frac{2}{n}\left(\Delta u+R-\frac{n}{\tau}-\frac{n}{2}\right)^2-\frac{2n}{\tau}+\frac{n}{2}.\end{aligned}$$

Adding $-n/4$ to H yields

$$\frac{\partial}{\partial \tau}\left(H-\frac{n}{4}\right)\le \Delta\left(H-\frac{n}{4}\right)-2\nabla\left(H-\frac{n}{4}\right)\cdot\nabla u-\left(\frac{2}{\tau}+2\right)\left(H-\frac{n}{4}\right) -\frac{2}{\tau}|\nabla u|^2-2\frac{R}{\tau}-\frac{2}{n}\left(\Delta u+R-\frac{n}{\tau}-\frac{n}{2}\right)^2-\frac{5n}{2\tau}. \tag{3-4}$$

Since $R \geq 0$, it is easy to see that $H - n/4 < 0$ for τ small enough. Applying the maximum principle to the evolution formula (3-4), we have $H - n/4 \leq 0$ for all τ, hence for all t. This finishes the proof of Theorem 1.9. □

We easily derive counterparts to Theorem 1.8 and Theorem 3.2:

Theorem 3.3. *Let* $(M, g(t))$, $t \in [0, T)$ (*where* $T < \infty$ *is the blow-up time*) *be a solution to the Ricci flow on a closed manifold of dimension n with nonnegative scalar curvature, and assume that g is of type I, that is, it satisfies* (1-14), *for some constant* d_0. *Let* f *be a positive solution to the nonlinear backward heat equation* (1-15), $u = -\ln f$, $\tau = T - t$ *and*

$$H = 2\Delta u - |\nabla u|^2 + R - d\frac{n}{\tau},$$

where $d = d(d_0, n) \geq 1$ *is some constant such that* $H(\tau) < 0$ *for small* τ. *Then, for all time* $t \in [0, T)$,

$$H \leq \frac{n}{4}.$$

Theorem 3.4. *Let* $(M, g(t))$, $t \in [0, T]$, *be a solution to the Ricci flow on a closed manifold of dimension n with nonnegative scalar curvature. Let* f *be a positive solution to the nonlinear backward heat equation* (1-15). *Assume that* (x_1, t_1) *and* (x_2, t_2), *with* $0 \leq t_1 < t_2 < T$, *are two points in* $M \times [0, T)$. *Then*

$$e^{t_2} \ln f(x_2, t_2) - e^{t_1} \ln f(x_1, t_1) \leq \frac{1}{2}\int_{t_1}^{t_2} e^{T-t}\left(|\dot{\gamma}|^2 + R + \frac{n}{4} + \frac{2n}{T-t}\right) dt,$$

where γ *is any space-time path joining* (x_1, t_1) *and* (x_2, t_2).

In the rest of this section, we will finish the proof of Theorem 1.10. The interesting feature of Theorem 1.10 is that the differential Harnack inequalities hold *without* any assumption on the scalar curvature R.

Proof of Theorem 1.10. We first compute that v satisfies

$$\frac{\partial}{\partial\tau} v = \Delta v - |\nabla v|^2 + R - \frac{n}{2\tau} - \left(v + \frac{n}{2}\ln(4\pi\tau)\right). \tag{3-5}$$

If we let

$$\tilde{P} := 2\Delta v - |\nabla v|^2 + R - 2\frac{n}{\tau},$$

then by adapting [Cao 2008, (3.7)], we have

$$\frac{\partial}{\partial\tau}\tilde{P} = \Delta\tilde{P} - 2\nabla\tilde{P}\cdot\nabla v - \frac{2}{\tau}\tilde{P} - \frac{2}{\tau}|\nabla v|^2 - 2\frac{R}{\tau} - 2\left|\nabla_i\nabla_j v + R_{ij} - \frac{1}{\tau}g_{ij}\right|^2 - 2(\Delta v - |\nabla v|^2).$$

Since $P = \tilde{P} - n/\tau$, we have

$$\text{(3-6)} \quad \frac{\partial}{\partial \tau} P = \Delta P - 2\nabla P \cdot \nabla v - \frac{2}{\tau} P - \frac{2}{\tau} |\nabla v|^2 - 2\frac{R}{\tau} - \frac{n}{\tau^2} \\ - 2\left|\nabla_i \nabla_j v + R_{ij} - \frac{1}{\tau} g_{ij}\right|^2 - 2(\Delta v - |\nabla v|^2).$$

According to the definition of P, we have

$$-2a(\Delta v - |\nabla v|^2) = -2P + 2\left(\Delta v + R - \frac{n}{\tau}\right) - \frac{4n}{\tau}.$$

Substituting this into (3-6), we get

$$\begin{aligned} \text{(3-7)} \quad \frac{\partial}{\partial \tau} P &\le \Delta P - 2\nabla P \cdot \nabla v - \left(\frac{2}{\tau} + 2\right) P - \frac{2}{\tau} |\nabla v|^2 - 2\frac{R}{\tau} - \frac{n}{\tau^2} \\ &\qquad - \frac{2}{n}\left(\Delta v + R - \frac{n}{\tau}\right)^2 + 2\left(\Delta v + R - \frac{n}{\tau}\right) - \frac{4n}{\tau} \\ &= \Delta P - 2\nabla P \cdot \nabla v - \left(\frac{2}{\tau} + 2\right) P - \frac{2}{\tau} |\nabla v|^2 - \frac{2}{\tau}\left(R + \frac{n}{2\tau}\right) \\ &\qquad - \frac{2}{n}\left(\Delta v + R - \frac{n}{\tau} - \frac{n}{2}\right)^2 - \frac{4n}{\tau} + \frac{n}{2}. \end{aligned}$$

Note that the evolution of scalar curvature under the Ricci flow is

$$\frac{\partial R}{\partial t} = \Delta R + 2|\mathrm{Rc}|^2 \ge \Delta R + \frac{2}{n} R^2.$$

Applying the maximum principle to this inequality yields $R \ge -n/(2t)$. Since $t \ge T/2$, we have $1/t \le 1/\tau$. Hence

$$R \ge -\frac{n}{2t} \ge -\frac{n}{2\tau},$$

that is,

$$R + \frac{n}{2\tau} \ge 0.$$

Combining this with (3-7), we have

$$\frac{\partial}{\partial \tau} P \le \Delta P - 2\nabla P \cdot \nabla v - \left(\frac{2}{\tau} + 2\right) P - \frac{4n}{\tau} + \frac{n}{2}.$$

Adding $-n/4$ to P, we get

$$\text{(3-8)} \quad \frac{\partial}{\partial \tau}\left(P - \frac{n}{4}\right) \le \Delta\left(P - \frac{n}{4}\right) - 2\nabla\left(P - \frac{n}{4}\right) \cdot \nabla v - \left(\frac{2}{\tau} + 2\right)\left(P - \frac{n}{4}\right) - \frac{9n}{2\tau}.$$

It is easy to see that $P - n/4 < 0$ for τ small enough. Applying the maximum principle to the evolution formula (3-8) yields

$$P - \frac{n}{4} \le 0$$

for all time $t \ge T/2$. Hence the theorem is proved. □

Remark 3.5. Motivated by Theorems 3.3 and 3.4, we can prove similar theorems by the standard argument from Theorem 1.10. We omit them in the interests of brevity.

4. Gradient estimates for nonlinear (backward) heat equations

In this section, on one hand we consider the positive solution $f(x,t) < 1$ to the nonlinear heat equation without any potential

$$\frac{\partial}{\partial t} f = \Delta f - f \ln f, \tag{4-1}$$

with the metric evolved by the Ricci flow (1-4) on a closed manifold M. This equation has been considered by S.-Y. Hsu [2011] and L. Ma [2010a]. If we let $u = -\ln f$, then

$$\frac{\partial}{\partial t} u = \Delta u - |\nabla u|^2 - u \tag{4-2}$$

and $u > 0$. Note that $0 < f < 1$ is preserved as time t evolves. In fact the initial assumption says that

$$-\ln \sup_M f(x,0) \le u(x,0) \le -\ln \inf_M f(x,0).$$

Applying the maximum principle to (4-2), we have

$$-e^{-t} \ln \sup_M f(x,0) \le u(x,t) \le -e^{-t} \ln \inf_M f(x,0)$$

and hence

$$0 < u(x,t) \le -\ln \inf_M f(x,0)$$

for all $x \in M$ and $t \in [0,T)$. Since $u = -\ln f$, this implies

$$0 < \inf_M f(x,0) \le f(x,t) < 1$$

for all $x \in M$ and $t \in [0,T)$.

Following the arguments of [Cao and Hamilton 2009], we let

$$H = |\nabla u|^2 - \frac{u}{t}.$$

Comparing with the equation (5.3) in the same reference, we have

$$\begin{aligned} \frac{\partial}{\partial t} H &= \Delta H - 2\nabla H \cdot \nabla u - \frac{1}{t} H - 2|\nabla\nabla u|^2 - 2|\nabla u|^2 + \frac{u}{t} \\ &= \Delta H - 2\nabla H \cdot \nabla u - \left(\frac{1}{t} + 1\right) H - 2|\nabla\nabla u|^2 - |\nabla u|^2. \end{aligned} \tag{4-3}$$

Notice that if t small enough, then $H < 0$. Then applying the maximum principle to (4-3), we obtain:

Theorem 4.1. *Let* $(M, g(t))$, $t \in [0, T)$, *be a solution to the Ricci flow on a closed manifold. Let* $f < 1$ *be a positive solution to the nonlinear heat equation* (4-1), $u = -\ln f$ *and*

$$H = |\nabla u|^2 - \frac{u}{t}.$$

Then, *for all time* $t \in (0, T)$,

$$H \leq 0.$$

Remark 4.2. Theorem 4.1 can be regarded as a nonlinear version of [Cao and Hamilton 2009, Theorem 5.1]. Recently, L. Ma [2010a, Theorem 3] has proved the same estimate as in Theorem 4.1 on a closed manifold with nonnegative Ricci curvature under a static metric. However, in our case, we do not need any curvature assumption.

On the other hand, we can also consider the positive solution $f(x, t) < 1$ to the nonlinear backward heat equation without any potential

$$\frac{\partial}{\partial t} f = -\Delta f + f \ln f, \tag{4-4}$$

with the metric evolved by the Ricci flow (1-4). Let $u = -\ln f$. Then we have

$$\frac{\partial}{\partial \tau} u = \Delta u - |\nabla u|^2 - u$$

and $u > 0$. Using the maximum principle, one can see that $0 < f < 1$ is also preserved under the Ricci flow. In fact from the initial assumption

$$0 < \inf_M f(x, T) \leq f(x, T) \leq \sup_M f(x, T) < 1,$$

one can also show that

$$0 < \inf_M f(x, T) \leq f(x, \tau) < 1$$

for all $x \in M$ and $\tau \in (0, T]$ in the same way as the above arguments.

Following the arguments of [Cao 2008], let

$$H = |\nabla u|^2 - \frac{u}{\tau}.$$

Comparing with the equation (5.3) in [Cao 2008], we have

$$\begin{aligned} \frac{\partial}{\partial \tau} H &= \Delta H - 2\nabla H \cdot \nabla u - \frac{1}{\tau} H - 2|\nabla\nabla u|^2 - 4R_{ij}u_i u_j - 2|\nabla u|^2 + \frac{u}{\tau} \\ &= \Delta H - 2\nabla H \cdot \nabla u - \left(\frac{1}{\tau} + 1\right) H - 2|\nabla\nabla u|^2 - 4R_{ij}u_i u_j - |\nabla u|^2. \end{aligned} \tag{4-5}$$

If we assume $R_{ij}(g(t)) \geq -K$, where $0 \leq K \leq \frac{1}{4}$, then

$$-4R_{ij}u_iu_j - |\nabla u|^2 \leq (4K-1)|\nabla u|^2 \leq 0.$$

Hence if τ small enough, then $H < 0$. Then applying the maximum principle to (4-5), we have a nonlinear version of [Cao 2008, Theorem 5.1].

Theorem 4.3. *Let* $(M, g(t))$, $t \in [0, T]$, *be a solution to the Ricci flow on a closed manifold with the Ricci curvature satisfying* $R_{ij}(g(t)) \geq -K$, *where* $0 \leq K \leq \frac{1}{4}$. *Let* $f < 1$ *be a positive solution to the nonlinear backward heat equation* (4-4), $u = -\ln f$, $\tau = T - t$ *and*

$$H = |\nabla u|^2 - \frac{u}{\tau}.$$

Then, for all time $t \in [0, T)$,

$$H \leq 0.$$

Acknowledgments

The author would like to express his gratitude to the referee for careful readings and many valuable suggestions.

References

[Andrews 1994] B. Andrews, "Harnack inequalities for evolving hypersurfaces", *Math. Z.* **217**:2 (1994), 179–197. MR 95j:58178 Zbl 0807.53044

[Aronson and Bénilan 1979] D. G. Aronson and P. Bénilan, "Régularité des solutions de l'équation des milieux poreux dans $\mathbb{R}^N$", *C. R. Acad. Sci. Paris Sér. A-B* **288**:2 (1979), A103–A105. MR 82i:35090

[Bailesteanu et al. 2010] M. Bailesteanu, X. Cao, and A. Pulemotov, "Gradient estimates for the heat equation under the Ricci flow", *J. Funct. Anal.* **258**:10 (2010), 3517–3542. MR 2011b:53153 Zbl 1193.53139

[Cao 1992] H. D. Cao, "On Harnack's inequalities for the Kähler–Ricci flow", *Invent. Math.* **109**:2 (1992), 247–263. MR 93f:58227 Zbl 0779.53043

[Cao 2008] X. Cao, "Differential Harnack estimates for backward heat equations with potentials under the Ricci flow", *J. Funct. Anal.* **255**:4 (2008), 1024–1038. MR 2009e:35121 Zbl 1146.58014

[Cao and Hamilton 2009] X. Cao and R. S. Hamilton, "Differential Harnack estimates for time-dependent heat equations with potentials", *Geom. Funct. Anal.* **19**:4 (2009), 989–1000. MR 2010j:53124 Zbl 1183.53059

[Cao and Ni 2005] H.-D. Cao and L. Ni, "Matrix Li–Yau–Hamilton estimates for the heat equation on Kähler manifolds", *Math. Ann.* **331**:4 (2005), 795–807. MR 2006k:53113 Zbl 1083.58024

[Cao and Zhang 2011a] X. Cao and Q. S. Zhang, "The conjugate heat equation and ancient solutions of the Ricci flow", *Adv. Math.* **228**:5 (2011), 2891–2919. MR 2838064 Zbl 05969510

[Cao and Zhang 2011b] X. Cao and Z. Zhang, "Differential Harnack estimates for parabolic equations", pp. 87–98 in *Complex and differential geometry* (Hannover, 2009), edited by W. Ebeling et al., Springer Proceedings in Mathematics **8**, Springer, Berlin, 2011. Zbl 1228.53078

[Chau et al. 2011] A. Chau, L.-F. Tam, and C. Yu, "Pseudolocality for the Ricci flow and applications", *Canad. J. Math.* **63**:1 (2011), 55–85. MR 2012g:53133 Zbl 1214.53053

[Chen and Chen 2009] L. Chen and W. Chen, "Gradient estimates for a nonlinear parabolic equation on complete non-compact Riemannian manifolds", *Ann. Global Anal. Geom.* **35**:4 (2009), 397–404. MR 2010k:35501 Zbl 1177.35040

[Cheng 2006] H.-B. Cheng, "A new Li–Yau–Hamilton estimate for the Ricci flow", *Comm. Anal. Geom.* **14**:3 (2006), 551–564. MR 2008h:53120 Zbl 1116.53039

[Chow 1991a] B. Chow, "The Ricci flow on the 2-sphere", *J. Differential Geom.* **33**:2 (1991), 325–334. MR 92d:53036 Zbl 0734.53033

[Chow 1991b] B. Chow, "On Harnack's inequality and entropy for the Gaussian curvature flow", *Comm. Pure Appl. Math.* **44**:4 (1991), 469–483. MR 93e:58032 Zbl 0734.53035

[Chow 1992] B. Chow, "The Yamabe flow on locally conformally flat manifolds with positive Ricci curvature", *Comm. Pure Appl. Math.* **45**:8 (1992), 1003–1014. MR 93d:53045 Zbl 0785.53027

[Chow 1998] B. Chow, "Interpolating between Li–Yau's and Hamilton's Harnack inequalities on a surface", *J. Partial Differ. Equ.* **11**:2 (1998), 137–140. MR 99h:58182 Zbl 0943.58017

[Chow and Hamilton 1997] B. Chow and R. S. Hamilton, "Constrained and linear Harnack inequalities for parabolic equations", *Invent. Math.* **129**:2 (1997), 213–238. MR 98i:53051 Zbl 0903.58054

[Chow and Knopf 2002] B. Chow and D. Knopf, "New Li–Yau–Hamilton inequalities for the Ricci flow via the space-time approach", *J. Differential Geom.* **60**:1 (2002), 1–54. MR 2003g:53116 Zbl 1048.53026

[Chow et al. 2006] B. Chow, P. Lu, and L. Ni, *Hamilton's Ricci flow*, Graduate Studies in Mathematics **77**, Amer. Math. Soc., Providence, RI, 2006. MR 2008a:53068 Zbl 1118.53001

[Chow et al. 2010] B. Chow, S.-C. Chu, D. Glickenstein, C. Guenther, J. Isenberg, T. Ivey, D. Knopf, P. Lu, F. Luo, and L. Ni, *The Ricci flow: techniques and applications, III: Geometric-analytic aspects*, Mathematical Surveys and Monographs **163**, Amer. Math. Soc., Providence, RI, 2010. MR 2011g:53142 Zbl 1216.53057

[Chung and Yau 1996] F. R. K. Chung and S.-T. Yau, "Logarithmic Harnack inequalities", *Math. Res. Lett.* **3**:6 (1996), 793–812. MR 97k:58182 Zbl 0880.58026

[Guenther 2002] C. M. Guenther, "The fundamental solution on manifolds with time-dependent metrics", *J. Geom. Anal.* **12**:3 (2002), 425–436. MR 2003a:58034 Zbl 1029.58018

[Hamilton 1988] R. S. Hamilton, "The Ricci flow on surfaces", pp. 237–262 in *Mathematics and general relativity* (Santa Cruz, CA, 1986), edited by J. A. Isenberg, Contemp. Math. **71**, Amer. Math. Soc., Providence, RI, 1988. MR 89i:53029 Zbl 0663.53031

[Hamilton 1993a] R. S. Hamilton, "The Harnack estimate for the Ricci flow", *J. Differential Geom.* **37**:1 (1993), 225–243. MR 93k:58052 Zbl 0804.53023

[Hamilton 1993b] R. S. Hamilton, "A matrix Harnack estimate for the heat equation", *Comm. Anal. Geom.* **1**:1 (1993), 113–126. MR 94g:58215 Zbl 0799.53048

[Hamilton 1995] R. S. Hamilton, "Harnack estimate for the mean curvature flow", *J. Differential Geom.* **41**:1 (1995), 215–226. MR 95m:53055 Zbl 0827.53006

[Hsu 2011] S.-Y. Hsu, "Gradient estimates for a nonlinear parabolic equation under Ricci flow", *Differential Integral Equations* **24**:7-8 (2011), 645–652. MR 2830313 Zbl 06033866

[Huang and Ma 2010] G. Huang and B. Ma, "Gradient estimates for a nonlinear parabolic equation on Riemannian manifolds", *Arch. Math.* (*Basel*) **94**:3 (2010), 265–275. MR 2011b:58054 Zbl 1194.58020

[Kuang and Zhang 2008] S. Kuang and Q. S. Zhang, "A gradient estimate for all positive solutions of the conjugate heat equation under Ricci flow", *J. Funct. Anal.* **255**:4 (2008), 1008–1023. MR 2009h:53150 Zbl 1146.58017

[Li and Xu 2011] J. Li and X. Xu, "Differential Harnack inequalities on Riemannian manifolds, I: Linear heat equation", *Adv. Math.* **226**:5 (2011), 4456–4491. MR 2770456 Zbl 1226.58009

[Li and Yau 1986] P. Li and S.-T. Yau, "On the parabolic kernel of the Schrödinger operator", *Acta Math.* **156**:3-4 (1986), 153–201. MR 87f:58156 Zbl 0611.58045

[Liu 2009] S. Liu, "Gradient estimates for solutions of the heat equation under Ricci flow", *Pacific J. Math.* **243**:1 (2009), 165–180. MR 2010g:53122 Zbl 1180.58017

[Ma 2006] L. Ma, "Gradient estimates for a simple elliptic equation on complete non-compact Riemannian manifolds", *J. Funct. Anal.* **241**:1 (2006), 374–382. MR 2007e:53034 Zbl 1112.58023

[Ma 2010a] L. Ma, "Hamilton type estimates for heat equations on manifolds", preprint, 2010. arXiv 1009.0603

[Ma 2010b] L. Ma, "Gradient estimates for a simple nonlinear heat equation on manifolds", preprint, 2010. arXiv 1009.0604

[Ni 2007] L. Ni, "A matrix Li–Yau–Hamilton estimate for Kähler–Ricci flow", *J. Differential Geom.* **75**:2 (2007), 303–358. MR 2008d:53093 Zbl 1120.53023

[Perelman 2002] G. Perelman, "The entropy formula for the Ricci flow and its geometric applications", preprint, 2002. Zbl 1130.53001 arXiv math.DG/0211159v1

[Wu 2010a] J.-Y. Wu, "Gradient estimates for a nonlinear diffusion equation on complete manifolds", *J. Partial Differ. Equ.* **23**:1 (2010), 68–79. MR 2011b:58056 Zbl 1224.58022

[Wu 2010b] J.-Y. Wu, "Li–Yau type estimates for a nonlinear parabolic equation on complete manifolds", *J. Math. Anal. Appl.* **369**:1 (2010), 400–407. MR 2011b:35432 Zbl 1211.58017

[Wu and Zheng 2010] J.-Y. Wu and Y. Zheng, "Interpolating between constrained Li–Yau and Chow–Hamilton Harnack inequalities on a surface", *Arch. Math.* (*Basel*) **94**:6 (2010), 591–600. MR 2011j:53127 Zbl 1198.53078

[Yang 2008] Y. Yang, "Gradient estimates for a nonlinear parabolic equation on Riemannian manifolds", *Proc. Amer. Math. Soc.* **136**:11 (2008), 4095–4102. MR 2009d:58048 Zbl 1151.58013

[Yau 1994] S.-T. Yau, "On the Harnack inequalities of partial differential equations", *Comm. Anal. Geom.* **2**:3 (1994), 431–450. MR 96f:58186 Zbl 0841.58059

[Yau 1995] S.-T. Yau, "Harnack inequality for non-self-adjoint evolution equations", *Math. Res. Lett.* **2**:4 (1995), 387–399. MR 96k:58211 Zbl 0884.58091

[Zhang 2006] Q. S. Zhang, "Some gradient estimates for the heat equation on domains and for an equation by Perelman", *Int. Math. Res. Not.* **2006** (2006), Art. ID 92314. MR 2007f:35116

Received June 25, 2011. Revised February 22, 2012.

JIA-YONG WU
DEPARTMENT OF MATHEMATICS
SHANGHAI MARITIME UNIVERSITY
HAIGANG AVENUE 1550
SHANGHAI 201306
CHINA

jywu81@yahoo.com

PACIFIC JOURNAL OF MATHEMATICS

http://pacificmath.org

Founded in 1951 by
E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

EDITORS

PRODUCTION

SUPPORTING INSTITUTIONS

See inside back cover or pacificmath.org for submission instructions.

The subscription price for 2012 is US $420/year for the electronic version, and $485/year for print and electronic.
Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. Prior back issues are obtainable from Periodicals Service Company, 11 Main Street, Germantown, NY 12526-5635. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and the Science Citation Index.

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 969 Evans Hall, Berkeley, CA 94720-3840, is published monthly except July and August. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW™ from Mathematical Sciences Publishers.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS
at the University of California, Berkeley 94720-3840
A NON-PROFIT CORPORATION
Typeset in LaTeX

PACIFIC JOURNAL OF MATHEMATICS

Volume 257 No. 1 May 2012

0030-8730(201205)257:1;1-7