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Let $D \subseteq E$ be an extension of integral domains, Γ a numerical semigroup with $\Gamma \subsetneq \mathbb{N}_0$, $\Gamma^* = \Gamma \setminus \{0\}$ and $R = D + E[\Gamma^*]$. In this paper, we completely characterize when R is a weakly Krull domain, an AWFD or a GWFD. We also prove that R is never a WFD.

Introduction

We first review some preliminaries. Let D be an integral domain with quotient field $qf(D)$ and let $\mathbf{F}(D)$ denote the set of nonzero fractional ideals of D. Recall that the *v-operation* on D is a star-operation on $\mathbf{F}(D)$ defined by $I \mapsto I_v := (I^{-1})^{-1}$, where $I^{-1} = \{x \in qf(D) \mid xI \subseteq D\}$. The *$t$-operation* on D is a star-operation defined by $I \mapsto I_t := \bigcup\{J_v \mid J \subseteq I$ with $J \in \mathbf{F}(D)$ finitely generated$\}$. An $I \in \mathbf{F}(D)$ is said to be a *v-ideal* if $I_v = I$, and a *t-ideal* if $I_t = I$. A v-ideal I is said to be of *finite type* if $I = J_v$ for some finitely generated fractional ideal J of D. A t-ideal M of D is called a *maximal t-ideal* if M is maximal among proper integral t-ideals of D. It is well known that maximal t-ideals are prime ideals. Let t-Max(D) be the set of maximal t-ideals of D. Then t-Max$(D) \neq \varnothing$ if D is not a field. An $I \in \mathbf{F}(D)$ is said to be *t-invertible* if $(II^{-1})_t = D$; equivalently, $II^{-1} \nsubseteq M$ for each $M \in t$-Max(D). Let $T(D)$ be the abelian group of t-invertible fractional t-ideals of D under the t-multiplication $I * J = (IJ)_t$, and let Inv(D) and Prin(D) be the subgroups of $T(D)$ consisting respectively of invertible fractional ideals of D and nonzero principal fractional ideals of D. Then it is clear that Prin$(D) \subseteq$ Inv$(D) \subseteq T(D)$. The *t-class group* of D is an abelian group Cl$(D) = T(D)/$Prin(D) and the *Picard group* Pic$(D) =$ Inv$(D)/$Prin(D) is a subgroup of Cl(D). The *local t-class group* $G(D)$ of D is defined by $G(D) =$ Cl$(D)/$Pic(D).

Let $X^1(D)$ stand for the set of height-one prime ideals of D. We say that D is a *weakly Krull domain* if $D = \bigcap_{P \in X^1(D)} D_P$ and this intersection has finite character, i.e., each nonzero element $d \in D$ is a unit in D_P for all but a finite number of P's in $X^1(D)$; D is a *weakly factorial domain* (WFD) if every nonzero nonunit element of D is a product of primary elements; D is an *almost weakly factorial domain*

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(AWFD) if for each nonzero nonunit element $d \in D$, there exists a positive integer $n = n(d)$ such that d^n is a product of primary elements; and D is a *generalized weakly factorial domain* (GWFD) if each nonzero prime ideal of D contains a primary element. (Recall that a nonzero nonunit $d \in D$ is called a *primary element* of D if (d) is a primary ideal of D.) It is well known that

$$\text{WFD} \Rightarrow \text{AWFD} \Rightarrow \text{GWFD} \Rightarrow \text{weakly Krull domain}$$

and a weakly Krull domain has t-dimension one. (The *t-dimension of D*, abbreviated t-dim(D), is the supremum of lengths of chains of prime t-ideals of D. Hence t-dim$(D) = 1$ if and only if each maximal t-ideal of D has height-one.) Also, it was shown in [Anderson and Zafrullah 1990, Theorem] that a weakly Krull domain D is a WFD if and only if $\mathrm{Cl}(D) = 0$, and in [Anderson et al. 1992, Theorem 3.4] that a weakly Krull domain D is an AWFD if and only if $\mathrm{Cl}(D)$ is torsion. We note that t-dim$(D[\Gamma]) = t$-dim$(D[X])$ for any numerical semigroup Γ [Chang et al. 2012, Theorem 1.5].

Let $\mathbb{N}_0$ (resp., $\mathbb{Z}$) be the set of nonnegative integers (resp., integers). A semigroup Γ is called a *numerical semigroup* if Γ is a subset of $\mathbb{N}_0$ containing 0 and generates $\mathbb{Z}$ as a group. It is known that if Γ is a numerical semigroup, then Γ is finitely generated and $\mathbb{N}_0 \setminus \Gamma$ is a finite set. Hence there exists the largest nonnegative integer which is not contained in Γ. This number is called the *Frobenius number* of Γ and is denoted by $F(\Gamma)$.

Throughout this article, $D \subseteq E$ denotes an extension of integral domains, $qf(D)$ (resp., $qf(E)$) is the quotient field of D (resp., E), $\bar{D}$ means the integral closure of D, X is an indeterminate over E, Γ is a numerical semigroup with $\Gamma \subsetneq \mathbb{N}_0$ and $D[\Gamma]$ is the numerical semigroup ring of Γ over D. Note that each element $f \in D[\Gamma]$ is uniquely expressible in the form $f = a_1X^{\alpha_1} + \cdots + a_kX^{\alpha_k}$, where $a_i \in D$ and $\alpha_i \in \Gamma$ with $\alpha_1 < \cdots < \alpha_k$. Let $\Gamma^* = \Gamma \setminus \{0\}$, $R = D + E[\Gamma^*]$, $T = D + XE[X]$ and $T_n = D + X^nE[X]$ for integers $n \geq 2$, i.e., $R = \{f \in E[\Gamma] \mid f(0) \in D\}$, $T = \{f \in E[X] \mid f(0) \in D\}$ and $T_n = R$ when $\Gamma = \{0\} \cup \{m \in \mathbb{N}_0 \mid m \geq n\}$. Then $D[\Gamma] \subseteq R \subseteq E[\Gamma]$ and $T_{F(\Gamma)+1} \subseteq R \subsetneq T \subseteq E[X]$. For an $f \in qf(D)[\Gamma]$, $c(f)$ means the fractional ideal of D generated by the coefficients of f. If I is an ideal of $D[\Gamma]$, then $c(I)$ denotes the ideal of D generated by the coefficients of all the polynomials in I.

In multiplicative ideal theory, the $D + E[\Gamma^*]$ construction has been extensively studied by several authors for its interest in constructing examples with prescribed properties. As a special kind of pullbacks, this has become so important that in recent years there have been many papers devoted to ring- and ideal-theoretic properties in this construction.

Anderson et al. [2003a; 2006] (see also [Anderson and Chang 2007]) studied when the domains $D[X^2, X^3]$, $D + XE[X]$ and $D + X^2E[X]$ are weakly Krull domains, WFDs, AWFDs or GWFDs. In fact, they showed that $D[X^2, X^3]$ is a weakly Krull domain if and only if D is a weakly Krull UMT-domain [Anderson et al. 2003a, Proposition 2.7]; if $\mathrm{char}(D) \neq 0$, then $D[X^2, X^3]$ is an AWFD if and only if $D[X^2, X^3]$ is a GWFD [Anderson and Chang 2007, Corollary 2.11]; $D+XE[X]$ is a weakly Krull domain if and only if $D+X^2E[X]$ is a weakly Krull domain [Anderson et al. 2006, Theorem 4.3]; and $D+XE[X]$ is an AWFD if and only if $D+XE[X]$ is a GWFD [Anderson and Chang 2007, Corollary 2.10]. The main purpose of this paper is to determine how certain properties of D, E and Γ influence those of R, and vice versa. This also extends the results for the domains $D[X^2, X^3]$, $D+XE[X]$ and $D+X^2E[X]$ to any composite numerical semigroup ring $D+E[\Gamma^*]$.

In Section 1, we investigate weakly Krull domains, AWFDs and GWFDs in the context of numerical semigroup rings $D[\Gamma]$ which coincide with the domains $R = D+E[\Gamma^*]$ when $D = E$. We prove that $D[\Gamma]$ is a weakly Krull domain if and only if D is a weakly Krull UMT-domain, and that if $\mathrm{char}(D) \neq 0$, then $D[\Gamma]$ is an AWFD if and only if $D[\Gamma]$ is a GWFD, if and only if D is an almost weakly factorial quasi-AGCD-domain, if and only if D is a generalized weakly factorial quasi-AGCD-domain.

In Section 2, we study when the domain $R = D+E[\Gamma^*]$ is a weakly Krull domain, an AWFD or a GWFD, where $D \subsetneq E$. We show that R is a weakly Krull domain if and only if $T = D+XE[X]$ is a weakly Krull domain, and that if $\mathrm{char}(E) \neq 0$, then R is an AWFD if and only if R is a GWFD, if and only if T is an AWFD, if and only if R is a GWFD. We also prove that R is never a WFD.

1. Weakly Krull domains as numerical semigroup rings

In this section, we characterize when the numerical semigroup ring $D[\Gamma]$ is a weakly Krull domain, an AWFD or a GWFD.

The first two lemmas are well known for the general semigroup rings, but we include their proofs for the convenience of the reader.

Lemma 1.1 [El Baghdadi et al. 2002, Lemma 2.3]. *Let D be an integral domain and Γ be a numerical semigroup. The following statements hold for an $I \in \mathbf{F}(D)$:*

(1) $(ID[\Gamma])^{-1} = I^{-1}D[\Gamma]$.

(2) $(ID[\Gamma])_v = I_vD[\Gamma]$.

(3) $(ID[\Gamma])_t = I_tD[\Gamma]$.

Proof. (1) Since $(ID[\Gamma])(I^{-1}D[\Gamma]) \subseteq D[\Gamma]$, $I^{-1}D[\Gamma] \subseteq (ID[\Gamma])^{-1}$. Conversely, let $f \in (ID[\Gamma])^{-1}$. Then $fID[\Gamma] \subseteq D[\Gamma]$ and hence $c(f)I \subseteq D$. Hence $c(f) \subseteq I^{-1}$, and therefore $f \in c(f)D[\Gamma] \subseteq I^{-1}D[\Gamma]$. Thus the equality holds.

(2) By (1), $(ID[\Gamma])_v = ((ID[\Gamma])^{-1})^{-1} = (I^{-1}D[\Gamma])^{-1} = I_vD[\Gamma]$.

(3) Let $f_1, \dots, f_n$ be nonzero elements of $ID[\Gamma]$. Then we have

$$\begin{aligned}((f_1, \dots, f_n)D[\Gamma])_v &\subseteq ((c(f_1), \dots, c(f_n))D[\Gamma])_v \\ &= (c(f_1), \dots, c(f_n))_v D[\Gamma] \\ &\subseteq I_t D[\Gamma]\end{aligned}$$

by (2), i.e., $(ID[\Gamma])_t \subseteq I_t D[\Gamma]$. For the reverse inclusion, let J be a nonzero finitely generated subideal of I. Then $J_v D[\Gamma] = (JD[\Gamma])_v \subseteq (ID[\Gamma])_t$ by (2). Hence $I_t D[\Gamma] \subseteq (ID[\Gamma])_t$. Thus we have the desired equality. □

Lemma 1.2 [Anderson and Chang 2005, Corollary 2.3]. *Let D be an integral domain, Γ be a numerical semigroup and let Q be a maximal t-ideal of $D[\Gamma]$ such that $Q \cap D \neq (0)$. Then $Q = (Q \cap D)D[\Gamma]$. In particular, $Q \cap D$ is a maximal t-ideal of D.*

Proof. The containment $(Q \cap D)D[\Gamma] \subseteq Q$ is obvious. For the converse, it suffices to show that $c(Q) \subseteq Q$. Suppose to the contrary that $c(Q) \nsubseteq Q$. Then

$$Q \subsetneq c(Q)D[\Gamma].$$

Since Q is a maximal t-ideal of $D[\Gamma]$, $(c(Q)D[\Gamma])_t = D[\Gamma]$. Therefore $c(Q)_t = D$ by Lemma 1.1(3), and hence $c(f)_v = D$ for some $f \in Q$. Let $0 \neq d \in Q \cap D$ and choose $0 \neq g \in (d, f)^{-1}$. Then $gd \in D[\Gamma]$ and hence $g \in qf(D)[\Gamma]$. Also, we have $fg \in D[\Gamma]$. Hence it follows from [Gilmer 1992, Theorem 28.1] that

$$c(g) \subseteq c(g)_v = (c(f)^{m+1}c(g))_v = (c(f^m)c(fg))_v = c(fg)_v \subseteq D,$$

where m is the degree of g. So $g \in c(g)D[\Gamma] \subseteq D[\Gamma]$, which implies that $(d, f)^{-1} = D[\Gamma]$. This contradicts the fact that Q is a maximal t-ideal of $D[\Gamma]$. Therefore $c(Q) \subseteq Q$, and thus $Q \subseteq (Q \cap D)D[\Gamma]$. The second assertion is an immediate consequence of Lemma 1.1(3). □

An integral domain B is said to be a *UMT-domain* if every upper to zero (a nonzero prime ideal of $B[X]$ which contracts to zero in B) Q of $B[X]$ is a maximal t-ideal (equivalently, is t-invertible). Now, we give the numerical semigroup ring version of [Anderson et al. 1993, Proposition 4.11].

Theorem 1.3. *Let D be an integral domain and Γ be a numerical semigroup with $\Gamma \subsetneq \mathbb{N}_0$. Then the following assertions are equivalent.*

(1) *$D[\Gamma]$ is a weakly Krull domain.*

(2) *$D[X]$ is a weakly Krull domain.*

(3) *D is a weakly Krull UMT-domain.*

Proof. (1) ⇒ (3) Assume $D[\Gamma]$ is a weakly Krull domain. Then $t\text{-dim}(D[\Gamma])=1$ [Anderson et al. 1992, Lemma 2.1]. Let P be a prime t-ideal of D. Then $PD[\Gamma]$ is a prime t-ideal of $D[\Gamma]$ by Lemma 1.1(3); so $\text{ht}_D(P)=\text{ht}_{D[\Gamma]}(PD[\Gamma])=1$; so $t\text{-dim}(D)=1$. Since $t\text{-dim}(D[\Gamma])=1$, we have $t\text{-dim}(D[X])=1$ by [Chang et al. 2012, Theorem 1.5]. Therefore every upper to zero in $D[X]$ is a maximal t-ideal, and thus D is a UMT-domain. Note that

$$D=\bigcap_{P\in X^1(D)} D_P$$

by [Kang 1989, Proposition 2.9]. To show that this intersection has finite character, let $d\in D\setminus\{0\}$. Since $D[\Gamma]$ is a weakly Krull domain, d belongs to only finitely many height-one prime ideals of $D[\Gamma]$, and hence there exists only a finite number of height-one prime ideals of D containing d. Thus D is a weakly Krull domain.

(3) ⇒ (1) Assume that D is a weakly Krull UMT-domain and let Q be a maximal t-ideal of $D[\Gamma]$ with $Q\cap D\neq(0)$. By Lemma 1.2, $Q=(Q\cap D)D[\Gamma]$ and $Q\cap D$ is a maximal t-ideal of D. Since $t\text{-dim}(D)=1$ [Anderson et al. 1992, Lemma 2.1], $\text{ht}_D(Q\cap D)=1$; so $\text{ht}_{D[\Gamma]}Q\leq 2$ (cf. [Kaplansky 1970, Theorem 37]). If $\text{ht}_{D[\Gamma]}Q=2$, then there exists a nonzero prime ideal $P\subsetneq Q$ which contracts to zero in D. Note that $P=M\cap D[\Gamma]$ for some prime ideal M of $D[X]$ [Chang et al. 2012, Proposition 1.1]. Since $M\cap D=(0)$ and D is a UMT-domain, M is a maximal t-ideal of $D[X]$. Hence P is a maximal t-ideal of $D[\Gamma]$ [Chang et al. 2012, Theorem 1.4]. This contradicts the choice of P. Thus $t\text{-dim}(D[\Gamma])=1$. By [Kang 1989, Proposition 2.9], we have $D[\Gamma]=\bigcap_{Q\in X^1(D[\Gamma])} D[\Gamma]_Q$. We claim that this intersection has finite character. Let $f\in D[\Gamma]\setminus\{0\}$ and set

$$\mathcal{S}=\{Q\in X^1(D[\Gamma])\mid f\in Q\},$$
$$\mathcal{S}_1=\{Q\in\mathcal{S}\mid Q\cap D\in X^1(D)\},\text{ and}$$
$$\mathcal{S}_2=\{Q\in\mathcal{S}\mid Q\cap D=(0)\}.$$

Then $\mathcal{S}=\mathcal{S}_1\cup\mathcal{S}_2$. If $\mathcal{S}_1$ is an infinite set, then $c(f)$ belongs to infinitely many height-one prime ideals of D by Lemma 1.2. This is absurd, because D is a weakly Krull domain. Hence $\mathcal{S}_1$ is a finite set. Note that $qf(D)[\Gamma]$ is a one-dimensional Noetherian domain; so $qf(D)[\Gamma]$ is a weakly Krull domain. Hence $\mathcal{S}_2$ is also a finite set. Therefore $\mathcal{S}$ is a finite set. Thus $D[\Gamma]$ is a weakly Krull domain.

(2) ⇔ (3) See [Anderson et al. 1993, Proposition 4.11]. □

Recall that if $D\subseteq E$ is an extension of integral domains, then E is said to be a *root extension* of D if for each $z\in E$, there is a positive integer $n=n(z)$ such that $z^n\in D$. A domain B is called an *almost Prüfer v-multiplication domain* (APvMD) (resp., *almost GCD-domain* (AGCD-domain)) if for each $0\neq a,b\in B$, there exists a positive integer $n=n(a,b)$ such that $(a^n,b^n)_v$ is t-invertible (resp., principal).

It is known that B is a weakly Krull PvMD if and only if $B[X]$ is weakly Krull and B is integrally closed [Anderson et al. 1993, Corollary 4.13]. We weaken the hypothesis and obtain the following result.

Corollary 1.4. *Let D be an integral domain and Γ be a numerical semigroup.*

(1) *D is a weakly Krull APvMD if and only if $D[\Gamma]$ is a weakly Krull domain and $D \subseteq \overline{D}$ is a root extension.*

(2) *D is an almost weakly factorial AGCD-domain if and only if $D[\Gamma]$ is a weakly Krull domain, $\mathrm{Cl}(D)$ is torsion and $D \subseteq \overline{D}$ is a root extension.*

Proof. (1) By [Li 2012, Theorem 3.8], a domain B is an APvMD if and only if B is a UMT-domain and $B \subseteq \overline{B}$ is a root extension. Thus the result follows from Theorem 1.3.

(2) By [Li 2012, Theorem 3.1], a domain B is an AGCD-domain if and only if B is an APvMD and $\mathrm{Cl}(B)$ is torsion. Also, by [Anderson et al. 1992, Theorem 3.4], B is an AWFD if and only if B is a weakly Krull domain and $\mathrm{Cl}(B)$ is torsion. Thus the result is an immediate consequence of Theorem 1.3 and (1). □

Let S be a saturated multiplicative subset of a domain B and let $N(S) = \{0 \neq b \in B \mid (b, s)_v = B \text{ for all } s \in S\}$ be the m-complement of S. We say that S is an *almost splitting set* if for each $0 \neq b \in B$, there exists a positive integer $n = n(b)$ such that $b^n = st$ for some $s \in S$ and $t \in N(S)$. Following [Anderson and Chang 2007], B is called a *quasi-AGCD-domain* if $B \setminus \{0\}$ is an almost splitting set in $B[X]$. It was shown that if B is integrally closed, then the notion of quasi-AGCD-domains coincides with that of AGCD-domains [Chang 2005, Proposition 2.6]. The next corollary characterizes when the numerical semigroup ring $D[\Gamma]$ is an AWFD or a GWFD.

Corollary 1.5. *Let D be an integral domain with $\mathrm{char}(D) \neq 0$ and Γ be a numerical semigroup with $\Gamma \subsetneq \mathbb{N}_0$. Then the following conditions are equivalent.*

(1) *$D[\Gamma]$ is an AWFD.*

(2) *$D[\Gamma]$ is a GWFD.*

(3) *$D[X]$ is an AWFD.*

(4) *$D[X]$ is a GWFD.*

(5) *D is an almost weakly factorial quasi-AGCD-domain.*

(6) *D is a generalized weakly factorial quasi-AGCD-domain.*

(7) *D is a weakly Krull quasi-AGCD-domain.*

Proof. Let $\mathrm{char}(D) = p$.

(1) $\Rightarrow$ (2) This is well known.

(1) $\Leftrightarrow$ (3) By [Anderson et al. 1992, Theorem 3.4], an integral domain B is an AWFD if and only if B is a weakly Krull domain and $\mathrm{Cl}(B)$ is torsion, and by Theorem 1.3, $D[\Gamma]$ is a weakly Krull domain if and only if $D[X]$ is a weakly Krull domain. By [Chang et al. 2012, Lemma 2.7], $\mathrm{Pic}(qf(D)[\Gamma])$ is torsion if and only if $\mathrm{char}(D) \neq 0$. Since $\mathrm{Cl}(D[\Gamma]) = \mathrm{Cl}(D[X]) \oplus \mathrm{Pic}(qf(D)[\Gamma])$ [Anderson and Chang 2004, Theorem 5], $\mathrm{Cl}(D[\Gamma])$ is torsion if and only if $\mathrm{Cl}(D[X])$ is torsion and $\mathrm{char}(D) \neq 0$. Thus this equivalence follows from these facts.

(4) $\Rightarrow$ (2) By [Anderson et al. 2003b, Theorem 2.2], a domain B is a GWFD if and only if t-$\dim(B) = 1$ and for each $P \in X^1(B)$, $P = \sqrt{bB}$ for some $b \in B$. Assume that $D[X]$ is a GWFD and let $P \in X^1(D[\Gamma])$. Since t-$\dim(D[\Gamma]) = t$-$\dim(D[X]) = 1$ [Chang et al. 2012, Theorem 1.5], it suffices to show that $P = \sqrt{fD[\Gamma]}$ for some $f \in D[\Gamma]$. If $P \cap D \neq (0)$, then $P = (P \cap D)D[\Gamma]$ by Lemma 1.2. Since $D[X]$ is a GWFD, $(P \cap D)D[X] = \sqrt{dD[X]}$ for some $d \in P \cap D$. It is easy to see that $P = \sqrt{dD[\Gamma]}$. Next, suppose that $P \cap D = (0)$. Then there exists a prime t-ideal Q of $D[X]$ such that $P = Q \cap D[\Gamma]$ [Chang et al. 2012, Theorem 1.5]. Since $D[X]$ is a GWFD, $Q = \sqrt{fD[X]}$ for some $f \in D[X]$. Also, since $\mathrm{char}(D) = p > 0$, there exists a positive integer n such that $f^{p^n} \in D[\Gamma]$. An easy calculation shows that $P = \sqrt{f^{p^n} D[\Gamma]}$. Thus $D[\Gamma]$ is a GWFD.

(2) $\Rightarrow$ (4) This direction is an easy modification of the proof of (4) $\Rightarrow$ (2).

(2) $\Rightarrow$ (5) See [Anderson and Chang 2007, Corollary 2.9].

(5) $\Rightarrow$ (6) $\Rightarrow$ (7) These implications are obvious.

(7) $\Rightarrow$ (1) Assume that D is a weakly Krull quasi-AGCD-domain. Then D is a UMT-domain and $\mathrm{Cl}(D[X])$ is torsion [Anderson and Chang 2007, Theorem 2.4]. Hence $D[\Gamma]$ is a weakly Krull domain by Theorem 1.3. Also, it follows from [Anderson and Chang 2004, Theorem 5; Chang et al. 2012, Lemma 2.7] that $\mathrm{Cl}(D[\Gamma])$ is torsion. Thus $D[\Gamma]$ is an AWFD [Anderson et al. 1992, Theorem 3.4]. □

We end this section by noting that $D[\Gamma]$ is never a WFD. We also show that $D[\Gamma]$ need not be an AWFD if $\mathrm{char}(D) = 0$.

Remark 1.6. (1) Let B be an integral domain with quotient field K. In [Gilmer and Martin 1990, Theorem 7], Gilmer and Martin showed that if B is a seminormal domain and $B + X^n B[X] \subseteq B[\Gamma]$, then $\mathrm{Pic}(B[\Gamma]) = \mathrm{Pic}(B) \oplus (W_n/L)$, where $L \subseteq W_n$ are the subgroups of the group $U(B[X]/X^n B[X])$ of units of $B[X]/X^n B[X]$ defined by $W_n = \{1 + Xf + X^n B[X] \mid f \in B[X]\}$ and $L = \{1 + Xf + X^n B[X] \mid 1 + Xf \in B[\Gamma]\}$. Note that $\mathrm{Cl}(B[\Gamma]) = \mathrm{Cl}(B[X]) \oplus \mathrm{Pic}(K[\Gamma])$ [Anderson and Chang 2004, Theorem 5] and that B is a WFD if and only if B is a weakly Krull domain and $\mathrm{Cl}(B) = 0$ [Anderson and Zafrullah 1990, Theorem]. If $D[\Gamma]$ is a WFD, then $\mathrm{Cl}(D[\Gamma]) = 0$, and hence $\mathrm{Pic}(qf(D)[\Gamma]) = 0$. Therefore $W_n = L$;

so $1 + X + X^n qf(D)[X] \in L$, which implies that $1 \in \Gamma$. Thus, if Γ is a proper numerical semigroup, then $D[\Gamma]$ is never a WFD.

(2) If $D[\Gamma]$ is an AWFD, then $\mathrm{Cl}(D[\Gamma])$ is torsion [Anderson et al. 1992, Theorem 3.4]; so $\mathrm{Pic}(qf(D)[\Gamma])$ is torsion [Anderson and Chang 2004, Theorem 5]. Hence $\mathrm{char}(D) \neq 0$ [Chang et al. 2012, Lemma 2.7]. This shows that the condition that $\mathrm{char}(D) \neq 0$ is essential in Corollary 1.5.

(3) It is known that a generalized unique factorization domain (GUFD) is a weakly factorial GCD-domain [Anderson et al. 1995, Theorem 7], and hence integrally closed. (See [Anderson et al. 1995] for the definition and some characterizations of a GUFD.) Thus, if Γ is a numerical semigroup with $\Gamma \subsetneq \mathbb{N}_0$, then $D[\Gamma]$ is not a GUFD by (1). In fact, $D[\Gamma]$ is not integrally closed; so $D[\Gamma]$ is never a GUFD.

2. Weakly Krull domains and the ring $D + E[\Gamma^*]$ when $D \subsetneq E$

For a domain A, $\mathrm{Spec}(A)$ stands for the set of prime ideals of A. Assume that $D \subsetneq E$ is an extension of integral domains, Γ is a numerical semigroup with $\Gamma \subsetneq \mathbb{N}_0$ and let $R = D + E[\Gamma^*]$, $T = D + XE[X]$, $T_n = D + X^n E[X]$ and $\Delta_n = \{0\} \cup \{m \in \mathbb{N}_0 \mid m \geq n\}$ for integers $n \geq 2$. Note that $D[\Gamma] \subsetneq R \subsetneq T$ and $T_n \subsetneq T$. In this section, we characterize when the domains R and T_n are weakly Krull domains, AWFDs or GWFDs. To do this, we need two lemmas.

Lemma 2.1. *Let $R = D + E[\Gamma^*]$ and $T = D + XE[X]$. If Q is a prime ideal of R, then there exists a unique prime ideal of T lying over Q. Thus the natural map $\phi : \mathrm{Spec}(T) \to \mathrm{Spec}(R)$, given by $P \mapsto P \cap R$, is an order-preserving bijection. In particular, $ht_T(XE[X]) = ht_R(E[\Gamma^*])$.*

Proof. Let Q be a prime ideal of R. Since T is an integral extension of R, there exists a prime ideal P of T such that $Q = P \cap R$ [Kaplansky 1970, Theorem 44]. Note that $E[\Gamma^*] \subseteq Q$ if and only if $XE[X] \subseteq P$. If $E[\Gamma^*] \subseteq Q$, then P is the unique prime ideal of T lying over Q because $R/XE[X] \cong D \cong R/E[\Gamma^*]$. If $E[\Gamma^*] \nsubseteq Q$, then $X^{F(\Gamma)+1} f \notin Q$ for some $f \in E[X]$; so

$$g = \frac{X^{F(\Gamma)+1} fg}{X^{F(\Gamma)+1} f} \in R_Q$$

for any $g \in T$. Hence $T_{QR_Q \cap T} = R_Q$. Thus $QR_Q \cap T$ is the unique prime ideal of T lying over Q. □

Let n be an integer ≥ 2. Then it is clear that if $\Gamma = \Delta_n$, then $R = T_n$. Hence Lemma 2.1 also shows that $\mathrm{ht}_T(XE[X]) = \mathrm{ht}_{T_n}(X^n E[X])$.

Remark 2.2. Let $\Gamma = \{\alpha_1, \ldots, \alpha_n\} \cup \Delta_{F(\Gamma)+1}$ with $1 < \alpha_1 < \cdots < \alpha_n < F(\Gamma) + 1$ and $R = D + E[\Gamma^*]$.

(1) Let $g \in (R : E[\Gamma^*])$. Then $gE[\Gamma^*] \subseteq R$; hence for each $\alpha \in \Gamma^*$, $gX^{\alpha} = a_{\alpha} + f_{\alpha}$ for some $a_{\alpha} \in D$ and $f_{\alpha} \in E[\Gamma^*]$. Therefore $gX^{\alpha+F(\Gamma)} = (a_{\alpha}+f_{\alpha})X^{F(\Gamma)} \in R$, which means that $a_{\alpha}=0$. Hence $gX^{\alpha} = f_{\alpha} \in E[\Gamma^*]$, and so $g \in \bigcap_{\alpha\in\Gamma^*}\{\frac{1}{X^{\alpha}} f \mid f \in E[\Gamma^*]\}$. The reverse containment is obvious. Thus we have

$$(R : E[\Gamma^*]) = \bigcap_{\alpha\in\Gamma^*} \left\{ \frac{1}{X^{\alpha}} f \mid f \in E[\Gamma^*] \right\}.$$

(2) It is clear that $E[\Gamma] \subsetneq (R : E[\Gamma^*])$ because $X^{F(\Gamma)} \in (R : E[\Gamma^*]) \setminus E[\Gamma]$. Let $g \in (R : E[\Gamma^*])$. Then $X^{F(\Gamma)+1} g \in R$; so we can write

$$X^{F(\Gamma)+1} g = \sum_{i=0}^{n} g_i X^{\alpha_i} + X^{F(\Gamma)+1} h$$

for some $g_i \in E$ and $h \in E[X]$. (For the sake of convenience, set $\alpha_0 = 0$.). Fix a $k \in \{1, \dots, n\}$. Then we have $X^{2F(\Gamma)-\alpha_k+1} g = \sum_{i=0}^{k-1} g_i X^{F(\Gamma)+\alpha_i-\alpha_k} + g_k X^{F(\Gamma)} + X^{F(\Gamma)+1}\big(\sum_{i=k+1}^{n} g_i X^{\alpha_i-\alpha_k-1} + h\big) \in R$; so $g_k = 0$ for all $k = 1, \dots, n$. Also, we have $X^{F(\Gamma)+2} g = g_0 X + X^{F(\Gamma)+2} h \in R$; so $g_0 = 0$. Therefore $X^{F(\Gamma)+1} g = X^{F(\Gamma)+1} h$, and hence $g = h \in E[X]$. Thus $E[\Gamma] \subsetneq (R : E[\Gamma^*]) \subseteq E[X]$. In particular, if $\Gamma = \Delta_{F(\Gamma)+1}$, then $E[X] \subseteq (R : E[\Gamma^*])$; so $(R : E[\Gamma^*]) = E[X]$.

(3) Lemma 4.2 of [Anderson et al. 2006] cannot be extended to any proper numerical semigroup, i.e., it may happen that $(R : E[\Gamma^*]) \subsetneq E[X]$ for some $\Gamma \subsetneq \mathbb{N}_0$. For instance, if $\Gamma = \{2\} \cup \Delta_4$, then $X \in E[X] \setminus (R : E[\Gamma^*])$.

Lemma 2.3. *The following statements hold for $R = D + E[\Gamma^*]$.*

(1) *$E[\Gamma^*]$ is a prime t-ideal of R.*

(2) *$E[\Gamma^*]$ is a maximal t-ideal of R if and only if $qf(D) \cap E = D$.*

Proof. (1) Let $\Gamma = \{\alpha_1, \dots, \alpha_k\} \cup \Delta_{F(\Gamma)+1}$ such that $0 < \alpha_1 < \cdots < \alpha_k < F(\Gamma)+1$. Since $R/E[\Gamma^*] \cong D$, $E[\Gamma^*]$ is a prime ideal of R. It suffices to show that $E[\Gamma^*]$ is a v-ideal of R, because each v-ideal is a t-ideal.

Case 1. $\{\alpha_1, \dots, \alpha_k\}$ is empty. In this case, $(R : E[\Gamma^*]) = E[X]$ by Remark 2.2(2); so we need to show that $(R : E[X]) = E[\Gamma^*]$. It is clear that $E[\Gamma^*] \subseteq (R : E[X])$. For the converse, let $f \in (R : E[X])$. Then $fE[X] \subseteq R$. Since $1 \in E[X]$, $f \in R$. Also, since $X \in E[X]$, $f(0) = 0$; so $f \in E[\Gamma^*]$.

Case 2. $\{\alpha_1, \dots, \alpha_k\}$ is nonempty. Deny the conclusion, and then there exists a polynomial $g = g_0 + \sum_{i=1}^{k} g_{\alpha_i} X^{\alpha_i} + \sum_{i=F(\Gamma)+1}^{l} g_i X^i \in (E[\Gamma^*])_v \setminus E[\Gamma^*]$. Hence $g(R : E[\Gamma^*]) \subseteq R$. Let $f \in (R : E[\Gamma^*])$. Then $f \in E[X]$ by Remark 2.2(2); so we can write $f = \sum_{i=0}^{m} f_i X^i$. Note that

$$fg = f_0 g_0 + g_0 \sum_{i=1}^{\alpha_1-1} f_i X^i + (f_0 g_{\alpha_1} + f_{\alpha_1} g_0) X^{\alpha_1} + X^{\alpha_1+1} h_1$$

for some $h_1 \in E[X]$. Since $fg \in R$ and $g_0 \neq 0$, $f_1 = \cdots = f_{\alpha_1 - 1} = 0$; so $f = f_0 + \sum_{i=\alpha_1}^{m} f_i X^i$. Note that $2\alpha_1 \in \Gamma^*$; so $2\alpha_1 \geq F(\Gamma)+1$ or $2\alpha_1 = \alpha_p$ for some $p \in \{2, \ldots, k\}$. If $2\alpha_1 \geq F(\Gamma)+1$, then we have

$$fg = f_0 g_0 + (f_0 g_{\alpha_1} + f_{\alpha_1} g_0) X^{\alpha_1} + g_0 \sum_{i=\alpha_1+1}^{\alpha_2-1} f_i X^i + (f_0 g_{\alpha_2} + f_{\alpha_2} g_0) X^{\alpha_2} + X^{\alpha_2+1} h_2$$

for some $h_2 \in E[X]$. Again, since $fg \in R$, $f_{\alpha_1+1} = \cdots = f_{\alpha_2-1} = 0$. By repeating this process, we have $f_i = 0$ for all $i \in \mathbb{N}_0 \setminus \Gamma$, and hence $f \in R$. Therefore $(R : E[\Gamma^*]) = R$. However, this is impossible because $X^{F(\Gamma)} \in (R : E[\Gamma^*]) \setminus R$. If $2\alpha_1 = \alpha_p$ for some $p \in \{2, \ldots, k\}$, a simple modification of the proof of the previous case leads to the same conclusion because $2\alpha_l \geq F(\Gamma)+1$ for some $l \leq k$.

In either case, $E[\Gamma^*]$ is a v-ideal, and thus $E[\Gamma^*]$ is a t-ideal of R.

(2) This appears in [Lim 2012, Lemma 1.2]. □

Now, we are ready to give a necessary and sufficient condition for the domain R to be a weakly Krull domain.

Theorem 2.4. *Let $R = D + E[\Gamma^*]$, $T = D + XE[X]$, $T_n = D + X^n E[X]$ and $\Delta_n = \{0\} \cup \{m \in \mathbb{N}_0 \mid m \geq n\}$ for integers $n \geq 2$. Then the following statements are equivalent.*

1. *R is a weakly Krull domain.*
2. *T is a weakly Krull domain.*
3. *T_n is a weakly Krull domain.*
4. *$X^n E[X]$ is a height-one maximal t-ideal of T_n and $E[\Delta_n]$ is a weakly Krull domain.*
5. *$E_{D\setminus\{0\}}$ is a field, $qf(D) \cap E = D$ and $E[X]$ is a weakly Krull domain.*

Proof. (2) ⇒ (1) Let T be a weakly Krull domain. Let $\Gamma = \{\alpha_1, \ldots, \alpha_k\} \cup \Delta_{F(\Gamma)+1}$ be such that $0 < \alpha_1 < \cdots < \alpha_k < F(\Gamma)+1$. Then $T = \bigcap_{P \in X^1(T)} T_P$ and this intersection has finite character. Note that $XE[X]$ is a height-one prime ideal of T [Anderson et al. 2006, Theorem 3.4]; so $E[\Gamma^*]$ is a height-one prime ideal of R by Lemma 2.1. We claim that $R = \bigcap_{P \cap R \in X^1(R)} R_{P \cap R}$, where P ranges over all height-one prime ideals of T. Suppose to the contrary that there exists an element f in $\bigcap_{P \cap R \in X^1(R)} R_{P \cap R} \setminus R$. Note that $f \in T$, and hence we can write $f = \sum_{i=0}^{m} f_i X^i$. Then there exists a polynomial $g \in R \setminus E[\Gamma^*]$ such that $fg \in R$. Since $g(0) \neq 0$, the same argument as in the proof of Lemma 2.3(1) shows that $f \in R$, which contradicts the choice of f. Thus the equality holds. Since $T = \bigcap_{P \in X^1(T)} T_P$ has finite character, it is clear that the intersection $R = \bigcap_{P \cap R \in X^1(R)} R_{P \cap R}$ also has finite character. Thus R is a weakly Krull domain.

(2) ⇒ (3) This implication was already shown in the proof of (2) ⇒ (1).

(3) ⇒ (4) Assume that T_n is a weakly Krull domain. Then t-$\dim(T_n)=1$ [Anderson et al. 1992, Lemma 2.1]; so $X^nE[X]$ is a maximal t-ideal of T_n by Lemma 2.3(1).

Let $S=\{X^m \mid m\in\Delta_n\}$. Then $E[\Delta_n]_S=E[X,X^{-1}]=(T_n)_S$ is a weakly Krull domain [Anderson et al. 1993, Proposition 4.7]. Note that $XE[X]$ is a height-one prime ideal of $E[X]$; so $X^nE[X]$ is a height-one prime ideal of $E[\Delta_n]$ [Chang et al. 2012, Proposition 1.1]; so $E[\Delta_n]_{X^nE[X]}$ is a one-dimensional quasi-local domain. Hence $E[\Delta_n]_{X^nE[X]}$ is a weakly Krull domain. We claim that $E[\Delta_n]=E[\Delta_n]_S\cap E[\Delta_n]_{X^nE[X]}$. Let $f=f_0+\sum_{i=n}^{k_1}f_iX^i$ and $h=h_0+\sum_{i=n}^{k_2}h_iX^i$ be nonzero elements of $E[\Delta_n]$ with $h(0)\neq 0$ and let $g=\sum_{i=0}^{k_3}g_iX^i\in E[X]\setminus\{0\}$ with $g(0)\neq 0$ satisfying $\frac{g}{X^m}=\frac{f}{h}\in E[\Delta_n]_S\cap E[\Delta_n]_{X^nE[X]}$ for some nonnegative integer m. Then $X^mf=gh$; so $m=0$. By comparing coefficients of f and gh, it is easy to see that $g_i=0$ for all $i=1,\ldots,n-1$. Hence $\frac{g}{X^m}\in E[\Delta_n]$. The reverse inclusion is clear. Thus $E[\Delta_n]$ is a weakly Krull domain.

(4) ⇒ (5) By [Zafrullah 2003, Lemma 2.6], $\mathrm{ht}_T(XE[X])=\dim(E_{D\setminus\{0\}}[X])$. By (4), $\mathrm{ht}_{T_n}(X^nE[X])=1$; so the comment before Remark 2.2 establishes that

$$\dim(E_{D\setminus\{0\}}[X])=1.$$

Thus $E_{D\setminus\{0\}}$ is a field. Also, since $X^nE[X]$ is a maximal t-ideal of T_n, $qf(D)\cap E=D$ by Lemma 2.3(2). Finally, it follows directly from Theorem 1.3 that $E[X]$ is a weakly Krull domain.

(5) ⇒ (2) [Anderson et al. 2006, Theorem 3.4].

(1) ⇒ (2) In the proof of (2) ⇔ (4), the integer $n\geq 2$ was arbitrary; so it suffices to show that $X^{F(\Gamma)+1}E[X]$ is a height-one maximal t-ideal of $T_{F(\Gamma)+1}$ and $E[\Delta_{F(\Gamma)+1}]$ is a weakly Krull domain. Assume that R is a weakly Krull domain. Since t-$\dim(R)=1$ [Anderson et al. 1992, Lemma 2.1], $E[\Gamma^*]$ is a height-one maximal t-ideal of R by Lemma 2.3(1); so $X^{F(\Gamma)+1}E[X]$ is a height-one maximal t-ideal of $T_{\Delta_{F(\Gamma)+1}}$ by Lemma 2.1 and the remark before Remark 2.2. Let $S_1=\{X^\alpha \mid \alpha\in\Delta_{F(\Gamma)+1}\}$ and $S_2=\{X^\alpha \mid \alpha\in\Gamma\}$. Then $E[\Delta_{F(\Gamma)+1}]_{S_1}=R_{S_2}$ is a weakly Krull domain [Anderson et al. 1993, Proposition 4.7]. Also, $E[\Delta_{F(\Gamma)+1}]_{X^{F(\Gamma)+1}E[X]}$ is a weakly Krull domain because it is one-dimensional quasi-local. Note that $E[\Delta_{F(\Gamma)+1}]=E[\Delta_{F(\Gamma)+1}]_{S_1}\cap E[\Delta_{F(\Gamma)+1}]_{X^{F(\Gamma)+1}E[X]}$ as in the proof of (3) ⇒ (4). Thus $E[\Delta_{F(\Gamma)+1}]$ is a weakly Krull domain. □

Corollary 2.5. *Let $R=D+E[\Gamma^*]$, $T=D+XE[X]$, $T_n=D+X^nE[X]$ and $\Delta_n=\{0\}\cup\{m\in\mathbb{N}_0 \mid m\geq n\}$ for integers $n\geq 2$. If $\mathrm{char}(E)\neq 0$, then the following statements are equivalent.*

(1) *R is an AWFD.*

(2) *R is a GWFD.*

(3) *T is an AWFD.*

(4) *T is a GWFD.*

(5) *T_n is an AWFD.*

(6) *T_n is a GWFD.*

(7) *$X^n E[X]$ is a maximal t-ideal of T_n, $E[\Delta_n]$ is an AWFD and for each $0 \neq e \in E$, there exist an integer $m = m(e) \geq 1$ and a unit u of E such that $ue^m \in D$.*

(8) *$X^n E[X]$ is a maximal t-ideal of T_n, $E[\Delta_n]$ is a GWFD and for each $0 \neq e \in E$, there exist an integer $m = m(e) \geq 1$ and a unit u of E such that $ue^m \in D$.*

(9) *$qf(D) \cap E = D$, $E[X]$ is an AWFD and for each $0 \neq e \in E$, there exist an integer $m = m(e) \geq 1$ and a unit u of E such that $ue^m \in D$.*

(10) *$qf(D) \cap E = D$, $E[X]$ is a GWFD and for each $0 \neq e \in E$, there exist an integer $m = m(e) \geq 1$ and a unit u of E such that $ue^m \in D$.*

Proof. (1) $\Rightarrow$ (2) and (5) $\Rightarrow$ (6) Their definitions lead to these implications.

(3) $\Leftrightarrow$ (9) [Anderson et al. 2006, Theorem 3.5].

(4) $\Leftrightarrow$ (10) [Anderson and Chang 2007, Corollary 2.10].

(7) $\Leftrightarrow$ (8) and (9) $\Leftrightarrow$ (10) See Corollary 1.5.

(7) $\Leftrightarrow$ (9) This equivalence follows from Corollary 1.5 and Lemma 2.3(2).

(3) $\Rightarrow$ (1) Assume that T is an AWFD. Then T is a weakly Krull domain [Anderson et al. 1992, Theorem 3.4]. Hence $E[X]$ is a weakly Krull domain by Theorem 2.4. Let $S = \{X^m \mid m \in \mathbb{N}_0\}$. Since X is a prime element of $E[X]$, $\mathrm{Cl}(E[X]) = \mathrm{Cl}(T_S)$ is torsion [Anderson et al. 1993, Corollary 4.9]; so $E[X]$ is an AWFD [Anderson et al. 1992, Theorem 3.4]. Let $f \in R \setminus \{0\}$. Then there exists an integer $m \geq 1$ such that $f^m = X^l f_1 \cdots f_r$ for some nonnegative positive integer l and primary elements $f_1, \dots, f_r$ of $E[X]$ with nonzero constant terms. Also, since $\mathrm{char}(E) \neq 0$, there exists an integer $k \geq F(\Gamma) + 1$ such that $f_i^k \in E[\Gamma]$ for all $i = 1, \dots, r$; so $f^{mk} = X^{lk} f_1^k \cdots f_r^k \in E[\Gamma]$. Fix an $i \in \{1, \dots, r\}$, and we claim that $\sqrt{f_i^k E[\Gamma]}$ is a prime ideal of $E[\Gamma]$ [Anderson et al. 2003b, Lemma 2.1]. Note that $\sqrt{f_i E[X]} = \sqrt{f_i^k E[X]}$. If $\sqrt{f_i^k E[X]} = XE[X]$, then an easy calculation using a similar method as in the proof of (2) $\Rightarrow$ (1) in Theorem 2.4 shows that $\sqrt{f_i^k E[\Gamma]} = E[\Gamma^*]$ is a prime ideal. Assume that $\sqrt{f_i^k E[X]} \neq XE[X]$. Since $f_i(0) \neq 0$, $f_i^k E[X, X^{-1}]$ is a primary ideal of $E[X, X^{-1}]$; so $f_i^k E[X, X^{-1}] \cap E[\Gamma]$ is primary in $E[\Gamma]$. It is easy to see that $\sqrt{f_i^k E[X, X^{-1}]} \cap E[\Gamma] = \sqrt{f_i^k E[\Gamma]}$. Hence $\sqrt{f_i^k E[\Gamma]}$ is a prime ideal. Therefore we may assume that $f_1, \dots, f_r$ are primary elements of $E[\Gamma]$ with nonzero constant terms and write $f^m = X^l f_1 \cdots f_r$ as above. Note that for each $i = 1, \dots, r$, there exist a unit u_i of E and an integer $a_i \geq F(\Gamma) + 1$ such that

$u_i f_i(0)^{a_i} \in D$ as in the proof of (3) ⇔ (9); so $u_i f_i^{a_i} \in R$. Let

$$a = a_1 \cdots a_r, \quad \hat{a}_i = \frac{a}{a_i}, \quad \text{and} \quad u = u_1^{\hat{a}_1} \cdots u_r^{\hat{a}_r}.$$

Then $uf^{am} = X^{al}(u_1 f_1^{a_1})^{\hat{a}_1} \cdots (u_r f_r^{a_r})^{\hat{a}_r}$ and $\sqrt{(u_i f_i^{a_i})^{\hat{a}_i} E[\Gamma]} = \sqrt{f_i E[\Gamma]}$ for each $i = 1, \dots, r$. Since t-dim$(E[\Gamma]) = 1$, $(u_i f_i^{a_i})^{\hat{a}_i} E[\Gamma]$ is a primary ideal of $E[\Gamma]$ [Anderson et al. 2003b, Lemma 2.1] for each $1 \le i \le r$.

Claim. *For each $1 \le i \le r$, $(u_i f_i^{a_i})^{\hat{a}_i} R$ is a primary ideal of R.*

Proof. Note that $(u_i f_i^{a_i})^{\hat{a}_i} \in R$ and fix an $i \in \{1, \dots, r\}$. We also note that t-dim$(R) = 1$ because R is a weakly Krull domain by Theorem 2.4. Hence, by [Anderson et al. 2003b, Lemma 2.1], it suffices to show that $\sqrt{(u_i f_i^{a_i})^{\hat{a}_i} R}$ is a prime ideal of R. If $\sqrt{(u_i f_i^{a_i})^{\hat{a}_i} E[\Gamma]} = E[\Gamma^*]$, then it is easy to see that $\sqrt{(u_i f_i^{a_i})^{\hat{a}_i} R} = E[\Gamma^*]$ is a prime ideal of R. Assume that $\sqrt{(u_i f_i^{a_i})^{\hat{a}_i} E[\Gamma]} \neq E[\Gamma^*]$. Then $(u_i f_i(0)^{a_i})^{\hat{a}_i} \neq 0$. Now, we show that $(u_i f_i^{a_i})^{\hat{a}_i} E[X, X^{-1}] \cap R = (u_i f_i^{a_i})^{\hat{a}_i} R$. Let $h \in (u_i f_i^{a_i})^{\hat{a}_i} E[X, X^{-1}] \cap R$. Note that we have

$$\begin{aligned}(u_i f_i^{a_i})^{\hat{a}_i} E[X, X^{-1}] \cap R &\subseteq (u_i f_i^{a_i})^{\hat{a}_i} E[X, X^{-1}] \cap E[\Gamma] \\ &= (u_i f_i^{a_i})^{\hat{a}_i} E[\Gamma]\end{aligned}$$

by adapting the proof of (2) ⇒ (1) in Theorem 2.4. So, we can write $h = (u_i f_i^{a_i})^{\hat{a}_i} g$ for some $g \in E[\Gamma]$. Then

$$g(0) = \frac{(u_i f_i(0)^{a_i})^{\hat{a}_i}}{h(0)} \in qf(D) \cap E = D$$

by Theorem 2.4; so $g \in R$. Therefore $h \in (u_i f_i^{a_i})^{\hat{a}_i} R$, and hence

$$(u_i f_i^{a_i})^{\hat{a}_i} E[X, X^{-1}] \cap R \subseteq (u_i f_i^{a_i})^{\hat{a}_i} R.$$

The reverse inclusion is clear, and hence $(u_i f_i^{a_i})^{\hat{a}_i} E[X, X^{-1}] \cap R = (u_i f_i^{a_i})^{\hat{a}_i} R$. Since $(u_i f_i^{a_i})^{\hat{a}_i} E[\Gamma]$ is a primary ideal of $E[\Gamma]$, $(u_i f_i^{a_i})^{\hat{a}_i} E[X, X^{-1}]$ is a primary ideal of $E[X, X^{-1}]$. Therefore $\sqrt{(u_i f_i^{a_i})^{\hat{a}_i} R} = \sqrt{(u_i f_i^{a_i})^{\hat{a}_i} E[X, X^{-1}]} \cap R$ is a prime ideal of R, and thus $(u_i f_i^{a_i})^{\hat{a}_i} R$ is a primary ideal of R. The claim is proved. □

If $l = 0$, then $uf(0)^{am} = (u_1 f_1(0)^{a_1})^{\hat{a}_1} \cdots (u_r f_r(0)^{a_r})^{\hat{a}_r} \in D$; so u is a unit of D because u is a unit of E. If $l \ge 1$, then $f^{am} = u^{-1} X^{al}(u_1 f_1^{a_1})^{\hat{a}_1} \cdots (u_r f_r^{a_r})^{\hat{a}_r}$. Since $u^{-1} X^{al} E[\Gamma]$ is a primary ideal of $E[\Gamma]$, $u^{-1} X^{al} R$ is a primary ideal of R by imitating the previous proof. Hence f^{am} is a product of primary elements of R, and thus R is an AWFD.

(2) ⇒ (8) Assume that R is a GWFD and fix an integer $n \ge 2$. Then R is a weakly Krull domain [Anderson et al. 2003b, Corollary 2.3]; so $X^n E[X]$ is a height-one maximal t-ideal of T_n by Theorem 2.4.

Next, we claim that $E[\Delta_n]$ is a GWFD. Let $S_1 = \{X^m \mid m \in \Delta_n\}$ and $S_2 = \{X^m \mid m \in \Gamma\}$. Then $E[\Delta_n]_{S_1} = E[X, X^{-1}] = R_{S_2}$ is a GWFD. Let Q be a nonzero prime ideal of $E[\Delta_n]$. If $Q \cap S_1 \neq \varnothing$, then Q contains a primary element X^n of $E[\Delta_n]$. If $Q \cap S_1 = \varnothing$, then $QE[\Delta_n]_{S_1}$ is a prime ideal of $E[\Delta_n]_{S_1}$; so $QE[\Delta_n]_{S_1}$ contains a primary element $f \in E[X, X^{-1}]$. Note that X is a unit of $E[X, X^{-1}]$ and $f^k \in E[\Delta_n]$ for some integer $k \geq 1$ because $\mathrm{char}(E) \neq 0$; so we may assume that $f \in E[\Delta_n]$ with $f(0) \neq 0$. Then

$$fE[\Delta_n] \subseteq fE[\Delta_n]_{S_1} \cap E[\Delta_n] \subseteq QE[\Delta_n]_{S_1} \cap E[\Delta_n] = Q;$$

so Q contains a primary element f. Hence $E[\Delta_n]$ is a GWFD.

In order to check the final condition, let $e \in E \setminus \{0\}$. If e is a unit of E, then we have nothing to prove. So, we assume that e is not a unit of E and let $h = e + X \in E[X]$. Since $c(h)_v = E$, $hE[X] = hqf(E)[X] \cap E[X]$ [Anderson and Chang 2007, Lemma 2.1(1)]; so $hE[X]$ is a height-one prime ideal. Let $P = hE[X] \cap R$. Since e is not a unit of E, $X^{F(\Gamma)+1} \notin P$; so $X^\alpha \notin P$ for all $\alpha \in \Gamma$. Therefore $hE[X, X^{-1}] = PR_{S_2} \subsetneq R_{S_2}$, and hence $\mathrm{ht}_R(P) = 1$. Since R is a GWFD, $P = \sqrt{gR}$ for some primary element $g \in R$ [Anderson et al. 2003b, Theorem 2.2]. Suppose to the contrary that $g(0) = 0$. Since $E_{D\setminus\{0\}}$ is a field by Theorem 2.4, $\frac{1}{e} = \frac{e'}{d}$ for some $0 \neq d \in D$ and $e' \in E$; so $e'h = d + e'X \in T$. Since $\mathrm{char}(E) \neq 0$, $(e'h)^k \in hE[X] \cap R = P$ for some integer $k \geq 1$. Hence $(e'h)^{kl} \in gR$ for some integer $l \geq 1$. However, this is impossible because $e \neq 0$. Therefore $g(0) \neq 0$. It is clear that gR_{S_2} is a primary ideal of R_{S_2}, $gR_{S_2} \cap E[X] = gE[X]$, $PR_{S_2} = \sqrt{gR_{S_2}}$ and $PR_{S_2} \cap E[X] = hE[X]$. Hence $gE[X]$ is a $hE[X]$-primary ideal. Therefore $g = uh^m$ for some $u \in qf(E)$ and some integer $m \geq 1$; so $ue^m = g(0) \in D$. Thus u is a unit of E.

(3) $\Rightarrow$ (5) and (6) $\Rightarrow$ (8) These implications can be obtained by applying $\Gamma = \Delta_n$ to the proofs of (3) $\Rightarrow$ (1) and (2) $\Rightarrow$ (8), respectively. ☐

We are closing this paper by showing that $R = D + E[\Gamma^*]$ is never a WFD and the assumption "$\mathrm{char}(E) = 0$" is essential in Corollary 2.5.

Remark 2.6. Assume that $R = D + E[\Gamma^*]$ is a WFD or an AWFD. Let $h = 1 + X \in E[X]$, $P = hE[X] \cap R$ and let M be a maximal t-ideal of R. If $M = E[\Gamma^*]$, then $PR_M = R_M$ because $1 + (-1)^{F(\Gamma)} X^{F(\Gamma)+1} \in P \setminus E[\Gamma^*]$. Assume that $M \neq E[\Gamma^*]$. Since $c(h)_v = E$, $hqf(E)[X] \cap E[X] = hE[X]$ [Anderson and Chang 2007, Lemma 2.1(1)]. Let $S = \{X^m \mid m \in \Gamma\}$. Then $PE[X, X^{-1}] = hE[X, X^{-1}]$; so $PR_M = hR_M$ is principal. Hence P is t-locally principal, and thus P is t-invertible [Anderson et al. 1992, Lemma 2.2].

(1) If R is a WFD, then $P = gR$ for some $g \in R$ with $g(0) \neq 0$ [Anderson and Zafrullah 1990, Theorem]. Note that $hE[X, X^{-1}] = gE[X, X^{-1}]$; so $g = uh$ for some unit u of E. Hence $uh \in R$, which is impossible. Thus R is not a WFD.

(2) Assume that R is an AWFD. Then $P^m = gR$ for some integer $m \geq 1$ and $g \in R$ with $g(0) \neq 0$ [Anderson et al. 1992, Theorem 3.4]. We note that

$$h^m E[X, X^{-1}] = gE[X, X^{-1}];$$

so $uh^m = g$ for some unit u of E. Hence $uh^m \in R$. However, this can not happen if $\operatorname{char}(E) = 0$. Thus R is never an AWFD whenever $\operatorname{char}(E) = 0$.

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References

[Anderson and Chang 2004] D. F. Anderson and G. W. Chang, "The class group of $D[\Gamma]$ for D an integral domain and Γ a numerical semigroup", *Comm. Algebra* **32**:2 (2004), 787–792. MR 2005g:13021 Zbl 1092.13017

[Anderson and Chang 2005] D. F. Anderson and G. W. Chang, "Homogeneous splitting sets of a graded integral domain", *J. Algebra* **288**:2 (2005), 527–544. MR 2006g:13001 Zbl 1084.13001

[Anderson and Chang 2007] D. F. Anderson and G. W. Chang, "Almost splitting sets in integral domains. II", *J. Pure Appl. Algebra* **208**:1 (2007), 351–359. MR 2007i:13004 Zbl 1171.13300

[Anderson and Zafrullah 1990] D. D. Anderson and M. Zafrullah, "Weakly factorial domains and groups of divisibility", *Proc. Amer. Math. Soc.* **109**:4 (1990), 907–913. MR 90k:13015 Zbl 0704.13008

[Anderson et al. 1992] D. D. Anderson, J. L. Mott, and M. Zafrullah, "Finite character representations for integral domains", *Boll. Un. Mat. Ital. B* (7) **6**:3 (1992), 613–630. MR 93k:13001 Zbl 0773.13004

[Anderson et al. 1993] D. D. Anderson, E. G. Houston, and M. Zafrullah, "t-linked extensions, the t-class group, and Nagata's theorem", *J. Pure Appl. Algebra* **86**:2 (1993), 109–124. MR 94e:13036 Zbl 0777.13002

[Anderson et al. 1995] D. D. Anderson, D. F. Anderson, and M. Zafrullah, "A generalization of unique factorization", *Boll. Un. Mat. Ital. A* (7) **9**:2 (1995), 401–413. MR 96d:13027 Zbl 0919.13001

[Anderson et al. 2003a] D. F. Anderson, G. W. Chang, and J. Park, "$D[X^2, X^3]$ over an integral domain D", pp. 1–14 in *Commutative ring theory and applications* (Fez, 2001), Lecture Notes in Pure and Appl. Math. **231**, Dekker, New York, 2003. MR 2004k:13011 Zbl 1080.13510

[Anderson et al. 2003b] D. F. Anderson, G. W. Chang, and J. Park, "Generalized weakly factorial domains", *Houston J. Math.* **29**:1 (2003), 1–13. MR 2004e:13003 Zbl 1029.13012

[Anderson et al. 2006] D. F. Anderson, G. W. Chang, and J. Park, "Weakly Krull and related domains of the form $D+M$, $A+XB[X]$ and $A+X^2B[X]$", *Rocky Mountain J. Math.* **36**:1 (2006), 1–22. MR 2007h:13027 Zbl 1133.13022

[Chang 2005] G. W. Chang, "Almost splitting sets in integral domains", *J. Pure Appl. Algebra* **197**:1-3 (2005), 279–292. MR 2005j:13005 Zbl 1091.13001

[Chang et al. 2012] G. W. Chang, H. Kim, and J. W. Lim, "Numerical semigroup rings and almost Prüfer v-multiplication domains", preprint, 2012. Accepted in *Comm. Algebra.*

[El Baghdadi et al. 2002] S. El Baghdadi, L. Izelgue, and S. Kabbaj, "On the class group of a graded domain", *J. Pure Appl. Algebra* **171**:2-3 (2002), 171–184. MR 2003d:13012 Zbl 1058.13006

[Gilmer 1992] R. Gilmer, *Multiplicative ideal theory*, Queen's Papers in Pure and Applied Mathematics **90**, Queen's University, Kingston, ON, 1992. MR 93j:13001 Zbl 0804.13001

[Gilmer and Martin 1990] R. Gilmer and M. B. Martin, "On the Picard group of a class of nonseminormal domains", *Comm. Algebra* **18**:10 (1990), 3263–3293. MR 91g:13017

[Kang 1989] B. G. Kang, "Prüfer v-multiplication domains and the ring $R[X]_{N_v}$", *J. Algebra* **123**:1 (1989), 151–170. MR 90e:13017 Zbl 0668.13002

[Kaplansky 1970] I. Kaplansky, *Commutative rings*, Allyn and Bacon, Boston, 1970. Reprinted Polygonal Publishing House, Washington, 1994.

[Li 2012] Q. Li, "On almost Prüfer v-multiplication domains", *Algebra Colloq.* (2012).

[Lim 2012] J. W. Lim, "Generalized Krull domains and the composite semigroup ring $D+E[\Gamma^*]$", *J. Algebra* **357** (2012), 20–25.

[Zafrullah 2003] M. Zafrullah, "Various facets of rings between $D[X]$ and $K[X]$", *Comm. Algebra* **31**:5 (2003), 2497–2540. MR 2004d:13029 Zbl 1052.13003

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