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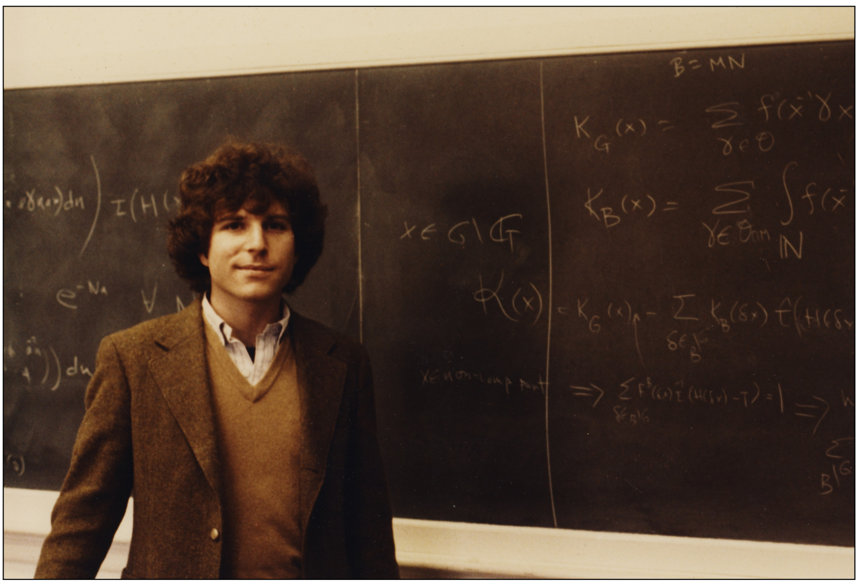
Special Issue

In memoriam

Jonathan Rogawski



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IN MEMORIAM
JONATHAN ROGAWSKI (1955–2011)

I do not ask to see the distant scene, one step enough for me.
John Henry Newman

Jonathan Rogawski is no more. In loving memory, his friends and collaborators and admirers have contributed to this garland of articles that you will find in the pages to follow. He was of course a gifted mathematician, one of the earliest workers on the Langlands Program, but much more. He wrote papers, textbooks, supervised students, and served for many years with great distinction as an editor for the *Pacific Journal of Mathematics*.

During the last 10 years or so of his life he had a fight with cancer which he faced with great equanimity, but which ended, as it so often does, in his passing away. He was a source of great inspiration to his friends. A few days before his death he was still conversing with friends, totally at peace with himself. To us, his life is epitomized by John Henry Newman's moving words from his hymn "Lead kindly light".

DON BLASIUS
UNIVERSITY OF CALIFORNIA, LOS ANGELES
blasius@math.ucla.edu

DINAKAR RAMAKRISHNAN
CALIFORNIA INSTITUTE OF TECHNOLOGY
dinakar@caltech.edu

V. S. VARADARAJAN
UNIVERSITY OF CALIFORNIA, LOS ANGELES
vsv@math.ucla.edu

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p-ADIC RANKIN L-SERIES AND RATIONAL POINTS ON CM ELLIPTIC CURVES

MASSIMO BERTOLINI, HENRI DARMON AND KARTIK PRASANNA

Dedicated to the memory of Jon Rogawski

This article presents a new proof of a theorem of Karl Rubin relating values of the Katz p-adic L-function of an imaginary quadratic field at certain points outside its range of classical interpolation to the formal group logarithms of rational points on CM elliptic curves. The approach presented here is based on the p-adic Gross–Zagier type formula proved by the three authors in previous work. As opposed to the original proof which relied on a comparison between Heegner points and elliptic units, it only makes use of Heegner points, and leads to a mild strengthening of Rubin's original result. A generalization to the case of modular abelian varieties with complex multiplication is also included.

1. Introduction

The aim of this article is to present a new proof of a theorem of Karl Rubin (see [Rubin 1992] and Theorem 1 below) relating values of the Katz p-adic L-function of an imaginary quadratic field at certain points outside its range of classical interpolation to the formal group logarithms of rational points on CM elliptic curves. This theorem has been seminal in providing a motivation for Perrin-Riou's formulation 1993; 2000 of the p-adic Beilinson conjectures. The new proof described in this work is based on the p-adic Gross–Zagier type formula of [Bertolini et al. 2012b], and only makes use of Heegner points as opposed to the original proof, which relied on a comparison between Heegner points and elliptic units. Hence, it should be adaptable to more general situations, for example to the setting of general CM fields.

Let A be an elliptic curve over $\mathbb{Q}$ with complex multiplication by the ring of integers of a quadratic imaginary field K. A classical result of Deuring identifies the Hasse–Weil L-series $L(A,s)$ of A with the L-series $L(\nu_A,s)$ attached to a

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Hecke character ν_A of K of infinity type $(1,0)$. When p is a prime which splits in K and does not divide the conductor of A, the Hecke L-function $L(\nu_A, s)$ has a p-adic analog, namely the Katz two-variable p-adic L-function attached to K. It is a p-adic analytic function, denoted by $\nu \mapsto \mathscr{L}_p(\nu)$, on the space of Hecke characters equipped with its natural p-adic analytic structure. Section 3A recalls the definition of this L-function: the values $\mathscr{L}_p(\nu)$ at Hecke characters of infinity type $(1+j_1, -j_2)$ with $j_1, j_2 \geq 0$ are defined by interpolation of the classical L-values $L(\nu^{-1}, 0)$. Letting $\nu^* := \nu \circ c$, where c denotes complex conjugation on the ideals of K, it is readily seen by comparing Euler factors that $L(\nu, s) = L(\nu^*, s)$. A similar equality need not hold in the p-adic setting, because the involution $\nu \mapsto \nu^*$ corresponds to the map $(j_1, j_2) \mapsto (j_2, j_1)$ on weight space and therefore does not preserve the lower right quadrant of weights of Hecke characters that lie in the range of classical interpolation. Since ν_A lies in the domain of classical interpolation, the p-adic L-value $\mathscr{L}_p(\nu_A)$ is a simple multiple of $L(\nu_A^{-1}, 0) = L(A, 1)$. Suppose that it vanishes. (The Birch and Swinnerton-Dyer conjecture predicts then that $A(\mathbb{Q})$ is infinite; this is known to be true when the order of vanishing is exactly one.) The value $\mathscr{L}_p(\nu_A^*)$ is a second, a priori more mysterious p-adic avatar of the leading term of $L(A, s)$ at $s = 1$. Rubin's theorem gives a formula for this quantity in the analytic rank-one case:

Theorem 1 [Rubin 1992]. *Let ν_A be a Hecke character of type $(1,0)$ attached to an elliptic curve $A/\mathbb{Q}$ with complex multiplication, and suppose that $L(A, s)$ vanishes to order one at the central point $s = 1$. Then there exists a global point $P \in A(\mathbb{Q})$ of infinite order such that*

$$\mathscr{L}_p(\nu_A^*) = \Omega_p(A)^{-1} \log_{\omega_A}(P)^2 \pmod{K^\times}, \tag{1-1}$$

where

- $\Omega_p(A)$ *is the p-adic period attached to A as in Section 2C*;
- $\omega_A \in \Omega^1(A/\mathbb{Q})$ *is a regular differential on A over $\mathbb{Q}$, and* $\log_{\omega_A} : A(\mathbb{Q}_p) \to \mathbb{Q}_p$ *denotes the p-adic formal group logarithm with respect to ω_A.*

(For a more precise statement without the $K^\times$ ambiguity, see [Rubin 1992].) Formula (1-1) is peculiar to the p-adic world and suggests that p-adic L-functions encode arithmetic information that is not readily apparent in their complex counterparts.

Rubin's proof of Theorem 1 breaks up naturally into three parts:

(1) He exploits the Euler system of elliptic units to construct a global cohomology class κ_A belonging to a pro-p Selmer group $\mathrm{Sel}_p(A/\mathbb{Q})$ attached to A. The close connection between elliptic units and the Katz p-adic L-function is then parlayed into the explicit evaluation of two natural p-adic invariants attached to κ_A: the p-adic

formal group logarithm $\log_{A,p}(\kappa_A)$ and the cyclotomic p-adic height $\langle \kappa_A, \kappa_A \rangle$:

$$\log_{A,p}(\kappa_A) = (1-\beta_p^{-1})^{-1}\mathscr{L}_p(\nu_A^*)\Omega_p(A), \tag{1-2}$$

$$\langle \kappa_A, \kappa_A \rangle = (1-\alpha_p^{-1})^{-2}\mathscr{L}'_p(\nu_A)\mathscr{L}_p(\nu_A^*), \tag{1-3}$$

where

- α_p and β_p denote the roots of the Hasse polynomial $x^2 - a_p(A)x + p$, ordered in such a way that $\mathrm{ord}_p(\alpha_p) = 0$ and $\mathrm{ord}_p(\beta_p) = 1$;
- the quantity $\mathscr{L}'_p(\nu_A)$ denotes the derivative of $\mathscr{L}_p$ at ν_A in the direction of the cyclotomic character.

(2) Independently of the construction of κ_A, the theory of Heegner points can be used to construct a canonical point $P \in A(\mathbb{Q})$, which is of infinite order when $L(\nu_A, s) = L(A, s)$ vanishes to exact order one at $s = 1$. The Selmer group $\mathrm{Sel}_p(A/\mathbb{Q}) \otimes \mathbb{Q}$ is of rank one (by the results of Kolyvagin) and the image $\kappa_P \in \mathrm{Sel}_p(A/\mathbb{Q})$ of P under the connecting homomorphism of Kummer theory supplies us with a generator for $\mathrm{Sel}_p(A/\mathbb{Q}) \otimes \mathbb{Q}$. Furthermore, the p-adic analog of the Gross–Zagier formula proved by Perrin-Riou [1987] shows that

$$\langle \kappa_P, \kappa_P \rangle = \mathscr{L}'_p(\nu_A)\Omega_p(A)^{-1} \pmod{K^\times}. \tag{1-4}$$

Finally, a theorem of Bertrand shows that the p-adic height pairing is nondegenerate in the above situation, that is, $\langle \kappa_P, \kappa_P \rangle \neq 0$. In particular, one concludes from (1-4) that $\mathscr{L}'_p(\nu_A) \neq 0$.

(3) Using that $\mathscr{L}'_p(\nu_A)$ is nonzero, Rubin shows that κ_A is nonzero in $\mathrm{Sel}_p(A/\mathbb{Q}) \otimes \mathbb{Q}$ and therefore is a second generator of this one-dimensional $\mathbb{Q}_p$-vector space. (See Theorem 8.1 and Corollary 8.3 of [Rubin 1992].) Equations (1-2) and (1-3) then show that $\mathscr{L}_p(\nu_A^*) \neq 0$, and further, for any generator κ of the $\mathbb{Q}_p$-vector space $\mathrm{Sel}_p(A/\mathbb{Q}) \otimes \mathbb{Q}$, one has

$$\frac{\log^2_{A,p}(\kappa)}{\langle \kappa, \kappa \rangle} = \frac{(1-\beta_p^{-1})^{-2}\mathscr{L}_p(\nu_A^*)\Omega_p(A)^2}{(1-\alpha_p^{-1})^{-2}\mathscr{L}'_p(\nu_A)}, \tag{1-5}$$

since the quantity on the left-hand side does not depend on the choice of κ. Rubin obtains Theorem 1 by setting $\kappa = \kappa_P$ in (1-5) and using (1-4) to eliminate the quantities $\langle \kappa_P, \kappa_P \rangle$ and $\mathscr{L}'_p(\nu_A)$.

The reader will note the key role that is played in Rubin's proof by both the Euler systems of elliptic units and of Heegner points. The new approach to Theorem 1 described in this paper relies solely on Heegner points, and requires neither elliptic units nor Perrin-Riou's p-adic height calculations. Instead, the key ingredient in this approach is the p-adic variant of the Gross–Zagier formula, arising from the results of [Bertolini et al. 2012b], which is stated in Theorem 3.12. This formula expresses p-adic logarithms of Heegner points in terms of the special values of a p-adic Rankin L-function attached to a cusp form f and an imaginary quadratic field K, and may be of some independent interest insofar as it exhibits a strong analogy with Rubin's formula but applies to arbitrary — not necessarily CM — elliptic curves over $\mathbb{Q}$. When f is the theta series attached to a Hecke character of K, Theorem 1 follows from the factorization of the associated p-adic Rankin L-function into a product of two Katz L-functions, a factorization which is a simple manifestation of the Artin formalism for these p-adic L-series.

One might expect that the statement of Theorem 1 should generalize to the setting where ν_A is replaced by an algebraic Hecke character ν of infinity type $(1, 0)$ of a quadratic imaginary field K (of arbitrary class number) satisfying

$$\nu|_{\mathbb{A}_\mathbb{Q}} = \varepsilon_K \cdot \boldsymbol{N}, \tag{1-6}$$

where ε_K is the quadratic Dirichlet character associated to $K/\mathbb{Q}$ and $\boldsymbol{N} : \mathbb{A}_\mathbb{Q}^\times \to \mathbb{R}^\times$ is the adelic norm character. Chapter 3 treats this more general setting, which (although probably amenable as well to the original methods of [Rubin 1992]) is not yet covered in the literature. Assumption (1-6) implies that the classical functional equation relates $L(\nu, s)$ to $L(\nu, 2-s)$. Assume further that the sign w_ν in this functional equation satisfies

$$w_\nu = -1, \tag{1-7}$$

so that $L(\nu, s)$ vanishes to odd order at $s = 1$. For technical reasons, it will also be convenient to make two further assumptions. Firstly, we assume that

$$\text{the discriminant } -D \text{ of } K \text{ is odd.} \tag{1-8}$$

Secondly, we note that assumption (1-6) implies that $\mathfrak{d}_K := \sqrt{-D}$ necessarily divides the conductor of ν, and we further restrict the setting by imposing the assumption that

$$\text{the conductor of } \nu \text{ is exactly divisible by } \mathfrak{d}_K. \tag{1-9}$$

The statement of Theorem 2 below requires some further notions, which we now introduce. Let E_ν be the subfield of $\mathbb{C}$ generated by the values of the Hecke character ν, and let T_ν be its ring of integers. A general construction which is recalled in Sections 2B and 3F attaches to ν an abelian variety B_ν over K of dimension $[E_\nu : K]$, equipped with inclusions

$$T_\nu \subset \operatorname{End}_K(B_\nu), \qquad E_\nu \subset \operatorname{End}_K(B_\nu) \otimes \mathbb{Q}.$$

Given $\lambda \in T_\nu$, denote by $[\lambda]$ the corresponding endomorphism of B_ν, and set

$$\Omega^1(B_\nu/E_\nu)^{T_\nu} := \{\omega \in \Omega^1(B_\nu/E_\nu) \mid [\lambda]^*\omega = \lambda\omega, \text{ for all } \lambda \in T_\nu\}, \tag{1-10}$$

$$(B_\nu(K) \otimes E_\nu)^{T_\nu} := \{P \in B_\nu(K) \otimes_{\mathbb{Z}} E_\nu \mid [\lambda]P = \lambda P, \text{ for all } \lambda \in T_\nu\}. \tag{1-11}$$

The vector space $\Omega^1(B_\nu/E_\nu)^{T_\nu}$ is one-dimensional over E_ν.

After fixing a p-adic embedding $K \subset \mathbb{Q}_p$, the formal group logarithm on B_ν gives rise to a bilinear pairing

$$\langle\ ,\ \rangle : \Omega^1(B_\nu/K) \times B_\nu(K) \to \mathbb{Q}_p,$$
$$(\omega, P) \mapsto \log_\omega P,$$

satisfying $\langle[\lambda]^*\omega, P\rangle = \langle\omega, [\lambda]P\rangle$ for all $\lambda \in T_\nu$. This pairing can be extended by E_ν-linearity to an $E_\nu \otimes \mathbb{Q}_p$-valued pairing between $\Omega^1(B_\nu/E_\nu)$ and $B_\nu(K) \otimes E_\nu$. When ω and P belong to these E_ν-vector spaces, we will continue to write $\log_\omega(P)$ for $\langle\omega, P\rangle$.

Theorem 2. *Let ν be an algebraic Hecke character of infinity type* $(1, 0)$ *satisfying* (1-6), (1-7), (1-8) *and* (1-9) *above. Then there exists $P_\nu \in B_\nu(K) \otimes \mathbb{Q}$ such that*

$$\mathscr{L}_p(\nu^*) = \Omega_p(\nu^*)^{-1} \log_{\omega_\nu}(P_\nu)^2 \pmod{E_\nu^\times},$$

where $\Omega_p(\nu^*) \in \mathbb{C}_p$ is the p-adic period attached to ν in Definition 2.13, and ω_ν is a nonzero element of $\Omega^1(B_\nu/E_\nu)^{T_\nu}$. The point P_ν is nonzero if and only if $L'(\nu, 1) \neq 0$.

Remark 3. Assumptions (1-8) and (1-9) could certainly be relaxed with more work. For instance, (1-8) is needed since the main theorem of [Bertolini et al. 2012b] is only proved for imaginary quadratic fields of odd discriminant. Likewise, removing (1-9) would require generalizing the cited result to the case of Shimura curves over $\mathbb{Q}$.

Remark 4. In [Bertolini et al. 2012c], we give a conjectural construction of rational points on CM elliptic curves (called Chow–Heegner points) using cycles on higher-dimensional varieties. While this construction of points is contingent on a certain case of the Tate conjecture, the corresponding construction at the level of cohomology classes can be made unconditionally. The results of this paper (especially Theorem 2 above) combined with those of [Bertolini et al. 2012b] are used in [Bertolini et al. 2012c] to establish that these cohomology classes indeed correspond to global points via the Kummer map.

The careful reader will notice that the hypothesis in Theorem 2 on the order of vanishing of $L(\nu, s)$ is weaker than that in [Rubin 1992], since $L(\nu, s)$ is only assumed to vanish to odd order rather than to exact order one. In the case that the order of vanishing is at least 3, the point P_ν (which comes from a Heegner construction) is torsion, so Theorem 2 just says that $\mathscr{L}_p(\nu^*) = 0$.

Corollary 5. *Let ν be an algebraic Hecke character of infinity type* (1, 0) *satisfying* (1-6), (1-7), (1-8) *and* (1-9) *above. Suppose that $\mathscr{L}_p(\nu^*) \neq 0$. Then*

(1) *$L(\nu, s)$ vanishes to exact order one at the center $s = 1$.*

(2) *$(B_\nu(K) \otimes E_\nu)^{T_\nu}$ is one-dimensional over E_ν.*

(3) *The Shafarevich–Tate group $\text{Ш}(B_\nu)$ is finite.*

Indeed, the nonvanishing of $\mathscr{L}_p(\nu^*)$ implies that a Heegner point on A_ν is nontorsion, and the conclusion then follows from results of Gross–Zagier and Kolyvagin (see [Kolyvagin 1990; Kolyvagin and Logachëv 1989]). Corollary 5 appears to be new; it would be interesting to see if it can also be obtained via the more indirect methods of [Rubin 1992].

Remark 6. The methods used in the proof of Theorem 2 also give information about the special values $\mathscr{L}_p(\nu^*)$ for Hecke characters ν of type $(1+j, -j)$ satisfying (1-6) with $j \geq 0$. A discussion of this point will be taken up in future work. (See [Bertolini et al. 2012a].)

The results of this article concern p-adic L-functions for the unitary group U(1); its proofs rely on p-adic L-functions for the unitary group U(2), as well as the theorem of Waldspurger relating periods of automorphic forms on U(2) along an embedded U(1) to central values of Rankin–Selberg L-functions, the latter being the main ingredient in the proof of the main result of [Bertolini et al. 2012b]. One should expect fruitful generalizations of the present work to the setting of higher-dimensional unitary groups. The authors are therefore pleased to dedicate this article to the memory of Jon Rogawski, whose deep ideas on automorphic forms, periods and L-functions for unitary groups are destined to play a key role in such eventual generalizations.

2. Hecke characters and periods

Throughout this article, all number fields that arise are viewed as being embedded in a fixed algebraic closure $\bar{\mathbb{Q}}$ of $\mathbb{Q}$. A complex embedding $\bar{\mathbb{Q}} \to \mathbb{C}$ and p-adic embeddings $\bar{\mathbb{Q}} \to \mathbb{C}_p$ for each rational prime p are also fixed from the outset, so that any finite extension of $\mathbb{Q}$ is simultaneously realized as a subfield of $\mathbb{C}$ and of $\mathbb{C}_p$.

2A. ***Algebraic Hecke characters.*** We will briefly recall some key definitions regarding algebraic Hecke characters, mainly to fix notation. The reader is referred to [Schappacher 1988, Chapter 0] for more details. Let K and E be number fields.

Given a $\mathbb{Z}$-linear combination

$$\phi = \sum_\sigma n_\sigma \sigma \in \mathbb{Z}[\mathrm{Hom}(K, \bar{\mathbb{Q}})]$$

of embeddings of K into $\bar{\mathbb{Q}}$, we define

$$\alpha^\phi := \prod_\sigma (\sigma\alpha)^{n_\sigma},$$

for all $\alpha \in K^\times$. Let $I_{\mathfrak{f}}$ denote the group of fractional ideals of K which are prime to a given integral ideal $\mathfrak{f}$ of K, and let

$$J_{\mathfrak{f}} := \{(\alpha) \mid \alpha \gg 0 \text{ and } \alpha - 1 \in \mathfrak{f}\} \subseteq I_{\mathfrak{f}}.$$

Definition 2.1. An E-valued *algebraic Hecke character* (or simply Hecke character) of K of infinity type ϕ and conductor *dividing* $\mathfrak{f}$ is a homomorphism

$$\chi : I_{\mathfrak{f}} \to E^\times$$

such that

$$\chi((\alpha)) = \alpha^\phi, \qquad \text{for all } (\alpha) \in J_{\mathfrak{f}}. \tag{2-1}$$

The smallest integral ideal $\mathfrak{g}$ such that χ can be extended to a Hecke character of conductor dividing $\mathfrak{g}$ is called the *conductor* of χ, and is denoted $\mathfrak{f}_\chi$.

The most basic examples of algebraic Hecke characters are the norm characters on $\mathbb{Q}$ and on K respectively, which are given by

$$\boldsymbol{N}((a)) = |a|, \qquad \boldsymbol{N}_K := \boldsymbol{N} \circ \mathrm{N}^K_{\mathbb{Q}}.$$

Note that the infinity type ϕ of a Hecke character χ must be trivial on all totally positive units congruent to 1 mod $\mathfrak{f}$. Hence, the existence of such a χ implies there is an integer $w(\chi)$ (called the *weight* of χ or of ϕ) such that for any choice of embedding of $\bar{\mathbb{Q}}$ into $\mathbb{C}$,

$$n_\sigma + n_{\bar\sigma} = w(\chi), \quad \text{for all } \sigma \in \mathrm{Hom}(K, \bar{\mathbb{Q}}).$$

Let $U_{\mathfrak{f}} \subset U'_{\mathfrak{f}} \subset \mathbb{A}_K^\times$ be the subgroups defined by

$$U'_{\mathfrak{f}} := \left\{ (x_v) \in \mathbb{A}_K^\times \;\middle|\; \begin{array}{ll} x_v \equiv 1 \pmod{\mathfrak{f}} & \text{for all } v \mid \mathfrak{f}, \\ x_v > 0 & \text{for all real } v \end{array} \right\}$$

and

$$U_{\mathfrak{f}} := \{(x_v) \in U'_{\mathfrak{f}} \mid x_v \in \mathcal{O}_{K_v}^\times, \text{ for all nonarchimedean } v\}.$$

A Hecke character χ of conductor dividing $\mathfrak{f}$ may also be viewed as a character on $\mathbb{A}_K^\times / U_{\mathfrak{f}}$ (denoted by the same symbol by a common abuse of notation),

$$\chi : \mathbb{A}_K^\times / U_{\mathfrak{f}} \to E^\times, \quad \text{satisfying } \chi|_{K^\times} = \phi. \tag{2-2}$$

To wit, given $x \in \mathbb{A}_K^\times$, we define $\chi(x)$ by choosing $\alpha \in K^\times$ such that αx belongs to $U'_{\mathfrak{f}}$, and setting

$$\chi(x) = \chi(i(\alpha x))\phi(\alpha)^{-1}, \tag{2-3}$$

where the symbol $i(x)$ denotes the fractional ideal of K associated to x. This definition is independent of the choice of α by (2-1). In the opposite direction, given a character χ as in (2-2), we can set

$$\chi(\mathfrak{a}) = \chi(x) \quad \text{for any } x \in U'_{\mathfrak{f}} \text{ such that } i(x) = \mathfrak{a}.$$

The subfield of E generated by the values of χ on $I_{\mathfrak{f}}$ is easily seen to be independent of the choice of $\mathfrak{f}$ and will be denoted E_χ.

Definition 2.2. The *central character* ε_η of a Hecke character η of K is the finite-order character of $\mathbb{Q}$ given by

$$\eta|_{\mathbb{A}_{\mathbb{Q}}^\times} = \varepsilon_\eta \cdot \boldsymbol{N}^{w(\eta)}.$$

The infinity type ϕ defines a homomorphism $\mathrm{Res}_{K/\mathbb{Q}}(\boldsymbol{G}_m) \to \mathrm{Res}_{E/\mathbb{Q}}(\boldsymbol{G}_m)$ of algebraic groups, and therefore induces a homomorphism

$$\phi_{\mathbb{A}} : \mathbb{A}_K^\times \to \mathbb{A}_E^\times$$

on adelic points. Given a Hecke character χ with values in E and a place λ of E (either finite or infinite), we may use $\phi_{\mathbb{A}}$ to define an idèle class character

$$\chi_\lambda : \mathbb{A}_K^\times / K^\times \to E_\lambda^\times,$$

by setting

$$\chi_\lambda(x) = \chi(x)/\phi_{\mathbb{A}}(x)_\lambda.$$

If λ is an infinite place, the character χ_λ is a Grossencharacter of K of type A_0. If λ is a finite place, then χ_λ factors through G_K^{ab} and gives a Galois character (denoted by $\rho_{\chi,\lambda}$) valued in $E_\lambda^\times$, satisfying

$$\rho_{\chi,\lambda}(\mathrm{Frob}_{\mathfrak{p}}) = \chi(\mathfrak{p})$$

for any prime ideal $\mathfrak{p}$ of K not dividing $\mathfrak{f}\lambda$.

Let $\mathfrak{g}$ be any integral ideal of K. The L-function and L-function with modulus $\mathfrak{g}$ attached to χ are defined by

$$L(\chi, s) = \prod_{\mathfrak{p}} \Bigl(1 - \frac{\chi(\mathfrak{p})}{\mathrm{N}\mathfrak{p}^s}\Bigr)^{-1}, \qquad L_{\mathfrak{g}}(\chi, s) = \prod_{\mathfrak{p}\nmid\mathfrak{g}} \Bigl(1 - \frac{\chi(\mathfrak{p})}{\mathrm{N}\mathfrak{p}^s}\Bigr)^{-1}.$$

Note that $L(\chi, s) = L_{\mathfrak{f}_\chi}(\chi, s)$.

The following definition will only be used in Section 3F.

Definition 2.3. Let $E = \prod_i E_i$ be a product of number fields. An *E-valued algebraic Hecke character* of conductor dividing $\mathfrak{f}$ is a character

$$\chi : I_{\mathfrak{f}} \to E^{\times}$$

whose projection to each component E_i is an algebraic Hecke character in the sense defined above.

2B. *Abelian varieties associated to characters of type* (1, 0). In this section, we limit the discussion to the case where K is an imaginary quadratic field. Let $\tau : K \mapsto \mathbb{C}$ be the given complex embedding of K. A Hecke character of infinity type $\phi = n_{\tau}\tau + n_{\bar{\tau}}\bar{\tau}$ will also be said to be of infinity type $(n_{\tau}, n_{\bar{\tau}})$.

Let ν be a Hecke character of K of infinity type $(1, 0)$ and conductor $\mathfrak{f}_{\nu}$, let $E_{\nu} \supset K$ denote the subfield of $\bar{\mathbb{Q}}$ generated by its values, and let T_{ν} be the ring of integers of E_{ν}. The Hecke character ν gives rise to a compatible system of one-dimensional ℓ-adic representations of G_K with values in $(E_{\nu} \otimes \mathbb{Q}_{\ell})^{\times}$, denoted $\rho_{\nu,\ell}$, satisfying

$$\rho_{\nu,\ell}(\sigma_{\mathfrak{a}}) = \nu(\mathfrak{a}), \quad \text{for all } \mathfrak{a} \in I_{\mathfrak{f}_{\nu}\ell},$$

where $\sigma_{\mathfrak{a}} \in \mathrm{Gal}(\bar{K}/K)$ denotes the Frobenius conjugacy class attached to $\mathfrak{a}$. The theory of complex multiplication realizes the representations $\rho_{\nu,\ell}$ on the division points of CM abelian varieties:

Definition 2.4. A *CM abelian variety* over K is a pair (B, E) where

(1) B is an abelian variety over K;

(2) E is a product of CM fields equipped with the structure of a K-algebra and an inclusion
$$i : E \to \mathrm{End}_K(B) \otimes \mathbb{Q}$$
satisfying $\dim_K(E) = \dim B$;

(3) for all $\lambda \in K \subset E$, the endomorphism $i(\lambda)$ acts on the cotangent space $\Omega^1(B/K)$ as multiplication by λ.

The abelian varieties (B, E) over K with complex multiplication by a fixed E form a category denoted $\mathscr{CM}_{K,E}$ in which a morphism from B_1 to B_2 is a morphism $j : B_1 \to B_2$ of abelian varieties over K for which the diagrams

$$\begin{tikzcd} B_1 \arrow[r, "j"] \arrow[d, "e"] & B_2 \arrow[d, "e"] \\ B_1 \arrow[r, "j"] & B_2 \end{tikzcd}$$

commute, for all $e \in E$ which belong to both $\mathrm{End}_K(B_1)$ and $\mathrm{End}_K(B_2)$. An isogeny in $\mathscr{CM}_{K,E}$ is simply a morphism in this category arising from an isogeny on the underlying abelian varieties.

If (B, E) is a CM abelian variety, its endomorphism ring over K contains a finite-index subring T^0 of the integral closure T of $\mathbb{Z}$ in E. After replacing B by the K-isogenous abelian variety $\mathrm{Hom}_{T_0}(T, B)$, we can assume that $\mathrm{End}_K(B)$ contains T. This assumption, which is occasionally convenient, will consistently be made from now on.

Let (B, E) be a CM-abelian variety with E a field, and let $E' \supset E$ be a finite extension of E with ring of integers T'. The abelian variety $B \otimes_T T'$ is defined to be the variety whose L-rational points, for any $L \supset K$, are given by

$$(B \otimes_T T')(L) = (B(\bar{\mathbb{Q}}) \otimes_T T')^{\mathrm{Gal}(\bar{\mathbb{Q}}/L)}.$$

This abelian variety is equipped with an action of T' by K-rational endomorphisms, described by multiplication on the right, and therefore $(B \otimes_T T', E')$ is an object of $\mathscr{CM}_{K,E'}$. Note that $B \otimes_T T'$ is isogenous to $t := \dim_E(E')$ copies of B, and that the action of T on $B \otimes_T T'$ agrees with the "diagonal" action of T on B^t.

Let ℓ be a rational prime. For each CM abelian variety (B, E), let

$$T_\ell(B) := \lim_{\leftarrow, n} B[\ell^n](\bar{K}), \qquad V_\ell(B) := T_\ell(B) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$

be the ℓ-adic Tate module and ℓ-adic representation of G_K attached to B. The $\mathbb{Q}_\ell$-vector space $V_\ell(B)$ is a free $E \otimes \mathbb{Q}_\ell$-module of rank one via the action of E by endomorphisms. The natural action of $G_K := \mathrm{Gal}(\bar{K}/K)$ on $V_\ell(B)$ commutes with this $E \otimes \mathbb{Q}_\ell$-action, and the collection $\{V_\ell(B)\}$ thus gives rise to a compatible system of one-dimensional ℓ-adic representations of G_K with values in $(E \otimes \mathbb{Q}_\ell)^\times$, denoted $\rho_{B,\ell}$. We note in passing that for any extension $E' \supset E$ where T' is the integral closure of T in E', we have

$$T_\ell(B \otimes_T T') = T_\ell(B) \otimes_T T', \qquad V_\ell(B \otimes_T T') = V_\ell(B) \otimes_E E'.$$

The following result is due to Casselman (cf. [Shimura 1971, Theorem 6]).

Theorem 2.5. *Let ν be a Hecke character of K of type $(1, 0)$ as above, and let $\rho_{\nu,\ell}$ be the associated one-dimensional ℓ-adic representation with values in $(E_\nu \otimes \mathbb{Q}_\ell)^\times$. Then*:

(1) *There exists a CM abelian variety (B_ν, E_ν) satisfying*

$$\rho_{B_\nu,\ell} \simeq \rho_{\nu,\ell}.$$

(2) *The CM abelian variety B_ν is unique up to isogeny over K. More generally, if (B, E) is any CM abelian variety with $E \supset E_\nu$ satisfying $\rho_{B,\ell} \simeq \rho_{\nu,\ell} \otimes_{E_\nu} E$ as $(E \otimes \mathbb{Q}_\ell)[G_K]$-modules, then there is an isogeny in $\mathscr{CM}_{K,E}$ from B to $B_\nu \otimes_{T_\nu} T$.*

Let ψ be a Hecke character of infinity type $(1,0)$, and let χ be a finite-order Hecke character of K, so that $\psi\chi^{-1}$ also has infinity type $(1,0)$. In comparing the abelian varieties B_ψ and $B_{\psi\chi^{-1}}$, it is useful to introduce a CM abelian variety $B_{\psi,\chi}$ over K, which we now describe.

Let E_χ denote the field generated by K and the values of χ. We denote by $E_{\psi,\chi}$ the compositum of E_ψ and E_χ, and by $T_{\psi,\chi} \subset E_{\psi,\chi}$ its ring of integers. We also write H_χ for the abelian extension of K which is cut out by χ viewed as a Galois character of G_K. Consider first the abelian variety over K with endomorphisms by $T_{\psi,\chi}$:

$$B^0_{\psi,\chi} := B_\psi \otimes_{T_\psi} T_{\psi,\chi}.$$

The natural inclusion $i_\psi : T_\psi \to T_{\psi,\chi}$ induces a morphism

$$i : B_\psi \to B^0_{\psi,\chi} \tag{2-4}$$

with finite kernel, which is compatible with the T_ψ-actions on both sides and is given by

$$i(P) = P \otimes 1.$$

Lemma 2.6. *Let F be any number field containing $E_{\psi,\chi}$. With notations as in Equation* (1-10) *of the Introduction*, *the restriction map i^* induces an isomorphism*

$$i^* : \Omega^1(B^0_{\psi,\chi}/F)^{T_{\psi,\chi}} \to \Omega^1(B_\psi/F)^{T_\psi} \tag{2-5}$$

of one-dimensional F-vector spaces.

Proof. The fact that B_ψ and $B^0_{\psi,\chi}$ are CM abelian varieties over F implies that the spaces $\Omega^1(B_\psi/F)$ and $\Omega^1(B^0_{\psi,\chi}/F)$ of regular differentials over F are free of rank one over $T_\psi \otimes_{\mathcal{O}_K} F$ and $T_{\psi,\chi} \otimes_{\mathcal{O}_K} F$ respectively. In particular, the source and target in (2-5) are both one-dimensional over F. The space $\Omega^1(B^0_{\psi,\chi}/F)$ is canonically identified with $\mathrm{Hom}_{T_\psi}(T_{\psi,\chi}, \Omega^1(B_\psi/F))$, and under this identification, the pullback

$$i^* : \Omega^1(B^0_{\psi,\chi}/F) \to \Omega^1(B_\psi/F)$$

corresponds to the natural restriction

$$\mathrm{Hom}_{T_\psi}(T_{\psi,\chi}, \Omega^1(B_\psi/F)) \to \Omega^1(B_\psi/F)$$

sending the function φ to $\varphi(1)$. (To see this, consider the map i_* on tangent spaces and dualize.) It follows directly from this description that

$$\ker(i^*) \cap \Omega^1(B^0_{\psi,\chi}/F)^{T_{\psi,\chi}} = 0;$$

hence the restriction of i^* to the one-dimensional F-vector space $\Omega^1(B^0_{\psi,\chi}/F)^{T_{\psi,\chi}}$ is injective. $\square$

Fix $\omega_\psi \in \Omega^1(B_\psi/E_\psi)^{T_\psi}$ and define $\omega^0_{\psi,\chi} \in \Omega^1(B^0_{\psi,\chi}/\bar{\mathbb{Q}})^{T_{\psi,\chi}}$ by

$$i^*(\omega^0_{\psi,\chi}) = \omega_\psi. \tag{2-6}$$

It follows from Lemma 2.6 that such an $\omega^0_{\psi,\chi}$ exists and is unique (once ω_ψ has been fixed), and further that $\omega^0_{\psi,\chi}$ belongs to $\Omega^1(B^0_{\psi,\chi}/E_{\psi,\chi})$.

The character $\chi^{-1} : \mathrm{Gal}(H_\chi/K) \to T_\chi^\times$ can be viewed as a one-cocycle in

$$H^1\big(\mathrm{Gal}(H_\chi/K), T^\times_{\psi,\chi}\big) \subset H^1\big(\mathrm{Gal}(H_\chi/K), \mathrm{Aut}_K(B^0_{\psi,\chi})\big).$$

Let

$$B_{\psi,\chi} := (B^0_{\psi,\chi})^{\chi^{-1}} \tag{2-7}$$

denote the twist of $B^0_{\psi,\chi}$ by this cocycle. There is a natural identification $B^0_{\psi,\chi}(\bar{K}) = B_{\psi,\chi}(\bar{K})$ of sets, arising from an isomorphism of varieties over H_χ, where H_χ is the extension of K cut out by χ. The actions of G_K on $B^0_{\psi,\chi}(\bar{K})$ and $B_{\psi,\chi}(\bar{K})$, denoted $*_0$ and $*$ respectively, are related by

$$\sigma * P = (\sigma *_0 P) \otimes \chi^{-1}(\sigma), \quad \text{for all } \sigma \in G_K. \tag{2-8}$$

In particular, for any $L \supset K$, we have

$$B_{\psi,\chi}(L) = \big\{ P \in B_\psi(\bar{\mathbb{Q}}) \otimes_{T_\psi} T_{\psi,\chi} \;\big|\; \sigma P = P \cdot \chi(\sigma), \text{ for all } \sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/L) \big\}. \tag{2-9}$$

Likewise, the natural actions of G_K on $\Omega^1(B^0_{\psi,\chi}/\bar{K})$ and on $\Omega^1(B_{\psi,\chi}/\bar{K})$ are related by

$$\sigma * \omega = [\chi^{-1}(\sigma)]^*(\sigma *_0 \omega) \quad \text{for all } \sigma \in G_K. \tag{2-10}$$

The isomorphism of $B^0_{\psi,\chi}$ and $B_{\psi,\chi}$ as CM abelian varieties over H_χ gives natural identifications

$$\Omega^1(B^0_{\psi,\chi}/H_\chi) = \Omega^1(B_{\psi,\chi}/H_\chi), \quad \Omega^1(B^0_{\psi,\chi}/E'_{\psi,\chi})^{T_{\psi,\chi}} = \Omega^1(B_{\psi,\chi}/E'_{\psi,\chi})^{T_{\psi,\chi}},$$

where $E'_{\psi,\chi}$ denotes the subfield of $\bar{\mathbb{Q}}$ generated by H_χ and $E_{\psi,\chi}$.

Let $\omega_{\psi,\chi}$ be any generator of $\Omega^1(B_{\psi,\chi}/E_{\psi,\chi})^{T_{\psi,\chi}}$ as an $E_{\psi,\chi}$-vector space. Since $\omega^0_{\psi,\chi}$ (defined in (2-6)) and $\omega_{\psi,\chi}$ both generate $\Omega^1(B_{\psi,\chi}/E'_{\psi,\chi})^{T_{\psi,\chi}}$ as an $E'_{\psi,\chi}$-vector space, they necessarily differ by a nonzero scalar in $E'_{\psi,\chi}$. To spell out the relation between $\omega^0_{\psi,\chi}$ and $\omega_{\psi,\chi}$ more precisely, it will be useful to introduce the notion of a *generalized Gauss sum* attached to any finite-order character χ of G_K. Given such a character, let

$$E\{\chi\} := \big\{ \lambda \in E_\chi H_\chi \;\big|\; \lambda^\sigma = \chi(\sigma)\lambda \text{ for all } \sigma \in \mathrm{Gal}(E_\chi H_\chi/E_\chi) \big\}.$$

This set is a one-dimensional E_χ-vector space in a natural way. It is not closed under multiplication, but

$$E\{\chi_1\} \cdot E\{\chi_2\} = E\{\chi_1\chi_2\} \pmod{(E_{\chi_1}E_{\chi_2})^\times}. \tag{2-11}$$

Definition 2.7. An E_χ-vector space generator of $E\{\chi\}$ is called a *Gauss sum* attached to the character χ, and is denoted $\mathfrak{g}(\chi)$.

By definition, the Gauss sum $\mathfrak{g}(\chi)$ belongs to $E\{\chi\} \cap (E_\chi H_\chi)^\times$, but is only well-defined up to multiplication by $E_\chi^\times$. It follows from (2-11) that

(2-12)

$$\mathfrak{g}(\chi_1\chi_2) = \mathfrak{g}(\chi_1)\mathfrak{g}(\chi_2) \pmod{(E_{\chi_1}E_{\chi_2})^\times}, \qquad \mathfrak{g}(\chi^{-1}) = \mathfrak{g}(\chi)^{-1} \pmod{E_\chi^\times}.$$

The following lemma pins down the relationship between the differentials $\omega^0_{\psi,\chi}$ and $\omega_{\psi,\chi}$.

Lemma 2.8. *For all Hecke characters ψ and χ as above,*

$$\omega_{\psi,\chi} = \mathfrak{g}(\chi)\omega^0_{\psi,\chi} \pmod{E^\times_{\psi,\chi}}.$$

Proof. Let $\lambda \in (H_\chi E_{\psi,\chi})^\times$ be the scalar satisfying

$$\omega_{\psi,\chi} = \lambda\omega^0_{\psi,\chi}. \tag{2-13}$$

Since $\omega_{\psi,\chi}$ is an $E_{\psi,\chi}$-rational differential on $B_{\psi,\chi}$, for all $\sigma \in \mathrm{Gal}(\bar{K}/E_{\psi,\chi})$ we have

$$\omega_{\psi,\chi} = \sigma * \omega_{\psi,\chi} = [\chi^{-1}(\sigma)]^*\sigma *_0 \omega_{\psi,\chi} = \chi^{-1}(\sigma)\lambda^\sigma\omega^0_{\psi,\chi}, \tag{2-14}$$

where the second equality follows from (2-10) and the last from the fact that the differential $\omega^0_{\psi,\chi}$ belongs to $\Omega^1(B^0_{\psi,\chi}/E_{\psi,\chi})^{T_{\psi,\chi}}$. Comparing (2-13) and (2-14) gives $\lambda^\sigma = \chi(\sigma)\lambda$, and hence $\lambda = \mathfrak{g}(\chi) \pmod{E^\times_{\psi,\chi}}$. □

The following lemma relates the abelian varieties $B_{\psi,\chi}$ and B_ν, where $\nu = \psi\chi^{-1}$.

Lemma 2.9. *There is an isogeny defined over K:*

$$i_\nu : B_{\psi,\chi} \to B_\nu \otimes_{T_\nu} T_{\psi,\chi},$$

which is compatible with the action of $T_{\psi,\chi}$ by endomorphisms on both sides.

Proof. The pair $(B^0_{\psi,\chi}, E_{\psi,\chi})$ is a CM abelian variety having ψ (viewed as taking values in $E_{\psi,\chi}$) as its associated Hecke character. The Hecke character attached to the Galois twist $B_{\psi,\chi}$ is therefore $\psi\chi^{-1} = \nu$. The second part of Theorem 2.5 implies that $B_{\psi,\chi}$ and $B_\nu \otimes_{T_\nu} T_{\psi,\chi}$ are isogenous over K as CM abelian varieties. □

2C. ***Complex periods and special values of L-functions.*** This section recalls certain periods attached to the quadratic imaginary field K and to Hecke characters of this field. We begin by fixing:

(1) An elliptic curve A with complex multiplication by $\mathcal{O}_K$, defined over a finite extension F of K. (Note that F necessarily contains the Hilbert class field of K.)

(2) A regular differential $\omega_A \in \Omega^1(A/F)$.

(3) A nonzero element γ of $H_1(A(\mathbb{C}), \mathbb{Q})$.

The complex period attached to this data is defined by

$$\Omega(A) := \frac{1}{2\pi i}\int_\gamma \omega_A. \tag{2-15}$$

Note that $\Omega(A)$ depends on the pair (ω_A, γ). A different choice of ω_A or γ has the effect of multiplying $\Omega(A)$ by a scalar in $F^\times$, and therefore $\Omega(A)$ can be viewed as a well-defined element of $\mathbb{C}^\times/F^\times$.

For any Hecke character ψ of K, recall that ψ^* is the Hecke character defined as in the Introduction by $\psi^*(x) = \psi(\bar{x})$. Suppose that ψ is of infinity type $(1, 0)$, and as before let $E_\psi \subset \bar{\mathbb{Q}} \subset \mathbb{C}$ denote the field generated by the values of ψ (or, equivalently, ψ^*). Choose (arbitrary) nonzero elements

$$\omega_\psi \in \Omega^1(B_\psi/E_\psi)^{T_\psi}, \quad \gamma \in H_1(B_\psi(\mathbb{C}), \mathbb{Q}),$$

with B_ψ the CM abelian variety attached to ψ by Theorem 2.5, and $\Omega^1(B_\psi/E_\psi)^{T_\psi}$ defined in (1-10). The period $\Omega(\psi^*)$ attached to ψ^* is defined by setting

$$\Omega(\psi^*) = \frac{1}{2\pi i}\int_\gamma \omega_\psi \pmod{E_\psi^\times}.$$

Note that the complex number $\Omega(\psi^*)$ does not depend, up to multiplication by $E_\psi^\times$, on the choices of ω_ψ and γ that were made to define it.

Lemma 2.10. *If ψ is a Hecke character of infinity type* $(1, 0)$, *and χ is a finite-order character*, *then*

$$\Omega(\psi^*\chi) = \Omega(\psi^*)\mathfrak{g}(\chi^*)^{-1} \pmod{E_{\psi,\chi}^\times}. \tag{2-16}$$

Proof. Choose a nonzero generator γ of $H_1(B^0_{\psi,\chi}(\mathbb{C}), \mathbb{Q}) = H_1(B_{\psi,\chi}(\mathbb{C}), \mathbb{Q})$ (viewed as a one-dimensional $E_{\psi,\chi}$ vector space via the endomorphism action). By definition,

$$\Omega\big((\psi\chi^{-1})^*\big) = \int_\gamma \omega_{\psi,\chi} = \mathfrak{g}(\chi)\int_\gamma \omega^0_{\psi,\chi} = \mathfrak{g}(\chi)\Omega(\psi^*) \pmod{E_{\psi,\chi}^\times},$$

where the second equality follows from Lemma 2.8. The result now follows after substituting χ^{*-1} for χ. $\square$

As in [Schappacher 1988, §1.8], one can also attach a period $\Omega(\psi)$ to an arbitrary Hecke character ψ of K; these satisfy the following relations:

Proposition 2.11. *Let ψ be a Hecke character of infinity type (k, j). Then*

(1) *The ratio*
$$\frac{\Omega(\psi^*)}{(2\pi i)^j \Omega(A)^{k-j}}$$
is algebraic.

(2) *For all ψ and ψ',*
$$\Omega(\psi\psi') = \Omega(\psi)\Omega(\psi') \pmod{E^\times_{\psi,\psi'}},$$
where $E_{\psi,\psi'}$ is the subfield of $\bar{\mathbb{Q}}$ generated by E_ψ and $E_{\psi'}$.

The following theorem is due to Goldstein and Schappacher [1981] in certain cases and Blasius [1986] in the general case (even CM fields).

Theorem 2.12. *Suppose that ψ has infinity type (k, j) with $k > j$, and that m is a critical integer for $L(\psi^{-1}, s)$. Then*
$$\frac{L(\psi^{-1}, m)}{(2\pi i)^m \Omega(\psi^*)} \text{ belongs to } E_\psi,$$
and for all $\tau \in \mathrm{Gal}(E_\psi/K)$,
$$\left(\frac{L(\psi^{-1}, m)}{(2\pi i)^m \Omega(\psi^*)}\right)^\tau = \frac{L\big((\psi^{-1})^\tau, m\big)}{(2\pi i)^m \Omega\big((\psi^*)^\tau\big)}.$$
Here the action of $\tau \in \mathrm{Gal}(E_\psi/K)$ on ψ^{-1} and ψ^* is defined by viewing them as E_χ-valued characters on ideals and applying τ to their values.

2D. ***p-adic periods.*** Fix a prime p that splits in K. We will need p-adic analogs of the periods $\Omega(A)$ and $\Omega(\nu^*)$. The p-adic analog $\Omega_p(A)$ of $\Omega(A)$ is obtained by considering the base change $A_{\mathbb{C}_p}$ of A to $\mathbb{C}_p$ (via our fixed embedding of F into $\mathbb{C}_p$). Assume that A has good reduction at the maximal ideal of $\mathcal{O}_{\mathbb{C}_p}$, that is, that $A_{\mathbb{C}_p}$ extends to a smooth proper model $A_{\mathcal{O}_{\mathbb{C}_p}}$ over $\mathcal{O}_{\mathbb{C}_p}$. The p-adic completion $\hat{A}_{\mathcal{O}_{\mathbb{C}_p}}$ of A along its special fiber is isomorphic to $\hat{\boldsymbol{G}}_m$. Following [de Shalit 1987, II, §4.4], choose an isomorphism $\iota_p : \hat{A} \to \hat{\boldsymbol{G}}_m$ over $\mathcal{O}_{\mathbb{C}_p}$, and define $\Omega_p(A) \in \mathbb{C}_p^\times$ by the rule

$$\omega_A = \Omega_p(A) \cdot \iota_p^*(du/u), \tag{2-17}$$

where u is the standard coordinate on $\hat{\boldsymbol{G}}_m$. The invariant $\Omega_p(A) \in \mathbb{C}_p^\times$ thus defined depends on the choices of ω_A and ι_p, but only up to multiplication by a scalar in $F^\times$. Observe also that $\Omega(A)$ and $\Omega_p(A)$ each depend linearly in the same way on the choice of the global differential ω_A.

The p-adic period $\Omega_p(A)$ can be used to define p-adic analogs of the complex periods that appear in the statement of Theorem 2.12.

Definition 2.13. Let ν be a Hecke character of K of type $(1,0)$. The p-adic period $\Omega_p(\nu^*)$ is defined by

$$\Omega_p(\nu^*) := \Omega_p(A) \cdot \frac{\Omega(\nu^*)}{\Omega(A)}.$$

More generally, for any character ν of infinity type (k, j), we define

$$\Omega_p(\nu^*) := \Omega_p(A)^{k-j} \cdot \frac{\Omega(\nu^*)}{(2\pi i)^j \Omega(A)^{k-j}}.$$

It can be seen from this definition that the period $\Omega_p(\nu^*)$, like its complex counterpart $\Omega(\nu^*)$, is well-defined up to multiplication by a scalar in $E_\nu^\times$. The following p-adic analog of Lemma 2.10 is a direct consequence of this lemma combined with the definition of $\Omega_p(\psi)$:

Lemma 2.14. *If ψ is a Hecke character of infinity type $(1,0)$, and χ is a finite order character, then*

$$\Omega_p(\psi^*\chi) = \Omega_p(\psi^*)\mathfrak{g}(\chi^*)^{-1} \pmod{E_{\psi,\chi}^\times}. \tag{2-18}$$

Likewise, Proposition 2.11 implies:

Proposition 2.15. *Let ψ be a Hecke character of infinity type (k, j). Then*:

(1) *The ratio*

$$\frac{\Omega_p(\psi^*)}{(2\pi i)^j \Omega_p(A)^{k-j}}$$

is algebraic.

(2) *For all ψ and ψ'*,

$$\Omega_p(\psi\psi') = \Omega_p(\psi)\Omega_p(\psi') \pmod{E_{\psi,\psi'}^\times}. \tag{2-19}$$

3. p-adic L-functions and Rubin's formula

3A. *The Katz p-adic L-function.* Throughout this chapter, we will fix a prime p that is split in K. Let $\mathfrak{c}$ be an integral ideal of K which is prime to p, and let $\Sigma(\mathfrak{c})$ denote the set of all Hecke characters of K of conductor dividing $\mathfrak{c}$. Denote by $\mathfrak{p}$ the prime above p corresponding to the chosen embedding $K \hookrightarrow \bar{\mathbb{Q}}_p$.

A character $\nu \in \Sigma(\mathfrak{c})$ is called a *critical character* if $L(\nu^{-1}, 0)$ is a critical value in the sense of Deligne, that is, if the Γ-factors that arise in the functional equation for $L(\nu^{-1}, s)$ are nonvanishing and have no poles at $s = 0$. The set $\Sigma_{\mathrm{crit}}(\mathfrak{c})$ of critical characters can be expressed as the disjoint union

$$\Sigma_{\mathrm{crit}}(\mathfrak{c}) = \Sigma_{\mathrm{crit}}^{(1)}(\mathfrak{c}) \cup \Sigma_{\mathrm{crit}}^{(2)}(\mathfrak{c}),$$

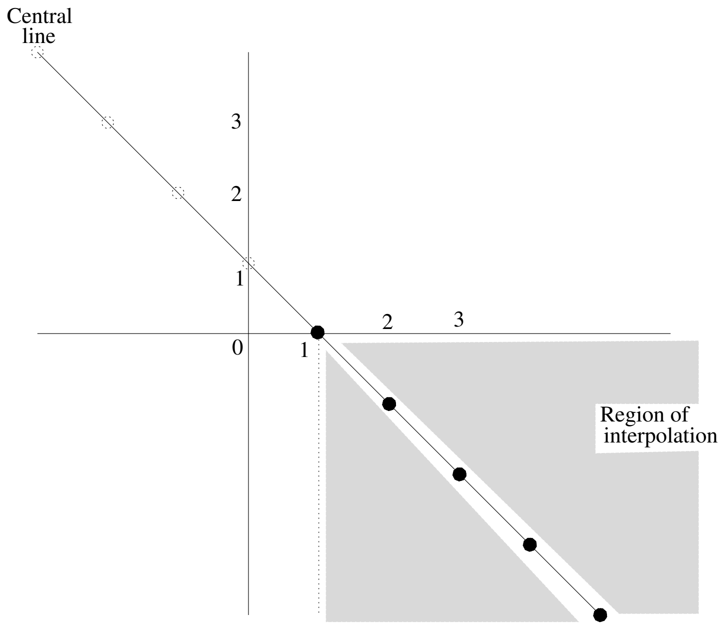

Figure 1. Critical infinity types for the Katz p-adic L-function.

where

$$\Sigma^{(1)}_{\mathrm{crit}}(\mathfrak{c}) = \{\nu \in \Sigma(\mathfrak{c}) \text{ of type } (\ell_1, \ell_2) \text{ with } \ell_1 \leq 0,\ \ell_2 \geq 1\},$$
$$\Sigma^{(2)}_{\mathrm{crit}}(\mathfrak{c}) = \{\nu \in \Sigma(\mathfrak{c}) \text{ of type } (\ell_1, \ell_2) \text{ with } \ell_1 \geq 1,\ \ell_2 \leq 0\}.$$

The possible infinity types of Hecke characters in $\Sigma^{(2)}_{\mathrm{crit}}(\mathfrak{c})$ lie in the shaded region in Figure 1 whose reflection about the principal diagonal corresponds likewise to $\Sigma^{(1)}_{\mathrm{crit}}(\mathfrak{c})$. Note in particular that when $\mathfrak{c} = \bar{\mathfrak{c}}$, the regions $\Sigma^{(1)}_{\mathrm{crit}}(\mathfrak{c})$ and $\Sigma^{(2)}_{\mathrm{crit}}(\mathfrak{c})$ are interchanged by the involution $\nu \mapsto \nu^*$. The set $\Sigma_{\mathrm{crit}}(\mathfrak{c})$ is endowed with a natural p-adic topology as described in Section 5.2 of [Bertolini et al. 2012b]. The subsets $\Sigma^{(1)}_{\mathrm{crit}}(\mathfrak{c})$ and $\Sigma^{(2)}_{\mathrm{crit}}(\mathfrak{c})$ are each dense in the completion $\hat{\Sigma}_{\mathrm{crit}}(\mathfrak{c})$ relative to this topology.

Recall that $\mathfrak{p}$ is the prime above p induced by our chosen embedding of K into $\mathbb{C}_p$. The following theorem on the existence of the p-adic L-function is due to Katz. The statement below is a restatement of [de Shalit 1987, II, Theorem 4.14] with a minor correction, and restricted to characters unramified at p. We remark that since our characters are unramified at p, the Gauss sum in the interpolation formula in [loc. cit.] is equal to 1.

Theorem 3.1. *There exists a p-adic analytic function $\nu \mapsto \mathscr{L}_{p,\mathfrak{c}}(\nu)$ (valued in $\mathbb{C}_p$) on $\hat{\Sigma}_{\mathrm{crit}}(\mathfrak{c})$ which is determined by the interpolation property*

$$\frac{\mathscr{L}_{p,\mathfrak{c}}(\nu)}{\Omega_p(A)^{\ell_1-\ell_2}} = \left(\frac{\sqrt{D}}{2\pi}\right)^{\ell_2}(\ell_1-1)!\,\big(1-\nu(\mathfrak{p})/p\big)\big(1-\nu^{-1}(\bar{\mathfrak{p}})\big)\frac{L_{\mathfrak{c}}(\nu^{-1},0)}{\Omega(A)^{\ell_1-\ell_2}}, \tag{3-1}$$

for all critical characters $\nu \in \Sigma^{(2)}_{\mathrm{crit}}(\mathfrak{c})$ *of infinity type* (ℓ_1, ℓ_2).

The right-hand side of (3-1) belongs to $\bar{\mathbb{Q}}$, by part (1) of Proposition 2.11 and Theorem 2.12 with $m = 0$. Equation (3-1) should be interpreted to mean that the left-hand side also belongs to $\bar{\mathbb{Q}}$, viewed as a subfield of $\mathbb{C}_p$ under the chosen embeddings, and agrees with the right-hand side. Note that although both sides of (3-1) depend on the choice of the differential ω_A that was made in the definition of the periods $\Omega(A)$ and $\Omega_p(A)$, the quantity $\mathscr{L}_{p,\mathfrak{c}}(\nu)$, just like its complex counterpart $L_{\mathfrak{c}}(\nu^{-1}, 0)$, does not depend on this choice.

Remark 3.2. Once a choice of $\mathfrak{c}$ is fixed, we shall often drop the subscript $\mathfrak{c}$ and simply write $\mathscr{L}_p$ for the p-adic L-function.

The following corollary is the p-adic counterpart of Theorem 2.12.

Corollary 3.3. *Suppose that* $\nu \in \Sigma^{(2)}_{\mathrm{crit}}(\mathfrak{c})$. *Then*

$$\frac{\mathscr{L}_{p,\mathfrak{c}}(\nu)}{\Omega_p(\nu^*)} \text{ belongs to } E_\nu.$$

Proof. Suppose that ν has infinity type (ℓ_1, ℓ_2). By the definition of $\Omega_p(\nu^*)$ and the interpolation property of the Katz p-adic L-function in Theorem 3.1, we have

$$\begin{aligned}\frac{\mathscr{L}_{p,\mathfrak{c}}(\nu)}{\Omega_p(\nu^*)} &= \frac{\mathscr{L}_{p,\mathfrak{c}}(\nu)}{\Omega_p(A)^{\ell_1-\ell_2}} \times \frac{(2\pi i)^{\ell_2}\Omega(A)^{\ell_1-\ell_2}}{\Omega(\nu^*)} \\ &= \sqrt{-D}^{\ell_2}\cdot(\ell_1-1)!\,\big(1-\nu(\mathfrak{p})\big)\big(1-\nu^{-1}(\bar{\mathfrak{p}})\big)\frac{L_{\mathfrak{c}}(\nu^{-1},0)}{\Omega(\nu^*)}.\end{aligned}$$

The result is now a direct consequence of Theorem 2.12 with $m = 0$. □

Corollary 3.3 expresses $\mathscr{L}_{p,\mathfrak{c}}(\nu)$ as an E_ν-multiple of a p-adic period $\Omega_p(\nu^*)$, when ν lies in the range $\Sigma^{(2)}_{\mathrm{crit}}(\mathfrak{c})$ of classical interpolation for the Katz p-adic L-function. On the other hand, the characters in $\Sigma^{(1)}_{\mathrm{crit}}(\mathfrak{c})$ are outside the range of interpolation, and so Corollary 3.3 does not directly say anything about these values, and indeed the main goal of this paper is to obtain analogous results for certain characters in $\Sigma^{(1)}_{\mathrm{crit}}(\mathfrak{c})$. It turns out that the methods of this paper only allow us to study $\mathscr{L}_{p,\mathfrak{c}}(\nu)$ for characters ν in $\Sigma^{(2)}_{\mathrm{crit}}(\mathfrak{c})$ satisfying the following auxiliary (but not unnatural) condition:

$$\nu \text{ is a } \textit{self-dual} \text{ Hecke character with } \varepsilon_\nu = \varepsilon_K. \tag{3-2}$$

For the benefit of the reader, we now recall this key definition.

Definition 3.4. A Hecke character $\nu \in \Sigma_{\mathrm{crit}}(\mathfrak{c})$ is said to be *self-dual* or *anticyclotomic* if

$$\nu\nu^* = \boldsymbol{N}_K.$$

The reason for the terminology in Definition 3.4 is that the functional equation for the L-series $L(\nu^{-1}, s)$ relates $L(\nu^{-1}, s)$ to $L(\nu^{-1}, -s)$, and therefore $s = 0$ is the *central critical point* for this complex L-series. Note that a self-dual character is necessarily of infinity type $(1 + j, -j)$ for some $j \in \mathbb{Z}$. Also the conductor of a self-dual character is clearly invariant under complex conjugation. If $\mathfrak{c}$ is an integral ideal such that $\mathfrak{c} = \bar{\mathfrak{c}}$, we denote by $\Sigma_{\mathrm{sd}}(\mathfrak{c})$ the set of self-dual Hecke characters of conductor *exactly* $\mathfrak{c}$, and write

$$\Sigma^{(1)}_{\mathrm{sd}}(\mathfrak{c}) = \Sigma^{(1)}_{\mathrm{crit}}(\mathfrak{c}) \cap \Sigma_{\mathrm{sd}}(\mathfrak{c}), \qquad \Sigma^{(2)}_{\mathrm{sd}}(\mathfrak{c}) = \Sigma^{(2)}_{\mathrm{crit}}(\mathfrak{c}) \cap \Sigma_{\mathrm{sd}}(\mathfrak{c}).$$

In particular, the possible infinity types of characters in $\Sigma^{(2)}_{\mathrm{sd}}(\mathfrak{c})$ correspond to the black dots in Figure 1.

For convenience, we restate Theorem 3.1 for self-dual characters.

Proposition 3.5. *For all characters $\nu \in \Sigma^{(2)}_{\mathrm{sd}}(\mathfrak{c})$ of infinity type $(1 + j, -j)$ with $j \geq 0$,*

$$\frac{\mathscr{L}_{p,\mathfrak{c}}(\nu)}{\Omega_p(A)^{1+2j}} = \bigl(1 - \nu^{-1}(\bar{\mathfrak{p}})\bigr)^2 \times \frac{j!\,(2\pi)^j L_{\mathfrak{c}}(\nu^{-1}, 0)}{\sqrt{D}^{\,j}\,\Omega(A)^{1+2j}}. \tag{3-3}$$

Remark 3.6. In the proposition above, we could equally write $L(\nu^{-1}, 0)$ instead of $L_{\mathfrak{c}}(\nu^{-1}, 0)$, since ν has conductor exactly equal to c.

Remark 3.7. The *central character* of such a ν is very restricted. Indeed, for any Hecke character ν, it is clear that $\varepsilon_{\bar{\nu}} = \overline{\varepsilon_\nu}$, while $\varepsilon_{\nu^*} = \varepsilon_\nu$. Further, if ν is a self-dual character, it follows that for any $x \in \mathbb{A}_K^\times$,

$$\nu\bigl(\mathrm{N}^K_{\mathbb{Q}}(x)\bigr) = \nu(x\bar{x}) = (\nu\nu^*)(x) = \boldsymbol{N}_K(x) = \boldsymbol{N}\bigl(\mathrm{N}^K_{\mathbb{Q}}(x)\bigr).$$

Hence

$$\nu|_{\mathrm{N}^K_{\mathbb{Q}}\mathbb{A}_K^\times} = \boldsymbol{N} \qquad \text{and} \qquad \varepsilon_\nu|_{\mathrm{N}^K_{\mathbb{Q}}\mathbb{A}_K^\times} = 1.$$

This implies that the central character ε_ν of a self-dual character ν is either 1 or ε_K, where ε_K denotes the quadratic Dirichlet character corresponding to the extension $K/\mathbb{Q}$. Conversely, it is easy to see that if ν is a Hecke character with $w(\nu) = 1$ and $\varepsilon_\nu = 1$ or ε_K, then ν is a self-dual character.

We define:

$$\Sigma_{\mathrm{sd}}(\mathfrak{c})^+ := \{\nu \in \Sigma_{\mathrm{sd}}(\mathfrak{c});\ \ \varepsilon_\nu = 1\}, \qquad \Sigma_{\mathrm{sd}}(\mathfrak{c})^- := \{\nu \in \Sigma_{\mathrm{sd}}(\mathfrak{c});\ \ \varepsilon_\nu = \varepsilon_K\}. \tag{3-4}$$

The sets $\Sigma^{(1)}_{\mathrm{sd}}(\mathfrak{c})^\pm$ and $\Sigma^{(2)}_{\mathrm{sd}}(\mathfrak{c})^\pm$ are defined similarly.

Our approach to studying $\mathscr{L}_{p,\mathfrak{c}}(\nu)$ for characters ν satisfying (3-2), that is, those ν lying in $\Sigma_{\mathrm{sd}}^{(1)}(\mathfrak{c})^-$ for some $\mathfrak{c}$, relies on a different kind of p-adic L-function. This latter p-adic L-function is attached to Rankin–Selberg L-series and is recalled in the following section.

3B. ***p-adic Rankin L-series.*** In this section, we consider p-adic L-functions obtained by interpolating special values of Rankin–Selberg L-series associated to modular forms and Hecke characters of a quadratic imaginary field K of odd discriminant. We briefly recall the definition of this p-adic L-function that is given in Section 5 of [Bertolini et al. 2012b], referring the reader to that work for a more detailed description.

Let $S_k(\Gamma_0(N), \varepsilon)$ denote the space of cusp forms of weight $k \geq 2$ and character ε on $\Gamma_0(N)$. Let $f \in S_k(\Gamma_0(N), \varepsilon)$ be a normalized newform and let E_f denote the subfield of $\mathbb{C}$ generated by its Fourier coefficients.

Definition 3.8. The pair (f, K) is said to satisfy the *Heegner hypothesis* if $\mathcal{O}_K$ contains a cyclic ideal of norm N, that is, an integral ideal $\mathfrak{N}$ of $\mathcal{O}_K$ with $\mathcal{O}_K/\mathfrak{N} = \mathbb{Z}/N\mathbb{Z}$.

Assume from now on that (f, K) satisfies the Heegner hypothesis, and let $\mathfrak{N}$ be a cyclic $\mathcal{O}_K$-ideal of norm N. We write $\mathfrak{N}_\varepsilon$ for the unique ideal dividing $\mathfrak{N}$ of norm N_ε, where N_ε is the conductor of ε.

Definition 3.9. A Hecke character χ of K of infinity type (ℓ_1, ℓ_2) is said to be *central critical* for f if

$$\ell_1 + \ell_2 = k \quad \text{and} \quad \varepsilon_\chi = \varepsilon.$$

The reason for the terminology of Definition 3.9 is that when χ satisfies these hypotheses, the complex Rankin L-series $L(f, \chi^{-1}, s)$ is self-dual and $s = 0$ is its central (critical) point.

Definition 3.10. Let c be a rational integer prime to pN. Then $\Sigma_{\mathrm{cc}}(c, \mathfrak{N}, \varepsilon)$ is defined to be the set of Hecke characters χ of K such that

(1) χ is central critical for f;

(2) $\mathfrak{f}_\chi = c \cdot \mathfrak{N}_\varepsilon$;

(3) the local sign $\varepsilon_q(f, \chi^{-1}) = +1$ for all finite primes q.

It is easily checked that this agrees with the definition of $\Sigma_{\mathrm{cc}}(c, \mathfrak{N}, \varepsilon)$ given in [Bertolini et al. 2012b, §4.1], where this is just denoted $\Sigma_{\mathrm{cc}}(\mathfrak{N})$. Further, as in [loc. cit.], given conditions (1) and (2) above, condition (3) is automatic except possibly for primes q lying in the set S_f defined by

$$S_f := \{q : q \mid (N, D), q \nmid N_\varepsilon\}.$$

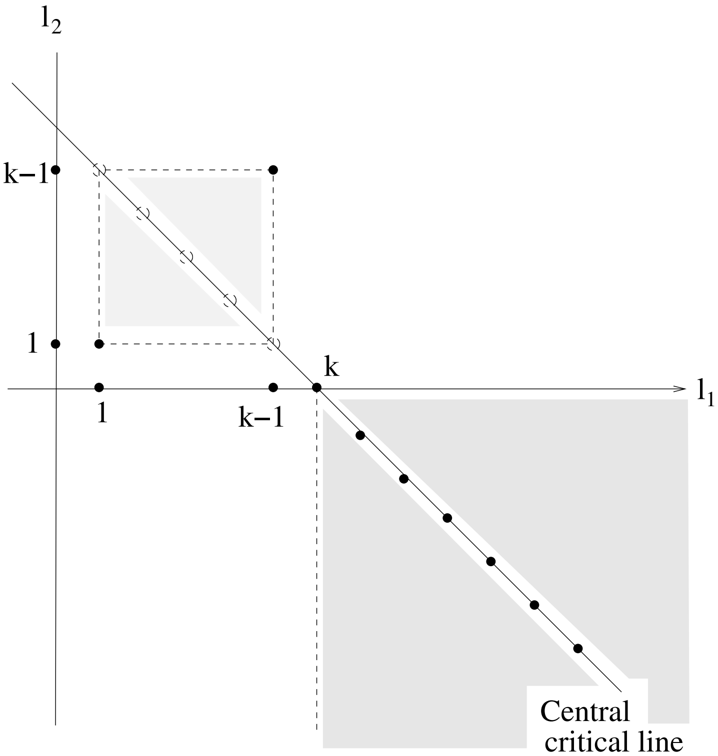

Figure 2. Critical infinity types for the p-adic Rankin L-function.

The set $\Sigma_{\rm cc}(c, \mathfrak{N}, \varepsilon)$ can be expressed as a disjoint union

$$\Sigma_{\rm cc}(c, \mathfrak{N}, \varepsilon) = \Sigma^{(1)}_{\rm cc}(c, \mathfrak{N}, \varepsilon) \cup \Sigma^{(2)}_{\rm cc}(c, \mathfrak{N}, \varepsilon),$$

where $\Sigma^{(1)}_{\rm cc}(c, \mathfrak{N}, \varepsilon)$ and $\Sigma^{(2)}_{\rm cc}(c, \mathfrak{N}, \varepsilon)$ denote the subsets consisting of characters of infinity type $(k + j, -j)$, with $1 - k \le j \le -1$ and $j \ge 0$ respectively. We shall also denote by $\hat{\Sigma}_{\rm cc}(c, \mathfrak{N}, \varepsilon)$ the completion of $\Sigma_{\rm cc}(c, \mathfrak{N}, \varepsilon)$ relative to the p-adic compact open topology on $\Sigma_{\rm cc}(c, \mathfrak{N}, \varepsilon)$, which is defined in Section 5.2 of [Bertolini et al. 2012b]. The infinity types of Hecke characters in $\Sigma^{(1)}_{cc}(c, \mathfrak{N}, \varepsilon)$ and $\Sigma^{(2)}_{cc}(c, \mathfrak{N}, \varepsilon)$ correspond respectively to the white and black dots in the shaded regions in Figure 2. We note that the set $\Sigma^{(2)}_{\rm cc}(c, \mathfrak{N}, \varepsilon)$ of classical central critical characters "of type 2" is dense in $\hat{\Sigma}_{\rm cc}(c, \mathfrak{N}, \varepsilon)$.

For all $\chi \in \Sigma^{(2)}_{\rm cc}(c, \mathfrak{N}, \varepsilon)$ of infinity type $(k + j, -j)$ with $j \ge 0$, let $E_{f,\chi}$ denote the subfield of $\mathbb{C}$ generated by E_f and the values of χ, and let $E_{f,\chi,\varepsilon}$ be the field generated by $E_{f,\chi}$ and by the abelian extension of $\mathbb{Q}$ cut out by ε. The *algebraic part* of $L(f, \chi^{-1}, 0)$ is defined by the rule

$$L_{\rm alg}(f, \chi^{-1}, 0) := w(f, \chi)^{-1} C(f, \chi, c) \cdot \frac{L(f, \chi^{-1}, 0)}{\Omega(A)^{2(k+2j)}}, \tag{3-5}$$

where $w(f,\chi)^{-1} \in E_{f,\chi,\varepsilon}$ and $C(f,\chi,c)$ are respectively the scalar (of complex norm 1) and the explicit real constant defined in [Bertolini et al. 2012b, (5.1.11) and Theorem 4.6]; we have

$$C(f,\chi,c) = \frac{2^{k+2j-2}\pi^{k+2j-1} j!\,(k+j-1)!\,w_K}{\sqrt{D}^{k+2j-1} c^{k+2j-1}} \prod_{q|c} \frac{q-\varepsilon_K(q)}{q-1}, \tag{3-6}$$

where $w_K = \#\mathcal{O}_K^\times$ is the number of roots of unity in K. Theorems 5.5 and 5.10 of [loc. cit.] show respectively that the values $L_{\mathrm{alg}}(f,\chi^{-1},0)$ belong to $\bar{\mathbb{Q}}$, and that they interpolate p-adically:

Proposition 3.11. *Let $\chi \mapsto L_p(f,\chi)$ be the function on $\Sigma_{cc}^{(2)}(c,\mathfrak{N},\varepsilon)$ defined by*

$$L_p(f,\chi) := \Omega_p(A)^{2(k+2j)}\big(1-\chi^{-1}(\bar{\mathfrak{p}})a_p(f)+\chi^{-2}(\bar{\mathfrak{p}})\varepsilon(p)p^{k-1}\big)^2 L_{\mathrm{alg}}(f,\chi^{-1},0), \tag{3-7}$$

for χ of infinity type $(k+j,-j)$ with $j \geq 0$. This function extends (uniquely) to a p-adically continuous function on $\hat{\Sigma}_{cc}(c,\mathfrak{N},\varepsilon)$.

The function $\chi \mapsto L_p(f,\chi)$ on $\hat{\Sigma}_{cc}(c,\mathfrak{N},\varepsilon)$ will be referred to as the *p-adic Rankin L-function* attached to the cusp form f.

3C. ***A p-adic Gross–Zagier formula.*** In this section, we specialize to the case where the newform f is of weight $k=2$, and assume that χ is a finite-order Hecke character of K satisfying

$$\chi \mathbf{N}_K \text{ belongs to } \Sigma_{cc}^{(1)}(c,\mathfrak{N},\varepsilon).$$

In particular, the character $\chi\mathbf{N}_K$ lies outside the domain $\Sigma_{cc}^{(2)}(c,\mathfrak{N},\varepsilon)$ of classical interpolation defining $L_p(f,-)$. The p-adic Gross–Zagier formula alluded to in the title of this section relates the special value $L_p(f,\chi\mathbf{N}_K)$ to the formal group logarithm of a Heegner point on the modular abelian variety attached to f.

The Eichler–Shimura construction associates to f an abelian variety B_f with endomorphism by an order in the ring of integers $T_f \subset E_f$, and a surjective morphism

$$\Phi_f : J_1(N) \to B_f$$

of abelian varieties over $\mathbb{Q}$, called the *modular parametrization*, which is well-defined up to a rational isogeny. Let

$$\omega_f = 2\pi i f(\tau)\,d\tau \in \Omega^1\big(X_1(N)/E_f\big)$$

be the differential form on $X_1(N)$ attached to f; we use the same symbol ω_f to denote the associated one-form on $J_1(N)$. Let $\omega_{B_f} \in \Omega^1(B_f/E_f)^{T_f}$ be the unique one-form satisfying

$$\Phi_f^*(\omega_{B_f}) = \omega_f. \tag{3-8}$$

Let A' be an elliptic curve with endomorphism ring isomorphic to the order $\mathbb{O}_c = \mathbb{Z} + c\mathbb{O}_K$ of conductor c, defined over the ring class field H_c of conductor c. The pair $(A', A'[\mathfrak{N}])$ corresponds to a point on $X_0(N)(H_c)$. Let t be any generator of $A'[\mathfrak{N}]$. Then the triple $(A', A'[\mathfrak{N}], t)$ corresponds to a point in $X_1(N)$, whose field of definition $H_{c,\mathfrak{N}}$ is an abelian extension of K, independent of the choice of t, and the finite-order Hecke character χ can be viewed as a character

$$\chi : \mathrm{Gal}(H_{c,\mathfrak{N}}/K) \to E_\chi.$$

Fix a cusp ∞ of $X_1(N)$ which is defined over $\mathbb{Q}$, and let

$$\Delta = [A', A'[\mathfrak{N}], t] - (\infty) \quad \in J_1(N)(H_{c,\mathfrak{N}}). \tag{3-9}$$

To the pair (f, χ) we associate a Heegner point by letting $G = \mathrm{Gal}(H_{c,\mathfrak{N}}/K)$ and setting

$$P_f(\chi) := \sum_{\sigma \in G} \chi^{-1}(\sigma)\Phi_f(\Delta^\sigma) \quad \in B_f(H_{c,\mathfrak{N}}) \otimes_{T_f} E_{f,\chi}. \tag{3-10}$$

Note that, since $P_f(\chi)^\sigma = P_f(\chi)$ for any $\sigma \in \mathrm{Gal}(H_{c,\mathfrak{N}}/H_\chi)$, the point $P_f(\chi)$ lies in the subspace $B_f(H_\chi) \otimes_{T_f} E_{f,\chi}$. The embedding of $\bar{\mathbb{Q}}$ into $\mathbb{C}_p$ that was fixed from the outset allows us to consider the formal group logarithm

$$\log_{\omega_{B_f}} : B_f(H_{c,\mathfrak{N}}) \to \mathbb{C}_p.$$

We extend this function to $B_f(H_{c,\mathfrak{N}}) \otimes_{T_f} E_{f,\chi}$ by $E_{f,\chi}$-linearity.

Theorem 3.12. *With notations and assumptions as above*,

$$L_p(f, \chi \boldsymbol{N}_K) = \left(1 - \chi^{-1}(\bar{\mathfrak{p}})p^{-1}a_p(f) + \chi^{-2}(\bar{\mathfrak{p}})\varepsilon(p)p^{-1}\right)^2 \log^2_{\omega_{B_f}}(P_f(\chi)).$$

Proof. Let

$$\mathscr{E}(f, \chi) := \left(1 - \chi^{-1}(\bar{\mathfrak{p}})p^{-1}a_p(f) + \chi^{-2}(\bar{\mathfrak{p}})\varepsilon(p)p^{-1}\right)^2 \quad \in E^\times_{f,\chi}$$

be the Euler factor appearing in the statement of Theorem 3.12. Let F' denote the p-adic completion of $H_{c,\mathfrak{N}}$. Theorem 5.13 of [Bertolini et al. 2012b] in the case $k = 2$ and $r = j = 0$, with χ replaced by $\chi \mathrm{N}_K$, gives

$$L_p(f, \chi \boldsymbol{N}_K) = \mathscr{E}(f, \chi) \times \left(\sum_{\sigma \in G} \chi^{-1}(\sigma) \cdot \mathrm{AJ}_{F'}(\Delta^\sigma)(\omega_f)\right)^2. \tag{3-11}$$

Note that in this context, the p-adic Abel–Jacobi map $\mathrm{AJ}_{F'}$ that appears in (3-11) is related to the formal group logarithm by

$$\mathrm{AJ}_{F'}(\Delta)(\omega_f) = \log_{\omega_f}(\Delta).$$

Therefore,

$$L_p(f, \chi N_K) = \mathscr{E}(f, \chi)\Big(\sum_{\sigma\in G} \chi^{-1}(\sigma)\log_{\omega_f}(\Delta^\sigma)\Big)^2. \tag{3-12}$$

Theorem 3.12 follows from this formula and the fact that, by (3-8),

$$\log_{\omega_f}(\Delta) = \log_{\Phi_f^*(\omega_{B_f})}(\Delta) = \log_{\omega_{B_f}}(\Phi_f(\Delta)). \qquad \square$$

In the special case where f has rational Fourier coefficients and $\chi = 1$ is the trivial character, the abelian variety B_f is an elliptic curve quotient of $J_0(N)$ and the Heegner point $P_f := P_f(1)$ belongs to $B_f(K)$. Theorem 3.12 implies in this case that

$$L_p(f, N_K) = \left(\frac{p+1-a_p(f)}{p}\right)^2 \log^2(P_f), \tag{3-13}$$

where $\log : B_f(K_\mathfrak{p}) \to K_\mathfrak{p}$ is the formal group logarithm attached to a rational differential on $B_f/\mathbb{Q}$. Equation (3-13) exhibits a strong analogy with Theorem 1 of the Introduction, although it applies to arbitrary (modular) elliptic curves and not just elliptic curves with complex multiplication.

The remainder of Chapter 3 explains how Theorem 3.12 can in fact be used to prove Theorems 1 and 2 of the Introduction. The key to this proof is a relation between the Katz p-adic L-function of Section 3A and the p-adic Rankin L-function $L_p(f, \chi)$ of Section 3B in the special case where f is a theta series attached to a Hecke character of the imaginary quadratic field K. This explicit relation is described in the following section.

3D. ***A factorization of the p-adic Rankin L-series.*** This section focuses on the Rankin L-function $L_p(f, \chi)$ of f and K in the special case where f is a theta series associated to a Hecke character of the *same* imaginary quadratic field K.

More precisely, let ψ be a fixed Hecke character of K of infinity type $(k-1, 0)$ with $k = r+2 \geq 2$. Consider the associated theta series

$$\theta_\psi := \sum_{\mathfrak{a}} \psi(\mathfrak{a}) q^{\mathrm{N}\mathfrak{a}} = \sum_{n=1}^{\infty} a_n(\theta_\psi) q^n,$$

where the first sum is taken over integral ideals of K. The Fourier coefficients of θ_ψ generate a number field E_{θ_ψ} which is clearly contained in E_ψ.

The following classical proposition is due to Hecke and Schoenberg. (See [Ogg 1969] or Section 3.2 of [Zagier 2008]).

Proposition 3.13. *The theta series θ_ψ belongs to $S_k(\Gamma_0(N), \varepsilon)$, where*

(1) *the level N is equal to DM, with $M = \mathrm{N}_{\mathbb{Q}}^K \mathfrak{f}_\psi$;*

(2) *the Nebentypus character ε is equal to $\varepsilon_K \varepsilon_\psi$.*

Lemma 3.14. *If the conductor $\mathfrak{f}_\psi$ of ψ is a cyclic ideal $\mathfrak{m}$ of norm M prime to D, then $f := \theta_\psi$ satisfies the Heegner hypothesis relative to K.*

Proof. In this case, the modular form θ_ψ is of level $N = DM$, by Proposition 3.13. But then the ideal

$$\mathfrak{N} := \mathfrak{d}_K \mathfrak{m}, \tag{3-14}$$

with $\mathfrak{d}_K := (\sqrt{-D_K})$, is a cyclic ideal of K of norm N. □

We will assume from now on that the condition in Lemma 3.14 is satisfied. Furthermore, we will always take $\mathfrak{N}$ to be the ideal in (3-14).

The goal of this section is to factor the p-adic Rankin L-function $L_p(\theta_\psi, \cdot)$ as a product of two Katz p-adic L-functions. As a preparation to stating the main result, we record the following two lemmas:

Lemma 3.15. *For $f := \theta_\psi$, the L-function $L(f, \chi^{-1}, s)$ factors as*

$$L(f, \chi^{-1}, s) = L(\psi\chi^{-1}, s) \cdot L(\psi^*\chi^{-1}, s).$$

Proof. Let $(\rho_{f,\ell})$ denote the compatible system of ℓ-adic Galois representations associated to f. The factorization above then follows from the fact that $L(f, \chi^{-1}, s)$ is the L-function of the compatible system of Galois representations

$$\rho_{f,\ell}|_{\mathrm{Gal}(\bar{K}/K)} \otimes \chi_\ell^{-1} = (\psi_\ell \oplus \psi_\ell^*) \otimes \chi_\ell^{-1} = \psi_\ell\chi_\ell^{-1} \oplus \psi_\ell^*\chi_\ell^{-1}. \qquad \square$$

Lemma 3.16. *Let c be an integer prime to pN and let χ be any character in $\Sigma_{\mathrm{cc}}(c, \mathfrak{N}, \varepsilon)$.*

(1) *If χ belongs to $\Sigma_{\mathrm{cc}}^{(2)}(c, \mathfrak{N}, \varepsilon)$, then $\psi^{-1}\chi$ belongs to $\Sigma_{\mathrm{sd}}^{(2)}(c\mathfrak{d}_K)^-$ and $\psi^{*-1}\chi$ belongs to $\Sigma_{\mathrm{sd}}^{(2)}(c\mathfrak{d}_K M)^-$.*

(2) *If χ belongs to $\Sigma_{\mathrm{cc}}^{(1)}(c, \mathfrak{N}, \varepsilon)$, then $\psi^{-1}\chi$ belongs to $\Sigma_{\mathrm{sd}}^{(1)}(c\mathfrak{d}_K)^-$ and $\psi^{*-1}\chi$ belongs to $\Sigma_{\mathrm{sd}}^{(2)}(c\mathfrak{d}_K M)^-$.*

Proof. We first note that when χ is of type $(k+j, -j)$, then $\psi^{-1}\chi$ is of infinity type $(1+j, -j)$ and $\psi^{*-1}\chi$ is of infinity type $\big(k+j, 1-(k+j)\big)$. Since $\chi \in \Sigma_{\mathrm{cc}}(c, \mathfrak{N}, \varepsilon)$, we have

$$\varepsilon_\chi = \varepsilon = \varepsilon_\psi \cdot \varepsilon_K.$$

Thus $\varepsilon_{\psi^{-1}\chi}$ equals ε_K and the same holds for $\varepsilon_{\psi^{*-1}\chi}$ since $\varepsilon_{\psi^*} = \varepsilon_\psi$. It follows then from Remark 3.7 that $\psi^{-1}\chi$ and $\psi^{*-1}\chi$ are self-dual characters.

Let q be a rational prime dividing M. Since $\mathfrak{m}$ is a cyclic $\mathcal{O}_K$-ideal, it follows that $q = \mathfrak{q}\bar{\mathfrak{q}}$ must be split in K, and exactly one of $\mathfrak{q}$, $\bar{\mathfrak{q}}$ divides $\mathfrak{m}$. From this, it is easy to see that ε_ψ has conductor exactly M, and hence ε has conductor exactly N and $\mathfrak{N}_\varepsilon = \mathfrak{N}$. Thus $\mathfrak{f}_\chi = c\mathfrak{N} = c\mathfrak{d}_K\mathfrak{m}$ and $\mathfrak{f}_{\psi^{*-1}\chi} = c\mathfrak{d}_K\mathfrak{m}\bar{\mathfrak{m}} = c\mathfrak{d}_K M$. On the other hand, since $\varepsilon_\chi = \varepsilon_\psi\varepsilon_K$, it follows that $\mathfrak{f}_{\psi^{-1}\chi} = c\mathfrak{d}_K$.

The preceding remarks imply that if χ is in $\Sigma_{\mathrm{cc}}(c, \mathfrak{N}, \varepsilon)$, then $\psi^{-1}\chi$ lies in $\Sigma_{\mathrm{sd}}(c\mathfrak{d}_K)^-$ and $\psi^{*-1}\chi$ lies in $\Sigma_{\mathrm{sd}}(c\mathfrak{d}_K M)^-$. To finish, we note that if $j \geq 0$, then both $\psi^{-1}\chi$ and $\psi^{*-1}\chi$ lie in $\Sigma_{\mathrm{sd}}^{(2)}$, while if $-(k-1) \leq j \leq -1$, then $\psi^{*-1}\chi$ is in $\Sigma_{\mathrm{sd}}^{(2)}$ while $\psi^{-1}\chi$ lies in $\Sigma_{\mathrm{sd}}^{(1)}$. $\square$

Theorem 3.17. *For all* $\chi \in \Sigma_{\mathrm{cc}}(c, \mathfrak{N}, \varepsilon)$,

$$L_p(\theta_\psi, \chi) = \frac{w(\theta_\psi, \chi)^{-1} w_K}{2c^{k+2j-1}} \prod_{q \mid c} \frac{q - \varepsilon_K(q)}{q-1} \times \mathscr{L}_{p,c\mathfrak{d}_K}(\psi^{-1}\chi) \times \mathscr{L}_{p,c\mathfrak{d}_K M}(\psi^{*-1}\chi). \tag{3-15}$$

Proof. Since $\Sigma_{\mathrm{cc}}^{(2)}(c, \mathfrak{N}, \varepsilon)$ is dense in $\hat{\Sigma}_{\mathrm{cc}}(c, \mathfrak{N}, \varepsilon)$, it suffices to prove the formula for the characters χ in this range, where it follows directly from the interpolation properties defining the respective p-adic L-functions. More precisely, by (3-7),

$$\frac{L_p(\theta_\psi, \chi)}{\Omega_p(A)^{2(k+2j)}} = \big((1 - \psi\chi^{-1}(\bar{\mathfrak{p}}))(1 - \psi^*\chi^{-1}(\bar{\mathfrak{p}}))\big)^2 L_{\mathrm{alg}}(\theta_\psi, \chi^{-1}, 0). \tag{3-16}$$

Let

$$\delta_c := \prod_{q \mid c} \frac{q - \varepsilon_K(q)}{q-1}.$$

By Lemma 3.15 and the definition of $L_{\mathrm{alg}}(\theta_\psi, \chi^{-1}, 0)$ given in (3-5) and (3-6),

$$\begin{aligned} L_{\mathrm{alg}}(\theta_\psi, \chi^{-1}, 0) &= w(\theta_\psi, \chi)^{-1} C(\theta_\psi, \chi, c) \frac{L(\theta_\psi, \chi^{-1}, 0)}{\Omega(A)^{2(k+2j)}} \\ &= w(\theta_\psi, \chi)^{-1} w_K \delta_c \frac{2^{r+2j} \pi^{k+2j-1} j!\,(k+j-1)!}{\sqrt{D}^{k+2j-1} c^{k+2j-1}} \\ &\quad \times \frac{L(\psi\chi^{-1}, 0) L(\psi^*\chi^{-1}, 0)}{\Omega(A)^{2(k+2j)}} \\ &= \frac{w(\theta_\psi, \chi)^{-1} w_K \delta_c}{2c^{k+2j-1}} \left(\frac{j!\,(2\pi)^j L(\psi\chi^{-1}, 0)}{\sqrt{D}^j \Omega(A)^{1+2j}} \right) \\ &\quad \times \left(\frac{(k+j-1)!\,(2\pi)^{k+j-1} L(\psi^*\chi^{-1}, 0)}{\sqrt{D}^{k+j-1} \Omega(A)^{1+2(k+j-1)}} \right). \end{aligned} \tag{3-17}$$

Combining (3-16) and (3-17) with the interpolation property of the Katz p-adic L-function given in Proposition 3.5, we obtain

$$\frac{L_p(\theta_\psi, \chi)}{\Omega_p(A)^{2(k+2j)}} = \frac{w(\theta_\psi, \chi)^{-1} w_K \delta_c}{2c^{k+2j-1}} \times \frac{\mathscr{L}_{p,c\mathfrak{d}_K}(\psi^{-1}\chi)}{\Omega_p(A)^{1+2j}} \times \frac{\mathscr{L}_{p,c\mathfrak{d}_K M}(\psi^{*-1}\chi)}{\Omega_p(A)^{1+2(k+j-1)}}. \tag{3-18}$$

Clearing the powers of $\Omega_p(A)$ on both sides gives the desired result. $\square$

The Nebentypus character ε can be viewed as a finite-order Galois character of $G_{\mathbb{Q}}$. Recall that $E_{\psi,\chi,\varepsilon}$ denotes the smallest extension of $E_{\psi,\chi}$ containing the field through which this character factors.

Corollary 3.18. *For all* $\chi \in \Sigma_{cc}(c, \mathfrak{N}, \varepsilon)$,

$$L_p(\theta_\psi, \chi) = \mathscr{L}_{p,c\mathfrak{d}_K}(\psi^{-1}\chi) \times \mathscr{L}_{p,c\mathfrak{d}_K M}(\psi^{*-1}\chi) \pmod{E^\times_{\psi,\chi,\varepsilon}}.$$

Proof. This follows from Theorem 3.17 in light of the fact that the constant that appears on the right-hand side of (3-15) belongs to $E^\times_{\psi,\chi,\varepsilon}$. □

3E. *Proof of Rubin's theorem.* The goal of this section is to prove Theorem 2 of the Introduction. Let $\mathfrak{c} = \bar{\mathfrak{c}}$ be an integral ideal in $\mathcal{O}_K$ invariant under complex conjugation and let $\nu \in \Sigma_{sd}(\mathfrak{c})^-$ be a Hecke character of K of infinity type $(1, 0)$. Since $\varepsilon_\nu = \varepsilon_K$, it follows that $\mathfrak{d}_K \mid \mathfrak{c}$. We will also assume that ν satisfies the following additional conditions:

(i) The sign w_ν of the functional equation of the L-function $L(\nu, s)$ is -1.

(ii) $\mathfrak{d}_K \parallel \mathfrak{c}$. Thus $\mathfrak{c} = (c)\mathfrak{d}_K$ for a unique positive rational integer c that is prime to D.

Let p be a rational prime split in K that is prime to c.

Definition 3.19. A pair (ψ, χ) of Hecke characters is said to be *good* for ν if it satisfies the following conditions.

(1) The character ψ is of type $(1, 0)$ and has conductor $\mathfrak{m}$, where $\mathfrak{m}$ is a cyclic $\mathcal{O}_K$-ideal prime to pD. Thus θ_ψ is a newform in $S_2(\Gamma_0(N), \varepsilon)$, where $N = MD$ and $\varepsilon = \varepsilon_\psi \varepsilon_K$ is a Dirichlet character of conductor exactly N. Let $\mathfrak{N} := \mathfrak{m}\mathfrak{d}_K$.

(2) The character χ is of finite order, and $\chi \boldsymbol{N}_K$ belongs to $\Sigma^{(1)}_{cc}(c, \mathfrak{N}, \varepsilon)$. This implies (on account of Lemma 3.16 applied to $\chi \boldsymbol{N}_K$) that $\psi^{-1}\chi \boldsymbol{N}_K$ lies in $\Sigma^{(1)}_{sd}(\mathfrak{c})$ and $\psi^{*-1}\chi \boldsymbol{N}_K$ lies in $\Sigma^{(2)}_{sd}(\mathfrak{c}M)$.

(3) The character $\psi\chi^{-1}$ is equal to ν, that is, $\psi^{-1}\chi \boldsymbol{N}_K = \nu^*$.

(4) The classical L-value $L(\psi^*\chi^{-1}\boldsymbol{N}_K^{-1}, 0) = L(\psi^*\chi^{-1}, 1)$ is nonzero, and hence $\mathscr{L}_{p,\mathfrak{c}M}(\psi^{*-1}\chi \boldsymbol{N}_K) \neq 0$.

Remark 3.20. Suppose that a pair (ψ, χ) satisfies (1) and (3) above with $\mathfrak{m}$ prime to $\mathfrak{c}$. Then such a pair automatically satisfies (2) also. Indeed, the character $\chi \boldsymbol{N}_K = \psi\nu^*$ is of type $(1, 1)$ and its central character is equal to

$$\varepsilon_\chi = \varepsilon_\psi \varepsilon_{\nu^*} = \varepsilon_\psi \varepsilon_K = \varepsilon,$$

where ε is the Nebentypus character attached to θ_ψ. Further, $\mathfrak{f}_\chi = \mathfrak{f}_\psi \mathfrak{f}_{\nu^*} = \mathfrak{m} \cdot c\mathfrak{d}_K$. It follows that the character $\chi \boldsymbol{N}_K$ belongs to $\Sigma_{cc}(c, \mathfrak{N}, \varepsilon)$, with $\mathfrak{N} = \mathfrak{d}_K \mathfrak{m}$. (The set S_{θ_ψ} in the discussion below Definition 3.10 is empty, since $D \mid N_\varepsilon$.)

Remark 3.21. Suppose that a pair (ψ, χ) satisfies conditions (1), (2) and (3) above. Since χN_K lies in $\Sigma^{(1)}_{\mathrm{cc}}(c, \mathfrak{N}, \varepsilon)$, the sign in the functional equation of $L(\theta_\psi, \chi^{-1}, s)$ is -1. As seen previously, this L-function factors as

$$\begin{aligned} L\big(\theta_\psi, (\chi N_K)^{-1}, s\big) &= L(\psi\chi^{-1}N_K^{-1}, s)L(\psi^*\chi^{-1}N_K^{-1}, s) \\ &= L(\nu N_K^{-1}, s)L(\psi^*\chi^{-1}N_K^{-1}, s). \end{aligned}$$

The normalization here is such that the central point is $s = 0$. Since the sign of $L(\nu, s)$ is -1, it follows that the sign of $L(\psi^*\chi^{-1}N_K^{-1}, s)$ is $+1$. Hence condition (4) would be expected to hold generically.

The modular abelian variety B_{θ_ψ} attached to ψ is a CM abelian variety in the sense of Definition 2.4. Hence it is K-isogenous to the CM abelian variety B_ψ constructed in Section 2B. In particular, the modular parametrization $\Phi_\psi := \Phi_{\theta_\psi}$ can be viewed as a surjective morphism of abelian varieties over K:

$$\Phi_\psi : J_1(N) \to B_\psi. \tag{3-19}$$

Given a good pair (ψ, χ), recall the Heegner divisor $\Delta \in J_1(N)(H_{c,\mathfrak{N}})$ that was constructed in Section 3C, and the Heegner point

$$P_\psi(\chi) := P_{\theta_\psi}(\chi) = \sum_{\sigma\in G} \Phi_\psi(\Delta^\sigma) \otimes \chi^{-1}(\sigma) \in B_\psi(H_\chi) \otimes_{T_\psi} E_{\psi,\chi} \tag{3-20}$$

that was defined in Equation (3-10) of that section with $f = \theta_\psi$. Recall also that ω_ψ is an E_ψ-vector space generator of $\Omega^1(B_\psi/E_\psi)^{T_\psi}$. Viewing the point $P_\psi(\chi)$ as a formal linear combination of elements of $B_\psi(H_\chi)$ with coefficients in $E_{\psi,\chi}$, we define the expression $\log_{\omega_\psi}(P_\psi(\chi))$ by $E_{\psi,\chi}$-linearity.

In the rest of this section, we will denote by $E'_{\psi,\chi}$ the subfield of $\bar{\mathbb{Q}}$ generated by E_ψ, E_χ, and the abelian extension H'_χ of K cut out by the finite-order characters χ and χ^*. The motivation for singling out good pairs for a special definition lies in the following proposition.

Proposition 3.22. *For any pair (ψ, χ) which is good for ν,*

$$\mathscr{L}_{p,\mathfrak{c}}(\nu^*) = \Omega_p(\nu^*)^{-1} \log^2_{\omega_\psi}(P_\psi(\chi)) \pmod{(E'_{\psi,\chi})^\times}, \tag{3-21}$$

where $\Omega_p(\nu^*)$ is the p-adic period from Definition 2.13.

Proof. By Theorem 3.12 applied to $f = \theta_\psi$,

$$L_p(\theta_\psi, \chi N_K) = \log^2_{\omega_\psi}(P_\psi(\chi)) \pmod{E^\times_{\psi,\chi}}. \tag{3-22}$$

On the other hand, since $E'_{\psi,\chi}$ contains $E_{\psi,\chi,\varepsilon}$, Corollary 3.18 implies that

$$\begin{aligned} L_p(\theta_\psi, \chi N_K) &= \mathscr{L}_{p,\mathfrak{c}}(\psi^{-1}\chi N_K)\mathscr{L}_{p,\mathfrak{c}M}(\psi^{*-1}\chi N_K) \pmod{(E'_{\psi,\chi})^\times} \\ &= \mathscr{L}_{p,\mathfrak{c}}(\nu^*)\mathscr{L}_{p,\mathfrak{c}M}(\psi^{*-1}\chi N_K) \pmod{(E'_{\psi,\chi})^\times}, \end{aligned} \tag{3-23}$$

where the second equality follows from condition (3) in the definition of a good pair. The value $\mathscr{L}_{p,\mathfrak{c}M}(\psi^{*-1}\chi\boldsymbol{N}_K)$ is nonzero by condition (4) in the definition of a good pair. Therefore, by Corollary 3.3,

$$\mathscr{L}_{p,\mathfrak{c}M}(\psi^{*-1}\chi\boldsymbol{N}_K)=\Omega_p(\psi^{-1}\chi^*\boldsymbol{N}_K) \pmod{E_{\psi,\chi}^\times}. \tag{3-24}$$

Since $\psi\chi^{-1}=\nu$, we have

$$\begin{aligned}\Omega_p(\psi^{-1}\chi^*\boldsymbol{N}_K)&=\Omega_p(\nu^{-1}\chi^{-1}\chi^*\boldsymbol{N}_K)\\&=\Omega_p(\nu^*\cdot\chi^*/\chi)=\Omega_p(\nu^*) \pmod{(E'_{\psi,\chi})^\times},\end{aligned} \tag{3-25}$$

where the last equality follows from Lemma 2.14. The proposition now follows by combining Equations (3-22) through (3-25). □

To go further, we will analyze the expression $\log_{\omega_\psi}(P_\psi(\chi))$ and relate it to quantities depending solely on ν and not on the good pair (ψ,χ). It will be useful to view the point $P_\psi(\chi)$ appearing in (3-21) as an element of $B^0_{\psi,\chi}(H_{c,\mathfrak{N}})$ or as a K-rational point on the abelian variety $B_{\psi,\chi}$ that was introduced in Section 2B. More precisely, after setting

$$\boldsymbol{P}_\psi(\chi):=\sum_{\sigma\in G}\Phi_\psi(\Delta^\sigma)\otimes\chi^{-1}(\sigma)\in B_\psi(\bar K)\otimes_{T_\psi}T_{\psi,\chi}=B^0_{\psi,\chi}(\bar K), \tag{3-26}$$

we observe that, for all $\tau\in\mathrm{Gal}(\bar K/K)$,

$$\tau *_0\boldsymbol{P}_\psi(\chi)=\sum_{\sigma\in G}\Phi_\psi(\Delta^{\tau\sigma})\otimes\chi^{-1}(\sigma)=\sum_{\sigma\in G}\Phi_\psi(\Delta^{\sigma})\otimes\chi^{-1}(\sigma\tau^{-1})=\boldsymbol{P}_\psi(\chi)\chi(\tau).$$

The point $\boldsymbol{P}_\psi(\chi)$ therefore belongs to $B_{\psi,\chi}(K)$ by (2-9). For the following lemmas, recall the differentials $\omega^0_{\psi,\chi}\in\Omega^1(B^0_{\psi,\chi}/E_{\psi,\chi})^{T_{\psi,\chi}}$ and $\omega_{\psi,\chi}\in\Omega^1(B_{\psi,\chi}/E_{\psi,\chi})^{T_{\psi,\chi}}$.

Lemma 3.23. *For all good pairs (ψ,χ) attached to $\nu=\psi\chi^{-1}$,*

$$\log_{\omega_\psi}(P_\psi(\chi))=\log_{\omega^0_{\psi,\chi}}(\boldsymbol{P}_\psi(\chi)).$$

Proof. Let $G=\mathrm{Gal}(H_{c,\mathfrak{N}}/K)$ and let $P=\Phi_\psi(\Delta)$. Also, let i be the map defined in (2-4). Then

$$\begin{aligned}\log_{\omega_\psi}(P_\psi(\chi))&=\sum_{\sigma\in G}\chi(\sigma)^{-1}\log_{\omega_\psi}(P^\sigma)=\sum_{\sigma\in G}\chi(\sigma)^{-1}\log_{i^*(\omega^0_{\psi,\chi})}(P^\sigma)\\&=\sum_{\sigma\in G}\chi(\sigma)^{-1}\log_{\omega^0_{\psi,\chi}}(P^\sigma\otimes 1)=\sum_{\sigma\in G}\log_{\chi(\sigma)^{-1}\omega^0_{\psi,\chi}}(P^\sigma\otimes 1)\\&=\sum_{\sigma\in G}\log_{\omega^0_{\psi,\chi}}(P^\sigma\otimes\chi(\sigma)^{-1})\\&=\log_{\omega^0_{\psi,\chi}}\left(\sum_{\sigma\in G}P^\sigma\otimes\chi(\sigma)^{-1}\right)=\log_{\omega^0_{\psi,\chi}}(\boldsymbol{P}_{\psi,\chi}).\end{aligned}$$

□

Lemma 3.24. $\log_{\omega^0_{\psi,\chi}}(\boldsymbol{P}_\psi(\chi)) = \log_{\omega_{\psi,\chi}}(\boldsymbol{P}_\psi(\chi)) \pmod{(E'_{\psi,\chi})^\times}$.

Proof. This follows from Lemma 2.8 since the Gauss sum $\mathfrak{g}(\chi)$ lies in $(E'_{\psi,\chi})^\times$. □

Lemma 3.25. *There exist $P_\nu \in B_\nu(K) \otimes \mathbb{Q}$ and $\omega_\nu \in \Omega^1(B_\nu/E_\nu)^{T_\nu}$ such that*

$$\log_{\omega_{\psi,\chi}}(\boldsymbol{P}_\psi(\chi)) = \log_{\omega_\nu}(P_\nu) \pmod{(E'_{\psi,\chi})^\times}.$$

The point P_ν is nonzero if and only if $L(\nu, s)$ vanishes to exact order one at $s = 1$.

Proof. Recall from Lemma 2.9 that there is a K-rational isogeny

$$B_\nu \otimes_{T_\nu} T_{\psi,\chi} \to B_{\psi,\chi}.$$

Composing it with the natural morphism $B_\nu \to B_\nu \otimes_{T_\nu} T_{\psi,\chi}$, we obtain a T_ν-equivariant morphism $j : B_\nu \to B_{\psi,\chi}$ defined over K with finite kernel. By the Gross–Zagier theorem (see [Gross and Zagier 1986; Yuan et al. 2011] and the remark below), the point $\boldsymbol{P}_\psi(\chi)$ is nonzero if and only if $L'(\theta_\psi, \chi^{-1}, 1) \neq 0$. By Remark 3.21, we have

$$L'(\theta_\psi, \chi^{-1}, 1) = L'(\nu, 1) \cdot L(\psi^*\chi^{-1}, 1).$$

Since $L(\psi^*\chi^{-1}, 1) \neq 0$, we see that $\boldsymbol{P}_\psi(\chi)$ is nonzero if and only if $L'(\nu, 1) \neq 0$. Thus, if $L(\nu, s)$ vanishes to order strictly greater than one (hence order ≥ 3), the lemma holds with $P_\nu := 0$.

We may assume therefore that $L(\nu, s)$ has a simple zero at $s = 1$. This implies (by [Gross and Zagier 1986] and [Kolyvagin 1990]; see also [Kolyvagin and Logachëv 1989]) that $B_\nu(K) \otimes \mathbb{Q}$ is one-dimensional over E_ν, and therefore that $B_{\psi,\chi}(K) \otimes \mathbb{Q}$ is one-dimensional over $E_{\psi,\chi}$. In particular, if P_ν is any generator of $B_\nu(K) \otimes \mathbb{Q}$, we may write

$$\boldsymbol{P}_\psi(\chi) = \lambda j(P_\nu)$$

for some nonzero scalar $\lambda \in E^\times_{\psi,\chi}$. But letting

$$\omega_\nu = j^*(\omega_{\psi,\chi}) \in \Omega^1(B_\nu/E'_{\psi,\chi})^{T_\nu},$$

we have

$$\begin{aligned}\log_{\omega_{\psi,\chi}}(\boldsymbol{P}_\psi(\chi)) = \log_{\omega_{\psi,\chi}}(\lambda j(P_\nu)) = \log_{\lambda^*\omega_{\psi,\chi}}(j(P_\nu)) = \lambda \log_{\omega_{\psi,\chi}}(j(P_\nu))\\ = \lambda \log_{j^*\omega_{\psi,\chi}}(P_\nu) = \lambda \log_{\omega_\nu}(P_\nu).\end{aligned}$$

The lemma now follows after multiplying ω_ν by an appropriate scalar in $(E'_{\psi,\chi})^\times$ so that it belongs to $\Omega^1(B_\nu/E_\nu)^{T_\nu}$. □

Remark 3.26. The original result of [Gross and Zagier 1986] is not general enough to include the situation above. However, Yuan et al. [2011] have proved a very general GZ formula with no assumptions on ramification. This formula relates the height of a Heegner point to a derivative of a Rankin–Selberg L-function, but involves some extra local integrals at bad places that depend on particular choices of test vectors. To deduce that the Heegner point is nontorsion from the nonvanishing of the derivative of the L-function, one needs in addition to know that the local zeta integrals are nonzero; this follows in our case from the computations of [Bertolini et al. 2012b, Section 4.6].

Proposition 3.27. *There exist $\omega_\nu \in \Omega^1(B_\nu/E_\nu)^{T_\nu}$ and $P_\nu \in B_\nu(K)\otimes\mathbb{Q}$ such that*

$$\mathscr{L}_{p,\mathfrak{c}}(\nu^*) = \Omega_p(\nu^*)^{-1}\log^2_{\omega_\nu}(P_\nu) \pmod{(E'_{\psi,\chi})^\times}, \tag{3-27}$$

for all good pairs (ψ,χ) attached to ν. The point P_ν is nonzero if and only if $L(\nu, s)$ vanishes to exact order one at $s = 1$.

Proof. This follows immediately from Proposition 3.22 and Lemmas 3.23 through 3.25. ☐

While Proposition 3.27 brings us close to Theorem 2 of the Introduction, it is somewhat more vague in that both sides of the purported equality may differ a priori by a nonzero element of the typically larger field $E'_{\psi,\chi}$. The alert reader will also notice that this proposition is potentially vacuous for now, because the existence of a good pair for ν has not yet been established! The next proposition repairs this omission, and directly implies Theorem 2 of the Introduction.

Proposition 3.28. *The set S_ν of pairs (ψ,χ) that are good for ν is nonempty. Furthermore,*

$$\bigcap_{(\psi,\chi)\in S_\nu} E'_{\psi,\chi} = E_\nu. \tag{3-28}$$

The proof of Proposition 3.28 rests crucially on a nonvanishing result of Rohrlich [1984] and Greenberg [1985] for the central critical values of Hecke L-series. In order to state it, we fix a rational prime ℓ which is split in K and let

$$K^-_\infty = \bigcup_{n\geq 0} K^-_n$$

be the so-called *anticyclotomic $\mathbb{Z}_\ell$ extension* of K; it is the unique $\mathbb{Z}_\ell$-extension of K which is Galois over $\mathbb{Q}$ and for which $\mathrm{Gal}(K^-_\infty/\mathbb{Q}) = \mathbb{Z}_\ell \rtimes (\mathbb{Z}/2\mathbb{Z})$ is a generalized dihedral group.

Lemma 3.29 [Greenberg 1985; Rohrlich 1984]. *Let ψ_0 be a self-dual Hecke character of K of infinity type $(1, 0)$. Assume that the sign w_{ψ_0} in the functional equation of $L(\psi_0, s)$ is equal to* 1. *Then there are infinitely many finite-order characters χ of* $\mathrm{Gal}(K^-_\infty/K)$ *for which $L(\psi_0\chi, 1) \neq 0$.*

Proof. Let $\mathfrak{c}'$ be the conductor of ψ_0. In light of the hypothesis that $w_{\psi_0} = 1$, Theorem 1 of [Greenberg 1985] implies that the Katz p-adic L-function (with $p = \ell$) does not vanish identically on any open ℓ-adic neighborhood of ψ_0 in

$\Sigma_{\mathrm{sd}}(\mathfrak{c}')$. (See the discussion in the first paragraph of the proof of Proposition 1 on p. 93 of [Greenberg 1985].) If U is any sufficiently small such neighborhood, then:

(1) The restriction to U of the Katz p-adic L-function is described by a power series with p-adically bounded coefficients, and therefore admits only finitely many zeros by the Weierstrass preparation theorem.

(2) The region U contains a dense subset of points of the form $\psi_0\chi$, where χ is a finite-order character of $\mathrm{Gal}(K_\infty^-/K)$.

Lemma 3.29 follows directly from these two facts. □

Proof of Proposition 3.28. Let $\bar{S}_\nu \supset S_\nu$ be the set of pairs satisfying conditions (1)–(3) in the definition of a good pair, but without necessarily requiring the more subtle fourth condition. The proof of Proposition 3.28 will be broken down into four steps.

Step 1. The set $\bar{S}_\nu$ is nonempty.

To see this, let ψ be any Hecke character of K of infinity type $(1, 0)$ and conductor $\mathfrak{m}$, where $\mathfrak{m}$ is a cyclic $\mathbb{O}_K$-ideal prime to $\mathfrak{c}$. Setting $\chi = \psi\nu^{-1}$, the pair (ψ, χ) satisfies conditions (1) and (3) by construction, and (2) as well on account of Remark 3.20. Therefore, the pair (ψ, χ) belongs to $\bar{S}_\nu$.

Step 2. Given $(\psi, \chi) \in \bar{S}_\nu$, there exist (ψ_1, χ_1) and $(\psi_2, \chi_2) \in S_\nu$ with

$$E'_{\psi_1,\chi_1} \cap E'_{\psi_2,\chi_2} \subset E'_{\psi,\chi}.$$

To see this, let $\ell = \lambda\bar{\lambda}$ be a rational prime which splits in K and is relatively prime to the class number of K and the conductors of ψ and χ, and which is unramified in $E'_{\psi,\chi}/\mathbb{Q}$. For such a prime, let

$$K_\infty = \bigcup_{n\geq 0} K_n, \qquad K'_\infty = \bigcup_{n\geq 0} K'_n$$

be the unique $\mathbb{Z}_\ell$-extensions of K which are unramified outside of λ and $\bar{\lambda}$ respectively, with $[K_n : K] = \ell^n$ and likewise for K'_n. The condition that ℓ does not divide the class number of K implies that the fields K_n and K'_n are totally ramified at λ and $\bar{\lambda}$ respectively. If α is any character of $\mathrm{Gal}(K_\infty/K)$, the pair $(\psi_1, \chi_1) := (\psi\alpha, \chi\alpha)$ still belongs to $\bar{S}_\nu$, with $\mathfrak{m}$ in condition 1 replaced by $\mathfrak{m}\lambda^n$ for a suitable $n \geq 0$. Furthermore,

$$L(\psi_1^*\chi_1^{-1}N_K^{-1}, 0) = L\big(\psi^*\chi^{-1}N_K^{-1}\cdot(\alpha^*/\alpha), 0\big). \tag{3-29}$$

The character α^*/α is an anticyclotomic character of K of ℓ-power order and conductor, and all such characters can be obtained by choosing α appropriately. The fact that (ψ, χ) satisfies conditions (1)–(3) of a good pair implies (see Remark 3.21) that the sign $w_{\psi^*\chi^{-1}}$ is equal to $+1$. Hence, by Lemma 3.29, there exists a choice of α for which the L-value appearing on the right of (3-29) is nonvanishing. The corresponding pair (ψ_1, χ_1) belongs to S_ν and satisfies

$$E'_{\psi_1,\chi_1} \subset E'_{\psi,\chi,\ell,n} := E'_{\psi,\chi}\mathbb{Q}(\zeta_{\ell^n})K_n K'_n$$

for some n. Note that the extension $E'_{\psi,\chi,\ell,n}/E'_{\psi,\chi}$ has degree dividing $\ell^\infty(\ell-1)$. Repeating the same construction with a different rational prime ℓ' in place of ℓ such that $\ell'-1$ is prime to ℓ yields a second pair $(\psi_2, \chi_2) \in S_\nu$ and a corresponding extension $E'_{\psi,\chi,\ell',n'}$, whose degree over $E'_{\psi,\chi}$ divides $\ell'^\infty(\ell'-1)$, and such that

$$E'_{\psi_2,\chi_2} \subset E'_{\psi,\chi,\ell',n'}.$$

Let

$$E'' := E'_{\psi,\chi,\ell,n} \cap E'_{\psi,\chi,\ell',n'}.$$

We see then using degrees that $E''/E'_{\psi,\chi}$ has degree dividing $(\ell-1)$, and hence $E'' \subseteq E'_{\psi,\chi}\mathbb{Q}(\zeta_\ell)$. Since ℓ is unramified in $E'_{\psi,\chi}/\mathbb{Q}$, the extension $E''/E'_{\psi,\chi}$ must be totally ramified at the primes above ℓ. On the other hand, being a subextension of $E'_{\psi,\chi,\ell',n'}/E'_{\psi,\chi}$, it is also unramified at the primes above ℓ, and hence $E'' = E'_{\psi,\chi}$. It follows that $E'_{\psi_1,\chi_1} \cap E'_{\psi_2,\chi_2} \subset E'_{\psi,\chi}$.

Thanks to Step 2, we are reduced to showing that

$$\bigcap_{(\psi,\chi)\in\bar{S}_\nu} E'_{\psi,\chi} = E_\nu. \tag{3-30}$$

The next step shows that the fields $E'_{\psi,\chi}$ can be replaced by $E_{\psi,\chi}$ in this equality.

Step 3. For all $(\psi, \chi) \in \bar{S}_\nu$, there exists a finite-order character α of G_K such that the pair $(\psi\alpha, \chi\alpha)$ belongs to $\bar{S}_\nu$ and

$$E'_{\psi,\chi} \cap E'_{\psi\alpha,\chi\alpha} \subseteq E_{\psi,\chi}. \tag{3-31}$$

To see this, note that the finite-order character χ has cyclic image, isomorphic to $\mathbb{Z}/n\mathbb{Z}$ say. Pick α such that conditions (i)–(iii) below are satisfied:

(i) α has order n and is ramified at a single prime λ of K which lies over a rational prime ℓ that is split in K.

(ii) λ is prime to the conductors of χ and χ^*.

(iii) ℓ is unramified in $E'_{\psi,\chi}/\mathbb{Q}$.

Conditions (i) and (ii) imply:

(iv) The field $H_{\chi\alpha}/K$ is totally ramified at λ and unramified at λ^* while $H_{\chi^*\alpha^*}$ is unramified at λ and totally ramified at λ^*.

Taking

$$L = E_{\psi,\chi}, \quad M_1 = H_\chi H_{\chi^*}, \quad M_2 = H_{\chi\alpha} H_{\chi^*\alpha^*},$$

we see from (iii) and (iv) that:

(v) LM_1/L is unramified at all primes above ℓ.

(vi) Any subextension of LM_2/L is ramified at some prime above λ or λ^*.

Thus, $LM_1 \cap LM_2 = L$. But $LM_1 = E_{\psi,\chi} H_\chi H_{\chi^*} = E'_{\psi,\chi}$. Also, since α has order n, we have $E_{\psi',\chi'} = E_{\psi,\chi}$ and

$$LM_2 = E_{\psi,\chi} H_{\chi\alpha} H_{\chi^*\alpha^*} = E_{\psi',\chi'} H_{\chi\alpha} H_{\chi^*\alpha^*} = E'_{\psi',\chi'},$$

so (3-31) follows.

Step 4. We are now reduced to showing

$$\bigcap_{(\psi,\chi)\in\bar{S}_\nu} E_{\psi,\chi} = E_\nu. \tag{3-32}$$

We will do this by showing:

(3-33) There exists a pair $(\psi, \chi) \in \bar{S}_\nu$ such that $E_{\psi,\chi} = E_\nu$.

We begin by choosing an ideal $\mathfrak{m}_0$ of $\mathcal{O}_K$ with the property that $\mathcal{O}_K/\mathfrak{m}_0 = \mathbb{Z}/M\mathbb{Z}$ is cyclic, and an odd quadratic Dirichlet character ε_M of conductor dividing M. Let ψ_0 be any Hecke character satisfying

$$\psi_0((a)) = \varepsilon_M(a \bmod \mathfrak{m}_0)a$$

on principal ideals (a) of K. Such a ψ_0 satisfies condition (1) in Definition 3.19, and therefore, after letting χ_0 be the finite-order character satisfying

$$\nu^* = \psi_0^{-1}\chi_0 \mathrm{N}_K,$$

it follows that (ψ_0, χ_0) belongs to $\bar{S}_\nu$. Furthermore, the restriction of ψ_0 to the group of principal ideals of K takes values in K, and therefore

$$\chi_0(\sigma) \in E_\nu, \quad \text{for all } \sigma \in G_H := \mathrm{Gal}(\bar{K}/H). \tag{3-34}$$

The character ψ_0 itself takes values in a CM field of degree $[H : K]$, denoted E_0, which need not be contained in E_ν in general. To remedy this problem, let H_0 be the abelian extension of the Hilbert class field H cut out by the character χ_0. Next, let H_0' be any abelian extension of K containing H such that:

(1) There is an isomorphism $u : \mathrm{Gal}(H_0'/K) \to \mathrm{Gal}(H_0/K)$ of abstract groups such that the diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Gal}(H_0'/H) \arrow[r] \arrow[d, dotted] & \mathrm{Gal}(H_0'/K) \arrow[r] \arrow[d, dotted] & \mathrm{Gal}(H/K) \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathrm{Gal}(H_0/H) \arrow[r] & \mathrm{Gal}(H_0/K) \arrow[r] & \mathrm{Gal}(H/K) \arrow[r] & 0 \end{tikzcd} \tag{3-35}$$

commutes, where the dotted arrows indicate the isomorphisms induced by u and the other arrows are the canonical maps of Galois theory.

(2) The relative discriminant of H_0' over K is relatively prime to its conjugate (and therefore to the discriminant of K, in particular).

If the bottom exact sequence of groups in (3-35) is split, then the extension H_0' is readily produced, using class field theory. To handle the general case, we follow an approach that is suggested by the proof of Proposition 2.1.7 in [Serre 1992]. Let $\tilde{\Phi} := \mathrm{Gal}(H_0/K)$ and let $\Psi : G_K \to \tilde{\Phi}$ be the homomorphism attached to the extension H_0. Since H is everywhere unramified over K, the restriction Ψ_v of Ψ to a decomposition group at any prime v of K maps the inertia subgroup I_v to $\mathfrak{C} := \mathrm{Gal}(H_0/H)$. After viewing $\mathfrak{C}$ as a module of finite cardinality endowed with the trivial action of G_K, let $H^1_S(K, \mathfrak{C}) := \mathrm{Hom}(G_{K,S}, \mathfrak{C})$ denote the group of homomorphisms from G_K to $\mathfrak{C}$ which are unramified outside a given finite set S of primes of K, and let $H^1_{[S]}(K, \mathfrak{C}^*)$ denote the *dual Selmer group* attached to $H^1_S(K, \mathfrak{C})$ in the sense of, for example, Theorem 2.18 of [Darmon et al. 1997]. Here $\mathfrak{C}^* := \mathrm{Hom}(\mathfrak{C}, \boldsymbol{G}_m)$ is the *Kummer dual* of $\mathfrak{C}$, which is isomorphic to μ_n when $\mathfrak{C} = \mathbb{Z}/n\mathbb{Z}$ is cyclic of order n. Kummer theory (along with the nondegeneracy of the local Tate pairing) identifies $H^1_{[S]}(K, \mu_n)$ with the subgroup of $K^\times/(K^\times)^n$ consisting of elements α for which

$$\mathrm{ord}_v(\alpha) = 0 \pmod n \text{ for all } v, \qquad \mathrm{res}_v(\alpha) \in (K_v^\times)^n \text{ for all } v \in S.$$

Let S be any finite set of primes of K at which Ψ is unramified, satisfying the further conditions

$$v \in S \Rightarrow \bar{v} \notin S \qquad \text{and} \qquad H^1_{[S]}(K, \mathfrak{C}^*) = 0. \tag{3-36}$$

The existence of such a set S follows from the statement that for any

$$\alpha \in K^\times - (K^\times)^n,$$

there is a set of primes v of K of positive Dirichlet density for which the image of α in $K_v^\times$ is not an n-th power. (This statement follows in turn from the Chebotarev density theorem applied to the extension $K(\mu_n, \alpha^{1/n})$.) Now let T be any finite set of places which is disjoint from S. Comparing the statement of Theorem 2.18 of

[Darmon et al. 1997] in the case $M = \mathfrak{C}$ and $\mathscr{L} = S$ and $\mathscr{L} = S \cup T$ respectively, and noting that both $H^1_{[S]}(K, \mathfrak{C}^*)$ and (a fortiori) $H^1_{[S\cup T]}(K, \mathfrak{C}^*)$ are trivial, gives

$$\frac{\# H^1_{S\cup T}(K, \mathfrak{C})}{\# H^1_S(K, \mathfrak{C})} = \prod_{v\in T} \frac{\# H^1(K_v, \mathfrak{C})}{\#\mathfrak{C}} = \prod_v \#\operatorname{Hom}(I_v, \mathfrak{C}).$$

It follows that the rightmost arrow in the tautological exact sequence

$$0 \to H^1_S(K, \mathfrak{C}) \to H^1_{S\cup T}(K, \mathfrak{C}) \to \prod_{c\in T} \operatorname{Hom}(I_v, \mathfrak{C})$$

is surjective. Letting T be the set of places at which Ψ is ramified, it follows that there is a homomorphism $\epsilon : G_K \to \mathfrak{C}$ satisfying

$$\epsilon_v = \Psi_v \quad \text{on } I_v, \quad \text{for all } v \notin S.$$

After possibly enlarging the set S satisfying (3-36) and translating ϵ by a suitable homomorphism unramified outside S, we may further assume that the homomorphism $\Psi\epsilon^{-1}$ maps G_K surjectively onto $\tilde{\Phi}$; the field H_0' can then be obtained as the fixed field of the kernel of the homomorphism $\Psi\epsilon^{-1}$.

With the extension H_0' in hand, let $\alpha : \operatorname{Gal}(H_0'/K) \to E_\chi^\times$ be the finite-order Hecke character given by

$$\alpha(\sigma) = \chi_0(u(\sigma))^{-1},$$

and set $(\psi, \chi) = (\psi_0\alpha, \chi_0\alpha)$. By construction, (ψ, χ) belongs to $\bar{S}_\nu$. We claim that χ and ψ take values in E_ν. Since $\nu^* = \psi^{-1}\chi\mathrm{N}_K$, it is enough to prove this statement for χ. Observe that, for all integral ideals $\mathfrak{a}$ prime to the conductors of χ_0, χ, and ψ, we have

$$\chi(\mathfrak{a}) = \chi_0(\sigma_\mathfrak{a})/\chi_0(u(\sigma_\mathfrak{a})) = \chi_0(\sigma_\mathfrak{a} u(\sigma_\mathfrak{a})^{-1}).$$

But the element $\sigma_\mathfrak{a} u(\sigma_\mathfrak{a})^{-1}$ belongs to $\operatorname{Gal}(H_0/H)$ by construction, and hence $\chi_0(\sigma_\mathfrak{a}^{-1} u(\sigma_\mathfrak{a}))$ belongs to E_ν by (3-34). It follows that ψ and χ are E_ν-valued, and therefore $E_{\psi,\chi} = E_\nu$, as claimed in (3-33). □

3F. *Elliptic curves with complex multiplication.* Theorem 2 of the Introduction admits an alternate formulation involving algebraic points on elliptic curves with complex multiplication rather than K-rational points on the CM abelian varieties B_ν of Theorem 2.5. The goal of this section is to describe this variant. As in the Introduction, we just write $\mathscr{L}_p$ for the p-adic L-function $\mathscr{L}_{p,\mathfrak{c}}$, where $\mathfrak{c}$ is the conductor of ν.

We begin by reviewing the explicit construction of B_ν in terms of CM elliptic curves. The reader is referred to §4 of [Goldstein and Schappacher 1981], whose treatment we largely follow, for a more detailed exposition. Let F be any abelian extension of K for which

$$\nu_F := \nu \circ \mathrm{N}_{F/K} \tag{3-37}$$

becomes K-valued. There exists an elliptic curve A/F with complex multiplication by $\mathbb{O}_K$ whose associated Grossencharacter is ν_F. (See Theorem 6 of [Shimura 1971] and its corollary on p. 512.) Let

$$B := \mathrm{Res}_{F/K}(A).$$

It is an abelian variety over K of dimension $d := [F : K]$. Let $G := \mathrm{Gal}(F/K) = \mathrm{Hom}_K(F, \bar{\mathbb{Q}})$, where the natural identification between these two sets arises from the distinguished embedding of F into $\bar{\mathbb{Q}}$ that was fixed from the outset. By definition of the restriction of scalars functor, there are natural isomorphisms

$$B_{/F} = \prod_{\sigma \in G} A^\sigma, \qquad B(\bar{K}) = A(\bar{K} \otimes_K F) = \prod_{\sigma \in G} A^\sigma(\bar{K})$$

of algebraic groups over F and abelian groups respectively. In particular, a point of $B(\bar{K})$ is described by a d-tuple $(P_\tau)_{\tau \in G}$, with $P_\tau \in A^\tau(\bar{K})$. Relative to this identification, the Galois group G_K acts on $B(\bar{K})$ on the left by the rule

$$\xi(P_\tau)_\tau = (\xi P_\tau)_{\xi\tau}, \quad \text{for all } \xi \in G_K.$$

Consider the "twisted group ring"

$$T := \bigoplus_{\sigma \in G} \mathrm{Hom}_F(A, A^\sigma) = \left\{ \sum_{\sigma \in G} a_\sigma \sigma \;\middle|\; a_\sigma \in \mathrm{Hom}_F(A, A^\sigma) \right\}, \tag{3-38}$$

with multiplication given by

$$(a_\sigma \sigma)(a_\tau \tau) = a_\sigma a_\tau^\sigma \sigma\tau, \tag{3-39}$$

where the isogeny a_τ^σ belongs to $\mathrm{Hom}_F(A^\sigma, A^{\sigma\tau})$ and the composition of isogenies in (3-39) is to be taken from left to right. The right action of T on $B(\bar{K})$ defined by

$$(P_\tau)_\tau * (a_\sigma \sigma) := (a_\sigma^\tau(P_\tau))_{\tau\sigma} \tag{3-40}$$

commutes with the Galois action described in Section 3F, and corresponds to a natural inclusion $T \hookrightarrow \mathrm{End}_K(B)$. The K-algebra $E := T \otimes_{\mathbb{Z}} \mathbb{Q}$ is isomorphic to a finite product

$$E = \prod_i E_i$$

of CM fields, and $\dim_K(E) = \dim(B)$. Therefore, the pair (B, E) is a CM abelian variety in the sense of Definition 2.4. The compatible system of ℓ-adic Galois representations attached to (B, E) corresponds to an E-valued algebraic Hecke character $\tilde{\nu}$ in the sense of Definition 2.3, satisfying the relation

$$\sigma_{\mathfrak{a}}(P) = P * \tilde{\nu}(\mathfrak{a}), \quad \text{for all } \mathfrak{a} \in I_{\mathfrak{f}\ell} \text{ and } P \in B(\bar{K})_{\ell^\infty}, \tag{3-41}$$

where $\sigma_{\mathfrak{a}} \in G_K^{\mathrm{ab}}$ denotes as before the Artin symbol attached to $\mathfrak{a} \in I_{\mathfrak{f}\ell}$.

The element $\tilde{\nu}(\mathfrak{a}) \in T$ is of the form $\varphi_{\mathfrak{a}}\sigma_{\mathfrak{a}}$, where

$$\varphi_{\mathfrak{a}} : A \to A^{\sigma_{\mathfrak{a}}} \tag{3-42}$$

is an isogeny of degree $\mathrm{N}\mathfrak{a}$ satisfying

$$\varphi_{\mathfrak{a}}(P) = P^{\sigma_{\mathfrak{a}}}, \tag{3-43}$$

for any $P \in A[\mathfrak{g}]$ with $(\mathfrak{g}, \mathfrak{a}) = 1$. Note that the isogenies $\varphi_{\mathfrak{a}}$ satisfy the following cocycle condition:

$$\varphi_{\mathfrak{a}\mathfrak{b}} = \varphi_{\mathfrak{b}}^{\sigma_{\mathfrak{a}}} \circ \varphi_{\mathfrak{a}}. \tag{3-44}$$

The following proposition relates the Hecke characters $\tilde{\nu}$ and ν.

Proposition 3.30. *Given any homomorphism $j \in \mathrm{Hom}_K(E, \mathbb{C})$, let $\nu_j := j \circ \tilde{\nu}$ be the corresponding $\mathbb{C}$-valued Hecke character of K of infinity type $(1, 0)$. The assignment $j \mapsto \nu_j$ gives a bijection from $\mathrm{Hom}_K(E, \mathbb{C})$ to the set $\Sigma_{\nu,F}$ of Hecke characters ν' of K (of infinity type $(1, 0)$) satisfying*

$$\nu' \circ \mathrm{N}_{F/K} = \nu \circ \mathrm{N}_{F/K}.$$

Proposition 3.30 implies that there is a unique homomorphism $j_\nu \in \mathrm{Hom}_K(E, \mathbb{C})$ satisfying $j_\nu \circ \tilde{\nu} = \nu$. In particular, j_ν maps E to E_ν and T to a finite-index subring of T_ν. The abelian variety B_ν attached to ν in Theorem 2.5 can now be defined as the quotient $B \otimes_{T, j_\nu} T_\nu$. In subsequent constructions, it turns out to be more useful to realize B_ν as a subvariety of B, which can be done by setting

$$B_\nu := B[\ker j_\nu]. \tag{3-45}$$

The natural action of T on B_ν factors through the quotient $T/\ker(j_\nu)$, an integral domain having E_ν as field of fractions.

Consider the inclusion

$$i_\nu : B_\nu(K) \hookrightarrow B(K) = A(F), \tag{3-46}$$

where the last identification arises from the functorial property of the restriction of scalars. The following proposition gives an explicit description of the image of $(B_\nu(K) \otimes E_\nu)^{T_\nu}$ in $A(F) \otimes_{\mathbb{O}_K} E_\nu$ under the inclusion i_ν obtained from (3-46).

Proposition 3.31. *Let $\tilde{E}$ be any field containing E_ν. The inclusion i_ν of* (3-46) *identifies $(B_\nu(K) \otimes \tilde{E})^{T_\nu}$ with*

$$(A(F) \otimes_{\mathbb{O}_K} \tilde{E})^\nu := \left\{ P \in A(F) \otimes_{\mathbb{O}_K} \tilde{E} \text{ such that } \varphi_{\mathfrak{a}}(P) = \nu(\mathfrak{a}) P^{\sigma_{\mathfrak{a}}},\ \text{for all } \mathfrak{a} \in I_{\mathfrak{f}} \right\}.$$

Proof. It follows from the definitions that $B(K)$ is identified with the set of (P_τ), with $P_\tau \in A^\tau(\bar{K})$ satisfying

$$\xi P_\tau = P_{\xi\tau}, \quad \text{for all } \xi \in G_K. \tag{3-47}$$

Furthermore, if such a (P_τ) belongs to $(B_\nu(K) \otimes E_\nu)^{T_\nu}$, then after setting $\tilde{\nu}(\mathfrak{a}) = \varphi_{\mathfrak{a}} \sigma_{\mathfrak{a}}$ as in (3-42), we also have

$$(\varphi_{\mathfrak{a}}^\tau(P_\tau))_{\tau\sigma_{\mathfrak{a}}} = (P_\tau)_\tau * \tilde{\nu}(\mathfrak{a}) = (\nu(\mathfrak{a}) P_\tau)_\tau. \tag{3-48}$$

Equating the $\sigma_{\mathfrak{a}}$-components of these two vectors gives

$$\varphi_{\mathfrak{a}}(P_1) = \nu(\mathfrak{a}) P_{\sigma_{\mathfrak{a}}} = \nu(\mathfrak{a}) \sigma_{\mathfrak{a}} P_1,$$

where 1 is the identity embedding of F and the last equality follows from (3-47). The proposition follows directly from this, after noting that the identification of $B(K)$ with $A(F)$ is simply the one sending $(P_\tau)_\tau$ to P_1. □

Given a global field F as in (3-37), let F_ν denote the subfield of $\bar{\mathbb{Q}}$ generated by F and E_ν. Recall that $\omega_A \in \Omega^1(A/F)$ is a nonzero differential and that $\Omega_p(A)$ is the associated p-adic period.

Theorem 3.32. *There exists a point $P_{A,\nu} \in (A(F) \otimes_{\mathbb{O}_K} E_\nu)^\nu$ such that*

$$\mathcal{L}_p(\nu^*) = \Omega_p(A)^{-1} \log^2_{\omega_A}(P_{A,\nu}) \pmod{F_\nu^\times}.$$

The point $P_{A,\nu}$ is nonzero if and only if $L'(\nu, 1) \neq 0$.

Proof. Theorem 2 of the Introduction asserts that

$$\mathcal{L}_p(\nu^*) = \Omega_p(\nu^*)^{-1} \log^2_{\omega_\nu}(P_\nu), \tag{3-49}$$

for some point $P_\nu \in B_\nu(K) \otimes \mathbb{Q}$ which is nontrivial if and only if $L'(\nu, 1) \neq 0$. By Lemma 2.14, we find

$$\Omega_p(\nu^*)^{-1} = \Omega_p(A)^{-1} \pmod{F_\nu^\times}. \tag{3-50}$$

Also, by Proposition 3.31, we can view P_ν as a point $P_{A,\nu} \in (A(F) \otimes_{\mathbb{O}_K} E_\nu)^\nu$, and we have

$$\log_{\omega_\nu}(P_\nu) = \log_{\omega_A}(P_{A,\nu}) \pmod{F_\nu^\times}. \tag{3-51}$$

Theorem 3.35 now follows by rewriting (3-49) using (3-50) and (3-51). □

3G. ***A special case.*** This section is devoted to a more detailed and precise treatment of Theorem 3.32 under the following special assumptions:

(1) The quadratic imaginary field K has class number one, odd discriminant, and unit group of order two. This implies that $K = \mathbb{Q}(\sqrt{-D})$, where $D := -\mathrm{Disc}(K)$ belongs to the finite set

$$S := \{7, 11, 19, 43, 67, 163\}.$$

(2) ψ_0 is the Hecke character of K of infinity type $(1,0)$ given by the formula

$$\psi_0((a)) = \varepsilon_K(a \bmod \mathfrak{d}_K)a. \tag{3-52}$$

The character ψ_0 determines (uniquely, up to an isogeny) an elliptic curve $A/\mathbb{Q}$ satisfying

$$\mathrm{End}_K(A) = \mathbb{O}_K, \qquad L(A/\mathbb{Q}, s) = L(\psi_0, s).$$

After fixing A, we will also write ψ_A instead of ψ_0. It can be checked that the conductor of ψ_A is equal to $\mathfrak{d}_K$, and that

$$\psi_A^* = \bar{\psi}_A, \qquad \psi_A\psi_A^* = \mathrm{N}_K, \qquad \varepsilon_{\psi_A} = \varepsilon_K.$$

Remark 3.33. The rather stringent assumptions on K that we have imposed exclude the arithmetically interesting, but somewhat idiosyncratic, cases where $K = \mathbb{Q}(\sqrt{-3})$, $\mathbb{Q}(i)$, and $\mathbb{Q}(\sqrt{-2})$.

With the above assumptions, the character ψ_A can be used to give an explicit description of the set $\Sigma_{\mathrm{sd}}(c\mathfrak{d}_K)$:

Lemma 3.34. *Let c be an integer prime to D, and let ν be a Hecke character in $\Sigma_{\mathrm{sd}}(c\mathfrak{d}_K)$. Then ν is of the form*

$$\nu = \psi_A\chi^{-1},$$

where χ is a finite-order ring class character of K of conductor c.

Proof. The fact that ν and ψ_A both have central character ε_K implies that χ is a ring class character that is unramified at $\mathfrak{d}_K$, and hence has conductor exactly c. □

Given a ring class character χ of conductor c as above with values in a field E_χ, let

$$\begin{aligned}(A(H_c)\otimes_{\mathbb{O}_K} E_\chi)^\chi \\ := \big\{P \in A(H_c)\otimes_{\mathbb{O}_K} E_\chi \text{ such that } \sigma P = \chi(\sigma)P, \text{ for all } \sigma \in \mathrm{Gal}(H_c/K)\big\}.\end{aligned} \tag{3-53}$$

Finally, choose a nonzero differential $\omega_A \in \Omega^1(A/K)$, and write $\Omega_p(A)$ for the p-adic period attached to this choice as in Section 3A. Since $A = B_{\psi_0}$ is the abelian variety attached to ψ_0, it follows that $\Omega_p(\psi_A^*) = \Omega_p(A)$.

The following theorem is a more precise variant of Theorem 3.32.

Theorem 3.35. *Let χ be a ring class character of K of conductor prime to $\mathfrak{d}_K$. Then there exists a point $P_A(\chi) \in \big(A(H_\chi) \otimes_{\mathbb{O}_K} E_\chi\big)^\chi$ such that*

$$\mathcal{L}_p(\psi_A^* \chi) = \Omega_p(A)^{-1} \mathfrak{g}(\chi) \log_{\omega_A}^2 (P_A(\chi)) \pmod{E_\chi^\times}.$$

The point $P_A(\chi)$ is nonzero if and only if $L'(\psi_A \chi^{-1}, 1) \neq 0$.

Proof. By Theorem 2 of the Introduction,

$$\mathcal{L}_p(\psi_A^* \chi) = \mathcal{L}_p(\nu^*) = \Omega_p(\nu^*)^{-1} \log_{\omega_\nu}^2 (P_\nu) \pmod{E_\nu^\times}, \tag{3-54}$$

for some point $P_\nu \in B_\nu(K) \otimes \mathbb{Q}$ which is nontrivial if and only if $L'(\psi_A \chi^{-1}, 1) \neq 0$. Since $\chi^{*-1} = \chi$ and $E_\nu = E_\chi$, we find from Lemma 2.14 that

$$\Omega_p(\nu^*)^{-1} = \Omega_p(\psi_A^* \chi^{*-1})^{-1} = \Omega_p(A)^{-1} \mathfrak{g}(\chi)^{-1} \pmod{E_\chi^\times}. \tag{3-55}$$

After noting that, as in (2-7), $B_\nu = B_{\psi,\chi} = (A \otimes_{\mathbb{O}_K} T_\chi)^{\chi^{-1}}$ as abelian varieties over K, we observe that $\omega_\nu = \omega_{\psi,\chi}$ and that the point $P_\nu \in B_\nu(K)$ can be written as

$$P_\nu = \sum_{\sigma \in G} P^\sigma \otimes \chi^{-1}(\sigma),$$

for some $P \in A(H_c) \otimes \mathbb{Q}$. Letting $P_{A,\chi}$ be the corresponding element in

$$A(H_c) \otimes_{\mathbb{O}_K} E_\chi$$

given by

$$P_{A,\chi} = \sum_{\sigma \in G} \chi^{-1}(\sigma) P^\sigma,$$

we have

$$\log_{\omega_\nu}^2 (P_\nu) = \log_{\omega_{\psi,\chi}}^2 (P_\nu) = \mathfrak{g}(\chi)^2 \log_{\omega_{\psi,\chi}^0}^2 (P_\nu) = \mathfrak{g}(\chi)^2 \log_{\omega_A}^2 (P_{A,\chi}) \pmod{E_\chi^\times}, \tag{3-56}$$

where the second equality follows from Lemma 2.8 and the last from Lemma 3.23. Theorem 3.35 now follows by rewriting (3-54) using (3-55) and (3-56). □

In the special case where χ is a quadratic ring class character of K, cutting out an extension $L = K(\sqrt{a})$ of K, we obtain

$$\mathcal{L}_p(\psi_A^* \chi) = \Omega_p(A)^{-1} \sqrt{a} \log_{\omega_A}^2 (P_{A,L}^-) \pmod{K^\times}, \tag{3-57}$$

where $P_{A,L}^-$ is a K-vector space generator of the trace 0 elements in $A(L) \otimes \mathbb{Q}$. Since in this case $\psi_A \chi$ is the Hecke character attached to a CM elliptic curve over $\mathbb{Q}$, from (3-57) one recovers Rubin's Theorem 1 of the Introduction.

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MASSIMO BERTOLINI
DIPARTIMENTO DI MATEMATICA
UNIVERSITÀ DEGLI STUDI DI MILANO
VIA CESARE SALDINI 50
I-20133 MILANO
ITALY
massimo.bertolini@unimi.it

HENRI DARMON
DEPARTMENT OF MATHEMATICS
MCGILL UNIVERSITY
MONTREAL, QC H3W 1Z4
CANADA
henri.darmon@mcgill.ca

KARTIK PRASANNA
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MICHIGAN
530 CHURCH STREET, 2074 EAST HALL
ANN ARBOR, MD 48109
UNITED STATES
kartikp@umich.edu

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THE SYNTOMIC REGULATOR FOR K_4 OF CURVES

AMNON BESSER AND ROB DE JEU

Dedicated to the memory of Jon Rogawski

Let C be a curve defined over a complete discrete valuation subfield of $\mathbb{C}_p$. Assuming that C has good reduction over the residue field, we compute the syntomic regulator on a certain part of $K_4^{(3)}(C)$. The result can be expressed in terms of p-adic polylogarithms and Coleman integration. We also compute the syntomic regulator on a certain part of $K_4^{(3)}(F)$ for the function field F of C. The result can be expressed in terms of p-adic polylogarithms and Coleman integration, or by using a trilinear map ("triple index") on certain functions.

1. Introduction

Let K be a complete discrete valuation field of characteristic zero, R its valuation ring, and κ its residue field. Assume κ is of positive characteristic p. If $\mathscr{X}/R$ is a scheme, smooth and of finite type, then, after tensoring with $\mathbb{Q}$, one can decompose the algebraic K-theory of $\mathscr{X}$ according to the Adams weight eigenspaces, that is,

$$K_n(\mathscr{X}) \otimes_{\mathbb{Z}} \mathbb{Q} = \bigoplus_j K_n^{(j)}(\mathscr{X}),$$

where $K_n^{(j)}(\mathscr{X})$ consists of those α in $K_n(\mathscr{X}) \otimes_{\mathbb{Z}} \mathbb{Q}$ such that $\psi^k(\alpha) = k^j \alpha$ for all Adams operators ψ^k; see [Soulé 1985, Proposition 5]. The cup product on $K_*(\mathscr{X})$ results in cup products $K_m^{(i)}(\mathscr{X}) \times K_n^{(j)}(\mathscr{X}) \to K_{m+n}^{(i+j)}(\mathscr{X})$. There is a regulator map

$$\mathrm{reg}_p : K_n^{(j)}(\mathscr{X}) \to H_{\mathrm{syn}}^{2j-n}(\mathscr{X}, j)\,;$$

see [Besser 2000b]. In many interesting cases the target group of the regulator is isomorphic to the rigid cohomology group of the special fiber $\mathscr{X}_\kappa$, in the sense of Berthelot, $H_{\mathrm{rig}}^{2j-n-1}(\mathscr{X}_\kappa/K)$. We shall be interested in the situation where $\mathscr{X}$ is a proper, irreducible, smooth curve $\mathscr{C}$ over R with a geometrically irreducible generic fiber C, and the K-group is $K_4^{(3)}(\mathscr{C})$. $K_4^{(3)}(C)$ is known to be isomorphic

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to $K_4^{(3)}(\mathscr{C})$ under localization; see Section 2.2. The target group for the regulator in this case is $H^1_{\mathrm{rig}}(\mathscr{C}_\kappa/K) \cong H^1_{\mathrm{dR}}(C/K)$. The cup product gives a pairing

$$H^1_{\mathrm{dR}}(C/K) \times H^1_{\mathrm{dR}}(C/K) \xrightarrow{\cup} H^2_{\mathrm{dR}}(C/K) \cong K,$$

where the last isomorphism is given by the trace map. We will denote this pairing by $\cup$ as well. If α is an element of $K_4^{(3)}(C)$ and ω is an element of $H^1_{\mathrm{dR}}(C/K)$, we want to compute $\omega \cup \mathrm{reg}_p(\alpha) \in K$.

To achieve this goal, we first of all need to be able to write elements in the above mentioned K-group. We do this using an integral version of the motivic complexes introduced by the second named author. The complex $\mathcal{M}_{(3)}(F)$ was defined in [de Jeu 1995, Section 3] for any field F of characteristic zero. It consists of three terms in cohomological degrees 1, 2 and 3:

$$M_3(F) \to M_2(F) \otimes F^*_{\mathbb{Q}} \to F^*_{\mathbb{Q}} \otimes \bigwedge^2 F^*_{\mathbb{Q}}, \tag{1.1}$$

with $F^*_{\mathbb{Q}} = F^* \otimes_{\mathbb{Z}} \mathbb{Q}$, and $M_n(F)$ a $\mathbb{Q}$-vector space on symbols $[x]_n$ for x in $F \setminus \{0, 1\}$, modulo nonexplicit relations depending on n. The maps in the complex are given by

$$\begin{aligned} \mathrm{d}[x]_3 &= [x]_2 \otimes x, \\ \mathrm{d}\big([x]_2 \otimes y\big) &= (1-x) \otimes (x \wedge y). \end{aligned} \tag{1.2}$$

There are maps

$$H^i(\mathcal{M}_{(3)}(F)) \to K^{(3)}_{6-i}(F)$$

for $i = 2, 3$, and for $i = 3$ this is an isomorphism. Quotienting out by a suitable subcomplex (see Section 2.4.2) one obtains the complex

$$\widetilde{\mathcal{M}}_{(3)}(F) : \widetilde{M}_3(F) \to \widetilde{M}_2(F) \otimes F^*_{\mathbb{Q}} \to \bigwedge^3 F^*_{\mathbb{Q}}, \tag{1.3}$$

which is quasiisomorphic to $\mathcal{M}_{(3)}(F)$ in degrees 2 and 3. Its shape is more in line with conjectures (see for instance [Goncharov 1994, Conjecture 2.1]) and it is easier to work with for explicit examples. Each $\widetilde{M}_i(F)$ is a quotient of $M_i(F)$, and the image of $[x]_i$ in $\widetilde{M}_i(F)$ is still denoted $[x]_i$.

We can apply this with F the function field $K(C)$ of C, but as the syntomic regulator needs some information over the residue field, we have to use an analogous complex.

Notation 1.4. For the curve $\mathscr{C}$ as above with generic fiber C/K, we let $\mathbb{O} \subset F$ be the local ring consisting of functions that are generically defined on the special fiber $\mathscr{C}_\kappa$.

In Section 2.5.2 we shall construct a complex

$$\mathcal{M}_{(3)}(\mathbb{O}) : M_3(\mathbb{O}) \to M_2(\mathbb{O}) \otimes \mathbb{O}^*_{\mathbb{Q}} \to \mathbb{O}^*_{\mathbb{Q}} \otimes \bigwedge^2 \mathbb{O}^*_{\mathbb{Q}}, \tag{1.5}$$

with, in this case, $M_n(\mathbb{O})$ a $\mathbb{Q}$-vector space on symbols $[u]_n$ for u in $\mathbb{O}^\flat$, the *special units* of $\mathbb{O}$, namely those u in $\mathbb{O}^*$ for which $1-u$ is also in $\mathbb{O}^*$, again modulo nonexplicit relations depending on n, and $\mathbb{O}^*_{\mathbb{Q}} = \mathbb{O}^* \otimes_{\mathbb{Z}} \mathbb{Q}$. The maps in the complex are given by (1.2) as before, and there is a natural map $\mathcal{M}_{(3)}(\mathbb{O}) \to \mathcal{M}_{(3)}(F)$ of complexes. In fact, one may view $M_2(\mathbb{O}) \subseteq M_2(F)$; see Remark 2.45. The complex comes with maps

$$H^i(\mathcal{M}_{(3)}(\mathbb{O})) \to K^{(3)}_{6-i}(\mathbb{O}) \tag{1.6}$$

for $i=2$ and 3.

Similar constructions, satisfying in particular (1.6), can be made in the following situation.

Notation 1.7. Suppose $k \subset K$ is a number field and let R' be the local ring $R \cap k$. For $\mathscr{C}'$ a smooth, proper, geometrically irreducible curve over R', let $\mathbb{O}'$ denote the local ring of rational functions on $\mathscr{C}'$ that are generically defined on the special fiber above the maximal ideal of R'.

In this case one has an additional map

$$M_2(\mathbb{O}') \otimes_{\mathbb{Q}} \mathbb{O}'^*_{\mathbb{Q}} \xrightarrow{\partial_1} \coprod_x \widetilde{M}_2(k(x)) ,$$

where $M_2(\mathbb{O}')$ is now a $\mathbb{Q}$-vector space on symbols $[u]_2$ with u in $\mathbb{O}'^*$ such that $1-u$ is also in $\mathbb{O}'^*$, the coproduct is over all closed points of the generic fiber $C' = \mathscr{C}' \otimes_{R'} k$, given by

$$\partial_{1,x}([g]_2 \otimes f) = \mathrm{ord}_x(f) \cdot [g(x)]_2 ,$$

with the convention that $[0]_2 = [1]_2 = [\infty]_2 = 0$.

To explain the terms in which the formula for the regulator will be expressed, we need to introduce Coleman integration theory (see Section 4). Coleman [1982; Coleman and de Shalit 1988] defined an integration theory on curves over $\mathbb{C}_p$ with good reduction and on certain rigid analytic subdomains of these, which he termed "wide open spaces". One first needs to choose a branch of the p-adic logarithm, that is, a group homomorphism $\log : \mathbb{C}_p^* \to \mathbb{C}_p$, such that around $z=1$, it is given by the usual power series expansions for $\log(1+y)$. This amounts to specifying $\log(p)$ in $\mathbb{C}_p$. Once this is done, the theory includes single valued iterated integrals on the appropriate domain, called "Coleman functions". In particular, we have the functions

$$\begin{aligned} \mathrm{Li}_2(z) &= -\textstyle\int_0^z \log(1-x)\,\mathrm{dlog}\,x, \\ \mathrm{L}_2(z) &= \mathrm{Li}_2(z) + \log(z)\log(1-z), \\ \mathrm{L}_{\mathrm{mod},2}(z) &= \mathrm{Li}_2(z) + \tfrac{1}{2}\log(z)\log(1-z). \end{aligned} \tag{1.8}$$

The function $\mathrm{Li}_2(z)$ is defined on $\mathbb{C}_p \setminus \{1\}$; see the beginning of Section VI in [Coleman 1982]. Consequently, all 3 functions are defined everywhere except possibly 0, 1, ∞. They, and other Coleman functions, can be assigned a value at these points as follows.

For every point $y \in \mathbb{P}^1(\mathbb{C}_p)$, the residue disc U_y is the collection of points reducing to the same point as y. For each such y, and in terms of a local parameter $z = z_y$ on U_y, a Coleman function G can be expanded as $G(z) = \sum_{i=0}^n f_i(z) \log^i(z)$, where all $f_i(z)$ are in $\mathbb{C}_p[\![z, z^{-1}]\!]$. We define the *constant term* $c_z(G)$ at y with respect to the parameter z as the constant term of f_0; see Definition 7.7. In general the constant term will depend on the choice of the local parameter z, but there are many Coleman functions for which the constant term is independent of this choice. In such a case we will write $G(y)$ for this constant term. In particular, this is the case at all points y for $\mathrm{L}_{\mathrm{mod},2}(z)$ and $\int \mathrm{L}_2(g)\omega$ for any rational function g (it is in fact sufficient that ω is holomorphic at y), as well as for $\mathrm{Li}_2(z)$ and $\mathrm{L}_2(z)$ at all points except ∞ (see Lemmas 10.7 and 10.9 as well as Corollary 10.8). We further define all three functions from (1.8) to be 0 at 0 and ∞ (this is the constant term with respect to the standard parameter). For any Coleman function G, which is the integral of a form η, and divisor $D = \sum n_i y_i$, we will define

$$G(D) = \int_D \eta := \sum n_i G(y_i),$$

where we will be assuming that either G is defined at each y_i, or its constant term there is independent of the parameter.

We note that $\mathrm{L}_{\mathrm{mod},2}(z) + \mathrm{L}_{\mathrm{mod},2}(z^{-1}) = 0$ for z in $\mathbb{C}_p \setminus \{0, 1\}$ [Coleman 1982, Proposition 6.4(ii)], and that this extends to all values using constant terms. Similarly we have $\mathrm{L}_2(z) + \mathrm{L}_2(z^{-1}) = \frac{1}{2}\log^2(z)$.

We shall state the theorems in the introduction in a way that allows comparison with similar results in the classical case over $\mathbb{C}$. The formulas in that case can be easily transformed by using Stokes' theorem, whereas it seems the formulas in the syntomic case are not as flexible. Consequently, in the syntomic case we have to state a larger number of theorems. In order to enable a comparison in Remark 10.14 of the syntomic formulas below (especially those in Theorems 1.12 and 1.13) with those in the classical case, we recall and reformulate some of the classical results in Section 3.

We are now ready to state the first main theorem. In it, and the remaining theorems in the introduction, we assume that K is a closed subfield of $\mathbb{C}_p$ and evaluate Coleman functions at closed points of C by working over a finite extension of K over which all such points are rational. The result will be in K by Galois equivariance of Coleman integration.

Theorem 1.9. *Suppose, in the situation of Notation 1.4, that ω is a holomorphic form on C.*

(1) *The assignment*

$$[g]_2 \otimes f \mapsto 2 \int_{(f)} \mathrm{L}_2(g)\,\omega$$

gives a well-defined map $\Psi_{p,\omega} : M_2(\mathbb{O}) \otimes \mathbb{O}^*_{\mathbb{Q}} \to K$, *and this induces a map* $\Psi_{p,\omega} : H^2(\mathcal{M}_{(3)}(\mathbb{O})) \to K$.

(2) *Suppose* $k \subset K$ *is a number field, and* $\mathscr{C}'$ *is a smooth, proper, geometrically connected curve over the local ring* $R' = R \cap k$. *Let* $\mathbb{O}'$ *be as in Notation 1.7 and put* $\mathscr{C} = \mathscr{C}' \otimes_{R'} R$. *Let* α' *in* $H^2(\mathcal{M}_{(3)}(\mathbb{O}'))$ *be such that* $\partial_1(\alpha') = 0$. *Then there exists a unique* β' *in* $K_4^{(3)}(\mathscr{C}')$ *whose image in* $K_4^{(3)}(\mathbb{O}')$ *under localization equals the image of* α' *under* (1.6) *modulo* $K_3^{(2)}(k) \cup \mathbb{O}'^*_{\mathbb{Q}}$. *If* β *is the image of* β' *under* $K_4^{(3)}(\mathscr{C}') \to K_4^{(3)}(\mathscr{C})$, *then we have*

$$\omega \cup \mathrm{reg}_p(\beta) = \Psi_{p,\omega}(\alpha),$$

where α *is the image of* α' *in* $H^2(\mathcal{M}_{(3)}(\mathbb{O}))$.

Remark 1.10. The reader should compare the above formula for the regulator with the formula obtained by Coleman and de Shalit [1988], which is known to be the syntomic regulator by [Besser 2000c]. There, the regulator is obtained by sending the symbol $\{f, g\}$ in $K_2(F)$ to $\int_{(f)} \log(g)\omega$. The similarity with the present formula should be clear.

The rest of our results concern the K-theory of open curves over R and not over a number field. Thus, they are more general on the one hand, but progressively harder to state. Indeed, the first theorem is special because we are able to simplify matters by taking account of boundary terms over number fields.

As we are now computing on an open scheme, we no longer have a nontrivial cup product pairing, so we first need to explain what it is that we are computing. Under the regulator, each element of $K_4^{(3)}(\mathbb{O})$ maps to $H^1_{\mathrm{dR}}(U/K)$ for some wide open space U in C in the terminology of Coleman. There exists a canonical projection $H^1_{\mathrm{dR}}(U/K) \to H^1_{\mathrm{dR}}(C/K)$, compatible with restriction to a smaller U; see [Besser 2000c, Proposition 4.8] and (9.13) below. We denote by reg'_p the composition

$$K_4^{(3)}(\mathbb{O}) \to H^1_{\mathrm{dR}}(U/K) \to H^1_{\mathrm{dR}}(C/K).$$

Theorem 1.11. *Suppose* ω *is a holomorphic form on* C. *The assignment*

$$[g]_2 \otimes f \mapsto 2 \int_{(f)} \mathrm{L}_2(g)\omega - 2 \sum_y \mathrm{ord}_y(f) F_\omega(y) \mathrm{L}_{\mathrm{mod},2}(g(y)),$$

where in the sum y *runs through the closed points of* C, *gives a well-defined map* $\Psi'_{p,\omega} : M_2(\mathbb{O}) \otimes \mathbb{O}^*_{\mathbb{Q}} \to K$. *It induces a map* $\Psi'_{p,\omega} : H^2(\mathcal{M}_{(3)}(\mathbb{O})) \to K$, *which* *coincides with the composition*

$$H^2(\mathcal{M}_{(3)}(\mathcal{O})) \to K_4^{(3)}(\mathcal{O}) \xrightarrow{\mathrm{reg}'_p} H^1_{\mathrm{dR}}(C/K) \xrightarrow{\omega\cup} K.$$

Over the complex numbers, it is known that computing the cup product of the regulator with holomorphic forms suffices to describe it completely in the case we are considering because those linear maps surject onto the dual of the target space of the regulator (see the beginning of Section 4 of [de Jeu 1996], especially Proposition 4.1). This is not true over the p-adics. It is therefore important to have formulas for the cup product of the regulator with a general cohomology class (such a class can be represented by a form of the second kind on C, that is, a meromorphic form all of whose residues are 0). This can be done at the cost of introducing further machinery — the notion of the triple index. It is a generalization of the "local index" that was introduced in [Besser 2000c, Section 4].

Informally speaking, working on an annulus e over $\mathbb{C}_p$, $e \cong \{r < |z| < 1\}$, the triple index associates to the integrals F, G and H of three rigid analytic 1-forms on e (in this case these forms are simply Laurent series converging on e multiplied by dz) together with choices of integrals for $F\,\mathrm{d}G$, $F\,\mathrm{d}H$ and $G\,\mathrm{d}F$, a number $\langle F, G; H\rangle_e$ in $\mathbb{C}_p$ that is supposed to be a generalization of $\mathrm{Res}_e\, FG\,\mathrm{d}H$. Note that the integrals appearing in the data for the triple index make perfect sense once one admits a log function to correspond to the integral of $\mathrm{d}z/z$, and are determined up to a constant by the form they integrate. Suppose now that $C/\mathbb{C}_p$ is a curve with good reduction and that C contains discs $D_i \cong \{|z| < 1\}$. The rigid analytic domain $U = C \setminus \bigcup_i (D_i - e_i)$, where $e_i \subset D_i$ is the annulus corresponding to $\{r < |z| < 1\}$, is called a wide open space by Coleman. The $e_i \subset U$ are called the ends of U. Suppose that F, G and H are Coleman functions defined on U such that restricted to the e_i they are of the type allowing us to compute the triple indices $\langle F|_{e_i}, G|_{e_i}; H|_{e_i}\rangle_{e_i}$. We may use auxiliary data composed of Coleman integrals restricted to e_i for computing these. It sometimes turns out that the sum of triple indices over all the e_i depends only on F, G, and H and not on the auxiliary data. This applies in particular to the sum of triple indices in the two theorems below. It is further known that this sum of triple indices behaves well with respect to shrinking the wide open space U. Finally, if everything is defined over a complete subfield K of $\mathbb{C}_p$ then this sum of triple indices is in K.

Theorem 1.12. *Let ω be a form of the second kind on C. The assignment*

$$[g]_2 \otimes f \mapsto \sum_e \left\langle \log(f), \log(g); \int F_\omega \,\mathrm{dlog}(1-g) \right\rangle_e,$$

where F_ω is any Coleman integral of ω, and the sum of triple indices is over all ends e of a wide open space U on which all f, g and $1-g$ are invertible and ω is *holomorphic, gives a well-defined map* $\Psi''_{p,\omega} : M_2(\mathbb{O}) \otimes \mathbb{O}^*_{\mathbb{Q}} \to K$. *It induces a map* $\Psi''_{p,\omega} : H^2(\mathcal{M}_{(3)}(\mathbb{O})) \to K$, *which coincides with the composition*

$$H^2(\mathcal{M}_{(3)}(\mathbb{O})) \to K_4^{(3)}(\mathbb{O}) \xrightarrow{\mathrm{reg}'_p} H^1_{\mathrm{dR}}(C/K) \xrightarrow{\omega\cup} K.$$

The complex $\widetilde{\mathcal{M}}_{(3)}(F)$ defined in (1.3) is easier to work with in explicit computations than the complex $\mathcal{M}_{(3)}(F)$. Therefore, just as in [de Jeu 1996, Remark 4.5], it is desirable to have a formula for the regulator using this complex. With that in mind, we define in Section 2.5.5 a complex

$$\widetilde{\mathcal{M}}_{(3)}(\mathbb{O}) : \widetilde{M}_3(\mathbb{O}) \to \widetilde{M}_2(\mathbb{O}) \otimes \mathbb{O}^*_{\mathbb{Q}} \to \textstyle\bigwedge^3 \mathbb{O}^*_{\mathbb{Q}}$$

such that its cohomology in degrees 2 and 3 is isomorphic to that of the complex $\mathcal{M}_{(3)}(\mathbb{O})$ in (1.5). There is a natural map $\widetilde{\mathcal{M}}_{(3)}(\mathbb{O}) \to \widetilde{\mathcal{M}}_{(3)}(F)$ of complexes, and one may view $\widetilde{M}_2(\mathbb{O}) \subseteq \widetilde{M}_2(F)$. Corresponding to the statements in Theorems 1.11 and 1.12 for $\mathcal{M}_{(3)}(\mathbb{O})$, we have the following two expressions for the regulator in this case.

Theorem 1.13. 1. *Let ω be a form of the second kind on C. The assignment*

$$[g]_2 \otimes f \mapsto \tfrac{2}{3} \sum_e \left\langle \log(f), \log(g); \int F_\omega \,\mathrm{dlog}(1-g) \right\rangle_e - \tfrac{2}{3} \sum_e \left\langle \log(f), \log(1-g); \int F_\omega \,\mathrm{dlog}(g) \right\rangle_e$$

gives a well-defined map $\Psi'''_{p,\omega} : \widetilde{M}_2(\mathbb{O}) \otimes \mathbb{O}^*_{\mathbb{Q}} \to K$. *It induces a map*

$$\Psi'''_{p,\omega} : H^2(\widetilde{\mathcal{M}}_{(3)}(\mathbb{O})) \to K,$$

which coincides with the composition of maps

$$H^2(\widetilde{\mathcal{M}}_{(3)}(\mathbb{O})) \xrightarrow{\simeq} H^2(\mathcal{M}_{(3)}(\mathbb{O})) \to K_4^{(3)}(\mathbb{O}) \xrightarrow{\mathrm{reg}'_p} H^1_{\mathrm{dR}}(C/K) \xrightarrow{\omega\cup} K,$$

with the leftmost map being the isomorphism alluded to before.

2. *If ω is a holomorphic form on C, then the same holds for the assignment*

$$[g]_2 \otimes f \mapsto \tfrac{2}{3} \int_{(f)} (3\mathrm{L}_2(g) - \log(1-g)\log(g))\omega + \tfrac{2}{3} \int_{(g)} \log(f)\log(1-g)\omega - 2\sum_y \mathrm{ord}_y(f) F_\omega(y) \mathrm{L}_{\mathrm{mod},2}(g(y)).$$

A key complex for doing computations is $\mathcal{C}^\bullet(\mathbb{O}) : \mathcal{C}^1(\mathbb{O}) \to \mathcal{C}^2(\mathbb{O})$ in cohomological degrees 1 and 2, which we shall construct in Section 2.5.4. The theorems in this introduction admit analogous results expressed in terms of this complex. We avoided these results for clarity in the introduction. However, they are very useful in applications since it is easier to find explicit examples to which these results apply, for instance, for certain elliptic curves; see [de Jeu 1996, Section 6].

We end the introduction with a conjecture. The regulator formulas that we obtain do not depend on any integrality assumptions. This is only required because the syntomic regulator is a map from the K-theory of an integral model. Thus we conjecture the following.

Conjecture 1.14. Theorems 1.9, 1.11, 1.12 and 1.13 hold, with the same formulas, with $\mathcal{O}$ replaced by F and $\mathcal{C}$ replaced by C.

Notation 1.15. Unless stated otherwise, throughout the paper, we will be working with the following notation.

K will be a discrete valuation field of characteristic zero with valuation ring R and residue field κ of positive characteristic p, which is a subfield of $\overline{\mathbb{F}}_p$. In various places, k will be a number field inside K. In that case we denote by $\mathbb{F} \subseteq \kappa$ the residue field of the local ring $R' = k \cap R$.

$\mathcal{C}$ will be a smooth, proper, geometrically irreducible curve over R. The generic fiber is denoted C, the special fiber is denoted $\mathcal{C}_\kappa$. We let $F = K(C)$, and $\mathcal{O} \subset F$ will be the valuation ring for the valuation on F corresponding to the generic point of $\mathcal{C}_\kappa$, which consists of those elements in F that are generically defined on $\mathcal{C}_\kappa$.

If $k \subset K$ is a number field, and $\mathcal{C}'$ is a smooth, proper, geometrically irreducible curve over $R' = R \cap k$, then the generic fiber is denoted C', the special fiber is denoted $\mathcal{C}'_{\mathbb{F}}$. We let $F' = k(C')$, and $\mathcal{O}' \subset F'$ will be the valuation ring for the valuation on F' corresponding to the generic point of $\mathcal{C}'_{\mathbb{F}}$. In particular, if $\mathcal{C} = \mathcal{C}' \otimes_{R'} R$, then $\mathcal{O}' = \mathcal{O} \cap F'$.

If S is a subset of a group, then we denote by $<S>$ the subgroup generated by S, and if S is a subset of a $\mathbb{Q}$-vector space, we denote by $<S>_{\mathbb{Q}}$ the $\mathbb{Q}$-vector subspace generated by S.

All tensor products will be over $\mathbb{Q}$, unless specified otherwise.

For the convenience of the reader, we give a commutative diagram that plays the role of a two-dimensional Leitfaden ("Leitteppich") for the proofs in this paper. In the left lower square we may also use $\mathcal{O}'$ instead of $\mathcal{O}$, in which case $C = C' \otimes_{R'} K$.

$$
\begin{tikzcd}
H^2(\mathcal{M}_{(3)}(\mathcal{C}')) \arrow[r] \arrow[d] & K_4^{(3)}(\mathcal{C}') \oplus K_3^{(2)}(k) \cup \mathcal{O}'^*_{\mathbb{Q}} \arrow[d] & \\
H^2(\mathcal{M}_{(3)}(\mathcal{O})) \arrow[r] \arrow[d] & K_4^{(3)}(\mathcal{O}) \arrow[r, "\mathrm{reg}'_p"] \arrow[d] & H^1_{\mathrm{dR}}(C/K) \arrow[d, "\omega\cup\cdot"] \\
H^1(\mathcal{C}^\bullet(\mathcal{O})) \arrow[r] & K_4^{(3)}(\mathcal{O})/K_3^{(2)}(\mathcal{O}) \cup \mathcal{O}^*_{\mathbb{Q}} \arrow[r, dotted] & K
\end{tikzcd}
\tag{1.16}
$$

The constructions in algebraic K-theory will be carried out in Section 2. The top left square comes from the natural map $\mathcal{M}_{(3)}(\mathscr{C}') \to \mathcal{M}_{(3)}(\mathbb{O}')$ (see Section 2.5.3), and is justified by (2.58), whereas the bottom left square is (2.67). The map

$$K_4^{(3)}(\mathbb{O}) \xrightarrow{\mathrm{reg}'_p} H^1_{\mathrm{dR}}(C/K)$$

already factorizes through the quotient map $K_4^{(3)}(\mathbb{O}) \to K_4^{(3)}(\mathbb{O})/K_3^{(2)}(\mathbb{O}) \cup \mathbb{O}^*_{\mathbb{Q}}$ (see Corollary 9.5). The resulting composition in the bottom line of (1.16) is then computed in Section 9, using the techniques developed in the preceding sections. In Section 10 we then finish the proofs of the theorems above, based on this calculation.

2. *K*-theory

2.1. ***Introduction.*** Consider a proper, smooth, geometrically irreducible curve $\mathscr{C}$ over R as in Notation 1.4, or $\mathscr{C}'$ over R' as in Notation 1.7. We shall construct various cohomological complexes whose cohomologies are related to that of F, $\mathbb{O}$, F' or $\mathbb{O}'$. The main idea is the same as in [de Jeu 1996], but the fact that we shall be working with a discrete valuation ring rather than a field gives rise to some complications. In order to highlight the idea we start with a more gentle exposition. For the proofs of the statements that are used in the construction, we refer the reader to [de Jeu 1995], especially Sections 2.1 through 2.3 and 3. There most of the work was done over $\mathbb{Q}$, but in fact the proofs hold over our base $\mathbb{O}$, a discrete valuation ring of characteristic zero, without any change.

It will be clear from the constructions that the complexes are natural in terms of F, F', $\mathbb{O}$ and $\mathbb{O}'$, which we shall use later in this paper. In particular, if we start with $\mathscr{C}'$ over R' and let $\mathscr{C} = \mathscr{C}' \otimes_{R'} R$, then there are natural maps from the complexes for F' to those for F, and from those for $\mathbb{O}'$ to those for $\mathbb{O}$.

If B is a Noetherian scheme of finite Krull dimension d, then according to [Soulé 1985, Proposition 5], one can write

$$K_n(B) \otimes_{\mathbb{Z}} \mathbb{Q} = \bigoplus_{j=\min\{2,n\}}^{n+d} K_n^{(j)}(B) \tag{2.1}$$

where $K_n^{(j)}(B)$ consists of all α in $K_n(B) \otimes_{\mathbb{Z}} \mathbb{Q}$ such that $\psi^k(\alpha) = k^j \alpha$ for all Adams operators ψ^k. (The regularity assumption at the beginning of Section 4 of [loc. cit.] is not necessary; see [Gillet and Soulé 1999, Proposition 8].) If in addition B is separated and regular, then the pullback $K_*(B) \to K_*(\mathbb{A}^1_B)$ is an isomorphism; see [Quillen 1973, §7]. The weight behaves naturally with respect to pullback, also giving us $K_m^{(j)}(B) \simeq K_m^{(j)}(\mathbb{A}^1_B)$ under pullback. And under suitable hypotheses for a closed embedding, there is a pushforward Gysin map with a shift in weights corresponding to the codimension; see, for instance, [de Jeu 1995, Proposition 2.3].

Let $X_B = \mathbb{P}^1_B \setminus \{t = 1\}$ with t the standard affine coordinate on $\mathbb{P}^1_B$. Write $\square^1_B$ for the closed subset $\{t = 0, \infty\}$ in $\mathbb{P}^1_B$. Then the relative exact sequence for the couple $(X_B; \square^1_B)$ gives us

$$\cdots \to K_{n+1}(X_B) \to K_{n+1}(\square^1_B) \to K_n(X_B; \square^1_B) \to K_n(X_B) \to K_n(\square^1_B) \to \cdots$$

for $n \geq 0$. Because the map pullback $K_{n+1}(B) \to K_{n+1}(X_B)$ is an isomorphism, combining it with the pullback $K_{n+1}(X_B) \to K_{n+1}(\square^1_B) = K_{n+1}(B)^2$ shows that the map $K_{n+1}(X_B) \to K_{n+1}(\square^1_B)$ corresponds to the diagonal embedding $K_{n+1}(B) \to K_{n+1}(B)^2$. As this holds for all $n \geq 0$, we get that we have an isomorphism $K_n(X_B; \square^1_B) \simeq K_{n+1}(B)$ for $n \geq 0$. Note that we have a choice of sign here in the isomorphism of the cokernel of $K_n(B) \to K_n(B)^2$ with $K_n(B)$. This results in similar choices of signs in the maps $H^i(\mathcal{M}_{(n)}(\mathbb{O})) \to K^{(n)}_{2n-i}(\mathbb{O})$ and $H^i(\widetilde{\mathcal{M}}_{(n)}(\mathbb{O})) \to K^{(n)}_{2n-i}(\mathbb{O})$ later on in this section.

We will have to go up one level in the relativity. If we let $\square^2_B$ be shorthand for $\{t_1 = 0, \infty\}; \{t_2 = 0, \infty\}$, then we can get a long exact sequence

$$\begin{aligned}&\cdots \to K_{n+1}(X^2_B; \{t_1 = 0, \infty\}) \to K_{n+1}(\{t_2 = 0, \infty\}; \{t_1 = 0, \infty\})\\ &\to K_n(X^2_B; \square^2_B) \to K_n(X^2_B; \{t_1 = 0, \infty\}) \to K_n(\{t_2 = 0, \infty\}; \{t_1 = 0, \infty\}) \to \cdots .\end{aligned}$$

The composition

$$\begin{aligned}K_{n+1}(X_B; \{t_1 = 0, \infty\}) &\xrightarrow{\simeq} K_{n+1}(X^2_B; \{t_1 = 0, \infty\})\\ &\to K_{n+1}(\{t_2 = 0, \infty\}; \{t_1 = 0, \infty\}) \simeq K_{n+1}(X_B; \{t_1 = 0, \infty\})^2\end{aligned}$$

(with the first the pullback along the projection $(t_1, t_2) \mapsto t_2$) is the diagonal embedding, hence we obtain an isomorphism $K_n(X^2_B; \square^2_B) \simeq K_{n+1}(X_B; \square^1_B)$ for $n \geq 0$. Therefore we get

$$K_n(X^2_B; \square^2_B) \simeq K_{n+1}(X_B; \square^1_B) \simeq K_{n+2}(B) \quad \text{for } n \geq 0.$$

A similar argument with weights gives us an isomorphism

$$K^{(j)}_n(X^2_B; \square^2_B) \simeq K^{(j)}_{n+2}(B) \quad \text{for } n \geq 0.$$

In order to get elements in $K_{n+2}(X^2_B; \square^2_B)$, we use localization sequences. We first explain the idea for $K_{n+1}(X_B; \square^1_B)$, because for $K_{n+2}(X^2_B; \square^2_B)$ the process involves a spectral sequence. If u is an element in our discrete valuation ring $\mathbb{O}$ such that both u and $1-u$ are units, then we get an exact localization sequence

$$\cdots \to K_m(\mathbb{O}) \to K_m(X_{\mathbb{O}}; \square^1_{\mathbb{O}}) \to K_m(X_{\mathbb{O},\mathrm{loc}}; \square^1_{\mathbb{O}}) \to K_{m-1}(\mathbb{O}) \to \cdots$$

where $X_{\mathbb{O},\mathrm{loc}} = X_{\mathbb{O}} \setminus \{t = u\}$ and we identified $\{t = u\} \subset X_{\mathbb{O}}$ with $\mathbb{O}$ (or rather $\mathrm{Spec}(\mathbb{O})$). We used here that u and $1-u$ are units in $\mathbb{O}$ so that $\{t = u\}$ does not meet $\square^1_{\mathbb{O}}$ or $\{t = 1\}$, and that $\mathbb{O}$ is regular in order to identify $K_m(\mathbb{O})$ with $K'_m(\mathbb{O})$. (If we want to leave out $\{t = u\}$ and $\{t = v\}$ simultaneously for two distinct elements u and v in $\mathbb{O}$ such that all of u, v, $1 - u$ and $1 - v$ are units, which we shall do below, this already becomes far more complicated and one is forced to use a spectral sequence.) The image of $K_2(\mathbb{O}) \to K_2(X_{\mathbb{O}}; \square^1_{\mathbb{O}})$ can be controlled by looking at the weights, which for the bit that we are interested in gives us

$$\cdots \to K_2^{(1)}(\mathbb{O}) \to K_2^{(2)}(X_{\mathbb{O}}; \square^1_{\mathbb{O}}) \to K_2^{(2)}(X_{\mathbb{O},\mathrm{loc}}; \square^1_{\mathbb{O}}) \to K_1^{(1)}(\mathbb{O}) \to \cdots .$$

Because of weights in K-theory, one knows that $K_2^{(1)}(\mathbb{O}) = 0$, so that

$$K_3^{(2)}(\mathbb{O}) \simeq \mathrm{Ker}(K_2^{(2)}(X_{\mathbb{O},\mathrm{loc}}; \square^1_{\mathbb{O}}) \to K_1^{(1)}(\mathbb{O})),$$

and we can analyze $K_2^{(2)}(X_{\mathbb{O}}; \square^1_{\mathbb{O}})$ as a subgroup of $K_2^{(2)}(X_{\mathbb{O},\mathrm{loc}}; \square^1_{\mathbb{O}})$. In [de Jeu 1995, Section 3.2] universal elements $[S]_n$ were constructed, of which we want to use $[S]_2$ here. It gives rise to an element $[u]_2$ in $K_2^{(2)}(X_{\mathbb{O},\mathrm{loc}}; \square^1_{\mathbb{O}})$ with boundary $(1-u)^{-1}$ in $K_1^{(1)}(\mathbb{O})$. If we use this for various u (suitably modifying the localization sequence above into a spectral sequence) and also consider elements coming from the cup product

$$K_1^{(1)}(X_{\mathbb{O},\mathrm{loc}}; \square^1_{\mathbb{O}}) \times K_1^{(1)}(\mathbb{O}) \to K_2^{(2)}(X_{\mathbb{O},\mathrm{loc}}; \square^1_{\mathbb{O}},)$$

we can get part of $K_2^{(2)}(X_{\mathbb{O}}; \square^1_{\mathbb{O}}) \simeq K_3^{(2)}(\mathbb{O})$ by intersecting the kernel of the map corresponding to $K_2^{(2)}(X_{\mathbb{O},\mathrm{loc}}; \square^1_{\mathbb{O}}) \to K_1^{(1)}(\mathbb{O})$ with the space generated by the symbols $[u]_2$ and the image $K_1^{(1)}(X_{\mathbb{O},\mathrm{loc}}; \square^1_{\mathbb{O}}) \cup K_1^{(1)}(\mathbb{O})$ of the cup product.

2.2. ***Preliminary material.*** We describe some basic facts about the various K-groups of F, $\mathbb{O}$, C and $\mathscr{C}$, or F', $\mathbb{O}'$, C' and $\mathscr{C}'$, including those mentioned in the introduction. The two cases are very similar so we shall treat them together.

We shall first consider the case where $F = k(C')$ for a smooth, projective curve $\mathscr{C}'$ over R' with geometrically irreducible generic fiber C'. Let $\mathscr{C}'_{\mathbb{F}}$ be the special fiber of $\mathscr{C}'$, which is a smooth, projective curve over the finite field $\mathbb{F}$. Because $\mathscr{C}'_{\mathbb{F}}$ is regular, there is an exact localization sequence

$$(2.2) \qquad \cdots \to K_4^{(2)}(\mathbb{F}(\mathscr{C}'_{\mathbb{F}})) \to K_4^{(3)}(\mathbb{O}') \to K_4^{(3)}(F') \to K_3^{(2)}(\mathbb{F}(\mathscr{C}'_{\mathbb{F}})) \to \cdots .$$

By [Harder 1977, Korollar 2.3.2], $K_n(L)$ is torsion for $n \geq 2$ for all function fields L of curves over finite fields, so in particular, $K_4^{(3)}(\mathbb{O}') \xrightarrow{\simeq} K_4^{(3)}(F')$. If $F = K(C)$, then we get

$$\cdots \to K_4^{(2)}(\kappa(\mathscr{C}_\kappa)) \to K_4^{(3)}(\mathbb{O}) \to K_3^{(2)}(F) \to K_3^{(2)}(\kappa(\mathscr{C}_\kappa)) \to \cdots .$$

By our assumptions (see Notation 1.15), $\kappa \subseteq \overline{\mathbb{F}}_p$. According to [Quillen 1973, Proposition 2.2] or [Srinivas 1996, Lemma 5.9], $K_n(\kappa(\mathscr{C}_\kappa))$ is the direct limit of K_n of function fields of curves over finite fields, hence is torsion as well, and we find $K_4^{(3)}(\mathbb{O}) \simeq K_4^{(3)}(F)$.

From the exact localization sequence

$$\cdots \to \coprod_{x\in\mathscr{C}_{\mathbb{F}}^{\prime(1)}} K_n^{(1)}(\mathbb{F}(x)) \to K_n^{(2)}(\mathscr{C}'_{\mathbb{F}}) \to K_n^{(2)}(\mathbb{F}(\mathscr{C}'_{\mathbb{F}})) \to \cdots$$

and the fact that $K_n^{(1)}(L)$ is zero for any field L for $n \geq 2$, we see that $K_n^{(2)}(\mathscr{C}'_{\mathbb{F}})$ is trivial for $n \geq 2$. From the exact localization sequence

$$\cdots \to K_4^{(2)}(\mathscr{C}'_{\mathbb{F}}) \to K_4^{(3)}(\mathscr{C}') \to K_4^{(3)}(C') \to K_3^{(2)}(\mathscr{C}'_{\mathbb{F}}) \to \cdots$$

we see that $K_n^{(2)}(\mathscr{C}'_{\mathbb{F}})$ is trivial for $n \geq 2$, hence $K_4^{(3)}(\mathscr{C}') \simeq K_4^{(3)}(C')$. Using a direct limit argument as before, we then see that $K_4^{(3)}(\mathscr{C}) \simeq K_4^{(3)}(C)$ as well.

Remark 2.3. We now have two identifications fitting into a commutative diagram

$$\begin{tikzcd} K_4^{(3)}(\mathscr{C}') \arrow[r] \arrow[d, equal] & K_4^{(3)}(\mathbb{O}') \arrow[d, equal] \\ K_4^{(3)}(C') \arrow[r] & K_4^{(3)}(F') \end{tikzcd}$$

and similarly for F, $\mathbb{O}$, $\mathscr{C}$ and C. From the exact localization sequence

$$\cdots \to \coprod_{x\in C'^{(1)}} K_4^{(2)}(k(x)) \to K_4^{(3)}(C') \to K_4^{(3)}(F') \xrightarrow{\partial} \coprod_{x\in C'^{(1)}} K_3^{(2)}(k(x)) \to \cdots$$

we see that the map $K_4^{(3)}(F') \to K_4^{(3)}(C')$ is injective because $K_4^{(2)}(L) = 0$ for any number field L. Hence the map $K_4^{(3)}(\mathscr{C}') \to K_4^{(3)}(\mathbb{O}')$ is also injective.

Remark 2.4. We have

$$K_4^{(3)}(C') \oplus K_3^{(2)}(k) \cup F'^{*}_{\mathbb{Q}} \quad \text{inside} \;\; K_4^{(3)}(F').$$

(This makes sense because $F'^{*}_{\mathbb{Q}} = K_1^{(1)}(F')$.) Namely, $K_4^{(3)}(C) = \mathrm{Ker}(\partial)$ in the localization sequence in Remark 2.3. On the other hand, for f in $F^*_{\mathbb{Q}}$ and α in $K_3^{(2)}(k)$, $\partial(\alpha \cup f) = \alpha \cup \mathrm{div}(f)$ in $\coprod_{x\in C^{(1)}} k(x)^*_{\mathbb{Q}}$, hence this is trivial only if f is in $k^*_{\mathbb{Q}}$. But

$$K_3^{(2)}(k) \cup k^*_{\mathbb{Q}} \subseteq K_4^{(3)}(k),$$

which is zero since k is a number field. Therefore $K_3^{(2)}(F) \cup F^*_{\mathbb{Q}}$ injects into $\coprod_{x\in C^{(1)}} k(x)^*_{\mathbb{Q}}$ under ∂.

Remark 2.5. Note that a local parameter of R' is also a local parameter for $\mathbb{O}'$, so F'^{*} is generated by $\mathbb{O}'^{*}$ and that local parameter. This implies that

$$K_3^{(2)}(k) \cup \mathbb{O}'^{*}_{\mathbb{Q}} = K_3^{(2)}(k) \cup F'^{*}_{\mathbb{Q}},$$

again because $K_3^{(2)}(k) \cup k^*_{\mathbb{Q}}$ is trivial.

We shall need the following result at several places later on.

Proposition 2.6. *For a discrete valuation ring $\mathbb{O}$, with residue field κ and field of fractions F, for all $n \geq 1$, the sequence $\mathbb{O}_{\mathbb{Q}}^{*\,\otimes n} \to K_n^{(n)}(F) \to K_{n-1}^{(n-1)}(\kappa) \to 0$ is exact.*

Proof. Since $K_n^{(n)}(L) \simeq K_n^M(L)_{\mathbb{Q}}$ for any field L by [Soulé 1985, Théorème 2], with $K_n^M(L)$ the Milnor K-theory of L, it suffices to show that

$$(\mathbb{O}^*)^{\otimes_{\mathbb{Z}} n} \to K_n^M(F) \to K_{n-1}^M(\kappa) \to 0$$

is exact. If π is a uniformizer of $\mathbb{O}$, then $K_n^M(F)$ is generated by symbols $\{u_1, \ldots, u_n\}$ and $\{u_1, \ldots, u_{n-1}, \pi\}$, with all u_j in $\mathbb{O}^*$. The map $K_n^M(F) \to K_{n-1}^M(\kappa)$ is the tame symbol, which is trivial on the first type of generator, and maps the second to $\{\bar{u}_1, \ldots, \bar{u}_{n-1}\}$. It is clearly surjective. So we only have to show that if α in $(\mathbb{O}^*)^{\otimes_{\mathbb{Z}}(n-1)}$ maps to the trivial element under the composition

$$(\mathbb{O}^*)^{\otimes_{\mathbb{Z}}(n-1)} \to (\kappa^*)^{\otimes_{\mathbb{Z}}(n-1)} \to K_{n-1}^M(\kappa),$$

then the image of $\alpha \otimes \pi$ in $K_n^M(F)$ is in the image of $(\mathbb{O}^*)^{\otimes_{\mathbb{Z}} n}$. Noticing that the Steinberg relations $\cdots \otimes x \otimes \cdots \otimes (1-x) \otimes \cdots$ in $(\mathbb{O}^*)^{\otimes_{\mathbb{Z}}(n-1)}$ surject onto those in $(\kappa^*)^{\otimes_{\mathbb{Z}}(n-1)}$, we see that we may assume that α is in the kernel of the map $(\mathbb{O}^*)^{\otimes_{\mathbb{Z}}(n-1)} \to (\kappa^*)^{\otimes_{\mathbb{Z}}(n-1)}$. From the exact sequence

$$1 \to 1 + \mathbb{O}\pi \to \mathbb{O}^* \to \kappa \to 1$$

and the fact that, if we have exact sequences $0 \to A_i \to B_i \to C_i \to 0$ $(i = 1, \ldots, m)$ of Abelian groups, then the kernel of $B_1 \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} B_m \to C_1 \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} C_m$ is the image of $A_1 \otimes_{\mathbb{Z}} B_2 \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} B_m + B_1 \otimes_{\mathbb{Z}} A_2 \otimes_{\mathbb{Z}} B_3 \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} B_m + \cdots$, we see α lies in the image of

$$(1 + \mathbb{O}\pi) \otimes_{\mathbb{Z}} \mathbb{O}^* \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} \mathbb{O}^* + \mathbb{O}^* \otimes_{\mathbb{Z}} (1 + \mathbb{O}\pi) \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} \mathbb{O}^* + \cdots .$$

But each element $\{u_1, \ldots, u_{n-1}, \pi\}$ with all u_i in $\mathbb{O}^*$ and at least one of them in $1 + \mathbb{O}\pi$ lies in the image of $(\mathbb{O}^*)^{\otimes_{\mathbb{Z}} n}$. Namely, an element in $1 + \mathbb{O}\pi$ is of the form $1 - \pi^d u$ for some u in $\mathbb{O}^*$, $d > 0$. If $d = 1$ we can rewrite $\{\ldots, 1 - \pi u, \ldots, \pi\} = -\{\ldots, 1 - \pi u, \ldots, u\}$. If $d > 1$, then using that

$$\frac{1-\pi^d u}{1-\pi} = 1 - \pi \frac{\pi^{d-1}u - 1}{1-\pi},$$

we find that $\{\ldots, 1 - \pi^d u, \ldots, \pi\} = \{\ldots, 1 - \pi \frac{\pi^{d-1}u-1}{1-\pi}, \ldots, \pi\}$, which reduces to the case $d = 1$ as $(\pi^{d-1}u - 1)/(1-\pi)$ is in $\mathbb{O}^*$. $\square$

Assumption 2.7. Throughout the construction of the complexes in the various subsections below, we let F be a field of characteristic zero. In the constructions for complexes for $\mathbb{O}$, $\mathbb{O}$ will be a discrete valuation ring, with residue field κ and field of fractions F, which we assume to be of characteristic zero. We shall always assume that $|\kappa| > 2$, so that $\mathcal{O}^\flat$ is nonempty and $\langle \mathcal{O}^\flat \rangle = \mathcal{O}^*$.

2.3. ***A few more preliminaries.*** It will be convenient to introduce the notation $F^\flat = F^* \setminus \{1\}$, as well as $\mathcal{O}^\flat = \{u \text{ in } \mathcal{O}^* \text{ such that } 1-u \text{ is in } \mathcal{O}^*\}$, and $\kappa^\flat = \kappa^* \setminus \{1\}$.

Throughout the remainder of Section 2, we shall let X_F^{loc} be the scheme obtained from $X_F = \mathbb{P}_F^1 \setminus \{t = 1\}$ by removing all points $t = u$ with u in $F^\flat$. We write $X_F^{2,\mathrm{loc}}$ for $(X_F^{\mathrm{loc}})^2$. Similarly, we let $X_\mathcal{O} = \mathbb{P}_\mathcal{O}^1 \setminus \{t = 1\}$, we write $X_\mathcal{O}^{\mathrm{loc}}$ for the scheme obtained from $X_\mathcal{O}$ by removing all subschemes $t = u$ with u in $\mathcal{O}^\flat$, and we write $X_\mathcal{O}^{2,\mathrm{loc}}$ for $(X_\mathcal{O}^{\mathrm{loc}})^2$. Finally, for κ, we let $K_\kappa = \mathbb{P}_\kappa^1 \setminus \{t = 1\}$, we write X_κ^{loc} for the scheme obtained from X_κ by removing all subschemes $t = u$ with u in $\kappa^\flat$, and we write $X_\kappa^{2,\mathrm{loc}}$ for $(X_\kappa^{\mathrm{loc}})^2$. (Of course, we would have to remove such a closed subscheme for only a finite set of u's first, and then take a direct limit. But by [Quillen 1973, Proposition 2.4] and some exact sequences in relative K-theory this will give us the K-theory of X_κ^{loc} anyway. Moreover, as such a direct limit over finite subsets of $\mathcal{O}^\flat$ or $F^\flat$ is clearly filtered, hence exact, this procedure will commute with taking spectral sequences, etc., below, so that we work directly in the direct limit.)

Since writing $\{t = 0, \infty\}$ or $\{t_1 = 0, \infty\}; \{t_2 = 0, \infty\}$ can be rather too long in places, we often abbreviate the first by writing $\square$, and the second by writing $\square^2$.

Let $(1+I)^* = K_1^{(1)}(X_F^{\mathrm{loc}}; \square)$. From the exact sequence

$$\cdots \to K_2^{(1)}(\square) \to K_1^{(1)}(X_F^{\mathrm{loc}}; \square) \to K_1^{(1)}(X_F^{\mathrm{loc}}) \to K_1^{(1)}(\square) \to \cdots$$

we see that $(1+I)^* \subset K_1^{(1)}(X_F^{\mathrm{loc}})$ as $K_2^{(1)}(\square) \simeq K_2^{(1)}(F)^{\oplus 2} = 0$. So we can describe $(1+I)^*$ explicitly as those elements in $K_1^{(1)}(X_F^{\mathrm{loc}})$ that restrict to 1 at $t = 0$ and $t = \infty$. Because $K_1(X_F^{\mathrm{loc}})$ is given by the units in the ring corresponding to a localization of the affine line, we find that

$$(1+I)^* = \left\{ \prod_j \left(\frac{t-u_j}{t-1} \right)^{n_j} \text{ with } u_j \text{ in } F^\flat,\, n_j \text{ in } \mathbb{Z}, \text{ such that } \prod_j u_j^{n_j} = 1 \right\} \otimes_\mathbb{Z} \mathbb{Q}\,.$$

Note that in particular the divisor map

$$(1+I)^* \to \coprod_{t \in F^\flat} K_0^{(0)}(F) \tag{2.8}$$

is an injection.

Note that, if A is any $\mathbb{Q}$-subspace of $K_n^{(l)}(X_F^{\mathrm{loc}}; \square)$, and we use the cup product $(1+I)^* \cup A \to K_{n+1}^{(l+1)}(X_F^{2,\mathrm{loc}}; \square^2)$ by pulling $(1+I)^*$ back along the first projection, and A along the second, then $\mathrm{d}((1+I)^* \cup A) = (\mathrm{d}(1+I)^*) \cup A - (1+I)^* \cup (\mathrm{d}A)$, and $\coprod_{t_1 \in F^\flat} A/(\mathrm{d}(1+I)^*) \cup A \simeq A \otimes F_\mathbb{Q}^*$ because $F^\flat$ generates F^*, and the functions in $(1+I)^*$ (without $\cdots \otimes_\mathbb{Z} \mathbb{Q}$) give exactly the multiplicative relations among the elements in $F^\flat$. Of course, by reversing the role of the projections we can do this with t_2 instead of t_1 instead. This will be used in order to change $\coprod_{t\in F^\flat}\cdots$ into $\cdots\otimes_{\mathbb{Q}} F^*_{\mathbb{Q}}$ in localization sequences or spectral sequences below.

Under Assumption 2.7, we can do the same for $\mathbb{O}$. Namely, define

$$(1+I)^*_{\mathbb{O}} = K_1^{(1)}(X^{\mathrm{loc}}_{\mathbb{O}};\square).$$

Because $K_2^{(1)}(\mathbb{O})=0$ and $K_1^{(1)}(\mathbb{O})=\mathbb{O}^*_{\mathbb{Q}}$, one sees by exactly the same argument as for $(1+I)^*$ that

$$(2.9)\quad (1+I)^*_{\mathbb{O}} = \Big\{\prod_j\Big(\frac{t-u_j}{t-1}\Big)^{n_j}\ \Big|\ u_j \text{ in } \mathbb{O}^\flat,\ n_j \text{ in } \mathbb{Z}, \text{ such that } \prod_j u_j^{n_j}=1\Big\}\otimes_{\mathbb{Z}}\mathbb{Q}\,.$$

In particular, we have $(1+I)^*_{\mathbb{O}}\subseteq(1+I)^*$ under localization of the base from $\mathbb{O}$ to F. Note that we used here that $(1+I)^*_{\mathbb{O}}$ gives us exactly the relations needed to turn $\coprod_{t\in\mathbb{O}^\flat}\cdots$ into $\cdots\otimes\mathbb{O}^*_{\mathbb{Q}}$, as $(1+I)^*_{\mathbb{O}}$ (without $\cdots\otimes_{\mathbb{Z}}\mathbb{Q}$) gives the multiplicative relations among elements in $\mathbb{O}^\flat$, and $\mathbb{O}^\flat$ generates $\mathbb{O}^*$.

Finally, we like to mention that for x in F, under the map

$$K_0^{(0)}(F)_{|t=x}\to K_0^{(1)}(X_F;\square)\simeq F^*_{\mathbb{Q}},$$

1 is mapped to $x^{\pm1}$; see [de Jeu 1995, Lemma 3.14]. The same holds for $\mathbb{O}$ instead of F, and this is compatible with products.

2.4. ***Construction of the complexes for F and C′.*** Several parts of the constructions of the complexes in this section and in Section 2.5 below were carried out in earlier papers [de Jeu 1995; 1996; Besser and de Jeu 2003], but we review them so that we can refer to the relevant details in some new constructions for $\mathbb{O}$ and in the calculations relating to regulators in later sections. Also, in various cases the constructions were carried out more generally, in which case they tend to become dependent on assumptions on weights in K-theory, and our exposition below will avoid such assumptions.

2.4.1. *Construction of the complexes $\mathcal{M}_{(2)}(F)$ and $\widetilde{\mathcal{M}}_{(2)}(F)$.* The principle of the construction of the complex $\mathcal{M}_{(2)}(F)$ was first used in Bloch's Irvine notes (finally published as [Bloch 2000]). The construction of $\mathcal{M}_{(2)}(F)$ and $\widetilde{\mathcal{M}}_{(2)}(F)$ can be found in [de Jeu 1995, Section 3].

We start with the localization sequence

$$(2.10)\quad \cdots\to\coprod_{t\in F^\flat}K_2^{(1)}(F)\to K_2^{(2)}(X_F;\square)\to K_2^{(2)}(X_F^{\mathrm{loc}};\square)\to \coprod_{t\in F^\flat}K_1^{(1)}(F)\to K_1^{(2)}(X_F;\square)\to\cdots.$$

Because $K_2^{(1)}(F)=0$ for any field F by (2.1), this means that the cohomological complex (in degrees 1 and 2)

$$RC_{(2)}(F) : K_2^{(2)}(X_F^{\mathrm{loc}}; \square) \to \coprod_{t \in F^\flat} K_1^{(1)}(F) \tag{2.11}$$

has cohomology groups $H^1(RC_{(2)}(F)) \simeq K_3^{(2)}(F)$ and $H^2(RC_{(2)}(F)) \simeq K_2^{(2)}(F)$.

In [de Jeu 1995, Section 3.2] (see also [Bloch 1990]), for every x in $F^\flat$ an element $[x]_2$ was constructed in $K_2^{(2)}(X_F^{\mathrm{loc}}; \square)$ with the property that its boundary in $\coprod K_1^{(1)}(F)$ is $(1-x)^{-1}_{|t=x}$. Let

$$\begin{aligned} \mathrm{Symb}_1(F) &= K_1^{(1)}(F) = F_{\mathbb{Q}}^*, \\ \mathrm{Symb}_2(F) &= \langle [x]_2 \text{ with } x \text{ in } F^\flat \rangle_{\mathbb{Q}} + (1+I)^* \cup \mathrm{Symb}_1(F). \end{aligned}$$

Then we get a subcomplex of (2.11):

$$\mathit{Symb}_2(F) : \mathrm{Symb}_2(F) \to \coprod_{t \in F^\flat} \mathrm{Symb}_1(F). \tag{2.12}$$

Letting $F_{\mathbb{Q}}^*$ act on the right in (2.8) gives the subcomplex

$$(1+I)^* \cup F_{\mathbb{Q}}^* \to \mathrm{d}(\cdots), \tag{2.13}$$

which is acyclic by [de Jeu 1995, Lemma 3.7]. Taking the quotient of (2.12) by (2.13), we obtain the complex

$$\mathcal{M}_{(2)}(F) : M_2(F) \to F_{\mathbb{Q}}^* \otimes F_{\mathbb{Q}}^*,$$

where we used that $\mathrm{d}(1+I)^*$ gives exactly the right relations to turn $\coprod_{t \in F^\flat} \cdots$ into $\cdots \otimes F_{\mathbb{Q}}^*$, as $F^\flat$ generates F^*, and

$$M_2(F) = \mathrm{Symb}_2(F)/(1+I)^* \cup \mathrm{Symb}_1(F) = \mathrm{Symb}_2(F)/(1+I)^* \cup F_{\mathbb{Q}}^*.$$

Then $M_2(F)$ is a $\mathbb{Q}$ -vector space generated by the $[x]_2$, x in $F^\flat$, and the boundary of $[x]_2$ is $(1-x) \otimes x$.

Note that from the maps $\mathcal{M}_{(2)}(F) \leftarrow \mathit{Symb}_2(F) \to RC_{(2)}(F)$, with the left one a quasiisomorphism, we obtain maps

$$H^i(\mathcal{M}_{(2)}(F)) \to K_{4-i}^{(2)}(F)$$

for $i = 1$ and 2. The map for $i = 1$ is an injection as the corresponding statement holds for $RC_{(2)}(F)$ and $\mathit{Symb}_2(F)$ is a subcomplex, and we are in the lowest degree. For $i = 2$ the map is an isomorphism because $K_2^{(2)}(F)$ is the quotient of $F_{\mathbb{Q}}^* \otimes F_{\mathbb{Q}}^*$ by $\langle x \otimes (1-x) \text{ with } x \text{ in } F^\flat \rangle$.

We shall quotient out the complex $\mathcal{M}_{(2)}(F)$ in order to end up with a second term $\bigwedge^2 F_{\mathbb{Q}}^*$ rather than $F_{\mathbb{Q}}^* \otimes F_{\mathbb{Q}}^*$. The shape of the quotient complexes $\widetilde{\mathcal{M}}_{(2)}(F)$ here and $\widetilde{\mathcal{M}}_{(3)}(F)$ in Section 2.4.2 is more in line with conjectures; see for instance

[Goncharov 1994, Conjecture 2.1]. Besides, the definition of complex $\mathcal{M}_{(3)}(C')$ depends on the complexes $\widetilde{\mathcal{M}}_{(2)}(L)$ for number fields L.

Namely, consider the subcomplex of $\mathcal{M}_{(2)}(F)$

$$N_2(F) \to \mathrm{d}(\cdots) \tag{2.14}$$

with

$$N_2(F) = \langle [u]_2 + [u^{-1}]_2 \text{ with } u \text{ in } F^\flat \rangle_{\mathbb{Q}} \subseteq M_2(F). \tag{2.15}$$

As $\mathrm{d}([x]_2 + [x^{-1}]_2) = x \otimes x$, the second term is in fact $\mathrm{Sym}^2(F^*_{\mathbb{Q}})$. By the proof of [de Jeu 1995, Corollary 3.22], (2.14) is acyclic. Taking the quotient complex we get

$$\widetilde{\mathcal{M}}_{(2)}(F) : \widetilde{M}_2(F) \to \bigwedge^2 F^*_{\mathbb{Q}}, \tag{2.16}$$

with $\widetilde{M}_2(F) = M_2(F)/N_2(F)$, and $\mathrm{d}[x]_2 = (1-x) \wedge x$.

Because $\widetilde{\mathcal{M}}_{(2)}(F)$ is quasiisomorphic to $\mathcal{M}_{(2)}(F)$ we have maps

$$H^i(\widetilde{\mathcal{M}}_{(2)}(F)) \to K^{(2)}_{4-i}(F). \tag{2.17}$$

Again this map is an injection for $i = 1$ and an isomorphism for $i = 2$.

There are essentially two ways of generalizing the complex $\mathcal{M}_{(2)}(F)$. The first one is to look at another part of the localization sequence (2.10), the other to replace X_F by X_F^n for $n \geq 2$, and use localization there, which will give a spectral sequence. The first will be used to construct the complex $\mathcal{C}^\bullet(F)$ in Section 2.4.4 below, the second (with $n = 2$) will be used for constructing the complex $\mathcal{M}_{(3)}(F)$ below.

2.4.2. *Construction of the complexes $\mathcal{M}_{(3)}(F)$ and $\widetilde{\mathcal{M}}_{(3)}(F)$.* Those complexes were also defined in [de Jeu 1995, Section 3]. The complex $\mathcal{M}_{(3)}(F)$ consists of three terms in cohomological degrees 1, 2 and 3,

$$M_3(F) \to M_2(F) \otimes F^*_{\mathbb{Q}} \to F^*_{\mathbb{Q}} \otimes \bigwedge^2 F^*_{\mathbb{Q}}, \tag{2.18}$$

and comes equipped with maps

$$H^2(\mathcal{M}_{(3)}(F)) \to K_4^{(3)}(F) \quad \text{and} \quad H^3(\mathcal{M}_{(3)}(F)) \to K_3^{(3)}(F).$$

The last of those two maps is in fact an isomorphism.

Although we shall need a similar complex $\mathcal{M}_{(3)}(\mathbb{O})$ in order to have information about the special fiber, we describe the complex $\mathcal{M}_{(3)}(F)$ first, as it is notationally easier. Moreover, in the part of the complex we are interested in, we can view $\mathcal{M}_{(3)}(\mathbb{O})$ as a subcomplex of $\mathcal{M}_{(3)}(F)$ (see Remark 2.45).

Consider the divisors on X_F^2 defined by putting $t_i = u_j$ for some u_j in $F^\flat$ for $i = 1$ or 2. Then there is a spectral sequence (see [de Jeu 1996, page 257; de Jeu

1995, page 221])

(2.19)
$$
\begin{array}{ccc}
\vdots & \vdots & \vdots \\
K_2^{(3)}(X_F^{2,\mathrm{loc}};\square^2) & \coprod_{t_1\in F^\flat} K_1^{(2)}(X_F^{\mathrm{loc}};\square) \amalg \coprod_{t_2\in F^\flat} K_1^{(2)}(X_F^{\mathrm{loc}};\square) & \coprod_{t_1,t_2\in F^\flat} K_0^{(1)}(F) \\
K_3^{(3)}(X_F^{2,\mathrm{loc}};\square^2) & \coprod_{t_1\in F^\flat} K_2^{(2)}(X_F^{\mathrm{loc}};\square) \amalg \coprod_{t_2\in F^\flat} K_2^{(2)}(X_F^{\mathrm{loc}};\square) & \coprod_{t_1,t_2\in F^\flat} K_1^{(1)}(F) \\
K_4^{(3)}(X_F^{2,\mathrm{loc}};\square^2) & \coprod_{t_1\in F^\flat} K_3^{(2)}(X_F^{\mathrm{loc}};\square) \amalg \coprod_{t_2\in F^\flat} K_3^{(2)}(X_F^{\mathrm{loc}};\square) & \coprod_{t_1,t_2\in F^\flat} K_2^{(1)}(F) \\
\vdots & \vdots & \vdots
\end{array}
$$

converging to $K_*^{(3)}(X_F^2;\square^2) \simeq K_{*+2}^{(3)}(F)$. The only terms in it that contribute to $K_4^{(3)}(F)$ are

$$
K_2^{(3)}(X_F^{2,\mathrm{loc}};\square^2) \quad \text{and} \quad \coprod_{t_1\in F^\flat} K_2^{(2)}(X_F^{\mathrm{loc}};\square) \amalg \coprod_{t_2\in F^\flat} K_2^{(2)}(X_F^{\mathrm{loc}};\square)
$$

because $\coprod_{t_1,t_2\in F^\flat} K_1^{(2)}(F)$ is trivial. Let $RC_{(3)}(F)$ be the cohomological complex in degrees 1, 2 and 3, consisting of the row in (2.19) that begins with $K_3^{(3)}(X_F^{2,\mathrm{loc}};\square^2)$:

$$
\begin{aligned}
(2.20)\quad RC_{(3)}(F) : K_3^{(3)}&(X_F^{2,\mathrm{loc}};\square^2) \\
&\to \coprod_{t_1\in F^\flat} K_2^{(2)}(X_F^{\mathrm{loc}};\square) \amalg \coprod_{t_2\in F^\flat} K_2^{(2)}(X_F^{\mathrm{loc}};\square) \to \coprod_{t_1,t_2\in F^\flat} K_1^{(1)}(F).
\end{aligned}
$$

This complex was denoted $C_{(3)}$ in [de Jeu 1995, Section 3.1], but considering the notational overload of the letter C in this paper, we prefer to think of it as a row complex rather than just a complex.

Note that $K_1^{(2)}(F)$ equals zero, so for $i = 2$ and 3 there is a map

$$
H^i(RC_{(3)}(F)) \to K_{6-i}^{(3)}(F)\,. \tag{2.21}
$$

For x in $F^\flat$, in addition to the element $[x]_2$ in $K_2^{(2)}(X_F^{\mathrm{loc}};\square)$ of Section 2.4.1, there is also an element $[x]_3$ in $K_3^{(3)}(X_F^{2,\mathrm{loc}};\square^2)$ (see [de Jeu 1995, Section 3.2]) with boundary

$$
-[x]_{2|t_1=x} + [x]_{2|t_2=x} \quad \text{in} \quad \coprod_{t_1\in F^\flat} K_2^{(2)}(X_F^{\mathrm{loc}};\square) \amalg \coprod_{t_2\in F^\flat} K_2^{(2)}(X_F^{\mathrm{loc}};\square)
$$

in (2.19). Let us define $\mathrm{Symb}_n(F) \subseteq K_n^{(n)}(X_F^{n-1,\mathrm{loc}}; \square^{n-1})$ for $n = 1$, 2 and 3 by setting

$$\begin{aligned}
\mathrm{Symb}_1(F) &= F^*_{\mathbb{Q}},\\
\mathrm{Symb}_2(F) &= \langle [u]_2 \text{ with } u \text{ in } F^\flat \rangle_{\mathbb{Q}} + (1+I)^* \cup \mathrm{Symb}_1(F),\\
\mathrm{Symb}_3(F) &= \langle [u]_3 \text{ with } u \text{ in } F^\flat \rangle_{\mathbb{Q}} + (1+I)^* \,\tilde{\cup}\, \mathrm{Symb}_2(F).
\end{aligned}$$

For $n \le 2$, those are the definitions given in Section 2.4.1, and for $n = 3$, by $\tilde{\cup}$ we mean the following. In the projection X_F^2 to X_F, we can use one of the factors to pull back $(1+I)^*$, the other to pull back $\mathrm{Symb}_2(F)$ and then take the product to land in $\mathrm{Symb}_3(F)$, giving us two cup products. The $\tilde{\cup}$ indicates that we take the sum of the images of both possibilities for those cup products.

Because, in (2.20), $\mathrm{d}[u]_2 = (1-u)^{-1}_{|t=u}$ and $\mathrm{d}[u]_3 = -[u]_{2|t_1=u} + [u]_{2|t_2=u}$, it follows that

$$\begin{aligned}
(2.22)\quad & \mathit{Symb}_{(3)}(F) : \mathrm{Symb}_3(F)\\
&\to \coprod_{t_1\in F^\flat} \mathrm{Symb}_2(F) \coprod \coprod_{t_2\in F^\flat} \mathrm{Symb}_2(F) \to \coprod_{t_1,t_2\in F^\flat} \mathrm{Symb}_1(F)
\end{aligned}$$

is a subcomplex of (2.20). It is shown in [de Jeu 1995, Lemma 3.9 and Remark 3.10] that the subcomplex

$$\begin{aligned}
(2.23)\quad & (1+I)^* \,\tilde{\cup}\, \mathrm{Symb}_2(F)\\
&\to \coprod_{t_1\in F^\flat} (1+I)^* \cup F^*_{\mathbb{Q}} \coprod \coprod_{t_2\in F^\flat} (1+I)^* \cup F^*_{\mathbb{Q}} + \mathrm{d}(\cdots) \to \mathrm{d}(\cdots)
\end{aligned}$$

of (2.22) is acyclic.

S_2 acts on the spectral sequence (2.19) by swapping t_1 and t_2. It therefore also acts on the complex (2.20) above. Because the symbol $[x]_3$ is alternating by construction (see [de Jeu 1995, Section 3.2]), we can take the alternating parts of (2.22) and (2.23), and form the quotient complex

$$\mathcal{M}_{(3)}(F) : M_3(F) \to M_2(F) \otimes F^*_{\mathbb{Q}} \to F^*_{\mathbb{Q}} \otimes \textstyle\bigwedge^2 F^*_{\mathbb{Q}},$$

where

$$\begin{aligned}
M_3(F) &= \mathrm{Symb}_3(F)/((1+I)^* \,\tilde{\cup}\, \mathrm{Symb}_2(F))^{\mathrm{alt}},\\
M_2(F) &= \mathrm{Symb}_2(F)/(1+I)^* \cup F^*_{\mathbb{Q}},
\end{aligned}$$

as before in Section 2.4.1. Note that, for $n = 2$ and 3, $M_n(F)$ is a $\mathbb{Q}$-vector space on symbols $[x]_n$ for x in $F^\flat$, modulo nonexplicit relations depending on n. The maps in the complex are given by $\mathrm{d}[x]_3 = [x]_2 \otimes x$ and

$$\mathrm{d}[x]_2 \otimes y = (1-x) \otimes (x \wedge y). \tag{2.24}$$

As before, we used here that $\mathrm{d}(1+I)^*$ gives exactly the right relations to turn $\coprod_{t\in F^\flat}\cdots$ into $\cdots\otimes F^*_{\mathbb{Q}}$, as $F^\flat$ generates F^*. As $Symb_{(3)}(F)$ is a subcomplex of $RC_{(3)}(F)$, this gives us maps

$$\mathcal{M}_{(3)}(F) \leftarrow Symb_{(3)}(F)^{\mathrm{alt}} \rightarrow RC_{(3)}(F)^{\mathrm{alt}} \rightarrow RC_{(3)}(F)$$

with the left map a quasiisomorphism. Combining this with (2.21) gives us a map

$$H^i(\mathcal{M}_{(3)}(F)) \to K^{(3)}_{6-i}(F) \tag{2.25}$$

for $i=2$ and 3. (For $i=1$, starting with $H^1(RC_{(3)}(F)) \to K_5^{(3)}(F)/K_4^{(2)}(F)\cup F^*_{\mathbb{Q}}$, we still obtain a map $H^1(\mathcal{M}_{(3)}(F)) \to K_5^{(3)}(F)/K_4^{(2)}(F)\cup F^*_{\mathbb{Q}}$.)

Finally, we quotient out $\mathcal{M}_{(3)}(F)$ in order to obtain $\widetilde{\mathcal{M}}_{(3)}(F)$, as follows. Let

$$N_3(F) = \langle [u]_3 - [u^{-1}]_3 \text{ with } u \text{ in } F^\flat\rangle_{\mathbb{Q}} \subseteq M_3(F)$$

(cf. (2.15); in general $N_n(F)$ is generated by the $[u]_n + (-1)^n[u^{-1}]_n$) and consider the subcomplex

$$N_3(F) \to N_2(F)\otimes F^*_{\mathbb{Q}} \to \mathrm{d}(\cdots) \tag{2.26}$$

of $\mathcal{M}_{(3)}(F)$. By the proofs of [de Jeu 1995, Proposition 3.20, Corollary 3.22] it is acyclic in degrees 2 and 3, hence for the quotient complex

$$\widetilde{\mathcal{M}}_{(3)}(F) : \widetilde{M}_3(F) \to \widetilde{M}_2(F)\otimes F^*_{\mathbb{Q}} \to \textstyle\bigwedge^3 F^*_{\mathbb{Q}}\,,$$

where $\widetilde{M}_3(F) = M_3(F)/N_3(F)$, we get a map

$$H^i(\widetilde{\mathcal{M}}_{(3)}(F)) \xleftarrow{\simeq} H^i(\mathcal{M}_{(3)}(F)) \to K^{(3)}_{6-i}(F)\,. \tag{2.27}$$

In $\widetilde{M}_3(F)$ we still denote the class of $[x]_i$ with $[x]_i$, so that the maps are now given by $\mathrm{d}[u]_3 = [u]_2\otimes u$ and $\mathrm{d}[u]_2\otimes v = (1-u)\wedge u\wedge v$.

The next remark, lemma, and corollary will be used in Section 10 to define the various maps in the theorems in the introduction.

Remark 2.28. Consider the map

$$\begin{aligned}\Phi : (F^*_{\mathbb{Q}})^{\otimes 3} &\to \mathrm{Sym}^2(F^*_{\mathbb{Q}})\otimes F^*_{\mathbb{Q}}\\ a\otimes b\otimes c &\mapsto \tfrac{2}{3}((a\cdot b)\otimes c - (a\cdot c)\otimes b),\end{aligned}$$

where $a_1\cdot a_2 = \frac{1}{2}(a_1\otimes a_2 + a_2\otimes a_1)$ in $\mathrm{Sym}^2(F^*_{\mathbb{Q}})$. Up to scaling, Φ is the composition of antisymmetrizing in the last two factors, followed by symmetrizing in the first two factors, so it is trivial on $F^*_{\mathbb{Q}}\otimes\mathrm{Sym}^2(F^*_{\mathbb{Q}})$. It is easy to check that $\Phi\circ\Phi=\Phi$ and Φ maps a generator $(a\cdot a)\otimes c$ of $\mathrm{Sym}^2(F^*_{\mathbb{Q}})\otimes F^*_{\mathbb{Q}}$ to itself modulo $\mathrm{Sym}^3(F^*_{\mathbb{Q}})$. In particular, $\mathrm{id}-\Phi$ maps $\mathrm{Sym}^2(F^*_{\mathbb{Q}})\otimes F^*_{\mathbb{Q}} + F^*_{\mathbb{Q}}\otimes\mathrm{Sym}^2(F^*_{\mathbb{Q}})$ to $F^*_{\mathbb{Q}}\otimes\mathrm{Sym}^2(F^*_{\mathbb{Q}})$.

For $\tilde{\alpha}$ in $\widetilde{M}_2(F)\otimes F^*_{\mathbb{Q}}$, let α be a lift of $\tilde{\alpha}$ to $M_2(F)\otimes F^*_{\mathbb{Q}}$, so that $(\mathrm{d}\otimes\mathrm{id})(\alpha)$ is in $(F^*_{\mathbb{Q}})^{\otimes 3}$. Because of the statements just after (2.14), there is a unique β_α in

$N_2(F)\otimes F_{\mathbb{Q}}^* \subset M_2(F)\otimes F_{\mathbb{Q}}^*$ with $\Phi\circ(\mathrm{d}\otimes\mathrm{id})(\alpha)=(\mathrm{d}\otimes\mathrm{id})(\beta_\alpha)$. By definition, α is unique up to adding β' in $N_2(F)\otimes F_{\mathbb{Q}}^*$. But

$$\Phi\circ(\mathrm{d}\otimes\mathrm{id})(\alpha+\beta')=\Phi\circ(\mathrm{d}\otimes\mathrm{id})(\alpha)+\Phi\circ(\mathrm{d}\otimes\mathrm{id})(\beta')=(\mathrm{d}\otimes\mathrm{id})(\beta_\alpha+\beta'+\gamma)$$

for some γ in $\mathrm{d}(N_3(F))=\langle([h]_2+[h^{-1}]_2)\otimes h\rangle\subset N_2(F)\otimes F_{\mathbb{Q}}^*$ as $(\mathrm{d}\otimes\mathrm{id})(\beta')$ is in $\mathrm{Sym}^2(F_{\mathbb{Q}}^*)\otimes F_{\mathbb{Q}}^*$, hence $(\Phi-\mathrm{id})\circ(\mathrm{d}\otimes\mathrm{id})(\beta')$ is in $\mathrm{Sym}^3(F_{\mathbb{Q}}^*)$. So $\beta_{\alpha+\beta'}=\beta_\alpha+\beta'+\gamma$, hence the class of $\alpha-\beta_\alpha$ is well-defined in $M_2(F)\otimes F_{\mathbb{Q}}^*/\mathrm{d}(N_3(F))$.

Let

$$\begin{aligned}\Xi:\widetilde{M}_2(F)\otimes F_{\mathbb{Q}}^* &\to M_2(F)\otimes F_{\mathbb{Q}}^*/\mathrm{d}(N_3(F))\\ \tilde\alpha &\mapsto \alpha-\beta_\alpha \text{ modulo } \mathrm{d}(N_3(F))\end{aligned}$$

be the resulting map, so α in $M_2(F)\otimes F_{\mathbb{Q}}^*$ lifts $\tilde\alpha$ and β_α in $N_2(F)\otimes F_{\mathbb{Q}}^*$ satisfies $\Phi\circ(\mathrm{d}\otimes\mathrm{id})(\alpha)=(\mathrm{d}\otimes\mathrm{id})(\beta_\alpha)$. Clearly, the quotient map $M_2(F)\otimes F_{\mathbb{Q}}^*\to\widetilde{M}_2(F)\otimes F_{\mathbb{Q}}^*$ gives a quotient map $M_2(F)\otimes F_{\mathbb{Q}}^*/\mathrm{d}(N_3(F))\to\widetilde{M}_2(F)\otimes F_{\mathbb{Q}}^*$, and Ξ is a section of the latter. Hence

$$M_2(F)\otimes F_{\mathbb{Q}}^*/\mathrm{d}(N_3(F))=\mathrm{im}(\Xi)\oplus N_2(F)\otimes F_{\mathbb{Q}}^*/\mathrm{d}(N_3(F)).$$

Now assume $\tilde\alpha$ is in the kernel of $\mathrm{d}:\widetilde{M}_2(F)\otimes F_{\mathbb{Q}}^*\to\bigwedge^3 F_{\mathbb{Q}}^*$. If α in $M_2(F)\otimes F_{\mathbb{Q}}^*$ lifts $\tilde\alpha$, then $(\mathrm{d}\otimes\mathrm{id})(\alpha)$ is in $\mathrm{Sym}^2(F_{\mathbb{Q}}^*)\otimes F_{\mathbb{Q}}^*+F_{\mathbb{Q}}^*\otimes\mathrm{Sym}^2(F_{\mathbb{Q}}^*)$. The same holds for $\eta=(\mathrm{d}\otimes\mathrm{id})(\alpha-\beta_\alpha)$ with $\alpha-\beta_\alpha$ any representative of $\Xi(\tilde\alpha)$, so that α lifts $\tilde\alpha$ and $(\mathrm{d}\otimes\mathrm{id})(\beta_\alpha)=\Phi\circ(\mathrm{d}\otimes\mathrm{id})(\alpha)$. Therefore $\eta-\Phi(\eta)$ is in $F_{\mathbb{Q}}^*\otimes\mathrm{Sym}^2(F_{\mathbb{Q}}^*)$. But $\Phi(\eta)=\Phi\circ(\mathrm{d}\otimes\mathrm{id})(\alpha)-\Phi\circ\Phi\circ(\mathrm{d}\otimes\mathrm{id})(\alpha)=0$, hence $\alpha-\beta_\alpha$ is in $(M_2(F)\otimes F_{\mathbb{Q}}^*)^{\mathrm{d}=0}$, and therefore Ξ maps $(\widetilde{M}_2(F)\otimes F_{\mathbb{Q}}^*)^{\mathrm{d}=0}$ to $(M_2\otimes F_{\mathbb{Q}}^*)^{\mathrm{d}=0}/\mathrm{d}(N_3(F))$. It is easy to check that $\Xi(\mathrm{d}(\widetilde{M}_3(F)))=\mathrm{d}(M_3(F))/\mathrm{d}(N_3(F))$, so that Ξ induces the inverse to the natural isomorphism $H^2(\mathcal{M}_{(3)}(F))\to H^2(\widetilde{\mathcal{M}}_{(3)}(F))$.

Lemma 2.29. *Let V be a $\mathbb{Q}$-vector space.*

(1) *Suppose we have a linear map $G:(F_{\mathbb{Q}}^*)^{\otimes 3}\to V$. Then the assignment*

$$[g]_2\otimes f\mapsto G((1-g)\otimes g\otimes f)$$

defines a linear map $\Psi:M_2(F)\otimes F_{\mathbb{Q}}^*\to V$.

(2) *If this Ψ is trivial on* $\mathrm{d}(N_3(F))$, *then $\Psi\circ\Xi$ maps $[g]_2\otimes f$ in $\widetilde{M}_2(F)\otimes F_{\mathbb{Q}}^*$ to*

$$G((1-g)\otimes g\otimes f)-\tfrac{2}{3}G(((1-g)\cdot g)\otimes f)+\tfrac{2}{3}G(((1-g)\cdot f)\otimes g),$$

where $a_1\cdot a_2=\frac{1}{2}(a_1\otimes a_2+a_2\otimes a_1)$ in $\mathrm{Sym}^2(F_{\mathbb{Q}}^*)\subset(F_{\mathbb{Q}}^*)^{\otimes 2}$.

(3) *Suppose that we have linear maps $\Psi:M_2(F)\otimes F_{\mathbb{Q}}^*/\mathrm{d}(N_3(F))\to V$ and $H:\mathrm{Sym}^2(F_{\mathbb{Q}}^*)\otimes F_{\mathbb{Q}}^*\to V$, such that $\Psi(([a]_2+[a^{-1}]_2)\otimes b)=H((a\cdot a)\otimes b)$. Then $\Psi\circ\Xi$ maps $[g]_2\otimes f$ in $\widetilde{M}_2(F)\otimes F_{\mathbb{Q}}^*$ to*

$$\Psi(g,f)-\tfrac{2}{3}H((1-g)\cdot g)\otimes f)+\tfrac{2}{3}H((1-g)\cdot f)\otimes g).$$

Proof. (1) The map Ψ is the composition of G with $\mathrm{d}\otimes\mathrm{id}$, with $\mathrm{d}: M_2(F) \to F_{\mathbb{Q}}^{*\otimes 2}$ the differential in $\mathcal{M}_{(2)}(F)$. For (2) and (3) we lift $\alpha = [g]_2 \otimes f$ in $\widetilde{M}_2(F) \otimes F_{\mathbb{Q}}^*$ to $[g]_2 \otimes f$ in $M_2(F) \otimes F_{\mathbb{Q}}^*$ to find $\Psi \circ \Xi([g]_2 \otimes f) = \Psi([g]_2 \otimes f) - \Psi(\beta)$, with $\beta = \beta_\alpha$, so it suffices to compute $\Psi(\beta)$. For (2) we find

$$\begin{aligned}\Psi(\beta) &= G(\mathrm{d}\otimes\mathrm{id}(\beta)) = G(\Phi(\mathrm{d}\otimes\mathrm{id}([g]_2 \otimes f))) = G(\Phi((1-g)\otimes g \otimes f)) \\ &= \tfrac{2}{3} G(((1-g)\cdot g)\otimes f) - \tfrac{2}{3} G(((1-g)\cdot f)\otimes g)\end{aligned}$$

For (3) we find the formula in a similar way by noting that β can be written as a sum of elements of the form $[a]_2 + [a^{-1}]_2$ and that $\mathrm{d}([a]_2 + [a^{-1}]_2) = a \cdot a$. $\square$

Corollary 2.30. *Under the assumptions in* (2) *and* (3) *of Lemma 2.29, the composition $H^2(\widetilde{\mathcal{M}}_{(3)}(F)) \to H^2(\mathcal{M}_{(3)}(F)) \to V$ is given by the corresponding formulas.*

2.4.3. *Construction of the complex $\mathcal{M}_{(3)}(C')$.* In this section we consider the situation where we have smooth, projective, geometrically irreducible curve C' over a number field k with function field $F' = k(C')$.

Because we are interested in finding elements in $K_4^{(3)}(C')$, we introduce yet another complex, $\mathcal{M}_{(3)}(C')$, which is the total complex associated to the double complex:

$$\begin{tikzcd} M_3(F') \arrow[r, "\mathrm{d}"] \arrow[d] & M_2(F') \otimes_{\mathbb{Q}} F'^*_{\mathbb{Q}} \arrow[r, "\mathrm{d}"] \arrow[d, "\partial_1"'] & F'^*_{\mathbb{Q}} \otimes \bigwedge^2 F'^*_{\mathbb{Q}} \arrow[d, "\partial_2"'] \\ 0 \arrow[r] & \coprod_x \widetilde{M}_2(k(x)) \arrow[r, "\mathrm{d}"] & \coprod_x \bigwedge^2 k(x)^*_{\mathbb{Q}} \end{tikzcd}$$

(Although not needed in this paper, one could define the complex $\widetilde{\mathcal{M}}_{(3)}(C')$ by using $\widetilde{\mathcal{M}}_{(3)}(F')$ in the top row.) Here the coproducts are over all closed points x of C'. The boundary maps are as follows. The d's in the top row are as in $\mathcal{M}_{(3)}(F')$. In the bottom row, $\mathrm{d}[z]_2 = (1-z)\wedge z$. For the vertical maps,

$$\partial_{1,x}([g]_2 \otimes f) = \mathrm{ord}_x(f)\cdot [g(x)]_2,$$

with the convention that $[0]_2 = [1]_2 = [\infty]_2 = 0$. Finally, $\partial_{2,x}$ described as follows. Let π be a uniformizer at x, u_j units at x. Then $\partial_{2,x}$ is determined by

$$\pi \wedge u_1 \wedge u_2 \mapsto u_1(x) \wedge u_2(x) \quad \text{and} \quad u_1 \wedge u_2 \wedge u_3 \mapsto 0\,.$$

Therefore, an element $\sum_i [g_i]_2 \otimes f_i$ in $H^2(\mathcal{M}_{(3)}(F'))$ satisfies

$$\sum_i (1-g_i) \otimes (g_i \wedge f_i) = 0$$

in $F'^*_{\mathbb{Q}} \otimes \bigwedge^2 F'^*_{\mathbb{Q}}$. The additional condition for it to lie in $H^2(\mathcal{M}_{(3)}(C'))$ is that $\sum_i \mathrm{ord}_x(f_i)[g_i(x)]_2 = 0$ in $\widetilde{M}_2(k(x))$ for all closed points x in C', with the convention that $[0]_2 = [1]_2 = [\infty]_2 = 0$.

We have an obvious map $\mathcal{M}_{(3)}(C') \to \mathcal{M}_{(3)}(F')$, corresponding to the localization map in (2.2). In [de Jeu 1996, Theorem 5.2], it is shown that this induces a commutative diagram:

$$\begin{tikzcd} H^2(\mathcal{M}_{(3)}(C')) \arrow[r] \arrow[d] & H^2(\mathcal{M}_{(3)}(F')) \arrow[d] \\ K_4^{(3)}(C') \oplus K_3^{(2)}(k) \cup F'^{*}_{\mathbb{Q}} \arrow[r] & K_4^{(3)}(F') \end{tikzcd} \tag{2.31}$$

Note that it was shown in Remark 2.4 that $K_4^{(3)}(C') \oplus K_3^{(2)}(k) \cup F'^{*}_{\mathbb{Q}}$ is indeed a direct sum, and that the lower horizontal map is an injection.

Remark 2.32. If k is totally real then $K_3^{(2)}(k)$ is zero. But in general we can use the projection

$$K_4^{(3)}(C') \oplus K_3^{(2)}(k) \cup F'^{*}_{\mathbb{Q}} \to K_4^{(3)}(C')$$

to get a map $H^2(\mathcal{M}_{(3)}(C')) \to K_4^{(3)}(C')$ as the composition

$$H^2(\mathcal{M}_{(3)}(C')) \to K_4^{(3)}(C') \oplus K_3^{(2)}(k) \cup F'^{*}_{\mathbb{Q}} \to K_4^{(3)}(C').$$

2.4.4. *Construction of the complex* $\mathscr{C}^\bullet(F)$. The complex $\mathscr{C}^\bullet(F)$ is described in [de Jeu 1996, Section 3], but it was first constructed in [Bloch 1990]. We recall its construction in order to clarify the construction of the corresponding complex for $\mathbb{O}$ in Section 2.5.4.

One starts with another part of the exact localization sequence (2.10) in relative K-theory.

$$\cdots \to \coprod_{t\in F^\flat} K_3^{(2)}(F) \to K_3^{(3)}(X_F;\square) \to K_3^{(3)}(X_F^{\mathrm{loc}};\square) \to \coprod_{t\in F^\flat} K_2^{(2)}(F) \to K_2^{(3)}(X_F;\square) \to \cdots . \tag{2.33}$$

Because $K_2^{(3)}((X_F;\square)) \simeq K_3^{(3)}(F) \simeq K_3^M(F)_{\mathbb{Q}}$, so that the map

$$\coprod_{t\in F^\flat} K_2^{(2)}(F) \to K_2^{(3)}(X_F;\square)$$

is surjective, this shows that the cohomological complex in degrees 1 and 2,

$$AC_{(3)}(F) : K_3^{(3)}(X_F^{\mathrm{loc}};\square) \to \coprod_{t\in F^\flat} K_2^{(2)}(F),$$

has maps $H^1(AC_{(3)}(F)) \simeq K_4^{(3)}(F)/K_3^{(2)}(F) \cup F^*_{\mathbb{Q}}$ and $H^2(AC_{(3)}(F)) \simeq K_3^{(3)}(F)$. (Here AC stands for "auxiliary complex".)

Again we have an acyclic subcomplex

$$(1+I)^* \cup K_2^{(2)}(F) \to \mathrm{d}(\cdots),$$

and therefore the quotient complex $\mathscr{C}^\bullet(F)$ is a cohomological complex in degree 1 and 2,

$$\mathscr{C}^\bullet(F) : \mathscr{C}^1(F) \to \mathscr{C}^2(F),$$

with

$$\mathscr{C}^1(F) = \frac{K_3^{(3)}(X_F^{\mathrm{loc}};\square)}{(1+I)^* \cup K_2^{(2)}(F)} \quad \text{and} \quad \mathscr{C}^2(F) = K_2^{(2)}(F) \otimes F_{\mathbb{Q}}^* .$$

It comes with maps

$$H^1(\mathscr{C}^\bullet(F)) \simeq K_4^{(3)}(F)/K_3^{(2)}(F) \cup F_{\mathbb{Q}}^* \tag{2.34}$$

and $H^2(\mathscr{C}^\bullet(F)) \simeq K_3^{(3)}(F.)$

Note that if g is in $F^\flat$, and f is in F^*, then $[g]_2 \cup f$ lies in $K_3^{(3)}(X_F^{\mathrm{loc}};\square)$. In fact, if we take the class of $[g]_2$ in $M_2(F)$ instead, then we do get a well-defined class in $\mathscr{C}^1(F)$, as $(1+I)^* \cup F_{\mathbb{Q}}^* \cup f$ goes to zero in $\mathscr{C}^1(F)$ by definition. Under the differential in the complex, $[g]_2 \cup (f)$ is mapped to

$$\{(1-g)^{-1}, f\} \otimes g = -\{1-g, f\} \otimes g,$$

so the condition for an element $\sum_i [g_i]_2 \cup (f_i)$ to be in $H^1(\mathscr{C}^\bullet(F))$ is that

$$\sum_i \{1-g_i, f_i\} \otimes g_i = 0 \quad \text{in } K_2^{(2)}(F) \otimes F_{\mathbb{Q}}^*.$$

The map $M_{(2)}(F) \otimes F_{\mathbb{Q}}^* \to \mathscr{C}^1(F)$ given by $[g]_2 \otimes f \mapsto [g]_2 \cup f$ fits into a commutative diagram

$$\begin{array}{ccccc} M_3(F) & \longrightarrow & M_2(F) \otimes F_{\mathbb{Q}}^* & \longrightarrow & F_{\mathbb{Q}}^* \otimes \bigwedge^2 F_{\mathbb{Q}}^* \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathscr{C}^1(F) & \longrightarrow & \mathscr{C}^2(F) \end{array} \tag{2.35}$$

where we map $f \otimes g \wedge h$ to $\{f, g\} \otimes h - \{f, h\} \otimes g$. Multiplying the map $H^2(\mathscr{M}_{(3)}(F)) \to K_4^{(3)}(F)$ by -1 if necessary, we obtain a commutative diagram (see [de Jeu 1996, Proposition 3.2]):

$$\begin{array}{ccc} H^2(\mathscr{M}_{(3)}(F)) & \longrightarrow & K_4^{(3)}(F) \\ \downarrow & & \downarrow \\ H^1(\mathscr{C}^\bullet(F)) & \longrightarrow & K_4^{(3)}(F)/K_3^{(2)}(F) \cup F_{\mathbb{Q}}^* \end{array} \tag{2.36}$$

2.5. *Construction of the complexes for $\mathbb{O}$ and $\mathscr{C}'$.*

Remark 2.37. At various stages there will be some properties of the complexes for $\mathbb{O}$ that depend on $K_3^{(2)}(\kappa)$ being trivial. Clearly, this applies to $\mathbb{O}$ as in Section 1 by our remarks about the K-groups of $\kappa(\mathscr{C}_\kappa)$ and $\mathbb{F}(\mathscr{C}'_{\mathbb{F}})$ in Section 2.2.

2.5.1. *Construction of the complex $\mathcal{M}_{(2)}(\mathbb{O})$.* When we try to imitate the localization sequence (2.10) for $\mathbb{O}$ rather than F, we are dealing with the two dimensional scheme $X_{\mathbb{O}}$, and we end up with a spectral sequence instead,

$$\begin{array}{ccc} \vdots & \vdots & \\ K_1^{(2)}(X_{\mathbb{O}}^{\mathrm{loc}};\square) & \coprod_{t\in\mathbb{O}^\flat} K_0^{(1)}(F) & \\ K_2^{(2)}(X_{\mathbb{O}}^{\mathrm{loc}};\square) & \coprod_{t\in\mathbb{O}^\flat} K_1^{(1)}(F) & \coprod_{t\in\kappa^\flat} K_0^{(0)}(\kappa) \\ K_3^{(2)}(X_{\mathbb{O}}^{\mathrm{loc}};\square) & \coprod_{t\in\mathbb{O}^\flat} K_2^{(1)}(F) & \coprod_{t\in\kappa^\flat} K_1^{(0)}(\kappa) \\ \vdots & \vdots & \vdots \end{array} \tag{2.38}$$

which converges to $K_*^{(2)}(X_{\mathbb{O}};\square)\simeq K_{*+1}^{(2)}(\mathbb{O})$.

Because $K_2^{(1)}(F)$, $K_1^{(0)}(\kappa)$ and $K_2^{(0)}(\kappa)$ are all trivial, if we let $RC_{(2)}(\mathbb{O})$ be the cohomological complex in degrees 1, 2 and 3, given by

$$K_2^{(2)}(X_{\mathbb{O}}^{\mathrm{loc}};\square)\to\coprod_{t\in\mathbb{O}^\flat} K_1^{(1)}(F)\to\coprod_{t\in\kappa^\flat} K_0^{(0)}(\kappa)\,, \tag{2.39}$$

then there are maps $H^1(RC_{(2)}(\mathbb{O}))\simeq K_3^{(2)}(\mathbb{O})$ and $H^2(RC_{(2)}(\mathbb{O}))\to K_2^{(2)}(\mathbb{O})$. The last map is surjective by Proposition 2.6 and the exact sequence

$$\cdots\to K_2^{(1)}(\kappa)\to K_2^{(2)}(\mathbb{O})\to K_2^{(2)}(F)\to K_1^{(1)}(\kappa)\to\cdots$$

as $K_2^{(1)}(\kappa)=0$. Note that the map $K_1^{(1)}(F)\to K_0^{(0)}(\kappa)$ is surjective, so that $H^3(RC_{(2)}(\mathbb{O}))$ is zero, as is $K_1^{(2)}(\mathbb{O})$.

Now let $A\subseteq K_2^{(2)}(X_{\mathbb{O}}^{\mathrm{loc}};\square)$ be the inverse image of $\coprod_{t\in\mathbb{O}^\flat}\mathbb{O}_{\mathbb{Q}}^*$ in $\coprod_{t\in\mathbb{O}^\flat} K_1^{(1)}(F)$. Because $K_1^{(1)}(\mathbb{O})=\mathbb{O}_{\mathbb{Q}}^*$ is equal to

$$\ker\bigl(K_1^{(1)}(F)\to K_0^{(0)}(\kappa)\bigr),$$

this means that the subcomplex

$$RC_{(2)}(\mathbb{O}): A\to\coprod_{t\in\mathbb{O}^\flat}\mathbb{O}_{\mathbb{Q}}^* \tag{2.40}$$

of (2.39) has maps $H^1(RC_{(2)}(\mathbb{O}))\to K_3^{(2)}(\mathbb{O})$ and $H^2(RC_{(2)}(\mathbb{O}))\to K_2^{(2)}(\mathbb{O})$.

We again use the element $[u]_2$ in $K_2^{(2)}(X_{\mathbb{O}}^{\mathrm{loc}};\square)$ for every u in $\mathbb{O}^\flat$, and put

$$\begin{aligned}\mathrm{Symb}_1(\mathbb{O}) &= K_1^{(1)}(\mathbb{O}) = \mathbb{O}_{\mathbb{Q}}^*\,,\\ \mathrm{Symb}_2(\mathbb{O}) &= \langle [u]_2 \text{ with } u \text{ in } \mathbb{O}^\flat\rangle_{\mathbb{Q}} + (1+I)_{\mathbb{O}}^* \cup \mathbb{O}_{\mathbb{Q}}^*\,.\end{aligned}$$

(See (2.9) for the definition of $(1+I)_{\mathbb{O}}^*$.) Observe that, if u is in $\mathbb{O}^\flat$ and v is in $\mathbb{O}_{\mathbb{Q}}^*$, then $[u]_2$ and $(1+I)_{\mathbb{O}}^* \cup v$ are in A, so we get a subcomplex of (2.40)

(2.41) $$\mathit{Symb}_2(\mathbb{O}) : \mathrm{Symb}_2(\mathbb{O}) \to \coprod_{t\in\mathbb{O}^\flat} \mathbb{O}_{\mathbb{Q}}^*\,,$$

containing the acyclic subcomplex

(2.42) $$(1+I)_{\mathbb{O}}^* \cup \mathbb{O}_{\mathbb{Q}}^* \to \mathrm{d}(\cdots).$$

We take the quotient complex of (2.41) by (2.42) to obtain the complex

(2.43) $$\mathcal{M}_{(2)}(\mathbb{O}) : M_2(\mathbb{O}) \to \mathbb{O}_{\mathbb{Q}}^* \otimes \mathbb{O}_{\mathbb{Q}}^*\,,$$

with $M_2(\mathbb{O}) = \mathrm{Sym}_2(\mathbb{O})/(1+I)^* \cup \mathbb{O}_{\mathbb{Q}}^*$. Then $M_2(\mathbb{O})$ is a $\mathbb{Q}$-vector space generated by the $[u]_2$, u in $\mathbb{O}^\flat$, and $\mathrm{d}[u]_2 = (1-u)\otimes u$. (Again, we used that $\mathrm{d}(1+I)_{\mathbb{O}}^* \cup \mathbb{O}_{\mathbb{Q}}^*$ gives us exactly the right relations to change $\coprod_{t\in\mathbb{O}^\flat} \mathbb{O}_{\mathbb{Q}}^*$ into $\mathbb{O}_{\mathbb{Q}}^* \otimes \mathbb{O}_{\mathbb{Q}}^*$ because $\mathbb{O}^\flat$ generates $\mathbb{O}^*$.) Note that we now have maps

$$\mathcal{M}_{(2)}(\mathbb{O}) \leftarrow \mathit{Symb}_2(\mathbb{O}) \to \mathit{RC}_{(2)}(\mathbb{O}),$$

with the left one a quasiisomorphism, so we obtain maps

(2.44) $$H^i(\mathcal{M}_{(2)}(\mathbb{O})) \to K_{4-i}^{(2)}(\mathbb{O})$$

for $i=1$ and 2. Again the map for $i=1$ is an injection (cf. (2.17)). For $i=2$ the map is a surjection by Proposition 2.6 because $K_2^{(2)}(\mathbb{O}) = \ker\big(K_2^{(2)}(F) \to K_1^{(1)}(\kappa)\big)$.

Localizing the base from $\mathbb{O}$ to F in (2.38) gives us (2.19), so that we get a map of complexes $M_2(\mathbb{O}) \to M_2(F)$ since the various steps in the constructions of the two complexes are compatible.

Remark 2.45. The map $M_2(\mathbb{O}) \to M_2(F)$ is injective. Namely, because the construction of the complexes for $\mathcal{M}_{(2)}(\mathbb{O})$ and $\mathcal{M}_{(2)}(F)$ is compatible with the localization from $\mathbb{O}$ to F in (2.38), we have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & H^1(\mathcal{M}_{(2)}(\mathbb{O})) \arrow[r] \arrow[d] & M_2(\mathbb{O}) \arrow[r] \arrow[d] & \mathbb{O}_{\mathbb{Q}}^* \otimes \mathbb{O}_{\mathbb{Q}}^* \arrow[d] \\ 0 \arrow[r] & H^1(\mathcal{M}_{(2)}(F)) \arrow[r] & M_2(F) \arrow[r] & F_{\mathbb{Q}}^* \otimes F_{\mathbb{Q}}^*\,, \end{tikzcd}$$

with $H^1(\mathcal{M}_{(2)}(\mathbb{O})) \subseteq K_3^{(2)}(\mathbb{O})$ and $H^1(\mathcal{M}_{(2)}(F)) \subseteq K_3^{(2)}(F)$. From the exact localization sequence

$$\cdots \to K_3^{(1)}(\kappa) \to K_3^{(2)}(\mathbb{O}) \to K_3^{(2)}(F) \to K_2^{(1)}(\kappa) \to \cdots$$

we see that $K_3^{(2)}(\mathbb{O}) \simeq K_3^{(2)}(F)$, so that the map on H^1's must be injective. As $\mathbb{O}_{\mathbb{Q}}^* \otimes \mathbb{O}_{\mathbb{Q}}^* \to F_{\mathbb{Q}}^* \otimes F_{\mathbb{Q}}^*$ is clearly injective, $M_2(\mathbb{O}) \to M_2(F)$ must be injective as well. So we may think of $M_2(\mathbb{O})$ as the subspace of $M_2(F)$ generated by the $[u]_2$ with u in $\mathbb{O}^\flat \subset F^\flat$.

2.5.2. *Construction of the complex $\mathcal{M}_{(3)}(\mathbb{O})$.* In this subsection, we shall be making Assumption 2.7.

If we now try to imitate the construction of $\mathcal{M}_{(3)}(F)$ using $\mathbb{O}$ instead of F, we see some differences. For example, in the construction of the spectral sequence, in codimension one, we shall end up with copies of $\{t_i = u\}$ for u in $\mathbb{O}^\flat$, which look like $X_{\mathbb{O}}$, out of which we have to remove the intersections with all other such pieces of codimension one of the form $\{t_i = v\}$ for $i = 1$ and 2, and v in $\mathbb{O}^\flat$. Note that, in particular, we also cut out $t_i = v$ with u and v different elements in $\mathbb{O}^\flat$, but reducing to the same in the residue field. Then $t_i = v$ cuts out the bit in the special fiber in $t_i = u$. We therefore end up with copies of $X_F^{'\mathrm{loc}} = X_F \setminus \{t = u \text{ with } u \text{ in } \mathbb{O}^\flat\}$.

So if we do this for $\mathbb{O}$, we end up with the following spectral sequence, converging to $K_*^{(3)}(X_{\mathbb{O}}^2; \square^2) \simeq K_{*+2}^{(3)}(\mathbb{O})$; see [Besser and de Jeu 2003, (3.7)]. For typographical reasons, let us introduce the following abbreviations:

$$K_{n,\mathbb{O}}^{(j),m} := K_n^{(j)}(X_{\mathbb{O}}^m; \square^m), \quad K_{n,F}^{(j),1} := K_n^{(j)}(X_F^{'\mathrm{loc}}; \square), \quad K_{n,\kappa}^{(j),1} := K_n^{(j)}(X_\kappa; \square).$$

Then the spectral sequence is

(2.46)

$$\begin{array}{cccc}
\vdots & \vdots & \vdots & \\
K_{2,\mathbb{O}}^{(3),2} & \Big(\coprod_{t\in\mathbb{O}^\flat} K_{1,F}^{(2),1}\Big)^2 & \coprod_{t_1,t_2\in\mathbb{O}^\flat} K_0^{(1)}(F) \coprod \Big(\coprod_{t\in\kappa^\flat} K_{0,\kappa}^{(1),1}\Big)^2 & \\
K_{3,\mathbb{O}}^{(3),2} & \Big(\coprod_{t\in\mathbb{O}^\flat} K_{2,F}^{(2),1}\Big)^2 & \coprod_{t_1,t_2\in\mathbb{O}^\flat} K_1^{(1)}(F) \coprod \Big(\coprod_{t\in\kappa^\flat} K_{1,\kappa}^{(1),1}\Big)^2 & \coprod_{t_1,t_2\in\kappa^\flat} K_0^{(0)}(\kappa) \\
K_{4,\mathbb{O}}^{(3),2} & \Big(\coprod_{t\in\mathbb{O}^\flat} K_{3,F}^{(2),1}\Big)^2 & \coprod_{t_1,t_2\in\mathbb{O}^\flat} K_2^{(1)}(F) \coprod \Big(\coprod_{t\in\kappa^\flat} K_{2,\kappa}^{(1),1}\Big)^2 & \coprod_{t_1,t_2\in\kappa^\flat} K_1^{(0)}(\kappa) \\
\vdots & \vdots & \vdots & \vdots
\end{array}$$

Here the $(\cdots)^2$ corresponds to two copies, corresponding to a coproduct over t_1 in $\mathbb{O}^\flat$ or $\kappa^\flat$, and t_2 in $\mathbb{O}^\flat$ or $\kappa^\flat$. As explained before, in order to obtain $X_F^{\prime\mathrm{loc}}$ out of X_F, we only remove $t_i = u_j$ with u_j in $\mathbb{O}^\flat$.

Now notice that all $K_j^{(0)}(\kappa)$ are zero for $j \geq 1$, that $K_j^{(1)}(F)$ is zero for $j \geq 2$, and finally that $K_j^{(1)}(X_\kappa^{\mathrm{loc}}; \square)$ is zero as well for $j \geq 2$: we consider the exact localization sequence

$$\cdots \to K_j^{(1)}(X_\kappa^1; \square) \to K_j^{(1)}(X_\kappa^{\mathrm{loc}}; \square) \to \coprod K_{j-1}^{(0)}(\kappa) \to \cdots,$$

and use that $K_j^{(1)}(X_\kappa^1; \square) \simeq K_{j+1}^{(1)}(\kappa)$, which is zero as $K_m^{(1)}(L) = 0$ for $m \geq 2$ for any field L, as well as that $K_{j-1}^{(0)}(\kappa) = 0$ because $j-1 \geq 1$. Therefore, with $RC_{(3)}(\mathbb{O})$ the following cohomological complex in degrees 1 through 4 (corresponding to the row in (2.46) starting with $K_3^{(3)}(X_{\mathbb{O},\mathrm{loc}}^2; \square^2)$):

$$RC_{(3)}(\mathbb{O}) : K_3^{(3)}(X_{\mathbb{O}}^{2,\mathrm{loc}}; \square^2) \to \Bigl(\coprod_{t \in \mathbb{O}^\flat} K_2^{(2)}(X_F^{\mathrm{loc}}; \square)\Bigr)^2$$
$$\to \coprod_{t_1, t_2 \in \mathbb{O}^\flat} K_1^{(1)}(F) \coprod \Bigl(\coprod_{t \in \kappa^\flat} K_1^{(1)}(X_\kappa^{\mathrm{loc}}; \square)\Bigr)^2 \to \coprod_{t_1, t_2 \in \kappa^\flat} K_0^{(0)}(\kappa) \tag{2.47}$$

has maps

$$H^i(RC_{(3)}(\mathbb{O})) \to K_{6-i}^{(3)}(\mathbb{O}) \tag{2.48}$$

for $i = 2$, 3 and 4.

Remark 2.49. Note that for $i = 4$ this statement is vacuous since from the localization sequence

$$\cdots \to K_3^{(3)}(F) \to K_2^{(2)}(\kappa) \to K_2^{(3)}(\mathbb{O}) \to K_2^{(3)}(F) \to \cdots$$

and the facts that $K_2^{(3)}(F)$ is trivial and $K_3^{(3)}(F) \to K_2^{(2)}(\kappa)$ is surjective (see Proposition 2.6), it follows that $K_2^{(3)}(\mathbb{O})$ is zero.

Remark 2.50. The map $K_2^{(2)}(X_{\mathbb{O}}^{\mathrm{loc}}; \square) \to K_2^{(2)}(X_F^{\prime\mathrm{loc}}; \square) \to K_2^{(2)}(X_F^{\mathrm{loc}}; \square)$ is injective. Namely, we have an exact localization sequence

$$\cdots \to K_2^{(1)}(X_\kappa^{\mathrm{loc}}; \square) \to K_2^{(2)}(X_{\mathbb{O}}^{\mathrm{loc}}; \square) \to K_2^{(2)}(X_F^{\prime\mathrm{loc}}; \square) \to \cdots,$$

and $K_2^{(1)}(X_\kappa^{\mathrm{loc}}; \square)$ equals zero, as seen above. Also, we have an exact localization sequence

$$\cdots \to \coprod_{t \in F^* \setminus F^\flat \bigcup \{1\}} K_2^{(1)}(F) \to K_2^{(2)}(X_F^{\prime\mathrm{loc}}; \square) \to K_2^{(2)}(X_F^{\mathrm{loc}}; \square) \to \cdots,$$

and again $K_2^{(1)}(F)$ is zero.

Remark 2.51. Note that, because we can localize $\mathbb{O}$ to F, we have a natural map of the spectral sequence in (2.46) to the one in (2.19), which, at the level of the complexes (2.20) and (2.47), simply forgets the terms over κ, includes a coproduct over $\mathbb{O}^\flat$ into the corresponding coproduct over $F^\flat$, and uses the maps $K_2^{(2)}(X_{\mathbb{O}}^{\mathrm{loc}};\square) \to K_2^{(2)}(X_F^{\mathrm{loc}};\square)$ and $K_3^{(3)}(X_{\mathbb{O}}^{2,\mathrm{loc}};\square^2) \to K_3^{(3)}(X_F'^{2,\mathrm{loc}};\square^2)$. By Remark 2.50, the first one is always injective, and the second is injective if $K_5^{(2)}(\kappa)$ and $K_4^{(2)}(F)$ are zero.

Let us try to create a jewel in the crown of the scary notation in (2.47). Define $\mathrm{Symb}_n(\mathbb{O}) \subseteq K_n^{(n)}(X_{\mathbb{O}}^{n-1,\mathrm{loc}};\square^{n-1})$ for $n = 1$, 2 and 3 by setting

$$\mathrm{Symb}_1(\mathbb{O}) = \mathbb{O}_{\mathbb{Q}}^*,$$
$$\mathrm{Symb}_2(\mathbb{O}) = \langle [u]_2 \text{ with } u \text{ in } \mathbb{O}^\flat\rangle_{\mathbb{Q}} + (1+I)_{\mathbb{O}}^* \cup \mathrm{Symb}_1(\mathbb{O}),$$

as before, and

$$\mathrm{Symb}_3(\mathbb{O}) = \langle [u]_3 \text{ with } u \text{ in } \mathbb{O}^\flat\rangle_{\mathbb{Q}} + (1+I)_{\mathbb{O}}^* \,\tilde{\cup}\, \mathrm{Symb}_2(\mathbb{O}).$$

Again, by $\tilde{\cup}$ we denote that we use both products, coming from the two ways of projecting $X_{\mathbb{O}}^2$ to $X_{\mathbb{O}}$.

Note that for $n = 1$, $\mathrm{Symb}_1(\mathbb{O}) = \mathbb{O}_{\mathbb{Q}}^* \subseteq \mathrm{Symb}_1(F) = F_{\mathbb{Q}}^*$, and that for $n = 2$, we can view $\mathrm{Symb}_2(\mathbb{O}) \subseteq \mathrm{Symb}_2(F)$ inside $K_2^{(2)}(X_F^{\mathrm{loc}};\square)$ by Remark 2.50, as

$$K_2^{(2)}(X_{\mathbb{O}}^{\mathrm{loc}};\square) \subseteq K_2^{(2)}(X_F^{\mathrm{loc}};\square).$$

Because $\mathrm{d}[u]_2 = (1-u)^{-1}_{|t=u}$, and $\mathrm{d}[u]_3 = -[u]_{2|t_1=u} + [u]_{2|t_2=u}$ (where both terms lie in a copy of $K_2^{(2)}(X_{\mathbb{O}}^{\mathrm{loc}};\square)$ inside $K_2^{(2)}(X_F^{\mathrm{loc}})$, again by Remark 2.50), it follows that

$$Symb_{(3)}(\mathbb{O}) : \mathrm{Symb}_3(\mathbb{O}) \to \Bigl(\coprod_{t\in\mathbb{O}^\flat} \mathrm{Symb}_2(\mathbb{O})\Bigr)^2 \to \coprod_{t_1,t_2\in\mathbb{O}^\flat} \mathbb{O}_{\mathbb{Q}}^* \tag{2.52}$$

is a subcomplex (in degrees 1, 2 and 3) of (2.47). Note that we used here that elements in $\mathbb{O}^\flat$ never give rise to a pole or zero over κ, so the map to $\coprod K_0^{(0)}(\kappa)$ is zero. Also, we used that an element $[u]_2$ with u in $\mathbb{O}^\flat$ under the localization (of its construction),

$$K_2^{(2)}(X_{\mathbb{O}} \setminus \{t=u\};\square) \to K_1^{(1)}(\mathbb{O}) \to \cdots$$

maps to $(1-u)^{-1}$, so under the boundary in (2.46) it never hits the $K_1^{(1)}(X_\kappa^{\mathrm{loc}};\square)$ components. Similarly, the elements in $(1+I)_{\mathbb{O}}^* \cup \mathbb{O}_{\mathbb{Q}}^*$ never hit these components.

Again, one shows that the subcomplex of (2.52) given by

$$(1+I)_{\mathbb{O}}^* \,\tilde{\cup}\, \mathrm{Symb}_2(\mathbb{O}) \to \Bigl(\coprod_t (1+I)_{\mathbb{O}}^* \cup \mathbb{O}_{\mathbb{Q}}^*\Bigr)^2 + (\cdots) \to \mathrm{d}(\cdots)$$

is acyclic; see [de Jeu 1995, Lemma 3.7 and Remark 3.10].

Taking the quotient complex and the alternating part for the action of S_2 under swapping the coordinates, we finally get a complex

$$M_3(\mathbb{O}) \to M_2(\mathbb{O}) \to \mathbb{O}^*_{\mathbb{Q}} \otimes \bigwedge^2 \mathbb{O}^*_{\mathbb{Q}} .$$

Here

$$M_3(\mathbb{O}) = \mathrm{Symb}_3(\mathbb{O}) / \big((1+I)^*_{\mathbb{O}} \,\tilde{\cup}\, \mathrm{Symb}_2(\mathbb{O})\big)^{\mathrm{alt}}$$

and, as before,

$$M_2(\mathbb{O}) = \mathrm{Symb}_2(\mathbb{O}) / (1+I)^*_{\mathbb{O}} \cup \mathbb{O}^*_{\mathbb{Q}} .$$

Note that $M_n(\mathbb{O})$ is a $\mathbb{Q}$-vector space on symbols $[u]_n$ for u in $\mathbb{O}^\flat$, modulo nonexplicit relations depending on n. The maps in the complex are given by $\mathrm{d}[u]_3 = [u]_2 \otimes u$ and $\mathrm{d}[u]_2 \otimes v = (1-u) \otimes (u \wedge v)$.

In particular, the condition for an element $\sum_i [u_i] \otimes v_i$ in $M_2(\mathbb{O}) \otimes \mathbb{O}^*_{\mathbb{Q}}$ to lie in $H^2(\mathcal{M}_{(3)}(\mathbb{O}))$ is that

$$\sum_i (1-u_i) \otimes (u_i \wedge v_i) = 0 \quad \text{in } \mathbb{O}^*_{\mathbb{Q}} \otimes \bigwedge^2 \mathbb{O}^*_{\mathbb{Q}} . \tag{2.53}$$

Again S_2 acts on the various complexes by swapping the coordinates, and we get maps

$$\mathcal{M}_{(3)}(\mathbb{O}) \leftarrow \mathit{Symb}_{(3)}(\mathbb{O})^{\mathrm{alt}} \to RC_{(3)}(\mathbb{O})^{\mathrm{alt}} \to RC_{(3)}(\mathbb{O})$$

with the left map a quasiisomorphism. Combining this with (2.48) gives us a map

$$H^i(\mathcal{M}_{(3)}(\mathbb{O})) \to K^{(3)}_{6-i}(\mathbb{O}) \tag{2.54}$$

for $i = 2$ and 3, where the map for $i = 3$ is a surjection if $K^{(2)}_3(\kappa) = 0$ by Proposition 2.6 and the localization sequence

$$\cdots \to K^{(2)}_3(\kappa) \to K^{(3)}_3(\mathbb{O}) \to K^{(3)}_3(F) \to K^{(2)}_2(\kappa) \to \cdots .$$

Remark 2.55. Notice that by construction (that is, by compatibility of everything we did with the localization of $\mathbb{O}$ to F), these maps for $i = 2$ or 3 fit into a commutative diagram:

$$\begin{array}{ccc} H^i(\mathcal{M}_{(3)}(\mathbb{O})) & \longrightarrow & K^{(3)}_{6-i}(\mathbb{O}) \\ \downarrow & & \downarrow \\ H^i(\mathcal{M}_{(3)}(F)) & \longrightarrow & K^{(3)}_{6-i}(F) \end{array} \tag{2.56}$$

We also note that it was proved in Remark 2.45 that the map $M_2(\mathbb{O}) \to M_2(F)$ is injective. Because we clearly have that $\mathbb{O}^*_{\mathbb{Q}} \to F^*_{\mathbb{Q}}$ is an injection, this means that, in degrees 2 and 3, $\mathcal{M}_{(3)}(\mathbb{O})$ injects into $\mathcal{M}_{(3)}(F)$.

2.5.3. *Construction of the complex* $\mathcal{M}_{(3)}(\mathscr{C}')$. In this subsection we imitate the definition of the complex $\mathcal{M}_{(3)}(C')$ in Section 2.4.3, but using the complex $\mathcal{M}_{(3)}(\mathcal{O}')$ rather than $\mathcal{M}_{(3)}(F')$ in the top row. The advantage of using the complex $\mathcal{M}_{(3)}(\mathscr{C}')$ (just like the advantage of using any $\mathcal{O}'$-complex over the corresponding F'-complex) is that the syntomic regulator gets the input it needs on the special fiber of $\mathscr{C}'$.

We therefore put ourselves in the situation of Notation 1.7, so assume we have a number field $k \subset K$, a proper, smooth, irreducible curve $\mathscr{C}'$ over $R' = \mathcal{O} \cap k$, and that the generic fiber $C' = \mathscr{C}' \otimes_{R'} k$ is geometrically irreducible. We put $F' = k(C')$, and $\mathcal{O}'$ the discrete valuation ring in F' corresponding to the generic point of the special fiber of $\mathscr{C}'$. We have a commutative diagram as follows:

$$\begin{tikzcd} M_3(\mathcal{O}') \arrow[r, "\mathrm{d}"] \arrow[d] & M_2(\mathcal{O}') \otimes_{\mathbb{Q}} \mathcal{O}'^*_{\mathbb{Q}} \arrow[r, "\mathrm{d}"] \arrow[d, "\partial_1"'] & \mathcal{O}'^*_{\mathbb{Q}} \otimes \bigwedge^2 \mathcal{O}'^*_{\mathbb{Q}} \arrow[d, "\partial_2"'] \\ 0 \arrow[r] & \coprod_x \widetilde{M}_2(k(x)) \arrow[r, "\mathrm{d}"] & \coprod_x \bigwedge^2 k(x)^*_{\mathbb{Q}}\,. \end{tikzcd}$$

The d's in the top row are as in $\mathcal{M}_{(3)}(\mathcal{O}')$. The vertical maps and the map in the bottom row are given by the same formulas as before (see 2.4.3), via the natural map $\mathcal{M}_{(3)}(\mathcal{O}') \to \mathcal{M}_{(3)}(F')$ corresponding to the localization from $\mathcal{O}'$ to F'.

We let $\mathcal{M}_{(3)}(\mathscr{C}')$ be the cohomological complex in degrees 1 through 4, given by the total complex associated to the double complex in the commutative diagram above. Note that therefore in particular, an element $\sum_i [u_i]_2 \otimes v_i$ in $M_2(\mathcal{O}') \otimes \mathcal{O}'^*_{\mathbb{Q}}$ is in $H^2(\mathcal{M}_{(3)}(\mathscr{C}'))$ if and only if it satisfies (2.53) as well as, for every closed point x in C',

$$\sum_i \mathrm{ord}_x(v_i)[u_i(x)]_2 = 0 \tag{2.57}$$

in $\widetilde{M}_2(k(x))$, with the convention that $[0]_2 = [1]_2 = [\infty]_2 = 0$.

The map to K-theory is similar to the map for $\mathcal{M}_{(3)}(F')$, but now we get

$$H^2(\mathcal{M}_{(3)}(\mathscr{C}')) \to H^2(\mathcal{M}_{(3)}(\mathcal{O}')) \to K_4^{(3)}(\mathcal{O}'),$$

where the first arrow corresponds to forgetting the bottom row in $\mathcal{M}_{(3)}(\mathscr{C}')$. In fact, because this is compatible with the localization to F' (that is, with the map $\mathcal{M}_{(3)}(\mathcal{O}') \to \mathcal{M}_{(3)}(F')$), from (2.31) we find that we have a commutative diagram

$$\begin{tikzcd} H^2(\mathcal{M}_{(3)}(\mathscr{C}')) \arrow[r] \arrow[d] & K_4^{(3)}(\mathscr{C}') \oplus K_3^{(2)}(k) \cup \mathcal{O}'^*_{\mathbb{Q}} \arrow[d, equal] \\ H^2(\mathcal{M}_{(3)}(C')) \arrow[r] & K_4^{(3)}(C') \oplus K_3^{(2)}(k) \cup F'^*_{\mathbb{Q}} \end{tikzcd} \tag{2.58}$$

where the group on the right is contained in $K_4^{(3)}(\mathbb{O}') = K_4^{(3)}(F')$, and we used that $K_4^{(3)}(\mathscr{C}') \oplus K_3^{(2)}(k) \cup F'^*_{\mathbb{Q}} = K_4^{(3)}(C') \oplus K_3^{(2)}(k) \cup \mathbb{O}'^*_{\mathbb{Q}}$ by Remarks 2.3 and 2.5. This proves that the top square in (1.16) exists and commutes.

Note that in Theorem 1.9(2), the condition $\partial_1(\alpha') = 0$ on α' in $H^2(\mathcal{M}_{(3)}(\mathbb{O}'))$ is exactly that α' satisfies (2.57), hence lies in the subspace $H^2(\mathcal{M}_{(3)}(\mathscr{C}'))$. Therefore we have proved the existence of β' in the theorem. Its uniqueness is clear because the direct sum above gives an injection $K_4^{(3)}(\mathscr{C}') \to K_4^{(3)}(\mathbb{O}')/K_3^{(2)}(k) \cup \mathbb{O}'^*_{\mathbb{Q}}$.

Remark 2.59. Just as in Remark 2.32, we can consider the projection

$$K_4^{(3)}(\mathscr{C}') \oplus K_3^{(2)}(k) \cup \mathbb{O}'^*_{\mathbb{Q}} \to K_4^{(3)}(\mathscr{C}')$$

to get a map $H^2(\mathcal{M}_{(3)}(\mathscr{C}')) \to K_4^{(3)}(\mathscr{C}')$ as the composition

$$H^2(\mathcal{M}_{(3)}(\mathscr{C}')) \to K_4^{(3)}(\mathscr{C}') \oplus K_3^{(2)}(k) \cup \mathbb{O}'^*_{\mathbb{Q}} \to K_4^{(3)}(\mathscr{C}').$$

2.5.4. *Construction of the complex* $\mathscr{C}^\bullet(\mathbb{O})$. The remainder of the theorems in the introduction will be proved in Section 10. The necessary calculations will in fact depend heavily on the analogue of $\mathscr{C}^\bullet(F)$ for $\mathbb{O}$, $\mathscr{C}^\bullet(\mathbb{O})$.

Because we are dealing with the two dimensional scheme $X_{\mathbb{O}}$, the localization sequence (2.33) becomes a spectral sequence (cf. (2.38)):

$$\begin{array}{ccc}
\vdots & \vdots & \vdots \\
K_2^{(3)}(X_{\mathbb{O}}^{\mathrm{loc}};\square) & \coprod_{t\in\mathbb{O}^\flat} K_1^{(2)}(F) & \coprod_{t\in\kappa^\flat} K_0^{(1)}(\kappa) \\
K_3^{(3)}(X_{\mathbb{O}}^{\mathrm{loc}};\square) & \coprod_{t\in\mathbb{O}^\flat} K_2^{(2)}(F) & \coprod_{t\in\kappa^\flat} K_1^{(1)}(\kappa) \\
K_4^{(3)}(X_{\mathbb{O}}^{\mathrm{loc}};\square) & \coprod_{t\in\mathbb{O}^\flat} K_3^{(2)}(F) & \coprod_{t\in\kappa^\flat} K_2^{(1)}(\kappa) \\
\vdots & \vdots & \vdots
\end{array} \tag{2.60}$$

converging to $K_*^{(3)}(X_{\mathbb{O}};\square) \simeq K_{*+1}^{(3)}(\mathbb{O})$. Let us notice that $K_2^{(1)}(\kappa)$ and $K_3^{(1)}(\kappa)$ are zero, and that the exact localization sequence

$$\cdots \to K_3^{(1)}(\kappa) \to K_3^{(2)}(\mathbb{O}) \to K_3^{(2)}(F) \to K_2^{(1)}(\kappa) \to K_2^{(2)}(\mathbb{O}) \to K_2^{(2)}(F) \to \cdots$$

tells us that $K_2^{(2)}(\mathbb{O}) \subseteq K_2^{(2)}(F)$ and $K_3^{(2)}(\mathbb{O}) \simeq K_3^{(2)}(F)$. Therefore we get an exact sequence

$$0 \to \frac{K_4^{(3)}(\mathbb{O})}{K_3^{(2)}(\mathbb{O}) \cup \mathbb{O}^*_{\mathbb{Q}}} \to K_3^{(3)}(X_{\mathbb{O}}^{\mathrm{loc}};\square) \to \ker\Bigl(\coprod_{t\in\mathbb{O}^\flat} K_2^{(2)}(F) \to \coprod_{t\in\kappa^\flat} K_1^{(1)}(\kappa)\Bigr).$$

In the middle row of the spectral sequence (2.60) above, let $B \subseteq K_3^{(3)}(X_{\mathbb{O}}^{\mathrm{loc}};\square)$ be the inverse image of $\coprod K_2^{(2)}(\mathbb{O})$ (with the coproduct over all of $\mathbb{O}^\flat$). Then we have a cohomological complex in degrees 1 and 2,

$$AC_{(3)}(\mathbb{O}) : B \to \coprod_{t\in\mathbb{O}^\flat} K_2^{(2)}(\mathbb{O}), \tag{2.61}$$

an isomorphism

$$H^1(AC_{(3)}(\mathbb{O})) \simeq \frac{K_4^{(3)}(\mathbb{O})}{K_3^{(2)}(\mathbb{O})\cup\mathbb{O}_{\mathbb{Q}}^*},$$

and a map $H^2(AC_{(3)}(\mathbb{O})) \to K_3^{(3)}(\mathbb{O})$.

Remark 2.62. If $K_3^{(2)}(\kappa)=0$, or more generally, the map $K_4^{(3)}(F)\to K_3^{(2)}(\kappa)$ is surjective, then from the exact localization sequence

$$\cdots \to K_4^{(3)}(F)\to K_3^{(2)}(\kappa)\to K_3^{(3)}(\mathbb{O})\to K_3^{(3)}(F)\to K_2^{(2)}(\kappa)\to\cdots,$$

Proposition 2.6 and (2.44), we see that the map $\coprod_{t\in\mathbb{O}^\flat} K_2^{(2)}(\mathbb{O})\to K_3^{(3)}(\mathbb{O})$, and hence the map $H^2(AC_{(3)}(\mathbb{O}))\to K_3^{(3)}(\mathbb{O})$, are surjective.

Remark 2.63. Because $K_1^{(2)}(F)$ and $K_2^{(1)}(\kappa)$ are zero, and $K_2^{(2)}(F)\to K_1^{(1)}(\kappa)$ is surjective, from (2.60) we get that there is an exact sequence

$$\mathrm{Ker}\Bigl(\coprod_{t\in\mathbb{O}^\flat} K_2^{(2)}(F)\to\coprod_{t\in\kappa^\flat} K_1^{(1)}(\kappa)\Bigr)\to K_2^{(3)}(X_{\mathbb{O}};\square)\to K_2^{(3)}(X_{\mathbb{O}}^{\mathrm{loc}};\square)\to 0.$$

If $K_3^{(2)}(\kappa)$ is zero, or, more generally, the map $K_4^{(3)}(F)\to K_3^{(2)}(\kappa)$ surjective, then Proposition 2.6 tells us that $\coprod_{t\in\mathbb{O}^\flat} K_2^{(2)}(\mathbb{O})$ surjects onto $K_2^{(3)}(X_{\mathbb{O}};\square)\simeq K_3^{(3)}(\mathbb{O})$, and we can conclude that $K_2^{(3)}(X_{\mathbb{O}}^{\mathrm{loc}};\square)$ is zero.

Now we consider the acyclic subcomplex $(1+I)_{\mathbb{O}}^*\cup K_2^{(2)}(\mathbb{O})\to \mathrm{d}(\cdots)$ of (2.61), and quotient out to find a complex $\mathscr{C}^\bullet(\mathbb{O}):\mathscr{C}^1(\mathbb{O})\to\mathscr{C}^2(\mathbb{O})$, where

$$\mathscr{C}^1(\mathbb{O}) = \frac{B}{(1+I)_{\mathbb{O}}^*\cup K_2^{(2)}(\mathbb{O})} \tag{2.64}$$

and $\mathscr{C}^2(\mathbb{O}) = K_2^{(2)}(\mathbb{O})\otimes\mathbb{O}_{\mathbb{Q}}^*$. We still have an isomorphism

$$H^1(\mathscr{C}^\bullet(\mathbb{O}))\simeq K_4^{(3)}(\mathbb{O})/K_3^{(2)}(\mathbb{O})\cup\mathbb{O}_{\mathbb{Q}}^* \tag{2.65}$$

and a map $H^2(\mathscr{C}^\bullet(\mathbb{O}))\to K_3^{(3)}(\mathbb{O})$, which by Proposition 2.6 and (2.44) is a surjection if $K_4^{(3)}(F)\to K_3^{(2)}(\kappa)$ is surjective, for example, if $K_3^{(2)}(\kappa)=0$.

Observe that if g is in $\mathbb{O}^\flat$, and f is in $\mathbb{O}^*_\mathbb{Q}$, then $[g]_2 \cup (f)$ is in $\mathscr{C}^1(\mathbb{O})$, and has boundary $\{(1-g)^{-1}, f\} \otimes g = -\{(1-g), f\} \otimes g$ in $\mathscr{C}^2(\mathbb{O})$. The condition for $\sum_i [g_i]_2 \cup (f_i)$ to be in $H^1(\mathscr{C}^\bullet(\mathbb{O}))$ is therefore that

$$\sum_i \{1-g_i, f_i\} \otimes g_i = 0 \quad \text{in } \mathscr{C}^2(\mathbb{O}) = K_2^{(2)}(\mathbb{O}) \otimes \mathbb{O}^*_\mathbb{Q}.$$

Note that because the construction of the spectral sequence in (2.60) is compatible with localizing the base from $\mathbb{O}$ to F and enlarging the coproduct from being over $\mathbb{O}^\flat$ to $F^\flat$ (in which case it becomes the localization sequence in (2.33)), and that $(1+I)^*_\mathbb{O}$ is contained in $(1+I)^*$, and $K_2^{(2)}(\mathbb{O}) \subseteq K_2^{(2)}(F)$, we have an obvious map of complexes, $\mathscr{C}^\bullet(\mathbb{O}) \to \mathscr{C}^\bullet(F)$, which fits into the commutative diagram

$$\begin{tikzcd} H^1(\mathscr{C}^\bullet(\mathbb{O})) \arrow[r] \arrow[d] & K_4^{(3)}(\mathbb{O})/K_3^{(2)}(\mathbb{O}) \cup \mathbb{O}^*_\mathbb{Q} \arrow[d] \\ H^1(\mathscr{C}^\bullet(F)) \arrow[r] & K_4^{(3)}(F)/K_3^{(2)}(F) \cup F^*_\mathbb{Q} \end{tikzcd} \tag{2.66}$$

and similarly for H^2.

Finally, we have a commutative diagram

$$\begin{tikzcd} M_3(\mathbb{O}) \arrow[r] \arrow[d] & M_2(\mathbb{O}) \otimes \mathbb{O}^*_\mathbb{Q} \arrow[r] \arrow[d] & \mathbb{O}^*_\mathbb{Q} \otimes \bigwedge^2 \mathbb{O}^*_\mathbb{Q} \arrow[d] \\ 0 \arrow[r] & \mathscr{C}^1(\mathbb{O}) \arrow[r] & \mathscr{C}^2(\mathbb{O}) \end{tikzcd}$$

as follows. We map $[u]_2 \otimes v$ to $[u]_2 \cup v$, and $u \otimes v \wedge w$ to $\{u, v\} \otimes w - \{u, w\} \otimes v$. This gives rise to a commutative diagram

$$\begin{tikzcd} H^2(\mathscr{M}_{(3)}(\mathbb{O})) \arrow[r] \arrow[d] & K_4^{(3)}(\mathbb{O}) \arrow[d] \\ H^1(\mathscr{C}^\bullet(\mathbb{O})) \arrow[r] & K_4^{(3)}(\mathbb{O})/K_3^{(2)}(\mathbb{O}) \cup \mathbb{O}^*_\mathbb{Q}, \end{tikzcd} \tag{2.67}$$

which is the bottom left square of (1.16). Obviously, the two diagrams above are compatible with (2.35) and (2.36) under the localization from $\mathbb{O}$ to F.

2.5.5. *Construction of the complexes* $\widetilde{\mathscr{M}}_{(2)}(\mathbb{O})$ *and* $\widetilde{\mathscr{M}}_{(3)}(\mathbb{O})$. For $n = 2$ and 3, let $N_n(\mathbb{O}) = \langle [u]_n + (-1)^n [u^{-1}]_n \text{ with } u \text{ in } \mathbb{O}^\flat \rangle_\mathbb{Q} \subseteq M_n(\mathbb{O})$. Consider the subcomplex of $\mathscr{M}_{(2)}(\mathbb{O})$ given by $N_2(\mathbb{O}) \to \mathrm{d}(\cdots)$. Because the corresponding subcomplex (2.14) of $\mathscr{M}_{(2)}(F)$ is acyclic and the natural map $M_2(\mathbb{O}) \to M_2(F)$ is an injection (see Remark 2.45), this subcomplex is acyclic. The second term is $\mathrm{Sym}^2(\mathbb{O}^*_\mathbb{Q})$, and the resulting quotient complex of $\mathscr{M}_{(2)}(\mathbb{O})$ is

$$\widetilde{\mathcal{M}}_{(2)}(\mathcal{O}) : \widetilde{M}_2(\mathcal{O}) \to \bigwedge^2 \mathcal{O}^*_{\mathbb{Q}}, \tag{2.68}$$

with $\widetilde{M}_2(\mathcal{O}) = M_2(\mathcal{O})/N_2(\mathcal{O})$, and $\mathrm{d}[u]_2 = (1-u) \wedge u$.

Because $\widetilde{\mathcal{M}}_{(2)}(\mathcal{O})$ is quasiisomorphic to $\mathcal{M}_{(2)}(\mathcal{O})$ we have maps

$$H^i(\widetilde{\mathcal{M}}_{(2)}(\mathcal{O})) \to K^{(2)}_{4-i}(\mathcal{O}).$$

For $i = 1$ this is again an injection. There is a map $\widetilde{\mathcal{M}}_{(2)}(\mathcal{O}) \to \widetilde{\mathcal{M}}_{(2)}(F)$ obtained by localizing the construction from $\mathcal{O}$ to F, and for $i = 1, 2$ a commutative diagram

$$\begin{tikzcd}
H^i(\widetilde{\mathcal{M}}_{(2)}(\mathcal{O})) \arrow[d] & H^i(\mathcal{M}_{(2)}(\mathcal{O})) \arrow[l, "\simeq"'] \arrow[r] \arrow[d] & K^{(2)}_{4-i}(\mathcal{O}) \arrow[d] \\
H^i(\widetilde{\mathcal{M}}_{(2)}(F)) & H^i(\mathcal{M}_{(2)}(F)) \arrow[l, "\simeq"'] \arrow[r] & K^{(2)}_{4-i}(F).
\end{tikzcd}$$

In this diagram for $i = 1$ the central vertical map is injective by the discussion in Remark 2.45. Hence the same holds for the map $H^1(\widetilde{\mathcal{M}}_{(2)}(\mathcal{O})) \to H^1(\widetilde{\mathcal{M}}_{(2)}(F))$, the map $\widetilde{M}_2(\mathcal{O}) \to \widetilde{M}_2(F)$ is an injection, and $\widetilde{\mathcal{M}}_{(2)}(\mathcal{O})$ is a subcomplex of $\widetilde{\mathcal{M}}_{(2)}(F)$.

By Remark 2.45, in the commutative diagram

$$\begin{tikzcd}
M_3(\mathcal{O}) \arrow[r] \arrow[d] & M_2(\mathcal{O}) \otimes \mathcal{O}^*_{\mathbb{Q}} \arrow[r] \arrow[d] & \mathcal{O}^*_{\mathbb{Q}} \otimes \bigwedge^2 \mathcal{O}^*_{\mathbb{Q}} \arrow[d] \\
M_3(F) \arrow[r] & M_2(F) \otimes F^*_{\mathbb{Q}} \arrow[r] & F^*_{\mathbb{Q}} \otimes \bigwedge^2 F^*_{\mathbb{Q}}
\end{tikzcd}$$

the two right-most maps are injective. (If we knew (as part of the rigidity conjecture) that $H^1(\mathcal{M}_{(3)}(\mathcal{O})) \to H^1(\mathcal{M}_{(3)}(F))$ were injective, then this would also hold for the left-most map.) We can quotient out the complex $\mathcal{M}_{(3)}(\mathcal{O})$ in the first row by the subcomplex

$$N_3(\mathcal{O}) \to N_2(\mathcal{O}) \otimes \mathcal{O}^*_{\mathbb{Q}} \to \mathrm{d}(\cdots),$$

which maps to the subcomplex (2.26) of the second row. We saw earlier that $d : N_2(\mathcal{O}) \to \mathrm{Sym}^2(\mathcal{O}^*_{\mathbb{Q}})$ is an isomorphism, so as in the proof of [de Jeu 1995, Corollary 3.22] one sees that this subcomplex is acyclic in degrees 2 and 3. The quotient complex is

$$\widetilde{\mathcal{M}}_{(3)}(\mathcal{O}) : \widetilde{M}_3(\mathcal{O}) \to \widetilde{M}_2(\mathcal{O}) \otimes \mathcal{O}^*_{\mathbb{Q}} \to \bigwedge^3 \mathcal{O}^*_{\mathbb{Q}},$$

where $\widetilde{M}_3(\mathcal{O}) = M_3(\mathcal{O})/N_3(\mathcal{O})$, and the natural map $\widetilde{\mathcal{M}}_{(3)}(\mathcal{O}) \to \widetilde{\mathcal{M}}_{(3)}(F)$ is an injection in degrees 2 and 3 because, as we saw earlier, $\widetilde{M}_2(\mathcal{O})$ injects into $\widetilde{M}_2(F)$. Still denoting the class of $[x]_i$ with $[x]_i$, the maps are now given by $\mathrm{d}[u]_3 = [u]_2 \otimes u$ and

$$\mathrm{d}[u]_2 \otimes v = (1-u) \wedge u \wedge v. \tag{2.69}$$

Using (2.54) we see that for $i = 2, 3$ we have a commutative diagram

$$\begin{tikzcd}
H^i(\widetilde{\mathcal{M}}_{(3)}(\mathbb{O})) \arrow[d] & H^i(\mathcal{M}_{(3)}(\mathbb{O})) \arrow[l, "\simeq"'] \arrow[r] \arrow[d] & K^{(3)}_{6-i}(\mathbb{O}) \arrow[d] \\
H^i(\widetilde{\mathcal{M}}_{(3)}(F)) & H^i(\mathcal{M}_{(3)}(F)) \arrow[l, "\simeq"'] \arrow[r] & K^{(3)}_{6-i}(F) \,.
\end{tikzcd}$$

Remark 2.70. Using the statements just before (2.68), the arguments in Remark 2.28 can also be given for $\mathbb{O}$ instead of F. This way we obtain a map

$$\widetilde{M}_2(\mathbb{O}) \otimes \mathbb{O}^*_{\mathbb{Q}} \to M_2(\mathbb{O}) \otimes \mathbb{O}^*_{\mathbb{Q}}/\mathrm{d}(N_3(\mathbb{O})),$$

which we still denote by Ξ. It yields a decomposition

$$M_2(\mathbb{O}) \otimes \mathbb{O}^*_{\mathbb{Q}}/\mathrm{d}(N_3(\mathbb{O})) = \mathrm{im}(\Xi) \oplus N_2(\mathbb{O}) \otimes \mathbb{O}^*_{\mathbb{Q}}/\mathrm{d}(N_3(\mathbb{O})),$$

and induces the inverse to the natural isomorphism $H^2(\mathcal{M}_{(3)}(\mathbb{O})) \to H^2(\widetilde{\mathcal{M}}_{(3)}(\mathbb{O}))$. The formulas in Lemma 2.29 and Corollary 2.30 apply in this case as well.

2.6. ***A diagram.*** For the convenience of the reader, we give in Figure 1 a commutative diagram summarizing the cohomology groups of most of the complexes introduced, and the maps. We have kept the layout of the diagram in the same spirit as the relativity in the plane. Note that the outer square is only relevant in the situation of Notation 1.7, and that we may replace F and $\mathbb{O}$ elsewhere in the diagram with F' and $\mathbb{O}'$ in this case.

The top half of this diagram is the top of the one in (1.16). The vertical maps correspond to the maps from constructions over $\mathbb{O}$ to the corresponding constructions over F. The horizontal maps are the maps on cohomology of complexes constructed in the previous subsections, and the diagonal maps correspond to the maps in (2.35), (2.56), (2.58) and (2.66).

Note that by Remarks 2.3 and 2.5 the rightmost vertical map is an isomorphism.

3. The classical case

In Proposition 3.1 below, we rephrase the results in Theorem 4.2 and Remarks 4.3 and 4.5 of [de Jeu 1996], which concern a curve C over $\mathbb{C}$ with function field $F = \mathbb{C}(C)$ and associated analytic manifold C_{an}, in a way that resembles the formulas in Theorems 1.12 and 1.13(1). (See Remark 10.14 for some thoughts on this comparison.) In fact, Sections 7 and 8 grew out of attempts to obtain syntomic analogues of those results of [loc. cit.], but the resulting formulas seem to be less flexible than the classical ones so we rephrase the latter.

In this section we let $H^1_{\mathrm{dR}}(F, \mathbb{R}(2)) = \lim_{\overrightarrow{U}} H^1_{\mathrm{dR}}(U, \mathbb{R}(2))$ where the limit is over U with $C_{\mathrm{an}} \setminus U$ finite, and similarly for other cohomology groups, or forms. Here

$$
\begin{tikzcd}
H^2(\mathcal{M}_{(3)}(\mathscr{C}')) \arrow[rrrrr] \arrow[ddddd] \arrow[dr] & & & & & K_4^{(3)}(\mathscr{C}') \oplus K_3^{(2)}(k) \cup \mathbb{O}'^*_{\mathbb{Q}} \arrow[dl] \arrow[ddddd] \\
& H^2(\mathcal{M}_{(3)}(\mathbb{O})) \arrow[rrr] \arrow[ddd] \arrow[dr] & & & K_4^{(3)}(\mathbb{O}) \arrow[dl] \arrow[ddd] & \\
& & H^1(\mathscr{C}^\bullet(\mathbb{O})) \arrow[r] \arrow[d] & \frac{K_4^{(3)}(\mathbb{O})}{K_3^{(2)}(\mathbb{O}) \cup \mathbb{O}^*_{\mathbb{Q}}} \arrow[d] & & \\
& & H^1(\mathscr{C}^\bullet(F)) \arrow[r] & \frac{K_4^{(3)}(F)}{K_3^{(2)}(F) \cup F^*_{\mathbb{Q}}} & & \\
& H^2(\mathcal{M}_{(3)}(F)) \arrow[rrr] \arrow[ur] & & & K_4^{(3)}(F) \arrow[ul] & \\
H^2(\mathcal{M}_{(3)}(C')) \arrow[rrrrr] \arrow[ur] & & & & & K_4^{(3)}(C') \oplus K_3^{(2)}(k) \cup F'^*_{\mathbb{Q}} \arrow[ul]
\end{tikzcd}
$$

Figure 1. Diagram summarizing cohomology groups (see previous page).

$\mathbb{R}(m) = (2\pi i)^m \mathbb{R} \subset \mathbb{C}$. If ω is holomorphic on C_{an}, then by [loc. cit., Proposition 4.6] one has a well-defined map $H^1_{\text{dR}}(F, \mathbb{R}(2)) \to \mathbb{C}$ by taking a representative β of a class in $H^1_{\text{dR}}(F, \mathbb{R}(2))$ satisfying [loc. cit., (9)], and computing $\int_{C_{\text{an}}} \omega \wedge \beta$.

The signs of the maps in the following proposition are normalized to be compatible with the ones in the theorems in the introduction (see Remark 3.3).

Proposition 3.1. *Let C be a smooth, proper, irreducible curve over $\mathbb{C}$ with function field $F = \mathbb{C}(C)$, and let C_{an} be the analytic manifold associated to $C(\mathbb{C})$. For a holomorphic 1-form ω on C_{an}, the maps*

$$
\begin{aligned}
\Psi''_{\infty,\omega} : M_2(F) \otimes F^*_{\mathbb{Q}} &\to \mathbb{C} \\
[g]_2 \otimes f &\mapsto 4 \int_{C_{\text{an}}} \log|f| \log|g| \operatorname{dlog}|1-g| \wedge \omega, \\
\Psi'''_{\infty,\omega} : \widetilde{M}_2(F) \otimes F^*_{\mathbb{Q}} &\to \mathbb{C} \\
[g]_2 \otimes f &\mapsto \tfrac{8}{3} \int_{C_{\text{an}}} \log|f| (\log|g| \operatorname{dlog}|1-g| - \log|1-g| \operatorname{dlog}|g|) \wedge \omega
\end{aligned}
$$

are well-defined, and induce maps $H^2(\mathcal{M}_{(3)}(F)) \to \mathbb{C}$ and $H^2(\widetilde{\mathcal{M}}_{(3)}(F)) \to \mathbb{C}$, respectively. Moreover, with $\operatorname{reg}_{\mathbb{C}} : K_4^{(3)}(F) \to H^2_{\mathscr{D}}(F, \mathbb{R}(3)) \simeq H^1_{\text{dR}}(F, \mathbb{R}(2))$ *the*

Beilinson regulator map, *the compositions*

$$H^2(\mathcal{M}_{(3)}(F)) \xrightarrow{(2.25)} K_4^{(3)}(F) \xrightarrow{\int_{C_{\mathrm{an}}} \omega\wedge \mathrm{reg}_{\mathbb{C}}(\cdot)} \mathbb{C},$$
$$H^2(\widetilde{\mathcal{M}}_{(3)}(F)) \xrightarrow{(2.27)} K_4^{(3)}(F) \xrightarrow{\int_{C_{\mathrm{an}}} \omega\wedge \mathrm{reg}_{\mathbb{C}}(\cdot)} \mathbb{C}$$

coincide with these induced maps.

Proof. Since $d\otimes\mathrm{id}: M_2(F)\otimes F_{\mathbb{Q}}^* \to F_{\mathbb{Q}}^*\otimes F_{\mathbb{Q}}^*\otimes F_{\mathbb{Q}}^*$ maps $[g]_2\otimes f$ to $(1-g)\otimes g\otimes f$, $\Psi''_{\infty,\omega}$ is well-defined. That it induces the stated map on $H^2(\mathcal{M}_{(3)}(F))$, and that this induced map has the stated property, follows from Proposition 3.2 and (the proof of) Theorem 4.2 of [de Jeu 1996], where we normalize the maps as explained in Remark 3.3 below. (The condition in [loc. cit.] that C is defined over a number field is not used in the proof of Theorem 4.2. The same holds for the condition with respect to complex conjugation on ω, which guaranteed only that the value of the integral was in $\mathbb{R}(1) \subset \mathbb{C}$.)

Applying Corollary 2.30 shows that $\Psi'''_{\infty,\omega}$ maps $[g]_2 \otimes f$ to

$$\tfrac{4}{3}\int_{C_{\mathrm{an}}} (3\log|f|\log|g|\,\mathrm{dlog}|1-g|+\log|1-g|(\log|g|\,\mathrm{dlog}|f|-\log|f|\,\mathrm{dlog}|g|))\wedge\omega.$$

Using a limit version of Stokes' theorem we may subtract $0 = \int_{C_{\mathrm{an}}} \mathrm{d}(\alpha\wedge\omega)$ for $\alpha = \frac{4}{3}\log|g|\log|1-g|\log|f|$, which gives the formula in the proposition. $\square$

Remark 3.2. The Bloch–Wigner dilogarithm $D(z): \mathbb{P}^1_{\mathbb{C}}\setminus\{0, 1, \infty\} \to (2\pi i)\mathbb{R} \subset \mathbb{C}$ satisfies $\mathrm{d}D(z) = \log|z|\mathrm{d}i\arg(1-z) - \log|1-z|\mathrm{d}i\arg(z)$ and extends to a continuous function on $\mathbb{P}^1_{\mathbb{C}}$. It is the function in the classical case that corresponds to $\mathrm{L}_{\mathrm{mod},2}(z)$ in the sense that they have similar functional equations, for example, $D(z) + D(z^{-1}) = 0$. Because $\mathrm{d}\log(g)\wedge\omega = \mathrm{d}\log(1-g)\wedge\omega = 0$, we find $\mathrm{d}(P_{2,\mathrm{Zag}}(g)\log|f|\omega)$ equals

$$P_{2,\mathrm{Zag}}(g)\,\mathrm{dlog}|f|\wedge\omega + \log|f|(\log|1-g|\,\mathrm{dlog}|g| - \log|g|\,\mathrm{dlog}|1-g|)\wedge\omega.$$

Hence $\Psi'''_{\infty,\omega}$ is also given by mapping $[g]_2\otimes f$ to $\frac{8}{3}\int_{C_{\mathrm{an}}}\log|f|D(g)\omega$.

Remark 3.3. The signs in Proposition 3.1 and Remark 3.2 are chosen in a way that is compatible with the ones in the p-adic case in Remark 5.25 below. In [de Jeu 1996] it is shown that, for a holomorphic 1-form ω, the map

$$K_4^{(3)}(F) \xrightarrow{\int_{C_{\mathrm{an}}} \omega\wedge \mathrm{reg}_{\mathbb{C}}(\cdot)} \mathbb{C}$$

factorizes through the quotient map $K_4^{(3)}(F) \to K_4^{(3)}(F)/K_3^{(2)}(F)\cup F_{\mathbb{Q}}^*$ in (2.36), giving maps $H^2(\mathcal{M}_{(3)}(F)) \to H^1(\mathscr{C}^\bullet(F)) \simeq K_4^{(3)}(F)/K_3^{(2)}(F)\cup F_{\mathbb{Q}}^* \to \mathbb{C}$. This composition is the one used in Proposition 3.1, and there is a choice of sign in the isomorphism here, which we normalize as follows.

The regulator map gives us

$$\mathrm{reg}_{\mathbb{C}} : K_3^{(3)}(X_F^{\mathrm{loc}}; \square) \to H^3_{\mathcal{D}}(X_F^{\mathrm{loc}}; \square; \mathbb{R}(3)) \simeq H^2_{\mathrm{dR}}(X_F^{\mathrm{loc}}; \square; \mathbb{R}(2)).$$

Computing the last cohomology group here as

$$\frac{\{(\varepsilon, \varepsilon_\infty, \varepsilon_0) \mid \varepsilon \in A^2(X_F^{\mathrm{loc}}), \varepsilon_s \in A^1(F), \mathrm{d}\varepsilon = 0, \mathrm{d}\varepsilon_s = \varepsilon_{|t=s} (s = 0, \infty)\}}{\{(\mathrm{d}\psi, \psi_{|t=\infty} + \mathrm{d}f_\infty, \psi_{|t=0} + \mathrm{d}f_0) \mid \psi \in A^1(X_F^{\mathrm{loc}}), f_s \in A^0(F)\}}, \tag{3.4}$$

we can map the class of $(\varepsilon, \varepsilon_\infty, \varepsilon_0)$ to

$$\frac{1}{2\pi i} \int_{X \times C_{\mathrm{an}}} \omega \wedge \mathrm{dlog}(t) \wedge \varepsilon - \int_{C_{\mathrm{an}}} \omega \wedge (\varepsilon_\infty - \varepsilon_0), \tag{3.5}$$

where the integral is taken with the product orientation on $X \times C_{\mathrm{an}}$, because this is trivial on $(\mathrm{d}\psi, \psi_{|t=\infty}, \psi_{|t=0})$ with ψ in $A^1(F)$. The calculations in [de Jeu 1996] are carried out using ε in $A^*(X_F^{\mathrm{loc}})$ that restrict to 0 for $t = 0$ or ∞, which yield the same cohomology group. The calculations in the proof of Proposition 3.1 therefore use the first term in (3.5).

The connecting map

$$H^1_{\mathrm{dR}}(F; \mathbb{R}(2))_{|t=\infty} \oplus H^1_{\mathrm{dR}}(F; \mathbb{R}(2))_{t=0} \to H^2_{\mathrm{dR}}(X_F; \square; \mathbb{R}(2))$$

in the long exact sequence for relative cohomology maps $(\varepsilon_\infty, \varepsilon_0)$ to $(0, \varepsilon_\infty, \varepsilon_0)$. The map in (3.5) therefore factorizes the composition

$$H^2_{\mathrm{dR}}(X_F; \square; \mathbb{R}(2)) \xrightarrow{\simeq} H^1_{\mathrm{dR}}(F; \mathbb{R}(2)) \xrightarrow{\int_{C_{\mathrm{an}}} \omega \wedge \cdot} \mathbb{C}$$

(with one of the two natural choices of isomorphism in the first map) over the localization map $H^2_{\mathrm{dR}}(X_F; \square; \mathbb{R}(2)) \to H^2_{\mathrm{dR}}(X_F^{\mathrm{loc}}; \square; \mathbb{R}(2))$.

For this choice of isomorphism we have a commutative diagram

$$\begin{tikzcd} K_3^{(3)}(X_C; \square) \arrow[r, "\simeq"] \arrow[d, "\mathrm{reg}_{\mathbb{C}}"] & K_4^{(3)}(C) \arrow[d, "\mathrm{reg}_{\mathbb{C}}"] \\ H^2_{\mathrm{dR}}(X_{C_{\mathrm{an}}}; \square; \mathbb{R}(2)) \arrow[r, "\simeq"] & H^1_{\mathrm{dR}}(C_{\mathrm{an}}; \mathbb{R}(2)) \end{tikzcd} \tag{3.6}$$

by normalizing the isomorphism at the top in the same way, and using the same convention in all localizations. This fixes the choice of sign in (2.34). Finally, there is a choice in the sign of the map $H^2(\mathcal{M}_{(3)}(F)) \to H^1(\mathscr{C}^\bullet(F))$ (see (2.36)), but we choose this so that the formulas in Proposition 3.1 hold.

For $\mathbb{O}$, one can give a similar discussion on the K-theory side using the diagram (2.67), and this is compatible with the one here by the commutativity of (2.66) and the compatibility of (2.67) with (2.36). In particular, the choices of signs on the K-theory side for $\mathbb{O}$ are compatible with those for F.

In Remark 5.25 we give a description of the maps on the p-adic side, using the description of syntomic cohomology in (5.5) that matches our description above. Comparing the sign of the term $\varepsilon_\infty - \varepsilon_0$ in both cases, and taking into account that we are ultimately cupping on the left with ω in the p-adic case as well (see Proposition 5.22), it is then clear that we have normalized the formulas in Proposition 3.1 and those in the theorems in the introduction in the same way.

4. Coleman integration

In this short section we briefly discuss Coleman's integration theory in the one-dimensional case only. The interested reader may refer to [Besser 2000b] for more details.

Coleman theory is done on wide open spaces in the sense of Coleman [1988]. In general these are the overconvergent spaces described in Section 5. In the one-dimensional case these can be described concretely in the following way. Let X be a curve over $\mathbb{C}_p$ with good reduction (there is a minor assumption that it is obtained by extension of coefficients from a curve over a complete discretely valued subfield, which will always be satisfied in our applications). The rigid analytic space $X(\mathbb{C}_p)$ is set-theoretically decomposed as the union $X = \bigcup_x U_x$ where x varies over the points in the reduction of X and U_x is the residue disc (tube in the language of Berthelot) of points reducing to x. By the assumption of good reduction each residue disc is isomorphic to a disc $|z| < 1$. A wide open space U is obtained from X by fixing a finite and nonempty set of points S in the reduction and throwing away the discs inside the residue discs U_x, $x \in S$, isomorphic to $|z| < r$ for arbitrarily large $r < 1$. The space U should be thought of as the inverse limit of the corresponding spaces U_r.

Coleman theory associates to U the $\mathbb{C}_p$-algebra $A_{\mathrm{col}}(U)$ and the $A_{\mathrm{col}}(U)$-modules $\Omega^i_{\mathrm{col}}(U)$ with differentials forming a complex. The key property is that this complex is exact at the one and zero forms, that is, there is an exact sequence

$$0 \to \mathbb{C}_p \to A_{\mathrm{col}}(U) \to \Omega^1_{\mathrm{col}}(U) \to \Omega^2_{\mathrm{col}}(U).$$

The space $\Omega^1_{\mathrm{col}}(U)$ contains the space $\Omega^1(U)$ of overconvergent forms on U, that is, those forms that are rigid analytic on some U_r. Similarly, the space $A_{\mathrm{col}}(U)$ contains the space $A(U)$ of overconvergent functions. The differential extends the usual differential on the subspaces.

The whole picture extends to higher dimensions. We shall only need the case where U is one-dimensional. In this case the space $\Omega^2_{\mathrm{col}}(U)$ is already 0.

Coleman functions may be interpreted as locally analytic functions on U. More precisely, again in the one-dimensional case, for $x \notin S$, the intersection of the residue disc U_x with U is U_x, while for $x \in S$ it is an annulus e_x isomorphic to an annulus of the form $r < |z| < 1$. A Coleman function is analytic on each disc U_x and is in the polynomial algebra $A(e_x)[\log(z)]$ where z is a local parameter on an annulus U_x (here, there is an implicit global choice of a branch of the p-adic logarithm).

We define the space $A_{\mathrm{col},1}(U)$ to be the inverse image of $\Omega^1(U) \subseteq \Omega^1_{\mathrm{col}}(U)$ under the differential d. The space of differentials $\Omega^1_{\mathrm{col},1}(U)$ is $A_{\mathrm{col},1}(U) \cdot \Omega^1(U)$.

If $\omega \in \Omega^1(U_r)$ and $y, z \in U_r$ the integral $\int_z^y \omega$ is clearly well-defined as $f(y) - f(z)$ where $f \in A_{\mathrm{col}}(U_r)$ and $\mathrm{d}f = \omega$. It is a basic property of Coleman integration that if X, U, ω, z, y are all defined over the complete subfield K, then so is the integral $\int_z^y \omega$.

For $f \in A(U)$ the function $\log(f)$ is in $A_{\mathrm{col},1}(U)$. Pullback by a rigid analytic endomorphism ϕ of U (such as the Frobenius endomorphisms that will appear in the next section) preserves Coleman functions and in particular $A_{\mathrm{col},1}(U)$.

5. Regulators

In this section we compute the regulator on $\mathscr{C}^1(\mathbb{O})$ in (modified) syntomic cohomology. In case the element lies in the subspace $H^1(\mathscr{C}^\bullet(\mathbb{O}))$, we also explain how we wish to interpret the cup product of this regulator with the cohomology class of a form ω of the second kind on C, and what are the obstacles for doing so, thus paving the way for constructions in the next sections.

We first write down the relevant spaces and the (modified) syntomic complexes computing their cohomology. For the full story the reader should consult [Besser 2000b].

We begin with a smooth proper relative curve $\mathscr{C}/R$. Related to that is the space $X_{\mathscr{C}} := \mathbb{P}^1_{\mathscr{C}} \setminus \{t = 1\}$. The superscript loc will denote various localizations, obtained by removing the image of a finite number of R-sections. We note that the computations in this section can be done after a finite base change, so we may easily get from more general localizations into this situation by further localization. We shall use localizations $\mathscr{C}^{\mathrm{loc}}$ of $\mathscr{C}$ or $X^{\mathrm{loc}}_{\mathscr{C}}$ of $X_{\mathscr{C}}$. If the localization is nontrivial, and we may and do assume this, then all localized schemes are affine.

Our goal is to compute the syntomic regulator $K_4^{(3)}(\mathscr{C}) \to H^2_{\mathrm{syn}}(\mathscr{C}, 3)$. According to [Besser 2000b, Proposition 8.6.3] there is an isomorphism, commuting with the regulator, $H^2_{\mathrm{syn}}(\mathscr{C}, 3) \xrightarrow{\simeq} \tilde{H}^2_{\mathrm{ms}}(\mathscr{C}, 3)$, where $\tilde{H}_{\mathrm{ms}}$ is the Gros style modified rigid syntomic cohomology, in the sense of loc. cit. From now on we shall therefore concentrate on modified syntomic cohomology. We shall refer to it simply as syntomic cohomology.

Let us recall one of the possible models for modified syntomic cohomology for affine schemes. Let A be an affine R-scheme. We assume we have an open embedding $A \hookrightarrow \bar{A}$, where $\bar{A}$ is proper. From the embedding $A \hookrightarrow \bar{A}$ one obtains the overconvergent space $A^\dagger$. This space can be made sense of in Grosse-Klönne's

[2000] theory of overconvergent spaces as the space whose affine ring, $\mathcal{O}(A^\dagger)$, is the weak completion, in the sense of Monsky–Washnitzer, of $\mathcal{O}(A)$. However, here we shall simply think of $A^\dagger$ formally as the inverse system of strict neighborhoods of the special fiber of A in that of $\bar{A}$.

We further assume that we have an R-linear endomorphism $\phi : A^\dagger \to A^\dagger$ whose reduction is a power of Frobenius, say of degree $q = p^r$. We call ϕ a Frobenius endomorphism. Standard results [Coleman 1985, Theorem A-1; van der Put 1986, Theorem 2.4.4.ii] imply one always has such ϕ.

With the above data, we have

$$\tilde{H}^n_{\mathrm{ms}}(A, j) = H^n(\mathrm{MF}(F^j\Omega^\bullet(A^\dagger) \xrightarrow{1-\phi^*/q^j} \Omega^\bullet(A^\dagger))).$$

Here, the filtration is the stupid filtration on the space of differentials and MF denotes the mapping fiber (cone shifted by -1). To be more precise, one really needs to take the limit of these cohomology groups with respect to powers of ϕ, in a way explained in [Besser 2000b], but it is also explained there that one can ignore this point.

The cohomology groups $\tilde{H}_{\mathrm{ms}}$ are in fact functorial with respect to arbitrary maps of schemes. This functoriality is not at all obvious from the definition except in the case where the maps extend to the dagger spaces and commute with ϕ. Fortunately, this will always be the case for us. In this situation, one may also construct relative cohomology in the obvious way (the reader is advised to look at [Besser and de Jeu 2003, Section 5] for constructions of complexes computing relative syntomic cohomology).

To end this general review we recall that the corresponding syntomic regulator is defined by the formula

$$f \in \mathcal{O}(A)^* \subset K_1(A) \mapsto (\mathrm{dlog}(f), \log(f_0)/q) \in \tilde{H}^1_{\mathrm{ms}}(A, 1), \tag{5.1}$$

where $f_0 = f^q/\phi^*(f)$ and has the property that $\log(f_0)$ is in $\mathcal{O}(A^\dagger)$. We also recall from[Besser 2000b, Definition 6.5] that the cup product

$$\tilde{H}^\bullet_{\mathrm{ms}}(A, i) \times \tilde{H}^\bullet_{\mathrm{ms}}(A, j) \to \tilde{H}^\bullet_{\mathrm{ms}}(A, i + j)$$

is given by

$$\begin{aligned}(5.2)\quad &(\omega_1, \varepsilon_1) \cup (\omega_2, \varepsilon_2)\\ &= \Big(\omega_1 \wedge \omega_2,\ \varepsilon_1 \wedge \Big(\gamma + (1-\gamma)\frac{\phi^*}{q^j}\Big)\omega_2 + (-1)^{\deg \omega_1}\Big(\Big((1-\gamma) + \gamma\frac{\phi^*}{q^i}\Big)\omega_1\Big) \wedge \varepsilon_2\Big)\end{aligned}$$

for some constant γ, which can be taken arbitrarily (producing homotopic products).

We now write these constructions for the affine schemes we are considering. To simplify notation we write U for $(\mathscr{C}^{\mathrm{loc}})^\dagger$, U' for $(X^{\mathrm{loc}}_{\mathscr{C}})^\dagger$, and X_U for $(X_{\mathscr{C}^{\mathrm{loc}}})^\dagger$. We may localize so that $U' \subset X_U$. We fix a Frobenius endomorphism $\phi : U \to U$. We can then take the Frobenius endomorphism for X_U to be the product of ϕ with the map $t \mapsto t^q$ and for U' the restriction of this endomorphism to U'. Since $t \mapsto t^q$ fixes 0 and ∞ we can use the embedding of U in U' at $t=0$ and $t=\infty$ to create the complex computing relative cohomology. With this we have the following models for syntomic cohomology.

$$(5.3)\quad \tilde{H}^i_{\mathrm{ms}}(X^{\mathrm{loc}}_{\mathscr{C}}, i) = \frac{\left\{(\omega,\varepsilon),\, \omega\in\Omega^i(U'),\, \varepsilon\in\Omega^{i-1}(U'),\, \mathrm{d}\omega=0,\, \mathrm{d}\varepsilon=\left(1-\frac{\phi^*}{q^i}\right)(\omega)\right\}}{\{(0,\mathrm{d}\varepsilon),\, \varepsilon\in\Omega^{i-2}(U')\}}$$

for $i=1,2$. Now, for relative syntomic cohomology one we can write, by throwing away terms which are forced to be 0,

$$(5.4)\quad \tilde{H}^2_{\mathrm{ms}}(X^{\mathrm{loc}}_{\mathscr{C}}, \square, 2) = \frac{\left\{(\omega,\varepsilon,\varepsilon_\infty,\varepsilon_0),\ \begin{array}{l}\omega\in\Omega^2(U'),\ \varepsilon\in\Omega^1(U'),\ \varepsilon_s\in\mathbb{O}(U),\, s=0,\infty,\\ \mathrm{d}\omega=0,\ \mathrm{d}\varepsilon=\left(1-\frac{\phi^*}{q^2}\right)(\omega),\ \mathrm{d}\varepsilon_s=\varepsilon_{|t=s},\ s=0,\infty\end{array}\right\}}{\left\{\left(0,\mathrm{d}\varepsilon,\varepsilon_{|t=\infty},\varepsilon_{|t=0}\right),\ \varepsilon\in\mathbb{O}(U')\right\}}.$$

The map between $\tilde{H}^2_{\mathrm{ms}}(X_{\mathscr{C}}, \square, 2)$ and $\tilde{H}^2_{\mathrm{ms}}(X_{\mathscr{C}}, 2)$ remembers only ω and ε. Since U' is two dimensional and therefore does not support forms of degree 3, we also have

$$(5.5)\quad \tilde{H}^3_{\mathrm{ms}}(X^{\mathrm{loc}}_{\mathscr{C}}, \square, 3) = \frac{\{(\varepsilon,\varepsilon_\infty,\varepsilon_0),\, \varepsilon\in\Omega^2(U'),\, \varepsilon_s\in\Omega^1(U),\, \mathrm{d}\varepsilon=0,\, \mathrm{d}\varepsilon_s=\varepsilon_{|t=s}(s=0,\infty)\}}{\{(\mathrm{d}\varepsilon,\, \varepsilon_{|t=\infty}+\mathrm{d}\varepsilon_\infty,\, \varepsilon_{|t=0}+\mathrm{d}\varepsilon_0),\, \varepsilon\in\Omega^1(U'),\, \varepsilon_\infty,\varepsilon_0\in\mathbb{O}(U')\}}.$$

If we replace U' by X_U we obtain a model for $\tilde{H}^3_{\mathrm{ms}}(X_{\mathscr{C}^{\mathrm{loc}}}, \square, 3)$.

The last model is

$$(5.6)\qquad \tilde{H}^2_{\mathrm{ms}}(\mathscr{C}^{\mathrm{loc}}, 3) = \frac{\{\varepsilon\in\Omega^1(U),\, \mathrm{d}\varepsilon=0\}}{\{\mathrm{d}\varepsilon,\, \varepsilon\in\mathbb{O}(U)\}}.$$

This is of course just the first de Rham cohomology of U. However, the "correct" isomorphism with this cohomology is not the obvious one but rather the one twisted by $1-\phi^*/q^3$, that is,

$$(5.7)\qquad H^1_{\mathrm{dR}}(U/K) \to \tilde{H}^2_{\mathrm{ms}}(\mathscr{C}^{\mathrm{loc}}, 3), \quad [\eta]\mapsto[(1-\phi^*/q^3)\eta]$$

(for an explanation of this see [Besser 2000b, Proposition 10.1.3]). Here, and in what follows, we denote the cohomology class of an element in square brackets.

At this point, we are able to make more precise the definition of the p-adic regulator for open curves that was hinted at in the introduction before stating Theorem 1.11. As explained there, for each U as above, one has a canonical projection $H^1_{\mathrm{dR}}(U/K) \xrightarrow{\mathbf{p}} H^1_{\mathrm{dR}}(C/K)$. This is the unique Frobenius equivariant splitting of the natural restriction map in the other direction. These projections are compatible in the obvious way when restricting to a smaller U.

Definition 5.8. The regulator map

$$\mathrm{reg}'_p : K_4^{(3)}(\mathscr{C}^{\mathrm{loc}}) \to H^1_{\mathrm{dR}}(C/K)$$

is the composition

$$K_4^{(3)}(\mathscr{C}^{\mathrm{loc}}) \to H^1_{\mathrm{dR}}(U/K) \xrightarrow{\mathbf{p}} H^1_{\mathrm{dR}}(C/K)\,.$$

Using the compatibility of the maps $\mathbf{p}$ mentioned above for all possible $\mathscr{C}^{\mathrm{loc}}$, from $K_4^{(3)}(\mathbb{O}) = \varinjlim_{\mathscr{C}^{\mathrm{loc}}} K_4^{(3)}(\mathscr{C}^{\mathrm{loc}})$ (see [Quillen 1973, Proposition 2.2; Srinivas 1996, Lemma 5.9]) we also obtain a well defined regulator map

$$\mathrm{reg}'_p : K_4^{(3)}(\mathbb{O}) \to H^1_{\mathrm{dR}}(C/K).$$

We need a formula for the cup product

$$\tilde{H}^2_{\mathrm{ms}}(X^{\mathrm{loc}}_{\mathscr{C}}, \square, 2) \times \tilde{H}^1_{\mathrm{ms}}(X^{\mathrm{loc}}_{\mathscr{C}}, 1) \to \tilde{H}^3_{\mathrm{ms}}(X^{\mathrm{loc}}_{\mathscr{C}}, \square, 3)$$

in terms of the models (5.4), (5.3) and (5.5) respectively. Using the formula for a cup product between a cone and a complex and (5.2) with $\gamma = 0$ we find the following formula:

$$(\omega, \varepsilon, \varepsilon_\infty, \varepsilon_0) \cup (\eta, h) = \Big(h\omega + \varepsilon \wedge \frac{\phi^*}{q}\eta,\, \varepsilon_\infty \eta,\, \varepsilon_0 \eta\Big). \tag{5.9}$$

Suppose now that f and g are in $\mathbb{O}^*(\mathscr{C}^{\mathrm{loc}})$ (see Section 2.5.4). To compute the regulator of $[g]_2 \cup (f)$ we start with $[g]_2$ in $K_2^{(2)}(X^{\mathrm{loc}}_{\mathscr{C}}, \square)$. It maps in $K_2^{(2)}(X^{\mathrm{loc}}_{\mathscr{C}})$ to $-((t-g)/(t-1)) \cup (1-g)$, by pulling back along g the corresponding result for the universal elements [Besser and de Jeu 2003, Proposition 6.7].

Lemma 5.10. *We have in* $\tilde{H}^2_{\mathrm{ms}}(X^{\mathrm{loc}}_{\mathscr{C}}, 2)$ *that*

$$-\,\mathrm{reg}_p\Big(\frac{t-g}{t-1} \cup (1-g)\Big) = (\omega_g, \varepsilon_g)$$

in the model (5.3) *with*

$$\begin{aligned}
\omega_g &= -\,\mathrm{dlog}\Big(\frac{t-g}{t-1}\Big) \wedge \mathrm{dlog}(1-g)\\
\varepsilon_g &= \frac{1}{q}\log(1-g)_0\,\mathrm{dlog}\Big(\frac{t-g}{t-1}\Big) - \frac{1}{q^2}\log\Big(\frac{t-g}{t-1}\Big)_0\,\mathrm{dlog}\,\phi^*(1-g)
\end{aligned}$$

Proof. This follows from the formula (5.1) for the regulators of functions, the compatibility of reg_p with cup products and the cup product formula (5.2). □

In what follows, the notation $[a_1, \ldots, a_i]$ will denote the class of $(a_1, \ldots, a_i)$ in (5.4) or (5.5), depending on whether $i = 3$ or 4.

Proposition 5.11. *We have in* $\tilde{H}^2_{\mathrm{ms}}(X^{\mathrm{loc}}_{\mathscr{C}}, \square, 2)$, *using the model* (5.4),

$$\mathrm{reg}_p([g]_2) = [\omega_g, \varepsilon_g, 0, \Theta(g)]$$

where

$$\mathrm{d}\Theta(g) = \varepsilon_g|_{t=0} = \frac{1}{q}\log(1-g)_0 \operatorname{dlog} g - \frac{1}{q^2}\log g_0 \operatorname{dlog} \phi^*(1-g). \tag{5.12}$$

Proof. We are looking for a closed four-tuple, whose first two coordinates represent the cohomology class of $(\omega_g, \varepsilon_g)$. It is easy to see that we may assume that the first two coordinates are indeed $(\omega_g, \varepsilon_g)$. Then the closedness condition implies that the differentials of the next two coordinates give the restrictions to $t = \infty$ and $t = 0$, respectively, of ε_g. These are, respectively, 0 and $\varepsilon_g|_{t=0}$, so the result is clear. □

Remark 5.13. 1. One can show that there exists a function Θ on $\mathbb{P}^1$ such that $\Theta(g)$ is indeed the composition of Θ and g, but we shall not need to use this.

2. The determination of the regulator at this stage is incomplete, since we have only determined $\Theta(g)$ up to a constant. It will turn out that for the regulator computation this is irrelevant. For the computation of the boundary this becomes much trickier. We in fact failed to determine the boundary of the regulator directly. When we need this towards the end of Section 10 for the proof of Theorem 1.9, we shall use a trick to overcome this difficulty, which in particular forces us to assume working over a number field at that stage.

Proposition 5.14. *The regulator of* $[g]_2 \cup (f)$ *in* $\tilde{H}^3_{\mathrm{ms}}(X^{\mathrm{loc}}_{\mathscr{C}}, \square, 3)$ *is represented by the following element in the model* (5.5),

$$\varepsilon(g, f) := \Big(\frac{1}{q}\log f_0 \omega_g + \frac{1}{q}\varepsilon_g \wedge \phi^* \operatorname{dlog} f, 0, \frac{1}{q}\Theta(g)\phi^* \operatorname{dlog} f\Big).$$

Proof. This follows again from the compatibility of the regulator with cup products and from the formulas for the cup product in relative syntomic cohomology (5.9). □

Suppose now that $\alpha = \sum_i [g_i]_2 \cup (f_i)$ belongs to

$$H^1(\mathscr{C}^\bullet(\mathbb{O})) \simeq K_4^{(3)}(\mathbb{O})/K_3^{(2)}(\mathbb{O}) \cup \mathbb{O}^*_{\mathbb{Q}};$$

see (2.65). Note that α is only determined up to an element in $(1+I)^*_{\mathbb{O}} \cup \mathbb{O}^*_{\mathbb{Q}}$; see (2.61) and (2.64). A term in the latter space consists explicitly of elements of the form

$$\delta = \sum_j \delta_{1,j} \cup \delta_{2,j}, \tag{5.15}$$

with $\delta_{1,j} \in K_1^{(1)}(X^{\mathrm{loc}}_{\mathscr{C}}, \square)$ and $\delta_{2,j} \in K_2^{(2)}(\mathscr{C}^{\mathrm{loc}})$, for all possible localizations. Therefore, for an appropriately chosen $\mathscr{C}^{\mathrm{loc}}$, there exists $\beta \in K_3^{(3)}(X_{\mathscr{C}^{\mathrm{loc}}}, \square)$ whose restriction to $(X^{\mathrm{loc}}_{\mathscr{C}}, \square)$ is $\alpha + \delta$, where δ is as in (5.15). If we write $\mathrm{reg}_p(\beta) = [\varepsilon, \varepsilon_\infty, \varepsilon_0]$, with the ε's on X_U, then we have $[\varepsilon, \varepsilon_\infty, \varepsilon_0]|_{(X_{\mathscr{C}}^{\mathrm{loc}}, \square)} = \sum[\varepsilon(g_i, f_i)] + \mathrm{reg}_p(\delta)$. Writing this explicitly, this means that

$$(\varepsilon, \varepsilon_\infty, \varepsilon_0)|_{(U', \square)} = \sum \varepsilon(g_i, f_i) + \mathrm{reg}_p(\delta) + (\mathrm{d}\lambda, \lambda_{|t=\infty}, \lambda_{|t=0})$$

for some $\lambda \in \Omega^1(U')$ and where now $\mathrm{reg}_p(\delta)$ means any form representing this class.

The isomorphism $T_0^\infty : \tilde{H}^3_{\mathrm{ms}}(X_{\mathscr{C}^{\mathrm{loc}}}, \square, 3) \cong \tilde{H}^2_{\mathrm{ms}}(\mathscr{C}^{\mathrm{loc}}, 3)$ is obtained by integration from 0 to ∞. More precisely it is given by

$$[\varepsilon, \varepsilon_\infty, \varepsilon_0] \mapsto \left[\left(\int_0^\infty \varepsilon\right) - (\varepsilon_\infty - \varepsilon_0)\right] \tag{5.16}$$

where the integration is only with respect to the variable t;

$$\int_0^\infty (f(x,t)\,\mathrm{d}t \wedge \mathrm{d}x) = \left(\int_0^\infty f(x,t)\,\mathrm{d}t\right)\mathrm{d}x. \tag{5.17}$$

Note that we are integrating forms on X_U. For forms on U' we may do Coleman integration instead (Section 4). This technique was introduced in [Besser and de Jeu 2003, Section 5]. Note that we only discussed Coleman integration over $\mathbb{C}_p$. The extension of scalars of U and the fibers of $U' \to U$, to $\mathbb{C}_p$ are wide open space in the sense of Coleman so one can do Coleman integration on them. By abuse of notation we shall continue to denote this extension of scalars by the same letters. Coleman integration will be the same as ordinary integration if the forms extend to X_U. The theory of Coleman integration is not sufficiently developed yet to tell us that what we do makes sense in general, so we must be careful to check that it makes sense for the particular forms we are working with.

Now we check what happens to the term $\varepsilon(g, f)$ under this integration, which we continue to denote by T_0^∞. The integral of the first term is

$$\begin{aligned}\int_0^\infty \frac{1}{q}\log f_0\omega_g + \frac{1}{q}\varepsilon_g \wedge \phi^*\,\mathrm{dlog}\, f &= \frac{1}{q}\log f_0 \int_0^\infty \omega_g + \frac{1}{q}\left(\int_0^\infty \varepsilon_g\right) \wedge \phi^*\,\mathrm{dlog}\, f\\ &= \frac{1}{q}\log f_0 \log g\,\mathrm{dlog}(1-g) - \frac{1}{q^2}\log(1-g)_0 \log g \phi^*\,\mathrm{dlog}\, f.\end{aligned}$$

The last equality follows because $\int_0^\infty \mathrm{dlog}((t-g)/(t-1)) = -\log g$ and the term involving $\log((t-g)/(t-1))_0$ vanishes because it does not involve a $\mathrm{d}t$. Subtracting the term $\varepsilon_\infty - \varepsilon_0$ we obtain

$$\begin{aligned}T_0^\infty \varepsilon(g, f) = \frac{1}{q}&\log f_0 \log g\,\mathrm{dlog}(1-g)\\ &-\frac{1}{q^2}\log(1-g)_0 \log g\phi^*\,\mathrm{dlog}\, f + \frac{1}{q}\Theta(g)\phi^*\,\mathrm{dlog}\, f.\end{aligned} \tag{5.18}$$

Note that this integral belongs to $\Omega^1_{\mathrm{col},1}(U)$, in the notation of Section 4.

Lemma 5.19. *For δ in $(1+I)^*_{\mathbb{O}} \cup K_2^{(2)}(\mathbb{O})$ we have $T_0^\infty(\mathrm{reg}_p(\delta)) = 0$.*

Proof. As in (5.15) δ is a sum of terms of the form $\delta_1 \cup \delta_2$ with δ_1 in $K_1^{(1)}(X_{\mathscr{C}}^{\mathrm{loc}}, \square)$ and δ_2 in $K_2^{(2)}(\mathscr{C}^{\mathrm{loc}})$. That T_0^∞ vanishes on these elements follows from the proof of [Besser and de Jeu 2003, Proposition 7.2]. □

Now we deal with the term $(\mathrm{d}\lambda, \lambda_{|t=\infty}, \lambda_{|t=0})$.

Proposition 5.20. *Suppose that $X_{\mathscr{C}}^{\mathrm{loc}}$ is obtained from $X_{\mathscr{C}^{\mathrm{loc}}}$ by removing the graphs of $t = h_j(x)$ for $j = 1, \ldots, n$. Assume further that the reductions of those graphs are either disjoint or identical (which we can achieve by shrinking $\mathscr{C}^{\mathrm{loc}}$). Then there are $a_j(x), a(x) \in \mathbb{O}(U)$ such that we have*

$$T_0^\infty(\mathrm{d}\lambda, \lambda_{|t=\infty}, \lambda_{|t=0}) = \mathrm{d}(a + \sum_j a_j \log(h_j)),$$

where, if there are two h_j with identical reduction, one may take just one of them. In particular, it belongs to $\Omega^1_{\mathrm{col},1}(U)$.

Proof. We have global coordinates x and t on U' so we can write $\lambda = f(x,t)\,\mathrm{d}x + g(x,t)\,\mathrm{d}t$. Then

$$\mathrm{d}\lambda = \Big(\frac{\partial f}{\partial t} - \frac{\partial g}{\partial x}\Big)\,\mathrm{d}t \wedge \mathrm{d}x.$$

Therefore

$$\int_{t=0}^{t=\infty} \mathrm{d}\lambda = (f(x,\infty) - f(x,0))\,\mathrm{d}x - \Big(\int_{t=0}^{t=\infty} \frac{\partial g}{\partial x}\,\mathrm{d}t\Big)\,\mathrm{d}x.$$

But the first term is exactly $\lambda|_{t=\infty} - \lambda|_{t=0}$ so we find

$$T_0^\infty\,[\mathrm{d}\lambda, \lambda_{|t=\infty}, \lambda_{|t=0}] = -\mathrm{d}\Big(\int_{t=0}^{t=\infty} g(x,t)\,\mathrm{d}t\Big).$$

Consider now the two-form $\gamma = g(x,t)\mathrm{d}x \wedge \mathrm{d}t \in \Omega^2(U')$. This is closed so represents a cohomology class in $H^2_{\mathrm{rig}}((X_{\mathscr{C}}^{\mathrm{loc}})_\kappa/K)$. We have a short exact sequence

$$H^2_{\mathrm{rig}}((X_{\mathscr{C}^{\mathrm{loc}}})_\kappa/K) \to H^2_{\mathrm{rig}}((X_{\mathscr{C}}^{\mathrm{loc}})_\kappa/K \xrightarrow{\mathrm{Res}} \oplus_i H^1_{\mathrm{rig}}((\mathscr{C}^{\mathrm{loc}})_\kappa/K),$$

where the map $\mathrm{Res} = \oplus_j \mathrm{Res}_j$ is the sum of the boundary maps on the reductions of $t = h_j(x)$, composed with the pullback under the isomorphisms of these graphs with $(\mathscr{C}^{\mathrm{loc}})_\kappa$. Suppose that $\mathrm{Res}_j(\gamma)$ is the cohomology class of $a_j(x)\,\mathrm{d}x \in \Omega^1(U)$. Let $\gamma_j := a_j(x)\,\mathrm{d}x \wedge \mathrm{dlog}(t - h_j(x))$. Clearly $\mathrm{Res}_l(\gamma_j) = 0$ if $l \neq j$. We claim that $\mathrm{Res}_j(\gamma_j) = \mathrm{Res}_j(\gamma)$. This can be seen easily by applying the map $(x,t) \to (x, t - h_j(x))$, transforming γ_j to $a_j(x)\,\mathrm{d}x \wedge \mathrm{dlog}(t)$. Thus, $\gamma - \sum_j \gamma_j$ extends to $H^2_{\mathrm{rig}}((X_{\mathscr{C}^{\mathrm{loc}}})_\kappa/K)$ and its integral is a holomorphic one form on U. Let this form be $a(x)\,\mathrm{d}x$. Since $\int_{t=0}^{t=\infty} \gamma_j = \pm a_j(x) \log(h_j(x))\mathrm{d}x$

we find $\pm\int_{t=0}^{t=\infty}\gamma = (a(x)+\sum a_j(x)\log(h_j(x)))\,\mathrm{d}x$ and dividing by $\mathrm{d}x$ we find $\int_{t=0}^{t=\infty} g(x,t)\,\mathrm{d}t = \pm(a(x)+\sum a_j(x)\log(h_j(x)))$. This completes the proof. □

These results give us a strategy for breaking the regulator into a sum of terms, each depending on the pairs (g_i, f_i), as follows. Suppose that ω is a form of the second kind on C and let $[\omega]$ be its cohomology class in $H^1_{\mathrm{dR}}(C/K)$.

Definition 5.21. A functional $L_\omega : \Omega^1_{\mathrm{col},1}(U) \to \mathbb{C}_p$ will be called good if it has the following properties:

- it kills terms of the forms $\mathrm{d}a$ and $\mathrm{d}(a\log f)$ for $a, f \in \mathcal{O}(U)$,
- if η is in $\Omega^1(U)$ then we have $L_\omega(\eta) = [\omega]\cup \mathbf{p}([\eta])$.

Proposition 5.22. *Suppose that an element β in $K_4^{(3)}(\mathscr{C}^{\mathrm{loc}})$ maps to $\sum_i [g_i]_2 \cup (f_i)$ in $H^1(\mathscr{C}^\bullet(\mathbb{O}))$ under the natural map*

$$K_4^{(3)}(\mathscr{C}^{\mathrm{loc}}) \to K_4^{(3)}(\mathbb{O}) \to K_4^{(3)}(\mathbb{O})/K_3^{(2)}(\mathbb{O})\cup \mathbb{O}^*_{\mathbb{Q}}$$

(*see* (2.65)), *and that* $\mathrm{reg}_p(\beta) = [\eta_0]$ *in the model* (5.6). *Then we have, for a good functional L_ω,*

$$[\omega]\cup \mathbf{p}([\eta_0]) = \sum_i L_\omega\big(T_0^\infty \varepsilon(g_i, f_i)\big).$$

Proof. We must first show that the map

$$K_4^{(3)}(\mathscr{C}^{\mathrm{loc}}) \xrightarrow{\mathrm{reg}_p} \tilde{H}^2_{\mathrm{ms}}(\mathscr{C}^{\mathrm{loc}}, 3) \xrightarrow{\eta_0 \mapsto L_\omega(\eta_0)} \mathbb{C}_p$$

factors via $K_4^{(3)}(\mathbb{O})/K_3^{(2)}(\mathbb{O})\cup \mathbb{O}^*_{\mathbb{Q}}$. By further localizing, it suffices to show that the map above vanishes on elements of the form $\gamma\cup f$ with $\gamma \in K_3^{(2)}(\mathscr{C}^{\mathrm{loc}})$ and $f \in \mathbb{O}^*(\mathscr{C}^{\mathrm{loc}})$. We have

$$\tilde{H}^1_{\mathrm{ms}}(\mathscr{C}^{\mathrm{loc}}, 2) = \{(0,\varepsilon), \varepsilon \in \mathcal{O}(U),\ \mathrm{d}\varepsilon = 0\} = \{(0,\varepsilon), \varepsilon\in K\}. \tag{5.23}$$

Thus $\mathrm{reg}_p(\gamma) = (0,\alpha)$ for some $\alpha \in K$. On the other hand, by (5.1) we have $\mathrm{reg}_p(f) = (\mathrm{dlog}\, f, \log(f_0)/q)$ (here it does not matter what f_0 is). Using (5.2) we obtain, in the model (5.6)

$$\mathrm{reg}_p(\gamma\cup f) = (0,\alpha)\cup(\mathrm{dlog}\, f, \log(f_0)/q) = \alpha\,\mathrm{dlog}\, f.$$

The factorization thus follows from first property of the good functional. Next, by Proposition 5.20 the first property also implies that L_ω kills all terms of the form $T_0^\infty[\mathrm{d}\lambda, \lambda_{|t=\infty}, \lambda_{|t=0}]$. The result now follows immediately from the discussion above. □

Remark 5.24. There is a final wrinkle here because of the normalization (5.7) for the syntomic regulator. For β as in the corollary, the regulator of β is in fact $[\eta]$ with $(1-(\phi^*/q^3))[\eta] = [\eta_0]$. Thus, once we have the functional L_ω we shall be able to compute $[\omega] \cup \mathbf{p}(\eta_0)$ but will in fact want $[\omega] \cup \mathbf{p}(\eta)$. Fortunately, it is easy to see (and will be explained) that if we know $[\omega] \cup \mathbf{p}(\eta_0)$ for *all* ω, then we also know $[\omega] \cup \mathbf{p}(\eta)$ for all ω. In fact, as in previous computations, the result with η is much simpler than with η_0, confirming the "correctness" of our normalization.

Remark 5.25. As with some of our previous works on syntomic regulators, one can ask about the sign compatibility between the p-adic and classical regulators; see [Besser et al. 2009, Remark 4.16]. As explained in Remark 3.3, the signs in the various isomorphisms induced by using relative K-theory and relative Deligne or de Rham cohomology are normalized by choosing one of the natural isomorphisms $H^2_{\mathrm{dR}}(X_{C_{\mathrm{an}}}; \square; \mathbb{R}(2)) \simeq H^1_{\mathrm{dR}}(C_{\mathrm{an}}; \square; \mathbb{R}(2))$, in this case by choosing (3.5), and then demanding that (3.6) commutes. The same approach works for the syntomic regulator, using (5.16) and the analogue of (3.6) for syntomic cohomology.

Because the descriptions of relative cohomology in 3.3 and (5.5) and the signs in front of the term $\varepsilon_\infty - \varepsilon_0$ in (3.5) and (5.16) are the same (note that just as in Section 3 we are ultimately cupping on the left with ω; see Proposition 5.22), we have chosen the "relativity isomorphism" for Deligne (or de Rham) cohomology and syntomic cohomology in a compatible way. Therefore (3.6) and its analogue for syntomic cohomology lead to the same sign for the K-theory (under the compatibility of the constructions for $\mathbb{C}$ and F as explained in Section 2).

6. Wishes

This section is highly speculative. It contains no formal proofs. Nevertheless, we feel it is vital for the understanding of a significant portion of the computations to come. It also suggests interesting research directions into a more canonical representation of syntomic cohomology, one that would make the computations in the syntomic case equivalent to the complex case.

We want to follow a strategy that proved very successful in computing syntomic regulators on K_2 of curves; see the discussion after Proposition 5.2 in [Besser 2000c]. We argue heuristically, in some make-believe world where syntomic cohomology looks much more like Deligne cohomology from the computational standpoint, and get a formula for the regulator. Then we try to relate this formula with the formula we obtained in the previous section and see what needs to be proved to show that the two formulas are equivalent. That the make-believe formula turns out to be correct is a strong indication that one should be able to turn the make-believe computation into a rigorous one.

The make-believe computation is based on the following assumptions:

- The "cohomology" is given by the pairs (ω, h) where ω is an i-form and h is an $i-1$ form with $\mathrm{d}h = \omega$. Of course h is not an actual form but something like a Coleman form, for example a Coleman function.

- The "regulator" of a function f is the pair $(\operatorname{dlog}(f), \log(f))$.
- The cup product is given by $(\omega_1, h_1) \cup (\omega_2, h_2) = (\omega_1 \wedge \omega_2, \omega_1 \wedge h_2 \text{ or } h_1 \wedge \omega_2)$.

With these rules, we can redo the computation from the previous section in this make-believe language: We have in $\tilde{H}^2_{\mathrm{ms}}(X^{\mathrm{loc}}_{\mathscr{C}}, 2)$ that

$$-\operatorname{reg}_p\Bigl(\frac{t-g}{t-1} \cup (1-g)\Bigr) = (\omega_g, \varepsilon_g)$$

with ω_g as in Lemma 5.10 and

$$\varepsilon_g = -\log(1-g)\operatorname{dlog}\Bigl(\frac{t-g}{t-1}\Bigr).$$

Since the restriction of ε_g to $t = 0$ is $-\log(1-g)\operatorname{dlog}(g) = \mathrm{d}\,\mathrm{Li}_2(g)$ we have, following the proof of Proposition 5.11, that

$$\operatorname{reg}_p([g]_2) \in \tilde{H}^2_{\mathrm{ms}}(X^{\mathrm{loc}}_{\mathscr{C}}, \square, 2) \quad \text{equals } [\omega_g, \varepsilon_g, 0, \mathrm{Li}_2(g)].$$

Cupping with $(\operatorname{dlog}(f), \log(f))$ we get

$$\tilde{\varepsilon}(g, f) := \operatorname{reg}_p([g]_2 \cup (f)) = \Bigl[-\log(f)\operatorname{dlog}\Bigl(\frac{t-g}{t-1}\Bigr) \wedge \operatorname{dlog}(1-g)), 0, 0\Bigr].$$

Applying T_0^∞ we find $T_0^\infty(\tilde{\varepsilon}(g, f)) = \log(f)\log(g)\operatorname{dlog}(1-g)$.

We now compare this with $T_0^\infty \varepsilon(g, f)$ of (5.18). Continuing to mimic the discussion of the K_2 in [Besser 2000c], the former version should be an untwisted version of the latter, that is, without the "twist" by $(1-(\phi^*/q^3))$. To see this, we use the formalism described in [Besser 2000c, Remark 3.1] to get

$$(6.1)\quad \Bigl(1-\frac{\phi^*}{q^3}\Bigr)[\log(f)\log(g)\operatorname{dlog}(1-g)] = \frac{1}{q}\log(f_0)\log(g)\operatorname{dlog}(1-g) + \frac{1}{q^2}\log\phi^*(f)\log(g)\operatorname{dlog}(1-g)_0 + \frac{1}{q^3}\log(g_0)\log\phi^*(f)\phi^*\operatorname{dlog}(1-g).$$

This already begins to look similar to $T_0^\infty \varepsilon(g, f)$, but there are differences. We want to argue that the difference is "exact". This cannot be taken to simply mean being the differential of something, since in Coleman's theory every form is integrable. Experience has shown that things are exact if they are the differential of a product of functions. We shall use two such assertions. Each one will correspond to a precise statement in the following sections, which will be justified by the techniques we shall introduce. To remind ourselves where these occurred, we shall call them "Wishes", and mark them explicitly.

Wish 6.2. We have in cohomology that $\Theta(g)\operatorname{dlog}\phi^*(f) = -\log\phi^*(f)\,\mathrm{d}\Theta(g)$.

Using this wish we can write the term $\frac{1}{q}\Theta(g)\,\mathrm{dlog}\,\phi^*(f)$ in (5.18) as

$$\begin{aligned}
&-\frac{1}{q}\mathrm{d}\Theta(g)\log\phi^*(f)\\
&\quad=-\frac{1}{q}\left(\frac{1}{q}\log(1-g)_0\,\mathrm{dlog}\,g-\frac{1}{q^2}\log g_0\,\mathrm{dlog}\,\phi^*(1-g)\right)\log\phi^*(f)\\
&\quad=-\frac{1}{q^2}\log(1-g)_0\,\mathrm{dlog}(g)\log\phi^*(f)+\frac{1}{q^3}\log(g_0)\,\mathrm{dlog}\,\phi^*(1-g)\log\phi^*(f),
\end{aligned}$$

so we obtain

$$\begin{aligned}
T_0^\infty\varepsilon(g,f)&=\frac{1}{q}\log(f_0)\log(g)\,\mathrm{dlog}(1-g)-\frac{1}{q^2}\log(1-g)_0\log(g)\phi^*\,\mathrm{dlog}(f)\\
&\quad-\frac{1}{q^2}\log(1-g)_0\,\mathrm{dlog}(g)\log\phi^*(f)+\frac{1}{q^3}\log(g_0)\,\mathrm{dlog}\,\phi^*(1-g)\log\phi^*(f).
\end{aligned}$$

Comparing this with $(1-(\phi^*/q^3))(\log(f)\log(g)\,\mathrm{dlog}(1-g))$ given in (6.1) we see that the first and last terms are the same, and that therefore we get our desired equality, "twisted" by $1-(\phi^*/q^3)$ if we get our second wish to come true.

Wish 6.3. We have in cohomology that

$$\begin{aligned}
&\log(1-g)_0\log(g)\phi^*(\mathrm{dlog}(f))+\log(1-g)_0\log\phi^*(f)\,\mathrm{dlog}(g)\\
&\qquad\qquad+\log(g)\log\phi^*(f)\,\mathrm{dlog}(1-g)_0
\end{aligned}$$

is trivial.

In Sections 7 and 8 we shall introduce triple indices. The wishes described above correspond to precise results stated in terms of triple indices, which we can indeed prove.

7. The triple index, local theory

We first briefly recall the theory of the "local index" from [Besser 2000c, Section 4]. In our new context this should be called the double index. To make things slightly simpler, we work in an algebraic context. The transition to working with annuli is straightforward.

Let K be a field of characteristic 0. We consider the algebra $A_{\log}:=K((z))[\log(z)]$ of polynomials over the formal variable $\log(z)$, over the field of finite to the left Laurent power series in z. We further consider the module of differentials $A_{\log}\cdot\mathrm{d}z$. It is an easy exercise in integration by parts to see that every form in $A_{\log}\cdot\mathrm{d}z$ has an integral in $A_{\log}$ in a unique way up to a constant. We distinguish in $A_{\log}$ the subfield $\mathsf{Mer}:=K((z))$ of meromorphic functions and the subspace $A_{\log,1}=\mathsf{Mer}+K\cdot\log(z)$ consisting exactly of all functions whose differential is in $\mathsf{Mer}\cdot\mathrm{d}z$. To $F\in A_{\log,1}$

we can associated the residue of its differential $\operatorname{Res} \mathrm{d}F \in K$. If $F \in A_{\log,1}$, then $F \in \mathsf{Mer}$ if and only if $\operatorname{Res} \mathrm{d}F = 0$.

Definition 7.1 [Besser 2000c, Proposition 4.5]. The double index is the unique antisymmetric bilinear form $\langle \cdot, \cdot \rangle : A_{\log,1} \times A_{\log,1} \to K$ such that $\langle F, G \rangle = \operatorname{Res} F \,\mathrm{d}G$ whenever this last expression makes sense.

We recall that the construction of this index is essentially trivial: one notices that the antisymmetry forces $\langle \log(z), \log(z) \rangle = 0$ and that $\langle F, G \rangle = -\operatorname{Res} G \,\mathrm{d}F$ whenever this expression makes sense. Then one writes $F = \alpha \log(z) + f$, $G = \beta \log(z) + g$ with $f, g \in \mathsf{Mer}$ and then one uses the bilinearity to write $\langle F, G \rangle$ as a sum of terms that can be computed.

The triple index turns out to be a bit more complicated. First of all we need to explain on which data it is evaluated:

- three functions F, G, H in $A_{\log,1}$,
- for each two functions R and S out of F, G, H a choice of $\int R \,\mathrm{d}S$ (that is, a function in $A_{\log}$ whose differential is $R \,\mathrm{d}S$) and of $\int S \,\mathrm{d}R$ in such a way that

$$\int R \,\mathrm{d}S + \int S \,\mathrm{d}R = RS. \tag{7.2}$$

As it will turn out this information is a bit redundant: clearly $\int R \,\mathrm{d}S$ determines $\int S \,\mathrm{d}R$. Also it will turn out that the index will be independent of $\int F \,\mathrm{d}G$. Still, these symmetric data are very convenient. To not carry around too much notation, we shall simply denote these data by $(F, G; H)$, where the additional choices should be understood from the context. In particular, any permutation of F, G, H induces an obvious permutation of the additional data. Also, if $(F_i, G; H)$, $i = 1, 2$ are given with all their additional data then there is a natural choice of data for $(F_1 + F_2, G; H)$, and similarly in the second and third positions. If we do need to indicate a change in the auxiliary data we shall write this as $(F, G; H | I_{F\mathrm{d}G}, \dots)$, where the subscript $F \,\mathrm{d}G$ indicates that I is an integral of $F \,\mathrm{d}G$.

Proposition 7.3. *There exists a unique function from data as above to K, denoted $(F, G; H) \mapsto \langle F, G; H \rangle$, called the triple index, such that the following conditions are satisfied.*

(1) *Trilinearity: the triple index is linear in each of the three variables, which means that $\langle \alpha_1 F_1 + \alpha_2 F_2, G; H \rangle = \alpha_1 \langle F_1, G; H \rangle + \alpha_2 \langle F_2, G; H \rangle$ provided that all auxiliary data are chosen in the way indicated above, and similarly for linearity in G and H.*

(2) *Symmetry: we have $\langle F, G; H \rangle = \langle G, F; H \rangle$, again with the choice of auxiliary data indicated above.*

(3) *Triple identity*: *we have*, *again with the obvious additional choices*,

$$\langle F, G; H\rangle + \langle F, H; G\rangle + \langle G, H; F\rangle = 0.$$

(4) *Reduction to the double index*: *if* $G \in \mathsf{Mer}$ *then* $\langle F, G; H\rangle = \langle F, \int G\,\mathrm{d}H\rangle$, *where* $\int G\,\mathrm{d}H$ *is taken from the auxiliary data and is in* $A_{\log,1}$ *because by assumption* $G\,\mathrm{d}H \in \mathsf{Mer}\cdot \mathrm{d}z$.

Proof. We first show that the dependency on the choices of integrals is forced by the properties of the triple index.

Lemma 7.4. *Suppose that the triple index exists. We then have the following change of constant formulas*:

(1) *If* C *is a constant*, *then*

$$\langle F, G; H|(I+C)_{G\mathrm{d}H}, (J-C)_{H\mathrm{d}G}\rangle = \langle F, G; H|I_{G\mathrm{d}H}, J_{H\mathrm{d}G}\rangle - C\cdot \operatorname{Res}\mathrm{d}F,$$

$$\langle F, G; H|(I+C)_{F\mathrm{d}H}, (J-C)_{H\mathrm{d}F}\rangle = \langle F, G; H|I_{F\mathrm{d}H}, J_{H\mathrm{d}F}\rangle - C\cdot \operatorname{Res}\mathrm{d}G.$$

(2) *The triple index is independent of the integral* $\int F\,\mathrm{d}G$.

Proof. We use the trilinearity. Consider the data $(F, 0; H)$, where the additional data are the same for F and H but we take the integral of $0\,\mathrm{d}H$ to be C, hence we are forced to take that of $H\,\mathrm{d}0$ to be $-C$. We take $\int 0\,\mathrm{d}F = 0$. The trilinearity implied that $\langle F, G; H\rangle$ and $\langle F, 0; H\rangle$ gives the left-hand side of the formula. But reduction to the double index means that $\langle F, 0; H\rangle = \langle F, C\rangle = -\operatorname{Res} C\,\mathrm{d}F$. An identical argument proves the second case. Finally, if in the above argument we take instead $\int 0\,\mathrm{d}F = D$ and $\int 0\,\mathrm{d}H = 0$, we see from exactly the same argument that the integral is independent of the auxiliary choice $\int F\,\mathrm{d}G$. □

We now check that the triple index is uniquely defined on all data where at least one of F, G, H is in Mer. Clearly in this case we can use reduction to the double index together with symmetry and the triple formula to compute the index, so it is clearly unique. The following lemma gives existence.

Lemma 7.5. *Consider the following recipe*:

(1) *if* $G \in \mathsf{Mer}$ *define* $\langle F, G; H\rangle = \langle F, \int G\mathrm{d}H\rangle$,

(2) *if* $F \in \mathsf{Mer}$ *define* $\langle F, G; H\rangle = \langle G, F; H\rangle$ *where the last expression is defined as in* (1),

(3) *if* $H \in \mathsf{Mer}$ *define* $\langle F, G; H\rangle = -(\langle F, H; G\rangle + \langle G, H; F\rangle)$ *where each of these terms is defined as in* 1.

Then this recipe gives a well-defined $\langle F, G; H\rangle$ *in all cases where at least one of* F, G *and* H *is in* Mer *and restricted to this subset it satisfies all properties of the triple index.*

Proof. To show that this expression is well-defined we need to consider what happens when two of F, G, H are in Mer: If $F, G \in \mathsf{Mer}$ we check that $\langle F, \int G\,\mathrm{d}H\rangle = \langle G, \int F\,\mathrm{d}H\rangle$. This follows because by the definition of the double index both expressions equal $\operatorname{Res} FG\,\mathrm{d}H$. Next we check that if $G, H \in \mathsf{Mer}$ then

$$\begin{aligned}
&\langle F, \textstyle\int G\,\mathrm{d}H\rangle + \langle F, \int H\,\mathrm{d}G\rangle + \langle G, \int H\,\mathrm{d}F\rangle \\
&\qquad = \langle F, GH\rangle + \langle G, \textstyle\int H\,\mathrm{d}F\rangle \quad \text{by bilinearity of the double index and (7.2)} \\
&\qquad = -\operatorname{Res} GH\,\mathrm{d}F + \operatorname{Res} GH\,\mathrm{d}F = 0.
\end{aligned}$$

Thus we find that we have a well-defined expression. We need to check that all properties of the expected triple index hold in this case. Trilinearity is essentially clear from the bilinearity of the double index. Symmetry is also easy: if F or G are in Mer then symmetry follows from the first two rules. If H is in Mer then the expression in (3) is clearly symmetric in F and G. The triple identity is forced by (3) and the reduction to the double index is an immediate consequence of our check that the triple index is well-defined. □

Note that the proof of Lemma 7.4 applies verbatim for this partial triple index, so we know the dependency on the choices of integrals.

To extend the triple index to all F, G and H we first check the case where $F = G = H = \log(z)$. Then we can arrange that all auxiliary data equal $\frac{1}{2}\log^2(z)$. The triple formula implies immediately that (with these data)

$$\langle \log(z), \log(z); \log(z)\rangle = 0. \tag{7.6}$$

We can now demonstrate uniqueness for the triple index. Suppose $F_i = \alpha_i \log(z) + f_i$, $i = 1, 2, 3$ where $\alpha_i \in K$ and $f_i \in \mathsf{Mer}$. Choose some auxiliary data $\int R\,\mathrm{d}S$ for any two R and S out of f_i and $\alpha_i \log(z)$, where we continue to take $\int \log(z)\,\mathrm{dlog}(z) = \frac{1}{2}\log^2(z)$. Using trilinearity and (7.6) we can write $\langle F_1, F_2; F_3\rangle$, with some choice of auxiliary data, as the sum with some coefficients of triple indices where at least one of the entries is in Mer, which are therefore computable by previous considerations. Now we can use change of constant to write $\langle F_1, F_2; F_3\rangle$ with arbitrary auxiliary data. This shows uniqueness and gives a formula for the general index. We need to check that this formula is well-defined, which, given the fact that all the summands are well-defined thanks to Lemma 7.5, amounts to checking independence of the choices of the auxiliary data. This is just a tedious formal check: suppose for example that we add C to $\int \alpha_1 \log(z)\,\mathrm{d}f_3$, and correspondingly subtract C from $\int f_3\alpha_1\,\mathrm{dlog}\,z$. This will have the effect that $\int F_1\,\mathrm{d}F_3$ will have C added to it and $\int F_3\mathrm{d}F_1$ will have C subtracted from it. This procedure will subtract $\alpha_2 C = C \operatorname{Res} \mathrm{d}F_2$ from $\langle \alpha_1, \alpha_2 \log(z); f_3\rangle$ and will not change any of the other indices. This shows that the change does not alter the index.

It remains to check that our formula satisfies all the properties for the triple index. First the change of constant formula of Lemma 7.4 is clear because we used it in the definition and we showed that the formula we get is well-defined. Now given change of constant it easy to see that it is enough to check trilinearity, symmetry and triple identity for one choice of auxiliary data. The derivation of these three formulas is then completely formal. Finally, reduction to the double index can only occur if at least one α_i is 0. But in this case we clearly get the triple index for the case where $F_i \in \mathsf{Mer}$ so we know this formula already. □

To compute the triple index in some concrete situations, which will be needed later, we introduce the notion of the constant term.

Definition 7.7. The constant term with respect to the variable z is the linear functional $c_z : A_{\log} \to K$, first defined on Mer by

$$c_z\Big(\sum a_n z^n\Big) = a_0,$$

and then in general by

$$c_z\bigg(\sum_{i=0}^{\infty} f_i(z)\log^i(z)\bigg) = c_z(f_0).$$

Note that the unlike the triple index, the constant term definitely depends on the choice of the local parameter z. For example, for $\alpha \in K$ and the function $f(z) = \log(z) = \log(\alpha z) - \log(\alpha)$ we have $c_z(f) = 0$ but $c_{\alpha z}(f) = -\log(\alpha)$.

Proposition 7.8. *Let F, G and H be 3 functions in $A_{\log,1}$ whose differentials (which are in $\mathsf{Mer}\,\mathrm{d}z$) have at most simple poles at 0. The choice of integrals $\int F\,\mathrm{d}H$ and $\int G\,\mathrm{d}H$ gives auxiliary data for the computation of $\langle F, G; H\rangle$ and with respect to this choice we have*

$$\langle F, G; H\rangle = c_z(F)\cdot c_z(G)\cdot \operatorname{Res}\mathrm{d}H - \operatorname{Res}\mathrm{d}F\cdot c_z\Big(\int G\,\mathrm{d}H\Big) - \operatorname{Res}\mathrm{d}G\cdot c_z\Big(\int F\,\mathrm{d}H\Big).$$

Proof. We have a bilinear map

$$(F, H) \to \int' F\,\mathrm{d}H := \text{ unique } \int F\,\mathrm{d}H \text{ with } c_z\Big(\int F\,\mathrm{d}H\Big) = 0.$$

Therefore, we see that the map

$$(F, G, H) \to \langle F, G; H\rangle' := \Big\langle F, G; H \,\Big|\, \int' F\,\mathrm{d}H_{F\mathrm{d}H}, \int' G\,\mathrm{d}H_{G\mathrm{d}H}\Big\rangle$$

is trilinear and symmetric in F and G. By Lemma 7.4 it suffices to prove that

$$\langle F, G, H\rangle' = c_z(F)\cdot c_z(G)\cdot \operatorname{Res}\mathrm{d}H\ , \tag{7.9}$$

and as both sides are trilinear and symmetric in F and G, and as $F = a\log(z) + f(z)$ with $f(z)$ holomorphic and similarly for G and H, it suffices to treat the following cases:

(1) When f, g and h are holomorphic we have

$$\langle f, g, h\rangle' = \operatorname{Res} fg\,\mathrm{d}h = 0 = c_z(f)c_z(g)\operatorname{Res}\mathrm{d}h$$

since $\operatorname{Res}\mathrm{d}h = 0$.

(2) Suppose $F = G = H = \log(z)$. Since $c_z(\log^2(z)/2) = 0$ we see that the local index computed with all auxiliary data set equal to $\log^2(z)/2$ is given by $\langle\log(z), \log(z); \log(z)\rangle'$, and this we know is 0 by (7.6). On the other hand, the right-hand side of (7.9) is also zero since $c_z(\log(z)) = 0$.

(3) If g and h are holomorphic we have

$$\langle\log(z), g; h\rangle' = \langle\log(z), \int' g\,\mathrm{d}h\rangle = -\operatorname{Res}\Bigl(\int' g\,\mathrm{d}h\Bigr)\operatorname{dlog} z = \Bigl(\int' g\,\mathrm{d}h\Bigr)(0) = 0,$$

which equals $c_z(\log(z))c_z(g)\operatorname{Res}\mathrm{d}h$ as required.

(4) If f and g are holomorphic we find

$$\langle f, g; \log(z)\rangle' = \operatorname{Res} fg\operatorname{dlog} z = fg(0) = c_z(f)c_z(g)\operatorname{Res}\operatorname{dlog} z\,.$$

(5) If g is holomorphic and $a = c_z(g)$ we see that

$$\int'(g-a)\operatorname{dlog} z = \int' g\operatorname{dlog} z - a\log(z).$$

Using this we find

$$\begin{aligned}\langle\log(z), g; \log(z)\rangle &= \Bigl\langle\log(z), \int' g\operatorname{dlog} z\Bigr\rangle = \Bigl\langle\log(z), \int'(g-a)\operatorname{dlog} z\Bigr\rangle\\ &= -\operatorname{Res}\Bigl(\int'(g-a)\operatorname{dlog} z\Bigr)\operatorname{dlog} z = 0,\end{aligned}$$

since $\int'(g-a)\operatorname{dlog} z$ is holomorphic and has constant term 0. This again equals the right-hand side.

(6) The final case is for $\langle\log(z), \log(z); h\rangle$ with h holomorphic. As $c_z(h\log(z)) = 0$, we have the equation $\int' h\operatorname{dlog} z + \int'\log(z)\,\mathrm{d}h = h\log(z)$. We therefore immediately deduce this case from the previous one and the triple identity. □

8. The triple index, global theory

At this point we shall switch for convenience to assuming that our ground field is $\mathbb{C}_p$. Suppose now that we consider an open annulus $V \cong \{r < |z| < s\}$ with a parameter z. Then exactly the same analysis as in Section 7 gives us a triple index on V. Note that while a parameter is used for proving the existence of the index, the uniqueness statement is parameter-free, hence so is the index.

The uniqueness of the triple index immediately implies the following result (cf. [Besser 2000c, Lemma 4.6]).

Lemma 8.1. *If $\phi : V \to V$ is an endomorphism of degree n, let $\phi^*(F, G; H)$ be defined in the obvious way, pulling back by ϕ all the auxiliary data. Denote these data simply by $(\phi^* F, \phi^* G; \phi^* H)$. Then we have the formula*

$$\langle \phi^* F, \phi^* G; \phi^* H \rangle = n \langle F, G; H \rangle.$$

Consider now a wide open space U over $\mathbb{C}_p$, with set of ends $\mathsf{End}(U)$. We shall denote the triple index with respect to the end e by the subscript e. When we are given Coleman functions F, G and H in $A_{\mathrm{col},1}(U)$, in other words, such that their differentials are in $\Omega^1(U)$, we may choose Coleman integrals for all forms $R\,\mathrm{d}S$ when R and S are among F, G and H, and we may do so in such a way that $\int R\,\mathrm{d}S + \int S\,\mathrm{d}R = RS$ globally. This allows us to compute $\langle F, G; H \rangle_e$ at each end e and we may consider the global triple index

$$\langle F, G; H \rangle_{\mathrm{gl}} = \sum_{e \in \mathsf{End}(U)} \langle F, G; H \rangle_e.$$

Lemma 8.2. *For $F, G, H \in A_{\mathrm{col},1}(U)$, the expression $\langle F, G; H \rangle_{\mathrm{gl}}$ is independent of the auxiliary choices, so depends only on F, G and H.*

Proof. Since the possible integrals differ from one another by a global constant, if we change for example $\int G\,\mathrm{d}H$ by a constant C, the change of constant formula implies that the global triple index changes by

$$\sum_e C \operatorname{Res}_e \mathrm{d}F = C \sum_e \operatorname{Res}_e \mathrm{d}F = C \cdot 0 = 0. \qquad \square$$

Unlike the global double index, the global triple index does not depend solely on the cohomology classes of $\mathrm{d}F, \ldots,$ and not even just on the differentials of the functions. For example, if C is a constant we have the formula

$$\langle F, C; H \rangle_{\mathrm{gl}} = \sum_e \left\langle F, \int C\,\mathrm{d}H \right\rangle_e = C \sum_e \langle F, H \rangle_e.$$

However, we do have the following.

Lemma 8.3. *If $F, G \in A_{\mathrm{col},1}(U)$ and C is a constant then $\langle F, G; C \rangle_{\mathrm{gl}} = 0$.*

Proof. Indeed,

$$\begin{aligned}\langle F, G; 1\rangle_{\mathrm{gl}} &= -\langle F, 1, G\rangle_{\mathrm{gl}} - \langle G, 1, F\rangle_{\mathrm{gl}} && \text{by the triple identity}\\ &= -\langle F, \textstyle\int \mathrm{d}G\rangle_{\mathrm{gl}} - \langle G, \textstyle\int \mathrm{d}F\rangle_{\mathrm{gl}} && \text{by reduction to the double index}\\ &= -\langle F, G\rangle_{\mathrm{gl}} - \langle G, F\rangle_{\mathrm{gl}} = 0,\end{aligned}$$

where the last two equalities follow because the global double index is independent of the choice of the integral and by the antisymmetry of the double index. □

The lemma suggests that the global triple index is quite an interesting creature. It deserves further study. For our purposes we only need the following results:

Proposition 8.4. *Let F, G, H in $A_{\mathrm{col}}(U)$ have $\mathrm{d}F, \mathrm{d}G, \mathrm{d}H$ in $\Omega^1(U)$, and suppose that the classes $[\mathrm{d}F]$ and $[\mathrm{d}G]$ in $H^1_{\mathrm{dR}}(U/K)$ are eigenvectors for Frobenius with eigenvalue q. Then $\langle F, G; H\rangle_{\mathrm{gl}} = 0$.*

Proof. We begin by establishing the following formulas. If $r \in A(U)$ then

$$\langle F, r, H\rangle_{\mathrm{gl}} = \sum_e \Big\langle F, \int r\,\mathrm{d}H\Big\rangle_e = 0, \tag{8.5}$$

where the last equality follows from [Besser 2000c, Corollary 4.11]. Similarly we find that if also $s \in A(U)$ then $\langle s, G, H\rangle_{\mathrm{gl}} = 0$. Now if $h \in A(U)$, then

$$\langle F, G; h\rangle_{\mathrm{gl}} = -\langle F, h; G\rangle_{\mathrm{gl}} - \langle G, h; F\rangle_{\mathrm{gl}} = 0,$$

by application of (8.5). This last formula shows that for fixed F and G the function $H \mapsto \langle F, G; H\rangle_{\mathrm{gl}}$ depends only on the cohomology class of $\mathrm{d}H$, $[\mathrm{d}H] \in H^1_{\mathrm{dR}}(U/K)$. Let ϕ be a Frobenius lift on U. The assumption on F and G implies the existence of $r, s \in A(U)$ such that $\phi^*F = qF + r$ and $\phi^*G = qG + s$. Using this we can compute

$$\begin{aligned}q\langle F, G; H\rangle_{\mathrm{gl}} &= \langle \phi^*F, \phi^*G; \phi^*H\rangle_{\mathrm{gl}}\\ &= \langle qF + r, qG + s; \phi^*H\rangle_{\mathrm{gl}} = q^2\langle F, G; \phi^*H\rangle_{\mathrm{gl}},\end{aligned}$$

using bilinearity and (8.5). This shows that the functional $[\mathrm{d}H] \mapsto \langle F, G; H\rangle_{\mathrm{gl}}$ is an eigenvector for the action of ϕ^* with eigenvalue $1/q$. Such a functional must be 0 because the eigenvalues of ϕ^* on $H^1_{\mathrm{dR}}(U/K)$ are either q or Weil numbers of weight 1. □

Note that this proposition applies in particular when F and G are of the form $r + \log(f)$ where $r, f \in A(U)$. This follows since by [Coleman and de Shalit 1988, Lemma 2.5.1], $\log(f^q/\phi^*(f))$ is in $A(U)$.

Proposition 8.6. *Suppose ω in $\Omega^1(U)$ has trivial residues on all ends, so that its Coleman integral F_ω is in fact analytic on the ends. Let F, G, H be Coleman functions on U whose differentials are holomorphic and represent eigenvectors for*

Frobenius with eigenvalue q on $H^1_{\mathrm{dR}}(U/K)$. Then, choosing the integrals globally as Coleman integrals,

(8.7)
$$\sum_e \left\langle F, G; \int F_\omega \, \mathrm{d}H \right\rangle_e + \sum_e \left\langle H, F; \int F_\omega \, \mathrm{d}G \right\rangle_e + \sum_e \left\langle G, H; \int F_\omega \, \mathrm{d}F \right\rangle_e = 0.$$

Proof. Note that the expression above makes sense since on each end e the form $F_\omega \, \mathrm{d}H$ is analytic, so the corresponding triple index is defined, and similarly with H replaced by F and G. Note also that this is of course not a global index in the sense of this section, since $F_\omega \, \mathrm{d}H$ is not holomorphic. The strategy for the proof is the same as for Proposition 8.4. First we notice that if F_ω is in fact holomorphic, then the identity holds by Proposition 8.4. It follows that the expression factors via the cohomology class $[\omega]$. Suppose now that we replace F by a holomorphic function u. We then have

$$\sum_e \left\langle u, G; \int F_\omega \, \mathrm{d}H \right\rangle_e = \sum_e \left\langle G, \int F_\omega u \, \mathrm{d}H \right\rangle_e,$$
$$\sum_e \left\langle u, H; \int F_\omega \, \mathrm{d}G \right\rangle_e = \sum_e \left\langle H, \int F_\omega u \, \mathrm{d}G \right\rangle_e,$$

by reduction to the double index, and

$$\begin{aligned}
&\sum_e \left\langle G, H; \int F_\omega \mathrm{d}u \right\rangle_e \\
&\qquad = \sum_e \left\langle G, H; F_\omega u - \int u\omega \right\rangle_e = \sum_e \langle G, H; F_\omega u \rangle_e && \text{by Proposition 8.4} \\
&\qquad = -\sum_e \langle G, F_\omega u; H \rangle - \sum_e \langle H, F_\omega u; G \rangle && \text{by the triple identity} \\
&\qquad = -\sum_e \left\langle G, \int F_\omega u \, \mathrm{d}H \right\rangle_e - \sum_e \left\langle H, \int F_\omega u \, \mathrm{d}G \right\rangle_e
\end{aligned}$$

by reducing to the double index again as F_ω is analytic. This shows that if we replace F by u in the formula to be proved we indeed get 0. Similarly we get the same result if we replace G by a holomorphic v, H by a holomorphic w, or if we do 2 or 3 of these replacements at the same time. Now, exactly as in the proof of Proposition 8.4, writing the left-hand side of (8.7) as $T(F, G, H, \omega)$, we easily get from the previous computation that

$$qT(F, G, H, \omega) = T(\phi^* F, \phi^* G, \phi^* H, \phi^* \omega) = q^3 T(F, G, H, \phi^* \omega).$$

Defining the functional γ by $\gamma([\omega]) = T(F, G, H, \omega)$, this shows that γ satisfies $\gamma(\phi^*[\omega]) = q^{-2}\gamma([\omega])$, so that $\gamma((q^2\phi^* - \mathrm{id})[\omega]) = 0$. By the theory of Weil numbers, it follows that $\gamma = 0$. This proves what we want. $\square$

9. A formula for the regulator

In this section we obtain our first explicit regulator formula, Theorem 9.10, using the theory of the triple index. For technical reasons, the syntomic regulator itself must be developed over a discretely valued field. However, since we have formulas for the regulator that make sense over $\mathbb{C}_p$ as well, we work from now until the end of this paper over $\mathbb{C}_p$.

Now that we have at our disposal the triple index, we can interpret our make-believe computation of Section 6 in such a way that it will become true. We continue with the notation of the previous section, so U is a wide open space over $\mathbb{C}_p$.

The first thing that the triple index allows us to do is to extend the cup product to some Coleman differential forms. We first need a lemma.

Lemma 9.1. *The map* $\Omega^1_{\mathrm{col},1}(U) \to H^1(U) \otimes \Omega^1(U)$ *given by*

$$\sum F_{\omega_i}\eta_i \mapsto \sum [\omega_i] \otimes \eta_i$$

is well-defined.

Proof. This is [Besser 2002, Corollary 6.2]. $\square$

Proposition 9.2. *There is a unique bilinear map*

$$\langle\!\langle \cdot\,, \cdot \rangle\!\rangle : A_{\mathrm{col},1}(U) \otimes \Omega^1_{\mathrm{col},1}(U) \to \mathbb{C}_p$$

such that we have, for any F, G, H *in* $A_{\mathrm{col},1}(U)$,

$$\langle\!\langle F, G\,\mathrm{d}H \rangle\!\rangle = \langle F, G; H\rangle_{\mathrm{gl}}. \tag{9.3}$$

Proof. By definition, $\Omega^1_{\mathrm{col},1}(U)$ is generated by forms like $G\,\mathrm{d}H$ so uniqueness is clear. To show the existence we first note that by Lemma 8.3 the right-hand side depends only on $\mathrm{d}H$. This shows that $\langle\!\langle \cdot\,, \cdot \rangle\!\rangle$ is well-defined as a map $A_{\mathrm{col},1}(U) \otimes A_{\mathrm{col},1}(U) \otimes \Omega^1(U) \to \mathbb{C}_p$, where the tensors are taken over $\mathbb{C}_p$. Lemma 9.1 shows that the kernel of the map $G \otimes \mathrm{d}H \to G\,\mathrm{d}H$ from $A_{\mathrm{col},1}(U) \otimes \Omega^1(U)$ to $\Omega^1_{\mathrm{col},1}(U)$ is contained in $A(U) \otimes \Omega^1(U)$ so it is enough to observe that if g in $A(U)$ then $\langle F, g; H\rangle_{\mathrm{gl}} = \langle F, \textstyle\int g\,\mathrm{d}H\rangle_{\mathrm{gl}}$ indeed depends only on the form $g\,\mathrm{d}H$. $\square$

The interest in the pairing $\langle\!\langle \cdot\,, \cdot \rangle\!\rangle$ is justified by the fact that its restriction to $A_{\mathrm{col},1}(U) \otimes \Omega^1(U)$ is given by $\langle\!\langle F, \mathrm{d}G \rangle\!\rangle = \langle F, G\rangle_{\mathrm{gl}}$. The pairing on the right was studied in [Besser 2000c].

Let us now fix ω in $\Omega^1(U)$ such that $[\omega]$ extends to C, or equivalently, that it has trivial residues on all ends, and let $F = F_\omega$ in $A_{\mathrm{col},1}(U)$ be a Coleman integral of ω. Let $\mathbf{p}([\omega])$ be the canonical projection of $[\omega]$ on $H^1_{\mathrm{dR}}(C/K)$.

Proposition 9.4. *The functional $L_{\mathbf{p}([\omega])}(\eta) = \langle\!\langle F, \eta \rangle\!\rangle$ on $\Omega^1_{\mathrm{col},1}(U)$ is good in the sense of Definition 5.21.*

Proof. Note that we are not claiming that this functional is independent of the choice of the constant of integration. We first need to prove that $L_{\mathbf{p}([\omega])}$ vanishes on forms of type $\mathrm{d}(a \log f)$, with a and f in $A(U)$. This is easily established:

$$\begin{aligned}\langle\!\langle F, \mathrm{d}(a \log f) \rangle\!\rangle &= \langle\!\langle F, a \,\mathrm{dlog}\, f \rangle\!\rangle + \langle\!\langle F, \log f \,\mathrm{d}a \rangle\!\rangle \\ &= \langle F, a; \log f \rangle_{\mathrm{gl}} + \langle F, \log f; a \rangle_{\mathrm{gl}} \\ &= \langle a, \log f; F \rangle_{\mathrm{gl}} = 0\end{aligned}$$

by Proposition 8.4. The second property of a good functional is immediate from the formula $\langle F, G \rangle_{\mathrm{gl}} = \mathbf{p}([\mathrm{d}F]) \cup \mathbf{p}([\mathrm{d}G])$ [Besser 2000c, Proposition 4.10]. □

We will henceforth denote the above functional simply by L_ω. This is literally the case if ω is of the second kind on C, as in this case $\mathbf{p}(\omega) = [\omega]$.

Corollary 9.5. *The p-adic regulator $K_4^{(3)}(\mathbb{O}) \xrightarrow{\mathrm{reg}'_p} H^1_{\mathrm{dR}}(C/K)$ factors through the quotient map $K_4^{(3)}(\mathbb{O}) \to K_4^{(3)}(\mathbb{O})/K_3^{(2)}(\mathbb{O}) \cup \mathbb{O}^*_{\mathbb{Q}}$.*

Proof. By Proposition 5.22 and the normalization (5.7), the fact that a good functional for any cohomology class $\alpha \in H^1_{\mathrm{dR}}(C/K)$ exists implies that the composition

$$K_4^{(3)}(\mathbb{O}) \xrightarrow{\mathrm{reg}'_p} H^1_{\mathrm{dR}}(C/K) \xrightarrow{1-\phi^*/q^3} H^1_{\mathrm{dR}}(C/K) \xrightarrow{\alpha\cup} K$$

factors. As this is true for any α it follows that

$$K_4^{(3)}(\mathbb{O}) \xrightarrow{\mathrm{reg}'_p} H^1_{\mathrm{dR}}(C/K) \xrightarrow{1-\phi^*/q^3} H^1_{\mathrm{dR}}(C/K)$$

factors, but $1 - \phi^*/q^3$ is invertible on $H^1_{\mathrm{dR}}(C/K)$ so the result follows. □

Propositions 9.4 and 5.22 suggest that in order to get an explicit formula for reg'_p we need to compute $\langle\!\langle F, T_0^\infty \varepsilon(g, f) \rangle\!\rangle$, where the $\varepsilon(g, f)$ are computed in (5.18). We shall manipulate this by "making our wishes come true" in the form of the following proposition.

Proposition 9.6. *Let F be as in Proposition 9.4 and let $g, f \in \mathbb{O}^*(\mathscr{C}^{\mathrm{loc}})$ with $g \neq 1$. Let $T_0^\infty \varepsilon(g, f)$ be as in* (5.18)*. Then we have*

$$\langle\!\langle F, T_0^\infty \varepsilon(g, f) \rangle\!\rangle = \sum_e \mathcal{T}(g, f, F)_e \,, \tag{9.7}$$

where (*choosing the integrals globally as Coleman integrals*)

$$
\begin{aligned}
(9.8)\quad \mathcal{T}(g,f,F)_e = \frac{1}{q}\,&\langle \log f_0, \log g; \textstyle\int F\,\mathrm{dlog}(1-g)\rangle_e \\
&+\frac{1}{q^2}\,\langle \log\phi^*(f), \log(g); \textstyle\int F\,\mathrm{dlog}(1-g)_0\rangle_e \\
&\quad+\frac{1}{q^3}\,\langle \log\phi^*(f), \log(g_0); \textstyle\int F\phi^*\,\mathrm{dlog}(1-g)\rangle_e\,.
\end{aligned}
$$

Proof. We have by (5.18) and (9.3)

$$
\begin{aligned}
&\langle\!\langle F, T_0^\infty \varepsilon(g,f)\rangle\!\rangle \\
&\quad=\sum_e\Big(\frac{1}{q}\,\langle F, \log g; \textstyle\int \log f_0\,\mathrm{dlog}(1-g)\rangle_e \\
&\qquad\quad -\frac{1}{q^2}\,\langle F, \log g; \textstyle\int \log(1-g)_0\,\mathrm{dlog}\,\phi^*(f)\rangle_e + \frac{1}{q}\langle F, \Theta(g); \log\phi^*(f)\rangle_e\Big).
\end{aligned}
$$

Note that $\mathrm{d}F=\omega$ is in $\Omega^1(U)$ and has trivial residues along all ends. It follows that F is holomorphic on each end.

At every annulus e we obtain the identities

$$
\begin{aligned}
\langle F, \log g; \textstyle\int \log f_0\,\mathrm{dlog}(1-g)\rangle_e &= \langle \log(g), \textstyle\int F\log f_0\,\mathrm{dlog}(1-g)\rangle_e \\
&= \langle \log f_0, \log g; \textstyle\int F\,\mathrm{dlog}(1-g)\rangle_e, \\
\langle F, \log g; \textstyle\int \log(1-g)_0\,\mathrm{dlog}\,\phi^*(f)\rangle_e &= \langle \log g, \textstyle\int F\log(1-g)_0\,\mathrm{dlog}\,\phi^*(f)\rangle_e \\
&= \langle \log g, F\log(1-g)_0; \log\phi^*(f)\rangle_e\,, \\
\langle F, \Theta(g); \log\phi^*(f)\rangle_e &= \mathrm{Res}_e\, F\Theta(g)\,\mathrm{dlog}\,\phi^*(f) \\
&= -\langle \log\phi^*(f), \Theta(g)F\rangle_e,
\end{aligned}
$$

so we obtain

$$
\begin{aligned}
&\langle\!\langle F, T_0^\infty \varepsilon(g,f)\rangle\!\rangle \\
&\quad=\sum_e\Big(\frac{1}{q}\langle \log f_0, \log g; \textstyle\int F\,\mathrm{dlog}(1-g)\rangle_e \\
&\qquad\quad -\frac{1}{q^2}\langle \log g, F\log(1-g)_0; \log\phi^*(f)\rangle_e - \frac{1}{q}\langle \log\phi^*(f), \Theta(g)F\rangle_e\Big).
\end{aligned}
$$

To equate this with the right-hand side of (9.7) we now realize our wishes one by one. First we notice that the first summands in each expression are identical. The realization of the first wish corresponds to the formula

$$\begin{aligned}
\sum_e &\langle \log\phi^*(f),\, \Theta(g)F\rangle_e \\
&= \sum_e \langle \log\phi^*(f),\, \textstyle\int F\, \mathrm{d}\Theta(g)\rangle_e + \sum_e \langle \log\phi^*(f),\, \textstyle\int \Theta(g)\, \mathrm{d}F\rangle_e \\
&= \sum_e \langle \log\phi^*(f),\, \textstyle\int F\, \mathrm{d}\Theta(g)\rangle_e\,,
\end{aligned}$$

as the second sum on the second line vanishes by [Besser 2000c, Corollary 4.11]. Now we may use the formula (5.12) for $\mathrm{d}\Theta(g)$ to write this as

$$\sum_e \Big(\frac{1}{q} \langle \log\phi^*(f),\, F\log(1-g)_0;\, \log(g)\rangle_e - \frac{1}{q^2} \langle \log\phi^*(f),\, \log(g_0);\, \textstyle\int F\phi^*\, \mathrm{dlog}(1-g)\rangle_e \Big),$$

so the left-hand side of (9.7) becomes

$$\begin{aligned}
\sum_e \Big(&\frac{1}{q}\, \langle \log f_0, \log g;\, \textstyle\int F\, \mathrm{dlog}(1-g)\rangle_e - \frac{1}{q^2} \langle \log g,\, F\log(1-g)_0;\, \log\phi^*(f)\rangle_e \\
&- \frac{1}{q^2} \langle \log\phi^*(f),\, F\log(1-g)_0;\, \log(g)\rangle_e \\
&+ \frac{1}{q^3}\, \langle \log\phi^*(f),\, \log(g_0);\, \textstyle\int F\phi^*\, \mathrm{dlog}(1-g)\rangle_e \Big)\,.
\end{aligned}$$

Now the first and last terms both agree with those on the right-hand side of (9.7) and we are left with verifying the realization of the second wish in the form of

$$\begin{aligned}
\sum_e \Big(&\langle \log g,\, F\log(1-g)_0;\, \log\phi^*(f)\rangle_e + \langle \log\phi^*(f),\, F\log(1-g)_0;\, \log(g)\rangle_e \\
&+ \langle \log\phi^*(f),\, \log(g);\, \textstyle\int F\, \mathrm{dlog}(1-g)_0\rangle_e \Big) = 0\,.
\end{aligned}$$

If we could replace the last triple index by

$$\langle \log\phi^*(f),\, \log(g);\, F\log(1-g)_0\rangle_e$$

the result would be an immediate consequence of the triple identity; indeed,

$$\begin{aligned}
\sum_e &\langle \log\phi^*(f),\, \log(g);\, \textstyle\int F\, \mathrm{dlog}(1-g)_0\rangle_e \\
&= \sum_e \langle \log\phi^*(f),\, \log(g);\, F\log(1-g)_0\rangle_e \\
&\qquad - \sum_e \langle \log\phi^*(f),\, \log(g);\, \textstyle\int \log(1-g)_0\, \mathrm{d}F\rangle_e,
\end{aligned}$$

and the last sum is 0 by Proposition 8.4. □

Proposition 9.9. *Let G be such that* $\mathrm{d}G$ *lies in* $\Omega^1(U)$ *and G is holomorphic on ends. Then, with the notation of Proposition 9.6, we have*

$$\mathcal{T}(g, f, \phi^* G)_e = \left\langle \log(f), \log(g); \int \left(\phi^* - \frac{1}{q^2}\right) G \,\mathrm{dlog}(1-g) \right\rangle_e .$$

Proof. Let $F = \phi^* G$. We replace each term of the form h_0 by $q \log(h) - \log \phi^*(h)$ in (9.8). Then we get

$$\begin{aligned}\mathcal{T}(g, f, F)_e = \frac{1}{q} &\langle q \log(f) - \log \phi^*(f), \log g; \textstyle\int F \,\mathrm{dlog}(1-g)\rangle_e \\ &+ \frac{1}{q^2} \langle \log \phi^*(f), \log(g); q \textstyle\int F \,\mathrm{dlog}(1-g) - \int F \,\mathrm{dlog}\, \phi^*(1-g)\rangle_e \\ &+ \frac{1}{q^3} \langle \log \phi^*(f), q \log(g) - \log \phi^*(g); \textstyle\int F \phi^* \,\mathrm{dlog}(1-g)\rangle_e ,\end{aligned}$$

which after some cancellations equals

$$\langle \log(f), \log(g); \textstyle\int F \,\mathrm{dlog}(1-g)\rangle_e - \frac{1}{q^3} \langle \log \phi^*(f), \log \phi^*(g); \int F \,\mathrm{dlog}\, \phi^*(1-g)\rangle_e .$$

After substituting $\phi^* G$ for F and noting that

$$\begin{aligned}&\langle \log \phi^*(f), \log \phi^*(g); \textstyle\int \phi^* G \,\mathrm{dlog}\, \phi^*(1-g)\rangle_e \\ &\qquad = q \,\langle \log(f), \log(g); \textstyle\int G \,\mathrm{dlog}(1-g)\rangle_e\end{aligned}$$

by Lemma 8.1, this becomes

$$\begin{aligned}&\langle \log(f), \log(g); \textstyle\int \phi^* G \,\mathrm{dlog}(1-g)\rangle_e - \frac{1}{q^2} \langle \log(f), \log(g); \int G \,\mathrm{dlog}(1-g)\rangle_e \\ &\qquad = \left\langle \log(f), \log(g); \int \left(\phi^* - \frac{1}{q^2}\right) G \,\mathrm{dlog}(1-g) \right\rangle_e ,\end{aligned}$$

as required. ☐

We now proceed to apply this theory to elements in K-theory.

Theorem 9.10. 1. *Suppose that an element* $\beta \in K_4^{(3)}(\mathscr{C}^{\mathrm{loc}})$ *maps to* $\sum_i [g_i]_2 \cup f_i$ *in* $H^1(\mathscr{C}^\bullet(\mathbb{O}))$ *under the composition (with the last isomorphism from* (2.65))

$$K_4^{(3)}(\mathscr{C}^{\mathrm{loc}}) \to K_4^{(3)}(\mathbb{O}) \to K_4^{(3)}(\mathbb{O})/K_3^{(2)}(\mathbb{O}) \cup \mathbb{O}_{\mathbb{Q}}^* \xrightarrow{\simeq} H^1(\mathscr{C}^\bullet(\mathbb{O})) , \tag{9.11}$$

and that $\mathrm{reg}_p(\beta) \in \tilde{H}^2_{\mathrm{ms}}(\mathscr{C}^{\mathrm{loc}}, 3)$ *is the image of* $[\eta] \in H^1_{\mathrm{dR}}(U/K)$ *under the isomorphism* (5.7). *Let ω in $\Omega^1(U)$ have trivial residues along all ends of U. Then*

$$\langle F_\omega, F_\eta \rangle_{\mathrm{gl}} = \sum_i \sum_e \langle \log(f_i), \log(g_i); \textstyle\int F_\omega \,\mathrm{dlog}(1-g_i)\rangle_e , \tag{9.12}$$

where F_ω and F_η are any Coleman integrals of ω and η respectively.

2. *In particular, the composition*

$$K_4^{(3)}(\mathscr{C}^{\mathrm{loc}}) \xrightarrow{\mathrm{reg}_p} \tilde{H}^2_{\mathrm{ms}}(\mathscr{C}^{\mathrm{loc}}, 3) \xrightarrow{[\eta]\mapsto\langle F_\omega, F_\eta\rangle_{\mathrm{gl}}} \mathbb{C}_p$$

factors via (9.11).

Proof. First one easily checks that the validity of the formula depends only on the cohomology class of ω. Since the operator $\phi^* - 1/q^2$ is invertible on $H^1(U)$ we can assume that $\omega = (\phi^* - 1/q^2)\mu$ with μ in $\Omega^1(U)$ and that $F_\omega = (\phi^* - 1/q^2)G$ with G a Coleman integral of μ. Notice that G satisfies the condition of Proposition 9.9. Let η_0 be $\mathrm{reg}_p(\beta) \in \tilde{H}^2_{\mathrm{ms}}(\mathscr{C}^{\mathrm{loc}}, 3)$ in the model (5.6) so that by (5.7) we have $\eta_0 = (1 - \phi^*/q^3)\eta$ (up to an exact form, but this is irrelevant for global index computations). We can take the Coleman integral of η_0 to be $F_{\eta_0} = (1 - \phi^*/q^3)F_\eta$. Let $F = \phi^* G$. By Proposition 9.4 the functional $L_\omega(\eta) = \langle\!\langle F, \eta\rangle\!\rangle$ is good in the sense of Definition 5.21. It follows that we may apply Proposition 5.22 to obtain

$$\begin{aligned}
\langle\!\langle F, \eta_0\rangle\!\rangle &= \sum_i \langle\!\langle F, T_0^\infty \varepsilon(g_i, f_i)\rangle\!\rangle \\
&= \sum_i \sum_e \mathscr{T}(g_i, f_i, F)_e \quad \text{by Proposition 9.6} \\
&= \sum_i \sum_e \Big\langle \log(f), \log(g); \int \Big(\phi^* - \frac{1}{q^2}\Big) G \,\mathrm{dlog}(1-g)\Big\rangle_e \\
&= \sum_i \sum_e \Big\langle \log(f), \log(g); \int F_\omega \,\mathrm{dlog}(1-g)\Big\rangle_e
\end{aligned}$$

by Proposition 9.9. On the other hand, we have

$$\begin{aligned}
\langle\!\langle F, \eta_0\rangle\!\rangle &= \langle F, F_{\eta_0}\rangle_{\mathrm{gl}} = \Big\langle F, \Big(1 - \frac{\phi^*}{q^3}\Big) F_\eta\Big\rangle_{\mathrm{gl}} = \Big\langle \phi^* G, F_\eta - \frac{\phi^*}{q^3} F_\eta\Big\rangle_{\mathrm{gl}} \\
&= \langle \phi^* G, F_\eta\rangle_{\mathrm{gl}} - \Big\langle \frac{1}{q^2} G, F_\eta\Big\rangle_{\mathrm{gl}} = \Big\langle \Big(\phi^* - \frac{1}{q^2}\Big) G, F_\eta\Big\rangle_{\mathrm{gl}} = \langle F_\omega, F_\eta\rangle_{\mathrm{gl}}\,.
\end{aligned}$$

The last two equations immediately give the result. □

We can restate the first part of Theorem 9.10 in a form that is more convenient for the rest of this paper. As explained in the introduction, one has a canonical projection $H^1_{\mathrm{dR}}(U/K) \xrightarrow{\mathbf{p}} H^1_{\mathrm{dR}}(C/K)$. This is the unique Frobenius equivariant splitting of the natural restriction map in the other direction.

Recall now the Definition 5.8 of the regulator map reg'_p, using the projection map $\mathbf{p}$. It follows from [Besser 2000c, Proposition 4.10] that $\mathbf{p}$ can be described in the following way. It is the unique map such that for any $\eta \in \Omega^1(U)$ and for any form of the second kind ω on C, which is holomorphic on U, one has

$$[\omega] \cup (\mathbf{p}\eta) = \langle F_\omega, F_\eta, \rangle_{\mathrm{gl}}\,. \tag{9.13}$$

Corollary 9.14. *Suppose that an element* $\beta \in K_4^{(3)}(\mathscr{C}^{\mathrm{loc}})$ *maps to* $\sum_i [g_i]_2 \cup f_i$ *in* $H^1(\mathscr{C}^\bullet(\mathbb{O}))$ *under* (9.11). *Let* ω *be a form of the second kind on* C *that is holomorphic on* U. *Then* $[\omega] \cup \mathrm{reg}'_p(\beta)$ *is given by the right-hand side of* (9.12).

10. End of the proofs

In this section we prove our main theorems. These will all follow from manipulations of Theorem 9.10 and Corollary 9.14.

Fix a form ω of the second kind on C and a Coleman integral F_ω of ω. We begin with the proof of Theorem 1.12.

Lemma 10.1. *The assignment*

$$[g]_2 \otimes f \mapsto \sum_e \left\langle \log(f), \log(g); \int F_\omega \,\mathrm{dlog}(1-g) \right\rangle_e$$

extends to a well-defined map $\Psi''_{p,\omega} : M_2(F) \otimes F^*_{\mathbb{Q}} \to K$.

Proof. For functions $f, g, h \in F$ the map

$$G(h, g, f) = \sum_e \left\langle \log(f), \log(g); \int F_\omega \,\mathrm{dlog}(h) \right\rangle_e \tag{10.2}$$

is trilinear by the properties of the triple index. The result follows from Lemma 2.29. ☐

Lemma 10.3. *The restriction of* $\Psi''_{p,\omega}$ *to* $(M_2(\mathbb{O}) \otimes \mathbb{O}^*_{\mathbb{Q}})^{\mathrm{d}=0}$ *coincides with the composition*

$$(M_2(\mathbb{O}) \otimes \mathbb{O}^*_{\mathbb{Q}})^{\mathrm{d}=0} \to H^2(\mathcal{M}_{(3)}(\mathbb{O})) \to K_4^{(3)}(\mathbb{O}) \xrightarrow{\mathrm{reg}'_p} H^1_{\mathrm{dR}}(C/K) \xrightarrow{\omega\cup} K.$$

Proof. This is an immediate consequence of diagram (2.67), noting the vertical map on the left there is $[g]_2 \otimes f \mapsto [g]_2 \cup f$, and of Corollary 9.14. ☐

Proof of Theorem 1.12. The only part of the theorem not proven already in Lemmas 10.1 and 10.3 is that the map $\Psi''_{p,\omega}$ factors via $H^2(\mathcal{M}_{(3)}(\mathbb{O}))$, but this follows immediately from Lemma 10.3. ☐

Proof of part 1 of Theorem 1.13. By Corollary 2.30, which applies with F replaced with $\mathbb{O}$ by Remark 2.70, the fact that $\Psi''_{p,\omega}$ factors via $H^2(\mathcal{M}_{(3)}(\mathbb{O}))$ implies that $\Psi''_{p,\omega} \circ \Xi : H^2(\widetilde{\mathcal{M}}_{(3)}(\mathbb{O})) \to K$ is induced by the following map, with G as in (10.2):

$$\begin{aligned}
[g]_2 \otimes f \mapsto G((1-g) \otimes g \otimes f) &- \tfrac{1}{3} G((1-g) \otimes g \otimes f) - \tfrac{1}{3} G(g \otimes (1-g) \otimes f) \\
&+ \tfrac{1}{3} G((1-g) \otimes f \otimes g) + \tfrac{1}{3} G(f \otimes (1-g) \otimes g) \\
= G((1-g) \otimes g \otimes f) &- \tfrac{1}{3} G(g \otimes (1-g) \otimes f) + \tfrac{1}{3} G(f \otimes (1-g) \otimes g) \\
= \tfrac{2}{3} G((1-g) \otimes g \otimes f) &- \tfrac{2}{3} G(g \otimes (1-g) \otimes f),
\end{aligned}$$

where we used that G is symmetric in the last two positions by Proposition 7.3(2), and that $G(f \otimes (1-g) \otimes g) = -G((1-g) \otimes g \otimes f) - G(g \otimes f \otimes (1-g))$ by Proposition 8.6. This is the formula in the first part of Theorem 1.13 by (10.2). $\square$

For the proofs of Theorems 1.11 and 1.9, as well as part 2 of Theorem 1.13, we now assume that ω is a holomorphic form on C.

Lemma 10.4. *The associations*

$$[g]_2 \otimes f \mapsto \int_{(1-g)} \log(g) F_\omega \,\mathrm{dlog}(f) - \int_{(g)} \log(1-g) F_\omega \,\mathrm{dlog}(f)$$

$$[g]_2 \otimes f \mapsto \int_{(f)} \mathrm{L}_2(g)\omega$$

$$[g]_2 \otimes f \mapsto \sum_y \mathrm{ord}_y(f) F_\omega(y) \,\mathrm{L}_{\mathrm{mod},2}(g(y))$$

induce well-defined maps on $\widetilde{M}_2(F) \otimes F^*_{\mathbb{Q}}$ *(first) and* $M_2(F) \otimes F^*_{\mathbb{Q}}$ *(last two).*

Proof. All three assertions follow from Lemma 2.29. This is essentially clear for the first association. For the second association, observe that $\mathrm{dL}_2 = \log(z)\,\mathrm{dlog}(1-z)$ by (1.8). Consider the association

$$(h, g, f) \mapsto \int_{(f)} \left(\omega \cdot \int \log(g)\,\mathrm{dlog}(h)\right).$$

Here, the integral $\int \log(g)\,\mathrm{dlog}(h)$ is a Coleman integral defined only up to a constant. However, if the constant changes, the entire expression changes by the same constant multiplied by $\int_{(f)} \omega$, which equals 0 as it is the p-adic Abel–Jacobi map applied to the principal divisor (f); see [Besser 2000a]. This association is therefore well-defined, clearly trilinear, and we obtain the required result again by Lemma 2.29. For the third association, one first needs to note that $\mathrm{L}_{\mathrm{mod},2}(g(y))$ is the value of $\mathrm{L}_{\mathrm{mod},2}(g)$ at y (this is not obvious in general because we are using the generalized way of assigning values to Coleman functions by taking constant terms, discussed in the introduction) as we shall see in Corollary 10.8, so the entire expression can be written as $F_\omega \cdot \mathrm{L}_{\mathrm{mod},2}(g)$ evaluated at the divisor of f. It is now possible to proceed as in the previous case, given that

$$\mathrm{dL}_{\mathrm{mod},2}(g) = (\log(g)\,\mathrm{dlog}(1-g) - \log(1-g)\,\mathrm{dlog}(g))/2,$$

by associating to f, g, h the value of $F_\omega \cdot \int(\log(g)\,\mathrm{dlog}(h) - \log(h)\,\mathrm{dlog}(g))$ at (f), where the constant of integration does not matter for exactly the same reason it did not in the previous case. $\square$

By Lemma 10.4, the maps $\Psi_{p,\omega}$ in Theorem 1.9 and $\Psi'_{p,\omega}$ in Theorem 1.11 from $M_2(\mathbb{O}) \otimes \mathbb{O}^*_{\mathbb{Q}}$ to K exist. (The existence of the maps in Theorem 1.13 will be deduced from those in Theorems 1.11 and 1.12 later.)

Next, we shall derive the formulas for the regulator. In all cases, we already have a formula for the regulator, expressed in terms of a sum of local indices on annuli. We can use the argument in the proof of [Besser 2000c, Proposition 5.5] using Proposition 8.4 to replace the sum over ends by a sum over points.

Let $\alpha = \sum_i [g_i]_2 \otimes f_i$ be an element of $(M_2(\mathbb{O}) \otimes \mathbb{O}^*_{\mathbb{Q}})^{\mathrm{d}=0}$. By the above we have

$$\Psi''_{p,\omega}(\alpha) = \sum_i \sum_{y \in C} \left\langle \log(f_i), \log(g_i); \int F_\omega \,\mathrm{dlog}(1-g_i) \right\rangle_y .$$

We again extend scalars to $\mathbb{C}_p$, so in particular points are $\mathbb{C}_p$ valued. Fix a local parameter at each point y, which we shall call z_y, or, whenever there is no risk of confusion, simply z. Consider a single point y in C. We recall that with respect to the local parameter z at y we define, for a rational function f, $\bar{f}(y) = (f/z^{\mathrm{ord}_y(f)})(y)$. For such a function f we have $c_z(\log(f)) = \log(\bar{f}(y))$. We also have $\mathrm{Res}_y(F_\omega \,\mathrm{dlog}(f)) = \mathrm{ord}_y(f) \cdot F_\omega(y)$. Thus, using Proposition 7.8, we obtain

$$\begin{aligned}\Psi''_{p,\omega}(\alpha) = \sum_i \sum_{y \in C} \Big[&\mathrm{ord}_y(1-g_i) F_\omega(y) \log \bar{f}_i(y) \log \bar{g}_i(y) \\ &- \mathrm{ord}_y(f_i) c_z\Big(\int \log(g_i) F_\omega \,\mathrm{dlog}(1-g_i)\Big) \\ &- \mathrm{ord}_y(g_i) c_z\Big(\int \log(f_i) F_\omega \,\mathrm{dlog}(1-g_i)\Big)\Big].\end{aligned} \tag{10.5}$$

Let A (respectively B) be the subgroup of $k(C)^*$ generated by the f_i and g_i (respectively by the $1-g_i$). By choosing bases for A and B and then choosing appropriate integrals we can arrange it so that for each f in A and h in B an integral $\int \log(f) F_\omega \,\mathrm{dlog}\, h$ is chosen such that the map $(f, h) \mapsto \int \log(f) F_\omega \,\mathrm{dlog}\, h$ is bilinear. Since the overall sum in (10.5) is independent of the choice of integrals, we may and do assume from now on that the integrals there are chosen as above.

Lemma 10.6. *If $\sum_i [g_i]_2 \otimes f_i$ is in $(M_2(F) \otimes F^*_{\mathbb{Q}})^{\mathrm{d}=0}$, then for every y in C we have*

$$\sum_i \mathrm{ord}_y(f_i)\, c_z\Big(\int \log(g_i) F_\omega \,\mathrm{dlog}(1-g_i)\Big) = \sum_i \mathrm{ord}_y(g_i)\, c_z\Big(\int \log(f_i) F_\omega \,\mathrm{dlog}(1-g_i)\Big).$$

Proof. With the choices above the map

$$(f, g, h) \mapsto \mathrm{ord}_y(f) c_z\Big(\int \log(g) F_\omega \,\mathrm{dlog}(h)\Big) - \mathrm{ord}_y(g) c_z\Big(\int \log(f) F_\omega \,\mathrm{dlog}(h)\Big)$$

is trilinear and antisymmetric with respect to f and g. The lemma follows since $\sum(1-g_i)\otimes(g_i\wedge f_i)=0$ by (2.24). $\square$

We recall that the function $\mathrm{L}_2(z)$ is defined by $\mathrm{L}_2(z)=\mathrm{Li}_2(z)+\log(z)\log(1-z)$ and that we have $\mathrm{dL}_2(z)=\log(z)\,\mathrm{dlog}(1-z)$. Note that this last form is holomorphic in the residue disc of 1 and as a consequence so is $\mathrm{L}_2(z)$.

Lemma 10.7. *Let g be a rational function. The constant term at y of $\mathrm{L}_2(g)$ equals $\mathrm{L}_2(g(y))$ if $g(y)\neq 0,\infty$, equals 0 if $g(y)=0$ or 1 and equals $\log^2(\bar{g}(y))/2$ if $g(y)=\infty$, where $\bar{g}$ is computed with respect to the same local parameter as the constant term. In addition, the expansion of $\mathrm{L}_2(g)$ with respect to any local parameter z contains no summands of the form $Const\cdot z^n$ with $n<0$.*

Proof. This is clear if $g(y)\neq 0,\infty$. Suppose $g(y)=0$. Since Li_2 is holomorphic near 0 and has value 0 there, we see that the constant term and terms of the form z^n for $n<0$ are the same as in $\log(g)\log(1-g)$. Near y, $\log(g(z))=\mathrm{ord}_y(g)\log(z)+$, a holomorphic function in z. Also, $\log(1-g)$ is holomorphic near y with value 0 there. Thus the result is clear. Finally, by [Coleman 1982, Proposition 6.4], we have $\mathrm{L}_2(g)+\mathrm{L}_2(1/g)=\log^2(g)/2$ (from which it also follows that $\mathrm{L}_2(1)=0$) so the result at $g(y)=\infty$ is deduced from that of $1/g$ when $g(y)=\infty$. $\square$

Corollary 10.8. *The constant term of $\mathrm{L}_{\mathrm{mod},2}(z)$ at 0, 1 and ∞ is 0, regardless of parameter. Furthermore, setting the value of $\mathrm{L}_{\mathrm{mod},2}$ at these points to be the above constant term, we have that for any rational function g the constant term of $\mathrm{L}_{\mathrm{mod},2}(g)$ at any point y equals $\mathrm{L}_{\mathrm{mod},2}(g(y))$.*

Proof. Since $\mathrm{L}_{\mathrm{mod},2}(z)=\mathrm{L}_2(z)-\log(z)\log(1-z)/2$ it is easy to check that the constant term of $\mathrm{L}_{\mathrm{mod},2}(g)$ is 0 at either $g(y)=0,1,\infty$, and the result easily follows. $\square$

Lemma 10.9. *For any point y in C and for any choice of a Coleman integral $\int \mathrm{L}_2(g)\omega$ the quantity $c_z(\int \mathrm{L}_2(g)\omega)$ is independent of the choice of the local parameter z at y.*

Proof. Let f_ω be the unique Coleman integral of ω that vanishes at y. We may choose a Coleman integral $\int f_\omega d\mathrm{L}_2(g)$ in such a way that the integration by parts formula

$$\int \mathrm{L}_2(g)\omega=\mathrm{L}_2(g)f_\omega-\int f_\omega\,\mathrm{dL}_2(g)$$

holds. It is therefore sufficient to show that the constant term of each of the summands on the right is independent of the parameter. From the last assertion in Lemma 10.7 and the fact that $f_\omega(y)=0$ it is easy to see that the constant term of the first summand is 0. For the second summand we have

$$\begin{aligned}\int f_\omega \, \mathrm{dL}_2(g) &= \int f_\omega \log(g)\, \mathrm{dlog}(1-g)\\ &= \log(g) \int f_\omega \, \mathrm{dlog}(1-g) - \int \Big(\int f_\omega \, \mathrm{dlog}(1-g) \Big) \, \mathrm{dlog}(g)\end{aligned}$$

for appropriate choices of integrals. As $f_\omega \, \mathrm{dlog}(1-g)$ is holomorphic at y, we may arrange it so that $\int f_\omega \, \mathrm{dlog}(1-g)$ vanishes at y. Then in the last formula the first term has constant term 0 while the second term is holomorphic at y hence its constant term is independent of z. □

Using the last lemma we may set

$$\int \mathrm{L}_2(g)\omega|_y := c_z\Big(\int \mathrm{L}_2(g)\omega \Big)$$

with respect to any parameter z at y. Using this we can define $\int_D \mathrm{L}_2(g)\omega$ for any divisor D of degree zero. If we change $\int \mathrm{L}_2(g)\omega$ by a constant, its value at y in the above sense will change by the same constant. Thus when D has degree 0 the integral $\int_D \mathrm{L}_2(g)\omega$ does not depend on the constant of integration even if D and the divisor of g have a common support. This explains the general definition of the integral in Theorem 1.9.

Lemma 10.10. *Choose integrals such that the integration by parts formula*

$$\int \log(g) F_\omega \, \mathrm{dlog}(1-g) = F_\omega \mathrm{L}_2(g) - \int \mathrm{L}_2(g)\omega$$

is satisfied. Then we have at a point y and with respect to the local parameter z,

$$c_z\Big(\int \log(g) F_\omega \, \mathrm{dlog}(1-g) \Big) = F_\omega(y) c_z(\mathrm{L}_2(g)) - \int \mathrm{L}_2(g)\omega|_y \, .$$

Proof. One just applies c_z to the integration by parts formula and observes that by Lemma 10.7 we have $c_z(F_\omega \mathrm{L}_2(g)) = F_\omega(y) c_z(\mathrm{L}_2(g))$. □

Proof of Theorem 1.11. We already saw that the association gives a well-defined map on $M_2(\mathbb{O}) \otimes \mathbb{O}^*_{\mathbb{Q}}$. It therefore suffices to show that it gives the same map on $(M_2(\mathbb{O}) \otimes \mathbb{O}^*_{\mathbb{Q}})^{\mathrm{d}=0}$ as $\Psi''_{p,\omega}$ in Theorem 1.12. Consider (10.5). By Lemma 10.6 we can choose our integrals such that for each point y the sum over i of each of the last two terms is identical. The term

$$\mathrm{ord}_y(f_i) c_z\big(\textstyle\int \log(g_i) F_\omega \, \mathrm{dlog}(1-g_i) \big)$$

is computed in Lemmas 10.10 and 10.7. Substituting the results we see that we have the equation

$$\Psi''_{p,\omega}(\alpha) = \sum_i \left[\sum_{y\in C} (\operatorname{ord}_y(1-g_i)F_\omega(y)\log \bar{f}_i(y)\log \bar{g}_i(y)) + 2\int_{(f_i)} \mathrm{L}_2(g_i)\omega \right.$$
$$\left. - \sum_{y\in C} \operatorname{ord}_y(f_i)F_\omega(y) \times \begin{cases} 0 & g_i(y)=0, \\ 2\mathrm{L}_2(g_i(y)) & g_i(y)\neq 0,\infty, \\ \log^2(\bar{g}_i(y)) & g_i(y)=\infty. \end{cases} \right].$$

In the first sum over y, only terms with $g_i(y)=\infty$ can be nonzero. Thus neither sum over y contributes for $g_i(y)=0$, and the right-hand side becomes

$$\sum_i \left[2\int_{(f_i)} \mathrm{L}_2(g_i)\omega - 2\sum_{g_i(y)\neq 0,\infty} \operatorname{ord}_y(f_i)F_\omega(y)\mathrm{L}_2(g_i(y)) \right.$$
$$\left. + \sum_{g_i(y)=\infty} F_\omega(y)\lambda_y(f_i,g_i) \right] \tag{10.11}$$

with

$$\begin{aligned} \lambda_y(f,g) &= \operatorname{ord}_y(1-g)\log \bar{f}(y)\log \bar{g}(y) - \operatorname{ord}_y(f)\log^2 \bar{g}(y) \\ &= \log \bar{g}(y)(\operatorname{ord}_y(1-g)\log \bar{f}(y) - \operatorname{ord}_y(f)\log \bar{g}(y)) \\ &= \log \overline{1-g}(y)(\operatorname{ord}_y(g)\log \bar{f}(y) - \operatorname{ord}_y(f)\log \bar{g}(y)) \end{aligned}$$

because $g(y)=\infty$ implies $\operatorname{ord}_y(1-g)=\operatorname{ord}_y(g)$ and $\bar{g}(y) = -\overline{1-g}(y)$.

For y in C, the function

$$\mu_y(f,g,h) = \log \bar{h}(y)(\operatorname{ord}_y(g)\log \bar{f}(y) - \operatorname{ord}_y(f)\log \bar{g}(y))$$

is trilinear in f, g and h and antisymmetric in f and g. As $\sum_i (1-g_i)\otimes(g_i\wedge f_i)=0$ by (2.53), we find

$$\sum_i \mu_y(f_i,g_i,1-g_i) = 0\,. \tag{10.12}$$

If $g_i(y)=0$ then $\mu_y(f_i,g_i,1-g_i)=0$, while if $g(y)\neq 0,\infty$ then

$$\mu_y(f_i,g_i,1-g_i) = -\operatorname{ord}_y(f_i)\log g_i(y)\log(1-g_i(y)),$$

where we set the value of $\log(y)\log(1-y)$ at 1 to be 0, which is its constant term. Thus, summing (10.12) multiplied by $F_\omega(y)$ over all y in C we see that

$$\sum_i \sum_{g_i(y)=\infty} F_\omega(y)\lambda_y(f_i,g_i) = \sum_i \sum_{g_i(y)\neq 0,\infty} \operatorname{ord}_y(f_i)F_\omega(y)\log g_i(y)\log(1-g_i(y)).$$

Substituting this into (10.11), and using that $\mathrm{L}_2(z)-\log(z)\log(1-z)/2=\mathrm{L}_{\mathrm{mod},2}(z)$ by definition, we obtain

$$\Psi''_{p,\omega}(\alpha)=2\sum_i\int_{(f_i)}\mathrm{L}_2(g_i)\omega-2\sum_i\sum_{g_i(y)\neq 0,\infty}\mathrm{ord}_y(f_i)F_\omega(y)\mathrm{L}_{\mathrm{mod},2}(g_i(y)).$$

This formula finishes the proof of Theorem 1.11 as $\mathrm{L}_{\mathrm{mod},2}(0)=\mathrm{L}_{\mathrm{mod},2}(\infty)=0$. $\square$

Proof of Theorem 1.9. That the assignment is well-defined is part of Lemma 10.4. In order to see that it vanishes on $[f]_2\otimes f$, we note that we already know this is true for the assignment in Theorem 1.11, and that the second term in that assignment is trivial on such terms because $\mathrm{L}_{\mathrm{mod},2}(z)$ vanishes at 0 and ∞.

For part (2), consider (1.16). That $\partial_1(\alpha')=0$ means that α' satisfies (2.57), which is equivalent with α' being in $H^2(\mathcal{M}_{(3)}(\mathscr{C}'))$ inside $H^2(\mathcal{M}_{(3)}(\mathbb{O}'))$ (recall from Section 2.5.3 that the two vertical maps at the top in this diagram are injections if we use $\mathbb{O}'$ instead of $\mathbb{O}$ everywhere). The existence and uniqueness of β' was therefore proven just after (2.58). In fact, β' is the $K_4^{(3)}(\mathscr{C}')$ component of the image of α' in $K_4^{(3)}(\mathscr{C}')\oplus K_3^{(2)}(k)\cup\mathbb{O}'^*_{\mathbb{Q}}$, and the images of α' and β' in $K_4^{(3)}(\mathbb{O}')$ differ by some γ' in the image of $K_3^{(2)}(k)\cup\mathbb{O}'^*_{\mathbb{Q}}$. But $\omega\cup\mathrm{reg}'_p(\gamma')=0$ by the commutativity of the bottom right square, so that, after extending from $\mathbb{O}'$ to $\mathbb{O}$, we have $\omega\cup\mathrm{reg}'_p(\beta)=\Psi'_{p,\omega}(\alpha)$ by Theorem 1.11. It therefore suffices to show that the contribution of each $\mathrm{ord}_y(f)F_\omega(g(y))\mathrm{L}_{\mathrm{mod},2}(g(y))$ in $\Psi'_{p,\omega}(\alpha)$ is trivial.

Note that in Theorem 1.11 this sum has to be computed after a suitable finite extension $\tilde{K}$ of K that makes the relevant y rational, but that further extending the field to $\mathbb{C}_p$ as we are using here gives the same result. In fact, because we start over the number field k, the relevant y become rational over some number field $L\subset\tilde{K}$ containing k. The $\widetilde{M}_2(\cdot)$ are compatible with field extensions, and clearly the same holds for ∂_1. Therefore (2.57) gives us that for each closed point y of C'_L, $\partial_{1,y}(\alpha')$ is trivial in $\widetilde{M}_2(L)$. Because $F_\omega(y)$ is just a constant, comparing with the definition of $\partial_{1,y}$ in Section 2.4.3, we see that it suffices to show that the map

$$\begin{aligned}H^1(\widetilde{\mathcal{M}}_{(2)}(L))&\to\tilde{K}\\ \sum_i[a_i]_2&\mapsto\sum_i\mathrm{L}_{\mathrm{mod},2}(a_i)\end{aligned}$$

is well-defined. It is conjectured in [Besser and de Jeu 2003, Conjecture 1.14] that this map is the syntomic regulator map on as composition (with $\mathbb{O}_L$ the ring of integers in L)

$$H^1(\widetilde{\mathcal{M}}_{(2)}(L))\to K_3^{(2)}(L)\simeq K_3^{(2)}(\mathbb{O}_L)\to H^1_{\mathrm{syn}}(\mathbb{O}_L,2)\simeq K,$$

which would imply what we need. However, extending the domain of the map, we can show by more basic means that the map

$$\begin{aligned} \widetilde{M}_2(L) &\to \tilde{K} \\ [a]_2 &\mapsto \mathrm{L}_{\mathrm{mod},2}(a) \end{aligned}$$

is well-defined, which will prove what we want.

Namely, for any field L of characteristic zero, let $B_2'(L)$ be the free $\mathbb{Q}$-vector space on elements $\{b\}_2$ with b in F, $b \neq 0$, 1, modulo the five term relation

$$\{b\}_2 + \{c\}_2 + \left\{\frac{1-b}{1-bc}\right\}_2 + \{1-bc\}_2 + \left\{\frac{1-c}{1-bc}\right\}_2 = 0. \tag{10.13}$$

It is shown in [de Jeu 2000, Lemma 5.2] that there is a map $B_2'(L) \to \widetilde{M}_2(L)$, given by sending $\{b\}_2$ to $[b]_2$. In the case where L is a number field, this was already done on page 240 of [de Jeu 1995] (where the relations were not made explicit and the group was called $B_2(L)$), and the map was shown to be an isomorphism in that case. Finally, in [Coleman 1982, Corollaries 6.4(ii), (iii) and 6.5b] Coleman shows that $\mathrm{L}_{\mathrm{mod},2}$ (which is called D there) satisfies

$$\begin{aligned} \mathrm{L}_{\mathrm{mod},2}(z^{-1}) &= -\mathrm{L}_{\mathrm{mod},2}(z) \\ \mathrm{L}_{\mathrm{mod},2}(1-z) &= -\mathrm{L}_{\mathrm{mod},2}(z) \end{aligned}$$

as well as (with signs corrected)

$$\mathrm{L}_{\mathrm{mod},2}(z_1z_2) = \mathrm{L}_{\mathrm{mod},2}(z_1) + \mathrm{L}_{\mathrm{mod},2}(z_2) + \mathrm{L}_{\mathrm{mod},2}\left(\frac{z_1(1-z_2)}{z_1-1}\right) + \mathrm{L}_{\mathrm{mod},2}\left(\frac{z_2(1-z_1)}{z_2-1}\right).$$

Substituting $z_1 = (bc)^{-1}$, $z_2 = c$ in the last relation and using the first two, one sees that $\mathrm{L}_{\mathrm{mod},2}$ satisfies the relation corresponding to (10.13). Therefore it induces a map

$$\widetilde{M}_2(L) \simeq B_2'(L) \to K$$

mapping $[b]_2$ to $\mathrm{L}_{\mathrm{mod},2}(b)$. This finishes the proof of Theorem 1.9. □

Proof of part 2 of Theorem 1.13. Since by Theorem 1.11, the map $\Psi'_{p,\omega}$ factors via $H^2(\mathcal{M}_{(3)}(\mathbb{O}))$, we may again use Corollary 2.30, which applies with F replaced by $\mathbb{O}$ by Remark 2.70. Recall that $\Psi'_{p,\omega}$ is induced by

$$[g]_2 \otimes f \mapsto 2\int_{(f)} \mathrm{L}_2(g)\omega - 2\sum_y \mathrm{ord}_y(f) F_\omega(y) \mathrm{L}_{\mathrm{mod},2}(g(y)).$$

Since $\mathrm{L}_{\mathrm{mod},2}(z) + \mathrm{L}_{\mathrm{mod},2}(z^{-1}) = 0$, while $\mathrm{L}_2(z) + \mathrm{L}_2(z^{-1}) = \frac{1}{2}\log^2(z)$, we see that we are in the situation of part (3) of Lemma 2.29 with $H(a\cdot b \otimes c) = \int_{(c)} \log(a)\log(b)\omega$.

Applying the corollary, the composition $\Psi'_{p,\omega}\circ\Xi : H^2(\widetilde{\mathcal{M}}_{(2)}(\mathbb{C}))\to K$ is given by

$$[g]_2\otimes f\mapsto 2\int_{(f)}\mathrm{L}_2(g)\omega-2\sum_y \mathrm{ord}_y(f)F_\omega(y)\mathrm{L}_{\mathrm{mod},2}(g(y))\\ -\tfrac{2}{3}\int_{(f)}\log(1-g)\log(g)\omega+\tfrac{2}{3}\int_{(g)}\log(f)\log(1-g)\omega,$$

as required. □

Remark 10.14. We would like to explain a bit of the heuristics suggesting that Theorem 1.13 gives a formula which is the p-adic analogue of the complex analytic formula for the regulator in Section 3.

Experience has taught us that complex surface integrals translate in the p-adic world to a similar formula involving local indices. For example, the complex analytic formula for the regulator of the symbol $\{f, g\}$ in $K_2(F)$,

$$\int_C \log|g|\,\overline{\mathrm{dlog}\, f}\wedge\omega = 2\int_C \log|g|\ \mathrm{dlog}\,|f|\wedge\omega,$$

where ω is holomorphic, translates in the p-adic world into the formula

$$\langle\log f, F_\omega; \log g\rangle_{\mathrm{gl}}.$$

Note that, using the rules for the triple index, this is the same as the formula $\sum_e\langle\log f, \int(F_\omega\,\mathrm{dlog}(g))\rangle_e$ obtained in [Besser 2000c, Propositon 5.1]. This corresponds to the regulator on an open curve using the same projection on $H^1_{\mathrm{dR}}(C/K)$ we have been using in this paper. For a sum $\{f_i, g_i\}$ in the kernel of the tame symbol, we may, for every pair $(f, g)=(f_i, g_i)$, replace $\langle\log f, F_\omega; \log g\rangle_{\mathrm{gl}}$ with $\int_{(f)}\log(g)\cdot\omega$, obtaining the formula of Coleman and de Shalit [1988, (1)]. This is similar to Theorem 1.11 specializing to Theorem 1.9.

Relying on these considerations, the maps $\Psi''_{p,\omega}$ and $\Psi'''_{p,\omega}$ in Theorems 1.12 and 1.13 are precise analogues, up to a factor of 4, of the maps $\Psi''_{\infty,\omega}$ and $\Psi'''_{\infty,\omega}$ in Proposition 3.1. Factors that are powers of 2 appear in comparison with other regulator formulas; see for example the introduction of [Besser 2012].

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AMNON BESSER
DEPARTMENT OF MATHEMATICS
BEN-GURION UNIVERSITY OF THE NEGEV
P.O.B. 653
84105 BE'ER-SHEVA
ISRAEL

bessera@math.bgu.ac.il
http://www.math.bgu.ac.il/~bessera/

ROB DE JEU
FACULTEIT DER EXACTE WETENSCHAPPEN, AFDELING WISKUNDE
VU UNIVERSITY AMSTERDAM
DE BOELELAAN 1081A
1081 HV AMSTERDAM
NETHERLANDS

r.m.h.de.jeu@vu.nl
http://www.few.vu.nl/~jeu/

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UNIQUE FUNCTIONALS AND REPRESENTATIONS OF HECKE ALGEBRAS

BENJAMIN BRUBAKER, DANIEL BUMP AND SOLOMON FRIEDBERG

To the memory of Jonathan David Rogawski

Rogawski (1985) used the affine Hecke algebra to model the intertwining operators of unramified principal series representations of p-adic groups. On the other hand, a representation of this Hecke algebra in which the standard generators act by Demazure–Lusztig operators was introduced by Lusztig (1989) and applied by Kazhdan and Lusztig (1987) to prove the Deligne–Langlands conjecture. These operators appear in various other contexts. Ion (2006) used them to express matrix coefficients of principal series representations in terms of nonsymmetric Macdonald polynomials, while Brubaker, Bump and Licata (2011) found essentially the same operators underlying recursive relationships for Whittaker functions. Here we explain the role of unique functionals and Hecke algebras in these contexts and revisit the results of Ion from the point of view of Brubaker et al.

1. Introduction

One of the innovations in [Rogawski 1985] was the use of the Hecke algebra to model the intertwining operators of unramified principal series representations of p-adic groups. His goal was the classification of the irreducible representations of the Hecke algebra, or equivalently, the irreducible representations of a p-adic group having an Iwahori-fixed vector. These had already been classified in the case of GL_n by Zelevinsky [1980]. It was known from [Zelevinsky 1981] that there were analogies between this problem and the decomposition of Verma modules of a semisimple Lie algebra into irreducibles, where deep connections with the theory of Hecke algebras had been found by Kazhdan and Lusztig [1979]. Rogawski sought to clarify the relationship between p-adic representation theory and Kazhdan–Lusztig

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theory. As part of this effort, he expressed intertwining integrals between principal series representations in terms of Kazhdan–Lusztig elements of the Hecke algebra.

Among the tools that had been brought to bear on the study of Verma modules, Jantzen [1979] had introduced a filtration of the Verma module based on the Shapovalov bilinear form. With the analogy between p-adic groups and Verma modules in mind, Rogawski gave an analog of the Jantzen filtration, and also reproved the results of Zelevinsky [1980] for GL_n using Hecke algebra methods.

After [Rogawski 1985] was written, Zelevinsky's results were generalized to arbitrary p-adic groups by Kazhdan and Lusztig [1985; 1987], who proved the Langlands–Deligne conjecture classifying the irreducible representations of a p-adic group G that have Iwahori-fixed vectors. They made use of a representation of the affine Hecke algebra $\mathcal{H}_J$ on the ring of rational functions on the maximal torus $\hat{T}$ of the L-group $\hat{G}$. In this action, the generators of the Hecke algebra act by certain operators known as *Demazure–Lusztig* operators, which resemble the well-known Demazure operators that occur in the cohomology of line bundles over Schubert varieties. Given data consisting of a pair (s, u) of elements of $\hat{G}$ such that s is semisimple, u is unipotent, and $sus^{-1} = u^q$, where q is the cardinality of the residue field, a subquotient of this representation can be found that gives an irreducible $\mathcal{H}_J$-module. This module may be identified with the space of Iwahori-fixed vectors in an irreducible representation of $G(F)$. To prove that this gives every irreducible representation of $G(F)$ uniquely, thus proving the Deligne–Lusztig conjecture, Kazhdan and Lusztig made use of the "coincidence" that the same representation of the Hecke algebra by Demazure–Lusztig operators also occurs in the K-theory of flag varieties, allowing statements about representations to be translated into algebraic geometry, where suitable methods are available.

The representation of the Hecke algebra by Demazure–Lusztig operators comes up in yet another context, namely the study of special functions on a p-adic group realized as matrix coefficients involving Iwahori-fixed vectors. Our first task, after some preliminaries, will be to briefly retrace Rogawski's steps and to discuss the relationship between the Hecke algebra and the intertwining operators. The intertwining operators involve different principal series representations, which must be taken together to obtain a representation of the Hecke algebra. We express this by saying that the principal series representation is *variable* in this representation of the Hecke algebra.

As we will explain in Section 3, this representation may be converted into something concrete by introducing a family of functionals on the principal series representations. We will consider two particular such families: the Whittaker functionals and the spherical functionals. The role of the unique functional is that it converts the Iwahori-fixed vectors in a variable principal series representation into a family of regular functions on $\hat{T}$. The problem of variability of the principal series representation disappears, and the action of the Hecke algebra generators is by some variant (depending on the functional) of the Demazure–Lusztig operator. The regularity of the functions obtained this way is related to Bernstein's method of showing that a unique functional defined (typically by a suitable integral) on an open subspace extends to a meromorphic function for all Langlands parameters. (See, for example, [Banks 1998] or [Sakellaridis 2006, Section 7] for Bernstein's method.)

Both the spherical and the Whittaker functionals are defined and nonzero on a Zariski dense set of $\hat{T}$. By contrast, one may consider functionals defined and nonzero on only a subset of $\hat{T}$ that is not Zariski dense. An example is the Shalika functional on representations of GL_n, proved unique by Jacquet and Rallis [1996], with a Casselman–Shalika type formula established by Sakellaridis [2006]. The Shalika functional, like the Bessel models for classical groups, exhibits characteristics of both the spherical and Whittaker functionals and can be studied using our methods. We hope to return to this in a later paper.

Shortly before [Rogawski 1985] appeared, intertwining operators had been used for another purpose: the computation of the spherical vector in each of our two archetypal models. For the spherical model, Casselman [1980] used intertwining operators to give a new proof of the Macdonald formula which expresses the values of the spherical function as specializations of Macdonald symmetric functions. This specialization is the Hall–Littlewood polynomial in the case of GL_n. Later, nonsymmetric Macdonald polynomials were defined in [Macdonald 2003; Opdam 1995; Cherednik 1995]. Generalizing the Macdonald formula, Ion [2006] showed that the Iwahori-fixed vectors in the model are expressible in terms of nonsymmetric Macdonald polynomials, making use of recursions that they satisfy involving Demazure–Lusztig operators. See also [Cherednik and Ostrik 2003] for earlier hints of this connection.

For the Whittaker functional, Casselman and Shalika [1980] used the intertwining operators to show that the values of the spherical Whittaker function are the irreducible characters of $\hat{G}$, multiplied by a factor which is a deformation of the denominator in the Weyl character formula. Regarding the more general Iwahori-fixed vectors in the model, Reeder [1992; 1993] used the Hecke algebra action to give recursions between these, usable for explicit computation. These relations can be understood in terms of Demazure–Lusztig operators as proved by Brubaker, Bump and Licata [Brubaker, Bump and Licata 2011].

In Section 2, we review the relation between the Iwahori Hecke algebra and the intertwining operators for principal series. Then, in Section 3, we show that, given a unique functional on a Zariski dense subset of $\hat{T}$, to each generator T_i of the Iwahori Hecke algebra we may attach a difference operator on a suitable ring of regular functions that is similar to a Demazure operator. This extends to an action of the Hecke algebra. In the case of the spherical functional, this gives a new perspective on the work of Ion [2006].

2. Hecke algebras and intertwiners of principal series

Let G be a reductive algebraic group defined and split over a nonarchimedean local field F with ring $\mathfrak{o}$ of integers and prime $\mathfrak{p}$. We may regard G as defined over $\mathfrak{o}$ in such a way that $K = G(\mathfrak{o})$ is a special maximal compact subgroup. Let $q = |\mathfrak{o}/\mathfrak{p}|$. Let T be a split maximal torus contained in Borel subgroup B, and let $W_0 = N(T)/T$ be the Weyl group. We will always choose representatives for W_0 from $N(T) \cap K$.

The connected L-group $\hat{G}$ is an algebraic group defined over $\mathbb{C}$ with a maximal torus $\hat{T}$ that is in duality with T in the sense that elements of $\hat{T}(\mathbb{C})$ are in bijection with the unramified characters of $T(F)$, that is, those that are trivial on $T(F) \cap K = T(\mathfrak{o})$. Let J be the Iwahori subgroup, which is the preimage in K of $B(\mathbb{F}_q)$ under the canonical homomorphism $G(\mathfrak{o}) \to G(\mathbb{F}_q)$.

Let $s_1, \dots, s_r$ be the simple reflections in the Weyl group W_0. The affine Weyl group W_{aff} is obtained by adjoining one more "affine" simple reflection s_0 [Bourbaki 1968, Section VI.2]. It is the semidirect product of W_0 by the root lattice $Q^\vee$ of $\hat{T}$, which is the coroot lattice of T. The groups W_0 and W_{aff} are Coxeter groups. The extended affine Weyl group W_{ext} is a slightly larger group that is the semidirect product of W_0 by the weight lattice $P^\vee$ of $\hat{T}$, which is the coweight lattice of T, isomorphic to $T(F)/T(\mathfrak{o})$. In this presentation, $P^\vee$ is a normal subgroup. We will denote by $\lambda \mapsto a_\lambda$ a map sending $\lambda \in P^\vee$ to a representative of the corresponding coset in $T(F)/T(\mathfrak{o})$.

The group W_{ext} is not a Coxeter group, but like Coxeter groups, it has a length function. The finite subgroup Ω of elements of length 0 is isomorphic to $P^\vee/Q^\vee$. For example, if G is semisimple, then Ω is isomorphic to the fundamental group of G. The group W_{ext} is the semidirect product of W_{aff} by Ω, with W_{aff} being a normal subgroup. Conjugation by an element of Ω permutes the s_i.

The (affine) Iwahori Hecke algebra $\mathcal{H}_J$ is the convolution algebra of compactly supported J-biinvariant functions on $G(F)$. Let $\mathcal{H}_0$ be the subring of functions with support in K. Then $\mathcal{H}_0$ and $\mathcal{H}_J$ have the following explicit description due to Iwahori and Matsumoto [1965]. If r is the rank of G, the ring $\mathcal{H}_0$ is generated by $T_1, \dots, T_r$, where each T_i is the characteristic function of Js_iJ. These T_i then satisfy quadratic relations

$$T_i^2 = (q-1)T_i + q \tag{1}$$

and the braid relations

$$T_iT_jT_i \dots = T_jT_iT_j \dots,$$

where the number of terms on each side is the order of $s_i s_j$. The affine Hecke algebra $\mathcal{H}_{\text{aff}}$ is obtained by adjoining an element T_0 satisfying the same quadratic and braid relations but allowing $i = 0$ corresponding to the affine simple reflection s_0. The algebra $\mathcal{H}_J$ is slightly larger than $\mathcal{H}_{\text{aff}}$, and is isomorphic as a vector space to $\mathcal{H}_{\text{aff}} \otimes \mathbb{C}[\Omega]$. The elements of Ω act on $\mathcal{H}_{\text{aff}}$ by conjugation, and this action corresponds to permuting the T_i, just as in W_{ext} conjugation by elements of Ω permutes the s_i.

The algebra $\mathcal{H}_J$ also has a presentation analogous to the presentation of W_{ext} as the semidirect product of $P^\vee$ by W_0. This presentation, sometimes known as the *Bernstein presentation*, was developed but not published by Bernstein and Zelevinsky. Possibly its first use in a published paper was in [Rogawski 1985], and a full treatment was given by Lusztig [1989]. In the Bernstein presentation, $\mathcal{H}_0$ is supplemented by a ring homomorphism $\theta : \mathbb{C}[P^\vee] \to \mathcal{H}_J$. As a vector space, $\mathcal{H}_J = \mathcal{H}_0 \otimes \mathbb{C}[P^\vee]$. To describe the ring structure, it is sufficient to give one relation. Let $1 \leqslant i \leqslant r$ and let $\lambda \in P^\vee$. Then

$$\theta(\lambda)T_i - T_i\theta(s_i\lambda) = (q-1)\frac{\theta(\lambda) - \theta(s_i\lambda)}{1 - \theta(-\alpha_i^\vee)},$$

where $\alpha_i^\vee$ is the coroot corresponding to i.

Let $z \in \hat{T}(\mathbb{C})$, and let $\tau = \tau_z : T(F) \to \mathbb{C}^\times$ be the corresponding unramified character. We extend it to $B(F)$ by letting the unipotent radical $N(F)$ be in the kernel. The principal series representation $M(\tau)$ consists of all locally constant maps $f : G(F) \to \mathbb{C}$ such that $f(bg) = (\delta^{1/2}\tau)(b)\, f(g)$ for $b \in B(F)$, $g \in G(F)$. The action of $G(F)$ is by right translation. The module $M(\tau)$ is irreducible if τ is in general position.

If (π, V) is an irreducible representation having a J-fixed vector, then V^J is a finite-dimensional irreducible $\mathcal{H}_J$ module and its isomorphism class determines π. Any such (π, V) with a J-fixed vector is a subquotient of $M(\tau)$ for some τ, and the category of smooth representations of finite length all of whose composition factors have J-fixed vectors is equivalent to the category of finite-dimensional $\mathcal{H}_J$-modules.

The Weyl group W_0 acts on T and hence on unramified characters. We will make this a right action, so $\tau w(a) = \tau(waw^{-1})$ for $a \in T(F)$. If $w \in W_0$, the modules $M(\tau)$ and $M(\tau w)$ are isomorphic if irreducible, and in any case have isomorphic semisimplifications. To see this, one may construct homomorphisms $A_w : M(\tau) \to M(\tau w)$ by means of *intertwining integrals*. By definition,

$$A_w f(g) = \int_{N \cap w^{-1} N_- w} f(wng)\, dn, \tag{2}$$

where N_- is the unipotent radical of the negative Borel. The integral is convergent if $|\tau(a_\lambda)| < 1$ for dominant weights λ. By the *singular set*, we mean the union of hyperplanes in $\hat{T}(\mathbb{C})$ that are the kernels of the coroots $\alpha^\vee$ (regarded as characters of $\hat{T}$). For arbitrary τ, the intertwining operator may be defined by analytic continuation, except that the A_w can have poles in the singular set.

Now we return to the point in question: why may Hecke algebras be used to model intertwining operators? The basic insight is that, to any ring R regarded as a left R-module, a left R-module homomorphism $\lambda : R \to R$ is necessarily of the form $\lambda(x) = x \cdot a$ for some $a \in R$. This is trivial to prove with $a = \lambda(1)$.

The first way of applying this is to note that the space $M(\tau)^J$ of J-invariants is $|W_0|$-dimensional. It has several natural bases indexed by Weyl group elements. A particular one is the basis $\Phi_w^\tau = \Phi_w$ ($w \in W_0$) defined by

$$\Phi_w^\tau(bk) = \begin{cases} \delta^{1/2}\tau(b) & \text{if } k \in BwJ, \\ 0 & \text{otherwise,} \end{cases}$$

for $b \in B(F)$ and $k \in K$.

We see that $\mathcal{H}_0$ and $M(\tau)^J$ are both $|W_0|$-dimensional $\mathcal{H}_0$-modules, and in fact they are isomorphic as left $\mathcal{H}_0$-modules. A particular isomorphism

$$\varrho_\tau : M(\tau)^J \to \mathcal{H}_0$$

is given by $\varrho_\tau(f) = F$, where

$$F(g) = \begin{cases} f(g^{-1}) & \text{if } g \in K, \\ 0 & \text{otherwise}. \end{cases}$$

It is not hard to check that this is an isomorphism of left $\mathcal{H}_0$-modules.

Composing with this isomorphism, the intertwining integral A_w thus gives rise to a homomorphism $\mathcal{H}_0 \to \mathcal{H}_0$. This can have poles (in the singular set) or zeros (if $\tau(a_{\alpha^\vee}) = q^{\pm 1}$ for some coroot $\alpha^\vee$), but if τ is in general position, it is an isomorphism and so it agrees with right multiplication by a particular element F_w of $\mathcal{H}_0$, which was identified by Rogawski [1985]. It is sufficient to describe it when $w = s_i$ is a simple reflection, and in this case

$$F_{s_i} = \frac{1}{q}(T_i + 1) - C_{\alpha_i}(\tau),$$

where $C_{\alpha_i}(\tau)$ with $\tau = \tau_z$ is the ubiquitous rational function

$$C_{\alpha_i}(\tau) = \frac{1 - q^{-1} z^{\alpha_i^\vee}}{1 - z^{\alpha_i^\vee}}. \tag{3}$$

In other words, the following diagram is commutative:

$$\begin{tikzcd} M(\tau)^J \arrow[r, "\varrho_\tau"] \arrow[d, "A_{s_i}"'] & \mathcal{H}_0 \arrow[d, "\cdot F_{s_i}"] \\ M(\tau s_i)^J \arrow[r, "\varrho_{\tau s_i}"'] & \mathcal{H}_0 \end{tikzcd} \tag{4}$$

See [Brubaker, Bump and Licata 2011, Lemma 24] for a proof.

Instead of $\mathcal{H}_0$, one may also use $\mathcal{H}_J$ to model the intertwining integrals. The factor $C_{\alpha_i}(\tau)$, which depends on the spectral parameter, may then be replaced by an element of $\theta(P^\vee)$. This point of view is taken in Haines, Kottwitz and Prasad [Haines et al. 2010]. In order to see how this should work, consider that since the intertwining operator permutes the spaces $M(\tau)^J$, one might consider each A_w to be an endomorphism of

$$\bigoplus_{w\in W_0} M(\tau w)^J. \tag{5}$$

The characters τ of $B(F)$ that are induced all have a common trivial restriction to the subgroup $B_0 = T(\mathfrak{o})N(F)$, so by Frobenius reciprocity, the module $M = \mathrm{ind}_{B_0}^{G(F)}(1)$ is the direct integral of the spaces (5), with τ running over $\hat{T}$ modulo the action of W_0. This is the *universal principal series*. It is almost true that the intertwining integrals are endomorphisms of $\mathrm{ind}_{B_0}^{G(F)}(1)$; the difficulty is that the operators are polar in the singular set, so one must restrict to the orthogonal complement in M of the spaces (5) with singular τ. Alternatively, one may consider the compact induction $M_c = \text{c-ind}_{B_0}^{G(F)}(1)$. Although this is no longer closed under the intertwining operators, at least for f in M_c the intertwining integral (2) is always convergent, and A_w is realized as a map $M_c \to M$.

We now come to the point, which is that as an $\mathcal{H}_J$-module, M_c is a free module of rank one. This is to be expected from the Bruhat decomposition, because every coset in $B_0\backslash G(F)/J$ has a unique representative in the extended affine Weyl group, and this is also in bijection with $J\backslash G(F)/J$. Thus the extended affine Weyl group parametrizes both a basis of M_c and a basis of $\mathcal{H}_J$. For a proof that the module M_c is one-dimensional, see [Chriss and Khuri-Makdisi 1998] or [Haines et al. 2010, Lemma 1.6.1].

If A_w were a map $M_c \to M_c$, we could then transfer A_w to a map $\mathcal{H}_J \to \mathcal{H}_J$ and conclude that it agreed with right multiplication by some element. Due to the poles of the intertwining operators, this does not quite work. What may be done is to consider a somewhat larger Hecke algebra. In the Bernstein presentation $\mathcal{H}_J \cong \mathcal{H}_0 \otimes \mathbb{C}[P^\vee]$, we can enlarge $\mathbb{C}[P^\vee]$ to any ring $\mathfrak{R}$ such that $\mathbb{C}[P^\vee] \subseteq \mathfrak{R} \subseteq \mathfrak{M}$, where $\mathfrak{M}$ is the field of fractions of $\mathbb{C}[P^\vee]$. We take $\mathfrak{R}$ to be the localization at the set of singular hyperplanes, that is, the ring obtained by adjoining $1-\alpha^\vee$ for all coroots $\alpha^\vee$. Let $\mathcal{H}'_J = \mathcal{H}_0 \otimes \mathfrak{R}$. Since the poles of the intertwining operators are contained in the singular locus, if M' denotes the submodule of M generated by the image of the A_w, then we have a commutative diagram

$$\begin{tikzcd} M_c \cong \mathcal{H}_J \arrow[r] \arrow[d, "A_w"'] & \mathcal{H}'_J \arrow[d] \\ M' \arrow[r] & \mathcal{H}'_J \end{tikzcd} \tag{6}$$

The top arrow is the natural inclusion and the bottom arrow extends the injection of M_c into $\mathcal{H}_J$. The vertical arrows are $\mathcal{H}'_J$-module homomorphisms. As before, if R is a ring, then a left R-module homomorphism $R \to R$ is multiplication by some element, and therefore A_w may be realized as multiplication by some element of $\mathcal{H}'_J$.

3. Hecke algebra modules from unique functionals

There are two different kinds of actions of the affine Hecke algebra that we need to consider. First, in any smooth representation, the Hecke algebra acts by convolution. The Hecke algebra action on the Iwahori-fixed vectors in an induced representation is an action of this type.

Second, with the notation as in the prior section, the affine Hecke algebra acts on the ring $\mathbb{O}(\hat{T})$ of rational functions on the maximal torus $\hat{T}$ in the L-group. This ring is isomorphic to the group algebra $\mathbb{C}[P^\vee]$ of the coweight lattice $P^\vee$ of T, which may be identified with the weight lattice of $\hat{T}$. The generators T_i of the finite Hecke algebra $\mathcal{H}_0$ will act by so-called *Demazure–Lusztig operators*. This action was introduced by Lusztig [1985] and has far-reaching consequences.

We have seen in the previous section how the Hecke algebra can be used to model intertwining integrals. In this section, we will show how we may translate this action of the Hecke algebra into an action on rational functions. We described two interpretations of intertwining operators via Hecke algebras in the diagrams (4) and (6); the simpler point of view in (4) will be sufficient for our purposes, as it was for Rogawski.

Let us consider, for every z in some Zariski-dense subset of $\hat{T}$, a linear functional L_z on $M(\tau)$. We shall suppose that L_z arises from a multiplicity-free representation in the following way. Let H be a subgroup of $G(F)$ and η a character of H such that the induced representation $\mathrm{ind}_H^{G(F)}(\eta)$ is multiplicity-free. Then by Frobenius reciprocity, a functional L_z (if it exists) is characterized up to scalar multiple by the property that $L_z(\pi(g)\phi) = \eta(g)L_z(\phi)$ for $g \in H$. As noted above, using the uniqueness of the functional, it is often possible to show by a method of Bernstein that $L_z(\phi)$ is a rational function of z on this set. Alternatively, this rationality may be proved for one specific vector, together with recursions that imply the rationality for all ϕ.

As particular examples, L_z could be the spherical functional

$$\mathcal{S}_z(f) = \int_K f(k)\,dk$$

or the Whittaker functional

$$\mathcal{W}_z(f) = \int_{N(F)} f(w_0 n g)\psi(n)\,dn,$$

where ψ is a nondegenerate character of $N(F)$, the unipotent radical of the Borel, and w_0 is the long element of the Weyl group W. The spherical and Whittaker functionals are both characterized up to scalar multiple by the uniqueness property described above. Indeed, since the spherical Hecke algebra of compactly supported K-biinvariant functions is commutative, K is a Gelfand subgroup of $G(F)$, so the functional L_z is determined (up to scalar) by the fact that $\mathcal{S}_z(\pi(k)\phi) = \mathcal{S}_z(\phi)$ for all $k \in K$. For the Whittaker functional, the corresponding uniqueness result was obtained by Gelfand and Graev, by Piatetski-Shapiro and by Shalika [1974]. The Whittaker integral, like the intertwining integral, is convergent for z in an open set, and has analytic continuation to all z.

It follows from these uniqueness results that for every pair z and w, the functional $L_{zw} \circ A_w$ is a constant multiple of L_z. For the two examples above, these constants were computed by Casselman [1980] and Casselman and Shalika [1980], who found that for the spherical functional

$$\mathcal{S}_{zw} \circ A_w = \prod_{\substack{\alpha \in \Delta^+ \\ w^{-1}\alpha \in \Delta^-}} \left(\frac{1 - q^{-1} z^{\alpha^\vee}}{1 - z^{\alpha^\vee}} \right) \mathcal{S}_z = \prod_{\substack{\alpha \in \Delta^+ \\ w^{-1}\alpha \in \Delta^-}} C_\alpha(\tau)\mathcal{S}_z, \tag{7}$$

with $C_\alpha(\tau)$ as in (3), and for the Whittaker functional

$$\mathcal{W}_{zw} \circ A_w = \prod_{\substack{\alpha \in \Delta^+ \\ w^{-1}\alpha \in \Delta^-}} \left(\frac{1 - q^{-1} z^{-\alpha^\vee}}{1 - z^{\alpha^\vee}} \right) \mathcal{W}_z. \tag{8}$$

Here Δ^+ and Δ^- are the positive and negative roots, and if α is a root, then $\alpha^\vee$ is the associated coroot.

Our goal is to describe a Hecke algebra action on $M(\tau)^J$ arising from each of these functionals and explain how this action gives a recursion for $L_z(\pi(g)\Phi_w)$ for any standard basis element $\Phi_w \in M(\tau)^J$. To describe the function $L_z(\pi(g)\Phi_w)$, it suffices to choose g from a set of representatives for $H\backslash G(F)/J$, where $H = K$ when $L = \mathcal{S}$ and $H = N$ when $L = \mathcal{W}$. This means that we may choose $g = a_\lambda$, where $\lambda \in P^\vee$, and in the Whittaker case we may assume λ is dominant, since otherwise $\mathcal{W}(\pi(a_\lambda)\phi) = 0$ for any $\phi \in M(\tau)$; see [Brubaker, Bump and Licata 2011, Lemma 5].

For both the spherical and Whittaker functionals, there is a standard basis vector Φ_w for which $L_z(\pi(a_\lambda)\Phi_w)$ has a particularly simple form. In the Whittaker case, for any dominant weight λ we have

$$\mathcal{W}_z(\pi(a_\lambda)\Phi^\tau_{w_0}) = \delta^{1/2}(a_\lambda) z^{w_0\lambda};$$

see [Brubaker, Bump and Licata 2011, Proposition 6]. In the spherical case, we have:

Proposition 1. *Let $\lambda \in P^\vee$. Then $\mathcal{S}_z(\pi_z(a_\lambda)\Phi^\tau_1) = c(\lambda) z^\lambda$, where the constant $c(\lambda)$ is independent of z.*

Proof. We have

$$\mathcal{S}_z(\pi_z(a_\lambda)\Phi^\tau_1) = \int_K \Phi_1(ka_\lambda)\, dk = (\delta^{1/2}\tau)(a_\lambda) \int_K \Phi_1(a_\lambda^{-1} k a_\lambda)\, dk.$$

The support of Φ_1 is $B(F)J$. It is easy to see that if $k \in K$ and $a_\lambda^{-1} k a_\lambda \in B(F)J$, then $a_\lambda^{-1} k a_\lambda \in B_0 J$, where B_0 is the kernel of $\tau_z : B(F) \to \mathbb{C}$. Hence the integral is a constant independent of z. □

For either functional, this choice of standard basis vector will be the starting point for our recursion. It remains to define the Hecke action. From Rogawski's perspective, $M(\tau)^J$ is understood abstractly as a Hecke algebra module via the regular representation, which is determined by

$$T_w \Phi_1 = \Phi_w \quad \text{and} \quad T_w \Phi_y = T_w T_y \Phi_1 \quad \text{for all } y, w \in W.$$

Remembering the quadratic relation given in (1), we see that for a simple reflection s,

$$T_s \Phi_w = \begin{cases} \Phi_{sw} & \text{if } sw > w, \\ q\Phi_{sw} + (q-1)\Phi_w & \text{if } sw < w. \end{cases}$$

But (4) gives a relation between the intertwining operator A_s and T_s. So combining these two ingredients and manipulating the terms, we obtain the following result.

Proposition 2. *Let $\tau = \tau_z$ and let $w \in W_0$. Let $s = s_i$ be a simple reflection. Then*

$$A_s \Phi^{\tau s}_w + C_\alpha(\tau)\Phi^\tau_w = \begin{cases} \Phi^\tau_w + \Phi^\tau_{sw} & \textit{if } sw < w, \\ q^{-1}(\Phi^\tau_w + \Phi^\tau_{sw}) & \textit{if } sw > w. \end{cases} \tag{9}$$

Proof. This is an easy consequence of [Casselman 1980, Theorem 3.4]; see Proposition 8 of [Brubaker, Bump and Licata 2011]. □

Thus it is reasonable to expect that we may obtain an action of the Hecke algebra on the ring $\mathcal{O}(\hat{T})$ of regular functions on $\hat{T}$ via the regular representation by applying the functional to both sides of (9).

The case where $L = \mathscr{W}$ is treated in detail in [Brubaker, Bump and Licata 2011], so we will focus on the case $L = \mathscr{S}$, and return at the end to make some remarks about the difference between the two cases. For $L = \mathscr{S}$, we intend to start our recursion at Φ_1 and move up in the Bruhat order. So let us rewrite (9) in the case $sw > w$:

$$qA_s\Phi_w^{\tau s} + (qC_\alpha(\tau) - 1)\Phi_w^\tau = \Phi_{sw}^\tau. \tag{10}$$

We will find that applying $\mathscr{S}_z$ to both sides gives a recursive identity for the matrix coefficient

$$F_{\lambda,w}(z) := \mathscr{S}_z(\pi(a_\lambda)\Phi_w^\tau)$$

of the principal series representation. To state this more precisely, let us introduce the *Demazure–Lusztig* operator $\mathscr{T}_i$ defined on an arbitrary function F on the dual torus $\hat{T}$ by

$$\mathscr{T}_i F(z) := qC_{-\alpha_i}(\tau)F(zs_i) + (qC_{\alpha_i}(\tau) - 1)F(z). \tag{11}$$

Here $1 \le i \le r$ corresponds to a simple reflection s_i of the finite Weyl group. After some algebra, this is equivalent to

$$\mathscr{T}_i F(z) = (z^{\alpha_i} - 1)^{-1}\big(F(z) - F(zs_i) - qF(z) + qz^{\alpha_i}F(zs_i)\big),$$

which is precisely the operator defined in [Lusztig 1985, (8.1)]. (This can also be defined if $i = 0$, but for this discussion we are excluding this case.)

The following result is equivalent to a result of Ion [2006, Proposition 5.8].

Theorem 1. *The Demazure–Lusztig operators satisfy the quadratic and braid relations, and hence generate a ring isomorphic to the Hecke algebra $\mathscr{H}_0$. If $w \in W_0$ and $s_i w > w$, then*

$$F_{\lambda,s_i w} = \mathscr{T}_i F_{\lambda,w}. \tag{12}$$

The fact that the Demazure–Lusztig operators satisfy the quadratic and braid relations is due to Lusztig [1985, Section 8]. However, checking the braid relations directly depends on a tedious computation for rank-2 root systems, so it may be of interest that we can avoid such computations using our methods.

Proof. Assume that $sw > w$. Apply $\pi(a_\lambda)$ to both sides of (10) and then apply $\mathscr{S}_z$. We obtain

$$F_{\lambda,sw}(z) = q\mathscr{S}_z^\tau A_s(\pi(a_\lambda)\Phi_w^{\tau s}) + (qC_\alpha(\tau) - 1)F_{\lambda,w}(z). \tag{13}$$

Now we use (7), replacing z by zs and remembering that $C_\alpha(\tau_{zs}) = C_{-\alpha}(\tau)$. Comparing with (11), the right-hand side of (13) is just $\mathscr{T}_i F_{\lambda,w}^\tau$, as desired.

We turn to the fact that the $\mathscr{T}_i$ satisfy the generating relations of $\mathscr{H}_0$. Let m be the order of $s_i s_j$ with $1 \leqslant i, j \leqslant r$, $i \neq j$. To show that $\mathscr{T}_i\mathscr{T}_j\mathscr{T}_i \ldots = \mathscr{T}_j\mathscr{T}_i\mathscr{T}_j \ldots$

(m factors on both sides), it is sufficient to show that they have the same effect on z^λ, where $\lambda \in P^\vee$. By Proposition 1, it is thus sufficient to show that

$$\mathcal{T}_i \mathcal{T}_j \mathcal{T}_i \dots F_{a_\lambda,1}(z) = \mathcal{T}_j \mathcal{T}_i \mathcal{T}_j \dots F_{a_\lambda,1}(z).$$

But applying (12), both sides equal $F_{a_\lambda,w}(z)$, where $w = s_i s_j s_i \dots = s_j s_i s_j \dots$ is the longest element of the dihedral group generated by s_i and s_j.

We next prove the quadratic relation. Assume now that $s_i w < w$. Then applying (12) to $F_{\lambda,s_i w}$, we have $F_{\lambda,s_i w} = \mathcal{T}_i^{-1} F_{\lambda,w}$. Now we can compute this by the first case of (9), and we find

$$\mathcal{T}_i^{-1} F(z) = C_{-\alpha} F(z s_i) + (C_\alpha(\tau) - 1) F(z).$$

This means that $\mathcal{T}_i^{-1} = q^{-1}\mathcal{T}_i + q^{-1} - 1$, which is equivalent to the quadratic relation $\mathcal{T}_i^2 = (q-1)\,\mathcal{T}_i + q$. □

This theorem guarantees that, given any reduced decomposition for the Weyl group element $w = s_{i_1} \dots s_{i_k}$, the operator

$$\mathcal{T}_w := T_{i_1} \dots \mathcal{T}_{i_k}$$

is well defined. As noted in [Ion 2008, Theorem 3.1], the Demazure–Lusztig operators applied to z^λ for λ dominant give a recursive definition for a certain limit of nonsymmetric Macdonald polynomials with weight $w \cdot \lambda \in P^\vee$. Thus our Iwahori-spherical functions are also limits of these polynomials for λ dominant.

We caution the reader that our conventions for Demazure–Lusztig operators and nonsymmetric Macdonald polynomials differ slightly from those of Ion. Instead, they more closely parallel those of Cherednik [1995]. In particular, our $\mathcal{T}_i$ essentially match those in [Cherednik 1995, (3.5)], which are then used to construct nonsymmetric Macdonald polynomials.

Returning to the Whittaker case, similar arguments to those presented above were given in [Brubaker, Bump and Licata 2011]. The resulting operators are not Demazure–Lusztig operators, but are related in a way that is made precise in Section 5 of that reference. The difference results from the fact that the starting point for the Whittaker recursion is the Iwahori-fixed vector Φ_{w_0}, rather than Φ_1. Furthermore, the constant of proportionality in (8) differs slightly from the spherical case given in (7). The resulting recursive operators for Whittaker functions are Demazure–Lusztig operators conjugated by $\theta(\rho^\vee)$ and with q replaced by q^{-1}. Here $\theta : \mathbb{C}[P^\vee] \to \mathcal{H}_J$ is as in the previous section and $\rho^\vee$ is half the sum of the positive coroots. In either the spherical or Whittaker cases, the resulting action of the finite Hecke algebra can be generalized to an action of the (extended) affine Hecke algebra where the elements in $\theta(P^\vee)$ act by translation. (See Theorem 28 of [Brubaker, Bump and Licata 2011] for the Whittaker case.)

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BENJAMIN BRUBAKER
SCHOOL OF MATHEMATICS
UNIVERSITY OF MINNESOTA
MINNEAPOLIS, MN 55455
UNITED STATES
brubaker@math.umn.edu

DANIEL BUMP
DEPARTMENT OF MATHEMATICS
STANFORD UNIVERSITY
BUILDING 380
STANFORD, CA 94305-2125
UNITED STATES
bump@math.stanford.edu

SOLOMON FRIEDBERG
DEPARTMENT OF MATHEMATICS
BOSTON COLLEGE
CHESTNUT HILL MA 02467-3806
UNITED STATES
friedber@bc.edu

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A RELATIVE TRACE FORMULA FOR PGL(2) IN THE LOCAL SETTING

BROOKE FEIGON

In memory of Jonathan Rogawski

We develop the local Kuznetsov trace formula on a unitary group in two variables for an unramified quadratic extension of local, non-Archimedean fields E/F and compare it to a local relative trace formula on PGL$(2, E)$. To define the local distributions for the relative trace formula, we define a regularized local period integral and prove that it is a PGL$(2, F)$-invariant linear functional. By comparison of the two local trace formulas, we get an equality between a local PGL$(2, F)$-period and local Whittaker functionals.

1. Introduction

Base change is an important type of functoriality which is useful in the study of automorphic forms by relating automorphic representations on different groups. Hervé Jacquet shed light on a new technique for attacking certain cases of Robert Langlands' important functoriality conjectures by comparing the relative and Kuznetsov trace formulas in the global setting. Jacquet's comparison of trace formulas leads to global identities that characterize the image of the base change map associating automorphic representations of a unitary group for a quadratic extension of number fields E/F to automorphic representations of GL$(2, \mathbb{A}_E)$ in terms of distinguished representations. While Jacquet's global identities factor, they do not give unique local identities.

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This paper uses techniques of James Arthur to define and develop a local Kuznetsov trace formula on U(2) and a local relative trace formula on GL(2). Both local trace formulas are expanded geometrically in terms of orbital integrals and spectrally in terms of local Bessel distributions and local relative Bessel distributions. The latter involve regularized local period integrals. We then carry out Jacquet's comparison in the local setting by relating these two local trace formulas for matching functions. This comparison yields identities between local Bessel distributions for automorphic representations on U(2) and local relative Bessel distributions for automorphic representations on GL(2).

Before we describe more precisely the local relative trace formula developed in this paper, let us recall the relative trace formula for GL(2). Take E/F to be a quadratic extension of number fields and $\mathbb{A}_F$ to be the adeles of F. Let ψ' be a character on $F\backslash\mathbb{A}_F \cong N(F)\backslash N(\mathbb{A}_F)$ where N is the upper triangular unipotent matrices of GL(2). Let $\psi = \psi' \circ \mathrm{tr}_{E/F}$.

A cuspidal automorphic representation π of GL$(2, \mathbb{A}_E)$ with central character trivial on GL$(2, \mathbb{A}_F)$ is *distinguished* by GL$(2, \mathbb{A}_F)$ if there exists a $\phi \in V_\pi$, the vector space associated to π, such that the *period integral*, $P(\phi)$, is nonzero:

$$P(\phi) := \int_{\mathrm{GL}(2,F)Z(\mathbb{A}_F)\backslash \mathrm{GL}(2,\mathbb{A}_F)} \phi(h)\, dh \neq 0.$$

Where π' is a cuspidal automorphic representation of the quasisplit unitary group $U(2, \mathbb{A}_F)$ and $\phi' \in V_{\pi'}$, let

$$W(\phi') = \int_{N(F)\backslash N(\mathbb{A}_F)} \phi'(n)\overline{\psi'(n)}\, dn \quad \text{and} \quad W(\phi) = \int_{N(E)\backslash N(\mathbb{A}_E)} \phi(n)\overline{\psi(n)}\, dn.$$

We define the *Bessel distribution* as

$$B'_{\pi'}(f') := \sum_i W'(\pi'(f')\phi'_i)\overline{W'(\phi'_i)},$$

and the *relative Bessel distribution* as

$$B_\pi(f) := \sum_j P(\pi(f)\phi_j)\overline{W(\phi_j)},$$

where the summations are over an orthonormal basis of $V_{\pi'}$ and V_π respectively. Flicker [1991], following related work of Jacquet and Lai [1985] and Ye [1989], showed that for "matching functions" f' on $U(2, \mathbb{A}_F)$ and f on GL$(2, \mathbb{A}_E)$, if π' maps to π under the unstable base change, then

$$\sum_i W'(\pi'(f')\phi'_i)\overline{W'(\phi'_i)} = \sum_j P(\pi(f)\phi_j)\overline{W(\phi_j)}. \tag{1-1}$$

In particular, this equality characterizes the image of the unstable base change lift associating every automorphic representation of $U(2, \mathbb{A}_F)$ to an automorphic representation of $\mathrm{GL}(2, \mathbb{A}_E)$ in terms of $\mathrm{GL}(2, \mathbb{A}_F)$ distinguished representations. The equality above is proved via the relative trace formula [Jacquet 2005], which tells us that for f and f' matching functions we have

$$\int_{(N(F)\backslash N(\mathbb{A}_F))^2} K_{f'}(n_1, n_2)\psi'(n_1^{-1}n_2)\, dn_1\, dn_2 = \int_{\mathrm{GL}(2,F)Z(\mathbb{A}_F)\backslash \mathrm{GL}(2,\mathbb{A}_F)} \int_{N(E)\backslash N(\mathbb{A}_E)} K_f(h, n)\psi(n)\, dn\, dh$$

where

$$K_f(x, y) = \sum_{\delta\in Z(E)\backslash \mathrm{GL}(2,E)} f(x^{-1}\delta y).$$

The distributions $B'_{\pi'}(f')$ and $B_\pi(f)$ occur in the spectral expansions of the respective trace formulas.

In a different direction, Arthur [1989; 1991] developed a local version of the classical Arthur–Selberg trace formula. Let G be a connected reductive algebraic group over a local field F of characteristic zero. Diagonally embed $G(F)$ into $G(F) \times G(F)$. Then $L^2(G(F))$ is isomorphic to $L^2(G(F)\backslash G(F) \times G(F))$ by

$$\phi \mapsto ((y_1, y_2) \mapsto \phi(y_1^{-1}y_2)).$$

For $\phi \in L^2(G(F))$, let $(\rho(g_1, g_2)\phi)(x) = \phi(g_1^{-1}xg_2)$. The right regular representation of $G(F) \times G(F)$ on $L^2(G(F)\backslash G(F) \times G(F))$ is equivalent to ρ of $G(F) \times G(F)$ on $L^2(G(F))$. Thus to develop the local trace formula we look at $\rho(f)$ where $f = f_1 \otimes f_2 \in C_c^\infty(G(F) \times G(F))$. Then

$$(\rho(f)\phi)(x) = \int_{G(F)} \int_{G(F)} f_1(g) f_2(y)\phi(g^{-1}xy)\, dg\, dy$$

is an integral operator on $L^2(G(F))$ with kernel

$$K_f(x, y) = \int_{G(F)} f_1(g) f_2(x^{-1}gy)\, dg.$$

The local trace formula develops an explicit formula for the regularized trace of $\rho(f)$.

The main result of this paper is that, when evaluated with matching functions, the two local trace formulas described in Theorems 1.3 and 1.4 below, that is the local Kuznetsov trace formula and the local relative trace formula, are equal. Thus there is an equality between their local distributions on the spectral sides. This equality is stated in Theorem 1.1. This is the natural local counterpart to the global comparison from (1-1). In order to develop the local relative trace formula stated in Theorem 1.4, we have to define a local regularized period integral, prove it is a $\mathrm{GL}(2, F) \times \mathrm{GL}(2, F)$-invariant linear functional and relate it to the truncated period integral that initially appears in the relative trace formula. We state these properties about the local regularized period integral in Proposition 1.2.

To describe our results more precisely we need to introduce some further notation. Let E/F now denote an unramified extension of local non-Archimedean fields of characteristic 0. Let $\mathcal{O}_F$ (respectively $\mathcal{O}_E$) denote the ring of integers in F (respectively E). Let $H = \mathrm{GL}(2)/F$, $G = \mathrm{Res}_{E/F} H$ and let

$$G' = \mathrm{U}(2, F) = \left\{ g \in G : {}^t\bar{g} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}.$$

Let N' and N be the upper triangular unipotent matrices of G' and G, respectively, and let M' and M be the diagonal subgroup of G' and G, respectively. Let Z and Z' denote the center of G and G', respectively. For any subgroup X of G let $\widetilde{X} = Z \cap X \backslash X$ and let $X_H = X \cap H$. Let ψ' be an additive character on F with conductor $\mathcal{O}_F$ and let $\psi(x) = \psi' \circ \mathrm{tr}_{E/F}$. Let $f = f_1 \otimes f_2 \in C_c^\infty(\widetilde{G}(F) \times \widetilde{G}(F))$ and $f' = f_1' \otimes f_2' \in C_c^\infty(\widetilde{G}'(F) \times \widetilde{G}'(F))$.

We define the local Kuznetsov trace formula as the equality between the geometric expansion (in terms of orbital integrals) and spectral expansion (in terms of representations) of

$$\lim_{t\to\infty} \int_{(N'\times N')(F)} K_{f'}(n_1, n_2)\psi'(n_1^{-1}n_2)u(n_1, t)u(n_2, t)\, dn_1\, dn_2$$

and the local relative trace formula as the equality between the expansions of

$$\lim_{t\to\infty} \int_{\widetilde{H}(F)} \int_{N(F)} K_f(h, n)\psi(n)u(h, t)u(n, t)\, dn\, dh.$$

In this local setting

$$K_f(x, y) = \int_{\widetilde{G}(F)} f_1(g) f_2(x^{-1}gy)\, dg, \qquad K_{f'}(x, y) = \int_{\widetilde{G}'(F)} f_1'(g) f_2'(x^{-1}gy)\, dg$$

and $u(n, t)$ and $u(h, t)$ are truncation parameters that are needed due to convergence issues. They are defined analogously to Arthur's truncation [1991, Section 3].

We use the following ideas in this paper to rewrite these local trace formulas in terms of orbital integrals and representations:

- methods of Arthur [1991] from the local trace formula,
- methods of Flicker [1991], Jacquet [2005] and Ye [1989] from the relative trace formula,
- Harish-Chandra's Plancherel formula [Harish-Chandra 1984; Waldspurger 2003],
- Jacquet, Lapid and Rogawski's methods for regularizing period integrals [Jacquet et al. 1999; Jacquet $\geq$ 2012].

The power of the two trace formulas lies in the comparison. For "matching functions", the geometric expansions of the two local relative trace formulas are equal. By comparing the spectral expansions in these two trace formulas, we get an analogue of (1-1), giving the following identity between local Bessel distributions for functions on $U(2)$ and local relative Bessel distributions for functions on GL(2, E), and therefore local periods and local Whittaker functionals:

Theorem 1.1. *If σ is a supercuspidal representation on $\widetilde{G}(F)$ that is the unstable base change lift of the supercuspidal representation σ' of $\widetilde{G}'(F)$, and*

$$f' = f_1' \otimes f_2' \in C_c^\infty(\widetilde{G}'(F) \times \widetilde{G}'(F)) \quad \text{and} \quad f = f_1 \otimes f_2 \in C_c^\infty(\widetilde{G}(F) \times \widetilde{G}(F))$$

are matching functions, then

$$\text{(1-2)} \quad d(\sigma') \sum_{S' \in \mathcal{B}(\sigma')} W'_{\sigma'}(\sigma'(f_2')S'\sigma'(f_1'^{\vee}))\overline{W'_{\sigma'}(S')} \\ = d(\sigma) \sum_{S \in \mathcal{B}(\sigma)} P_\sigma(\sigma(f_2)S\sigma(f_1^{\vee}))\overline{W_\sigma(S)},$$

where $d(\sigma)$ is the formal degree of σ, $\mathcal{B}(\sigma)$ is an orthonormal basis of the Hilbert space of Hilbert–Schmidt operators on V_σ,

$$W'_{\sigma'}(S') = \int_{N'(F)} \mathrm{tr}(\sigma'(n)S')\psi'(n^{-1})\,dn,$$
$$W_\sigma(S) = \int_{N(F)} \mathrm{tr}(\sigma(n)S)\psi(n^{-1})\,dn,$$
$$P_\sigma(S) = \int_{\widetilde{H}(F)} \mathrm{tr}(\sigma(h)S)\,dh.$$

The Bessel and relative Bessel distributions $B'_{\pi'}(f')$ and $B_\pi(f)$ factor into local (relative) Bessel distributions $B'_{\pi'_v}(f'_v)$ and $B_{\pi_v}(f_v)$, but it is not clear how to normalize the local distributions. The distributions on the left and right-hand side of (1-2) are each the product of two local distributions and (1-2) can be restated as

$$d(\sigma')B'_{\sigma'}(f_2')B'_{\sigma'^*}(f_1') = d(\sigma)B_\sigma(f_2)B_{\sigma^*}(f_1).$$

We note that the local period integral $P_\sigma(S)$ is not a convergent integral if σ is not a discrete series representation. To develop the local relative trace formula we have to define a local regularized period integral. Let $K = G(\mathbb{O}_F)$ and let $P = NM$. For $\lambda \in \mathbb{C}$ and $m = \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \beta \end{smallmatrix}\right)$ let $e^{\lambda H_M(m)} = |\alpha/\beta|_E$ where $|\cdot|_E$ denotes the normalized valuation on E. For a principal series representation π of $\widetilde{G}$ and $u, v \in \pi$ we define the matrix coefficient $f_{u,v}(g) = \langle \pi(g)u, v\rangle$. Asymptotically on M, $f_{u,v}$ will equal a finite sum of functions of the form $e^{\lambda H_M(m)}$. We define the regularized period integral as:

$$\int^{*}_{\widetilde{H}(F)} f_{u,v}(h)\,dh := \int_{\widetilde{H}(F)} f_{u,v}(h)u(h,t)\,dh + \int_{\widetilde{K}_H\times\widetilde{K}_H}\int^{\sharp}_{\widetilde{M}^+_H(F)} D_{P_H}(m) f_{u,v}(k_1mk_2)(1-u(m,t))\,dm\,dk_1\,dk_2$$

where

$$\int^{\sharp}_{\widetilde{M}^+_H(F)} e^{\lambda H_M(m)}(1-u(m,t))\,dm$$

is the meromorphic continuation at $\nu=0$ of

$$\int_{\widetilde{M}^+_H(F)} e^{(\nu+\lambda)H_M(m)}(1-u(m,t))\,dm,$$

which is absolutely convergent for $\mathrm{Re}(\nu)\ll 0$.

We prove that the regularized period integral is an $H(F)\times H(F)$-invariant linear functional, and we relate it to the truncated period integral that initially appears in the local relative trace formula as follows. By abuse of notation we identify a character χ of $\widetilde{M}(F)$ with a character χ of $E^\times$ by letting $\chi\left(\begin{smallmatrix} a & 0\\ 0 & b\end{smallmatrix}\right)=\chi(a)\chi^{-1}(b)$. For $\lambda\in\mathbb{C}$ we let $\chi_\lambda(m)=\chi(m)e^{\lambda(H_M(m))}$. We let $I_P(\chi_\lambda)$ be the parabolically induced normalized representation acting on the Hilbert space $\mathcal{H}_P(\chi)$. Then for $S\in\mathcal{B}_P(\chi)$,

$$\mathrm{tr}(I_P(\chi_\lambda,k_1gk_2)S)=E_P(g,\Psi_S,\lambda)_{k_1,k_2},$$

where $E_P(g,\Psi,\lambda)$ is the Eisenstein integral and

$$(C^P E_P)(m,\psi,\lambda)=(c_{P|P}(1,\lambda)\psi)(m)e^{\lambda H_M(m)}+(c_{P|P}(w,\lambda)\psi)(m)e^{-\lambda H_M(m)}.$$

We fix a uniformizer ϖ in F (and E) and $q^{-1}=|\varpi|_F$.

Proposition 1.2. *Fix a character χ of $E^\times$ such that $\chi(\varpi)=1$. Then for $t\gg 0$,*

$$\int_{\widetilde{H}(F)}\mathrm{tr}(I_P(\chi_\lambda,h)S)u(h,t)\,dh=\int^{*}_{\widetilde{H}(F)}\mathrm{tr}(I_P(\chi_\lambda,h)S)\,dh -\delta(\chi)(1+q^{-1})\left(\frac{q^{2\lambda(t+1)}}{1-q^{2\lambda}}\int_{\widetilde{K}_H\times\widetilde{K}_H} c(1,\lambda)\Psi_S(1)_{k_1,k_2}\,dk_1\,dk_2 +\frac{q^{-2\lambda(t+1)}}{1-q^{-2\lambda}}\int_{\widetilde{K}_H\times\widetilde{K}_H} c(w,\lambda)\Psi_S(1)_{k_1,k_2}\,dk_1\,dk_2\right),$$

where $\delta(\chi)=1$ if $\chi|_{\mathbb{O}_F^\times}=1$ and $\delta(\chi)=0$ if $\chi|_{\mathbb{O}_F^\times}\neq 1$.

Denote the action of the nontrivial element in $\mathrm{Gal}(E/F)$ on $x\in E$ by $\bar{x}$. Denote by $N_{E/F}$ the norm map from $E^\times$ to $F^\times$. Let $E^1=\{x\in E^\times : N_{E/F}(x)=1\}$. Let η denote an element in $G(F)$ such that $\overline{\eta}^{-1}\eta=\left(\begin{smallmatrix} 0 & 1\\ 1 & 0\end{smallmatrix}\right)$.

We define

$$D'_{\chi'_\lambda}(f') = \sum_{S' \in \mathcal{B}_P(\chi')} W'_{\chi'_\lambda}(S'_\lambda[f'])\overline{W'_{\chi'_\lambda}(S')} \quad \text{and} \quad D_{\chi_\lambda}(f) = \sum_{S \in \mathcal{B}_P(\chi)} P_{\chi_\lambda}(S_\lambda[f])\overline{W_{\chi_\lambda}(S)},$$

where

$$\begin{aligned}
W'_{\chi'_\lambda}(S') &= \lim_{t\to\infty} \int_{N'(F)} \operatorname{tr}(I_{P'}(\chi'_\lambda, n)S')\psi'(n^{-1})u(n,t)\,dn,\\
W_{\chi_\lambda}(S) &= \lim_{t\to\infty} \int_{N(F)} \operatorname{tr}(I_P(\chi_\lambda, n)S)\psi(n^{-1})u(n,t)\,dn,\\
P_{\chi_\lambda}(S) &= \int^*_{\widetilde{H}(F)} \operatorname{tr}(I_P(\chi_\lambda, h)S)\,dh,\\
S_\lambda[f] &= I_P(\chi_\lambda, f_2)SI_P(\chi_\lambda, f_1^\vee).
\end{aligned}$$

We let $\Pi_2(\widetilde{G}'(F))$ be a set of equivalence classes of irreducible, tempered square integrable representations of $\widetilde{G}'(F)$. We identify unitary characters on $\widetilde{M}'(F)$ with characters on $E^\times$ that are trivial on E^1. We let $\{\Pi_2(\widetilde{M}'(F))\}$ be a set of representatives of unitary characters χ' on $\widetilde{M}'(F)$ such that $\chi'(\varpi) = 1$. We let $\mu(\chi'_\lambda)$ be Harish-Chandra's μ-function. We take the analogous definitions for $\widetilde{G}(F)$.

Theorem 1.3 (local Kuznetsov trace formula). *For any*

$$f' = f'_1 \otimes f'_2 \in C_c^\infty(\widetilde{G}'(F) \times \widetilde{G}'(F)),$$

we have

$$\begin{aligned}
\lim_{t\to\infty} &\int_{N'(F)}\int_{N'(F)} K_{f'}(n_1, n_2)\psi'(n_1^{-1}n_2)u(n_1,t)u(n_2,t)\,dn_1\,dn_2\\
&= \int_{a\in E^\times/E^1} O'\big(f_1, \psi', a\big)O'\big(f'_2, \bar{\psi}', a\big)\,|a|_E\,d^\times a\\
&= \sum_{\sigma'\in\Pi_2(\widetilde{G}'(F))} d(\sigma')D'_{\sigma'}(f') + \tfrac{1}{2}\sum_{\chi'\in\{\Pi_2(\widetilde{M}'(F))\}} d(\chi')\int_0^{\pi i/\log q} \mu(\chi'_\lambda)D'_{\chi'_\lambda}(f')\,d\lambda,
\end{aligned}$$

where

$$O'(f'_i, \psi', a) = \int_{N'(F)}\int_{N'(F)} f'_i\big(n_1^{-1}\big(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\big)\big(\begin{smallmatrix}a & 0\\ 0 & \bar{a}^{-1}\end{smallmatrix}\big)n_2\big)\overline{\psi'(n_1^{-1}n_2)}\,dn_1\,dn_2.$$

Theorem 1.4 (local relative trace formula). *For any*

$$f = f_1 \otimes f_2 \in C_c^\infty(\widetilde{G}(F) \times \widetilde{G}(F)),$$

we have

$$\begin{aligned}\lim_{t\to\infty}\int_{\widetilde{H}(F)}\int_{N(F)} & K_f(h,n)\psi(n)u(h,t)u(n,t)\,dn\,dh\\ &= \int_{a\in E^\times/E^1} O\big(f_1,\psi,a\big)O\big(f_2,\bar{\psi},a\big)|a|_E\,d^\times a\\ &= \sum_{\sigma\in\Pi_2(\widetilde{G}(F))} d(\sigma)D_\sigma(f)+\tfrac{1}{2}\sum_{\substack{\chi\in\{\Pi_2(\widetilde{M}(F))\}\\ \chi^2\neq 1,\chi|_{F^\times}=1}}\widetilde{D}_\chi(f)\\ &\qquad +\tfrac{1}{2}\sum_{\substack{\chi\in\{\Pi_2(\widetilde{M}(F))\}\\ \chi|_{E^1}=1}} d(\chi)\int_0^{\pi i/\log q}\mu(\chi_\lambda)D_{\chi_\lambda}(f)\,d\lambda\end{aligned}$$

where

$$O(f_i,\psi,a)=\int_{\widetilde{H}(F)}\int_{N(F)} f_i\left(h^{-1}\eta\begin{pmatrix}a&0\\0&1\end{pmatrix}n\right)\overline{\psi(n)}\,dn\,dh.$$

The representations that occur on the right-hand side of Theorem 1.4 are exactly the representations that are in the image of the unstable base change lift on $\widetilde{G}'(F)$. The additional discrete term $\widetilde{D}_\chi(f)$ corresponds to the representations that lift from discrete series on $\widetilde{G}'(F)$ to principal series on $\widetilde{G}(F)$.

In addition to the spectral comparison, these local trace formulas also have applications on the geometric side. If we define the inner product of two functions g_1, g_2 on $E^\times/E^1$ by

$$\langle g_1,g_2\rangle=\int_{a\in E^\times/E^1} g_1(a)g_2(a)|a|_E\,d^\times a,$$

then:

Proposition 1.5 (orthogonality relations). *For f_1 and f_2 matrix coefficients of the supercuspidal representations σ_1 and σ_2 of $\widetilde{G}(F)$ and f_1' and f_2' matrix coefficients of the supercuspidal representations σ_1' and σ_2' of $\widetilde{G}'(F)$,*

$$\big\langle O'(f_1',\psi',\cdot),\,O'(f_2',\psi'^{-1},\cdot)\big\rangle\neq 0 \iff \sigma_1'\sim\sigma_2',$$
$$\big\langle O(f_1,\psi,\cdot),\,O(f_2,\psi^{-1},\cdot)\big\rangle\neq 0 \iff \sigma_1\sim\sigma_2.$$

The rest of this paper is organized as follows. In Section 2 we define notation and give normalizations of measures. In Section 3 we develop the local Kuznetsov trace formula. For the geometric expansion we rewrite our trace formula in terms of orbital integrals corresponding to the $N'\backslash G'/N'$ double cosets. The orbital integrals for f_1' and f_2' initially depend on the truncation and are intertwined. It is only through the multiplication of the two orbital integrals, integration over the space of double cosets, and the nontriviality of the character ψ', that we are able to untangle the orbital integral for f_1' from the orbital integral for f_2'. For the spectral expansion we apply Harish-Chandra's Plancherel formula to rewrite the local kernel in terms of representations. We are left with truncated integrals over the unipotent subgroup of matrix coefficients against the character ψ'. By the smoothness of the matrix coefficients and the appearance of the character, we show these distributions stabilize for t large.

In Section 4 we develop the local relative trace formula of $H\backslash G/N$. In the spectral expansion we have truncated integrals of matrix coefficients over H that do not converge without the truncation. We define the regularized period integral $P_{\chi_\lambda}(S)$. We use the asymptotics of matrix coefficients of tempered representations to prove the truncated integral is a polynomial exponential function in the truncation parameter t. We define the regularized integral as the constant term of this polynomial, and prove that this is an $H \times H$ invariant linear functional and the relevant term in the local relative trace formula.

In Section 5 we compare our two local trace formulas. There is a bijection between the "admissible" $N'\backslash G'/N'$ cosets and the "admissible" $H\backslash G/N$ cosets and both of these sets can be parametrized by $E^\times/E^1$. This bijection allows us to compare the geometric sides. By work of Ye and Flicker, we know that for any f' there is an f such that the orbital integrals are equal for corresponding cosets. Thus, by their geometric expansions, our local trace formulas are equal for matching functions. This gives an equality of the spectral expansions and of local distributions.

This paper would not have come into being had it not been for my teacher and advisor, Jonathan Rogawski. These thoughts originated as my PhD thesis under his direction, and his ideas, support, and guidance were critical to its completion. I am fortunate and will be forever grateful to have had him as a mentor. He could explain complicated math in a clear and simple way that aimed at the heart of the problem. He served, and continues to serve, as the role model of the inquisitive, patient, and approachable mathematician.

2. Notation

Let F be a non-Archimedean local field of characteristic 0 and odd residual characteristic q. Let E be an unramified quadratic extension of F. Let $\mathbb{O}_F$ and $\mathbb{O}_E$ denote the rings of integers in F and E, respectively. Let ϖ denote a uniformizer in the maximal ideal of $\mathbb{O}_F$. Thus ϖ is also a uniformizer in E. Let $v(\cdot)$ denote the valuation on F, extended to E. Let $|\cdot|_F$ and $|\cdot|_E$ denote the normalized valuations on F E, respectively. Thus for $a \in F^\times$, $|a|_E = |a|_F^2$. Denote the action of the nontrivial element in $\mathrm{Gal}(E/F)$ on $x \in E$ by $\bar{x}$. Denote by $N_{E/F}$ the norm map from $E^\times$ to $F^\times$. Let $E^1 = \{a \in E^\times : N_{E/F}(a) = 1\}$.

Let $H = \mathrm{GL}(2)/F$ and let $G = \mathrm{Res}_{E/F} H$, the restriction of scalars of GL(2) from E to F. Thus $G(F) = \mathrm{GL}(2, E)$. Let

$$G' = \mathrm{U}(2, F) = \left\{ g \in G : {}^t\bar{g} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}.$$

We note that by defining the quasisplit unitary group in this way, $SL(2, F) \subset G'(F)$. Let N' and N be the upper triangular unipotent matrices of G' and G, respectively. Let M' and M be the diagonal subgroups of G' and G, respectively. That is,

$$M'(F) = \left\{ \begin{pmatrix} a & 0 \\ 0 & \bar{a}^{-1} \end{pmatrix} : a \in E^\times \right\} \quad \text{and} \quad M(F) = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a, b \in E^\times \right\}.$$

Occasionally by abuse of notation we let $n = \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)$ and $a = \left(\begin{smallmatrix} a & 0 \\ 0 & \bar{a}^{-1} \end{smallmatrix}\right)$. Let $P = NM$ and $P' = N'M'$. Let $K = G(\mathbb{O}_F)$ and $K' = G'(\mathbb{O}_F)$. Let Z and Z' denote the centers of G and G', respectively. For any subgroup X of G let $\widetilde{X} = Z \cap X \backslash X$ and $X_H = X \cap H$. By abuse of notation we identify a character χ of $\widetilde{M}(F)$ with a character χ of $E^\times$ by letting $\chi\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right) = \chi(a)\chi^{-1}(b)$.

Let ψ' be an additive character on F with conductor $\mathbb{O}_F$. Let ψ be the additive character on E defined by $\psi(x) = \psi'(x + \bar{x})$. By abuse of notation we will also denote by ψ and ψ' the corresponding characters on $N(F)$ and $N'(F)$, respectively. Let $f = f_1 \otimes f_2 \in C_c^\infty(\widetilde{G}(F) \times \widetilde{G}(F))$ and $f' = f_1' \otimes f_2' \in C_c^\infty(\widetilde{G}'(F) \times \widetilde{G}'(F))$. For a function f on G, let $f^\vee(g) = f(g^{-1})$.

To define the local Kuznetsov trace formula and local relative trace formula we first multiply our function by the characteristic function of a large compact subset of $\widetilde{G}(F)$ via Arthur's local truncation [1991, §3], and then take the limit of the integral of the truncated function. For $g \in G(F)$, $t \in \mathbb{Z}^+$, let

$$u(g, t) = \begin{cases} 1 & \text{if } g = zk_1\left(\begin{smallmatrix} 1 & 0 \\ 0 & \alpha \end{smallmatrix}\right)k_2, \text{ for some } k_1, k_2 \in K, z \in Z(F), 0 \le v(\alpha) \le t, \\ 0 & \text{otherwise.} \end{cases}$$

We note that $u(\,\cdot\,, t)$ is well-defined on $\widetilde{G}(F)$ and

$$u\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, t \right) = \begin{cases} 1 & \text{if } x \in \varpi^{[-t/2]}\mathbb{O}_E, \\ 0 & \text{otherwise,} \end{cases}$$

where $[x]$ is the integral part of x.

If X is a closed subgroup of $\widetilde{G}(F)$ with the subgroup topology, $\mathrm{supp}(u(\,\cdot\,, t)) \cap X$ is a compact set.

We normalize the Haar measure dx on F so that $\mathrm{vol}(\mathbb{O}_F) = 1$. We define the multiplicative measure $d^\times x$ on $F^\times$ as

$$d^\times x = \frac{1}{1 - q^{-1}} \frac{1}{|x|_F} \, dx.$$

Thus $\mathrm{vol}(\mathbb{O}_F^\times) = 1$. We let $N(F)$ and $M(F)$ have the measures induced by dx and $d^\times x$. We normalize the Haar measure dk on K so that $\mathrm{vol}(K) = 1$. We define the measure dg on $G(F)$ by

$$\int_{G(F)} f(g)\,dg = \int_{M(F)} \int_{N(F)} \int_K f(mnk)\,dk\,dn\,dm.$$

We define dg' on $G'(F)$ similarly. We normalize Haar measure on $\widetilde{K}$ by taking $\mathrm{vol}(\widetilde{K}) = 1$.

We let $d^\times a$ be the unique Haar measure on $E^\times/E^1$ such that

$$\mathrm{vol}(\mathbb{O}_E^\times/E^1) = \frac{1}{1+q^{-1}}.$$

3. The local Kuznetsov trace formula for U(2)

In this section we develop a local Kuznetsov trace formula for the quasisplit unitary group in two variables. We expand this local Kuznetsov trace formula geometrically in terms of separate orbital integrals for f_1' and f_2'. Then we use Harish-Chandra's Plancherel formula to rewrite this expression spectrally in terms of representations.

We define the local Kuznetsov trace formula for

$$f' = f_1' \otimes f_2' \in C_c^\infty(\widetilde{G}'(F) \times \widetilde{G}'(F))$$

as the equality between the geometric and spectral expansions of

$$\lim_{t\to\infty} \int_{(N'\times N')(F)} K_{f'}(n_1, n_2)\psi(n_1^{-1}n_2)u(n_1, t)u(n_2, t)\,dn_1\,dn_2$$

where

$$K_{f'}(n_1, n_2) = \int_{\widetilde{G}'(F)} f_1'(g) f_2'(n_1^{-1} g n_2)\,dg.$$

We will show that for a fixed f' this limit stabilizes, that is, there exists a T such that for all $t' \geq T$,

$$\int_{(N'\times N')(F)} K_{f'}(n_1, n_2)\psi'(n_1^{-1}n_2)u(n_1, t')u(n_2, t')\,dn_1\,dn_2$$
$$= \lim_{t\to\infty} \int_{(N'\times N')(F)} K_{f'}(n_1, n_2)\psi(n_1^{-1}n_2)u(n_1, t)u(n_2, t)\,dn_1\,dn_2.$$

3A. ***The geometric expansion.*** In this subsection we rewrite

$$\lim_{t\to\infty} \int_{N'(F)} \int_{N'(F)} K_{f'}(n_1, n_2)\psi'(n_1^{-1}n_2)u(n_1, t)u(n_2, t)\,dn_1\,dn_2$$

as an integral over admissible cosets of a product of an orbital integral for f_1' and an orbital integral for f_2'.

3A1. *Integration formula.* Let $w = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. For $a \in E^\times$, let $\beta_a = \left(\begin{smallmatrix} a & 0 \\ 0 & \bar{a}^{-1} \end{smallmatrix}\right)$ and $\gamma_a = w\left(\begin{smallmatrix} a & 0 \\ 0 & \bar{a}^{-1} \end{smallmatrix}\right)$. By the Bruhat decomposition, $G' = P' \sqcup P'wP'$. Thus

$$\left\{\beta_a : \begin{array}{l} a \text{ is in a set of} \\ \text{representatives for } E^\times/E^1 \end{array}\right\} \bigcup \left\{\gamma_a : \begin{array}{l} a \text{ is in a set of} \\ \text{representatives for } E^\times/E^1 \end{array}\right\}$$

is a set of representatives for the double cosets of $N'(F)\backslash\widetilde{G}'(F)/N'(F)$.

For $g \in G'(F)$ let

$$C_g(N'(F) \times N'(F)) \\ = \{(n_1, n_2) \in N'(F) \times N'(F) : n_1^{-1} g n_2 = zg \text{ for some } z \in Z'(F)\}.$$

Definition 3.1. An element $g \in \widetilde{G}'(F)$ and its corresponding orbit are called *admissible* if the map

$$C_g(N'(F) \times N'(F)) \to \mathbb{C} : (n_1, n_2) \mapsto \psi'(n_1^{-1} n_2)$$

is trivial.

By a simple calculation we see that

$$C_{\beta_a}(N'(F) \times N'(F)) = \left\{\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & \frac{x}{a\bar{a}} \\ 0 & 1 \end{pmatrix}\right) : x \in F\right\},$$
$$C_{\gamma_a}(N'(F) \times N'(F)) = 1.$$

Thus the orbits represented by $\{\beta_1\} \cup \{\gamma_a : a \in E^\times/E^1\}$ are admissible.

We use the following integration formula to rewrite $K_{f'}(n_1, n_2)$ as an integral over the admissible cosets. Unlike in the global case the trivial admissible coset, β_1, will not contribute to the trace formula.

For any $F \in C_c(\widetilde{G}'(F))$,

$$\int_{\widetilde{G}'(F)} F(g)\, dg = \int_{E^\times/E^1} \int_{(N' \times N')(F)} F(n_1^{-1} \gamma_a n_2)\, dn_1\, dn_2 |a|_E\, d^\times a. \tag{3-1}$$

3A2. *Separating the orbital integrals.* Let

$$K^t(f') = \int_{N'(F)} \int_{N'(F)} K_{f'}(n_1, n_2) \psi(n_1^{-1} n_2) u(n_1, t) u(n_2, t)\, dn_1\, dn_2.$$

Clearly $K^t(f')$ is absolutely convergent because f_1' and $u(\cdot, t)$ have compact support on $\widetilde{G}'(F)$ and $N'(F)$ respectively. By changing the order of integration and using (3-1), we see that $K^t(f')$ equals

$$\int_{E^\times/E^1} \int_{(N' \times N')(F)} \int_{(N' \times N')(F)} f_1'(\hat{n}_1^{-1} \gamma_a \hat{n}_2) f_2'(n_1^{-1} \hat{n}_1^{-1} \gamma_a \hat{n}_2 n_2) \\ \times \psi'(n_1^{-1} n_2) u(n_1, t) u(n_2, t)\, dn_1\, dn_2\, d\hat{n}_1\, d\hat{n}_2 |a|_E\, d^\times a.$$

This integral is absolutely convergent because the map

$$N'(F) \times E^\times/E^1 \times N'(F) \to \widetilde{G}'(F)$$

defined by

$$(n_1, a, n_2) \mapsto n_1^{-1}\gamma_a n_2$$

is injective and f_1' has compact support. By a change of variables we have

$$K^t(f') = \int_{E^\times/E^1} K^t(\gamma_a, f')|a|_E \, d^\times a,$$

where

$$K^t(\gamma_a, f') = \int_{(N'\times N')(F)} \int_{(N'\times N')(F)} f_1'(\hat{n}_1^{-1}\gamma_a\hat{n}_2) f_2'(n_1^{-1}\gamma_a n_2)\psi'(n_1^{-1}\hat{n}_1\hat{n}_2^{-1}n_2) \\ \times u(\hat{n}_1^{-1}n_1, t)u(\hat{n}_2^{-1}n_2, t)\, dn_1\, dn_2\, d\hat{n}_1\, d\hat{n}_2.$$

To complete the geometric expansion of the local Kuznetsov trace formula we rewrite $K^t(f')$ for $t \gg 0$ as an integral of two separate orbital integrals. We begin by examining the dependence of the integrand on the truncation.

Lemma 3.2. *Let $f_1', f_2' \in C_c(\widetilde{G}'(F))$. For each $t_0 > 0$ there exists a $T > 0$ such that for all $t \geq T$,*

$$f_1'(x_1^{-1}\gamma y_1) f_2'(x_2^{-1}\gamma y_2)u(x_1^{-1}x_2, t)u(y_1^{-1}y_2, t_0) \\ = f_1'(x_1^{-1}\gamma y_1) f_2'(x_2^{-1}\gamma y_2)u(y_1^{-1}y_2, t_0)$$

for all $x_1, x_2, y_1, y_2, \gamma \in \widetilde{G}'(F)$.

Proof. Let

$$\Omega_1 = \operatorname{supp}(f_1'), \quad \Omega_2 = \operatorname{supp}(f_2'), \quad \Omega_3 = \operatorname{supp}(u(\,\cdot\,, t_0)) \cap \widetilde{G}'(F).$$

These sets are all compact on $\widetilde{G}'(F)$. If $f_1'(x_1^{-1}\gamma y_1) f_2'(x_2^{-1}\gamma y_2)u(y_1^{-1}y_2, t_0) \neq 0$, then the following conditions must hold:

- $x_1^{-1} \in \Omega_1 y_1^{-1}\gamma^{-1}$.
- $x_2 \in \gamma y_2 \Omega_2^{-1}$.
- $y_1^{-1}y_2 \in \Omega_3$.

Thus if $f_1'(x_1^{-1}\gamma y_1) f_2'(x_2^{-1}\gamma y_2)u(y_1^{-1}y_2, t_0) \neq 0$, then $x_1^{-1}x_2 \in \Omega_1\Omega_3\Omega_2^{-1}$. Because this is a compact set, there exists a $T > 0$ such that $\Omega_1\Omega_3\Omega_2^{-1} \subseteq \operatorname{supp}(u(g, T))$. The lemma now follows. □

Now we use this lemma, along with the character ψ', to separate the two orbital integrals. By abuse of notation, in the proof of the following lemma we let

$$\varpi^n = \begin{pmatrix} \varpi^n & 0 \\ 0 & \varpi^{-n} \end{pmatrix} \quad \text{and} \quad a = \begin{pmatrix} a & 0 \\ 0 & \bar{a}^{-1} \end{pmatrix}.$$

Lemma 3.3. *For $f' = f'_1 \otimes f'_2 \in C_c^\infty(\widetilde{G}'(F) \times \widetilde{G}'(F))$, there exists a T such that for all $t \geq T$ and $n \in \mathbb{Z}$,*

$$\int_{a \in \varpi^n \mathbb{O}_E^\times / E^1} K^t(\gamma_a, f')\, d^\times a = \int_{a \in \varpi^n \mathbb{O}_E^\times / E^1} O'(f'_1, \psi', a)\, O'(f'_2, \bar{\psi}', a)\, d^\times a, \tag{3-2}$$

where

$$O'(f', \psi', a) = \int_{N'(F)} \int_{N'(F)} f'(n_1^{-1} \gamma_a n_2) \overline{\psi'(n_1^{-1} n_2)}\, dn_1\, dn_2.$$

Proof. We show that there is a hidden truncation on the right-hand side of (3-2) that comes from the fact that the two orbital integrals are simultaneously evaluated at the same γ_a. Let K_1 be an open compact subgroup of $\widetilde{G}'(F)$ such that f'_1 and f'_2 are bi-K_1-invariant. There exists a positive constant c such that

$$\begin{pmatrix} a & 0 \\ 0 & \bar{a}^{-1} \end{pmatrix} \in K_1 \quad \text{for all } a \in (1 + \varpi^c \mathbb{O}_E) E^1.$$

By definition

$$\begin{aligned}
&\int_{a \in \varpi^n \mathbb{O}_E^\times / E^1} K^t(\gamma_a, f')\, d^\times a \\
&= \int_{a \in \mathbb{O}_E^\times / E^1} \int_{(N' \times N')(F)} f'_1(\hat{n}_1^{-1} w \varpi^n a \hat{n}_2) \psi'(\hat{n}_1 \hat{n}_2^{-1}) \int_{(N' \times N')(F)} f'_2(n_1^{-1} w \varpi^n a n_2) \\
&\qquad\qquad \times \psi'(n_1^{-1} n_2) u(\hat{n}_1^{-1} n_1, t) u(\hat{n}_2^{-1} n_2, t)\, dn_2\, dn_1\, d\hat{n}_2\, d\hat{n}_1\, d^\times a \\
&= \sum_{\eta \in \mathbb{O}_E^\times / (1 + \varpi^c \mathbb{O}_E) E^1} \int_{a \in (1 + \varpi^c \mathbb{O}_E) E^1 / E^1} \int_{(N' \times N')(F)} f'_1(\hat{n}_1^{-1} w \varpi^n \eta a \hat{n}_2) \psi'(\hat{n}_1 \hat{n}_2^{-1}) \\
&\qquad \times \int_{(N' \times N')(F)} f'_2(n_1^{-1} w \varpi^n \eta a n_2) \psi'(n_1^{-1} n_2) u(\hat{n}_1^{-1} n_1, t) \\
&\qquad\qquad \times u(\hat{n}_2^{-1} n_2, t)\, dn_2\, dn_1\, d\hat{n}_2\, d\hat{n}_1\, d^\times a.
\end{aligned}$$

By a change of variables and the fact that f' is locally constant the right-hand side of this equation is equal to

$$\begin{aligned}
&\sum_{\eta \in \mathbb{O}_E^\times / (1 + \varpi^c \mathbb{O}_E) E^1} \int_{(N' \times N')(F)} f'_1(\hat{n}_1^{-1} w \varpi^n \eta \hat{n}_2) \psi'(\hat{n}_1) \\
&\qquad \times \int_{(N' \times N')(F)} f'_2(n_1^{-1} w \varpi^n \eta n_2) \psi'(n_1^{-1}) u(\hat{n}_1^{-1} n_1, t)\, dn_1\, d\hat{n}_1 \\
&\qquad\qquad \times \int_{a \in (1 + \varpi^c \mathbb{O}_E) E^1 / E^1} \psi'(a^{-1} \hat{n}_2^{-1} n_2 a) u(a^{-1} \hat{n}_2^{-1} n_2 a, t)\, d^\times a\, dn_2\, d\hat{n}_2.
\end{aligned}$$

We can rewrite the inner integral as

$$
\begin{aligned}
u(\hat n_2^{-1}n_2,t)&\int_{a\in(1+\varpi^c\mathbb{O}_E)E^1/E^1}\psi'((n_2-\hat n_2)(a\bar a)^{-1})\,d^\times a\\
&=u(\hat n_2^{-1}n_2,t)\int_{b\in 1+\varpi^c\mathbb{O}_F}\psi'(b(n_2-\hat n_2))\,d^\times b\\
&=u(\hat n_2^{-1}n_2,t)\,\frac{1}{1-q^{-1}}\,\psi'(n_2-\hat n_2)\int_{b\in\varpi^c\mathbb{O}_F}\psi'(b(n_2-\hat n_2))\,db\\
&=u(\hat n_2^{-1}n_2,t)u(\hat n_2^{-1}n_2,2c)\,\frac{\mathrm{vol}(\varpi^c\mathbb{O}_F)}{1-q^{-1}}\,\psi'(\hat n_2^{-1}n_2).
\end{aligned}
$$

Thus for $t\ge 2c$,

$$
\begin{aligned}
\int_{a\in\varpi^n\mathbb{O}_E^\times/E^1}K^t(\gamma_a,f')\,d^\times a&=\int_{a\in\varpi^n\mathbb{O}_E^\times/E^1}\int_{(N'\times N')(F)}f_1'(\hat n_1^{-1}\gamma_a\hat n_2)\psi'(\hat n_1\hat n_2^{-1})\\
\times\int_{(N'\times N')(F)}&f_2'(n_1^{-1}\gamma_a n_2)\psi'(n_1^{-1}n_2)u(\hat n_1^{-1}n_1,t)u(\hat n_2^{-1}n_2,2c)\,dn_2\,dn_1\,d\hat n_2 d\hat n_1 d^\times a.
\end{aligned}
$$

By Lemma 3.2 there exists a $T>0$ such that for all $t\ge\max\{T,2c\}$,

$$
\begin{aligned}
\int_{a\in\varpi^n\mathbb{O}_E^\times/E^1}&K^t(\gamma_a,f')\,d^\times a\\
&=\int_{a\in\varpi^n\mathbb{O}_E^\times/E^1}\int_{(N'\times N')(F)}f_1'(\hat n_1^{-1}\gamma_a\hat n_2)\psi'(\hat n_1\hat n_2^{-1})\,d\hat n_1\\
&\qquad\times\int_{(N'\times N')(F)}f_2'(n_1^{-1}\gamma_a n_2)\psi'(n_1^{-1}n_2)u(\hat n_2^{-1}n_2,2c)\,dn_2\,dn_1\,d\hat n_2\,d^\times a\\
&=\int_{a\in\varpi^n\mathbb{O}_E^\times/E^1}\int_{(N'\times N')(F)}f_1'(\hat n_1^{-1}\gamma_a\hat n_2)\psi'(\hat n_1\hat n_2^{-1})\,d\hat n_2\,d\hat n_1\\
&\qquad\qquad\times\int_{(N'\times N')(F)}f_2'(n_1^{-1}\gamma_a n_2)\psi'(n_1^{-1}n_2)\,dn_2\,dn_1\,d^\times a. \qquad\square
\end{aligned}
$$

We have shown that the truncated local Kuznetsov trace formula stabilizes.

Proposition 3.4. *For any $f'=f_1'\otimes f_2'\in C_c^\infty(\widetilde G'(F)\times\widetilde G'(F))$ and $t\gg 0$,*

$$
\begin{aligned}
\int_{N'(F)}\int_{N'(F)}&K_{f'}(n_1,n_2)\psi'(n_1^{-1}n_2)u(n_1,t)u(n_2,t)\,dn_1\,dn_2\\
&=\int_{a\in E^\times/E^1}O'(f_1',\psi',a)O'(f_2',\bar\psi',a)|a|_E\,d^\times a.
\end{aligned}
$$

3B. ***The spectral expansion.*** Now we derive a spectral expansion for the local Kuznetsov trace formula,

$$\lim_{t\to\infty}\int_{N'(F)}\int_{N'(F)} K_{f'}(n_1,n_2)\psi'(n_1^{-1}n_2)u(n_1,t)u(n_2,t)\,dn_1\,dn_2.$$

Our main tool is the Plancherel formula for p-adic groups, which was first stated, with an outlined proof, by Harish-Chandra [1984]. Silberger [1996] later filled in an important proof of one of the steps in the theorem. More recently Waldspurger [2003] provided a complete proof.

As in [Arthur 1991, §2], we begin by rewriting $K_{f'}(x,y)$ using the Plancherel formula. First we introduce some additional notation. For an irreducible representation (σ, V_σ) of $G'(F)$ let $\mathcal{B}(\sigma)$ be the Hilbert space of Hilbert–Schmidt operators on V_σ. The inner product on $\mathcal{B}(\sigma)$ is defined as

$$\langle S, S'\rangle := \operatorname{tr}(SS'^*)$$

for $S, S' \in \mathcal{B}(\sigma)$, where $\operatorname{tr}(SS'^*) = \sum_{\text{o.n.b.}V_\sigma}\langle SS'^* u_i, u_i\rangle$ and this sum converges absolutely and does not depend on the basis. For a discrete series representation σ of a group G let $d(\sigma)$ be the formal degree of σ.

Let $\Pi_2(\widetilde{G}'(F))$ be a set of representatives for the equivalence classes of irreducible, tempered square integrable representations of $\widetilde{G}'(F)$ and let $\{\Pi_2(\widetilde{M}'(F))\}$ be a set of representatives of unitary characters χ on $\widetilde{M}'(F)$ such that $\chi(\varpi)=1$. For a character χ of $M'(F)$ and $\lambda\in\mathbb{C}$, let $\chi_\lambda(m)=\chi(m)e^{\lambda(H_{P'}(m))}$. For $\chi\in\{\Pi_2(\widetilde{M}'(F))\}$, $I_{P'}^{G'}(\chi_\lambda) = I_{P'}(\chi_\lambda)$ is the normalized induced representation of $\widetilde{G}'(F)$ acting on a Hilbert space $\mathcal{H}_{P'}(\chi)$ of vector-valued functions on K'. Let $\mathcal{B}_{P'}(\chi)$ be a fixed K'-finite orthonormal basis of the Hilbert space of Hilbert–Schmidt operators on $\mathcal{H}_{P'}(\chi)$.

Let $m(\sigma)$ be the Plancherel density. We normalize our measures following [Arthur 1991, §1]. The Plancherel density satisfies $m(\chi_\lambda) = d(\chi)\mu(\chi_\lambda)$, where $\mu(\chi_\lambda)$ is Harish-Chandra's μ-function.

For a fixed $x \in G'(F)$, let

$$h(v) = \int_{\widetilde{G}'(F)} f_1'(xu)f_2'(uvx)\,du.$$

Then $h\in C_c^\infty(\widetilde{G}'(F))$ and $K_{f'}(x,y) = h(yx^{-1})$, so by the Plancherel formula,

$$K_{f'}(x,y) = \sum_{\sigma\in\Pi_2(\widetilde{G}'(F))} d(\sigma)\operatorname{tr}(\sigma(R(yx^{-1})h)) + \tfrac{1}{2}\sum_{\chi\in\{\Pi_2(\widetilde{M}'(F))\}}\int_0^{\frac{\pi i}{\log q}}\operatorname{tr}(I_{P'}(\chi, R(yx^{-1})h))m(\chi_\lambda)\,d\lambda.$$

Because $I_{P'}(\chi_\lambda, R(yx^{-1})h) = I_{P'}(\chi_\lambda, f_1'^{\vee})I_{P'}(\chi_\lambda, x)I_{P'}(\chi_\lambda, f_2')(I_{P'}(\chi_\lambda, y))^*$, we have

$$\begin{aligned}
&\operatorname{tr}(I_{P'}(\chi_\lambda, R(yx^{-1})h)) \\
&\qquad = \sum_{S\in\mathscr{B}_{P'}(\chi)} (I_{P'}(\chi_\lambda, f_1'^{\vee})I_{P'}(\chi_\lambda, x)I_{P'}(\chi_\lambda, f_2'), S^*)\overline{(I_{P'}(\chi_\lambda, y), S^*)} \\
&\qquad = \sum_{S\in\mathscr{B}_{P'}(\chi)} \operatorname{tr}(I_{P'}(\chi_\lambda, f_1'^{\vee})I_{P'}(\chi_\lambda, x)I_{P'}(\chi_\lambda, f_2')S)\overline{\operatorname{tr}(I_{P'}(\chi_\lambda, y)S)} \\
&\qquad = \sum_{S\in\mathscr{B}_{P'}(\chi)} \operatorname{tr}(I_{P'}(\chi_\lambda, x)S_\lambda[f'])\overline{\operatorname{tr}(I_{P'}(\chi_\lambda, y)S)},
\end{aligned}$$

where $S_\lambda[f'] = I_{P'}(\chi_\lambda, f_2')SI_{P'}(\chi_\lambda, f_1'^{\vee})$.

For $f' \in C_c^\infty(\widetilde{G}'(F))$, π an admissible representation, $\pi(f')$ has finite rank. Thus the sum over S is a finite sum of an orthonormal basis of operators on $\mathscr{H}_P(\chi)^{K_0}$ for some open compact K_0.

Putting everything together we have

$$\begin{aligned}
&\int_{N'(F)}\int_{N'(F)} K_{f_1'\otimes f_2'}(n_1, n_2)\psi'(n_1^{-1}n_2)u(n_1, t)u(n_2, t)\,dn_1\,dn_2 \\
&= \sum_{\sigma\in\Pi_2(\widetilde{G}'(F))} d(\sigma) \sum_{S\in\mathscr{B}(\sigma)} \biggl(\int_{N'(F)} \operatorname{tr}(\sigma(n)(\sigma(f_2')S\sigma(f_1'^{\vee})))\psi'(n^{-1})u(n, t)\,dn \\
&\qquad\qquad \times\overline{\int_{N'(F)} \operatorname{tr}(\sigma(n)S)\psi'(n^{-1})u(n, t)\,dn}\biggr) \\
&+\tfrac{1}{2} \sum_{\chi\in\{\Pi_2(\widetilde{M}'(F))\}} d(\chi)\times\int_0^{\frac{\pi i}{\log q}} \biggl(\sum_{S\in\mathscr{B}_{P'}(\chi)} \int_{N'(F)} \operatorname{tr}(I_{P'}(\chi_\lambda, n)S_\lambda[f'])\psi'(n^{-1})u(n, t)\,dn \\
&\qquad\qquad \times\overline{\int_{N'(F)} \operatorname{tr}(I_{P'}(\chi_\lambda, n)S)\psi'(n^{-1})u(n, t)\,dn}\biggr)\mu(\chi_\lambda)\,d\lambda.
\end{aligned}$$

To finish the spectral expansion we show that the above unipotent integrals stabilize. We first note that in the discrete series case the above integrals are absolutely convergent without any truncation for reasons similar to those in Section 4B1.

Lemma 3.5 (spectral stabilization). *For any complex-valued function ϕ on $\widetilde{G}'(F)$ that is biinvariant under a fixed open compact subgroup, there exists a positive integer c such that for all $t \geq c$,*

$$\int_{N'(F)} \phi(n)\psi'(n)u(n, t)\,dn = \int_{N'(F)} \phi(n)\psi'(n)u(n, c)\,dn.$$

This c only depends on the open compact subgroup under which ϕ is biinvariant.

Proof. Let K_1 be an open compact subgroup of $\widetilde{G}'(F)$ under which ϕ is biinvariant. K_1 must contain a neighborhood of the identity, so there exists a positive integer c' such that

$$\begin{pmatrix} a & 0 \\ 0 & \bar{a}^{-1} \end{pmatrix} \in K_1 \quad \text{for all } a \in (1+\varpi^{c'}\mathbb{O}_E)E^1.$$

We show that for $m > c'$,

$$\int_{\varpi^{-m}\mathbb{O}_F^\times} \phi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)\psi'(x)\,dx = 0.$$

We note that

$$\begin{pmatrix} a & 0 \\ 0 & \bar{a}^{-1} \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a^{-1} & 0 \\ 0 & \bar{a} \end{pmatrix} = \begin{pmatrix} 1 & a\bar{a}x \\ 0 & 1 \end{pmatrix}.$$

Thus for $x' \in 1+\varpi^{c'}\mathbb{O}_F$,

$$\phi\left(\begin{pmatrix} 1 & x'x \\ 0 & 1 \end{pmatrix}\right) = \phi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right).$$

Hence

$$\begin{aligned}
&\int_{\varpi^{-m}\mathbb{O}_F^\times} \phi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)\psi'(x)\,dx \\
&\qquad = \sum_{\alpha \in \mathbb{O}_F^\times/(1+\varpi^{c'}\mathbb{O}_F)} \int_{\varpi^{-m}(1+\varpi^{c'}\mathbb{O}_F)} \phi\left(\begin{pmatrix} 1 & \alpha x \\ 0 & 1 \end{pmatrix}\right)\psi'(\alpha x)\,dx \\
&\qquad = \sum_{\alpha \in \mathbb{O}_F^\times/(1+\varpi^{c'}\mathbb{O}_F)} \phi\left(\begin{pmatrix} 1 & \varpi^{-m}\alpha \\ 0 & 1 \end{pmatrix}\right)\psi'(\varpi^{-m}\alpha)\int_{\varpi^{c'-m}\mathbb{O}_F}\psi'(x)\,dx.
\end{aligned}$$

The last line equals 0 for $m > c'$. Thus for $t > 2c'$,

$$\int_{N'(F)} \phi(n)\psi'(n)u(n,t)\,dn = \int_{N'(F)} \phi(n)\psi'(n)u(n,2c')\,dn. \qquad \square$$

We have now proved the following.

Proposition 3.6. *For any $f' = f_1' \otimes f_2' \in C_c^\infty(\widetilde{G}'(F) \times \widetilde{G}'(F))$,*

$$\begin{aligned}
&\lim_{t\to\infty} \int_{N'(F)}\int_{N'(F)} K_{f'}(n_1,n_2)\psi'(n_1^{-1}n_2)u(n_1,t)u(n_2,t)\,dn_1\,dn_2 \\
&\qquad = \sum_{\sigma' \in \Pi_2(\widetilde{G}'(F))} d(\sigma')D'_{\sigma'}(f') + \tfrac{1}{2}\sum_{\chi' \in \{\Pi_2(\widetilde{M}'(F))\}} d(\chi')\int_0^{\frac{\pi i}{\log q}} D'_{\chi'_\lambda}(f')\mu(\chi'_\lambda)\,d\lambda,
\end{aligned}$$

where

$$D'_{\sigma'}(f') = \sum_{S\in\mathcal{B}(\sigma')} W'_{\sigma'}(\sigma'(f'_2)S\sigma'(f_1'^{\vee}))\overline{W'_{\sigma'}(S)},$$

$$W'_{\sigma'}(S) = \int_{N'(F)} \operatorname{tr}(\sigma'(n)S)\psi'(n^{-1})\,dn,$$

$$D'_{\chi'_\lambda}(f') = \sum_{S\in\mathcal{B}_{P'}(\chi')} W'_{\chi'_\lambda}(I_{P'}(\chi'_\lambda, f'_2)SI_{P'}(\chi'_\lambda, f_1'^{\vee}))\overline{W'_{\chi'_\lambda}(S)},$$

$$W'_{\chi'_\lambda}(S) = \lim_{t\to\infty}\int_{N'(F)} \operatorname{tr}(I_{P'}(\chi'_\lambda, n)S)\psi'(n^{-1})u(n,t)\,dn.$$

We note that Theorem 1.3 now follows from the results of Propositions 3.4 and 3.6.

4. The local relative trace formula and periods for PGL(2)

In this section we define a local relative trace formula for PGL(2). We expand this local relative trace formula geometrically in terms of separate orbital integrals of f_1 and f_2. Then we use Harish-Chandra's Plancherel formula to rewrite this expression spectrally in terms of representations. We define a regularized period integral, show that it is an $H\times H$-invariant linear functional and that it is the term that appears in the spectral expansion of the local relative trace formula.

We define the local relative trace formula for $f = f_1\otimes f_2 \in C_c^\infty(\widetilde{G}(F)\times\widetilde{G}(F))$ as the equality between the geometric and spectral expansions of

$$\lim_{t\to\infty}\int_{\widetilde{H}(F)}\int_{N(F)} K_f(h,n)\psi(n)u(h,t)u(n,t)\,dn\,dh$$

where

$$K_f(h,n) = \int_{\widetilde{G}(F)} f_1(g)f_2(h^{-1}gn)\,dg.$$

As we did with the local Kuznetsov trace formula, we show that for a fixed f this limit stabilizes.

4A. ***The geometric expansion.*** We will rewrite

$$\lim_{t\to\infty}\int_{\widetilde{H}(F)}\int_{N(F)} K_f(h,n)\psi(n)u(h,t)u(n,t)\,dn\,dh$$

as an integral over admissible cosets of a product of an orbital integral for f_1 and an orbital integral f_2.

4A1. *Integration formula.* As pointed out in [Jacquet et al. 1999, §VI.13], by [Springer 1985], $G(F) = H(F)P(F)\sqcup H(F)\eta P(F)$, where η is any element in $G(F)$ such that $\overline{\eta}^{-1}\eta = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Let $\eta_a = \eta\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $\gamma_\alpha = \left(\begin{smallmatrix} 1 & 0 \\ 0 & \alpha+\sqrt{\tau} \end{smallmatrix}\right)$, where $E = F(\sqrt{\tau})$. Then

$$\left\{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)\right\} \cup \{\gamma_\alpha : \alpha \in F\} \cup \{\eta_a : a \text{ is in a set of representatives for } E^\times/E^1\}$$

is a set of representatives for the double cosets of $\widetilde{H}(F)\backslash\widetilde{G}(F)/N(F)$.

For $g \in G(F)$, let

$$C_g(\widetilde{H}(F) \times N(F)) = \{(h, n) \in \widetilde{H}(F) \times N(F) : h^{-1}gn = zg \text{ for some } z \in Z(F)\}.$$

Definition 4.1. An element $g \in \widetilde{G}(F)$ and its corresponding orbit is called *admissible* if the map $C_g(\widetilde{H}(F) \times N(F)) \to \mathbb{C} : (h, n) \mapsto \psi(n)$ is trivial.

By a short calculation we see that

$$\begin{aligned} C_{\gamma_\alpha}(\widetilde{H}(F) \times N(F)) &= \left\{\left(\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & y(\alpha+\sqrt{\tau}) \\ 0 & 1 \end{pmatrix}\right) : y \in F\right\}, \\ C_{\eta_a}(\widetilde{H}(F) \times N(F)) &= 1. \end{aligned}$$

Thus the orbits represented by $\{\eta_a : a \in E^\times/E^1\} \cup \{\gamma_0\}$ are admissible.

We have the following integration formula. For any $F \in C_c(\widetilde{G}(F))$,

$$\int_{\widetilde{G}(F)} F(g)\,dg = \int_{E^\times/E^1} \int_{\widetilde{H}(F)\times N(F)} F(h^{-1}\eta_a n)\,dn\,dh|a|_E\,d^\times a. \tag{4-1}$$

4A2. *Separating the orbital integrals.* Let

$$R^t(f) = \int_{\widetilde{H}(F)} \int_{N(F)} K_f(h, n)\psi(n)u(h, t)u(n, t)\,dn\,dh.$$

$R^t(f)$ is absolutely convergent because $f_1(g)$, $u(h, t)$ and $u(n, t)$ have compact support on $\widetilde{G}(F)$, $\widetilde{H}(F)$ and $N(F)$ respectively. By changing the order of integration and applying (4-1) we see that $R^t(f)$ equals

$$\begin{aligned} \int_{E^\times/E^1} \int_{\widetilde{H}(F)\times N(F)} \int_{\widetilde{H}(F)\times N(F)} & f_1(h_1^{-1}\eta_a n_1) f_2(h_2^{-1}h_1^{-1}\eta_a n_1 n_2) \\ & \times \psi(n_2)u(h_2, t)u(n_2, t)\,dn_2\,dh_2\,dn_1\,dh_1|a|_E\,d^\times a. \end{aligned}$$

By a change of variables we have

$$R^t(f) = \int_{E^\times/E^1} R^t(\eta_a, f)|a|_E\,d^\times a,$$

where

$$\begin{aligned} R^t(\eta_a, f) = \int_{\widetilde{H}(F)\times N(F)} \int_{\widetilde{H}(F)\times N(F)} & f_1(h_1^{-1}\eta_a n_1) f_2(h_2^{-1}\eta_a n_2)\psi(n_1^{-1}n_2) \\ & \times u(h_1^{-1}h_2, t)u(n_1^{-1}n_2, t)\,dn_2\,dh_2\,dn_1\,dh_1. \end{aligned}$$

To complete the geometric expansion of the local relative trace formula we rewrite $R^t(f)$ for $t \gg 0$ as an integral of a product of two separate orbital integrals that are not truncated. We omit the proof the lemma below as it is very similar to the proof of Lemma 3.3.

Lemma 4.2. *For $f \in C_c^\infty(\widetilde{G}(F)\times\widetilde{G}(F))$, there exists a $T > 0$ such that for all $t \geq T$ and $n \in \mathbb{Z}$,*

$$\int_{\varpi^n \mathbb{O}_E^\times/E^1} R^t(\eta_a, f)\, d^\times a = \int_{\varpi^n \mathbb{O}_E^\times/E^1} O(f_1, \psi, a) O(f_2, \overline{\psi}, a)\, d^\times a$$

where

$$O(f, \psi, a) = \int_{\widetilde{H}(F)} \int_{N(F)} f(h^{-1}\eta_a n)\overline{\psi(n)}\, dn\, dh.$$

We have proved the following proposition.

Proposition 4.3. *For any $f = f_1 \otimes f_2 \in C_c^\infty(\widetilde{G}(F) \times \widetilde{G}(F))$,*

$$\lim_{t\to\infty} \int_{\widetilde{H}(F)} \int_{N(F)} K(h, n)\psi(n)u(h, t)u(n, t)\, dn\, dh$$
$$= \int_{a\in E^\times/E^1} O(f_1, \psi, a)\, O(f_2, \overline{\psi}, a)|a|_E\, d^\times a.$$

Here, as in the local Kuznetsov trace formula, we have actually shown that the limit of the truncated local relative trace formula stabilizes.

4B. *The spectral expansion and period integrals.* We want to develop a spectral expansion for the local relative trace formula,

$$\lim_{t\to\infty} \int_{\widetilde{H}(F)} \int_{N(F)} K_f(h, n)\psi(n)u(h, t)u(n, t)\, dn\, dh.$$

As in the previous section, we expand the kernel via the Plancherel formula:

$$\text{(4-2)} \quad \int_{\widetilde{H}(F)} \int_{N(F)} K(h, n)\psi(n)u(h, t)u(n, t)\, dn\, dh$$
$$= \sum_{\sigma\in\Pi_2(\widetilde{G}(F))} d(\sigma) D_\sigma^t(f) + \tfrac{1}{2} \sum_{\chi\in\{\Pi_2(\widetilde{M}(F))\}} d(\chi) \int_0^{\frac{\pi i}{\log q}} \mu(\chi_\lambda) D_{\chi_\lambda}^t(f)\, d\lambda$$

where

$$\begin{aligned}
D^t_\sigma(f) &= \sum_{S\in\mathcal{B}(\sigma)} P^t_\sigma(\sigma(f_2)S\sigma(f_1^\vee))\overline{W^t_\sigma(S)},\\
D^t_{\chi_\lambda}(f) &= \sum_{S\in\mathcal{B}_P(\chi)} P^t_{I_P(\chi_\lambda)}(I_P(\chi_\lambda, f_2)SI_P(\chi_\lambda, f_1^\vee))\overline{W^t_{I_P(\chi_\lambda)}(S)},\\
W^t_\pi(S) &= \int_{N(F)} \operatorname{tr}(\pi(n)S)\psi(n^{-1})u(n,t)\,dn,\\
P^t_\pi(S) &= \int_{\widetilde{H}(F)} \operatorname{tr}(\pi(h)S)u(h,t)\,dh.
\end{aligned}$$

By Lemma 3.5, there exists a positive integer c, such that for $t > c$,

$$W^t_\pi(S) = \int_{N(F)} \operatorname{tr}(\pi(n)S)\psi(n^{-1})u(n,c)\,dn.$$

Thus as in the previous section, we define

$$\begin{aligned}
W_\sigma(S) &= \int_{N(F)} \operatorname{tr}(\sigma(n)S)\psi(n^{-1})\,dn,\\
W_{\chi_\lambda}(S) &= \lim_{t\to\infty}\int_{N(F)} \operatorname{tr}(I_P(\chi_\lambda, n)S)\psi(n^{-1})u(n,t)\,dn.
\end{aligned}$$

To finish the spectral expansion of the local relative trace formula we need to define the regularized integral

$$\int^*_{\widetilde{H}(F)} \operatorname{tr}(I_P(\chi_\lambda, h)S)\,dh$$

because $\operatorname{tr}(I_P(\chi_\lambda, -)S)$ is not integrable over $\widetilde{H}(F)$.

Many of the techniques in this section are inspired by the work of Jacquet, Lapid and Rogawski in [Jacquet et al. 1999]. In that paper they define a regularized period integral for an automorphic form ϕ on $G(\mathbb{A})$ integrated over H where G is a reductive group over a number field F and H is the fixed point set of an involution of G. They focus on the case $G = \operatorname{Res}_{E/F} H$ where E/F is a quadratic extension and they obtain explicit results for $G = \mathrm{GL}(n, E)$, $H = \mathrm{GL}(n, F)$.

For $\lambda \in \mathbb{C}$ and $m = \left(\begin{smallmatrix}\alpha & 0\\ 0 & \beta\end{smallmatrix}\right) \in M(F)$, let $e^{\lambda H_M(m)} = |\alpha/\beta|^\lambda_E$. If $g = m(g)n(g)k(g)$, $m(g) \in M(F)$, $n(g) \in N(F)$, $k(g) \in K$, we let $e^{\lambda H_P(g)} = e^{\lambda H_M(m(g))}$. Let $\delta_P(m) = e^{H_M(m)}$. We give analogous definitions for $e^{\lambda H_{M_H}}$ and δ_{P_H} so that for $m \in M_H(F)$, $e^{\lambda H_M(m)} = e^{2\lambda H_{M_H}(m)}$.

We recall the Cartan decomposition $H(F) = K_H M^+_H(F) K_H$, where

$$M^+_H(F) = \left\{\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix} \in M(F) : v\Big(\frac{\alpha}{\beta}\Big) \le 0\right\}.$$

Then for any absolutely integrable function f

$$\int_{\widetilde{H}(F)} f(h)\,dh = \int_{\widetilde{K}_H}\int_{\widetilde{K}_H}\int_{\widetilde{M}_H^+(F)} D_{P_H}(m) f(k_1 m k_2)\,dm\,dk_2\,dk_1,$$

where

$$D_{P_H}\left(\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}\right) = \begin{cases} |\alpha/\beta|_F(1+|\varpi|_F) & v(\alpha/\beta)\le 0,\\ 0 & v(\alpha/\beta)>0.\end{cases}$$

To define the regularized integral, we begin by defining a regularized integral on $M_H^+(F)$. We note that

$$1-u\left(\begin{pmatrix}1 & \\ & \alpha\end{pmatrix}, t\right) = \begin{cases} 0 & 0\le v(\alpha)\le t,\\ 1 & v(\alpha)>t.\end{cases}$$

For $\operatorname{Re}\nu < -\operatorname{Re}\lambda$,

$$\int_{\widetilde{M}_H^+(F)} e^{(\nu+\lambda)H_M(m)}(1-u(m,t))\,dm = \sum_{n=t+1}^{\infty} q^{2n(\nu+\lambda)} = \frac{q^{(t+1)2(\nu+\lambda)}}{1-q^{2(\nu+\lambda)}}. \tag{4-3}$$

We write

$$\int^{\sharp}_{\widetilde{M}_H^+(F)} e^{\lambda H_M(m)}(1-u(m,t))\,dm$$

to denote the meromorphic continuation at $\nu=0$ of (4-3). This is well-defined so long as $\lambda\ne 0$. Let

$$\begin{aligned}\phi(k_1 m k_2) &= \sum_{i=1}^{r}\phi_i(k_1,k_2) f_i(m) e^{\lambda_i H_M(m)}, \quad k_1,k_2\in K_H,\\ m &= \begin{pmatrix}1 & \\ & \varpi^n\end{pmatrix}, \quad n\ge 0, \quad f_i\in C_c(\widetilde{M}(F))\end{aligned} \tag{4-4}$$

with $\lambda_i \ne -\frac{1}{2}$. We define for $t\gg 0$,

$$\begin{aligned}&\int^{\sharp}_{\widetilde{H}(F)} \phi(h)(1-u(h,t))\,dh\\ &\quad= \sum_{i=1}^{r}\int_{\widetilde{K}_H\times\widetilde{K}_H}\phi_i(k_1,k_2)\int^{\sharp}_{\widetilde{M}_H^+(F)} D_{P_H}(m) e^{\lambda_i H_M(m)}(1-u(m,t))\,dm\\ &\quad= (1+q^{-1})\sum_{i=1}^{r}\int_{\widetilde{K}_H\times\widetilde{K}_H}\phi_i(k_1,k_2)\int^{\sharp}_{\widetilde{M}_H^+(F)} e^{(\lambda_i+1/2)H_M(m)}(1-u(m,t))\,dm.\end{aligned}$$

If ϕ is a matrix coefficient of $I_P(\chi_\lambda)$ where $\chi(\varpi)=1$ then by smoothness and the asymptotics of matrix coefficients there exists a function $C^P\phi$ of the form in

(4-4) with $\lambda_i \in \{\lambda - \frac{1}{2}, -\lambda - \frac{1}{2}\}$ and for $n \gg 0$,

$$C^P\phi\left(k_1 \begin{pmatrix} 1 & \\ & \varpi^n \end{pmatrix} k_2\right) = \phi\left(k_1 \begin{pmatrix} 1 & \\ & \varpi^n \end{pmatrix} k_2\right).$$

Note that the condition for the regularized integral to exist is now that $\lambda \neq 0$.

Definition 4.4. For any matrix coefficient ϕ of $I_P(\chi_\lambda)$ such that $\chi(\varpi) = 1$ and $\lambda \neq 0$,

$$\int^*_{\widetilde{H}(F)} \phi(h)\,dh := \int_{\widetilde{H}(F)} \phi(h)u(h,t)\,dh + \int^\sharp_{\widetilde{H}(F)} \phi(h)(1-u(h,t))\,dh$$

for $t \gg 0$.

One can check that this definition of the regularized integral is independent of t and agrees with the usual integral if we start with something that is integrable. Now we will prove that it is H-invariant and then we will explicitly relate the regularized period to the truncated period that occurs in the local trace formula.

Let $\phi^{h_0}(x) = \phi(xh_0)$ for $h_0 \in \widetilde{H}$. Note that if ϕ is a matrix coefficient of π then ϕ^{h_0} is as well.

Lemma 4.5. *Fix $h_0 \in H$, $\lambda \neq 0$ and a character χ of $E^\times$ with $\chi(\varpi) = 1$. Then for any matrix coefficient ϕ of $I_P(\chi_\lambda)$ and $t \gg 0$,*

$$\int_{\widetilde{K}_H \times \widetilde{K}_H} \int^\sharp_{\widetilde{M}_H(F)} D_{P_H}(m)\phi^{h_0}(k_1mk_2)(1-u(k_1mk_2h_0,t))\,dm\,dk_1\,dk_2$$
$$= \int_{\widetilde{K}_H \times \widetilde{K}_H} \int^\sharp_{\widetilde{M}_H(F)} D_{P_H}(m)\phi(k_1mk_2)(1-u(m,t))\,dm\,dk_1\,dk_2.$$

Proof. For $g \in G(F)$ let $\mathcal{M}(g) \in M^+(F)$ be such that $g = k_1\mathcal{M}(g)k_2$, $k_1, k_2 \in K$. For $\operatorname{Re}\nu \ll 0$ and $t \gg 0$,

$$\begin{aligned}
&\int_{\widetilde{K}_H \times \widetilde{K}_H} \int_{\widetilde{M}_H(F)} D_{P_H}(m)\phi^{h_0}(k_1mk_2) \\
&\qquad\qquad \times e^{\nu(H_M(\mathcal{M}(k_1mk_2h_0)))}(1-u(k_1mk_2h_0,t))\,dm\,dk_1\,dk_2 \\
&\quad = \int_{\widetilde{H}(F)} \phi(hh_0)e^{\nu(H_M(\mathcal{M}(hh_0)))}(1-u(hh_0,t))\,dh \\
&\quad = \int_{\widetilde{H}(F)} \phi(h)e^{\nu(H_M(\mathcal{M}(h)))}(1-u(h,t))\,dh \\
&\quad = \int_{\widetilde{K}_H \times \widetilde{K}_H} \int_{\widetilde{M}_H(F)} D_{P_H}(m)\phi(k_1mk_2)e^{\nu(H_M(m))}(1-u(m,t))\,dm\,dk_1\,dk_2
\end{aligned}$$

by the invariance of Haar measure, since both sides are absolutely convergent. For $t \gg 0$, if $h \in \operatorname{supp}(1-u(\cdot\,,t))$, then $\mathcal{M}(hh_0) = \mathcal{M}(h)\mathcal{M}(k_2h_0)$. Thus both sides of the equation above have a meromorphic continuation whose value at $\nu = 0$ gives the statement of the lemma. □

Proposition 4.6 (H-invariance). *Let ϕ be a matrix coefficient of $I_P(\chi_\lambda)$, where $\chi(\varpi)=1$ and $\lambda\neq 0$, and let $h_0\in H(F)$. Then*

$$\int^*_{\widetilde{H}(F)} \phi^{h_0}(h)\,dh = \int^*_{\widetilde{H}(F)} \phi(h)\,dh.$$

Proof. By the definition of the regularized integrals, the statement of the proposition will follow once we prove the following equality:

$$\begin{aligned}\int_{\widetilde{K}_H\times\widetilde{K}_H}\int^{\sharp}_{\widetilde{M}^+_H(F)} & D_{P_H}(m)\phi^{h_0}(k_1mk_2)(1-u(m,t))\,dm\,dk_1\,dk_2\\ &-\int_{\widetilde{K}_H\times\widetilde{K}_H}\int^{\sharp}_{\widetilde{M}^+_H(F)} D_{P_H}(m)\phi(k_1mk_2)(1-u(m,t))\,dm\,dk_1\,dk_2\\ &=\int_{\widetilde{H}(F)}\phi(h)u(h,t)\,dh-\int_{\widetilde{H}(F)}\phi^{h_0}(h)u(h,t)\,dh.\end{aligned}$$

First we note that by Lemma 4.5

$$\begin{aligned}\int_{\widetilde{K}_H\times\widetilde{K}_H}\int^{\sharp}_{\widetilde{M}^+_H(F)} & D_{P_H}(m)\phi^{h_0}(k_1mk_2)(1-u(m,t))\,dm\,dk_1\,dk_2\\ &-\int_{\widetilde{K}_H\times\widetilde{K}_H}\int^{\sharp}_{\widetilde{M}^+_H(F)} D_{P_H}(m)\phi(k_1mk_2)(1-u(m,t))\,dm\,dk_1\,dk_2\\ =\int_{\widetilde{K}_H\times\widetilde{K}_H}\int^{\sharp}_{\widetilde{M}^+_H(F)} & D_{P_H}(m)\phi(k_1mk_2)(1-u(mk_2h_0^{-1},t)))\,dm\,dk_1\,dk_2\\ &-\int_{\widetilde{K}_H\times\widetilde{K}_H}\int^{\sharp}_{\widetilde{M}^+_H(F)} D_{P_H}(m)\phi(k_1mk_2)(1-u(m,t))\,dm\,dk_1\,dk_2.\end{aligned}$$

For fixed h_0 and t sufficiently large, $u(\,\cdot\,h_0^{-1},t)-u(\,\cdot\,,t)$ has support contained in an annulus. From this fact one can easily check that the previous line is equal to the convergent integral

$$\begin{aligned}\int_{\widetilde{K}_H\times\widetilde{K}_H}\int_{\widetilde{M}^+_H(F)} & D_{P_H}(m)\phi(k_1mk_2)[u(m,t)-u(mk_2h_0^{-1},t)]\,dm\,dk_1\,dk_2.\\ &=\int_{\widetilde{H}(F)}\phi(h)[u(h,t)-u(hh_0^{-1},t)]\,dh\\ &=\int_{\widetilde{H}(F)}\phi(h)u(h,t)\,dh-\int_{\widetilde{H}(F)}\phi^{h_0}(h)u(h,t)\,dh. \qquad\square\end{aligned}$$

We note that Proposition 4.6 also holds if we replace ϕ^{h_0} with $\phi(h_0-)$ so our regularized integral is $H\times H$ invariant.

Now we derive an explicit formula relating regularized periods to truncated periods for the matrix coefficients that appear in the trace formula. We begin by recalling some definitions of Harish-Chandra's. For σ an admissible, tempered representation of G, $\mathcal{A}_\sigma(G)$ is the space of functions on G spanned by K-finite matrix coefficients of σ, $\mathcal{A}_{\mathrm{temp}}(G)$ is the sum of $\mathcal{A}_\sigma(G)$ over all admissible tempered representations of G and $\mathcal{A}_2(G)$ is the sum of $\mathcal{A}_\sigma(G)$ over all unitary, square integrable representations. For τ a finite dimensional, unitary, two-sided representation of K,

$$\mathcal{A}_\sigma(G,\tau)=\{f\in\mathcal{A}_\sigma(G)\otimes V_\tau : f(k_1gk_2)=\tau(k_1)f(g)\tau(k_2),\ g\in G,\ k_1,k_2\in K\}.$$

Then $\mathcal{A}_{\mathrm{temp}}(G,\tau)$ and $\mathcal{A}_2(G,\tau)$ are defined similarly.

Let $\tau_M=\tau|_{K\cap M}$. By [Harish-Chandra 1984, §3] for $f\in\mathcal{A}_\sigma(G,\tau)$ there exists a unique function $C^Pf\in\mathcal{A}(M,\tau_M)$ such that

$$\lim_{|\frac{\alpha}{\beta}|_E\to\infty}\left|\delta_P\left(\begin{pmatrix}\alpha&0\\0&\beta\end{pmatrix}\right)^{\frac12} f\left(\begin{pmatrix}\alpha&0\\0&\beta\end{pmatrix}\right)-(C^Pf)\left(\begin{pmatrix}\alpha&0\\0&\beta\end{pmatrix}\right)\right|=0.$$

We call C^Pf the weak constant term of f.

For two parabolics P_1, P_2 with Levi component M, let

$$V_{P_1|P_2}=\{v\in V:\tau(n_1)v\tau(n_2)=v,\ n_1\in N_{P_1}\cap K,\ n_2\in N_{P_2}\cap K\}$$

and let $\tau_{P_1|P_2}$ be the subrepresentation of τ_M on $V_{P_1|P_2}$. For $\Psi\in\mathcal{A}_2(M,\tau_{P|P})$ and $\lambda\in[0,\pi i/\log q]$, the Eisenstein integral $E_P(g,\Psi,\lambda)\in\mathcal{A}_{\mathrm{temp}}(G,\tau)$ is defined as

$$E_P(g,\Psi,\lambda)=\int_K\tau(k)^{-1}\Psi_P(kg)e^{(\lambda+1/2)(H_M(kg))}\,dk$$

where Ψ_P extends Ψ to G by $\Psi_P(nmk)=\Psi(m)\tau(k)$ for $n\in N$, $m\in M$, $k\in K$.

The weak constant term of the Eisenstein integral uniquely defines Harish-Chandra's c-functions [1984, §6]. For each element w in the Weyl group W of $\widetilde{G}$, the c-function $c_{P|P}(w,\lambda)$ is a linear map from $\mathcal{A}_2(M,\tau_{P|P})$ to $\mathcal{A}_2(M,\tau_{P|\overline{P}})$ such that

$$(C^PE_P)(m,\Psi,\lambda)=(c_{P|P}(1,\lambda)\Psi)(m)e^{\lambda H_M(m)}+(c_{P|P}(w,\lambda)\Psi)(m)e^{-\lambda H_M(m)}$$

where w is a representative for the nontrivial element in the Weyl group of $\widetilde{G}$. Let $c_{P|P}(s,\lambda)_\chi$ denote the restriction of $c_{P|P}(s,\lambda)$ to $\mathcal{A}_\chi(M,(\tau_\Gamma)_{P|P})$. We have

$$\mu(\chi_\lambda)^{-1}=c_{P|P}(s,\lambda)^*_\chi c_{P|P}(s,\lambda)_\chi.$$

For the rest of this section we let $c(1,\lambda)=c_{P|P}(1,\lambda)_\chi$ and $c(w,\lambda)=c_{P|P}(w,\lambda)_\chi$.

We note that the S we consider are actually in $\mathcal{H}_P(\chi)^{K_0}$ for some open compact K_0. Harish-Chandra [1976, §7] gives an isomorphism $S\to\Psi_S$ from $\mathrm{End}(\mathcal{H}_P(\chi)^K)$ onto $\mathcal{A}_\chi(M,(\tau)_{P|P})$ where V_τ is a particular subspace of $L^2(K\times K)$ such that

$$\mathrm{tr}(I_P(\chi_\lambda,k_1gk_2)S)=E_P(g,\Psi_S,\lambda)_{k_1,k_2}.$$

We can now relate the regularized integral to what appears in the local relative trace formula.

Proposition 4.7. *For $\chi = (\chi, \chi^{-1}) \in \{\Pi_2(\widetilde{M}(F)\}$, $\chi(\varpi) = 1$, $\lambda \neq 0$, $t \gg 0$,*

$$\begin{aligned}
&\int^{*}_{\widetilde{H}(F)} \operatorname{tr}(I_P(\chi_\lambda, h)S)\, dh \\
&= \int_{\widetilde{H}(F)} \operatorname{tr}(I_P(\chi_\lambda, h)S) u(h,t)\, dh \\
&\qquad + \delta(\chi)(1+q^{-1}) \biggl(\frac{q^{2\lambda(t+1)}}{1-q^{2\lambda}} \int_{\widetilde{K}_H \times \widetilde{K}_H} c(1,\lambda)\Psi_S(1)_{k_1,k_2}\, dk_1\, dk_2 \\
&\qquad\qquad + \frac{q^{-2\lambda(t+1)}}{1-q^{-2\lambda}} \int_{\widetilde{K}_H \times \widetilde{K}_H} c(w,\lambda)\Psi_S(1)_{k_1,k_2}\, dk_1\, dk_2 \biggr),
\end{aligned}$$

where $\delta(\chi) = 1$ if $\chi|_{\mathbb{O}_F^\times} = 1$ and $\delta(\chi) = 0$ if $\chi|_{\mathbb{O}_F^\times} \neq 1$.

Proof. For $S \in \mathcal{B}_P(\chi)$, $\Psi_S \in \mathcal{A}_\chi(M, (\tau_\Gamma)_{P|P})$ and

$$c(1,\lambda)\Psi_S,\ c(w,\lambda)\Psi_S \in \mathcal{A}_\chi(M, (\tau_\Gamma)_{P|\overline{P}}).$$

Therefore $\Psi = \Psi_S$ can be written as a sum of matrix coefficients of χ. Thus

$$\begin{aligned}
&C^P E_P(m, \Psi, \lambda)_{k_1,k_2} \\
&\qquad = c(1,\lambda)\Psi(m)_{k_1,k_2} e^{\lambda H_M(m)} + c(w,\lambda)\Psi(m)_{k_1,k_2} e^{-\lambda H_M(m)} \\
&\qquad = \chi(m)\bigl[(c(1,\lambda)\Psi)(1)_{k_1,k_2} e^{\lambda H_M(m)} + (c(w,\lambda)\Psi)(1)_{k_1,k_2} e^{-\lambda H_M(m)}\bigr]
\end{aligned}$$

where $\chi(m) \in \mathbb{C}^\times$. Hence for $t \gg 0$,

$$\begin{aligned}
&\int_{\widetilde{M}_H^+(F)} D_{P_H}(m) \operatorname{tr}(I_P(\chi_\lambda, k_1 m k_2)S) e^{\nu(H_M(m))}(1-u(m,t))\, dm \\
&= \int_{\widetilde{M}_H^+(F)} D_{P_H}(m)\delta_P^{-\frac12}(m)(c(1,\lambda)\Psi)(1)_{k_1,k_2} e^{(\lambda+\nu)(H_M(m))}\chi(m)(1-u(m,t))\, dm \\
&\quad + \int_{\widetilde{M}_H^+(F)} D_{P_H}(m)\delta_P^{-\frac12}(m)(c(w,\lambda)\Psi)(1)_{k_1,k_2} e^{(-\lambda+\nu)(H_M(m))}\chi(m)(1-u(m,t))\, dm \\
&= (1+q^{-1})(c(1,\lambda)\Psi)(1)_{k_1,k_2} \int_{\widetilde{M}_H^+(F)} e^{(\lambda+\nu)(H_M(m))}\chi(m)(1-u(m,t))\, dm \\
&\qquad + (1+q^{-1})(c(w,\lambda)\Psi)(1)_{k_1,k_2} \int_{\widetilde{M}_H^+(F)} e^{(-\lambda+\nu)(H_M(m))}\chi(m)(1-u(m,t))\, dm \\
&= (1+q^{-1}) \int_{\mathbb{O}_F^\times} \chi(\alpha)\, d^\times\alpha \sum_{n=t+1}^{\infty} \bigl[(c(1,\lambda)\Psi)(1)_{k_1,k_2} q^{2(\lambda+\nu)n} \\
&\qquad\qquad + (c(w,\lambda)\Psi)(1)_{k_1,k_2} q^{2(-\lambda+\nu)n}\bigr].
\end{aligned}$$

Clearly $\int_{\mathbb{O}_F^\times} \chi(\alpha)\, d^\times\alpha = 0$ unless $\chi|_{\mathbb{O}_F^\times} = 1$. If $\chi|_{\mathbb{O}_F^\times} = 1$, the previous line equals

$$(1+q^{-1})\biggl(\frac{q^{2(\lambda+\nu)(t+1)}}{1-q^{2(\lambda+\nu)}}c(1,\lambda)\Psi(1)_{k_1,k_2} + \frac{q^{2(-\lambda+\nu)(t+1)}}{1-q^{2(-\lambda+\nu)}}c(w,\lambda)\Psi(1)_{k_1,k_2}\biggr).$$

Therefore for $t \gg 0$

$$\int_{\widetilde{M}_H^+(F)}^{\sharp} D_{P_H}(m)\operatorname{tr}(I_P(\chi_\lambda, k_1 m k_2)S)(1-u(m,t))\, dm$$
$$= \delta(\chi)(1+q^{-1})\biggl(\frac{q^{2\lambda(t+1)}}{1-q^{2\lambda}}c(1,\lambda)\Psi_S(1)_{k_1,k_2} + \frac{q^{-2\lambda(t+1)}}{1-q^{-2\lambda}}c(w,\lambda)\Psi_S(1)_{k_1,k_2}\biggr)$$

and the proposition now follows. □

Lemma 4.8. *Let $\chi = (\chi, \chi^{-1})$ where χ is a character of $E^\times$ such that $\chi(\varpi) = 1$. Then*

(1) *If $\chi|_{F^\times} \neq 1$ and $\chi|_{E^1} \neq 1$, then*

$$\int_{\widetilde{H}(F)}^{*} \operatorname{tr}(I_P(\chi_\lambda, h)S)\, dh = \int_{\widetilde{H}(F)} \operatorname{tr}(I_P(\chi_\lambda, h)u(h,t)\, dh = 0.$$

(2) *If $\chi|_{F^\times} \neq 1$ and $\chi|_{E^1} = 1$, then for $t \gg 0$,*

$$\int_{\widetilde{H}(F)}^{*} \operatorname{tr}(I_P(\chi_\lambda, h)S)\, dh = \int_{\widetilde{H}(F)} \operatorname{tr}(I_P(\chi_\lambda, h)S)u(h,t)\, dh.$$

(3) *If $\chi|_{F^\times} = 1$ and $\chi|_{E^1} \neq 1$, then $\int_{\widetilde{H}}^{*} \operatorname{tr}(I_P(\chi_\lambda, h)S)\, dh$ is 0 whenever defined and*

$$\int_{\widetilde{K}_H \times \widetilde{K}_H} c(1,\lambda)\Psi_S(1)_{k_1,k_2}\, dk_1\, dk_2 = \int_{\widetilde{K}_H \times \widetilde{K}_H} c(s,\lambda)\Psi_S(1)_{k_1,k_2}\, dk_1\, dk_2$$

at $\lambda = 0$.

(4) *If $\chi|_{F^\times} = 1$ and $\chi|_{E^1} = 1$, then $\chi^2 = 1$. In this case $c(1,\lambda)$ and $c(s,\lambda)$ have a simple pole at $\lambda = 0$ and so $\mu(\chi_\lambda)$ has a zero of order two at $\lambda = 0$ and $\mu(\chi_\lambda)c(1,\lambda) = \mu(\chi_\lambda)c(s,\lambda) = 0$ at $\lambda = 0$.*

In all cases,

$$\mu(\chi_\lambda)\int_{\widetilde{H}(F)}^{*} \operatorname{tr}(I_P(\chi_\lambda, h)S)\, dh$$

is holomorphic for all $\lambda \in i\mathbb{R}$, $S \in \mathscr{B}_P(\chi)$.

Proof. In this proof we follow the techniques of [Jacquet ≥ 2012]. Case 2 is obvious from the above work. Case 1 is obvious from the above work and the H-invariance of $\int_{\widetilde{H}}^{*} \operatorname{tr}(I_P(\chi_\lambda, h)S)\, dh$ [Jacquet et al. 1999, Proposition 22].

The vanishing of the regularized period for $\lambda \neq 0$ in case 3 also follows from H-invariance. Then by the previous proposition we know that for $\lambda \neq 0$,

$$\int_{\widetilde{H}(F)} \operatorname{tr}(I_P(\chi_\lambda, h)S)u(h,t)\,dh \\ = -(1+q^{-1})\biggl(\frac{q^{2\lambda(t+1)}}{1-q^{2\lambda}}\int_{\widetilde{K}_H\times\widetilde{K}_H} c(1,\lambda)\Psi_S(1)_{k_1,k_2}\,dk_1\,dk_2 \\ + \frac{q^{-2\lambda(t+1)}}{1-q^{-2\lambda}}\int_{\widetilde{K}_H\times\widetilde{K}_H} c(w,\lambda)\Psi_S(1)_{k_1,k_2}\,dk_1\,dk_2\biggr).$$

Both sides are holomorphic and the left-hand side is also defined and holomorphic for $\lambda = 0$. As

$$\operatorname{Res}_{\lambda=0}\frac{q^{2\lambda(t+1)}}{1-q^{2\lambda}} = \frac{-1}{2\log q} \quad\text{and}\quad \operatorname{Res}_{\lambda=0}\frac{q^{-2\lambda(t+1)}}{1-q^{-2\lambda}} = \frac{1}{2\log q},$$

we must have that

$$\int_{\widetilde{K}_H\times\widetilde{K}_H} c(1,0)\Psi(1)_{k_1,k_2}\,dk_1\,dk_2 = \int_{\widetilde{K}_H\times\widetilde{K}_H} c(w,0)\Psi(1)_{k_1,k_2}\,dk_1\,dk_2.$$

In case 4 the poles and zeros are well-known and can also be seen by explicit computations of the intertwining operators. We have that

$$\mu(\chi_\lambda)\int_{\widetilde{H}(F)} \operatorname{tr}(I_P(\chi_\lambda, h)S)u(h,t)\,dh \\ = \mu(\chi_\lambda)\int^*_{\widetilde{H}(F)} \operatorname{tr}(I_P(\chi_\lambda, h)S)\,dh \\ -(1+q^{-1})\mu(\chi_\lambda)\biggl(\frac{q^{2\lambda(t+1)}}{1-q^{2\lambda}}\int_{\widetilde{K}_H\times\widetilde{K}_H} c(1,\lambda)\Psi_S(1)_{k_1,k_2}\,dk_1\,dk_2 \\ + \frac{q^{-2\lambda(t+1)}}{1-q^{-2\lambda}}\int_{\widetilde{K}_H\times\widetilde{K}_H} c(w,\lambda)\Psi_S(1)_{k_1,k_2}\,dk_1\,dk_2\biggr).$$

The left-hand side is 0 at $\lambda = 0$ and the last two terms are holomorphic at $\lambda = 0$ so the first term must be holomorphic at $\lambda = 0$. □

Let

$$D_{\chi_\lambda}(f) = \sum_{S\in\mathcal{B}_P(\chi)} P_{\chi_\lambda}(S_\lambda[f])\overline{W_{\chi_\lambda}(S)},$$

$$P_{\chi_\lambda}(S) = \int^*_{\widetilde{H}(F)} \operatorname{tr}(I_P(\chi_\lambda, h)S)\,dh,$$

$$S_\lambda[f] = I_P(\chi_\lambda, f_2)SI_P(\chi_\lambda, f_1^\vee),$$

$$\widetilde{D}_\chi(f) = (1+q^{-1})\mu(\chi_0)\sum_{S\in\mathcal{B}_P(\chi)}\overline{W_{\chi_0}(S)}\int_{\widetilde{K}_H\times\widetilde{K}_H} c(1,0)\psi_{S_0[f]}(1)_{k_1,k_2}\,dk_1\,dk_2.$$

We now relate the distributions above to the truncated distributions from (4-2).

Lemma 4.9. *Let $\chi = (\chi, \chi^{-1})$ where χ is a character of $E^\times$ such that $\chi(\varpi) = 1$.*

(1) *If $\chi|_{F^\times} \neq 1$ and $\chi|_{E^1} \neq 1$, then*

$$\lim_{t\to\infty} \int_0^{\frac{\pi i}{\log q}} \mu(\chi_\lambda) D^t_{\chi_\lambda}(f)\, d\lambda = 0.$$

(2) *If $\chi|_{F^\times} \neq 1$ and $\chi|_{E^1} = 1$, then*

$$\lim_{t\to\infty} \int_0^{\frac{\pi i}{\log q}} \mu(\chi_\lambda) D^t_{\chi_\lambda}(f)\, d\lambda = \int_0^{\frac{\pi i}{\log q}} \mu(\chi_\lambda) D_{\chi_\lambda}(f)\, d\lambda.$$

(3) *If $\chi|_{F^\times} = 1$ and $\chi|_{E^1} \neq 1$, then*

$$\lim_{t\to\infty} \int_0^{\frac{\pi i}{\log q}} \mu(\chi_\lambda) D^t_{\chi_\lambda}(f)\, d\lambda = \widetilde{D}_\chi(f).$$

(4) *If $\chi|_{F^\times} = 1$ and $\chi|_{E^1} = 1$, then*

$$\lim_{t\to\infty} \int_0^{\frac{\pi i}{\log q}} \mu(\chi_\lambda) D^t_{\chi_\lambda}(f)\, d\lambda = \int_0^{\frac{\pi i}{\log q}} \mu(\chi_\lambda) D_{\chi_\lambda}(f)\, d\lambda.$$

Proof. First we note that

$$\begin{aligned}
\int_0^{\frac{\pi i}{\log q}} \mu(\chi_\lambda) D^t_{\chi_\lambda}(f)\, d\lambda &= \int_0^{\frac{\pi i}{\log q}} \mu(\chi_\lambda) \sum_{S\in\mathcal{B}_P(\chi)} P^t_{I_P(\chi_\lambda)}(S_\lambda[f]) \overline{W_{\chi_\lambda}(S)}\, d\lambda \\
&= \sum_{S\in\mathcal{B}_P(\chi)} \int_0^{\frac{\pi i}{\log q}} \left(\overline{\int_{N(F)} \operatorname{tr}(I_P(\chi_\lambda, n)S)\psi(n^{-1})u(n,t)\, dn} \right) \\
&\qquad \times \mu(\chi_\lambda) \int_{\widetilde{H}(F)} \operatorname{tr}(I_P(\chi_\lambda, h)S_\lambda[f])u(h,t)\, dh\, d\lambda.
\end{aligned}$$

Cases 1 and 2 now follow directly from Lemma 4.8. For the remaining cases we note that by Proposition 4.7 for $t \gg 0$,

$$\begin{aligned}
&\int_{\widetilde{H}(F)} \operatorname{tr}(I_P(\chi_\lambda, h)S_\lambda[f])u(h,t)\, dh \\
&\quad = \int^*_{\widetilde{H}(F)} \operatorname{tr}(I_P(\chi_\lambda, h)S_\lambda[f])\, dh \\
&\qquad + \delta(\chi) \frac{1+q^{-1}}{q^\lambda - q^{-\lambda}} \biggl(q^{2\lambda(t+\frac{1}{2})} \int_{\widetilde{K}_H\times\widetilde{K}_H} c(1,\lambda)\Psi_{S_\lambda[f]}(1)_{k_1,k_2}\, dk_1\, dk_2 \\
&\qquad\qquad - q^{-2\lambda(t+\frac{1}{2})} \int_{\widetilde{K}_H\times\widetilde{K}_H} c(w,\lambda)\Psi_{S_\lambda[f]}(1)_{k_1,k_2}\, dk_1\, dk_2 \biggr).
\end{aligned}$$

In case 3, by Lemma 4.8 the regularized period vanishes and we are left computing

$$(1+q^{-1}) \lim_{t\to\infty} \int_0^{\frac{\pi i}{\log q}} \mu(\chi_\lambda)\overline{W_{\chi_\lambda}(S)} \Bigg(\frac{q^{2\lambda(t+1/2)}}{q^{\lambda}-q^{-\lambda}} \int_{\widetilde{K}_H\times\widetilde{K}_H} c(1,\lambda)\Psi_{S_\lambda[f]}(1)_{k_1,k_2}\,dk_1\,dk_2 - \frac{q^{-2\lambda(t+1/2)}}{q^{\lambda}-q^{-\lambda}} \int_{\widetilde{K}_H\times\widetilde{K}_H} c(w,\lambda)\Psi_{S_\lambda[f]}(1)_{k_1,k_2}\,dk_1\,dk_2 \Bigg)\,d\lambda. \tag{4-5}$$

Let

$$f_1(\lambda) = \frac{1+q^{-1}}{2}\mu(\chi_\lambda)\overline{W_{\chi_\lambda}(S)}\Bigg(\int_{\widetilde{K}_H\times\widetilde{K}_H} c(1,\lambda)\Psi_{S_\lambda[f]}(1)_{k_1,k_2}\,dk_1\,dk_2 - \int_{\widetilde{K}_H\times\widetilde{K}_H} c(w,\lambda)\Psi_{S_\lambda[f]}(1)_{k_1,k_2}\,dk_1\,dk_2\Bigg),$$

$$f_2(\lambda) = \frac{1+q^{-1}}{2}\mu(\chi_\lambda)\overline{W_{\chi_\lambda}(S)}\Bigg(\int_{\widetilde{K}_H\times\widetilde{K}_H} c(1,\lambda)\Psi_{S_\lambda[f]}(1)_{k_1,k_2}\,dk_1\,dk_2 + \int_{\widetilde{K}_H\times\widetilde{K}_H} c(w,\lambda)\Psi_{S_\lambda[f]}(1)_{k_1,k_2}\,dk_1\,dk_2\Bigg).$$

Then (4-5) equals

$$\lim_{t\to\infty}\int_0^{\frac{\pi i}{\log q}} \frac{f_1(\lambda)(q^{2\lambda(t+\frac12)}+q^{-2\lambda(t+\frac12)})}{q^{\lambda}-q^{-\lambda}} + \lim_{t\to\infty}\int_0^{\frac{\pi i}{\log q}} \frac{f_2(\lambda)(q^{2\lambda(t+\frac12)}-q^{-2\lambda(t+\frac12)})}{q^{\lambda}-q^{-\lambda}}\,d\lambda.$$

By Lemma 4.8, $f_1(0)=0$. Hence by Fourier analysis the first integral will vanish. The limit of the second integral will be $f_2(0)$, which, by the identity in case 3 of Lemma 4.8, equals

$$(1+q^{-1})\mu(\chi_0)\overline{W_{\chi_0}(S)}\int_{\widetilde{K}_H\times\widetilde{K}_H} c(1,0)\Psi_{S_0[f]}(1)_{k_1,k_2}\,dk_1\,dk_2.$$

For case 4 by Lemma 4.8 when multiplied by $\mu(\chi_\lambda)\overline{W_{\chi_\lambda}(S)}$, $f_1(\lambda)$ and $f_2(\lambda)$ are holomorphic functions of λ and vanish at $\lambda=0$, thus by similar analysis as above the last two terms vanish in the limit and we are left with the statement of the lemma. □

4B1. *Discrete series representations.* Because the matrix coefficient of a supercuspidal representation σ has compact support it is obvious that

$$\lim_{t\to\infty}\int_{\widetilde{H}(F)} \mathrm{tr}(\sigma(h)S)u(h,t)\,dh = \int_{\widetilde{H}(F)} \mathrm{tr}(\sigma(h)S)\,dh.$$

Now we will prove that this is also true for Steinberg representations.

Lemma 4.10. *For $\sigma = St(\chi)$, $\chi^2 = 1$, the matrix coefficients are absolutely convergent over $\widetilde{H}(F)$. Thus the limit*

$$\lim_{t\to\infty} \int_{\widetilde{H}(F)} \mathrm{tr}(\sigma(h)S)u(h,t)\,dh$$

exists and equals

$$\int_{\widetilde{H}(F)} \mathrm{tr}(\sigma(h)S\,dh.$$

Proof. By [Borel and Wallach 1980, XI.4.3; Casselman 1995, 4.2.3], a matrix coefficient for σ evaluated at $\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)$ is equal to a matrix coefficient for the Jacquet functor σ_N, evaluated at the same value, for $\left|\frac{a}{b}\right|_E$ sufficiently small. The Jacquet functor of σ is δ_P. Thus outside some compact set, our original matrix coefficient will behave like δ_P on $M_H^-(F)$. When we integrate over $\widetilde{H}(F)$, using the $K_H M_H^-(F) K_H$ decomposition, we get a measure factor of $\delta_P^{-1/2}$. Thus outside a set of compact support our integral will look like $\int_{|a|<c} |a|_F \, d^\times a$ for some $c > 0$. □

Putting everything together we have proved the following.

Proposition 4.11. *For any $f \in C_c^\infty(\widetilde{G}(F) \times \widetilde{G}(F))$,*

$$\begin{aligned}\lim_{t\to\infty} \int_{\widetilde{H}(F)} \int_{N(F)} & K_f(h,n)\psi(n)u(h,t)u(n,t)\,dn\,dh \\ &= \sum_{\sigma\in\Pi_2(\widetilde{G}(F))} d(\sigma)D_\sigma(f) + \tfrac{1}{2} \sum_{\substack{\chi\in\{\Pi_2(\widetilde{M}(F))\} \\ \chi^2\neq 1,\, \chi|_{F^\times}=1}} \widetilde{D}_\chi(f) \\ &\qquad + \tfrac{1}{2} \sum_{\substack{\chi\in\{\Pi_2(\widetilde{M}(F))\} \\ \chi|_{E^1}=1}} d(\chi) \int_0^{\frac{\pi i}{\log q}} \mu(\chi_\lambda) D_{\chi_\lambda}(f)\,d\lambda,\end{aligned}$$

where

$$D_{\chi_\lambda}(f) = \sum_{S\in\mathcal{B}_P(\chi)} P_{\chi_\lambda}(S_\lambda[f])\overline{W_{\chi_\lambda}(S)},$$

$$P_{\chi_\lambda}(S) = \int_{\widetilde{H}(F)}^{*} \mathrm{tr}(I_P(\chi_\lambda, h)S)\,dh,$$

$$\widetilde{D}_\chi(f) = (1+q^{-1})\mu(\chi_0) \sum_{S\in\mathcal{B}_P(\chi)} \overline{W_{\chi_0}(S)} \int_{\widetilde{K}_H\times\widetilde{K}_H} c(1,0)\psi_{S_0[f]}(1)_{k_1,k_2}\,dk_1\,dk_2,$$

$$D_\sigma(f) = \sum_{S\in\mathcal{B}(\sigma)} P_\sigma(\sigma(f_2)S\sigma(f_1^\vee))\overline{W_\sigma(S)},$$

$$P_\sigma(S) = \int_{\widetilde{H}(F)} \mathrm{tr}(\sigma(h)S)\,dh.$$

This proposition combined with Proposition 4.3 proves Theorem 1.4.

5. Comparison of local trace formulas and applications

We now combine the results of the previous two sections to compare the two trace formulas. Let $\omega_{E/F}$ be the quadratic character of $F^\times$ associated to E/F and let ω denote its trivial extension to $E^\times$.

Definition 5.1. We say that $f' \in C_c^\infty(\widetilde{G}'(F))$ and $f \in C_c^\infty(\widetilde{G}(F))$ are *matching functions* if $O'(f', \psi', a) = \omega(a)O(f, \psi, a)$ for all $a \in E^\times$.

By work of Ye [1989] and Flicker [1991, Proposition 3], we know that for any $f' \in C_c^\infty(\widetilde{G}'(F))$ there exists a matching $f \in C_c^\infty(\widetilde{G}(F))$ and vice versa. In fact, by the Fundamental Lemma, for f' spherical, we know that f is the corresponding function from the base change map between their Hecke algebras. Thus by the geometric expansion of the trace formulas in Propositions 3.4 and 4.3 we have the following statement.

Proposition 5.2. *For $f_i' \in C_c^\infty(\widetilde{G}'(F))$ and $f_i \in C_c^\infty(\widetilde{G}(F))$ matching functions for $i = 1, 2$,*

$$\lim_{t\to\infty} \int_{N'(F)} \int_{N'(F)} K_{f_1'\otimes f_2'}(n_1, n_2)\psi'(n_1^{-1}n_2)u(n_1, t)u(n_2, t)\, dn_1\, dn_2$$
$$= \lim_{t\to\infty} \int_{\widetilde{H}(F)} \int_{N(F)} K_{f_1\otimes f_2}(h, n)\psi(n)u(h, t)u(n, t)\, dn\, dh.$$

Now we use the equality of the trace formulas to compare the spectral expansions. By Propositions 3.6, 4.11 and 5.2 we have the following result.

Theorem 5.3. *For f_i and f_i' matching functions for $i = 1, 2$,*

$$\sum_{\sigma'\in\Pi_2(\widetilde{G}'(F))} d(\sigma')D'_{\sigma'}(f_1'\otimes f_2') + \tfrac{1}{2} \sum_{\chi'\in\{\Pi_2(\widetilde{M}'(F))\}} d(\chi') \int_0^{\frac{\pi i}{\log q}} \mu(\chi'_\lambda)D'_{\chi'_\lambda}(f_1'\otimes f_2')\, d\lambda$$
$$= \sum_{\sigma\in\Pi_2(\widetilde{G}(F))} d(\sigma)D_\sigma(f_1\otimes f_2) + \tfrac{1}{2} \sum_{\substack{\chi\in\{\Pi_2(\widetilde{M}(F))\}\\ \chi^2\neq 1,\, \chi|_{F^\times}=1}} \widetilde{D}_\chi(f_1\otimes f_2)$$
$$+ \tfrac{1}{2} \sum_{\substack{\chi\in\{\Pi_2(\widetilde{M}(F))\}\\ \chi|_{E^1}=1}} d(\chi) \int_0^{\frac{\pi i}{\log q}} \mu(\chi_\lambda)D_{\chi_\lambda}(f_1\otimes f_2)\, d\lambda.$$

The unstable base change map associated to ω lifts principal series representations of $\widetilde{G}'$ to principal series representations $I_P(\chi)$ of $\widetilde{G}$ such that $\chi|_{E^1} = 1$. It also lifts certain square integrable representations of $\widetilde{G}'$ to the principal series representations of $\widetilde{G}$ defined by $I_P(\chi\omega)$ such that $\chi^2 \neq 1$, $\chi|_{F^\times} = 1$. It lifts the remaining square integrable representations of $\widetilde{G}'$ to square integral representations of $\widetilde{G}$ [Rogawski 1990; Flicker 1982]. Thus we could rephrase the right-hand side of Theorem 5.3 in terms of summing over the representations of $\widetilde{G}$ that are the unstable base change lifts of representations of $\widetilde{G}'$. The extra discrete term $\widetilde{W}_\chi(f)$ corresponds exactly to the representations that lift from the discrete series of $\widetilde{G}'$ to the principal series of $\widetilde{G}$.

We also note that the only representations that appear on the right-hand side of Theorem 5.3 are those σ or $I_P(\chi_\lambda)$ for which there is a matrix coefficient such that the regularized integral over H is nonzero. This gives us a more explicit description of the nonvanishing H invariant linear functional that characterizes the image of the unstable base change map.

We would like to relate our distributions to the local factors in the Bessel and relative Bessel distributions. Recall from the introduction that Jacquet's global relative trace formula tells us that for f' on $U(2, \mathbb{A}_F)$ and f on $\mathrm{GL}(2, \mathbb{A}_E)$ matching functions, if a cuspidal representation π' of $U(2, \mathbb{A}_F)$ maps to π of $\mathrm{GL}(2, \mathbb{A}_E)$ under unstable base change, then

$$B'_{\pi'}(f') = B_\pi(f)$$

where

$$\begin{aligned} B'_{\pi'}(f') &= \sum_{\phi' \in \text{o.n.b.}(V_{\pi'})} W'(\pi'(f')\phi')\overline{W'(\phi')}, \\ B_\pi(f) &= \sum_{\phi \in \text{o.n.b.}(V_\pi)} P(\pi(f)\phi)\overline{W(\phi)}, \\ W'(\phi') &= \int_{N'(F)\backslash N'(\mathbb{A}_F)} \phi'(n)\overline{\psi'(n)}\, dn, \\ W(\phi) &= \int_{N(E)\backslash N(\mathbb{A}_E)} \phi(n)\overline{\psi(n)}\, dn, \\ P(\phi) &= \int_{\mathrm{GL}(2,F)Z(\mathbb{A}_F)\backslash \mathrm{GL}(2,\mathbb{A}_F)} \phi(h)\, dh \neq 0. \end{aligned}$$

While $B'_{\pi'}(f')$ and $B_\pi(f)$ factor into local Bessel distributions $B'_{\pi'_v}(f'_v)$ and $B_{\pi_v}(f_v)$, it is not clear how to normalize the local Bessel distributions. We can rewrite our local distributions as a product of two local Bessel (or local relative Bessel) distributions:

Lemma 5.4. (1) *For σ' an irreducible supercuspidal representation of $\widetilde{G}'(F)$, there exists a local Bessel distribution $B'_{\sigma'}$, unique up to a constant of absolute value* 1, *such that*

$$D'_{\sigma'}(f'_1 \otimes f'_2) = B'_{\sigma'}(f'_2)B'_{\sigma'^*}(f'_1).$$

(2) *For σ an irreducible supercuspidal representation of $\widetilde{G}(F)$, there exists a local relative Bessel distribution B_σ, unique up to a constant of absolute value* 1,

such that

$$D_\sigma(f_1 \otimes f_2) = B_\sigma(f_2) B_{\sigma^*}(f_1).$$

Proof. We recall that

$$D'_{\sigma'}(f') = \sum_{S' \in \mathcal{B}(\sigma')} \int_{N'(F)} \operatorname{tr}(\sigma'(n_1)\sigma'(f'_2) S' \sigma'^*(f'_1)) \psi'(n_1)^{-1}\, dn_1 \overline{\int_{N'(F)} \operatorname{tr}(\sigma'(n_2) S') \psi'(n_2)^{-1}\, dn_2}.$$

Let $V = V_{\sigma'}$. As S' is an endomorphism on V there exist $v \in V$, $v^* \in V^*$ such that $S' = v \otimes v^*$. Then the linear functional on $V \otimes V^*$ that acts by

$$v \otimes v^* \mapsto \int_{N'(F)} \operatorname{tr}(\sigma'(n) v \otimes v^*) \psi'(n)^{-1}\, dn$$

transforms under n on v and v^* by ψ'. Thus it is a Whittaker functional on $V \otimes V^*$. By the uniqueness of Whittaker models,

$$\int_{N'(F)} \operatorname{tr}(\sigma'(n) S') \psi'(n)^{-1}\, dn = W'(v) W'(v^*).$$

Thus

$$\begin{aligned} D'_{\sigma'}(f') &= \sum_{v \otimes v^*} W'(\sigma'(f'_2)v)\overline{W'(v)}\, W'(\sigma'^*(f'_1)v^*)\overline{W'(v^*)} \\ &= B'_{\sigma'}(f'_2) B'_{\sigma'^*}(f'_1). \end{aligned}$$

We note that if we change $B'_{\sigma'}$ by a constant c, then $B'_{\sigma'^*}$ will change by $\bar{c}$.

The proof for the local relative Bessel distributions is similar, using the uniqueness of the H-invariant linear functional [Hakim 1991; Flicker 1991, Proposition 11]. □

We can also describe matching functions by an equality of all the Bessel distributions.

Lemma 5.5 (density). (1) *If $f'_1 \in C_c^\infty(\widetilde{G}'(F))$ is such that $D'_{\sigma'}(f'_1 \otimes f'_2) = 0$ for all irreducible tempered representations σ' of $\widetilde{G}'(F)$ and all f'_2, then $O'(f'_1, \psi'^{-1}, a) = 0$ for all $a \in E^\times$.*

(2) *If $f_1 \in C_c^\infty(\widetilde{G}(F))$ is such that $D_\sigma(f_1 \otimes f_2) = 0$ and $\widetilde{D}_\sigma(f_1 \otimes f_2) = 0$ for all irreducible tempered representations σ of $\widetilde{G}(F)$ and f_2, then $O(f_1, \psi^{-1}, a) = 0$ for all $a \in E^\times$.*

Proof. If $D'_{\sigma'}(f'_1 \otimes f'_2) = 0$ for all σ', then by Theorem 1.3,

$$\int_{a \in E^\times / E^1} |a|_E\, O'(f'_1, \psi'^{-1}, a) O'(f'_2, \psi', a)\, d^\times a = 0$$

for all $f'_2 \in C_c^\infty(\widetilde{G}'(F))$. As $O'(f'_1, \psi^{-1}, a)$ is a locally constant function of a there exists some open compact U such that $O'(f'_1, \psi'^{-1}, a)$ is biinvariant under it. Then by choosing f'_2 such that $O'(f'_2, \psi'^{-1}, a)$ has support contained in U we see that $O'(f'_1, \psi'^{-1}, a) = 0$. The second case follows from the first one. $\square$

Combining Theorem 5.3 with Lemma 5.4 and the global relative trace formula, we have the following result:

Corollary 5.6. *If σ is the supercuspidal representation of $\widetilde{G}(F)$ that is the unstable base change lift of the supercuspidal representation σ' on $\widetilde{G}'(F)$, and f'_i and f_i are matching functions for $i = 1, 2$, then*

$$d(\sigma')D_{\sigma'}(f'_1 \otimes f'_2) = d(\sigma)D_\sigma(f_1 \otimes f_2).$$

Proof. From the global comparison of relative trace formulas [Flicker 1991; Lapid 2006; Ye 1989] and a standard globalization argument we know there exists a constant c_σ such that $B'_{\sigma'}(f'_i) = c_\sigma B_\sigma(f_i)$ for all matching f_i, f'_i. Take f'_1 and f_2 to be matrix coefficients of σ' and σ such that $B'_{\sigma'}(f'_1) \neq 0$ and $B_\sigma(f_2) \neq 0$. Take f'_2 a matching function to f_2 and f_1 a matching function to f'_1. Then by Theorem 5.3,

$$d(\sigma')D'_{\sigma'}(f') = d(\sigma)D_\sigma(f). \qquad \square$$

In addition to the spectral comparison, these local trace formulas also have applications on the geometric side. If we define the inner product of two functions g_1, g_2 on $E^\times/E^1$ by

$$\langle g_1, g_2\rangle = \int_{a \in E^\times/E^1} g_1(a)g_2(a)|a|_E \, d^\times a,$$

then:

Corollary 5.7 (orthogonality relations). *For f_1 and f_2 matrix coefficients of the supercuspidal representations σ_1 and σ_2 of $\widetilde{G}(F)$,*

$$\big\langle O(f_1, \psi, \cdot), O(f_2, \psi^{-1}, \cdot)\big\rangle \neq 0 \iff \sigma_1 \sim \sigma_2.$$

For f'_1 and f'_2 matrix coefficients of the supercuspidal representations σ'_1 and σ'_2 of $\widetilde{G}'(F)$,

$$\big\langle O'(f'_1, \psi', \cdot), O'(f'_2, \psi'^{-1}, \cdot)\big\rangle \neq 0 \iff \sigma'_1 \sim \sigma'_2.$$

Proof. This follows directly from the local Kuznetsov and local relative trace formulas. $\square$

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BROOKE FEIGON
DEPARTMENT OF MATHEMATICS
THE CITY COLLEGE OF NEW YORK, CUNY
NAC 8/133
NEW YORK, NY 10031
UNITED STATES
bfeigon@ccny.cuny.edu

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ON THE DEGREES OF MATRIX COEFFICIENTS OF INTERTWINING OPERATORS

TOBIAS FINIS, EREZ LAPID AND WERNER MÜLLER

To the memory of Jonathan Rogawski

We state and discuss a general conjectural bound on the degrees of matrix coefficients of intertwining operators for reductive groups over p-adic fields and a supplementary uniformity conjecture for reductive groups over number fields. We prove both conjectures for the groups $\mathrm{GL}(r)$ and obtain partial results for other groups.

1. Introduction

Let $\boldsymbol{G}$ be a reductive algebraic group defined over a p-adic field F with residue field $\mathbb{F}_q$ and $G = \boldsymbol{G}(F)$. Fix a special maximal compact subgroup K_0 of G. For a maximal parabolic subgroup $\boldsymbol{P} = \boldsymbol{M}\boldsymbol{U}$ of $\boldsymbol{G}$ and a smooth irreducible representation π of $M = \boldsymbol{M}(F)$, we consider the family of induced representations $I_P(\pi, s)$, $s \in \mathbb{C}$, which extend the fixed K_0-representation $I_{P\cap K_0}^{K_0}(\pi|_{M\cap K_0})$, and the associated intertwining operators $M(s) = M_{\bar{P}|P}(\pi, s) : I_P(\pi, s) \to I_{\bar{P}}(\pi, -s)$. For any open subgroup K of K_0, the restriction

$$M(s)^K : I_{P\cap K_0}^{K_0}(\pi|_{M\cap K_0})^K = I_P(\pi, s)^K \to I_{\bar{P}}(\pi, -s)^K = I_{\bar{P}\cap K_0}^{K_0}(\pi|_{M\cap K_0})^K$$

of $M(s)$ to the space of K-fixed vectors is a family of linear maps between finite-dimensional vector spaces which do not depend on s. It is well known that the

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matrix coefficients of the linear operators $M(s)^K$ are rational functions of q^{-s}, whose denominators can be controlled explicitly (see, e.g., [Waldspurger 2003, IV.1.1, IV.1.2]). In particular, their degrees are bounded independently of K and π.

What can be said about the degrees of the numerators? In this note, we propose the following conjecture, which should provide a bound of the correct order of magnitude. Let $\boldsymbol{G}'$ be the derived group of $\boldsymbol{G}$ and set $G' = \boldsymbol{G}'(F)$. Note that $K_0' = K_0 \cap G'$ is a special maximal compact subgroup of G'.

Conjecture 1. *There exist constants $c > 0$ and d, depending only on $\boldsymbol{G}$, such that for any open subgroup $K \subset K_0$, the degrees of the numerators of the matrix coefficients of $M(s)^K$ are bounded by $c \log_q [K_0' : K'] + d$, where $K' = K \cap G'$.*

We also propose the following supplement in a global situation, where we consider a reductive group $\boldsymbol{G}$ defined over a number field k and its base change to $F = k_v$ for all nonarchimedean places v of k. Let $K_{0,v}$ be a special maximal compact subgroup of $\boldsymbol{G}(k_v)$.

Conjecture 2. *In the global situation, assume $K_{0,v}$ to be hyperspecial for almost all places v of k. Then Conjecture 1 is true for all pairs of local groups $\boldsymbol{G}(k_v)$ and $K_{0,v}$, with uniform values of c and d.*

It is equivalent to consider the normalized intertwining operators $R(s)$ defined by Arthur [1989]. We discuss this modification and some other simple variants in Section 3 below.

The main result of this paper is the following.

Theorem 1. *Conjectures 1 and 2 are true for the groups $\boldsymbol{G} = \mathrm{GL}(r)$. More precisely, the constants c and d in Conjecture 1 depend only on r and $[F : \mathbb{Q}_p]$.*

An important motivation for our paper is provided by the analysis of limit multiplicities for noncompact quotients of $\boldsymbol{G}(\mathbb{R})$, where in order to deal with the spectral side of Arthur's trace formula, it is crucial to bound the degrees of the matrix coefficients of local intertwining operators. This application (for $\boldsymbol{G} = \mathrm{GL}(r)$) is discussed in [Finis et al. 2012]. We opted to single out our conjectures and results on local intertwining operators as a separate paper, since they may be of interest in their own right.

A natural analog of Conjecture 1 in the archimedean case ($F = \mathbb{R}$ or $\mathbb{C}$) has been obtained in [Lapid 2004]. To explain it, fix a maximal compact subgroup K_0 of G (it is well known to be unique up to conjugation). For any K_0-module V and $\sigma \in \widehat{K_0}$, let V^σ denote the σ-isotypic part of V. Let $R(\pi, s) : I_P(\pi, s) \to I_{\bar{P}}(\pi, -s)$ be the normalized intertwining operators and $R(\pi, s)^\sigma$ their restrictions to linear maps between the finite-dimensional vector spaces $I_P(\pi, s)^\sigma$ and $I_{\bar{P}}(\pi, -s)^\sigma$ which do not depend on s. The matrix coefficients of the operators $R(\pi, s)$ are rational functions of s [Arthur 1989, Theorem 2.1]. We denote by $\|\sigma\|$ the maximum of the norms of the highest weights of σ (with respect to a fixed choice of norm on the vector space spanned by the lattice of characters of a maximal torus of the connected component of the identity of K_0). Then we can formulate the following direct consequence of [Lapid 2004, Proposition A.2].

Theorem 2. *There exists a constant $c > 0$, depending only on $\boldsymbol{G}$ and the norm $\|\cdot\|$, such that for any maximal parabolic subgroup $\boldsymbol{P} = \boldsymbol{M}\boldsymbol{U}$ of $\boldsymbol{G}$, any irreducible representation π of M, and any K_0-type $\sigma \in \widehat{K}_0$, the degrees of the matrix coefficients of $R(\pi, s)^\sigma$ are bounded by $c\|\sigma\|$.*

Let us now make a few comments about the proof of Theorem 1, at the same time outlining the partial results that we can prove for general groups $\boldsymbol{G}$. By a standard argument, we can reduce to the case where π is supercuspidal. Furthermore, a result of Lubotzky (quoted as Proposition 3 below) allows us to assume that K' is a *principal* congruence subgroup of G'. After these preliminary reductions, there are two main ingredients. First, assuming the widely believed conjecture that supercuspidal representations of G are induced from open subgroups which are compact modulo the center,[1] we can deduce a good bound for the support of matrix coefficients of these representations (property (PSC) of Definition 7 below). This inference is an explication of an argument which goes back to [Jacquet 1971] (cf. [Bushnell 1990]). The classification of supercuspidals needed for our argument has been proven for $\boldsymbol{G} = \mathrm{GL}(r)$ by Bushnell and Kutzko [1993a]. It is also known in many other cases, most notably for classical groups of odd residual characteristic [Stevens 2008] and for any group in large residual characteristic [Kim 2007]. Therefore, property (PSC) is true in these cases.

The second part of the main argument is a simple proof of the rationality of intertwining operators for parabolic subgroups $\boldsymbol{P}$ with abelian unipotent radical,[2] which allows us to control the degrees of the rational functions involved (Proposition 16 and Theorem 21). For $\boldsymbol{G} = \mathrm{GL}(r)$, this fortunately covers all cases, thereby completing the proof of Theorem 1. The technical geometric property that is needed for our argument is explicated in Definition 15 below. It is unfortunately not satisfied for all maximal parabolic subgroups, even in the case of classical groups (see Remark 18). It is conceivable that a more elaborate argument will work in general.

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[1] In fact, it suffices to assume that every supercuspidal representation is *contained* in such an induced representation of finite length (see Section 4 below for more details).

[2] We also make the additional technical assumption that the group $\boldsymbol{G}$ is split over F.

2. The setup

Let F be a p-adic field with normalized absolute value $|\cdot|$, ring of integers $\mathcal{O}$, and uniformizer ϖ. Let q be the cardinality of the residue field of F.

As a rule, we write $X = \boldsymbol{X}(F)$ whenever $\boldsymbol{X}$ is a variety over F. Let $\boldsymbol{G}$ be a connected reductive algebraic group defined over F with center $\boldsymbol{Z}$. All algebraic subgroups that will be considered in the sequel are implicitly assumed to be defined over F. Let $\boldsymbol{G}'$ be the derived group of $\boldsymbol{G}$ and for any subgroup $K \subset G$, write $K' = K \cap G'$. Fix a maximal F-split torus $\boldsymbol{T}_0$ and a minimal parabolic subgroup $\boldsymbol{P}_0 = \boldsymbol{M}_0\boldsymbol{U}_0 \supset \boldsymbol{T}_0$ of $\boldsymbol{G}$, where $\boldsymbol{M}_0 = C_{\boldsymbol{G}}(\boldsymbol{T}_0)$ is a minimal Levi subgroup of $\boldsymbol{G}$. Let $\Phi = R(\boldsymbol{T}_0, \boldsymbol{G})$ be the set of roots of $\boldsymbol{T}_0$. The choice of $\boldsymbol{P}_0$ fixes a set of positive roots $R(\boldsymbol{T}_0, \boldsymbol{U}_0) \subset \Phi$. Let $\Delta_0 \subset \Phi$ be the corresponding subset of simple roots. The standard maximal parabolic subgroups of $\boldsymbol{G}$ correspond bijectively to the simple roots, and for $\alpha \in \Delta_0$, we denote by $\boldsymbol{P}^\alpha = \boldsymbol{M}^\alpha\boldsymbol{U}^\alpha$ the unique standard maximal parabolic subgroup with $\alpha \in R(\boldsymbol{T}_0, \boldsymbol{U}^\alpha)$. For any Levi subgroup $\boldsymbol{M}$, we denote by $\mathcal{P}(\boldsymbol{M})$ the (finite) set of all parabolic subgroups of $\boldsymbol{G}$ with Levi part $\boldsymbol{M}$. For any standard parabolic subgroup $\boldsymbol{P}$ of $\boldsymbol{G}$ with standard Levi decomposition $\boldsymbol{P} = \boldsymbol{M}\boldsymbol{U}$, we denote by $\bar{\boldsymbol{P}} = \boldsymbol{M}\bar{\boldsymbol{U}}$ the opposite parabolic subgroup.

Fix a special maximal compact subgroup K_0 of G (more precisely, the stabilizer of a special point in the apartment associated to $\boldsymbol{T}_0$), so that we have the Iwasawa decomposition $P_0K_0 = G$. In addition, we have the Cartan decomposition $G = K_0M_0^+K_0$, where M_0^+ is the set of all $m \in M_0$ with $|\alpha(m)| \geq 1$ for all $\alpha \in \Delta_0$ [Tits 1979, §3.3]. Also, for any parabolic subgroup $\boldsymbol{P} = \boldsymbol{M}\boldsymbol{U}$ with Levi subgroup $\boldsymbol{M} \supset \boldsymbol{M}_0$, we have $(P \cap K_0) = (M \cap K_0)(U \cap K_0)$. We take a representative $w_0 \in K_0$ for the longest Weyl element. Fix a faithful representation $\rho : \boldsymbol{G} \to \mathrm{GL}(V)$ and an $\mathcal{O}$-lattice Λ_V in the representation space V such that $K_0 = \{g \in G : \rho(g)\Lambda_V = \Lambda_V\}$, and for $n = 1, 2, \ldots$, let

$$K_n = \{g \in G : \rho(g)v \equiv v \pmod{\varpi^n \Lambda_V},\ v \in \Lambda_V\}$$

be the associated principal congruence subgroups of K_0. Note that a more natural filtration of K_0 has been defined in terms of the Bruhat–Tits building of G' in [Schneider and Stuhler 1997, Chapter I].

Suppose now that $\boldsymbol{P} = \boldsymbol{M}\boldsymbol{U}$ is a standard maximal parabolic subgroup. Let χ_P be the fundamental weight of $\boldsymbol{P}$. Some integral power of χ_P defines a rational character of $\boldsymbol{P}$ trivial on $\boldsymbol{U}$. Therefore $|\chi_P|$ defines a character $|\chi_P| : P \to \mathbb{R}_{>0}$ and we can extend this character uniquely to a right-K_0-invariant function, still denoted by $|\chi_P|$, on G. Let (π, V_π) be an irreducible (smooth) representation of M. Let δ_P be the modulus function of P. Consider the family of induced representations $I_P(\pi, s)$, $s \in \mathbb{C}$, of G which extend the K_0-representation $I_{P\cap K_0}^{K_0}(\pi|_{M\cap K_0})$. Namely, $I_P(\pi, s)$ is the space of all smooth functions $\varphi : G \to V_\pi$ with

$$\varphi(pg) = |\chi_P|(p)^s \delta_P(p)^{1/2} \pi(p)\varphi(g)$$

for all $p \in P$, $g \in G$, where π is extended to P via the canonical projection $P \to M$, and the G-action is given by right translations. Any smooth function $\varphi : K_0 \to V_\pi$ with $\varphi(pk) = \pi(p)\varphi(k)$ for all $k \in P \cap K_0$ extends uniquely to a function $\varphi_s \in I_P(\pi, s)$. Let $\pi^\vee$ be the contragredient of π and denote the pairing between V_π and $V_{\pi^\vee}$ by $(\cdot\,,\cdot)$. Then

$$(\varphi, \varphi^\vee) = \int_{K_0} (\varphi(k), \varphi^\vee(k))\, dk$$

defines a pairing between $I_P(\pi, s)$ and $I_P(\pi^\vee, -s)$. Fix a choice of Haar measure on $\bar{U}$. The intertwining operators $M(s) = M_{\bar{P}|P}(\pi, -s) : I_P(\pi, s) \to I_{\bar{P}}(\pi, s)$,[3] which are defined by the meromorphic continuation of the integrals

$$(M(s)\varphi)(g) = \int_{\bar{U}} \varphi(\bar{u}g)\, d\bar{u}, \quad \varphi \in I_P(\pi, s),$$

were first studied in this generality by Harish-Chandra. (See [Waldspurger 2003, Section IV] for a self-contained treatment.) It is known that the matrix coefficients $(M(s)\varphi_s, \varphi_s^\vee)$ for $\varphi \in I_{P\cap K_0}^{K_0}(\pi|_{M\cap K_0})$ and $\varphi^\vee \in I_{\bar{P}\cap K_0}^{K_0}(\pi^\vee|_{M\cap K_0})$ are rational functions of q^{-s} [Waldspurger 2003, IV.1.1] and that the degree of the denominator is bounded in terms of $\boldsymbol{G}$ only [Waldspurger 2003, IV.1.2]; see also [Shahidi 1981, Theorems 2.2.1, 2.2.2; Silberger 1979]. It is often advantageous to work instead with the normalized intertwining operators $R(s) = R_{\bar{P}|P}(\pi, s) : I_P(s) \to I_{\bar{P}}(-s)$ defined in [Arthur 1989], which differ from $M(s)$ by a certain rational function of q^{-s} depending on π whose degree is bounded in terms of G only. Thus, the matrix coefficients of $R(s)$ are also rational functions in q^{-s} and the degree of the denominator is bounded in terms of G.

Occasionally we will also consider intertwining operators for general (nonmaximal) parabolic subgroups containing $\boldsymbol{T}_0$. For this, let $\boldsymbol{M} \supset \boldsymbol{M}_0$ be a Levi subgroup of $\boldsymbol{G}$ and set $\mathfrak{a}^*_{M,\mathbb{C}} = X^*(\boldsymbol{M}) \otimes \mathbb{C}$, where $X^*(\boldsymbol{M})$ denotes the group of (F-rational) characters of $\boldsymbol{M}$. Then for any smooth irreducible representation π of M, we have the families of induced representations $I_P(\pi, \lambda)$, $\boldsymbol{P} \in \mathcal{P}(\boldsymbol{M})$, $\lambda \in \mathfrak{a}^*_{M,\mathbb{C}}$, and the associated intertwining operators $M_{P_2|P_1}(\pi, \lambda) : I_{P_1}(\pi, \lambda) \to I_{P_2}(\pi, \lambda)$ for pairs of parabolic subgroups $\boldsymbol{P}_1, \boldsymbol{P}_2 \in \mathcal{P}(\boldsymbol{M})$ [Waldspurger 2003, p. 278]. We can extend arbitrary functions $\varphi \in I_{P_1\cap K_0}^{K_0}(\pi|_{M\cap K_0})$ and $\varphi^\vee \in I_{P_2\cap K_0}^{K_0}(\pi^\vee|_{M\cap K_0})$ uniquely to functions $\varphi_\lambda \in I_{P_1}(\pi, \lambda)$ and $\varphi^\vee_{-\lambda} \in I_{P_2}(\pi^\vee, -\lambda)$, respectively, and the matrix coefficients $(M(\lambda)\varphi_\lambda, \varphi^\vee_{-\lambda})$ are rational functions of the variables $q^{-\langle\lambda,\alpha^\vee\rangle}$, $\alpha \in \Delta_{\boldsymbol{P}}$. Here $\Delta_{\boldsymbol{P}}$ is the set of simple roots of $\boldsymbol{U}$. The degree of the denominator is bounded in terms of $\boldsymbol{G}$ only. The normalized intertwining operator $R_{P_2|P_1}(\pi, \lambda)$ differs from

[3] Note that $I_{\bar{P}}(\pi, -s)$ is defined using $\chi_{\bar{P}}$ and $\delta_{\bar{P}}$ and that $\chi_{\bar{P}}|_M = \chi_P^{-1}|_M$ and $\delta_{\bar{P}}|_M = \delta_P^{-1}|_M$.

the operator $M_{P_2|P_1}(\pi, \lambda)$ by a normalizing scalar which is a rational function of $q^{-\langle\lambda,\alpha^\vee\rangle}$ of degree bounded in terms of $\boldsymbol{G}$ only.

Let $\mathfrak{g} = \operatorname{Lie} \boldsymbol{G}$ and denote by $\operatorname{Ad} : \boldsymbol{G} \to \operatorname{GL}(\mathfrak{g})$ the adjoint representation. Fix an $\mathcal{O}$-lattice $\Lambda \subset \mathfrak{g}$ stabilized by the operators $\operatorname{Ad}(k)$, $k \in K_0$, and define a norm on $\mathfrak{g}$ by $\|\sum_{i=1}^d t_i X_i\|_{\mathfrak{g}} = \max_{1\le i\le d}|t_i|$ for an (arbitrary) $\mathcal{O}$-basis $X_1, \dots, X_d$ of Λ. This defines a norm $\|\cdot\|_{\operatorname{End}(\mathfrak{g})}$ on $\operatorname{End}(\mathfrak{g})$; namely, $\|A\|_{\operatorname{End}(\mathfrak{g})}$ is the maximum of the absolute values of the matrix coefficients of A with respect to the basis $X_1, \dots, X_d$. For any $g \in G$, we write $\|g\|_G = \|\operatorname{Ad}(g)\|_{\operatorname{End}(\mathfrak{g})}$, and for any real number R we set $\mathcal{B}^G(R) = \{g \in G : \|g\|_G \le q^R\}$, which is a compact set modulo Z. We often omit the index G from $\|\cdot\|_G$ and $\mathcal{B}^G(R)$ if it is clear from the context.

In the global situation of a reductive group $\boldsymbol{G}$ defined over a number field k, we need of course to fix analogous global data that induce the local data pertaining to $\boldsymbol{G}(k_v)$ for the nonarchimedean places v of k. In particular, we fix an $\mathcal{O}_k$-lattice $\Lambda \subset \mathfrak{g}$ to define the local norms $\|\cdot\|_{\boldsymbol{G}(k_v)}$ via base change to $\mathcal{O}_{k_v}$. In the same way, we obtain the representation ρ_v and the lattice $\Lambda_{V_v} \subset V_v$ intervening in the definition of the groups $K_{n,v}$ from a representation $\rho : \boldsymbol{G} \to \operatorname{GL}(V)$ defined over k and an $\mathcal{O}_k$-lattice Λ_V in the k-vector space V. It is well known that $K_{0,v}$ is then hyperspecial for almost all v.

We write $A \ll B$ (or $B \gg A$) if there exists a constant c (independent of other quantities) such that $A \le cB$.

3. Variants of the conjectures

In this section we discuss some simple variants of Conjectures 1 and 2. In studying our conjectures, it is useful to restrict attention to the principal congruence subgroups K'_n of K'_0. This is possible by the following statement, which is a special case of [Lubotzky 1995, Lemma 1.6].

Proposition 3 (Lubotzky). *There exist constants c_0 and d_0 such that any open subgroup K of K_0 contains the principal congruence subgroup K'_n of G' for $n = \lfloor c_0 \log_q [K'_0 : K'] + d_0 \rfloor$. Moreover, if $\boldsymbol{G}$ is defined over a number field k and for any finite place v, $K_{0,v}$ is a special maximal compact subgroup of $\boldsymbol{G}(k_v)$, which is hyperspecial for almost all v, then for the pairs $(\boldsymbol{G}(k_v), K_{0,v})$, one may take uniform values of c_0 and d_0 (in fact, $c_0 = [k_v : \mathbb{Q}_p]$ works for almost all v).*

Remark 4. Note that in [Lubotzky 1995] it is assumed that $\boldsymbol{G}'$ is simply connected, and one can then take $d_0 = 0$. The general case follows easily by passing to the simply connected covering group of $\boldsymbol{G}'$.

Proposition 3 implies that equivalent forms of Conjectures 1 and 2 are obtained by replacing the index $[K'_0 : K']$ by the level of K', which is defined as

$$\operatorname{level}(K') := q^n,$$

where $n \geq 0$ is the smallest integer with $K' \supset K'_n$.

We now consider the generalization of our conjectures to arbitrary parabolic subgroups and the associated intertwining operators.

Proposition 5. *Suppose that Conjecture 1 is true for any Levi subgroup $\boldsymbol{L} \supset \boldsymbol{M}_0$ in place of $\boldsymbol{G}$. Then there exist constants $c > 0$ and d, depending only on $\boldsymbol{G}$, such that for any open subgroup $K \subset K_0$, the degrees of the numerators of the matrix coefficients of $M_{P_2|P_1}(\lambda)^K$, as rational functions of the variables $q^{-\langle\lambda,\alpha^\vee\rangle}$, $\alpha \in \Delta_{\boldsymbol{P}}$, are bounded by $c \log_q [K'_0 : K'] + d$.*

In the global situation of a reductive group $\boldsymbol{G}$ defined over a number field k, suppose that Conjecture 2 is true for all $\boldsymbol{L} \supset \boldsymbol{M}_0$. Then the degree bound above holds for the local groups $\boldsymbol{G}(k_v)$ and $K_{0,v}$ with uniform values of c and d as v ranges over the nonarchimedean places of k.

Proof. Let $\boldsymbol{P}_1 = \boldsymbol{Q}_0, \boldsymbol{Q}_1, \dots, \boldsymbol{Q}_l = \boldsymbol{P}_2$ be a sequence of adjacent parabolic subgroups from $\boldsymbol{P}_1$ to $\boldsymbol{P}_2$ and let $\Delta_{\boldsymbol{Q}_i} \cap \Delta_{\boldsymbol{Q}_{i+1}} = \{\alpha_i\}$. We can decompose $M_{P_2|P_1}(\pi, \lambda)$ into a product of rank-one intertwining operators $M_{Q_{i+1}|Q_i}(\pi, \langle\lambda, \alpha_i^\vee\rangle)$. Thus, it is enough to consider the degrees of the matrix coefficients of $M_{Q_{i+1}|Q_i}(\sigma, \langle\lambda, \alpha_i^\vee\rangle)^K$, $i = 0, \dots, l-1$. Fix i and let $\boldsymbol{R} = \boldsymbol{M_R N_R}$ be the parabolic subgroup generated by $\boldsymbol{Q}_i$ and $\boldsymbol{Q}_{i+1}$. Let $\boldsymbol{Q}' = \boldsymbol{M_R} \cap \boldsymbol{Q}_i$ and $\boldsymbol{Q}'' = \boldsymbol{M_R} \cap \boldsymbol{Q}_{i+1}$. Then $\boldsymbol{Q}'$ and $\boldsymbol{Q}''$ are maximal parabolic subgroups of $\boldsymbol{M_R}$ with Levi subgroup $\boldsymbol{M}$ and $\boldsymbol{Q}'' = \overline{\boldsymbol{Q}'}$. By [Waldspurger 2003, p. 284, (14)], the matrix coefficients of $M_{Q_{i+1}|Q_i}(\sigma, \langle\lambda, \alpha_i^\vee\rangle)^K$ are given by those of $M_{\overline{Q'}|Q'}(\sigma, \langle\lambda, \alpha_i^\vee\rangle)^{K\cap M_R}$, and the degrees of the latter coefficients satisfy by assumption the bounds of Conjectures 1 and 2. □

Finally, it is clear that we can replace the intertwining operators $M(s)$ and $M(\lambda)$ by the normalized intertwining operators $R(s)$ and $R(\lambda)$ in Conjectures 1 and 2 and Proposition 5. In fact, we can obtain slightly stronger statements for the normalized operators. If we replace $M(s)$ by $R(s)$ in Conjecture 1, and in addition $\boldsymbol{G}$ is unramified and K_0 hyperspecial, then we may take $d = 0$, since any representation which admits a K'_0-fixed vector is a twist by a character of G/G' of an unramified representation of G. Similarly, by Remark 4, we may take $d = 0$ in the analog of Conjecture 2 for $R(s)$, if $\boldsymbol{G}'$ is simply connected and we omit the finitely many places v where $\boldsymbol{G}(k_v)$ is ramified or $K_{0,v}$ not hyperspecial. The same remarks apply to Proposition 5. If we consider here $\operatorname{level}(K')$ instead of $[K'_0 : K']$, then we do not need to make any additional assumption on G', since trivially $\log_q \operatorname{level}(K') \geq 1$ whenever $K' \neq K'_0$. We record the resulting variant of Proposition 5 explicitly, since we intend to use the statement in another paper.

Proposition 6. *Suppose that Conjecture 1 is true for any Levi subgroup $\boldsymbol{L} \supset \boldsymbol{M}_0$ of $\boldsymbol{G}$. Then there exists a constant $c > 0$, depending only on $\boldsymbol{G}$, such that for any open subgroup $K \subset K_0$, the degrees of the numerators of the matrix coefficients of $R_{P_2|P_1}(\lambda)^K$, as rational functions of the variables $q^{-\langle\lambda,\alpha^\vee\rangle}$, $\alpha \in \Delta_{\boldsymbol{P}}$, are* *bounded by* $c \log_q \operatorname{level}(K')$ *if* $\boldsymbol{G}$ *is unramified and* K_0 *is hyperspecial*, *and by* $c(\log_q \operatorname{level}(K') + 1)$ *otherwise.*

In the global situation of a reductive group $\boldsymbol{G}$ *defined over a number field* k, *suppose that Conjecture 2 is true for all* $\boldsymbol{L} \supset \boldsymbol{M}_0$. *Then the degree bound above for the numerators of the matrix coefficients of* $R_{P_2|P_1}(\lambda)^K$ *holds with a uniform value of* c *for all local groups* $\boldsymbol{G}(k_v)$ *and* $K_{0,v}$ *as* v *ranges over the nonarchimedean places of* k.

4. Matrix coefficients of supercuspidal representations

Definition 7. We say that G has *polynomially bounded support of supercuspidal matrix coefficients* (PSC) if there exist constants c and d such that for every open subgroup $K \subset K_0$ and any supercuspidal representation π of G, the support of the matrix coefficients $(\pi(g)v, v^\vee)$, $v \in \pi^K$, $v^\vee \in (\pi^\vee)^K$, is contained in $\mathcal{B}(c \log_q [K_0' : K'] + d)$.

Note that property (PSC) is independent of the choice of K_0, which could be replaced by an arbitrary open compact subgroup of G. However, the possible values of the constants c and d will depend on K_0 (and the norm $\|\cdot\|_G$ on $\mathfrak{g}$).

Conjecture 3. *Every* p*-adic reductive group* G *has property* (PSC).

We will show that this conjecture is true in a large number of cases. In addition, we will obtain a global uniformity statement for the constants c and d for reductive groups $\boldsymbol{G}$ defined over number fields k and almost all of the associated local groups $\boldsymbol{G}(k_v)$ (see Corollary 13 below).

Let L be an open subgroup of G containing Z such that L/Z is compact. We refer to such subgroups as *open compact modulo center (ocmc)* for short. We say that a finite-dimensional representation σ of L is *cuspidal* if for every proper parabolic subgroup $\boldsymbol{P}$ of $\boldsymbol{G}$ with unipotent radical $\boldsymbol{U}$, we have $\sigma^{L \cap U} = 0$. Here, it clearly suffices to consider only maximal parabolic subgroups. By [Bushnell 1990, Theorem 1 supp.], this condition is necessary (and in fact also sufficient, by Lemma 8 below) for $\operatorname{Ind}_L^G \sigma$ to be of finite length, in which case it is the direct sum of finitely many irreducible supercuspidal representations. Note that if σ is cuspidal, then its contragredient $\sigma^\vee$ is cuspidal as well. We say that a supercuspidal representation π of G is *induced from an ocmc*, if there exists a pair (L, σ) where L is an ocmc and $\sigma \in \hat{L}$, necessarily cuspidal, such that $\pi = \operatorname{Ind}_L^G \sigma$.

It is widely believed that every irreducible supercuspidal representation π is induced from an ocmc,[4] and in fact this is known in many cases (see [Bushnell and Kutzko 1993a; Kim 2007; Stevens 2008; Yu 2001], and earlier work by Howe,

[4]We were unable to trace back who precisely formulated the conjecture in this generality, but it certainly goes back to the early days of the representation theory of p-adic groups.

Morris, Moy and others). For our purposes it suffices to know that π is a constituent of $\operatorname{Ind}_L^G \sigma$ for some cuspidal σ.

Lemma 8. *Let L be an ocmc. Then there exist constants c, depending only on G, and d, depending on L, such that for any cuspidal $\sigma \in \hat{L}$, any open subgroup $K \subset K_0$ and any $f \in (\operatorname{Ind}_L^G \sigma)^K$ we have* $\operatorname{supp}(f) \subset \mathcal{B}(c \log_q [K_0' : K'] + d)$.

Proof. Note first that the assertion is trivial if $\boldsymbol{G}'$ is anisotropic, since G/Z is then compact. So, we may assume that the F-rank of $\boldsymbol{G}'$ is nonzero. By Lubotzky's result (Proposition 3 above), we may assume without loss of generality that K' is a principal congruence subgroup K_n' of G'. In particular, K' is normal in K_0.

Let $g \in G$ and write its Cartan decomposition as $g = k_1 a k_2 \in G$ with $k_1, k_2 \in K_0$ and $a \in M_0^+$. We first show that there are constants c and d such that $\|g\| > q^{cn+d}$ implies the existence of a standard maximal parabolic subgroup $\boldsymbol{P} = \boldsymbol{M}\boldsymbol{U}$ of $\boldsymbol{G}$ satisfying

$$U \cap k^{-1} L k \subset a(U \cap K)a^{-1} \quad \text{for all } k \in K_0. \tag{1}$$

Assume that $\|g\| = \|a\| > q^{cn+d}$ for some $c > 0$ and d which will be specified later. Note first that there are only finitely many K_0-conjugates of the group L, and that their intersections with U_0 generate an open compact subgroup $V_0(L)$ of U_0. Using the exponential map, we can identify $\boldsymbol{U}_0$ with its Lie algebra, which is an affine space. Fixing a norm on U_0, we let $U_0(n)$ be the lattice consisting of the elements of U_0 of norm bounded by q^n and set $U(n) = U_0(n) \cap U$ for any standard parabolic subgroup $P = MU$ of G. Clearly, there exists a constant $n_0 = n_0(L)$ such that $V_0(L)$ is contained in $U_0(n_0)$, and therefore the left-hand side of (1) is contained in $U(n_0)$ for all $k \in K_0$.

Let $\beta \in \Delta_0$ with $|\beta(a)| = \max_{\alpha\in\Delta_0} |\alpha(a)|$. There exist constants $c_1 > 0$ and n_1 such that $\max_{\alpha\in\Delta_0\cup-\Delta_0} |\alpha(b)| \geq q^{-n_1}\|b\|^{c_1}$ for any $b \in M_0$. Therefore, we obtain from $|\alpha(a)| \geq 1$, $\alpha \in \Delta_0$, and $\|a\| > q^{cn+d}$ that $|\beta(a)| > q^{c_1cn+c_1d-n_1}$, which implies in turn that $|\alpha(a)| > q^{c_1cn+c_1d-n_1}$ for all roots $\alpha \in R(\boldsymbol{T}_0, \boldsymbol{U}^\beta)$. There also exists a constant n_2 such that $U^\beta \cap K = U^\beta \cap K_n'$ contains $U^\beta(-n-n_2)$, which implies that $a(U^\beta \cap K)a^{-1}$ contains $U^\beta(c_1cn + c_1d - n_1 - n - n_2)$. It is therefore sufficient to take $c = c_1^{-1}$ and $d = c_1^{-1}(n_0 + n_1 + n_2)$ to obtain (1) for $P = P^\beta$.

Let now $\pi = \operatorname{Ind}_L^G \sigma$. For an arbitrary element $f \in \pi^K$, set $f_2 = \pi(k_2) f \in \pi^{K'}$. For any $u \in U \cap K = U \cap K'$, we have

$$f(g) = f_2(k_1 a) = f_2(k_1 a u) = f_2(u' k_1 a),$$

where $u' = k_1 a u a^{-1} k_1^{-1}$. If in addition $u' \in k_1 U k_1^{-1} \cap L$, then we get $f(g) = \sigma(u') f_2(k_1 a) = \sigma(u') f(g)$. Using (1) and the cuspidality of σ, we conclude that $f(g) \in \sigma^{k_1 U k_1^{-1} \cap L} = 0$. □

Remark 9. The qualitative statement that in the situation of the lemma any element of $\mathrm{Ind}_L^G \sigma$ has compact support modulo the center is contained in [Bushnell 1990, Theorem 1 supp.] in the case $\boldsymbol{G} = \mathrm{GL}(r)$. The argument is originally due to Jacquet [1971].

Corollary 10. *There exist constants c' and d' with the following property. Let L be an ocmc of G, σ be a cuspidal representation of L, and $\pi = \mathrm{Ind}_L^G \sigma$. Let $K \subset K_0$ be open and let $v \in \pi^K$ and $v^\vee \in (\pi^\vee)^K$. Then the support of $(\pi(g)v, v^\vee)$ is contained in $\mathcal{B}(c' \log_q [K_0' : K'] + d')$.*

Proof. Clearly, if σ is a cuspidal representation of an ocmc L_1 and $L \supset L_1$ is a larger ocmc, then $\mathrm{Ind}_{L_1}^L \sigma$ is a cuspidal representation of L [Bushnell 1990]. We can therefore assume that L is a maximal ocmc. In other words, denoting by $\boldsymbol{T_G}$ the maximal F-split torus of $\boldsymbol{Z}$, L is the inverse image under the projection $G \to G/T_G$ of a maximal compact subgroup of G/T_G, which is also the group of F-points of the algebraic group $\boldsymbol{G}/\boldsymbol{T_G}$, since the first Galois cohomology group of $\boldsymbol{T_G}$ is trivial. There are finitely many such subgroups L up to G-conjugation [Tits 1979, §3.2]. It follows from the previous lemma that for suitable positive constants c and d, the supports S and $S^\vee$ of $v \in \pi^K$ and $v^\vee \in (\pi^\vee)^K$, respectively, are both contained in $\mathcal{B}(c \log_q [K_0' : K'] + d)$. However, $(\pi(g)v, v^\vee) = 0$ whenever the support of $\pi(g)v$ is disjoint from the support of $v^\vee$, or equivalently whenever $g \notin (S^\vee)^{-1} S$. Observing that there exists a positive constant c_1 such that $\mathcal{B}(N)^{-1}\mathcal{B}(N) \subset \mathcal{B}(c_1 N)$ for all $N > 0$, we conclude that the support of the matrix coefficient $(\pi(g)v, v^\vee)$ is contained in $\mathcal{B}(c_1 c \log_q [K_0' : K'] + c_1 d)$. □

Remark 11. The proof shows also that in the global situation of a reductive group $\boldsymbol{G}$ defined over a number field k, there exist uniform constants c and d such that the assertion of the corollary is true for all local groups $\boldsymbol{G}(k_v)$, v a nonarchimedean place of k, and maximal compact subgroups $K_{0,v}$ that are hyperspecial for almost all v. One only needs to observe that every maximal compact subgroup of G/T_G is conjugate to a maximal compact subgroup $\tilde{L}$ containing a fixed Iwahori subgroup I [Tits 1979, §3.7]. Moreover, the index $[\tilde{L} : I]$ is bounded by q^N, where N does not depend on v. From this, we deduce that the constant n_0 in the proof of Lemma 8 can be bounded independently of v, if the norm on $U_0 = \boldsymbol{U}_0(k_v)$ used in the proof is induced from the choice of a fixed $\mathcal{O}_k$-lattice in the Lie algebra of $\boldsymbol{U}_0$. The boundedness of all other constants is clear.

Remark 12. The maximal ocmcs of $\mathrm{GL}(r, F)$ are (up to conjugation) parametrized by divisors of r. They can be realized as stabilizers of sequences L_i, $i \in \mathbb{Z}$, of $\mathcal{O}$-lattices in F^r such that $L_{i+l} = \varpi L_i$ and $\dim_{\mathbb{F}_q} L_i / L_{i+1} = k$ for all i, where k is a divisor of r and $kl = r$. Note that this stabilizer is the semidirect product of the parahoric subgroup of type $(k, \dots, k)$ with the cyclic group generated by an element z_l of $\mathrm{GL}(r, F)$ with $z_l L_i = L_{i+1}$ [Carayol 1984].

Corollary 13. *Assume that every supercuspidal representation of G is contained in a representation induced from a cuspidal representation of an ocmc. Then G has property* (PSC). *In particular, the following groups have property* (PSC):

(1) $G = \mathrm{GL}(r, F)$ [Bushnell and Kutzko 1993a],

(2) $G = \mathrm{SL}(r, F)$ [Bushnell and Kutzko 1993b],

(3) $\boldsymbol{G}(F)$ *for classical groups* $\boldsymbol{G}$, *provided* $p \neq 2$ [Stevens 2008], *and*

(4) $\boldsymbol{G}(k_v)$ *for any reductive group* $\boldsymbol{G}$ *defined over a number field k and almost all nonarchimedean places v of k* [Kim 2007]. *Moreover, if the maximal compact subgroups $K_{0,v}$ of $\boldsymbol{G}(k_v)$ are hyperspecial for almost all v, then there are uniform constants c and d for which $\boldsymbol{G}(k_v)$ has property* (PSC) *with respect to $K_{0,v}$ for almost all v.*

Remark 14. A general finiteness theorem of Bernstein [Bernstein 1974] (see also [Bernstein and Zelevinskii 1976; Bushnell 1990, p. 110]) shows (without appealing to any classification results) that for any open subgroup K of K_0, there are, up to twisting by unramified characters, only finitely many supercuspidal representations π of G with a nontrivial K-fixed vector. Therefore, there necessarily exists a number $N = N(K)$ such that the support of all matrix coefficients $(\pi(g)v, v^\vee)$, $v \in \pi^K$, $v^\vee \in (\pi^\vee)^K$, is contained in $\mathcal{B}(N)$. To prove property (PSC) predicted by Conjecture 3 this way, it seems necessary to obtain an effective version of Bernstein's stabilization theorem (see [Bushnell 2001, Theorem 1]) with a realistic bound for the exponent n_K, namely a bound that is logarithmic in $[K_0' : K']$.

5. A class of parabolic subgroups

Definition 15. We say that a maximal parabolic subgroup $\boldsymbol{P} = \boldsymbol{M}\boldsymbol{U}$ is *nice* if there exists a positive constant c such that for all $n > 0$, we have

$$\bar{U} \cap UZ(M)\mathcal{B}(n) \subset \begin{cases} \mathcal{B}(cn) \cup Pw_0K_n & \text{if } w_0\boldsymbol{M}w_0^{-1} = \boldsymbol{M}, \\ \mathcal{B}(cn), & \text{otherwise.} \end{cases} \tag{2}$$

In other words, $\boldsymbol{P}$ is nice if in a precise quantitative sense, for a compact subset Ω of G, either $\bar{U} \cap UZ(M)\Omega$ is bounded in terms of Ω, or $\boldsymbol{P}^{w_0} = \bar{\boldsymbol{P}}$ and for a small open compact subgroup $K = K(\Omega)$ of G the set $\bar{U} \cap UZ(M)\Omega \setminus Pw_0K$ is bounded in terms of Ω.

Our main result concerning this property is the following.

Proposition 16. *Suppose that $\boldsymbol{G}$ is split and $\boldsymbol{U}$ is abelian. Then $\boldsymbol{P}$ is nice. Moreover, if $\boldsymbol{G}$ is defined and split over a number field k, then there is a uniform constant $c > 0$ such that* (2) *is satisfied for all local groups $\boldsymbol{G}(k_v)$, where v is a nonarchimedean place of k.*

The assumption that $\boldsymbol{G}$ is split is mainly for convenience and can probably be suppressed. For the convenience of the reader, we first present a proof in the case of $\boldsymbol{G} = \mathrm{GL}(r)$, where we can simplify the argument by direct matrix computations. The general case will be dealt with in Section 7 below.

Lemma 17. *For $\boldsymbol{G} = \mathrm{GL}(r)$, all maximal parabolic subgroups are nice.*

Proof. To fix ideas, we define the norm of elements of G and the sets $\mathcal{B}(n)$ with respect to the standard $\mathbb{O}$-lattice in $\mathfrak{g}$ spanned by the elementary matrices. With this normalization, we will obtain (2) for $c = 2(r+1)$. For a matrix X over F we write $\|X\|$ (to be distinguished from $\|g\|_G$ for invertible g) for the standard norm of X, that is, the maximum of the absolute values of its entries.

Let P be of type (m', m). We may assume without loss of generality that $m \geq m'$, for otherwise we can apply the automorphism $g \mapsto w_0 {}^t g^{-1} w_0$ of $\boldsymbol{G}$. Let

$$\bar{u} = \begin{pmatrix} I_{m'} & \\ X & I_m \end{pmatrix}$$

and suppose that

$$\bar{u} = \begin{pmatrix} \lambda I_{m'} & * \\ & \mu^{-1} I_m \end{pmatrix} g, \quad \lambda, \mu \in F^*, \quad g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathcal{B}(n).$$

Note that $\|\bar{u}\|_G \leq \|X\|^2$. Modifying g by a central element (and modifying λ and μ accordingly), we can assume that $1 \leq |\det g| < q^r$. Then it is easy to see that the absolute values of the entries of g are bounded by q^n. Note that $\gamma = \mu X$ and $\delta = \mu I_m$. In particular, we have $\|X\| \leq q^n |\mu|^{-1}$.

Suppose first that $m > m'$. Expanding $\det g$ as an alternating sum of products of entries of g, we see that each product contains at least one entry (in fact, at least $m - m'$ entries) from δ as a factor. Thus $1 \leq |\det g| \leq q^{(r-1)n} |\mu|$, which implies $|\mu| \geq q^{-(r-1)n}$, and therefore $\|X\| \leq q^{rn}$ and $\|\bar{u}\|_G \leq q^{2rn}$.

Suppose now that $m = m'$. We distinguish the two cases $|\mu| > q^{-rn}$ and $|\mu| \leq q^{-rn}$. In the first case, we have $\|X\| \leq q^{(r+1)n}$ and $\|\bar{u}\|_G \leq q^{2(r+1)n}$. Assume therefore that $|\mu| \leq q^{-rn}$. The products in the expansion of $\det g$ which do not contain an entry from δ as a factor add up to $(-1)^m \det \beta \det \gamma$. Therefore,

$$\left| \det g - (-1)^m \det \beta\gamma \right| \leq |\mu| q^{(r-1)n} \leq q^{-n}.$$

On the other hand, we have $|\det g| \geq 1$. Therefore $|\det g| = |\det \beta\gamma|$. In particular, γ is invertible and

$$|\det \gamma|^{-1} = |\det \beta\gamma|^{-1} |\det \beta| \leq |\det g|^{-1} q^{mn} \leq q^{mn}.$$

It follows that X is invertible and

$$\|X^{-1}\| = |\mu| \|\gamma^{-1}\| \leq |\mu| |\det \gamma|^{-1} \|\gamma\|^{m-1} \leq |\mu| q^{(r-1)n} \leq q^{-n}.$$

Finally, the identity

$$\bar{u} = \begin{pmatrix} X^{-1} & I_m \\ & X \end{pmatrix} \begin{pmatrix} & -I_m \\ I_m & \end{pmatrix} \begin{pmatrix} I_m & X^{-1} \\ & I_m \end{pmatrix}$$

shows that $\bar{u} \in P w_0 K_n$. □

Remark 18. While there are other cases of nice parabolic subgroups (for example, the maximal parabolic subgroups of Sp(4)), unfortunately not all maximal parabolic subgroups are nice. As an example, consider

$$\boldsymbol{G} = \mathrm{Sp}(6) = \left\{ g \in \mathrm{GL}(6) : g \begin{pmatrix} &&&&&1 \\ &&&&1& \\ &&&1&& \\ &&-1&&& \\ &-1&&&& \\ -1&&&&& \end{pmatrix} g^t = \begin{pmatrix} &&&&&1 \\ &&&&1& \\ &&&1&& \\ &&-1&&& \\ &-1&&&& \\ -1&&&&& \end{pmatrix} \right\}$$

and let $\boldsymbol{P}$ be the maximal parabolic subgroup of the form $\boldsymbol{P} = \left\{ \begin{pmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * & * \end{pmatrix} \in \boldsymbol{G} \right\}$.

The equality

$$\begin{pmatrix} 1 &&&&& \\ & 1 &&&& \\ && 1 &&& \\ a &&& 1 && \\ -a &&&& 1 & \\ & -a & a &&& 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 &&&& -a^{-1} & \\ & 1 & 1 &&& -a^{-1} \\ && 1 &&& \\ &&& 1 & -1 & \\ &&&& 1 & \\ &&&&& 1 \end{pmatrix} \begin{pmatrix} a^{-1} &&&&& \\ & a^{-1} &&&& \\ && 1 &&& \\ &&& 1 && \\ &&&& a & \\ &&&&& a \end{pmatrix} \begin{pmatrix} 1 &&&&& \\ & 1 &&&& \\ && 1 &&& \\ 1 &&& 1 && \\ a^{-1} &&&& 1 & \\ & a^{-1} & 1 &&& 1 \end{pmatrix} \begin{pmatrix} &&&& 1 & \\ &&&&& 1 \\ && 1 &&& \\ &&& 1 && \\ -1 &&&&& \\ & -1 &&&& \end{pmatrix}$$

shows that

$$\begin{pmatrix} 1 &&&&& \\ & 1 &&&& \\ && 1 &&& \\ a &&& 1 && \\ -a &&&& 1 & \\ & -a & a &&& 1 \end{pmatrix} \in \bar{U} \cap U Z(M) K_0$$

for all $a \in F$. However, if $\begin{pmatrix} * & * & * \\ * & * & * \\ A & B & C \end{pmatrix} \in P w_0 K_n$ (with blocks of size 2×2), then $\|A^{-1}B\| \le q^{-n}$.

6. Matrix coefficients of intertwining operators

We now consider Conjectures 1 and 2 stated in the introduction, and prove some results in this direction. In particular, we prove Theorem 1.

Definition 19. Let $\boldsymbol{P}$ be a maximal parabolic subgroup of $\boldsymbol{G}$. We say G has *polynomial growth of matrix coefficients of intertwining operators* (PIO) with respect to P if there exist constants c and d such that for any open subgroup $K \subset K_0$ and any irreducible representation π of M, the degrees of the numerators of the linear operators $M_{\bar{P}|P}(\pi, s)^K$ are bounded by $c \log_q [K_0' : K'] + d$.

If this property is satisfied for all *supercuspidal* irreducible representations π of M, we say that G has *polynomial growth of supercuspidal matrix coefficients of intertwining operators* (PSIO) with respect to P.

Conjecture 1 amounts to the assertion that every p-adic reductive group G satisfies property (PIO). It is easy to see that we can replace (PIO) by the weaker condition (PSIO). More precisely, we have the following.

Lemma 20. *Suppose that any Levi subgroup $L \supset M_0$ of G (including G itself) satisfies* (PSIO). *Then G satisfies (PIO).*

Proof. We argue as in the proof of Proposition 5. Let π be an irreducible representation of M. By the Jacquet subrepresentation theorem, we can embed π in an induced representation $I^M_{Q\cap M}(\sigma)$ for a parabolic subgroup $\boldsymbol{Q} \subset \boldsymbol{P}$ of $\boldsymbol{G}$ with Levi subgroup $\boldsymbol{L} \subset \boldsymbol{M}$ and an irreducible supercuspidal representation σ of L. Consider the intertwining operators $M_{S_2|S_1}(\sigma, \lambda) : I_{S_1}(\sigma, \lambda) \to I_{S_2}(\sigma, \lambda)$, $\lambda \in \mathfrak{a}^*_{L,\mathbb{C}}$, for parabolic subgroups $\boldsymbol{S}_1$, $\boldsymbol{S}_2 \in \mathcal{P}(\boldsymbol{L})$. The embedding of π into $I^M_{Q\cap M}(\sigma)$ gives rise to an embedding of $I_P(\pi, s)$ into $I_Q(\sigma, s\chi_P)$, and the restriction of $M_{\overline{Q}|Q}(\sigma, s\chi_P)$ to $I_P(\pi, s)$ becomes $M(\pi, s)$. We will bound the degrees of the matrix coefficients of $M(\sigma, s\chi_P)^K$. Let $\boldsymbol{Q} = \boldsymbol{Q}_0, \boldsymbol{Q}_1, \dots, \boldsymbol{Q}_l = \overline{\boldsymbol{Q}}$ be a sequence of adjacent parabolic subgroups from $\boldsymbol{Q}$ to $\overline{\boldsymbol{Q}}$, and suppose that $\Delta_{\boldsymbol{Q}_i} \cap \Delta_{\boldsymbol{Q}_{i+1}} = \{\alpha_i\}$. We can decompose $M(\sigma, s\chi_P)$ into a product of rank-one intertwining operators $M_{Q_{i+1}|Q_i}(\sigma, s\langle\chi_P, \alpha_i^\vee\rangle)$. Therefore, it is enough to consider the degrees of the matrix coefficients of $M_{Q_{i+1}|Q_i}(\sigma, s\langle\chi_P, \alpha_i^\vee\rangle)^K$, $i = 0, \dots, l-1$. Fix i and let $\boldsymbol{R} = \boldsymbol{M_R N_R}$ be the parabolic subgroup generated by $\boldsymbol{Q}_i$ and $\boldsymbol{Q}_{i+1}$. Let $\boldsymbol{Q}' = \boldsymbol{M_R} \cap \boldsymbol{Q}_i$ and $\boldsymbol{Q}'' = \boldsymbol{M_R} \cap \boldsymbol{Q}_{i+1}$. Then $\boldsymbol{Q}'$ and $\boldsymbol{Q}''$ are maximal parabolic subgroups of $\boldsymbol{M_R}$ with Levi subgroup $\boldsymbol{L}$ and $\boldsymbol{Q}'' = \overline{\boldsymbol{Q}'}$. By [Waldspurger 2003, p. 284, (14)], the matrix coefficients of $M_{Q_{i+1}|Q_i}(\sigma, s\langle\chi_P, \alpha_i^\vee\rangle)^K$ are given by those of $M_{\overline{Q'}|Q'}(\sigma, s\langle\chi_P, \alpha_i^\vee\rangle)^{K\cap M_R}$. The lemma follows. $\square$

Theorem 21. *Suppose that $\boldsymbol{P} = \boldsymbol{MU}$ is a nice maximal parabolic subgroup of $\boldsymbol{G}$ and that M satisfies property* (PSC). *Then G satisfies* (PSIO) *with respect to P.*

Proof. Let π be a supercuspidal representation of M. Assume that $K' = K'_n$, $n > 0$, a normal subgroup of K_0. Let

$$\varphi \in I^{K_0}_{P\cap K_0}(\pi|_{M\cap K_0})^{K'_n} \quad \text{and} \quad \varphi^\vee \in I^{K_0}_{\bar{P}\cap K_0}(\pi^\vee|_{M\cap K_0})^{K'_n}.$$

This is equivalent to $\varphi(k) \in \pi^{M\cap K'_n}$ and $\varphi^\vee(k) \in (\pi^\vee)^{M\cap K'_n}$ for all $k \in K_0$. We extend these functions to functions $\varphi_s \in I_P(\pi, s)$ and $\varphi_s^\vee \in I_{\bar{P}}(\pi^\vee, s)$. Then the matrix coefficient $(M(\pi, s)\varphi_s, \varphi_s^\vee)$ can be computed as

$$\big(M(\pi, s)\varphi_s, \varphi_s^\vee\big) = \int_{K_0} \big((M(\pi, s)\varphi_s)(k), \varphi^\vee(k)\big)\, dk = \int_{\bar{U}} |\chi_P|(\bar{u})^s f(\bar{u})\, d\bar{u},$$

with

$$f(\bar{u}) = \int_{K_0} \big(\varphi_0(\bar{u}k), \varphi^\vee(k)\big)\, dk.$$

Note that f is right $\bar{U} \cap K_n'$-invariant. Since M satisfies property (PSC), there is a constant $c_1 > 0$ such that the matrix coefficients $(\pi(m)\varphi(k'), \varphi^\vee(k))$, $m \in M$, $k, k' \in K_0$, all vanish for $m \notin \mathcal{B}^M(c_1 n)$. Furthermore, there exists a constant $c_2 > 0$ with $\mathcal{B}^M(l) \subset Z(M)\mathcal{B}(c_2 l)$ for all $l > 0$. Applying the Iwasawa decomposition to $\bar{u}$, it follows that the support of f is contained in $\bar{U} \cap UZ(M)\mathcal{B}(c_1c_2n)$. Consider first the case where $\boldsymbol{P}^{w_0} \neq \bar{\boldsymbol{P}}$. Because $\boldsymbol{P}$ is nice, we conclude from the above that the support of f is contained in $\bar{U} \cap \mathcal{B}(cc_1c_2n)$ for the constant c of Definition 15. Thus, up to a constant, the integral becomes a finite sum

$$\sum_{\bar{u} \in \bar{U} \cap \mathcal{B}(cc_1c_2n)/\bar{U} \cap K_n'} |\chi_P|(\bar{u})^s f(\bar{u}),$$

which is a polynomial in q^{-s} of degree at most $-\log_q \min_{\bar{U} \cap \mathcal{B}(cc_1c_2n)} |\chi_P| \ll n$.

We still need to consider the case $\boldsymbol{P}^{w_0} = \bar{\boldsymbol{P}}$. Let ω_π be the central character of π. We take an element $a \in Z(M)$ as follows. If

$$\omega_\pi\big|_{Z(M)^1} \not\equiv \omega_{w_0\pi}\big|_{Z(M)^1} \tag{3}$$

then we take any $a \in Z(M)^1 = Z(M) \cap K_0$ such that $\omega_\pi(a) \neq \omega_\pi(b)$ where $b = w_0^{-1} a w_0 \in Z(M)$. Otherwise we take a which generates $T_0 \cap Z(M)$ modulo $Z(G)Z(M)^1$ and for which $|\chi_P|(a) = |\alpha(a)|^{\frac{1}{2}} = q^{-m} < 1$. We have $m \in \frac{1}{2}\mathbb{Z}_{>0}$.

We take $n_0 \geq 0$ such that $K_n' \cap bK_n'b^{-1} \supset K_{n+n_0}'$ and $Z(G)K_n' \supset Z(G)K_{n+n_0}$ for all n.

Note that under the action of K_0 the space $I_{\bar{P} \cap K_0}^{K_0}(\pi^\vee|_{M \cap K_0})^{K_{n+n_0}'}$ is spanned by functions $\varphi^\vee$ with support $(P \cap K_0)K_{n+n_0}'$. Thus, we can assume that $\varphi^\vee$ has this property. Hence, $\varphi^\vee$ is determined by its value at the identity and

$$(M(\pi, s)\varphi_s, \varphi_s^\vee) = c(M(\pi, s)\varphi_s(e), \varphi^\vee(e)) = c \int_{\bar{U}} |\chi_P|(\bar{u})^s (\varphi_0(\bar{u}), \varphi^\vee(e))\, d\bar{u}$$

for some constant c. If φ vanishes at w_0, then the last integrand vanishes on $\bar{U} \cap Pw_0K_n' \supset \bar{U} \cap Pw_0K_{n+n_0}$, and we can argue as in the case $\boldsymbol{P}^{w_0} \neq \bar{\boldsymbol{P}}$ above.

Otherwise, observe that

$$\begin{aligned}
(M(\pi, s)I_P(b, s)\varphi_s, \varphi_s^\vee) &= (I_{\bar{P}}(b, -s)M(\pi, s)\varphi_s, \varphi_s^\vee) \\
&= c(M(\pi, s)\varphi_s(b), \varphi^\vee(e)) = \delta_{\bar{P}}^{\frac{1}{2}}(b)\omega_s(b)(M(\pi, s)\varphi_s, \varphi_s^\vee) \\
&= \delta_P^{\frac{1}{2}}(a)\omega_s(b)(M(\pi, s)\varphi_s, \varphi_s^\vee),
\end{aligned}$$

where ω_s is the character $\omega_\pi |\chi_{\bar{P}}|^{-s} = \omega_\pi |\chi_P|^s$ of $Z(M)$. Thus, if we consider the operator

$$\Delta_{a,s} = \omega_s(b^{-1})\delta_P^{-\frac{1}{2}}(a) I(b,s) - \omega_s(b^{-1}a)\,\mathrm{Id}$$

on $I_P(\pi, s)$ then $\Delta_{a,s}\varphi_s$ vanishes at w_0, while

$$(M(\pi,s)\Delta_{a,s}\varphi_s, \varphi_s^\vee) = (1 - \omega_s(b^{-1}a))(M(\pi,s)\varphi_s, \varphi_s^\vee).$$

If condition (3) holds then

$$(M(\pi,s)\varphi_s, \varphi_{-s}^\vee) = (1 - \omega_\pi(b^{-1}a))^{-1}(M(\pi,s)\Delta_{a,s}\varphi_s, \varphi_s^\vee),$$

and since $\Delta_{a,s}\varphi_s \in I_P(\pi,s)^{K'_n}$, we reduce to the previous case. Otherwise,

$$(M(\pi,s)\varphi_s, \varphi_{-s}^\vee) = (1 - \omega_\pi(b^{-1}a)q^{-2ms})^{-1}(M(\pi,s)\Delta_{a,s}\varphi_s, \varphi_s^\vee)$$

and $\Delta_{a,s}\varphi_s \in I_P(\pi,s)^{K'_{n+n_0}}$. So once again, we reduce to the previous case. □

Remark 22. The argument also gives a simple proof of the rationality of $M(\pi, s)$ for supercuspidal π and nice $\boldsymbol{P}$. More precisely, it shows that $M(\pi, s)$ is a polynomial in q^{-s} if either $\boldsymbol{P}^{w_0} \neq \bar{\boldsymbol{P}}$ or $\omega_\pi \omega_{w_0\pi}^{-1}|_{Z(M)^1} \neq 1$. Otherwise,

$$\left(1 - \omega_\pi(w_0^{-1}a^{-1}w_0 a)q^{-2ms}\right)$$

is a polynomial in q^{-s}, where a and m are as above.

Remark 23. In the global situation of Conjecture 2, the proof shows that the constants c and d appearing in the definition of property (PSIO) can be chosen independently of the nonarchimedean place v, if this is the case for the constants appearing in Definition 7 (definition of property (PSC)) and Definition 15. By the fourth part of Corollary 13, for property (PSC) this uniformity statement is always satisfied after omitting finitely many places. Uniformity of the constant in Definition 15 is satisfied in the cases covered by Proposition 16.

Proof of Theorem 1. Lemma 17 and Corollary 13 show that in the case of $\boldsymbol{G} = \mathrm{GL}(r)$, the conditions of Theorem 21 hold for all maximal parabolic subgroups of $\boldsymbol{G}$. Therefore, G satisfies property (PSIO). Lemma 20 finishes the argument. The assertion on the constants c and d is clear. □

7. Parabolic subgroups with abelian unipotent radical

In this section, we prove Proposition 16 in general. Parabolic subgroups with Abelian unipotent radical and the associated action of their Levi subgroup on the radical have been studied by Richardson, Röhrle and Steinberg [1992]. We recall their results and extend them as necessary.

Let $\boldsymbol{G}$ be a split reductive group over F. It will be convenient to write $\mathfrak{g}$ in terms of a Chevalley basis [Serre 2001]. Namely, choose $X_\alpha \in \mathfrak{g}_\alpha$, $\alpha \in \Phi = R(\boldsymbol{T}_0, \boldsymbol{G})$, such that

$$[X_\alpha, X_\beta] = \begin{cases} N_{\alpha,\beta} X_{\alpha+\beta} & \text{if } \alpha+\beta \in \Phi, \\ H_\alpha & \text{if } \alpha = -\beta, \\ 0 & \text{otherwise.} \end{cases}$$

Here, the structure constants $N_{\alpha,\beta}$, α, β, $\alpha+\beta \in \Phi$, satisfy $N_{\alpha,\beta} = \pm(p+1)$, where p is the largest integer with $\beta - p\alpha \in \Phi$.

Obviously, to prove Proposition 16 we can pass to the adjoint group, which is a direct product of simple groups. Therefore, suppose from now on that $\boldsymbol{G}$ is simple and adjoint, $\boldsymbol{P}$ is maximal, and $\boldsymbol{U}$ is abelian. (Actually, the maximality of $\boldsymbol{P}$ is then automatic.) Let K_0 be the stabilizer of the $\mathcal{O}$-lattice spanned by the Chevalley basis, which is a hyperspecial maximal compact subgroup of G. Let α be the simple root defining $\boldsymbol{P}$. Write $\mathfrak{m} = \operatorname{Lie} M$, $\mathfrak{u} = \operatorname{Lie} U$, and $\bar{\mathfrak{u}} = \operatorname{Lie} \overline{U}$, so that $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{m} \oplus \bar{\mathfrak{u}}$. Denote by $\Phi_U = R(\boldsymbol{T}_0, \boldsymbol{U})$ the roots in $\mathfrak{u}$, namely the roots whose α-coefficient in the expansion with respect to Δ_0 is positive. (Since $\boldsymbol{U}$ is abelian, this coefficient is necessarily 1.) Let ρ be the highest root. We have α, $\rho \in \Phi_U$. The roots orthogonal to ρ form a parabolic root subsystem Φ_1 which contains a unique irreducible constituent $\Phi_1' \supset \Phi_U \cap \Phi_1$. If $\boldsymbol{G}$ is not simply laced, we write ρ_s for the highest short root and $\delta = \rho - \rho_s = -s_\rho \rho_s \in \Phi$. We have ρ_s, $2\rho_s - \rho = -s_{\rho_s}\rho \in \Phi_U$.

Lemma 24. *Suppose that $\boldsymbol{G}$ is not simply laced and let ρ, ρ_s and δ be as before. Then the following conditions are equivalent for $\gamma \in \Phi_U$:*

(1) $\gamma + \delta, \gamma + 2\delta \in \Phi_U$.

(2) *γ is long and* $\langle \delta, \gamma^\vee \rangle = -1$.

(3) *γ is long,* $\langle \rho, \gamma^\vee \rangle = 0$, *and* $\langle \rho_s, \gamma^\vee \rangle = 1$.

(4) *γ is the highest root in Φ_1'.*

(5) $\gamma = 2\rho_s - \rho$.

Proof. The first three conditions are clearly equivalent and they hold for $\gamma = 2\rho_s - \rho$. It remains to consider the cases of B_n and C_n. In the B_n case $\rho = 2\epsilon_1$, $\rho_s = \epsilon_1 + \epsilon_2$, $\delta = \epsilon_1 - \epsilon_2$, $\gamma = 2\epsilon_2$. In the C_n case $\rho = \epsilon_1 + \epsilon_2$, $\rho_s = \epsilon_1$, $\delta = \epsilon_2$, $\gamma = \epsilon_1 - \epsilon_2$. □

We fix once and for all a tuple $(\beta_1, \dots, \beta_r)$ of mutually orthogonal long roots in Φ_U with r maximal.

Theorem 25 [Richardson et al. 1992, Theorem 2.1]. (1) *For any $0 \le s \le r$, the Weyl group of $\boldsymbol{M}$ acts transitively on the set of s-tuples of mutually orthogonal long roots in Φ_U.*

(2) *Fix $u_i \in U_{\beta_i} \setminus \{0\}$. Then $\left\{\prod_{i=1}^{s} u_i\right\}_{s=0}^{r}$ is a set of representatives for the $\boldsymbol{M}$-orbits in $\boldsymbol{U}$ under the conjugation action.* (*The integer s is called the* rank *of the orbit.*)

The orbit corresponding to $s = r$ is the open orbit of the $\boldsymbol{M}$-action on $\boldsymbol{U}$. It is the intersection with $\boldsymbol{U}$ of the Richardson orbit associated to $\boldsymbol{P}$. The orbit corresponding to $s = 0$ is the zero orbit.

Remark 26. The possibilities (up to isogeny) for $\boldsymbol{G}$ and $\boldsymbol{P}$ have been enumerated in [Richardson et al. 1992, Remark 2.3], and the corresponding values of r are listed in [Richardson et al. 1992, Table 1]. We can explicate the orbit classification of Theorem 25 case by case.

In the cases where $\boldsymbol{G} = \mathrm{GL}(m)$, $\boldsymbol{M} = \mathrm{GL}(k) \times \mathrm{GL}(m-k)$, $\boldsymbol{U}$ is the space of $k \times (m-k)$ matrices, and $0 < k < m$, or $\boldsymbol{G} = \mathrm{Sp}(2m)$, $\boldsymbol{M} = \mathrm{GL}(m)$, and $\boldsymbol{U}$ is the space of symmetric $m \times m$ matrices, the notion of rank given by Theorem 25 coincides with the usual notion for matrices. In the case $\boldsymbol{G} = \mathrm{SO}(2m)$, $\boldsymbol{M} = \mathrm{GL}(m)$, and $\boldsymbol{U}$ is the space of antisymmetric $m \times m$ matrices, the rank in our sense is one half of the rank of the matrix. In the case $\boldsymbol{G} = \mathrm{SO}(m)$, $\boldsymbol{M} = \mathrm{GL}(1) \times \mathrm{SO}(m-2)$, and $\boldsymbol{U}$ is a quadratic space of dimension $m - 2$, the rank is one for a nonzero isotropic vector and two for anisotropic vectors.

There are (up to automorphisms of $\boldsymbol{G}$) two exceptional cases. For $\boldsymbol{G} = E_6$, $\boldsymbol{M} = \mathrm{GSpin}(10)$, and $\boldsymbol{U}$ one of the 16-dimensional half-spin representations of $\boldsymbol{M}$, we have $r = 2$. The nonzero pure spinors (i.e., the spinors in the orbit of 1, the unit element of the exterior algebra) have rank one, and the remaining nonzero spinors have rank two. The orbit dimensions are 0, 11, and 16, respectively [Igusa 1970, Proposition 2]. For $\boldsymbol{G} = E_7$, $\boldsymbol{M} = GE_6$, and $\boldsymbol{U}$ the 27-dimensional representation of $\boldsymbol{M}$, we have $r = 3$. The derived group of $\boldsymbol{M}$ leaves a nonzero cubic form f on $\boldsymbol{U}$ invariant, and this form is unique up to a scalar. The rank is one for the nonzero vectors in the singular locus of the hypersurface $f = 0$, two for the remaining nonzero vectors with $f = 0$, and three for the vectors with $f \neq 0$ [Chevalley 1951]. The orbit dimensions are 0, 17, 26, and 27, respectively [Richardson et al. 1992, Table 2].

Note that the second part of Theorem 25 does not apply to the M-orbits in U. However, the proof of [Richardson et al. 1992, Theorem 2.1] (see also [loc. cit., Theorem 5.3]) shows that fixing $\beta_1, \dots, \beta_r$ as above, it is still true that any M-orbit in U of rank s contains a representative of the form $\prod_{i=1}^{s} u_i$ for *some* $u_i \in U_{\beta_i} \setminus \{0\}$. More precisely, we have:

Lemma 27. *Let $\beta_1, \dots, \beta_r$ be as above. Then there exists a compact set $\omega \subset M$ with the following property*: *for all $X \in \mathfrak{u}$, there is $m \in \omega$ such that* $\mathrm{Ad}(m)X$ *is a linear combination of $X_{\beta_1}, \dots, X_{\beta_r}$. If either $\boldsymbol{G}$ is simply laced or $p \neq 2$, then we can take $\omega = K_M = M \cap K_0$.*

Proof. Write $X=\sum_{\beta\in\Phi_U} c_\beta(X)X_\beta$. Let $\rho\in\Phi_U$ be the highest root. We follow the argument of [Richardson et al. 1992, Proposition 2.13]. The proof is by induction on the rank of $\boldsymbol{G}$. The case $X=0$ is trivial, so we assume that $X\neq 0$. The first step is to show that in the $\operatorname{Ad} K_M$-orbit of X, we can choose X' such that $|c_\beta(X')|\le D|c_\rho(X')|$ for all $\beta\in\Phi_U$, where D is a fixed constant which can be taken to be 1 if $p\neq 2$ or if $\boldsymbol{G}$ is simply laced. This is done as follows. Let $\beta_0\in\Phi_U$ be such that $|c_{\beta_0}(X)|$ is maximal. Applying a Weyl element of $\boldsymbol{M}$, we can assume that either $\beta_0=\rho$ or $\beta_0=\rho_s$ (in the nonsimply laced case). If $|c_\rho(X)|=|c_{\beta_0}(X)|$ (and in particular, if $\boldsymbol{G}$ is simply laced), then we are done. Assume that this is not the case and let $\delta=\rho-\rho_s$ and $X'=\operatorname{Ad}(u_\delta(t))X$ with $t\in\mathcal{O}$. It follows from Lemma 24 and the commutation relations that

$$c_\gamma(X')=\begin{cases} c_\rho(X)\pm 2tc_{\rho_s}(X)+t^2c_{2\rho_s-\rho}(X) & \text{if } \gamma=\rho,\\ c_\gamma(X)\pm tc_{\gamma-\delta}(X) & \text{if } \gamma\neq\rho \text{ and } \gamma-\delta\in\Phi,\\ c_\gamma(X) & \text{if } \gamma-\delta\notin\Phi.\end{cases}$$

Therefore, we can choose $t\in\mathcal{O}^*$ such that $|c_\rho(X')|=\max_{\beta\in\Phi_U}|c_\beta(X')|$ if $p\neq 2$ and $|c_\rho(X')|\ge\frac{1}{2}|2|\max_{\beta\in\Phi_U}|c_\beta(X')|$ if $p=2$.

The second step is to clear the coefficients of all roots which are not orthogonal to ρ by conjugating by suitable unipotent elements. This is done as in [Richardson et al. 1992, p. 655], except that our condition on X' guarantees that the conjugating elements are taken from K_M (or at least from a bounded set, if $p=2$ and $\boldsymbol{G}$ is not simply laced). The rest of the proof (the induction step) follows [loc. cit.]. □

Let $w=s_{\beta_1}\dots s_{\beta_r}$. Note that the reflections s_{β_i} commute with each other, since the roots β_i are mutually orthogonal. For any $\beta\in\Phi_U$, let $\mathcal{N}(\beta)$ be the multiset

$$\mathcal{N}(\beta)=\begin{cases}\{\beta_i:\langle\beta,\beta_i^\vee\rangle=1\} & \text{if } \beta\neq\beta_1,\dots,\beta_r,\\ \{\beta_i,\beta_i\} & \text{if } \beta=\beta_i.\end{cases}$$

Thus, $\mathcal{N}(\beta)$ consists of the roots β_i which are not orthogonal to β, counted with multiplicity $\langle\beta,\beta_i^\vee\rangle$. Note that $w\beta=\beta-\sum\mathcal{N}(\beta)$ for any $\beta\in\Phi_U$. Also, for any $\beta\in\Phi_U$,

$$|\mathcal{N}(\beta)|=\sum_{i=1}^r\langle\beta,\beta_i^\vee\rangle, \tag{4}$$

and by [Richardson et al. 1992, Lemma 2.10], we have $1\le|\mathcal{N}(\beta)|\le 2$.

Suppose that $\beta,\gamma\in\Phi_U$ are distinct and β is long. Then the following conditions are equivalent:

(1) $\langle\gamma,\beta^\vee\rangle\neq 0$,

(2) $\langle\gamma,\beta^\vee\rangle=1$,

(3) $\gamma - \beta \in \Phi$, and

(4) $\gamma - \beta = s_\beta(\gamma)$.

For any $X \in \mathfrak{u}$, denote by D_X the double commutator map

$$D_X = \tfrac{1}{2} \operatorname{ad} X\big|_{\mathfrak{m}} \circ \operatorname{ad} X\big|_{\bar{\mathfrak{u}}} \in \operatorname{Hom}_F(\bar{\mathfrak{u}}, \mathfrak{u}).$$

Analogously, for $\bar{X} \in \bar{\mathfrak{u}}$, we denote by $\bar{D}_{\bar{X}}$ the double commutator map

$$\bar{D}_{\bar{X}} = \tfrac{1}{2} \operatorname{ad} \bar{X}\big|_{\mathfrak{m}} \circ \operatorname{ad} \bar{X}\big|_{\mathfrak{u}} \in \operatorname{Hom}_F(\mathfrak{u}, \bar{\mathfrak{u}}).$$

Lemma 28. *Let $X = \sum_{i=1}^r t_i X_{\beta_i}$. Then*

$$D_X X_{-\beta} = \begin{cases} 0 & \text{if } |\mathcal{N}(\beta)| = 1, \\ t_i t_j X_{-w\beta} & \text{if } \mathcal{N}(\beta) = \{\beta_i, \beta_j\}. \end{cases}$$

Proof. The statement is clear if $\beta = \beta_i$, since $\beta_i - \beta_j \notin \Phi$ for all j.

Now suppose that $\beta \neq \beta_1, \dots, \beta_r$. Then

$$\operatorname{ad} X(X_{-\beta}) = \sum_{i:\beta_i \in \mathcal{N}(\beta)} t_i X_{\beta_i - \beta},$$

and therefore

$$D_X(X_{-\beta}) = \frac{1}{2} \sum_{\substack{i,j:\, \beta_i \in \mathcal{N}(\beta), \\ \beta_i + \beta_j - \beta \in \Phi_U}} t_i t_j X_{\beta_i + \beta_j - \beta}.$$

Note that if $\beta_i \in \mathcal{N}(\beta)$ and $\delta = \beta_i + \beta_j - \beta \in \Phi_U$, then $i \neq j$, since β_i is long. If we set $\gamma = \beta_i - \beta = -s_{\beta_i}\beta$, then $\delta = \beta_j + \gamma$ and $s_{\beta_i}\delta = \beta_j - \beta \in \Phi$. Thus, $\beta_j \in \mathcal{N}(\beta)$ and $\delta = -w\beta$. ☐

Corollary 29. *For any $X \in \mathfrak{u}$, we have $\|D_X\|_{\operatorname{Hom}(\bar{\mathfrak{u}},\mathfrak{u})} \gg \|X\|^2$.*

Lemma 30. *The following conditions are equivalent*:

(1) *$\boldsymbol{P}$ is conjugate to $\bar{\boldsymbol{P}}$.*

(2) *$\boldsymbol{P}^{w_0} = \bar{\boldsymbol{P}}$.*

(3) *$\boldsymbol{P}^{w} = \bar{\boldsymbol{P}}$.*

(4) *$|\mathcal{N}(\beta)| = 2$ for all $\beta \in \Phi_U$.*

(5) *$\frac{1}{2}\sum_{i=1}^r \beta_i^\vee$ is the fundamental coweight with respect to $\boldsymbol{P}$.*

(6) *$\frac{1}{2}\sum_{i=1}^r \beta_i$ is the fundamental weight with respect to $\boldsymbol{P}$.*

(7) *There exists $X \in \mathfrak{u}$ such that D_X is invertible.*

If these conditions are satisfied, then D_X is invertible if and only if X belongs to the open Ad $\boldsymbol{M}$*-orbit in* $\mathfrak{u}$.

Proof. The equivalence of the first four conditions follows from [Richardson et al. 1992, Proposition 3.12]. The equivalence of the last and the fourth conditions, as well as the last assertion of the lemma, follows from Lemma 28. The equivalence between the fourth and fifth conditions follows from (4). Finally, the equivalence between the fifth and the sixth conditions is immediate, since α is a long root. □

Let H be the central element of $\mathfrak{m}$ such that $\operatorname{ad} H\big|_{\mathfrak{u}} = 2\,\mathrm{Id}_{\mathfrak{u}}$.

Lemma 31. *Suppose that $\boldsymbol{P}^{w_0} = \overline{\boldsymbol{P}}$. Then*

(1) *We have $H = \sum_{i=1}^{r} H_{\beta_i}$.*

(2) *The open (P, P) Bruhat cell is Pw_0U.*

(3) *We have*
$$Pw_0U = \{g \in G : \mathrm{proj}_{\bar{\mathfrak{u}}} \circ \mathrm{Ad}(g)\big|_{\mathfrak{u}} \textit{ is invertible}\}.$$

(4) *For any $g \in Pw_0U$, the U-part in the Bruhat decomposition is given by* $\exp Y$, *where* $2Y = \big(\mathrm{proj}_{\bar{\mathfrak{u}}} \circ \mathrm{Ad}(g)\big|_{\mathfrak{u}}\big)^{-1}(\mathrm{proj}_{\bar{\mathfrak{u}}}(\mathrm{Ad}(g)H))$.

(5) *In particular, for $\overline{X} \in \bar{\mathfrak{u}}$, we have* $\exp \overline{X} \in Pw_0U$ *if and only if $\overline{X}$ lies in the open* $\operatorname{Ad} \boldsymbol{M}$*-orbit, and in this case the U-part of* $\exp \overline{X}$ *is* $\exp Y$ *for* $Y = \overline{D}_{\overline{X}}^{-1}(\overline{X})$.

Proof. The first part follows from the previous lemma. The second part is clear. Let $\mathscr{C} = \big\{g \in G : \mathrm{proj}_{\bar{\mathfrak{u}}} \circ \mathrm{Ad}(g)\big|_{\mathfrak{u}} \text{ is invertible}\big\}$. Clearly, $\mathscr{C}$ is left and right P-invariant and $w_0 \in \mathscr{C}$. Therefore $\mathscr{C}$ is a union of (P, P) double cosets and $Pw_0U \subset \mathscr{C}$. The fourth part is also clear by direct computation. By [Richardson et al. 1992, Theorem 1.1], every (P, P) double coset intersects $\overline{U}$ in (the set of F-rational points of) a single $\boldsymbol{M}$-orbit under conjugation. Thus, in order to show that $\mathscr{C} = Pw_0U$, it is enough to show that $\mathscr{C} \cap \overline{U}$ is (the set of F-rational points of) an $\boldsymbol{M}$-orbit. However, $\mathscr{C} \cap \overline{U} = \{\exp \overline{X} : \overline{D}_{\overline{X}} \text{ is invertible}\}$. Therefore, the statement follows from Lemma 30. □

Corollary 32. *Let θ be the Cartan involution of $\boldsymbol{G}$ and set $d = \#\{\beta \in \Phi_U : \beta_i \in \mathcal{N}(\beta)\}$, which is independent of i. If $\boldsymbol{P}^{w_0} = \overline{\boldsymbol{P}}$, then $d = 2\dim U/r$. For $X = \sum_{i=1}^{r} t_i X_{\beta_i}$, we have*
$$\det(\theta \circ D_X) = \begin{cases} (t_1 \dots t_r)^d & \textit{if } \boldsymbol{P}^{w} = \overline{\boldsymbol{P}}, \\ 0 & \textit{otherwise.} \end{cases}$$

Remark 33. Suppose that $\boldsymbol{P}^{w_0} = \overline{\boldsymbol{P}}$. The character $\prod_{i=1}^{r} \beta_i$ of $\boldsymbol{T}_0$ is trivial on $\boldsymbol{M}'$ and therefore extends to a rational character ψ of $\boldsymbol{M}$. The polynomial
$$\sum_{i=1}^{r} t_i X_{\beta_i} \mapsto t_1 \dots t_r$$
extends to an irreducible $(\operatorname{Ad} M, \psi)$-equivariant polynomial Δ on $\mathfrak{u}$.

For $n \in N_M(T)$ representing $w \in W^M$ and $\beta \in \Phi_U$, let $f_{n,\beta}$ be the scalar so that $\mathrm{Ad}(n)X_\beta = f_{n,\beta} X_{w\beta}$. Clearly $f_{nt,\beta} = \beta(t) f_{n,\beta}$. In the simply laced case, we have

$$\Delta\Big(\sum_{\beta\in\Phi_U} c_\beta X_\beta\Big) = \sum_{w\in N_M(T_0)/T_0} \psi(n_w)\frac{c_{w\beta_1}}{f_{n_w,\beta_1}} \cdots \frac{c_{w\beta_r}}{f_{n_w,\beta_r}},$$

where n_w is any representative of w in M. The polynomial Δ is the determinant in the $\mathrm{GL}(m)$ or $\mathrm{Sp}(2m)$ case, the Pfaffian in the $\mathrm{SO}(4m)$ case, the canonical quadratic form in the $\mathrm{SO}(m)$ case, and the relatively invariant cubic form in the E_7 case.

Corollary 34. *Assume that $\boldsymbol{P}^{w_0} = \bar{\boldsymbol{P}}$.*

(1) *The open orbit in $\mathfrak{u}$ is the principal open set defined by* $\det\theta \circ D_X$.

(2) *Assume that $X \in \mathfrak{u}$ is in the open orbit. Then the Jacobson–Morozov parabolic subgroup of X is $\boldsymbol{P}$.*

(3) *Assume that $X = \sum_{i=1}^r t_i X_{\beta_i}$ with $t_1, \dots, t_r \neq 0$. Let $\bar{X} = \sum_{i=1}^r t_i^{-1} X_{-\beta_i}$. Then $(X, H, \bar{X})$ is an* $\mathrm{SL}(2)$*-triple.*

Remark 35. In [Kac 1980], the double commutator map has been used to obtain relatively invariant polynomials in a more general situation.

Finally, we are ready to prove Proposition 16.

Proof of Proposition 16. Suppose that $\bar{u} \in \bar{U} \cap Z(M)U\mathfrak{B}(n)$ and write $\bar{u} = zub$, where $z \in Z(M)$, $u \in U$, and $b \in \mathfrak{B}(n)$. Let $\lambda \in F^*$ be such that $\mathrm{Ad}(z)\big|_{\mathfrak{u}} = \lambda\,\mathrm{Id}_{\mathfrak{u}}$. Also write $\bar{u} = \exp \bar{X}$, where $\bar{X} \in \bar{\mathfrak{u}}$. As $\mathrm{Ad}(\exp\bar{X}) = \sum_{m=0}^\infty (1/m!)(\mathrm{ad}\,\bar{X})^m$, we have

$$\mathrm{Id}_{\mathfrak{u}} - \mathrm{ad}\,\bar{X}\big|_{\mathfrak{u}} + \bar{D}_{\bar{X}} = \mathrm{Ad}(\bar{u}^{-1})\big|_{\mathfrak{u}} = \mathrm{Ad}(b^{-1})\,\mathrm{Ad}(zu)^{-1}\big|_{\mathfrak{u}} = \lambda^{-1}\,\mathrm{Ad}(b^{-1})\big|_{\mathfrak{u}}. \tag{5}$$

It follows that $\max(1, \|\bar{D}_{\bar{X}}\|) \le |\lambda|^{-1}\|b\|$, and therefore by Corollary 29 (applied to $\bar{P}$) that $\max(1, \|\bar{X}\|)^2 \ll |\lambda|^{-1}\|b\|$, or equivalently,

$$|\lambda|\|b\| \max(1, \|\bar{X}\|) \ll \|b\|^2 \max(1, \|\bar{X}\|)^{-1}.$$

We can write (5) in the form

$$\lambda\,\mathrm{Ad}(b) \circ \bar{D}_{\bar{X}} = (\mathrm{Id}_{\mathfrak{g}} - \Delta)\big|_{\mathfrak{u}},$$

where $\Delta = \lambda\,\mathrm{Ad}(b) \circ (\mathrm{Id} - \mathrm{ad}\,\bar{X}) \in \mathrm{End}(\mathfrak{g})$. Suppose that $\|\bar{X}\| \gg \|b\|^2$. Then $\|\Delta\| \ll |\lambda|\|b\| \max(1, \|\bar{X}\|) < 1$, and therefore $\mathrm{Id} - \Delta$ is invertible and $\|(\mathrm{Id} - \Delta)^{-1}\| = 1$. It follows that $\bar{D}_{\bar{X}}$ is invertible, and therefore by Lemma 30 we infer that $\boldsymbol{P}^{w_0} = \bar{\boldsymbol{P}}$. Moreover, $\bar{D}_{\bar{X}}^{-1} = \lambda(\mathrm{Id} - \Delta)^{-1} \circ \mathrm{Ad}(b)\big|_{\bar{\mathfrak{u}}}$, and therefore $\|\bar{D}_{\bar{X}}^{-1}\| \le |\lambda|\|b\|$. By Lemma 31, we get $\bar{u} \in P w_0 U$ and the U-part in the Bruhat decomposition of $\bar{u}$ is $\exp Y$ for $Y = \bar{D}_{\bar{X}}^{-1}(\bar{X})$. Hence $\|Y\| \le |\lambda|\|b\|\|\bar{X}\| \ll \|\bar{X}\|^{-1}\|b\|^2$. This immediately implies Proposition 16. $\square$

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TOBIAS FINIS
INSTITUT FÜR MATHEMATIK
FREIE UNIVERSITÄT BERLIN
ARNIMALLEE 3
D-14195 BERLIN
GERMANY
finis@math.fu-berlin.de

EREZ LAPID
EINSTEIN INSTITUTE OF MATHEMATICS
THE HEBREW UNIVERSITY OF JERUSALEM
91904 JERUSALEM
ISRAEL
erezla@math.huji.ac.il

WERNER MÜLLER
MATHEMATISCHES INSTITUT
RHEINISCHE FRIEDRICH-WILHELMS-UNIVERSITÄT BONN
ENDENICHER ALLEE 60
D-53115 BONN
GERMANY
mueller@math.uni-bonn.de

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COMPARISON OF COMPACT INDUCTION WITH PARABOLIC INDUCTION

GUY HENNIART AND MARIE-FRANCE VIGNERAS

In memory of Jon Rogawski

I (GH) have very fond memories of an extraordinary expedition to the Grand Canyon with Jon. I have in mind these happy remembrances, and others of a more mathematical kind, in dedicating the present paper to him.

I (MFV) first met Jon in Paris when he was a student visiting École Normale Supérieure; at that time, he was already gifted, charming, fluent in French, and full of music and mathematics. During another visit that Jon made to ENS, we discussed in detail his preprint on the Jacquet–Langlands local correspondence for division algebras; my collaboration with Bernstein, Deligne and Kazhdan emerged from this. Afterwards, I always looked forward to seeing him at scientific meetings, and I was enriched by being welcomed by his wife Julie and their four children to their home in Los Angeles.

We and our French colleagues all share a sense of great loss.

Let F be a nonarchimedean locally compact field of residual characteristic p, let G be a connected reductive F-group, and let K be a special parahoric subgroup of $G(F)$. We choose a parabolic F-subgroup P of G with Levi decomposition $P = MN$ in good position with respect to K. Let C be an algebraically closed field of characteristic p, and V an irreducible smooth C-representation of K. We investigate the natural intertwiner from the compact induced representation c-$\mathrm{Ind}_K^{G(F)} V$ to the parabolic induced representation $\mathrm{Ind}_{P(F)}^{G(F)}\big(\text{c-}\mathrm{Ind}_{M(F)\cap K}^{M(F)} V_{N(F)\cap K}\big)$. Under a regularity condition on V, we show that the intertwiner becomes an isomorphism after localization at a specific Hecke operator. When F has characteristic 0, G is F-split and K is hyperspecial, the result was essentially proved by Herzig. We define the notion of K-supersingularity for an irreducible smooth C-representation of $G(F)$ which extends Herzig's definition for admissible irreducible representations and we give a list of irreducible representations which are neither supercuspidal nor K-supersingular.

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1. Introduction

Let F be a nonarchimedean locally compact field of residual characteristic p, let G be a reductive connected F-group, and let C be an algebraically closed field of characteristic p. We are interested in smooth admissible C-representations of $G(F)$. Two induction techniques are available: compact induction $\text{c-Ind}_K^{G(F)}$ from a compact open subgroup K of $G(F)$ and parabolic induction $\text{Ind}_{P(F)}^{G(F)}$ from a parabolic subgroup $P(F)$ with Levi decomposition $P(F)=M(F)N(F)$. Here we want to investigate the interaction between the two inductions.

More specifically, assume that $G(F)=P(F)K$ and

$$P(F)\cap K=(M(F)\cap K)(N(F)\cap K).$$

We construct (Definition 2.1), for any finite-dimensional smooth C-representation V of K, a canonical intertwiner

$$I_V:\text{c-Ind}_K^{G(F)}V\to \text{Ind}_{P(F)}^{G(F)}\big(\text{c-Ind}_{M(F)\cap K}^{M(F)}V_{N(F)\cap K}\big),$$

where $V_{N(F)\cap K}$ stands for the $N(F)\cap K$-coinvariants in V, and a canonical algebra homomorphism

$$\mathcal{S}':\mathcal{H}(G(F),K,V)\to \mathcal{H}\big(M(F),M(F)\cap K,V_{N(F)\cap K}\big),$$

where, as in [Henniart and Vigneras 2011], the Hecke algebra $\mathcal{H}(G(F),K,V)$ is $\text{End}_{G(F)}\,\text{c-Ind}_K^{G(F)}V$ seen as an algebra of double cosets of K in G, and similarly for

$$\mathcal{H}\big(M(F),M(F)\cap K,V_{N(F)\cap K}\big).$$

By construction,

$$\big(I_V(\Phi(f))\big)(g)=\mathcal{S}'(\Phi)\big(I_V(f)(g)\big),$$

for $f\in \text{c-Ind}_K^{G(F)}V$, $\Phi\in\mathcal{H}(G(F),K,V)$, $g\in G(F)$.

Let V^* be the contragredient representation of V. We constructed in [Henniart and Vigneras 2011] a Satake homomorphism

$$\mathcal{S} : \mathcal{H}(G(F), K, V^*) \to \mathcal{H}\big(M(F), M(F) \cap K, (V^*)^{N(F)\cap K}\big).$$

Here we show that $\mathcal{S}'$ and $\mathcal{S}$ are related by a natural anti-isomorphism of Hecke algebras (Proposition 2.3).

We study I_V further in the particular case where K is a special parahoric subgroup and V is irreducible. Such a V is trivial on the pro-p-radical K_+ of K. The quotient K/K_+ is the group of k-points of a connected reductive k-group G_k, so that we can use the theory of finite reductive groups in natural characteristic. We write $K/K_+ = G(k)$. The image of $P(F) \cap K = P_0$ in $G(k)$ is the group of k-points of a parabolic subgroup of G_k. We write $P_0/P_0 \cap K_+ = P(k)$, and we use similar notations for M and N, for the opposite parabolic subgroup $\overline{P} = M\overline{N}$ (Section 4A), and for a minimal parabolic F-subgroup $\overline{B}$ of G contained in $\overline{P}$, of Levi decomposition $\overline{B} = Z\overline{U}$.

We say that V is $\overline{P}$-regular when the stabilizer $\overline{P}_V(k)$ in $G(k)$ of the line $V^{\overline{U}(k)}$ is contained in $\overline{P}(k)$ (this does not depend on the choice of $\overline{B}$). An equivalent definition is that, for $h \in K$ which does not belong to $P_0\overline{P}_0$, the kernel of the quotient map $V \to V_{N(k)}$ contains $hV^{\overline{N}(k)}$ (Definition 3.6 and Corollary 3.19).

We choose a maximal F-split torus S in M such that K stabilizes a special vertex in the apartment of $G(F)$ associated to S. We choose an element $s \in S(F)$ which is central in $M(F)$ and strictly N-positive, in the sense that conjugation by s strictly contracts the compact subgroups of $N(F)$. There is a unique Hecke operator T_M in $\mathcal{H}\big(M(F), M_0, V_{N(k)}\big)$ with support in $M_0 s$ and value at s the identity morphism of $V_{N(k)}$. We prove (Proposition 4.5):

Proposition 1.1. *The map $\mathcal{S}'$ is a localization at T_M.*

This means that $\mathcal{S}'$ is injective, that T_M belongs to the image of $\mathcal{S}'$ and is central and invertible in $\mathcal{H}\big(M(F), M_0, V_{N(k)}\big)$, and that

$$\mathcal{H}\big(M(F), M_0, V_{N(k)}\big) = \mathcal{S}'\big(\mathcal{H}(G(F), K, V)\big)[T_M^{-1}].$$

This is a consequence of the analogous property of $\mathcal{S}$ proved in [Henniart and Vigneras 2011].

In this particular case, following a suggestion of Abe, we show that I_V is injective. We introduce the localization Θ of I_V at T_M. As I_V is injective, its localization Θ is injective. Our main theorem is this:

Theorem 1.2 (Theorem 4.6). *The map*

$$\Theta : \mathcal{H}\big(M(F), M_0, V_{N(k)}\big) \otimes_{\mathcal{H}(G(F),K,V),\mathcal{S}'} \text{c-Ind}_K^{G(F)} V \to \text{Ind}_{P(F)}^{G(F)}\big(\text{c-Ind}_{M(F)\cap K}^{M(F)} V_{N(k)}\big)$$

is bijective if V is $\overline{P}$-regular.

The special case when F has characteristic 0, G is F-split, and K is hyperspecial was proved in [Herzig 2011] (see also [Abe 2011]). In this case the Hecke algebras are commutative.

Writing $\mathscr{Z}_G(V)$ for the center of $\mathscr{H}(G(F), K, V)$ and $\mathscr{Z}_M(V_{N(k)})$ for the center of $\mathscr{H}\big(M(F), M(F)\cap K, V_{N(k)}\big)$, the theorem implies by specialization:

Corollary 1.3. *If V is $\overline{P}$-regular, for any right $\mathscr{Z}_M(V_{N(k)})$-module χ, the representations of $G(F)$*

$$\chi \otimes_{\mathscr{Z}_G(V),\mathscr{S}'} \text{c-Ind}_K^{G(F)} V \quad \textit{and} \quad \text{Ind}_{P(F)}^{G(F)}\big(\chi \otimes_{\mathscr{Z}_M(V_{N(k)})} \text{c-Ind}_{M(F)\cap K}^{M(F)} V_{N(k)}\big)$$

are isomorphic.

To prove the theorem, we follow the method of Herzig and decompose I_V as the composite $I_V = \zeta \circ \xi$ of two $G(F)$-equivariant maps: the natural inclusion ξ of $\text{c-Ind}_K^{G(F)} V$ in $\text{c-Ind}_K^{G(F)}\big(\text{c-Ind}_{P(k)}^{G(k)} V_{N(k)}\big)$ and the natural map

$$\zeta : \text{c-Ind}_K^{G(F)}\big(\text{c-Ind}_{P(k)}^{G(k)} V_{N(k)}\big) \to \text{Ind}_{P(F)}^{G(F)}\big(\text{c-Ind}_{M(F)\cap K}^{M(F)} V_{N(k)}\big),$$

associated to the quotient map $\text{c-Ind}_{P(k)}^{G(k)} V_{N(k)} \to V_{N(k)}$ (see (2) below). We write $\mathscr{P}$ for the parahoric subgroup inverse image of $P(k)$ in K and $T_{\mathscr{P}}$ for the Hecke operator in $\mathscr{H}(G(F), \mathscr{P}, V_{N(k)})$ with support $\mathscr{P}s\mathscr{P}$ and value at s the identity of $V_{N(k)}$. With no regularity assumption on V, we prove

$$\zeta \circ T_{\mathscr{P}} = T_M \circ \zeta.$$

Seeing $\text{c-Ind}_K^{G(F)}\big(\text{c-Ind}_{P(k)}^{G(k)} V\big) = \text{c-Ind}_{\mathscr{P}}^{G(F)} V_{N(k)}$ and $\text{Ind}_{P(F)}^{G(F)}\big(\text{c-Ind}_{M(F)\cap K}^{M(F)} V_{N(k)}\big)$ as $C[T]$-modules via $T_{\mathscr{P}}$ and T_M, the map ζ is $C[T]$-linear, and using Corollary 6.5 we prove:

Theorem 1.4. *The localization at T of ζ is an isomorphism.*

To study ξ, we consider the Hecke operator T_G in $\mathscr{H}(G(F), K, V)$ with support KsK and value at s the natural projector $V \to V^{\overline{N}(k)}$, and the Hecke operator $T_{K,\mathscr{P}}$ from $\text{c-Ind}_{\mathscr{P}}^{G(F)} V_{N(k)}$ to $\text{c-Ind}_K^{G(F)} V$ with support $Ks\mathscr{P}$ and value at s given by the natural isomorphism $V_{N(k)} \to V^{\overline{N}(k)}$. With no regularity assumption on V, we prove

$$T_{K,\mathscr{P}} \circ \xi = T_G.$$

Assuming that V is $\overline{P}$-regular, we prove

$$\xi \circ T_{K,\mathscr{P}} = T_{\mathscr{P}},$$
$$\mathscr{S}'(T_G) = T_M.$$

Seeing $\text{c-Ind}_K^{G(F)} V$ as a $C[T]$-module via $T_G = (\mathscr{S}')^{-1}(T_M)$, the map ξ is $C[T]$-linear and:

Theorem 1.5. *The localization at T of ξ is an isomorphism when V is $\overline{P}$-regular.*

These two theorems imply that Θ is an isomorphism when V is $\overline{P}$-regular.

Following Herzig and Abe, we define the notion of K-supersingularity.

Definition 1.6. We say that an irreducible smooth C-representation π of $G(F)$ is K-supersingular when

$$\mathscr{H}\big(M(F), M_0, V_{N(k)}\big) \otimes_{\mathscr{H}(G(F),K,V),\mathscr{S}'} \mathrm{Hom}_{G(F)}\big(\text{c-Ind}_K^{G(F)} V, \pi\big) = 0$$

for any irreducible smooth C-representation V of K and any standard Levi subgroup $M \neq G$.

If π is a smooth irreducible C-representation of $G(F)$, we say that π is supercuspidal if π is not a subquotient of a proper parabolically induced representation $\mathrm{Ind}_{P(F)}^{G(F)} \tau$, $P \neq G$, from an irreducible smooth C-representation τ. Note that for an admissible π, our requirement for supercuspidality is stronger than the one used in [Herzig 2011, Definition 9.12]: he only asks that π not be a subquotient of a proper parabolically induced representation from an irreducible admissible C-representation. In their context and with Herzig's definition, Herzig and Abe [Abe 2011, Corollary 5.10] show that, for admissible π, K-supersingularity is equivalent to supercuspidality. We expect that the same is true, for admissible π, in our more general context and with our definition. Here are the partial results we have in that direction:

Theorem 1.7. *Let π be an irreducible smooth C-representation of $G(F)$.*

i. *If π is isomorphic to a subrepresentation or is an admissible quotient of $\mathrm{Ind}_{P(F)}^{G(F)} \tau$ as above, then π is not K-supersingular.*

ii. *If π is admissible and*

$$\mathscr{H}\big(M(F), M_0, V_{N(k)}\big) \otimes_{\mathscr{H}(G(F),K,V),\mathscr{S}'} \mathrm{Hom}_{G(F)}\big(\text{c-Ind}_K^{G(F)} V, \pi\big) \neq 0 \tag{1}$$

for some $\overline{Q}$-regular irreducible subrepresentation V of $\pi|_K$ and some standard parabolic subgroups $P = MN \subset Q = LN' \neq G$, then π is a quotient of $\mathrm{Ind}_{Q(F)}^{G(F)} \tau$ for an admissible irreducible smooth C-representation τ of $L(F)$.

2. Generalities on the Satake homomorphisms

In this chapter we give a functorial construction of Herzig's Satake transform $\mathscr{S}'$ in a rather general situation. Let C be a field, G a locally profinite group, K an open subgroup of G, and P a closed subgroup of G satisfying the "Iwasawa decomposition" $G = KP$. We choose a smooth $C[K]$-module V. As in [Henniart and Vigneras 2011], we assume that P is the semidirect product of a closed normal subgroup N and of a closed subgroup M, and that $K \cap P$ is the semidirect product of $N \cap K$ by $M \cap K$. We also impose:

A1) Each double coset KgK in G is the union of a finite number of cosets Kg' and the union of a finite number of cosets $g''K$ (the first condition for g is equivalent to the second for g^{-1}).

A2) V is a finite-dimensional C-vector space.

The smooth $C[K]$-module V gives rise to a compactly induced representation $\text{c-Ind}_K^G V$ and a smooth $C[P]$-module W gives rise to the full smooth induced representation $\text{Ind}_P^G W$. We consider the space of intertwiners

$$\mathcal{J} := \text{Hom}_G\big(\text{c-Ind}_K^G V, \text{Ind}_P^G W\big).$$

By Frobenius reciprocity for compact induction (as K is open in G), the C-module $\mathcal{J}$ is canonically isomorphic to $\text{Hom}_K\big(V, \text{Res}_K^G \text{Ind}_P^G W\big)$; to an intertwiner I we associate the function $v \mapsto I[1, v]_K$, where $[1, v]_K$ is the function in $\text{c-Ind}_K^G V$ with support K and value v at 1. By the Iwasawa decomposition and the hypothesis that K is open in G, we get by restricting functions to K an isomorphism of $C[K]$-modules from $\text{Res}_K^G \text{Ind}_P^G W$ onto $\text{Ind}_{P\cap K}^K(\text{Res}_{P\cap K}^P W)$. Using now Frobenius reciprocity for the full smooth induction $\text{Ind}_{P\cap K}^K$ from $P \cap K$ to K, we finally get a canonical C-linear isomorphism

$$\mathcal{J} \simeq \text{Hom}_{P\cap K}(V, W)$$

(we now omit mentioning the obvious restriction functors in the notation); this map associates to an intertwiner I the function $v \mapsto (I[1, v]_K)(1)$.

We could have proceeded differently, first applying Frobenius reciprocity to $\text{Ind}_P^G W$, getting $\mathcal{J} \simeq \text{Hom}_P\big(\text{c-Ind}_K^G V, W\big)$, then identifying $\text{Res}_P^G \text{c-Ind}_K^G V$ with $\text{c-Ind}_{P\cap K}^P V$, and finally applying Frobenius reciprocity to $\text{c-Ind}_{P\cap K}^P V$. In this way we also obtain an isomorphism of $\mathcal{J}$ onto $\text{Hom}_{P\cap K}(V, W)$, which is readily checked to be the same as the preceding one.

Assume also that W is a smooth $C[M]$-module, seen as a smooth $C[P]$-module by inflation. Then $\text{Ind}_P^G W$ is the *parabolic induction* of W, and $\text{Hom}_{P\cap K}(V, W)$ identifies with $\text{Hom}_{M\cap K}(V_{N\cap K}, W)$, where $V_{N\cap K}$ is the space of coinvariants of $N \cap K$ in V. With that identification, an intertwiner I is sent to the map from $V_{N\cap K}$ to W sending the image $\bar{v}$ of $v \in V$ in $V_{N\cap K}$ to $(I[1, v]_K)(1)$. By Frobenius reciprocity again, $\text{Hom}_{M\cap K}(V_{N\cap K}, W)$ is isomorphic to $\text{Hom}_M\big(\text{c-Ind}_{M\cap K}^M V_{N\cap K}, W\big)$, so overall we obtain an isomorphism

$$j : \mathcal{J} = \text{Hom}_G\big(\text{c-Ind}_K^G V, \text{Ind}_P^G W\big) \to \text{Hom}_M\big(\text{c-Ind}_{M\cap K}^M V_{N\cap K}, W\big), \tag{2}$$

which associates to $I \in \mathcal{J}$ the $C[M]$-linear map sending $[1, \bar{v}]_{M\cap K}$ to $(I[1, v]_K)(1)$. The reciprocal isomorphism sends $I' \in \text{Hom}_M\big(\text{c-Ind}_{M\cap K}^M V_{N\cap K}, W\big)$ to the element in $\text{Hom}_G\big(\text{c-Ind}_K^G V, \text{Ind}_P^G W\big)$ which, for $v \in V$, sends $[1, v]_K$ to the unique function with value $I'([1, \overline{kv}]_{M\cap K})$ at $k \in K$.

For $W = \text{c-Ind}_{M\cap K}^{M} V_{N\cap K}$, the isomorphism j is written j_V:

$$j_V : \mathrm{Hom}_G\big(\text{c-Ind}_K^G V, \mathrm{Ind}_P^G\big(\text{c-Ind}_{M\cap K}^{M} V_{N\cap K}\big)\big) \to \mathrm{End}_M\big(\text{c-Ind}_{M\cap K}^{M} V_{N\cap K}\big).$$

Definition 2.1. We define I_V in $\mathrm{Hom}_G\big(\text{c-Ind}_K^G V, \mathrm{Ind}_P^G\big(\text{c-Ind}_{M\cap K}^{M} V_{N\cap K}\big)\big)$ such that $j_V(I_V)$ is the unit element of $\mathrm{End}_M\big(\text{c-Ind}_{M\cap K}^{M} V_{N\cap K}\big)$. The intertwiner I_V is determined by the condition

$$(I_V[1, v]_K)(1) = [1, \bar{v}]_{M\cap K} \tag{3}$$

for all $v \in V$.

The isomorphism j is natural in V and W. The functor

$$\mathcal{F}_V : W \mapsto \mathrm{Hom}_G\big(\text{c-Ind}_K^G V, \mathrm{Ind}_P^G W\big)$$

from the category of smooth $C[M]$-modules to the category of sets is representable by $\text{c-Ind}_{M\cap K}^{M} V_{N\cap K}$. Let now V' be another finite-dimensional smooth $C[K]$-module. Any G-intertwiner

$$b : \text{c-Ind}_K^G V \to \text{c-Ind}_K^G V'$$

gives a morphism of functors $\mathcal{F}_{V'} \to \mathcal{F}_V$. By the representability of $\mathcal{F}_V$ and $\mathcal{F}_{V'}$, there is then a unique $C[M]$-morphism

$$\mathcal{S}'(b) : \text{c-Ind}_{M\cap K}^{M} V_{N\cap K} \to \text{c-Ind}_{M\cap K}^{M} V'_{N\cap K}$$

such that the diagram

$$\begin{tikzcd}
\mathrm{Hom}_G\big(\text{c-Ind}_K^G V', \mathrm{Ind}_P^G W\big) \arrow[r, "j'"] \arrow[d, "I' \mapsto I' \circ b"'] & \mathrm{Hom}_M\big(\text{c-Ind}_{M\cap K}^{M} V'_{N\cap K}, W\big) \arrow[d, "I' \mapsto I' \circ \mathcal{S}'(b)"'] \\
\mathrm{Hom}_G\big(\text{c-Ind}_K^G V, \mathrm{Ind}_P^G W\big) \arrow[r, "j"'] & \mathrm{Hom}_M\big(\text{c-Ind}_{M\cap K}^{M} V_{N\cap K}, W\big)
\end{tikzcd} \tag{4}$$

is commutative for all smooth $C[M]$-modules W (the horizontal maps are the canonical isomorphisms constructed above, and the vertical maps are given by composition with b or with $\mathcal{S}'(b)$). Taking $W = \text{c-Ind}_{M\cap K}^{M} V_{N\cap K}$, we get

$$\mathcal{S}'(b) = j(I_{V'} \circ b), \tag{5}$$

when $V_{N\cap K} = V'_{N\cap K}$ as a representation of $M \cap K$. If V' is a third finite-dimensional smooth $C[K]$-module and

$$b' : \text{c-Ind}_K^G V' \to \text{c-Ind}_K^G V''$$

is a G-intertwiner, then $b' \circ b : \text{c-Ind}_K^G V \to \text{c-Ind}_K^G V''$ is a G-intertwiner and we have obviously

$$\mathcal{S}'(b' \circ b) = \mathcal{S}'(b') \circ \mathcal{S}'(b). \tag{6}$$

Taking $V = V' = V''$, we get an algebra homomorphism

$$\mathcal{S}' : \text{End}_G\big(\text{c-Ind}_K^G V\big) \to \text{End}_M\big(\text{c-Ind}_{M\cap K}^M V_{N\cap K}\big)$$

such that

$$j(I \circ b) = j(I) \circ \mathcal{S}'(b)$$

for I in $\text{Hom}_G\big(\text{c-Ind}_K^G V, \text{Ind}_P^G W\big)$.

By the naturality of j in W, for any homomorphism $\alpha : W' \to W$ of smooth $C[M]$-modules, we have a commutative diagram

$$\begin{array}{ccc}
\text{Hom}_G\big(\text{c-Ind}_K^G V, \text{Ind}_P^G W'\big) & \xrightarrow{\ j\ } & \text{Hom}_M\big(\text{c-Ind}_{M\cap K}^M V_{N\cap K}, W'\big) \\
\Big\downarrow{\scriptstyle \text{Ind}(\alpha)} & & \Big\downarrow{\scriptstyle \alpha} \\
\text{Hom}_G\big(\text{c-Ind}_K^G V, \text{Ind}_P^G W\big) & \xrightarrow[\ j\ ]{} & \text{Hom}_M\big(\text{c-Ind}_{M\cap K}^M V_{N\cap K}, W\big)
\end{array}$$

for any V. For $W = W'$, we obtain $j\big((\text{Ind}_P^G \alpha) \circ I\big) = \alpha \circ j(I)$ for $\alpha \in \text{End}_M(W)$. We have

$$j\big((\text{Ind}_P^G \alpha) \circ I_V\big) = \alpha$$

for all α in $\text{Hom}_M\big(\text{c-Ind}_{M\cap K}^M V_{N\cap K}, W\big)$. For $W = W' = \text{c-Ind}_{M\cap K}^M V_{N\cap K}$, we deduce

$$I_V \circ b = \big(\text{Ind}_P^G \mathcal{S}'(b)\big) \circ I_V$$

for $b \in \text{End}_G(\text{c-Ind}_K^G V)$, by applying j_V^{-1} to (5).

We now want to interpret the previous results in terms of actions of Hecke algebras. By Frobenius reciprocity, $\text{Hom}_G\big(\text{c-Ind}_K^G V, \text{c-Ind}_K^G V'\big)$ identifies with $\text{Hom}_K\big(V, \text{Res}_K^G \text{c-Ind}_K^G V'\big)$, as a C-module; to a G-intertwiner b we associate the map $v \mapsto b_v := b([1, v]_K)$. From such a b, we get a map

$$\Phi_b : G \to \text{Hom}_C(V, V'), \quad g \mapsto (v \mapsto b_v(g)).$$

Thus we identify $\text{Hom}_G\big(\text{c-Ind}_K^G V, \text{c-Ind}_K^G V'\big)$ with the space $\mathcal{H}(G, K, V, V')$ of functions Φ from G to $\text{Hom}_C(V, V')$ such that:

(i) $\Phi(k'gk) = k' \circ \Phi(g) \circ k$ for k, k' in K, g in G, where we have written k, k' for the endomorphisms $v \mapsto kv,\ v' \mapsto k'v'$ of V and of V';

(ii) The support of Φ is a finite union of double cosets KgK.

The natural map $\mathcal{H}(G, K, V, V') \times \text{c-Ind}_K^G V \to \text{c-Ind}_K^G V'$ is given by convolution

$$(\Phi * f)(g) = \sum_{h \in G/K} \Phi(h)(f(h^{-1}g)) = \sum_{h \in K\backslash G} \Phi(gh^{-1})(f(h)).$$

The composition

$$\mathcal{H}(G, K, V', V'') \times \mathcal{H}(G, K, V, V') \to \mathcal{H}(G, K, V, V'')$$

corresponding to the composition of intertwiners is given by convolution

$$(\Phi * \Psi)(g) = \sum_{h \in G/K} \Phi(h)\Psi(h^{-1}g) = \sum_{h \in K\backslash G} \Phi(gh^{-1})\Psi(h)$$

(the term $\Phi(h)\Psi(h^{-1}g)(v)$ vanishes, for fixed $g \in G$ and $v \in V$, outside finitely many cosets Kh, so that the sum makes sense). The map

$$\text{Hom}_G\big(\text{c-Ind}_K^G V, \text{c-Ind}_K^G V'\big) \to \text{Hom}_M\big(\text{c-Ind}_{M\cap K}^M V_{N\cap K}, \text{c-Ind}_{M\cap K}^M V'_{N\cap K}\big)$$

taking b to $\mathcal{S}'(b)$ translates into a map

$$\mathcal{S}' : \mathcal{H}(G, K, V, V') \to \mathcal{H}(M, M \cap K, V_{N\cap K}, V'_{N\cap K}).$$

The next proposition shows that our definition of $\mathcal{S}'$ is equivalent to Herzig's.

Proposition 2.2. *The homomorphism*

$$\mathcal{S}' : \mathcal{H}(G, K, V, V') \to \mathcal{H}\big(M, M \cap K, V_{N\cap K}, V'_{N\cap K}\big)$$

is given by

$$\mathcal{S}'(\Phi)(m)(\bar{v}) = \sum_{n \in (N\cap K)\backslash N} \overline{\Phi(nm)(v)} \quad \text{for } m \in M, v \in V,$$

where bars indicate the image in $V_{N\cap K}$ of elements in V and similarly for V'.

Proof. Let $b \in \text{Hom}_G\big(\text{c-Ind}_K^G V, \text{c-Ind}_K^G V'\big)$ and $\Phi_b \in \mathcal{H}(G, K, V, V')$ corresponding to b. We have, by (5),

$$\mathcal{S}'(\Phi_b) = \Phi_{\mathcal{S}'(b)} = \Phi_{j(I_{V'} \circ b)}.$$

For $g \in G$, $v \in V$, $m \in M$, we have $\Phi_b(g)(v) = b([1, v]_K)(g)$ in V' and

$$\mathcal{S}'(\Phi_b)(m)(\bar{v}) = \big(j(I_{V'} \circ b)\big)([1, \bar{v}]_{M\cap K})(m) = \big((I_{V'} \circ b)([1, v]_K)(1)\big)(m)$$

in $V'_{N\cap K}$. Using the Iwasawa decomposition, we write in $\text{c-Ind}_K^G V$

$$b([1, v]_K) = \sum_h h^{-1}[1, \Phi_b(h)(v)]_K$$

for h running over a system of representatives of $(P \cap K)\backslash P$.

We compute now the element $I_{V'}\big(h^{-1}[1, \Phi_b(h)(v)]_K\big)(1)$ of $\text{c-Ind}_{M\cap K}^{M} V'_{N\cap K}$. As $I_{V'}$ is G-equivariant, we have in $\operatorname{Ind}_P^G\big(\text{c-Ind}_{M\cap K}^{M} V'_{N\cap K}\big)$

$$I_{V'}\big(h^{-1}[1, \Phi_b(h)(v)]_K\big) = h^{-1} I_{V'}\big([1, \Phi_b(h)(v)]_K\big).$$

Taking the value at the unit element 1 of G, we obtain

$$\begin{aligned}\big(h^{-1} I_{V'}\big([1, \Phi_b(h)(v)]_K\big)\big)(1) &= I_{V'}\big([1, \Phi_b(h)(v)]_K\big)(h^{-1}) \\ &= h^{-1}\big(I_{V'}([1, \Phi_b(h)(v)]_K)(1)\big).\end{aligned}$$

Recalling (3), this is equal to

$$h^{-1}[1, \overline{\Phi_b(h)(v)}]_{M\cap K} = m_{h^{-1}}[1, \overline{\Phi_b(h)(v)}]_{M\cap K} = m_h^{-1}[1, \overline{\Phi_b(h)(v)}]_{M\cap K},$$

where m_h is the image of h in M. We deduce

$$(I_{V'} \circ b)([1, v]_K)(1) = \sum_h m_h^{-1}[1, \overline{\Phi_b(h)(v)}]_{M\cap K}.$$

For m in a system of representatives of $(M\cap K)\backslash M$, and n in a system of representatives of $(N\cap K)\backslash N$, the elements nm form a system of representatives of $(P\cap K)\backslash P$. We obtain

$$\begin{aligned}(I_{V'} \circ b)([1, v]_K)(1) &= \sum_{m\in(M\cap K)\backslash M} m^{-1}[1, w_m]_{M\cap K}, \\ w_m &:= \sum_{n\in(N\cap K)\backslash N} \overline{\Phi_b(nm)(v)}.\end{aligned} \qquad \square$$

In [Henniart and Vigneras 2011] we constructed a Satake homomorphism

$$\begin{aligned}\mathcal{S} : \mathcal{H}(G, K, V, V') &\to \mathcal{H}(M, M\cap K, V^{N\cap K}, V'^{N\cap K}), \\ \mathcal{S}(\Phi)(m)(v) &= \sum_{n\in N/(N\cap K)} \Phi(mn)(v),\end{aligned}$$

for $v \in V^{N\cap K}$. To compare $\mathcal{S}'$ with $\mathcal{S}$ we need to take the dual. Remark that K acts on the dual space $V^* = \operatorname{Hom}_C(V, C)$ of V via the contragredient representation, and that the dual of V^* is isomorphic to V by our finiteness hypothesis on V. It is straightforward to verify that the map

$$\iota : \mathcal{H}(G, K, V'^*, V^*) \to \mathcal{H}(G, K, V, V'), \quad \iota(\Phi)(g) := (\Phi(g^{-1}))^t,$$

where the upper index t indicates the transpose, is a C-isomorphism, and satisfies

$$\iota(\Phi * \Psi) = \iota(\Psi) * \iota(\Phi)$$

for $\Phi \in \mathcal{H}(G, K, V'^*, V^*)$, $\Psi \in \mathcal{H}(G, K, V''^*, V'^*)$.

The linear forms on V which are $(N \cap K)$-fixed identify with the linear forms on $V_{N\cap K}$,

$$(V_{N\cap K})^* \simeq (V^*)^{N\cap K},$$

and similarly for V' and V''. This leads to a natural C-linear isomorphism

$$\iota_M : \mathcal{H}\big(M, M\cap K, (V'^*)^{N\cap K}, (V^*)^{N\cap K}\big) \to \mathcal{H}\big(M, M\cap K, V_{N\cap K}, V'_{N\cap K}\big).$$

The following proposition describes the relation between the Satake homomorphisms $\mathcal{S}$ attached to V'^*, V^* and $\mathcal{S}'$ attached to V, V'.

Proposition 2.3. *The following diagram is commutative*:

$$\begin{tikzcd} \mathcal{H}(G, K, V'^*, V^*) \arrow[r, "\mathcal{S}"] \arrow[d, "\iota"'] & \mathcal{H}\big(M, M\cap K, (V'^*)^{N\cap K}, (V^*)^{N\cap K}\big) \arrow[d, "\iota_M"'] \\ \mathcal{H}(G, K, V, V') \arrow[r, "\mathcal{S}'"] & \mathcal{H}\big(M, M\cap K, V_{N\cap K}, V'_{N\cap K}\big). \end{tikzcd}$$

Proof. For $\Phi \in \mathcal{H}(G, K, V'^*, V^*)$, $m \in M$ and $v \in V$ of image $\bar{v}$ in $V_{N\cap K}$, we have:

$$\begin{aligned} \big((\iota_M \circ \mathcal{S})\Phi\big)(m)(\bar{v}) &= \big(\mathcal{S}(\Phi)(m^{-1})\big)^t(\bar{v}) \\ &= \sum_{n\in N/(N\cap K)} \overline{\big(\Phi(m^{-1}n)\big)^t(v)} \\ &= \sum_{n\in (N\cap K)\backslash N} \overline{\big(\Phi((nm)^{-1})\big)^t(v)} \\ &= \sum_{n\in (N\cap K)\backslash N} \overline{\iota(\Phi)(nm)(v)} = \big((\mathcal{S}'\circ\iota)\Phi\big)(m)(\bar{v}). \qquad \square \end{aligned}$$

By this proposition, the Satake map $\mathcal{S}$ is injective if and only if the map $\mathcal{S}'$ is injective because the maps ι and ι_M are isomorphisms.

Proposition 2.4. *Let V be a finite-dimensional smooth C-representation of K. If the homomorphisms $\mathcal{S}' : \mathcal{H}(G, K, V', V) \to \mathcal{H}\big(M, M\cap K, V'_{N\cap K}, V_{N\cap K}\big)$ are injective for all irreducible C-smooth representations V' of K, then the intertwiner*

$$I_V : \text{c-Ind}_K^G V \to \text{Ind}_P^G\big(\text{c-Ind}_{M\cap K}^M V_{N\cap K}\big)$$

is injective.

Proof. Assume that I_V is not injective. Then the kernel of I_V is a nonzero subrepresentation of $\text{c-Ind}_K^G V$, and contains an irreducible smooth $C[K]$-representation V'. By Frobenius reciprocity, we get a nonzero intertwiner

$$b \in \text{Hom}_G\big(\text{c-Ind}_K^G V', \text{c-Ind}_K^G V\big)$$

such that $I_V \circ b = 0$. By assumption, the map

$$\mathcal{S}' : \mathcal{H}(G, K, V', V) \to \mathcal{H}\big(M, M \cap K, V'_{N\cap K}, V_{N\cap K}\big)$$

is injective. By the relation (5), this means that the map

$$\mathrm{Hom}_G\big(\text{c-Ind}_K^G V', \text{c-Ind}_K^G V\big) \to \mathrm{Hom}_M\big(\text{c-Ind}_{M\cap K}^M V'_{N\cap K}, \text{c-Ind}_{M\cap K}^M V_{N\cap K}\big)$$

taking b to $j(I_V \circ b)$ is injective, which gives a contradiction. □

This criterion for the injectivity of I_V was communicated to us by Noriyuki Abe.

3. Representations of $G(k)$

Let C be an algebraically closed field of positive characteristic p, let k be a finite field of the same characteristic p and of cardinality q, and let G be a connected reductive group over k. We fix a minimal parabolic k-subgroup B of G with unipotent radical U and maximal k-subtorus T. Let S be the maximal k-split subtorus of T, let $W = W_G = W(S, G)$ be the Weyl group, let $\Phi = \Phi_G$ be the roots of S with respect to U (called positive), and let $\Delta \subset \Phi$ be the subset of simple roots. For $a \in \Phi$, let U_a be the unipotent subgroup denoted in [Bruhat and Tits 1984, 5.1] by $U_{(a)}$. A parabolic k-subgroup P of G containing B is called standard, and has a unique Levi decomposition $P = MN$ with Levi subgroup M (called standard) containing T. The standard parabolic subgroup $P = MU = UM$ is determined by M. There exists a unique subset $\Delta_M \subset \Delta$ such that M is generated by T, U_a, U_{-a} for a in the subset Φ_M of Φ generated by Δ_M. This determines a bijection between the subsets of Δ and the standard parabolic k-subgroups of G.

Let $\bar{B} = T\overline{U}$ be the opposite of $B = TU$, and $\bar{P} = M\bar{N}$ the opposite of P. We have $\bar{B} = w_0 B w_0^{-1}$, where $w_0 = w_0^{-1}$ is the longest element of W. The roots of S with respect to $\overline{U}$, that is, the positive roots for $\overline{U}$, are the negative roots for U. The simple roots for $\overline{U}$ are the roots $-a$ for $a \in \Delta$.

For $a \in \Delta$, let $G_{a,k} \subset G(k)$ be the subgroup generated by the unipotent subgroups $U_a(k)$ and $U_{-a}(k)$, and let $T_{a,k} := G_{a,k} \cap T(k)$.

Definition 3.1. Let $\psi : T(k) \to C^*$ be a C-character of $T(k)$. We denote by

$$\Delta_\psi := \{a \in \Delta \mid \psi(T_{a,k}) = 1\}$$

the set of simple roots a such that ψ is trivial on $T_{a,k}$.

Example 3.2. $G = \mathrm{GL}(n)$ and S is the diagonal group. Then $T = S$ and the groups T_a for $a \in \Delta$ are the subgroups $T_i \subset T$ for $1 \le i \le n-1$, with coefficients $x_i = x_{i+1}^{-1}$ and $x_j = 1$ otherwise. When $k = \mathbb{F}_2$ is the field with 2 elements, $T(k)$ is the trivial group.

Let V be an irreducible C-representation of $G(k)$. When $P = MN$ is a standard parabolic subgroup of G, we recall that the natural action of $M(k)$ on $V^{N(k)}$ is irreducible [Cabanes and Enguehard 2004, Theorem 6.12]. In particular, taking the Borel subgroup $B = TU$, the dimension of the vector space $V^{U(k)}$ is 1 and the group $T(k)$ acts on $V^{U(k)}$ by a character ψ_V.

Proposition 3.3. *The stabilizer in $G(k)$ of the line $V^{U(k)}$ is $P_V(k)$, where $P_V = M_V N_V$ is a standard parabolic subgroup of G associated to a subset $\Delta_V \subset \Delta_{\psi_V}$.*

Proof. The stabilizer of $V^{U(k)}$ contains $B(k)$, and hence is of the form $P_V(k)$ for a standard parabolic subgroup P_V of G associated to the set Δ_V of simple roots $a \in \Delta$ such that $U_{-a}(k)$ acts trivially on $V^{U(k)}$. When $U_{-a}(k)$ acts trivially on $V^{U(k)}$, so does $G_{a,k}$ by definition of this group, implying that a belongs to Δ_{ψ_V}, by definition of this set. □

Corollary 3.4. *The dimension of V is 1 if and only if $P_V = G$.*

Proof. If the dimension of V is 1, then $V = V^{U(k)}$ and $P_V = G$. Conversely, if $P_V = G$, the line $V^{U(k)}$ is stable by $G(k)$, and hence is equal to V because V is irreducible. □

Corollary 3.5. *When $P = MN$ is a standard parabolic subgroup of G, the dimension of $V^{N(k)}$ is equal to 1 if and only if $P \subset P_V$.*

The group P_V measures the irregularity of V. A 1-dimensional representation V is as little regular as possible ($P_V = G$), and in general V is as regular as possible when $P_V = B$.

Definition 3.6. Let P be any parabolic k-subgroup of G. We say that V is P-regular when the stabilizer in $G(k)$ of the line $V^{U(k)}$ is contained in $P(k)$, where U is the unipotent radical of a minimal parabolic k-subgroup of G contained in P.

The definition depends only on P and not on the choice of U. The reason is that for a parabolic k-subgroup $P' \subset P$ of G and $g \in G(k)$, we have $gP'g^{-1} \subset P$ if and only if $g \in P(k)$. As in the proof of [Borel and Tits 1965, Proposition 4.4 a)], the inclusion $P' \subset g^{-1}Pg \cap P$ implies $g^{-1}Pg = P$, and $g \in P(k)$ because P is equal to its own normalizer and is conjugate to a unique k-subgroup containing P'.

We recall the classification of the irreducible C-representations V of $G(k)$.

Theorem 3.7. *The isomorphism class of V is characterized by ψ_V and $\Delta_V \subset \Delta_{\psi_V}$. For each C-character ψ of $T(k)$ and each subset $J \subset \Delta_\psi$, there exists a C-irreducible representation V of $G(k)$ such that $\psi_V = \psi$, $\Delta_V = J$.*

Proof. [Curtis 1970, Theorem 5.7]. □

Definition 3.8. (ψ_V, Δ_V) are called the standard parameters of V.

Example 3.9. The irreducible representations V with $\psi_V = 1$ are classified by the subsets of Δ. They are sometimes called the special representations or the generalized Steinberg representations. We denote by Sp_P the representation such that $\Delta_V = \Delta_M$ for a standard parabolic group $P = MN$. The representation Sp_G is the trivial character and Sp_B is the classical Steinberg representation.

Let $P = MN$ be a standard parabolic k-subgroup of G. For V an irreducible C-representation of $G(k)$ with standard parameters (ψ_V, Δ_V), the C-representation $V^{N(k)}$ of $M(k)$ is irreducible of standard parameters $(\psi_V, \Delta_V \cap \Delta_M)$ [Henniart and Vigneras 2011, 5.7(i)].

Proposition 3.10. *The P-regular irreducible C-representations V of $G(k)$ are in bijection with the irreducible representations of $M(k)$ by the map $V \mapsto V^{N(k)}$. Those representations V with $M_V = M$ correspond to the characters of $M(k)$.*

Proof. Fix an irreducible representation W of $M(k)$ with standard parameters (ψ_W, Δ_W). For an irreducible representation V of $G(k)$ with standard parameters (ψ_V, Δ_V), we have $V^{N(k)} \simeq W$ if and only if $\psi_V = \psi_W$ and $\Delta_W = \Delta_V \cap \Delta_M$. Moreover, V is P-regular if and only if $\Delta_V \subset \Delta_M$. This implies the first claim, and the second one follows from Corollary 3.5. $\square$

If instead of choosing B, we choose the Borel subgroup $\bar{B}$ opposite to B, then V has other parameters that we call antistandard and write $(\overline{\psi}_V, \overline{\Delta}_V)$.

Lemma 3.11. *The antistandard parameters of V are $\overline{\psi}_V = w_0(\psi_V)$, $\overline{\Delta}_V = w_0(\Delta_V)$.*

Proof. As $\bar{B} = w_0 B w_0^{-1}$, the torus $T(k)$ acts by the character $w_0(\psi_V)$ on the line $V^{\overline{U}(k)}$ and $\bar{P}_V = w_0 P_V w_0^{-1}$ is the stabilizer of the line $V^{\overline{U}(k)}$. Hence, the subset $\overline{\Delta}_V$ of simple roots is equal to $w_0(\Delta_V) \subset -\Delta$. $\square$

The contragredient representation V^* of V is irreducible and its standard parameters are:

Lemma 3.12. $\psi_{V^*} = w_0(\psi_V)^{-1}$, $\Delta_{V^*} = -w_0(\Delta_V)$.

Proof. By Lemma 3.11, it is equivalent to describe the antistandard parameters $(\overline{\psi}_{V^*}, \overline{\Delta}_{V^*})$ of V^*. The direct decomposition $V = V^{U(k)} \oplus (1 - \overline{U}(k))V$ (see Proposition 3.14 below) gives a $T(k)$-equivariant isomorphism:

$$(V^*)^{\overline{U}(k)} = (V_{\overline{U}(k)})^* \simeq (V^{U(k)})^*.$$

The group $T(k)$ acts on the line $V^{U(k)}$ by the character ψ_V and on $(V^{U(k)})^*$ by the character ψ_V^{-1}. Hence $\overline{\psi}_{V^*} = \psi_V^{-1}$.

The space $(V^*)^{\overline{U}(k)}$ is the subspace of elements on V^* vanishing on $(1 - \overline{U}(k))V$. This space is stable by $M_V(k)$ because the direct decomposition of V for B is the same as for P_V (Remark 3.15 below). Hence $M_V \overline{U} \subset \bar{P}_{V^*}$, or equivalently,

$-\Delta_V \subset \overline{\Delta}_{V^*} = w_0(\Delta_{V^*})$. As V is isomorphic to the contragredient of V^* and $-w_0$ is an involution on Δ, we have also the inclusion in the other direction. $\square$

Remark 3.13. In general, $-w_0$ does not act trivially on Δ (for example for $G = \mathrm{GL}(3)$), and hence the stabilizer $\overline{P}_V$ of $V^{\overline{U}(k)}$ in $G(k)$ is not the opposite of P_V, and the P-regularity of V is not equivalent to the $\overline{P}$-regularity of V. The P-regularity of V is equivalent to the $\overline{P}$-regularity of V^*.

For a subgroup $H \subset G(k)$ and a subspace $W \subset V$, the notation $(1-H)W$ denotes the subspace of V linearly generated by the elements $v - hv$ for all $h \in H$ and $v \in W$.

Proposition 3.14. *We have the $M(k)$-equivariant direct decomposition*

$$V = V^{N(k)} \oplus (1 - \overline{N}(k))V^{N(k)} = V^{N(k)} \oplus (1 - \overline{N}(k))V,$$

which gives an $M(k)$-isomorphism $V^{N(k)} \to V_{\overline{N}(k)}$.

Proof. [Cabanes and Enguehard 2004, Theorem 6.12]. $\square$

Remark 3.15. The decompositions of V for $P = P_V$ and for $P = B$ are the same, because $V^{U(k)} = V^{N_V(k)}$ by the definition of P_V.

Proposition 3.16. *For $g \in G(k)$, the image of $gV^{U(k)}$ in $V_{\overline{N}(k)}$ is not 0 if and only if $g \in \overline{P}(k)P_V(k)$.*

Proof. It is clear that the nonvanishing condition on g depends only on $\overline{P}(k)gP_V(k)$ and that the image is not 0 when $g = 1$ as $V^{U(k)} \subset V^{N(k)} \simeq V_{\overline{N}(k)}$ (Proposition 3.14). We prove that the image of $gV^{U(k)}$ in $V_{\overline{N}(k)}$ is 0 when g does not belong to $\overline{P}(k)P_V(k)$. For convenience, we write in this proof $P_V = P' = M'N'$.

a) We reduce to the case where G_{der} is simply connected by choosing a z-extension defined over k,

$$1 \to R \to G_1 \to G \to 1,$$

where $R \subset G_1$ is a central induced k-subtorus and G_1 is a connected reductive k-group with $G_{1,\mathrm{der}}$ simply connected. The sequence of rational points

$$1 \to R(k) \to G_1(k) \to G(k) \to 1$$

is exact. The parabolic subgroups of G_1 inflated from P, P' are $P_1 = M_1N$, $P_1' = M_1'N'$, where $1 \to R \to M_1 \to M \to 1$ and $1 \to R \to M_1' \to M' \to 1$ are z-extensions defined over k. We consider V as an irreducible representation of $G_1(k)$ where $R(k)$ acts trivially. The image of $G_1(k) - \overline{P}_1(k)P_1'(k)$ in $G(k)$ is $G(k) - \overline{P}(k)P'(k)$. For $g_1 \in G_1(k) - \overline{P}_1(k)P_1'(k)$ of image $g \in G(k) - \overline{P}(k)P'(k)$, the image of $g_1V^{N'(k)}$ in $V_{\overline{N}(k)}$ is 0 if and only if the image of $gV^{N'(k)}$ in $V_{\overline{N}(k)}$ is 0.

b) The proposition can be reformulated in terms of Weyl groups because the equality depends only on the image of g in $\overline{P}(k)\backslash G(k)/P'(k) = W_M\backslash W/W_{M'}$. We denote by $\dot{w}$ a representative of $w \in W$ in $G(k)$. The proposition says that the image of $\dot{w}V^{N'(k)}$ in $V_{\overline{N}(k)}$ is 0 if $w \in W$ does not belong to $W_M W_{M'}$.

c) Given a), we now suppose that G_{der} is simply connected. In this case, V is the restriction of an irreducible algebraic representation $F(\nu)$ of G with highest weight ν equal to a q-restricted character of T [Herzig 2009, Appendix 1.3]. The stabilizer W_ν of ν in W is $W_{M'}$, the irreducible algebraic representation $F(\nu)$ of M with highest weight ν is $F(\nu)^N$, and $F(\nu)^N$ is equal to the sum of all weight spaces $F(\nu)_\mu$ with $\nu - \mu \in \mathbb{Z}\Phi_M$; for $w \in W$, $w\nu$ is a weight of $F(\nu)^N$ if and only if $w \in W_M W_{M'}$ [Herzig 2011, Lemma 2.3, and proof of Lemma 2.17 in the split case]. The quotient map $t : F(\nu) \to F(\nu)_{\overline{N}}$ restricts to an M-equivariant isomorphism $F(\nu)^N \to F(\nu)_{\overline{N}}$. We deduce that the weights of $F(\nu)_{\overline{N}}$ are the weights of $F(\nu)^N$ and are disjoint from the weights of the kernel of the quotient map t. In particular, for $w \in W$, the space $w(F(\nu)^U)$ is not in the kernel of t if and only if $w \in W_M W_{M'}$.

The space $V^{N(k)}$ is the restriction to $M(k)$ of $F(\nu)^N$ and the space $V_{\overline{N}(k)}$ is the restriction to $M(k)$ of $F(\nu)_{\overline{N}}$. This implies the proposition under the form given in b). ☐

Corollary 3.17. *Let $P' = M'N'$ be another standard parabolic subgroup. The image of $gV^{N'(k)}$ in $V_{\overline{N}(k)}$ is not 0 if and only if $g \in \overline{P}(k)P_V(k)P'(k)$.*

Proof. We have $V^{N'(k)} = \sum_{h \in M'(k)} hV^{U(k)}$ because the right-hand side is $N'(k)$-stable and $V^{N'(k)}$ is an irreducible representation of $M'(k)$. ☐

Remark 3.18. The equality $\overline{P}P_V P' = \overline{P}P'$ is equivalent to $P_V \subset \overline{P}P'$. The latter inclusion is obviously true when V is P-regular or P'-regular.

In our study of Hecke operators, we will use the following particular case:

Corollary 3.19. *For $g \in G(k)$, the image of $gV^{\overline{N}(k)}$ in $V_{N(k)}$ is not 0 if and only if $g \in P(k)\overline{P}_V(k)\overline{P}(k)$.*

4. Representations of $G(F)$

4A. ***Notation.*** Let C be an algebraically closed field of positive characteristic p, let F be a local nonarchimedean field of finite residue field k of characteristic p and of cardinality q, of ring of integers o_F and uniformizer p_F, and let G be a reductive connected group over F. We fix a minimal parabolic F-subgroup B of G with unipotent radical U and maximal F-split F-subtorus S. The group B has the Levi decomposition $B = ZU$, where Z is the G-centralizer of S. Let $\Phi(S, U)$ be the set of roots of S in U (called positive for U) and let $\Delta \subset \Phi(S, U)$ be the subset of simple roots. A parabolic k-subgroup P of G containing B is called standard (for U), and has a unique Levi decomposition $P = MN$ with Levi subgroup M containing Z (called standard), and unipotent radical N. The group $(M \cap B) = Z(M \cap U)$

is a minimal parabolic F-subgroup of M, and $\Delta_M = \Delta \cap \Phi(S, M \cap U)$ is the set of simple roots of $\Phi(S, M \cap U)$. This procedure determines bijections between the subsets of Δ, the standard parabolic k-subgroups of G, and their standard Levi subgroups.

The natural homomorphism $v : S(F) \to \mathrm{Hom}(X^*(S), \mathbb{Z})$, where $X^*(S)$ is the group of F-characters of S, extends uniquely to a homomorphism $v : Z(F) \to \mathrm{Hom}(X^*(S), \mathbb{Q})$; its kernel is the maximal compact subgroup of $Z(F)$. For a standard Levi subgroup M, we denote by $Z(F)^{+N}$ the monoid of elements z in $Z(F)$ which are N-positive, that is,

$$a(v_Z(z)) \geq 0 \quad \text{for all } a \in \Delta - \Delta_M.$$

When these inequalities are strict, z is called strictly N-positive. We denote by $Z(F)^{+\overline{N}}$ the monoid of elements in $Z(F)$ which are $\overline{N}$-positive, that is, N-negative,

$$a(v_Z(z)) \leq 0 \quad \text{for all } a \in \Delta - \Delta_M.$$

When $N = U$, we write $Z(F)^+ := Z(F)^{+U}$ and $Z(F)^- := Z(F)^{+\overline{U}}$, and if the inequalities are strict, z is called strictly positive or strictly negative. These notations extend to M; we write $Z(F)^{+M} = Z(F)^{+(U \cap M)}$.

In the building of the adjoint group G_{ad} over F, we choose a special vertex in the apartment attached to S and we write K for the corresponding special parahoric subgroup, as in [Henniart and Vigneras 2011, 6.1]. The quotient of K by its pro-p-radical K_+ is the group of k-points of a connected reductive k-group G_k. The group K/K_+ is $G_k(k)$. For $H = B, S, U, Z, P, M, N$, the image in $G_k(k)$ of $H(F) \cap K$ is the group of k-points of a connected k-group H_k. Note that B_k is a minimal parabolic subgroup of G_k, S_k is a maximal k-split torus in B_k, Z_k (being the centralizer of S_k in G_k) is a maximal k-subtorus of B_k, and $B_k = Z_k U_k$ is a Levi decomposition; moreover, there is a bijection between Δ and the set Δ_k of simple roots of S_k (with respect to U_k), P_k is a standard parabolic subgroup of G_k of standard Levi subgroup M_k and unipotent radical N_k, and the set Δ_{k,M_k} of simple roots of S_k in M_k is the image of Δ_M by the bijection above. We shall usually suppress the indices k from the notation, write $H_0 = H(F) \cap K$, and identify a character of $Z(k)$ (with the notations in the chapter on representations of $G(k)$ we have $T(k) = Z(k)$) with a smooth character of Z_0.

We now fix an irreducible C-representation V of $G(k)$ with parameters (ψ_V, Δ_V) (Definition 3.8), *a proper standard parabolic subgroup $P = MN$ of G, and an element $s \in S(F)$ central in $M(F)$ and strictly N-positive* (*and hence U-positive*).

4B. ***$\mathcal{S}'$ is a localization.*** We also see V as a smooth C-representation of K, trivial on K_+. We apply the generalities of the Satake homomorphisms to the group $G(F)$, the compact subgroup K, and the closed subgroup $P(F) = M(F)N(F)$. As K is a special parahoric subgroup, the Iwasawa decomposition $G(F)=P(F)K$ is valid. We have a $G(F)$-equivariant linear map (Definition 2.1)

$$I_V \;:\; \text{c-Ind}_K^{G(F)} V \;\to\; \text{Ind}_{P(F)}^{G(F)}\big(\text{c-Ind}_{M_0}^{M(F)} V_{N(k)}\big) \tag{7}$$

and an algebra homomorphism (Proposition 2.2)

$$\mathcal{S}'=\mathcal{S}'_{M,G}:\mathcal{H}\big(G(F),K,V\big)\to\mathcal{H}\big(M(F),M_0,V_{N(k)}\big), \tag{8}$$

related by $I_V(bf)=\mathcal{S}'(b)I_V(f)$ for $b\in\mathcal{H}(G(F),K,V)$ and f in $\text{c-Ind}_K^{G(F)} V$.

Proposition 4.1. *The intertwiner I_V and the algebra homomorphism $\mathcal{S}'$ are injective.*

Proof. Apply Proposition 2.4 and [Henniart and Vigneras 2011, 7.9], giving the injectivity of the Satake homomorphism $\mathcal{S}$ appearing in Proposition 2.3 when V, V' are irreducible smooth representations of K over a field of characteristic p. □

We write $\mathcal{S}'_G=\mathcal{S}'_{Z,G}$ and denote by $\mathcal{S}_G$ the corresponding Satake homomorphisms appearing in Proposition 2.3 when $M=Z$. We analogously define $\mathcal{S}'_M$ and $\mathcal{S}_M$ with a commutative diagram of algebra homomorphisms:

$$\begin{array}{ccc}
\mathcal{H}\big(M,M_0,(V^*)^{N(k)}\big) & \xrightarrow{\mathcal{S}_M} & \mathcal{H}\big(Z,Z_0,(V^*)^{U(k)}\big) \\
{\scriptstyle \iota_M}\big\downarrow & & \big\downarrow{\scriptstyle \iota_Z} \\
\mathcal{H}\big(M,M_0,V_{N(k)}\big)^0 & \xrightarrow{\mathcal{S}'^{\,0}_M} & \mathcal{H}\big(Z,Z_0,V_{U(k)}\big)^0,
\end{array}$$

where $(\iota_*(\Phi))(g)=\Phi(g^{-1})^t$ for $*=M$ or Z (definition before Proposition 2.3). In this diagram, A^0 denotes the opposite of an algebra A and $f^0:A^0\to B^0$ is the algebra homomorphism $a\mapsto f^0(a)=f(a)$ associated to an algebra homomorphism $f:A\to B$. By the transitivity relation of the Satake homomorphisms [Henniart and Vigneras 2011, Proposition 2.8] and by Proposition 2.3, we have

$$\mathcal{S}'_G=\mathcal{S}'_M\circ\mathcal{S}'. \tag{9}$$

Recalling the standard parameters $(\psi_{V^*},\Delta_{V^*})$ of V^*, we identify ψ_{V^*} with a smooth character of Z_0, and we denote by

$$Z_{V^*}=\{z\in Z(F)\mid \psi_{V^*}(zxz^{-1})=\psi_{V^*}(x)\text{ for all }x\in Z_0\}$$

the stabilizer of ψ_{V^*} in $Z(F)$. As $\psi_{V^*}=w_0(\psi_V)^{-1}$ (Lemma 3.12), we have $Z_{V^*}=w_0(Z_V)$.

Proposition 4.2. *The image of the map*

$$\mathcal{S}'_G:\mathcal{H}(G(F),K,V)\to\mathcal{H}(Z(F),Z_0,V_{U(k)})$$

is equal to $\mathcal{H}\big(Z(F)^+\cap Z_{V^*},Z_0,V_{U(k)}\big)$.

Proof. The support of a Hecke operator in $\mathcal{H}\big(Z(F), Z_0, (V^*)^{U(k)}\big)$ is contained in Z_{V^*}. By [Henniart and Vigneras 2011, Theorem 1.8], the image of $\mathcal{S}_G$ consists of the Hecke operators with negative support, that is, with support contained in $Z(F)^- \cap Z_{V^*}$. The image of $\iota_Z \circ \mathcal{S}_G$ consists of the Hecke operators with positive support, that is, of support in $Z(F)^+ \cap Z_{V^*}$, because the inverse map permutes the monoids $Z(F)^+$ and $Z(F)^-$ and respects Z_{V^*}. □

Analogously, the image of $\mathcal{S}'_M$ is $\mathcal{H}\big(Z(F)^{+_M} \cap Z_{V^*}, Z_0, V_{U(k)}\big)$.

Definition 4.3. A ring morphism $f : A \to B$ is a localization at $b \in B$ if f is injective, $b \in f(A)$ is central and invertible in B, and $B = \cup_{n \in \mathbb{N}} f(A) b^{-n}$.

There exists a unique Hecke operator T_Z central in $\mathcal{H}\big(Z(F)^+ \cap Z_{V^*}, Z_0, V_{U(k)}\big)$ with support $Z_0 s$ such that $T_Z(s) = 1$, because s is U-positive and belongs to $S(F)$ contained in Z_{V^*}.

The algebra $\mathcal{H}\big(Z(F)^{+_M} \cap Z_{V^*}, Z_0, V_{U(k)}\big)$ is the localization of

$$\mathcal{H}\big(Z(F)^+ \cap Z_{V^*}, Z_0, V_{U(k)}\big)$$

at T_Z because, for any $U \cap M$-positive element $z \in Z(F)$, there exists a positive integer n such that $s^n z$ belongs to $Z(F)^+$, because $s \in S(F)$ is strictly N-positive.

Definition 4.4. As s is central in $M(F)$ and contained in Z_{V^*}, there exists a unique Hecke operator T_M in $\mathcal{H}(M(F), M_0, V_{N(k)})$ with support $M_0 s$ with value $\mathrm{id}_{V_{N(k)}}$ at s.

The Hecke operator T_M is central and invertible in $\mathcal{H}(M(F), M_0, V_{N(k)})$; it acts on $\text{c-Ind}_{M_0}^{M(F)} V_{\overline{N}(k)}$ by $T_M([1, \bar{v}]_{M_0}) = s^{-1}[1, \bar{v}]_{M_0}$ for $v \in V$. We also denote by T_M the $G(F)$-endomorphism of

$$\mathrm{Ind}_{P(F)}^{G(F)}\big(\text{c-Ind}_{M_0}^{M(F)} V_{N(k)}\big),$$

such that $T_M(f)(g) = T_M(f(g))$ for $f \in \mathrm{Ind}_{P(F)}^{G(F)}\big(\text{c-Ind}_{M_0}^{M(F)} V_{N(k)}\big)$ and $g \in G(F)$. Using Proposition 2.2, we see that

$$\mathcal{S}'_M(T_M) = T_Z, \tag{10}$$

because $(U \cap M)(F) z \cap M_0 s = \big((U \cap M)(F) z s^{-1} \cap M_0\big) s = (U_0 \cap M_0) z$ if $z s^{-1} \in Z_0$ and is 0 otherwise. The Hecke operator T_M belongs to the image of $\mathcal{S}'$, because T_Z belongs to the image of $\mathcal{S}'_G$ by construction, $\mathcal{S}'$ is injective, and we have (10), (9). We have shown:

Proposition 4.5. *The map $\mathcal{S}'$ is a localization at T_M.*

In (7), we consider the map I_V as a $C[T]$-linear map, T acting on the left side by $(\mathcal{S}')^{-1}(T_M)$ and on the right side by T_M. By Proposition 4.5, the localization of

I_V at T is the $\big(G(F), \mathcal{H}(M(F), M_0, V_{N(k)})\big)$-equivariant map

$$(11) \quad \Theta : \mathcal{H}\big(M(F), M_0, V_{N(k)}\big) \otimes_{\mathcal{H}(G(F),K,V),\mathcal{S}'} \text{c-Ind}_K^{G(F)} V \\ \to \text{Ind}_{P(F)}^{G(F)}\big(\text{c-Ind}_{M_0}^{M(F)} V_{N(k)}\big).$$

The map Θ is injective because I_V is injective (Proposition 4.1). Our main theorem is:

Theorem 4.6. *Θ is surjective if V is $\overline{P}$-regular.*

The theorem will follow from Corollary 6.5 and Proposition 5.4.

4C. ***Decomposition of the intertwiner.*** Following Herzig, we write the intertwiner I_V as a composite of two $G(F)$-equivariant linear maps

$$(12) \quad \begin{tikzcd} & \text{c-Ind}_{\mathcal{P}}^{G(F)} V_{N(k)} \arrow[dr, "\zeta"] & \\ \text{c-Ind}_K^{G(F)} V \arrow[ur, "\xi"] \arrow[rr, "I_V"'] & & \text{Ind}_{P(F)}^{G(F)}\big(\text{c-Ind}_{M_0}^{M(F)} V_{N(k)}\big), \end{tikzcd}$$

which we now define. In this diagram, $\mathcal{P}$ is the inverse image in K of $P(k)$. The image of $\mathcal{P}$ in $G(k)$ is $P(k)$ by [Bruhat and Tits 1984, 5.1.22]; $\mathcal{P}$ is a parahoric subgroup of $G(F)$.

Lemma 4.7. *The parahoric subgroup $\mathcal{P}$ admits an Iwahori decomposition with respect to M,*

$$(13) \qquad \mathcal{P} = N_0 M_0 \overline{N}_{0+}, \quad \overline{N}_{0,+} := \overline{N}(F) \cap K_+,$$

with any order of the factors.

Proof. This decomposition is well known, but at the referee's suggestion, we outline a proof. By [Bruhat and Tits 1984, 4.6.4 and 5.1.31], $K_+ = U_{0+} Z_{0+} \overline{U}_{0+}$, with the sign $+$ indicating the intersection with K_+ as above. As M_0 is the parahoric subgroup of $M(F)$ fixing our special point, we have $M_{0+} = (U_{0+} \cap M_0) Z_{0+} (\overline{U}_{0+} \cap M_0)$. It follows that $K_+ = N_{0+} M_{0+} \overline{N}_{0+}$. From [Henniart and Vigneras 2011, Theorem 6.5], we have $\mathcal{P} = N_0 M_0 K_+$, and so $\mathcal{P} = N_0 M_0 N_{0+} M_{0+} \overline{N}_{0+}$. As M_0 normalizes N_0, $\overline{N}_0$ and K_+, it normalizes also N_{0+} and $\overline{N}_{0+}$, and we have the decomposition $\mathcal{P} = N_0 M_0 \overline{N}_{0+}$ with any order of the factors. $\square$

The transitivity of compact induction implies that

$$(14) \qquad \text{c-Ind}_{\mathcal{P}}^{G(F)} V_{N(k)} \simeq \text{c-Ind}_K^{G(F)}\big(\text{c-Ind}_{P(k)}^{G(k)} V_{N(k)}\big).$$

Definition 4.8. The map ξ is the image by the compact induction functor c-Ind_K^G of the natural embedding $V \to \text{c-Ind}_{P(k)}^{G(k)} V_{N(k)}$.

For $v \in V$, $\xi([1, v]_K)$ is the function in $\text{c-Ind}_{\mathcal{P}}^{G(F)} V_{N(k)}$ with support contained in K and value $\overline{kv}$ at $k \in K$.

Proposition 4.9. *There is a unique $G(F)$-equivariant map*

$$\zeta : \text{c-Ind}_{\mathcal{P}}^{G(F)} V_{N(k)} \to \text{Ind}_{P(F)}^{G(F)} \big(\text{c-Ind}_{M_0}^{M(F)} V_{N(k)}\big),$$

which for $v \in V$, sends $[1, \bar{v}]_{\mathcal{P}}$ to the function $f_{\bar{v}}$ with support contained in

$$P(F)\mathcal{P} = P(F)\overline{N}_{0,+}$$

and constant value $[1, \bar{v}]_{M_0}$ on $\overline{N}_{0,+}$.

Proof. The uniqueness is clear because the functions $[1, \bar{v}]_{\mathcal{P}}$ for $v \in V$ generate the representation $\text{c-Ind}_{\mathcal{P}}^{G(F)} V_{N(k)}$. The existence can be proved directly, but we can also apply the considerations of the beginning of Section 2 with $V' := \text{c-Ind}_{\mathcal{P}}^{K} (V_{N(k)})$ instead of V and $W = \text{c-Ind}_{M_0}^{M(F)} V_{N(k)}$.

The value at 1 from V' to $V_{N(k)}$ factorizes through the quotient map $v' \mapsto \overline{v'}$ from V' to $V'_{N(k)}$ and defines an M_0-equivariant map $r : V'_{N(k)} \to V_{N(k)}$, such that $r(\overline{v'}) = v'(1)$ for all $v' \in V'$. The image of r by the compact induction functor from M_0 to $M(F)$ is an element in

$$\text{Hom}_{M(F)}\big(\text{c-Ind}_{M_0}^{M(F)} V'_{N(k)}, \text{c-Ind}_{M_0}^{M(F)} V_{N(k)}\big)$$

which corresponds by the isomorphism (2) to an element in

$$\text{Hom}_{G(F)}\big(\text{c-Ind}_{K}^{G(F)} V', \text{Ind}_{P(F)}^{G(F)}\big(\text{c-Ind}_{M_0}^{M(F)} V_{N(k)}\big)\big),$$

sending $[1, v']_K$ to the unique function $\varphi_{v'}$ with value on $k \in K$ equal to

$$[1, r(\overline{kv'})]_{M_0} = [1, v'(k))]_{M_0},$$

for all $v' \in V'$. Applying the transitivity of the compact induction functor to $\text{c-Ind}_{K}^{G(F)} V'$, we obtain the element

$$\zeta \in \text{Hom}_{G(F)}\big(\text{c-Ind}_{\mathcal{P}}^{G(F)} V_{N(k)}, \text{Ind}_{P(F)}^{G(F)}\big(\text{c-Ind}_{M_0}^{M(F)} V_{N(k)}\big)\big)$$

of the proposition. For $v \in V$ with image $\bar{v}$ in $N_{N(k)}$, the morphism ζ sends $[1, \bar{v}]_{\mathcal{P}}$ to $\varphi_{v'}$, where $v' \in V'$ is the function on K of support $\mathcal{P}$ and equal to $\bar{v}$ at 1. It remains to check that $\varphi_{v'}$ is equal to the function $f_{\bar{v}}$ given in the proposition. Indeed, the support of the function $\varphi_{v'} \in \text{c-Ind}_{\mathcal{P}}^{G(F)} V_{N(k)}$ is contained in $P(F)\mathcal{P}$, we have $P(F)\mathcal{P} = P(F)\overline{N}_{0,+}$ by the Iwahori decomposition of $\mathcal{P}$, and for $k \in \overline{N}_{0,+}$ we have $v'(k) = \bar{v}$. □

Remark 4.10. Later we will use that, for $g \in G(F)$, $\zeta(g^{-1}[1, \bar{v}]_{\mathcal{P}})$ has support in $P(F)\mathcal{P}g$ which contains 1 if and only if $g \in \mathcal{P}P(F)$. Thus, for $f \in \text{c-Ind}_{\mathcal{P}}^{G(F)} V_{N(k)}$, the element $\zeta(f)(1)$ depends only on the restriction of f to $\mathcal{P}P(F)$.

Lemma 4.11. $I_V = \zeta \circ \xi$.

Proof. This is clear from the definitions of I_V, ξ, ζ. □

Remark 4.12. The map ξ is injective because I_V is injective (Proposition 4.1). We can give a direct proof: As V is irreducible and $V_{N(k)} \neq 0$, the map $V \to \text{c-Ind}_{P(k)}^{G(k)} V_{N(k)}$ is injective. As the functor c-Ind_K^G is exact, the map ξ is injective.

The map ξ is not surjective because the map $V \to \text{c-Ind}_{P(k)}^{G(k)} V_{N(k)}$ is not surjective, as $P \neq G$ by our running hypothesis. This can be seen by taking fixed points under $U(k)$.

5. Hecke operators

In this chapter, we introduce some Hecke operators associated to our fixed element $s \in S(F)$ central in $M(F)$ and strictly N-positive, and we show the compatibility of these Hecke operators with the maps $\xi, \zeta, \mathcal{S}'$ (sometimes we need to suppose that V is $\bar{P}$-regular).

The space of $G(F)$-equivariant homomorphisms from $\text{c-Ind}_K^{G(F)} V$ to

$$\text{c-Ind}_{\mathcal{P}}^{G(F)} V_{N(k)},$$

is isomorphic to the space $\mathcal{H}\big(G(F), K, \mathcal{P}, V, V_{N(k)}\big)$ of functions $\Phi : G(F) \to \text{Hom}_C(V, V_{N(k)})$ satisfying

(i) $\Phi(jgj') = j \circ \Phi(g) \circ j'$ for $j \in \mathcal{P}$, $j' \in K$,

(ii) Φ vanishes outside finitely many double cosets $\mathcal{P}gK$.

We call Φ a Hecke operator. We shall usually use the same notation for the Hecke operator and for the corresponding $G(F)$-equivariant homomorphism, defined by: for all $v \in V$,

$$[1, v]_K \to \sum_{g \in \mathcal{P}\backslash G(F)} g^{-1}[1, \Phi(g)(v)]_{\mathcal{P}}. \tag{15}$$

The map ξ corresponds to the Hecke operator with support K and value at 1 the projection $V \to V_{N(k)}$ given by $v \mapsto \bar{v}$.

In the same way, the space of $G(F)$-equivariant homomorphisms

$$\text{c-Ind}_{\mathcal{P}}^{G(F)} V_{N(k)} \to \text{c-Ind}_K^{G(F)} V$$

corresponds to a space $\mathcal{H}\big(G(F), \mathcal{P}, K, V_{N(k)}, V\big)$ of functions from $G(F)$ to

$$\text{Hom}_C(V_{N(k)}, V).$$

5A. ***Definition of Hecke operators.*** Recall (Proposition 3.14) that the quotient map $v \mapsto \bar{v}$ from V to $V_{N(k)}$ induces an isomorphism $V^{\overline{N}(k)} \to V_{N(k)}$. We write $\varphi : V_{N(k)} \to V^{\overline{N}(k)}$ for the reciprocal isomorphism. Since $s \in S(F)$ is U-positive and belongs to Z_{V^*}, we deduce from [Henniart and Vigneras 2011, 7.3 Lemma 1]:

Proposition 5.1. *There exists a unique Hecke operator T_G in $\mathcal{H}(G(F), K, V)$ with support KsK such that $T_G(s) \in \mathrm{End}_C(V)$ sends $v \in V$ to $\varphi(\bar{v})$.*

The Hecke operator T_M (Definition 4.4) could have been defined in the same way. We shall prove later that $\mathcal{S}'(T_G) = T_M$ when V is $\overline{P}$-regular. We define now Hecke operators $T_{\mathcal{P}}$ and $T_{K,\mathcal{P}}$ generalizing T_G and T_M.

Proposition 5.2. (i) *There is a unique Hecke operator $T_{\mathcal{P}}$ in $\mathcal{H}\big(G(F), \mathcal{P}, V_{N(k)}\big)$ with support $\mathcal{P}s\mathcal{P}$ and value at s the identity of $V_{N(k)}$.*

(ii) *There is a unique Hecke operator $T_{K,\mathcal{P}}$ in $\mathcal{H}\big(G(F), \mathcal{P}, K, V_{N(k)}, V\big)$ with support $Ks\mathcal{P}$ such that $T_{K,\mathcal{P}}(s) : V_{N(k)} \to V$ sends $\bar{v}$ to $\varphi(\bar{v})$.*

Proof. (i) By the condition (i) for Hecke operators, we have to check that for $h, h' \in \mathcal{P}$, the relation $h's = sh$ implies that the actions of h and of h' on $V_{N(k)}$ are the same. We use the Iwahori decomposition (13):

$$\mathcal{P} = \overline{N}_{0+} M_0 N_0.$$

Decomposing $h = \bar{n}mn$, we have $h' = s\bar{n}s^{-1}msns^{-1}$, since s is central in $M(F)$. Because s is N-positive, $sns^{-1} \in N_0$ and the condition $h' \in \mathcal{P}$ means that $s\bar{n}s^{-1} \in \overline{N}_{0+}$. Consequently, both h and h' act as m on $V_{N(k)}$.

(ii) We now have to check that for $h' \in K$, $h \in \mathcal{P}$, the relation $h's = sh$ implies that $h'\varphi(\bar{v}) = \varphi(\overline{hv})$ for all $v \in V$. Writing as above $h = \bar{n}mn$, the condition $h' \in K$ means $s\bar{n}s^{-1} \in \overline{N}(F) \cap K = \overline{N}_0$, so that $\bar{n}$ belongs to $\overline{N}_{0+}$ because s is strictly N-positive. Then $\varphi(\overline{hv}) = \varphi(\overline{mnv}) = m\varphi(\overline{nv}) = m\varphi(\bar{v})$. But sns^{-1} is in N_{0+} again because s is strictly N-positive and $h'\varphi(\bar{v}) = m\varphi(\bar{v})$ too. $\square$

Remark 5.3. We note that, for $v \in V$:

- $T_{\mathcal{P}}([1, \bar{v}]_{\mathcal{P}})$ is the function in $\text{c-Ind}_{\mathcal{P}}^{G(F)} V_{N(k)}$ with support $\mathcal{P}s\mathcal{P}$ and value $\bar{v}$ on $s\overline{N}_{0+}$.
- $T_{K,\mathcal{P}}([1, \bar{v}]_{\mathcal{P}})$ is the function in $\text{c-Ind}_K^{G(F)} V$ with support $Ks\mathcal{P}$ and value $\varphi(v)$ on $s\overline{N}_{0+}$.
- $T_G([1, v]_K)$ is the function in $\text{c-Ind}_K^{G(F)} V$ with support contained in KsK and value $\varphi(\overline{hv})$ on sh for all $h \in K$.

5B. ***Compatibilities between Hecke operators.*** In this section, following Herzig's method, we prove:

Proposition 5.4. (i) *The diagram on the left*

$$\text{(16)}\qquad \begin{tikzcd} \text{c-Ind}_K^{G(F)} V \arrow[r, "\xi"] \arrow[d, "T_G"'] & \text{c-Ind}_{\mathcal{P}}^{G(F)} V_{N(k)} \arrow[dl, "T_{K,\mathcal{P}}"'] \\ \text{c-Ind}_K^{G(F)} V & \end{tikzcd} \qquad \begin{tikzcd} & \text{c-Ind}_{\mathcal{P}}^{G(F)} V_{N(k)} \arrow[dl, "T_{K,\mathcal{P}}"'] \arrow[d, "T_{\mathcal{P}}"] \\ \text{c-Ind}_K^{G(F)} V \arrow[r, "\xi"'] & \text{c-Ind}_{\mathcal{P}}^{G(F)} V_{N(k)} \end{tikzcd}$$

is commutative; *the diagram on the right is commutative when V is $\overline{P}$-regular.*

(ii) *The diagram*

$$\begin{tikzcd} \text{c-Ind}_{\mathcal{P}}^{G(F)} V_{N(k)} \arrow[r, "\zeta"] \arrow[d, "T_{\mathcal{P}}"'] & \text{Ind}_{P(F)}^{G(F)}\big(\text{c-Ind}_{M_0}^{M(F)} V_{N(k)}\big) \arrow[d, "T_M"] \\ \text{c-Ind}_{\mathcal{P}}^{G(F)} V_{N(k)} \arrow[r, "\zeta"'] & \text{Ind}_{P(F)}^{G(F)}\big(\text{c-Ind}_{M_0}^{M(F)} V_{N(k)}\big) \end{tikzcd}$$

is commutative.

(iii) $\mathcal{S}'(T_G) = T_M$ *when V is $\overline{P}$-regular.*

By (15), the $G(F)$-homomorphisms corresponding to ξ, T_G, $T_{\mathcal{P}}$ and $T_{K,\mathcal{P}}$ are characterized by the following formulas, for $v \in V$:

$$\xi : [1, v]_K \mapsto \sum_{g \in \mathcal{P}\backslash K} g^{-1}[1, \overline{gv}]_{\mathcal{P}},$$

$$T_G : [1, v]_K \mapsto \sum_{g \in K\backslash KsK} g^{-1}[1, T_G(g)(v)]_K,$$

$$T_{\mathcal{P}} : [1, \bar{v}]_{\mathcal{P}} \to \sum_{g \in \mathcal{P}\backslash \mathcal{P}s\mathcal{P}} g^{-1}[1, T_{\mathcal{P}}(g)(\bar{v})]_{\mathcal{P}},$$

$$T_{K,\mathcal{P}} : [1, \bar{v}]_{\mathcal{P}} \mapsto \sum_{g \in K\backslash Ks\mathcal{P}} g^{-1}[1, T_{K,\mathcal{P}}(g)(\bar{v})]_K.$$

To prove the proposition, it is useful first to simplify these formulas.

Lemma 5.5. *We have*

$$\text{(17)}\qquad T_{\mathcal{P}} : [1, \bar{v}]_{\mathcal{P}} \mapsto \sum_{\bar{n} \in s^{-1}\overline{N}_{0+}s\backslash \overline{N}_{0+}} \bar{n}^{-1}s^{-1}[1, \bar{v}]_{\mathcal{P}},$$

$$\text{(18)}\qquad T_{K,\mathcal{P}} : [1, \bar{v}]_{\mathcal{P}} \mapsto \sum_{\bar{n} \in s^{-1}\overline{N}_{0}s\backslash \overline{N}_{0+}} \bar{n}^{-1}s^{-1}[1, \varphi(\bar{v})]_K,$$

$$\text{(19)}\qquad T_G : [1, v]_K \mapsto \sum_{h \in \mathcal{P}\backslash K} h^{-1} \sum_{\bar{n} \in s^{-1}\overline{N}_{0}s\backslash \overline{N}_{0+}} \bar{n}^{-1}s^{-1}[1, \varphi(\overline{hv})]_K.$$

Proof. By the Iwahori decomposition $\mathcal{P} = N_0 M_0 \overline{N}_{0+}$, we get that $\mathcal{P}s\mathcal{P} = \mathcal{P}s\overline{N}_{0+}$, because $sN_0s^{-1} \subset N_0$ and $sM_0s^{-1} = M_0$. Consequently, the map $\bar{n} \mapsto s\bar{n}$ induces a bijection of $s^{-1}\overline{N}_{0+}s\backslash\overline{N}_{0+}$ onto $\mathcal{P}\backslash\mathcal{P}s\mathcal{P}$. Since $\overline{N}_{0+}$ acts trivially on $V_{N(k)}$, we get the formula for $T_{\mathcal{P}}$.

A similar reasoning gives that $Ks\mathcal{P} = Ks\overline{N}_{0+}$ and that $\bar{n} \mapsto s\bar{n}$ induces a bijection of $s^{-1}\overline{N}_0 s\backslash\overline{N}_{0+}$ onto $K\backslash Ks\mathcal{P}$. This implies the formula for $T_{K,\mathcal{P}}$.

To simplify the formula for T_G, we note that the map $h \mapsto sh$ induces a bijection from $(K \cap s^{-1}Ks)\backslash K$ onto $K\backslash KsK$. But $K \cap s^{-1}Ks$ is contained in $\mathcal{P}$ by [Henniart and Vigneras 2011, Proposition 6.13], so that we can perform the sum in T_G as a sum over $(K \cap s^{-1}Ks)\backslash\mathcal{P}$ followed by a sum over $\mathcal{P}\backslash K$. By what we said in the previous paragraph, the inclusion $\overline{N}_{0+} \subset \mathcal{P}$ induces a bijection of $s^{-1}\overline{N}_0 s\backslash\overline{N}_{0,+}$ onto $(K \cap s^{-1}Ks)\backslash\mathcal{P}$, so that we finally get the formula for T_G. □

We now give the proof of Proposition 5.4.

Proof. From the formulas for T_G, $T_{K,\mathcal{P}}$ in Lemma 5.5 and the formula for ξ, we immediately get

$$T_G = T_{K,\mathcal{P}} \circ \xi, \tag{20}$$

so that the left diagram in Proposition 5.4(i) is indeed commutative.

The elements $[1, \bar{v}]_{\mathcal{P}}$ for $v \in V$ generate the representation $\text{c-Ind}_{\mathcal{P}}^{G(F)} V_{N(k)}$, and to prove the commutativity of the diagram in Proposition 5.4(ii), it thus suffices to prove for $v \in V$ the equality

$$(T_M \circ \zeta)([1, \bar{v}]_{\mathcal{P}}) = (\zeta \circ T_{\mathcal{P}})([1, \bar{v}]_{\mathcal{P}}).$$

From the value of $\zeta([1, \bar{v}]_{\mathcal{P}})$ for $v \in V$ given in Proposition 4.9 and from

$$T_M([1, \bar{v}]_{M_0}) = s^{-1}[1, \bar{v}]_{M_0},$$

we see that the function $(T_M \circ \zeta)([1, \bar{v}]_{\mathcal{P}})$ vanishes outside $P\overline{N}_{0+}$ and has constant value $s^{-1}[1, \bar{v}]_{M_0}$ on $\overline{N}_{0+}$. From the formula for $T_{\mathcal{P}}$ in Lemma 5.5, we have

$$(\zeta \circ T_{\mathcal{P}})([1, \bar{v}]_{\mathcal{P}}) = \sum_{\bar{n} \in s^{-1}\overline{N}_{0+}s\backslash\overline{N}_{0+}} \bar{n}^{-1}s^{-1}\zeta([1, \bar{v}]_{\mathcal{P}}),$$

and with the value of $\zeta([1, \bar{v}]_{\mathcal{P}})$, we see that this function is indeed the function $(T_M \circ \zeta)([1, \bar{v}]_{\mathcal{P}})$ described above, so that the diagram in Proposition 5.4(ii) is commutative.

Let us turn to the proof of the commutativity of the diagram on the right in Proposition 5.4(i). We now assume that V is $\overline{P}$-regular. From the formulas for $T_{K,\mathcal{P}}$ in Lemma 5.5, we have, for $v \in V$,

$$\xi \circ T_{K,\mathcal{P}} : [1, \bar{v}]_{\mathcal{P}} \mapsto \sum_{\bar{n} \in s^{-1}\overline{N}_0 s\backslash\overline{N}_{0+}} \bar{n}^{-1}s^{-1} \sum_{h \in \mathcal{P}\backslash K} h^{-1}[1, \overline{h\varphi(\bar{v})}]_{\mathcal{P}}.$$

We have seen that for $h \in K$, the image of $hV^{\overline{N}(k)}$ in $V_{N(k)}$ is 0 unless h belongs to $\mathcal{P}\overline{\mathcal{P}}$ (Corollary 3.19), so that the inner sum can be restricted to $h \in \overline{N}_{0,+}\backslash\overline{N}_0$. Now $\bar{n}^{-1}s^{-1}h^{-1} = \bar{n}^{-1}s^{-1}h^{-1}ss^{-1}$ and $s^{-1}hs$ runs through $s^{-1}\overline{N}_{0,+}s\backslash s^{-1}\overline{N}_0 s$, which gives the result

$$\xi \circ T_{K,\mathcal{P}}([1, \bar{v}]_{\mathcal{P}}) = T_{\mathcal{P}}([1, \bar{v}]_{\mathcal{P}}).$$

We finally prove $\mathcal{S}'(T_G) = T_M$, still assuming that V is $\overline{P}$-regular. We have just proved $\xi \circ T_{K,\mathcal{P}} = T_{\mathcal{P}}$ and previously we got $T_{K,\mathcal{P}} \circ \xi = T_G$, so we deduce $\xi \circ T_G = T_{\mathcal{P}} \circ \xi$. We also proved $\zeta \circ T_{\mathcal{P}} = T_M \circ \zeta$, so we obtain

$$\zeta \circ \xi \circ T_G = \zeta \circ T_{\mathcal{P}} \circ \xi = T_M \circ \zeta \circ \xi,$$

that is, $I_V \circ T_G = T_M \circ I_V$. Applying j_V and Definition 2.1, this implies $\mathcal{S}'(T_G) = T_M$. □

Note that the trivial representation V is not $\overline{P}$-regular, as $M \neq G$ by our running hypothesis; however, we can still have $\mathcal{S}'(T_G) = T_M$ when the representation V is the trivial representation. We now present some examples of that phenomenon (the referee remarks that even more examples result from [Herzig 2011, Proposition 5.1]).

Example 5.6. Take $G = \mathrm{GL}(2, -)$, Z_G the center, M the diagonal group, $B = NM$ the upper triangular subgroup, $K = \mathrm{GL}(2, o_F)$, and

$$s_p := \begin{pmatrix} p_F & 0 \\ 0 & 1 \end{pmatrix}.$$

The monoid of strictly positive elements in $M(F)$ is $\cup_{n\geq 1} s_p^n Z_G(F) M_0$, where $M_0 = M(F) \cap K$. An irreducible smooth C-representation V of K is B-regular if and only if it is $\overline{B}$-regular if and only if it is not 1-dimensional. For $g \in G(F)$, we denote by T_g the characteristic function of KgK in the Hecke C-algebra $\mathcal{H}(G(F), K, C) \simeq C[K\backslash G(F)/K]$ of (the trivial C-representation of) K in $G(F)$. For $t \in M(F)$, we denote by τ_t the characteristic function of tM_0 in the Hecke algebra

$$\mathcal{H}(M(F), M_0, C) \simeq C[M(F)/M_0].$$

Claim. *When $s \in M(F)$ is strictly positive, we have $\mathcal{S}'(T_s) = \tau_s$ if and only if $s \in s_p Z_G M_0$.*

Proof. By [Barthel and Livné 1994, Proposition 8], the characteristic function T_n of $Z_G(F) K s_p^n K$ in the Hecke algebra $\mathcal{H}\big(G(F), Z_G(F)K, C\big)$ satisfies the relations

$$T_n = T_1^n - T_1^{n-2} \quad \text{for } n \geq 2. \tag{21}$$

The natural surjective $G(F)$-equivariant map

$$\sigma : \text{c-Ind}_K^{G(F)} C \to \text{c-Ind}_{Z_G(F)K}^{G(F)} C, \quad 1_K \mapsto 1_{Z_G K},$$

satisfies $\sigma \circ T_s = T_n \circ \sigma$ when $s \in s_p^n p_F^{\mathbb{Z}_G} M_0$, $n \geq 1$.

Indeed, we write KsK as a disjoint union of cosets $Kb_i p_F^r$, where $s \in s_p^n p_F^r M_0$ and $b_i \in B(F)$. For $f \in \text{c-Ind}_K^{G(F)} C$, we have

$$T_s(f) = \sum_i p_F^{-r} b_i^{-1} f \quad \text{and} \quad (\sigma \circ T_s)(f) = \sum_i b_i^{-1} \sigma(f).$$

The double coset $Z_G(F)Ks_p^n K$ is the union of the cosets $Z_G(F)Kb_i$. The union remains disjoint because the equality of cosets $Z_G(F)Kb_i = Z_G(F)Kb_j$, equivalent to $b_j b_i^{-1} k = z$ for some $z \in Z_G(F)$, $k \in K$, implies that the determinant of z is a unit. When this holds, $z \in M_0 \cap Z_G$ and $Kb_i = Kb_j$. For $\varphi \in \text{c-Ind}_{Z_G(F)K}^{G(F)} C$, we have

$$T_n(\varphi) = \sum_i b_i^{-1} \varphi \quad \text{and} \quad (T_n \circ \sigma)(f) = \sum_i b_i^{-1} \sigma(f).$$

We deduce $\sigma \circ T_s = T_n \circ \sigma$. Then the relation (21) implies that $T_{s_p^n}$ is different from $T_{s_p}^n$ when $n \geq 2$.

The value of $\mathcal{S}'(T_s)$ at $t \in M(F)$ is the image in C of the number of $b \in F/o_F$ such that $n_b t \in KsK$, where

$$n_b := \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}.$$

The double coset Ks_pK is the disjoint union of Ks_p and of $K\left(\begin{smallmatrix} 1 & a \\ 0 & p_F \end{smallmatrix}\right)$ for a in a system of representatives of $o_F/p_F o_F$. The characteristic of C being p, we deduce that $\mathcal{S}'(T_{s_p}) = \tau_{s_p}$. Then we obtain $\mathcal{S}'(T_{s_p^n}) \neq \tau_{s_p^n}$ when $n \geq 2$, because $\mathcal{S}'$ is an injective algebra homomorphism and $T_{s_p^n} \neq T_{s_p}^n$. Our claim is proved for $s = s_p^n$ and $n \geq 1$. The general case $s = s_p^n p_F^r t_0$ with $r \in \mathbb{Z}$, $t_0 \in M_0$, reduces easily to this case. $\square$

Example 5.7. Let D be a quaternion division algebra over F. We write $\mathbb{O}$ for the ring of integers of D, and v for its normalized valuation; we choose a uniformizer p_D of D such that $p_D^2 = p_F$ is a uniformizer of F; the residue field k_D of $\mathbb{O}$ is a quadratic extension of the residue field k of F. We take for G the group such that $G(F) = \mathrm{GL}(2, D)$, for S the group such that $S(F)$ is the group of diagonal matrices with coefficients in F^*, and for $B = MN$ the groups such that $M(F)$ is the group of diagonal matrices and $B(F)$ is the upper triangular subgroup of $\mathrm{GL}(2, D)$.

Let $K = \mathrm{GL}(2, \mathbb{O})$; the quotient of K by its pro-p-radical is isomorphic to $\mathrm{GL}(2, k_D)$. The Cartan decomposition says that $G(F)$ is the disjoint union of the double cosets $Kd_{a,b}K$, for integers $a, b \in \mathbb{Z}$ with $a \geq b$, where $d_{a,b}$ is the diagonal matrix with entries p_D^a and p_D^b down the diagonal. The strictly positive elements of $M(F)$ are those of the form $s = m_0 d_{a,b}$, for $a, b \in \mathbb{Z}$ with $a > b$ and $m_0 \in M_0 = M(F) \cap K$.

An irreducible C-representation of $\mathrm{GL}(2, k_D)$ which is not $\bar{B}$-regular has dimension 1 and is given by a character $g \mapsto (\epsilon \circ \det)(g)$, where $\epsilon : k_D^* \to C^*$ is a character. We identify $\epsilon \circ \det$ with an irreducible smooth C-character of K and ϵ with a smooth C-character of $\mathbb{O}^*$.

The reduction of the conjugation by p_D on $\mathbb{O}$ induces the nontrivial automorphism σ of k_D/k. The character ϵ of $\mathbb{O}^*$ extends to a character of D^* exactly when ϵ is invariant under σ. In that case, the Hecke algebra $\mathcal{H}(D^*, \mathbb{O}^*, \epsilon)$ has support D^* (the support of the Hecke algebra is the union of the supports of its elements). This implies that the Hecke algebra $\mathcal{H}\big(M(F), M_0, \epsilon \otimes \epsilon\big)$ has support $M(F)$, and by the Satake isomorphism, that the Hecke algebra $\mathcal{H}\big(G(F), K, \epsilon \circ \det\big)$ has support $G(F)$.

Assume now that ϵ is not invariant under σ. Then the support of the Hecke algebra $\mathcal{H}(D^*, \mathbb{O}^*, \epsilon)$ is the set of elements in D^* of even normalized valuation. This implies that the support of $\mathcal{H}\big(M(F), M_0, \epsilon \otimes \epsilon\big)$ is the union of the cosets $M_0 d_{2a,2b}$ for $a, b \in \mathbb{Z}$, and that the support of the Hecke algebra $\mathcal{H}\big(G(F), K, \epsilon \circ \det\big)$ is the union of the double cosets $K d_{2a,2b} K$, for $a, b \in \mathbb{Z}$ and $a \geq b$.

For a positive element s in the support of $\mathcal{H}\big(M(F), M_0, \epsilon \otimes \epsilon\big)$, let τ_s be the Hecke operator in $\mathcal{H}\big(M(F), M_0, \epsilon \otimes \epsilon\big)$ of support $M_0 s$ and value 1 at s, and let T_s be the Hecke operator in $\mathcal{H}\big(G(F), K, \epsilon \circ \det\big)$ of support $K s K$ and value 1 at s.

Claim. *$\mathcal{S}'(T_s) = \tau_s$ for any choice of strictly positive $s \in S(F)$.*

Proof. It suffices to prove the claim for $s = d_{2a,2b} \in S(F)$ with $a > b$. We compute $\mathcal{S}'(T_s)$ on $d_{2\alpha,2\beta}$ with $\alpha \geq \beta$ in $\mathbb{Z}$,

$$\mathcal{S}'(T_s)(d_{2\alpha,2\beta}) = \sum_{x \in D/\mathbb{O}} T_s\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} p_F^\alpha & 0 \\ 0 & p_F^\beta \end{pmatrix}\right).$$

The matrix

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} p_F^\alpha & 0 \\ 0 & p_F^\beta \end{pmatrix} = \begin{pmatrix} p_F^\alpha & x p_F^\beta \\ 0 & p_F^\beta \end{pmatrix}$$

belongs to $K d_{2\alpha,2\beta} K$ when $x \in \mathbb{O}$.

If $x \notin \mathbb{O}$, then putting $v(x) = -\gamma$, $\gamma > 0$, we have

$$\begin{pmatrix} p_F^\alpha & x p_F^\beta \\ 0 & p_F^\beta \end{pmatrix} = \begin{pmatrix} 0 & x p_D^\gamma \\ -x^{-1} p_D^{-\gamma} & p_D^\gamma \end{pmatrix}\begin{pmatrix} p_D^{2\alpha+\gamma} & 0 \\ 0 & p_D^{2\beta-\gamma} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ x^{-1} p_F^{\alpha-\beta} & 1 \end{pmatrix},$$

which consequently belongs to $K d_{2\alpha+\gamma,2\beta-\gamma} K$.

If $(2\alpha, 2\beta) = (2a, 2b)$, we see that only $x \in \mathbb{O}$ contributes to $\mathcal{S}'(T_s)(d_{2\alpha,2\beta})$ and that this contribution is 1. Hence $\mathcal{S}'(T_s)(d_{2\alpha,2\beta}) = 1$.

If $(2\alpha, 2\beta) = (2a - \gamma, 2b + \gamma)$ with $\gamma > 0$, we see that the only x contributing to $\mathcal{S}'(T_s)(d_{2\alpha,2\beta})$ are those with $v(x) = -\gamma$ and that this contribution is 1. Therefore

$\mathcal{S}'(T_s)(d_{2\alpha,2\beta})$ is the number of $x \in D/\mathbb{O}$ of valuation $-\gamma$, and hence

$$\mathcal{S}'(T_s)(d_{2\alpha,2\beta}) = q^{2\gamma} - q^{2(\gamma-1)}.$$

However, γ has to be even, so that $\mathcal{S}'(T_s)(d_{2\alpha,2\beta}) = 0$.

For the other values of α, β in $\mathbb{Z}$, we see that $\mathcal{S}'(T_s)(d_{2\alpha,2\beta}) = 0$. □

6. Proof of the main theorem

We give three lemmas which will help us to study the map ζ (Proposition 4.9).

Lemma 6.1. *The map ζ is injective on the set of functions $f \in \text{c-Ind}_{\mathcal{P}}^{G(F)} V_{N(k)}$ with support in $\mathcal{P}Z(F)^{+N}K$.*

Proof. Let f be such that $\zeta(f) = 0$ with support in $\mathcal{P}Z(F)^{+N}K$. We claim that $f = 0$ on $\mathcal{P}P(F)$. This implies that $f = 0$, because $G(F) = P(F)K$ and for $k \in K$, the function $k^{-1}f$ satisfies the same conditions as f. To prove the claim, we use only that $\zeta(f)(1) = 0$ in $\text{c-Ind}_{M_0}^{M(F)} V_{N(k)}$. As $\zeta(f)(1)$ depends only on the restriction of f to $\mathcal{P}P(F)$ (Remark 4.10), we assume, as we may, that the support of f is contained in $\mathcal{P}P(F)$. The support of f is a finite disjoint union of $\mathcal{P}z_ik_i$ for $z_i \in Z(F)^{+N}$ and $k_i \in K$, with $z_ik_i \in \mathcal{P}P(F)$. We have $\mathcal{P}P(F) = \overline{N}_{0,+}P(F)$, and hence $k_i \in z_i^{-1}\overline{N}_{0,+}z_iP(F)$. As z_i is positive, $z_i^{-1}\overline{N}_{0,+}z_i \subset \overline{N}_{0,+}$. This implies that we can suppose $k_i \in P(F) \cap K$. As $P(F) \cap K = N_0M_0$ and z_i is positive, we can suppose $k_i \in M_0$. We proved that the support of f is a finite disjoint union of $\mathcal{P}z_ik_i$ for $z_i \in Z(F)^{+N}$ and $k_i \in M_0$. Taking the intersection with $M(F)$, the sets $M(F) \cap \mathcal{P}z_ik_i$ are also disjoint. Writing

$$f = \sum_i (z_ik_i)^{-1}[1, \bar{v}_i]_{\mathcal{P}},$$

we have $\zeta(f)(1) = \sum_i (z_ik_i)^{-1}[1, \bar{v}_i]_{M_0}$, and $\zeta(f)(1) = 0$ is equivalent to $\bar{v}_i = 0$ for all i. □

Lemma 6.2. (i) *A compact space $P(F)\backslash G(F)$ is given by the $G(F)$-translates of $P(F)\backslash P(F)\overline{N}_{0,+}s^n$, for all $n \in \mathbb{N}$.*

(ii) *For any subset $X \subset G(F)$ with finite image in $\mathcal{P}\backslash G(F)$, there exists a large integer $n \in \mathbb{N}$ such that $s^nX \subset \mathcal{P}Z(F)^{+N}K$.*

Proof [Herzig 2011, Lemma 2.20]. (i) The compact space $P(F)\backslash G(F)$ is the union of the right $G(F)$-translates of the big cell $P(F)\backslash P(F)\overline{N}(F)$, which is open, and the $s^{-n}\overline{N}_{0,+}s^n$ for $n \in \mathbb{N}$ form a decreasing sequence of open subgroups of $\overline{N}(F)$ converging to 1.

(ii) Let $\mathcal{N}$ be the normalizer of S in G and let $\mathcal{B}$ be the inverse image of $B(k)$ in K (an Iwahori subgroup). Then $(G(F), \mathcal{B}, \mathcal{N}(F))$ is a generalized Tits system [Morris 1993, 3.12]. We have:

a) $G(F)=\mathcal{B}\mathcal{N}(F)\mathcal{B}$.

b) For $\nu\in\mathcal{N}(F)$, there is a finite subset X_ν in $\mathcal{N}(F)$ such that, for all $\nu'\in\mathcal{N}(F)$, we have
$$\nu'\mathcal{B}\nu\subset\cup_{x\in X_\nu}\mathcal{B}\nu'x\mathcal{B}.$$

c) As the parahoric group K is special, it contains representatives of the Weyl group, so for any $\nu\in\mathcal{N}(F)$, there exists $z\in Z(F)$ such that $\nu K=zK$.

We deduce from a) and c) that $G(F)=\mathcal{B}Z(F)K$. We may assume that X is a finite union $X=\cup_i\mathcal{P}z_ik_i$ with $z_i\in Z(F)$, $k_i\in K$. We deduce from b) that, for any index i, there are finitely many $n_{i,j}\in\mathcal{N}(F)$ such that $z\mathcal{B}z_i\subset\cup_j\mathcal{B}zn_{i,j}\mathcal{B}$ for all $z\in Z(F)$. It follows that for $n\in\mathbb{N}$ and $z=s^n$, we have
$$z\mathcal{P}z_ik_i\subset P_0z\overline{N}_{0,+}z_ik_i\subset\cup_j\mathcal{P}zn_{i,j}K,$$
as $\overline{N}_{0,+}\subset\mathcal{B}$. We choose $z_{i,j}\in Z(F)$ such that $z_{i,j}K=n_{i,j}K$, as we may by c). There exists $n\in\mathbb{N}$ such that $s^nz_{i,j}\in Z(F)^{+N}$ for all i,j. Hence $s^nX\subset\cup_j\mathcal{P}s^nz_{i,j}K\subset\mathcal{P}Z(F)^{+N}K$. □

Let σ be a smooth C-representation of $M(F)$. For any nonzero vector y in the space of σ, there exists a function $f_y\in\operatorname{Ind}_{P(F)}^{G(F)}\sigma$ with support $P(F)\overline{N}_{0,+}$ and value y on $\overline{N}_{0+}$ because the multiplication $P(F)\times\overline{N}_{0+}\to P(F)\overline{N}_{0,+}$ is a homeomorphism.

Lemma 6.3. *Let σ be a smooth C-representation of $M(F)$ generated by an element x. Then the representation $\operatorname{Ind}_{P(F)}^{G(F)}\sigma$ is generated by the functions $f_{s^{-n}x}$, $n\in\mathbb{Z}$, with support $P(F)\overline{N}_{0,+}$ and value $s^{-n}x$ on $\overline{N}_{0+}$.*

Proof. By Lemma 6.2, it is enough to show that any function $f_{n,mx}\in\operatorname{Ind}_{P(F)}^{G(F)}\sigma$ with support contained in $P(F)\overline{N}_{0,+}s^n$ and value mx on $\overline{N}_{0+}s^n$, for $n\in\mathbb{N}$ and $m\in M(F)$, is contained in the subrepresentation generated by $f_{s^{-r}x}$ for all $r\in\mathbb{Z}$. The function $m^{-1}f_{n,mx}$ has support in $P(F)\backslash P(F)\overline{N}_{0+}s^nm$ and value $s^{-n}x$ on the compact open subset $m^{-1}s^{-n}\overline{N}_{0+}s^nm$ of $\overline{N}(F)$; for some $n'\in\mathbb{N}$, this set is a disjoint union of $s^{-n'}\overline{N}_{0+}s^{n'}\bar{n}$ for $\bar{n}$ running through a finite subset of $\overline{N}(F)$. For a nonzero y in the space of σ, the function $(s^{n'}\bar{n})^{-1}f_y\in\operatorname{Ind}_{P(F)}^{G(F)}\sigma$ has support $P(F)\overline{N}_{0+}s^{n'}\bar{n}$ and value $s^{-n'}y$ on $s^{-n'}\overline{N}_{0+}s^{n'}\bar{n}$. The sum over $\bar{n}$ of $(s^{n'}\overline{n})^{-1}f_{s^{n'-n}x}$ is equal to $m^{-1}f_{n,mx}$. □

Proposition 6.4. (i) *The image of ξ contains $T_{\mathcal{P}}\big(\text{c-}\operatorname{Ind}_{\mathcal{P}}^{G(F)}V_{N(k)}\big)$ when V is $\overline{P}$-regular.*

(ii) *The kernel of the map ζ is the $T_{\mathcal{P}}^{\infty}$-torsion part of* $\text{c-}\operatorname{Ind}_{\mathcal{P}}^{G(F)}V_{N(k)}$, *and the representation*
$$\text{c-}\operatorname{Ind}_{P(F)}^{G(F)}\big(\text{c-}\operatorname{Ind}_{M_0}^{M(F)}V_{N(k)}\big)$$

is generated by

$$(T_M^{-n} \circ \zeta)([1, \overline{v}]_{\mathcal{P}}), \quad \textit{for all } n \in \mathbb{Z},$$

for any fixed nonzero element $\overline{v} \in V_{N(k)}$.

Proof. (i) This follows from Proposition 5.4(i).

(ii) We fix a nonzero $\bar{v} \in V_{N(k)}$; then $x = [1, \bar{v}]_{M_0}$ generates the representation $\sigma = \text{c-Ind}_{M_0}^{M(F)} V_{N(k)}$. We note that for $n \in \mathbb{Z}$, by Definition 4.4 and 4.8,

$$(T_M^n \circ \zeta)([1, \bar{v}]_{\overline{\mathcal{P}}}) = f_{s^{-n}x}.$$

We obtain by Lemma 6.3 that the representation $\text{Ind}_{P(F)}^{G(F)}\big(\text{c-Ind}_{M_0}^{M(F)} V_{N(k)}\big)$ is generated by the elements $(T_M^n \circ \zeta)([1, \bar{v}]_{\overline{\mathcal{P}}})$, when n runs through $\mathbb{Z}$.

We now consider an element f in the kernel of ζ. The function f vanishes outside of a compact set X with finite image in $\mathcal{P}\backslash G(F)$. We choose an integer $n \in \mathbb{N}$ such that $s^n X \subset \mathcal{P} Z(F)^{+N} K$ (Lemma 6.2(ii)). The support of $T_{\mathcal{P}}^n$ is $\mathcal{P} s^n \mathcal{P}$ by (13) and the positivity of s. The support of $T_{\mathcal{P}}^n(f)$ is contained in $\mathcal{P} s^n X$, and hence in $\mathcal{P} Z(F)^{+N} K$. By Lemma 6.1, we conclude that $T_{\mathcal{P}}^n(f) = 0$. The converse follows from Proposition 5.4(ii). □

In the diagram (12), the representations are $C[T]$-modules, where T acts as on the middle space by $T_{K,\mathcal{P}}$, on the right space by T_M, and on the left space by $(\mathcal{S}')^{-1}(T_M)$. Proposition 5.4 tells us that:

- The map ζ is $C[T]$-linear.
- When V is $\bar{P}$-regular, the map ξ is $C[T]$-linear and $(\mathcal{S}')^{-1}(T_M) = T_G$.

Corollary 6.5. (i) *The T-localization ζ_T of ζ is an isomorphism.*

(ii) *When V is $\bar{P}$-regular, the T-localization ξ_T of ξ is an isomorphism.*

The map Θ is the T-localization of $I_V = \zeta \circ \xi$. By Corollary 6.5(ii), the map $\Theta = \zeta_T \circ \xi_T$ is surjective when V is $\bar{P}$-regular.

Remark 6.6. We suppose that V is given by a character ϵ of K, and that there exists a character ϵ_M of $M(F)$ equal to ϵ on M_0 (such a character ϵ_M does not always exist). We consider the composite of I_V with the surjective natural map

$$\psi : \text{Ind}_{P(F)}^{G(F)}\big(\text{c-Ind}_{M_0}^{M(F)} \epsilon\big) \to \text{Ind}_{P(F)}^{G(F)} \epsilon_M.$$

If ϵ_M extends to a character ϵ_G of $G(F)$, the image of $\psi \circ I_V$ is the subrepresentation ϵ_G of dimension 1 of $\text{Ind}_{P(F)}^{G(F)} \epsilon_M$, and the map $\psi \circ \Theta$ is nonsurjective.

But in the case where ϵ_M does not extend to a character ϵ_G of $G(F)$, the map $\psi \circ \Theta$ can be surjective. For example, $\psi \circ \Theta$ is surjective when $\text{Ind}_{P(F)}^{G(F)} \epsilon_M$ is irreducible. This is the case, for any choice of ϵ_M, when $G = U(2, 1)$ with respect to an unramified quadratic extension of F, B is a Borel subgroup, and K is a special nonhyperspecial parahoric subgroup [Abdellatif 2011]; this is also the case when $G(F) = \mathrm{GL}(2, D)$ with D a quaternion skew field over F, B is the upper triangular subgroup, and $K = \mathrm{GL}(2, O_D)$ [Ly $\geq$ 2012].

7. Supersingular representations of $G(F)$

We introduce first the notion of K-supersingularity for an irreducible smooth representation π of $G(F)$. Then we recall the notion of supercuspidality [Henniart and Vigneras 2011, 1.7 footnote]. We expect that supercuspidality is equivalent to K-supersingularity, at least for admissible representations. We will give some partial results in this direction. Finally, when π is admissible, we give an equivalent definition of K-supersingularity which coincides with the definition given by Herzig and Abe when G is F-split, K is hyperspecial, and the characteristic of F is 0.

Let π be an irreducible smooth C-representation of $G(F)$. For any smooth irreducible C-representation V of K, we consider

$$\mathrm{Hom}_{G(F)}\big(\text{c-Ind}_K^{G(F)} V, \pi\big)$$

as a right module for the Hecke algebra $\mathcal{H}(G(F), K, V)$.

Remark 7.1. Given π, there exists an irreducible representation V of K such that $\mathrm{Hom}_{G(F)}\big(\text{c-Ind}_K^{G(F)} V, \pi\big) \neq 0$. Indeed, a nonzero element $v \in \pi$ being fixed by an open subgroup of K generates a K-stable subspace W of finite dimension; if V is an irreducible subrepresentation of W, we have $\mathrm{Hom}_K(V, \pi) \neq 0$, and hence the result by Frobenius reciprocity.

For any standard parabolic subgroup $P = MN$, we consider the Satake map

$$\mathcal{S}' = \mathcal{S}'_{M,G} : \mathcal{H}\big(G(F), K, V\big) \to \mathcal{H}\big(M(F), M_0, V_{N(k)}\big).$$

We recall that $\mathcal{S}'$ is a localization at some element T_M (Proposition 4.5).

Definition 7.2. An irreducible smooth C-representation π of $G(F)$ is called K-supersingular when

$$\mathcal{H}\big(M(F), M_0, V_{N(k)}\big) \otimes_{\mathcal{H}(G(F),K,V),\mathcal{S}'} \mathrm{Hom}_{G(F)}\big(\text{c-Ind}_K^{G(F)} V, \pi\big) = 0,$$

for all irreducible smooth C-representations V of K and all standard Levi subgroups $M \neq G$.

The condition means that the localization of the right $\mathcal{H}(G(F), K, V)$-module

$$\mathrm{Hom}_{G(F)}\big(\text{c-Ind}_K^{G(F)} V, \pi\big)$$

at T_M is 0, that is, for any nonzero $f \in \mathrm{Hom}_{G(F)}\big(\text{c-Ind}_K^{G(F)} V, \pi\big)$, there is $n \in \mathbb{N}$ such that $\mathcal{S}'^{-1}(T_M^n)(f) = 0$. If the space $\mathrm{Hom}_{G(F)}\big(\text{c-Ind}_K^{G(F)} V, \pi\big)$ is finite-dimensional, this means that the eigenvalues of $\mathcal{S}'^{-1}(T_M)$ on this space are 0, or equivalently, that the characters of $\mathscr{Z}(G(F), K, V)$ appearing in

$$\operatorname{Hom}_{G(F)}\big(\text{c-Ind}_K^{G(F)} V, \pi\big)$$

vanish at $\mathscr{S}'^{-1}(T_M)$. For admissible representations, our definition is equivalent to the one given by Herzig [2011, Definition 4.7] and Abe [2011, Definition 5.1].

Definition 7.3. An irreducible smooth C-representation π of $G(F)$ is called supercuspidal if π is not isomorphic to a subquotient of $\text{c-Ind}_{P(F)}^{G(F)} \tau$ for a proper standard parabolic subgroup $P = MN$ of G and for an irreducible smooth C-representation τ of $M(F)$.

The definition, which is valid for any field C, does not depend on the minimal parabolic F-subgroup B of G used to define the standard parabolic subgroups, as all such B's are conjugate in $G(F)$. Consequently, we get an equivalent definition if we let P be any parabolic subgroup different from G.

Let V be an irreducible smooth C-representation of K, let $P = MN$ be a proper standard parabolic subgroup of G, and let σ be a smooth C-representation of $M(F)$. Our first result concerns the T_M-localization of the right $\mathscr{H}(G(F), K, V)$-module

$$\operatorname{Hom}_{G(F)}\big(\text{c-Ind}_K^{G(F)} V, \operatorname{Ind}_{P(F)}^{G(F)} \sigma\big).$$

Proposition 7.4. (i) *$V \subset \big(\operatorname{Ind}_{P(F)}^{G(F)} \sigma\big)|_K$ if and only if $V_{N(k)} \subset \sigma|_{M_0}$.*

(ii) *In this case, the action of $\mathscr{S}'^{-1}(T_M)$ on $\operatorname{Hom}_{G(F)}\big(\text{c-Ind}_K^{G(F)} V, \operatorname{Ind}_{P(F)}^{G(F)} \sigma\big)$ is invertible.*

Proof. (See [Herzig 2011, p. 416].) (i) This follows from the isomorphism (2).

(ii) By (4), we have isomorphisms of $\mathscr{H}(G(F), K, V)$-modules

$$\operatorname{Hom}_{G(F)}\big(\text{c-Ind}_K^{G(F)} V, \operatorname{Ind}_{P(F)}^{G(F)} \sigma\big) \simeq \operatorname{Hom}_K\big(V, \operatorname{Ind}_{P(F)}^{G(F)} \sigma\big) \simeq \operatorname{Hom}_{M_0}(V_{N(k)}, \sigma),$$

where $\mathscr{H}(G(F), K, V)$ acts on the final term by $\mathscr{S}'$; the last isomorphism follows from Frobenius reciprocity and $K \cap P(F) = (K \cap M(F))(K \cap N(F))$. The claim follows since $\mathscr{S}'$ is a localization map at T_M, by Proposition 4.5. $\square$

Our results on the comparison between non-K-supersingular and nonsupercuspidal irreducible smooth C-representations of $G(F)$ are:

Proposition 7.5. *Let τ be an irreducible smooth C-representation of $M(F)$.*

(i) *An irreducible subrepresentation of $\operatorname{Ind}_{P(F)}^{G(F)} \tau$ is not K-supersingular.*

(ii) *An admissible irreducible quotient of $\operatorname{Ind}_{P(F)}^{G(F)} \tau$ is not K-supersingular.*

This proposition claims that certain nonsupercuspidal irreducible representations of $G(F)$ are non-K-supersingular. The next proposition claims that certain non-K-supersingular admissible irreducible representations of $G(F)$ are nonsupercuspidal.

Proposition 7.6. *Let π be an admissible irreducible smooth C-representation of $G(F)$, let $P = MN \subset Q = LN'$ be two standard parabolic F-subgroups different from G, and let V be a $\bar{Q}$-regular irreducible smooth C-representation of K such that the localization of the right $\mathscr{H}(G(F), K, V)$-module*

$$\operatorname{Hom}_{G(F)}\bigl(\text{c-Ind}_K^{G(F)} V, \pi\bigr)$$

at T_M is not 0. *Then π is a quotient of* $\operatorname{Ind}_{Q(F)}^{G(F)} \tau$ *for an admissible irreducible smooth C-representation τ of $L(F)$.*

Proof of Proposition 7.5. (i) Proposition 7.4 implies that an irreducible subrepresentation of $\operatorname{Ind}_{P(F)}^{G(F)} \tau$ is not K-supersingular.

(ii) Let π be an irreducible quotient of $\operatorname{Ind}_{P(F)}^{G(F)} \tau$. We choose an irreducible smooth C-representation W of M_0 such that the irreducible representation τ is a quotient of $\text{c-Ind}_{M_0}^{M(F)} W$. Then π is a quotient of $\operatorname{Ind}_{P(F)}^{G(F)}\bigl(\text{c-Ind}_{M_0}^{M(F)} W\bigr)$. We consider the unique irreducible $\bar{P}$-regular representation V of $G(k)$ such that $V_{N(k)} \simeq W$ (Proposition 3.10). By our main theorem (Theorem 4.6),

$$\operatorname{Ind}_{P(F)}^{G(F)}\bigl(\text{c-Ind}_{M(F)\cap K}^{M(F)} W\bigr) \simeq \mathscr{H}\bigl(M(F), M_0, V_{N(k)}\bigr) \otimes_{\mathscr{H}(G(F),K,V),\mathscr{S}'} \text{c-Ind}_K^{G(F)} V.$$

We deduce:

$$\operatorname{Hom}_{G(F)}\bigl(\mathscr{H}\bigl(M(F), M_0, V_{N(k)}\bigr) \otimes_{\mathscr{H}(G(F),K,V),\mathscr{S}'} \text{c-Ind}_K^{G(F)} V, \pi\bigr) \neq 0.$$

If π is admissible, we will show

$$\mathscr{H}\bigl(M(F), M_0, V_{N(k)}\bigr) \otimes_{\mathscr{H}(G(F),K,V),\mathscr{S}'} \operatorname{Hom}_{G(F)}\bigl(\text{c-Ind}_K^{G(F)} V, \pi\bigr) \neq 0. \tag{22}$$

This implies that π is not K-supersingular.

To prove (22), we write $X := \text{c-Ind}_K^{G(F)} V$, $T := T_M \in A := \mathscr{H}(G(F), K, V)$, $B = A[T^{-1}]$. Our assumption is

$$\operatorname{Hom}_G(B \otimes_A X, \pi) \neq 0,$$

and we want to prove that $B \otimes_A \operatorname{Hom}_G(X, \pi) \neq 0$, provided that $\operatorname{Hom}_G(X, \pi)$ is finite-dimensional (which is the case if π is admissible).

We consider the natural linear map

$$r : \operatorname{Hom}_G(B \otimes_A X, \pi) \to \operatorname{Hom}_G(X, \pi), \quad \varphi \mapsto \bigl(x \mapsto \varphi(1 \otimes x)\bigr).$$

The space $\mathrm{Hom}_G(B \otimes_A X, \pi)$ is naturally a right B-module, and hence a right A-module by restriction. The map r is A-linear:

$$r(\varphi a)(x) = (\varphi a)(1 \otimes x) = \varphi(a \otimes x) = \varphi(1 \otimes ax) = r(\varphi)(ax) = (r(\varphi)a)(x),$$

for $a \in A$, $x \in X$, $\varphi \in \mathrm{Hom}_G(B \otimes_A X, \pi)$. Consequently, the image $\mathrm{Im}(r)$ is an A-submodule of $\mathrm{Hom}_G(X, \pi)$. We remark that $\mathrm{Im}(r)T = \mathrm{Im}(r)$ because $r(\varphi) = r(\varphi T^{-1})T$ for $\varphi \in \mathrm{Hom}_G(B \otimes_A X, \pi)$.

We show now that our hypothesis implies that $\mathrm{Im}(r)$ is not 0. Indeed, let $\varphi \neq 0$ in $\mathrm{Hom}_G(B \otimes_A X, \pi)$. There exist $b \in B$ and $x \in X$ such that $\varphi(b \otimes x) \neq 0$. Writing $b = T^{-n}a$ with $n \in \mathbb{N}$ and $a \in A$, we get $\varphi(T^{-n}a \otimes x) = \varphi T^{-n}(1 \otimes ax) \neq 0$ so that $r(\varphi T^{-n}) \neq 0$.

Assume now that $\mathrm{Hom}_G(X, \pi)$ is finite-dimensional. Then $\mathrm{Im}(r)$ is also finite-dimensional and as $\mathrm{Im}(r)T = \mathrm{Im}(r)$, T induces an automorphism of $\mathrm{Im}(r)$ so that $B \otimes_A \mathrm{Im}(r) \neq 0$. The localization being an exact functor, we have

$$B \otimes_A \mathrm{Hom}_G(X, \pi) \neq 0. \qquad \square$$

We state a useful general lemma before proving Proposition 7.6.

Let R be a commutative ring, let $\mathcal{H}$ be an R-algebra, let W be a left $\mathcal{H}$-module with a smooth $\mathcal{H}$-linear action of $M(F)$, and let $\mathcal{N}$ be a right $\mathcal{H}$-module. Then $\mathcal{N} \otimes_{\mathcal{H}} W$ is a smooth R-representation of $M(F)$ and we can form $\mathrm{Ind}_{P(F)}^{G(F)}(\mathcal{N} \otimes_{\mathcal{H}} W)$. We can also form $\mathcal{N} \otimes_{\mathcal{H}} \mathrm{Ind}_{P(F)}^{G(F)}(W)$, where the structure of left $\mathcal{H}$-module on $\mathrm{Ind}_{P(F)}^{G(F)}(W)$ is given by $(h, f) \to hf : g \to h(f(g))$. The canonical map

$$\iota_{\mathcal{N}} : \mathcal{N} \otimes_{\mathcal{H}} \mathrm{Ind}_{P(F)}^{G(F)}(W) \to \mathrm{Ind}_{P(F)}^{G(F)}(\mathcal{N} \otimes_{\mathcal{H}} W)$$

is clearly $G(F)$-equivariant.

Lemma 7.7. *The map $\iota_{\mathcal{N}}$ is an isomorphism.*

Proof. It is well known that the quotient map $G(F) \to P(F)\backslash G(F)$ admits a continuous section and that the module $C^\infty\big(P(F)\backslash G(F), R\big)$ is free. This implies that the parabolic induction functor $\mathrm{Ind}_{P(F)}^{G(F)} -$ for smooth R-representations is exact and commutes with infinite direct sums, and that $\mathrm{Ind}_{P(F)}^{G(F)}(W)$ identifies with $C^\infty\big(G(F)/P(F), R\big) \otimes W$ as R-modules, for any smooth R-representation W of $M(F)$.

We choose a resolution of $\mathcal{N}$ by free right $\mathcal{H}$-modules

$$\mathcal{F}_1 \to \mathcal{F}_0 \to \mathcal{N} \to 0.$$

We have a commutative diagram

$$\begin{tikzcd}
\mathcal{F}_1 \otimes_{\mathcal{H}} \mathrm{Ind}_{P(F)}^{G(F)}(W) \arrow[r] \arrow[d, "\iota_{\mathcal{F}_1}"'] & \mathcal{F}_0 \otimes_{\mathcal{H}} \mathrm{Ind}_{P(F)}^{G(F)}(W) \arrow[r] \arrow[d, "\iota_{\mathcal{F}_0}"'] & \mathcal{N} \otimes_{\mathcal{H}} \mathrm{Ind}_{P(F)}^{G(F)}(W) \arrow[r] \arrow[d, "\iota_{\mathcal{N}}"'] & 0 \\
\mathrm{Ind}_{P(F)}^{G(F)}(\mathcal{F}_1 \otimes_{\mathcal{H}} W) \arrow[r] & \mathrm{Ind}_{P(F)}^{G(F)}(\mathcal{F}_0 \otimes_{\mathcal{H}} W) \arrow[r] & \mathrm{Ind}_{P(F)}^{G(F)}(\mathcal{N} \otimes_{\mathcal{H}} W) \arrow[r] & 0,
\end{tikzcd}$$

where the lines are exact, the second one because $\mathrm{Ind}_{P(F)}^{G(F)}$ is an exact functor. The functor $\mathrm{Ind}_{P(F)}^{G(F)}$ being compatible with direct sums, the maps $\iota_{\mathcal{F}_1}$ and $\iota_{\mathcal{F}_0}$ are isomorphisms. It follows that $\iota_{\mathcal{N}}$ is an isomorphism. $\square$

Remark 7.8. When π is an admissible smooth C-representation of G, then

$$\mathrm{Hom}_{G(F)}\big(\text{c-Ind}_K^{G(F)} V, \pi\big)$$

is finite-dimensional, and hence it is 0 or contains a simple $\mathcal{H}(G(F), K, V)$-module.

An irreducible smooth C-representation π of $G(F)$ such that

$$\mathrm{Hom}_{G(F)}\big(\text{c-Ind}_K^{G(F)} V, \pi\big)$$

contains a simple $\mathcal{H}(G(F), K, V)$-module $\mathcal{N}$ has a central character. This follows from:

1. The center of $\mathcal{H}(G(F), K, V)$ acts on $\mathcal{N}$ by a character [Vigneras 2007].
2. π is a quotient of $\mathcal{N} \otimes_{\mathcal{H}(G(F),K,V)} \text{c-Ind}_K^{G(F)} V$.

Proof of Proposition 7.6. Put

$$\mathcal{H}_{L,V,\pi} := \mathcal{H}\big(L(F), L_0, V_{N'(k)}\big) \otimes_{\mathcal{H}(G(F),K,V),\mathcal{S}'_{L,G}} \mathrm{Hom}_{G(F)}\big(\text{c-Ind}_K^{G(F)} V, \pi\big),$$

and similarly define $\mathcal{H}_{M,V,\pi}$. From the transitivity $\mathcal{S}'_{M,G} = \mathcal{S}'_{M,L} \circ \mathcal{S}'_{L,G}$, we deduce

$$\mathcal{H}_{M,V,\pi} = \mathcal{H}\big(M(F), M_0, V_{N(k)}\big) \otimes_{\mathcal{H}(L(F),L_0,V_{N'(k)}),\mathcal{S}'_{M,L}} \mathcal{H}_{L,V,\pi}.$$

Hence $\mathcal{H}_{L,V,\pi}$ is not 0, because $\mathcal{H}_{M,V,\pi} \neq 0$. The space

$$\mathrm{Hom}_{G(F)}\big(\text{c-Ind}_K^{G(F)} V, \pi\big)$$

is finite-dimensional because π is admissible, and we have just seen that its localization at T_L is not 0. Therefore T_L has a nonzero eigenvalue α. The corresponding eigenspace is a nonzero $\mathcal{H}\big(G(F), K, V\big)$-submodule, and hence contains a simple right $\mathcal{H}\big(G(F), K, V\big)$-submodule $\mathcal{N}$, which we consider as a simple $\mathcal{H}\big(L(F), L_0, V_{N'(k)}\big)$-module with T_L acting by α. The irreducible representation π is a quotient of

$$\mathcal{N} \otimes_{\mathcal{H}(G(F),K,V)} \text{c-Ind}_K^{G(F)} V. \tag{23}$$

As V is $\overline{Q}$-regular, the representation (23) is isomorphic to

$$\mathcal{N} \otimes_{\mathcal{H}(L(F),L_0,V_{N'(k)})} \mathrm{Ind}_{Q(F)}^{G(F)}\big(\text{c-Ind}_{L_0}^{L(F)} V_{N'(k)}\big) \tag{24}$$

by Theorem 4.6. By Lemma 7.7, this last representation is isomorphic to $\mathrm{Ind}_{Q(F)}^{G(F)} \sigma$, where

$$\sigma := \mathcal{N} \otimes_{\mathcal{H}(L(F), L_0, V_{N'(k)})} \text{c-Ind}_{L_0}^{L(F)} V_{N'(k)} \tag{25}$$

is a smooth representation of $L(F)$. The center of $L(F)$ embeds naturally in the center of the Hecke algebra $\mathcal{H}\big(L(F), L_0, V_{N'(k)}\big)$ and acts by a character on the simple $\mathcal{H}\big(L(F), L_0, V_{N'(k)}\big)$-module $\mathcal{N}$. Hence σ has a central character.

The admissible irreducible representation π is a quotient of $\mathrm{Ind}_{Q(F)}^{G(F)} \sigma$, where σ has a central character. By Proposition 7.9 below, π is a quotient of $\mathrm{Ind}_{Q(F)}^{G(F)} \tau$ for an admissible irreducible smooth C-representation τ of $L(F)$. □

Proposition 7.9. *Let π be an admissible irreducible smooth C-representation of $G(F)$ which is a quotient of $\mathrm{Ind}_{P(F)}^{G(F)} \sigma$ for a smooth C-representation σ of $M(F)$ with a central character. Then there exists an admissible irreducible smooth C-representation τ of $M(F)$ such that π is a quotient of $\mathrm{Ind}_{P(F)}^{G(F)} \tau$.*

When the characteristic of F is 0, Herzig [2011, Lemma 9.9] proved this proposition using the $\overline{P}$-ordinary functor $\mathrm{Ord}_{\overline{P}}$ introduced by Emerton [2010]. His proof contains four steps:

1. As σ is locally Z_M-finite, we have
$$\mathrm{Hom}\big(\mathrm{Ind}_{P(F)}^{G(F)} \sigma, \pi\big) \simeq \mathrm{Hom}_{M(F)}(\sigma, \mathrm{Ord}_{\overline{P}}\pi).$$
2. As π is admissible, $\mathrm{Ord}_{\overline{P}}\pi$ is admissible.
3. As $\mathrm{Ord}_{\overline{P}}\pi$ is admissible and nonzero, it contains an admissible irreducible subrepresentation τ.
4. As $\mathrm{Ord}_{\overline{P}}$ is the right adjoint of $\mathrm{Ind}_{P(F)}^{G(F)}$ in the category of admissible representations, π is a quotient of $\mathrm{Ind}_{P(F)}^{G(F)} \tau$.

The proof is valid without hypothesis on the characteristic of F: we checked carefully that Emerton's proof of steps 1, 2, 4 never uses the characteristic of F. Only the proof of step 3 given by Herzig has to be replaced by a characteristic-free proof.

Lemma 7.10. *A nonzero admissible smooth C-representation of $G(F)$ contains an admissible irreducible subrepresentation.*

Proof. Let π be a nonzero admissible smooth C-representation of $G(F)$, and H an open pro-p-subgroup of $G(F)$. The dimension of π^H is a positive integer. Choose a subrepresentation π_1 of π such that π_1^H has minimal positive dimension; then the subrepresentation generated by π_1^H is irreducible. □

This ends the proof Proposition 7.9, and hence of Proposition 7.6.

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GUY HENNIART
LABORATOIRE DE MATHÉMATIQUES D'ORSAY
UNIVERSITÉ PARIS-SUD
CNRS, UMR 8628, ORSAY CEDEX F-91405
FRANCE
guy.henniart@math.u-psud.fr

MARIE-FRANCE VIGNERAS
INSTITUT DE MATHÉMATIQUES DE JUSSIEU
UNIVERSITÉ DE PARIS 7
175 RUE DU CHEVALERET
PARIS 75013
FRANCE
vigneras@math.jussieu.fr

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THE FUNCTIONAL EQUATION AND BEYOND ENDOSCOPY

P. EDWARD HERMAN

Dedicated to the memory of Jonathan Rogawski

In his paper "Beyond endoscopy," Langlands tries to understand functoriality via poles of L-functions. This paper further investigates the analytic continuation of an L-function associated to a GL_2 automorphic form through the trace formula. Though the usual way to obtain the analytic continuation of an L-function is through its functional equation, this paper shows that by simply assuming the trace formula, the functional equation of the L-function may be recovered. This paper is a step towards understanding the analytic continuation of the L-function at the same time as capturing information about functoriality.

From the perspective of analytic number theory, obtaining the functional equation from the trace formula implies that Voronoi summation should in general be also a consequence of the trace formula.

1. Beyond endoscopy

Let $\mathbb{A}_{\mathbb{Q}}$ be the ring of adeles of $\mathbb{Q}$, and π be an automorphic cuspidal representation of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$. We define $m(\pi, \rho)$ to be the order of the pole at $s = 1$ of $L(s, \pi, \rho)$, where ρ is a representation of the dual group $\mathrm{GL}_2(\mathbb{C})$.

Langlands proposes the study of

$$\lim_{X\to\infty} \sum_{\pi} \frac{1}{X} \operatorname{tr}(\pi)(f) \sum_{p\le X} \log(p) a(p, \pi, \rho). \tag{1-1}$$

Here f is a nice test function on $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$, $\operatorname{tr}(\pi)(f)$ is the trace of the operator defined by f on π, and $a(p, \pi, \rho)$ is the p-th Dirichlet coefficient of $L(s, \pi, \rho)$. The quantity

$$\lim_{X\to\infty} \frac{1}{X} \sum_{p\le X} \log(p) a(p, \pi, \rho)$$

is equal to $m(\pi, \rho)$.

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Therefore, summing over the range of representations π will project only on to the ones which have nontrivial multiplicity. The tool used to study this sum over the spectrum of forms π is the trace formula. Ultimately, one gets from use of the trace formula a sum over primes and conjugacy classes, and hopes by analytic number theory techniques to take the limit. One hopes that after getting the limit, one can decipher and construct the L-functions having nontrivial multiplicity of the pole at $s = 1$. Sarnak [2001] addresses (1-1) for $\rho = std$, the standard representation. He points out that such a computation can be done, but the tools used for the study of sums of primes is limited, and this problem is perhaps more tractable if rather studied over the sum of integers.

Sarnak's idea then is to evaluate

$$\lim_{X\to\infty} \sum_{\pi} \frac{1}{X} \operatorname{tr}(\pi)(f) \sum_{n\le X} a(n, \pi, \rho). \tag{1-2}$$

This should "detect," rather than the multiplicities of the poles, the residue of the poles of the associated L-functions. As well, instead of using the Arthur–Selberg trace formula, he uses the Petersson–Kuznetsov trace formula, which is a special case of the relative trace formula [Knightly and Li 2006a]. One advantage of the relative trace formula is that the spectrum contains only generic representations, so we avoid the task of excising the trivial representation as in [Frenkel et al. 2010]. As well, the geometric side of the relative trace formula has a nice "streamlined" appearance as a sum of Kloosterman sums. This is in comparison to the Arthur–Selberg trace formula, which has orbital integrals associated to different conjugacy classes for which the analysis of each class could be different.

The disadvantage to the relative trace formula is that each automorphic representation π on the spectral side of the trace formula is weighted by a factor $L(1, \pi, ad)^{-1}$, which is the adjoint representation of π evaluated at $s = 1$. This can perhaps make matching two different trace formulas more difficult. Another disadvantage of using the relative trace formula is that the Arthur–Selberg trace formula is in much better shape to generalize to other groups. Namely, one now has full use of the stable trace formula due to the proof of the "fundamental lemma" by Ngô [2010]. With the stable trace formula, one can compare stable conjugacy classes for different groups (specifically endoscopic groups), from which one can then compare automorphic representations for the respective groups.

However in our case of studying GL_2, the disadvantages seem minimal, and in fact the crucial exponential sums one encounters in either trace formula are the same. Sarnak [2001] made some points on the essential differences of the geometric sides of the two trace formulas. Also, in the case of GL_2, the stable trace formula is the same as the Arthur–Selberg trace formula, so one should not expect an advantage of one trace formula over another.

1.1. ***Sarnak's analysis for $\rho = std$.*** The obvious first example to test Langlands's beyond endoscopy idea on is for the standard representation. In this case we do not expect the L-functions to have any poles except for the continuous spectrum, but in this case there are not any poles as the spectrum is not spectrally isolated. So we expect in the case of $\rho = std$ that

$$\lim_{X\to\infty} \sum_{\pi} \frac{1}{X} \operatorname{tr}(\pi)(f) \sum_{n\le X} a(n,\pi,std) = 0. \tag{1-3}$$

Sarnak uses the classic Petersson–Kuznetsov trace formula instead of using the adelic language. To go from (1-3) to a classic approach, one can follow the great expository article of Rogawski [1994] or the book of Knightly and Li [2006b]. Then for an automorphic form f with normalized Fourier coefficients $a_n(f)$ associated to a representation π, Sarnak [2001] showed, up to some weight factors needed in the trace formula, that

$$\sum_{n\le X} \sum_{f} a_n(f) g(n/X) = O(X^{-A}) \tag{1-4}$$

for any $A > 0$. Here X is a large fixed parameter and $g \in C_0^\infty(\mathbb{R}^+)$ is used for "smoothing" the n-sum. Why is this smoothing needed? It is certainly not essential, but when one goes to the geometric side of the trace formula to get the bound (1-4), one requires freedom to apply analytic manipulations (interchanging sums, Fourier transforms, and so on). With the smoothing function g, these problems are removed and one can focus on the central issue of the arithmetic, which is the true difficulty in these problems. One can recover the left hand side of (1-3) by applying techniques in [Iwaniec 1984]. For completeness, we will reproduce Sarnak's argument in the appendix.

1.2. ***Results of the paper.*** Clearly (1-4) is a stronger result than (1-3), and up to using Hecke operators, is equivalent to $L(s,f) = \sum_{n=1}^{\infty} a_n(f)/n^s$ having analytic continuation to the complex plane. We see the analytic continuation of the left hand side of (1-4) by Mellin inversion. By applying Mellin inversion to (1-4) we get

$$\frac{1}{X}\sum_{f}\sum_{n\le X} g(n/X) a_n(f) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} G(s)\Bigl[\sum_{f} L(s,f)\Bigr] X^s\,ds, \tag{1-5}$$

where $G(s) = \int_0^\infty g(x)x^{s-1}dx$ is the Mellin transform with $\sigma > 2$ to ensure the convergence of the integral. Using the right hand side of (1-4) we know that the contour in (1-5) can be shifted (using decay properties of $G(s)$) to $\sigma = -A$, $A > 0$. So in Sarnak's application of the trace formula to get (1-4) we indirectly applied a functional equation of the L-function for each automorphic form f in our spectral sum. Can we actually see directly the functional equation via manipulations on the geometric side of the trace formula? In other words, can we show directly via the trace formula that

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} G(s)\Big[\sum_f L(s,f)\Big]X^s\,ds = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} G(s)\Big[\sum_f \frac{i^k\gamma(f,1-s)L(f,1-s)}{\gamma(f,s)}\Big]X^s\,ds?$$

We will prove this equality and get the functional equation for a fixed automorphic form f in this note.

There are two other methods we mention that also get the analytic continuation of an automorphic form on GL_n; both use integral representations. The first method is associated to Jacquet and Langlands [1970] (who followed Hecke [1918; 1920]), and expresses the standard L-function as an adelic integral of an explicitly chosen vector in the space of the associated automorphic representation. The second method is of a certain integral representation constructed by Godement and Jacquet [1972], which is inspired by Tate's construction [1967] for GL_1.

One can consider these two methods as easier ways to get the functional equation for a GL_2 automorphic form, but in consideration of Langlands's beyond endoscopy idea, a trace formula approach seems the most systematic way to get analytic continuation for all L-functions $L(s,\pi,\rho)$ associated to a dual group representation ρ of an automorphic representation π of a group G. For example, currently there is no general procedure of using integral representations to get the analytic continuation for the symmetric power L-functions. From the beyond endoscopy perspective, asking for the analytic continuation is certainly a more difficult question than investigating whether the L-function has a pole at $s=1$ or not. The question requires a deeper understanding of the geometric side of the trace formula, and this paper is just the first step in that direction.

Voronoi summation. If one can always recover the functional equation from the trace formula, then from the perspective of analytic number theory, the Voronoi summation should be implied also from the trace formula. For example in [Kowalski et al. 2000; 2002], an application of a trace formula and a Voronoi summation are used to get results on subconvexity. Could one avoid Voronoi summation and just apply the trace formula? In [Herman $\geq$ 2012a], we do just that to get subconvexity for the Rankin–Selberg L-function in both levels by applying a double trace formula instead of a Voronoi summation and a single trace formula.

1.3. ***Key steps in proof.*** As for the proof of the main theorem, one sees the role of the sum over the Kloosterman sums on the geometric side of the trace formula interacting with the averaging coming from the Dirichlet series for the standard L-function.

To see the functional equation of a GL_2 L-function, the Dirichlet series sum exchanges roles with the sum of Kloosterman sums. There are two important steps in this switching of roles of parametrization. One is elementary reciprocity,

$$\frac{\overline{A}}{B}+\frac{\overline{B}}{A}\equiv\frac{1}{AB}(1),$$

which allows one to invert the modulus of exponential sums. This simple reciprocity seems to come up several times in these beyond endoscopy calculations (see [Herman 2012; ≥ 2012b]). The second important tool is the integral representation

$$\int_0^\infty \exp(-\alpha x)J_\nu(2\beta\sqrt{x})J_\nu(2\gamma\sqrt{x})\,dx=\frac{1}{\alpha}I_\nu\Big(\frac{2\beta\gamma}{\alpha}\Big)\exp\Big(\frac{-(\beta^2+\gamma^2)}{\alpha}\Big)\,dx.$$

Given that Bessel functions are the archimedean version of Kloosterman sums, this representation implies that a Fourier transform of a product of Kloosterman sums is another Kloosterman sum times an exponential sum. It would be nice to see how these two steps are generalized for higher rank or for a relative trace formula for other groups.

2. Preliminaries

We recall the functional equation for a cusp form. Let D be a squarefree integer, χ be a primitive Dirichlet character modulo D, and $k\ge 2$, $k\in 2\mathbb{Z}$. Let $f\in S_k(D,\chi)$, where $S_k(D,\chi)$ is the space of holomorphic modular forms of weight k and level D with nebentypus χ; see [Iwaniec and Kowalski 2004]. In this case the space $S_k(D,\chi)$ can be spanned by an orthonormal basis of primitive newforms which we label $B_k(D,\chi)$. We note the Fourier coefficients $c_n(f)n^{(k-1)/2}$ of a form f in $B_k(D,\chi)$ satisfy

$$c_n(f)c_l(f)=\sum_{r\mid(n,l)}\chi(r)c_{(nl)/r^2}(f)$$

for $(nl,D)=1$, and also that $|c_D(f)|=1$.

Let $L(f,s)=\sum_{n=1}^\infty c_n(f)/n^s$, and define $\Lambda(f,s)=\gamma(f,s)L(f,s)$, where

$$\gamma(f,s)=\Big(\frac{\sqrt{D}}{2\pi}\Big)^s\Gamma\Big(\frac{s+\frac{k-1}{2}}{2}\Big)\Gamma\Big(\frac{s+\frac{k+1}{2}}{2}\Big).$$

The functional equation then says $\Lambda(f,s)=i^k\Lambda(f,1-s)$.

The trace formula we use is Petersson's formula, which is a variant of the relative trace formula [Knightly and Li 2006a]. This formula requires a normalization of the Fourier coefficients. For $c_n(f)$ above, define

$$a_n(f) := \frac{\sqrt{\pi^{-k}\Gamma(k)}}{2^{k-1}} c_n(f).$$

Petersson's formula states

$$\sum_{f \in B_k(D,\chi)} a_n(f)\overline{a_l(f)} = \delta_{n,l} + 2\pi i^{-k} \sum_{c \equiv 0(D)}^{\infty} \frac{S_\chi(n,l,c)}{c} J_{k-1}\Big(\frac{4\pi\sqrt{nl}}{c}\Big). \tag{2-1}$$

Here

$$S_\chi(a,b,c) = \sum_{x(c)^*} \overline{\chi(x)} \Big(\frac{ax + b\bar{x}}{c}\Big),$$

where $x\bar{x} \equiv 1(c)$, $e(x) := \exp(2\pi i x)$ and $J_t(x)$ is the J-Bessel function with index t.

To relate the functional equation to the geometric side of the trace formula, we need an equivalent version of the functional equation for a form $f \in B_k(D,\chi)$, which is called Voronoi summation. The Voronoi summation needed is proved in the appendix of [Kowalski et al. 2002], and states this:

Theorem 2.1. *Let $g \in C_0^\infty(\mathbb{R}^+)$ and $f \in B_k(D,\chi)$, then for integers a, c such that $(aD, c) = 1$,*

$$\begin{aligned} &\sum_{n \geq 1} a_n(f) e\Big(\frac{an}{c}\Big) g(n) \\ &\quad = \frac{2\pi i^k \eta(f)\chi(-c)}{c\sqrt{D}} \sum_{n \geq 1} a_n(f_D) e\Big(\frac{-n\overline{aD}}{c}\Big) \int_0^\infty g(x) J_{k-1}\Big(\frac{4\pi\sqrt{nx}}{\sqrt{D}c}\Big)\, dx, \end{aligned} \tag{2-2}$$

where $a\bar{a} \equiv 1(c)$. Here $\eta(f) = \tau(\chi)/(a_D(f)\sqrt{D})$, with $\tau(\chi)$ denoting the Gauss sum associated to χ, and

$$a_n(f_D) = \begin{cases} \overline{\chi(n)}\, a_n(f) & \text{if } (n,D) = 1, \\ \overline{a_n(f)} & \text{if } n \mid D^\infty. \end{cases}$$

In our case, we only take $a = c = 1$. If so, the functional equation of the L-function $L(f,s)$ is equivalent to the Voronoi summation by using Mellin inversion on the left hand side of (2-2), then applying the functional equation to $L(f,s)$ and using the integral representation

$$J_{k-1}(x) = \frac{1}{4\pi i} \int_{(\sigma)} \Big(\frac{x}{2}\Big)^{-s} \frac{\Gamma\Big(\frac{1}{2}\Big(s + \frac{k-1}{2}\Big)\Big)}{\Gamma\Big(\frac{1}{2}\Big(1 - \frac{s}{2} + \frac{k-1}{2}\Big)\Big)} ds$$

for $0 < \sigma < 1$, along with the duplication formula for the gamma function.

3. Main theorem

The main theorem of the paper is:

Theorem 3.1. *Let* $g \in C_0^\infty(\mathbb{R}^+)$ *satisfy* $|x^j g^{(j)}(x)| \ll (1 + |\log x|)$. *Then, for any* $l \in \mathbb{N}$ *with* $(l, D) = 1$, *and assuming Petersson's formula above*, *one gets*

$$\sum_{f \in B_k(D,\chi)} \overline{a_l(f)} \sum_{n\geq 1} a_n(f) g(n) = \sum_{f \in B_k(D,\chi)} \overline{a_l(f)} \left[\frac{2\pi i^k \eta(f)}{\sqrt{D}} \sum_n a_n(f_D) \int_0^\infty g(x) J_{k-1}\left(\frac{4\pi\sqrt{nx}}{\sqrt{D}}\right) dx \right]. \tag{3-1}$$

Using Hecke theory one gets:

Corollary 3.2. *For a modular form* $f \in B_k(D, \chi)$,

$$L(f, s) = \frac{i^k \gamma(f, 1-s) L(f, 1-s)}{\gamma(f, s)},$$

or,

$$\Lambda(f, s) = \Lambda(f, 1-s).$$

Proof of Theorem 3.1. Using Petersson's trace formula on the left hand side of (3-1) one gets

$$\sum_n g(n)\left[\delta_{n,l} + 2\pi i^{-k} \sum_{c=1}^\infty \frac{S_\chi(n, l, Dc)}{Dc} J_{k-1}\left(\frac{4\pi\sqrt{nl}}{Dc}\right)\right] = g(l) + 2\pi i^{-k} \sum_{c=1}^\infty \sum_n g(n) \frac{S_\chi(n, l, Dc)}{Dc} J_{k-1}\left(\frac{4\pi\sqrt{nl}}{Dc}\right). \tag{3-2}$$

We can interchange the c-sum and n-sum as the latter is compactly supported.

For now we will ignore the term $g(l)$, and come back to it later. Opening up the Kloosterman sum and gathering the n-sum together, we apply Poisson summation on it in arithmetic progressions modulo c, getting

$$2\pi i^{-k} \sum_{c=1}^\infty \frac{1}{(Dc)^2} \sum_{x(Dc)^*} \chi(x) e\left(\frac{\bar{x}l}{Dc}\right) \sum_{m\in\mathbb{Z}} \sum_{k(Dc)} e\left(\frac{xk+mk}{Dc}\right) \times \int_{-\infty}^\infty g(t) J_{k-1}\left(\frac{4\pi\sqrt{tl}}{Dc}\right) e\left(\frac{-mt}{Dc}\right) dt.$$

Using

$$\sum_{a(c)} e\left(\frac{ax}{c}\right) = \begin{cases} c & \text{if } x \equiv 0(c), \\ 0 & \text{else}, \end{cases} \tag{3-3}$$

one gets

$$2\pi i^{-k}\sum_{c=1}^{\infty}\frac{1}{Dc}\sum_{\substack{m\neq 0\in\mathbb{Z}\\(m,Dc)=1}}\overline{\chi(m)}\,e\Big(\frac{-l\overline{m}}{Dc}\Big)\int_{-\infty}^{\infty}g(t)J_{k-1}\Big(\frac{4\pi\sqrt{tl}}{Dc}\Big)e\Big(\frac{-mt}{Dc}\Big)\,dt. \tag{3-4}$$

Note the $m = 0$ disappears.

Now the interesting part of the argument is that the c-sum and n-sum swap roles, in that the c-sum will become part of the averaging coming from the L-function.

We use the elementary reciprocity

$$\frac{\overline{A}}{B}+\frac{\overline{B}}{A}\equiv\frac{1}{AB}(1),$$

to get

$$2\pi i^{-k}\sum_{c=1}^{\infty}\frac{1}{Dc}\sum_{\substack{m\neq 0\in\mathbb{Z}\\(m,Dc)=1}}\overline{\chi(m)}e\Big(\frac{l\bar{c}}{Dm}\Big)e\Big(\frac{-l}{mDc}\Big) \times\int_{-\infty}^{\infty}g(t)J_{k-1}\Big(\frac{4\pi\sqrt{tl}}{Dc}\Big)e\Big(\frac{-mt}{Dc}\Big)\,dt. \tag{3-5}$$

Also, the terms $m < 0$ we write as $-m$, $m \in \mathbb{N}$, and exchange sign to the c-sum. This can be clearly done everywhere except for in the J-Bessel function and the $\frac{1}{c}$ term. Using the fact that $J_{k-1}(-x) = -J_{k-1}(x)$, we can rewrite (3-5) as

$$2\pi i^{-k}\sum_{c\neq 0,c\in\mathbb{Z}}^{\infty}\frac{1}{Dc}\sum_{\substack{m=1\\(m,c)=1}}\overline{\chi(m)}e\Big(\frac{l\bar{c}}{Dm}\Big)e\Big(\frac{-l}{mDc}\Big) \times\int_{-\infty}^{\infty}g(t)J_{k-1}\Big(\frac{4\pi\sqrt{tl}}{Dc}\Big)e\Big(\frac{-mt}{Dc}\Big)\,dt. \tag{3-6}$$

The rearrangement of the m-sum is accomplished by using a standard integration by parts argument in the t-integral and the estimate in the appendix of [Kowalski et al. 2002],

$$|z^k J_\nu(z)| \ll_{k,\nu} \frac{1}{(1+z)^{1/2}}$$

for $\Re\nu \geq 0$.

We also interchange the c-sum and m-sum. To justify the rearrangement, note for c large, and by using the power series expansion, we have the estimate

$$J_{k-1}\Big(\frac{4\pi\sqrt{tl}}{Dc}\Big) \ll \frac{1}{c^{k-1}}.$$

Therefore, for N sufficiently large, by estimating the exponentials and integral trivially and noting that $k \geq 2$, we get

$$\sum_{c>N} \frac{1}{Dc} \sum_{\substack{m=1\\(m,c)=1}} \overline{\chi(m)} e\Big(\frac{l\bar{c}}{Dm}\Big) e\Big(\frac{-l}{mDc}\Big) \int_{-\infty}^{\infty} g(t) J_{k-1}\Big(\frac{4\pi\sqrt{tl}}{Dc}\Big) e\Big(\frac{-mt}{Dc}\Big)\, dt \ll L(0,\overline{\chi}) \sum_{c>N} \frac{1}{c^k} < \infty. \tag{3-7}$$

Clearly, the c-sum up to N is finite and is not a problem, and the sums can be interchanged.

Now we need a integral representation from [Gradshteyn and Ryzhik 2000, 6.615],

$$\int_0^{\infty} \exp(-\alpha x) J_\nu(2\beta\sqrt{x}) J_\nu(2\gamma\sqrt{x})\, dx = \frac{1}{\alpha} I_\nu\Big(\frac{2\beta\gamma}{\alpha}\Big) \exp\Big(\frac{-(\beta^2+\gamma^2)}{\alpha}\Big) \tag{3-8}$$

for $\Re\nu > -1$.

We rewrite (3-6) as

$$(2\pi i)(2\pi i^{-k}) \sum_m \frac{1}{m} \sum_{\substack{c\neq 0, c\in Z\\(c,m)=1}} \overline{\chi(m)} e\Big(\frac{l\bar{c}}{Dm}\Big) \times \int_{-\infty}^{\infty} g(t) \Big[\frac{m}{2\pi i Dc} J_{k-1}\Big(\frac{4\pi\sqrt{tl}}{Dc}\Big) e\Big(\frac{-l}{mDc}\Big) e\Big(\frac{-mt}{Dc}\Big)\Big]\, dt.$$

Note the term in brackets is equal to the right hand side of (3-8) times i^{k-1} for $\alpha = 2\pi i Dc/m$, $\beta = 2\pi\sqrt{l}/m$, and $\gamma = 2\pi\sqrt{t}$ by using the fact that for $k-1$ odd, $J_{k-1}(z) = i^{k-1} I_{k-1}(-iy)$.

Using this integral representation, one has

$$4\pi^2 \sum_m \frac{1}{m} \sum_{\substack{c\neq 0, c\in Z\\(c,m)=1}} \overline{\chi(m)} e\Big(\frac{l\bar{c}}{Dm}\Big) \times \int_{-\infty}^{\infty} g(t) \Big[\int_0^{\infty} J_{k-1}\Big(\frac{4\pi\sqrt{ly}}{m}\Big) J_{k-1}(4\pi\sqrt{ty}) e\Big(\frac{-Dcy}{m}\Big)\, dy\Big]\, dt. \tag{3-9}$$

We make a change of variables $y \to y/D$ to get

$$4\pi^2 \sum_m \frac{1}{Dm} \sum_{\substack{c\neq 0, c\in Z\\(c,m)=1}} \overline{\chi(m)} e\Big(\frac{l\bar{c}}{Dm}\Big) \times \int_{-\infty}^{\infty} g(t) \Big[\int_0^{\infty} J_{k-1}\Big(\frac{4\pi\sqrt{lDy}}{Dm}\Big) J_{k-1}\Big(\frac{4\pi\sqrt{ty}}{\sqrt{D}}\Big) e\Big(\frac{-cy}{m}\Big)\, dy\Big]\, dt. \tag{3-10}$$

Using $\tau(\chi)\tau(\overline{\chi})/D = 1$, we get

$$\tag{3-11} \frac{4\pi^2\tau(\chi)}{D}\sum_{m=1}\frac{1}{Dm}\sum_{c\in\mathbb{Z}}\overline{\chi(m)}\tau(\overline{\chi})e\Big(\frac{l\bar{c}}{Dm}\Big) \\ \times\int_{-\infty}^{\infty}J_{k-1}\Big(\frac{4\pi\sqrt{lDy}}{Dm}\Big)\Big[\int_0^{\infty}g(t)J_{k-1}\Big(\frac{4\pi\sqrt{ty}}{\sqrt{D}}\Big)\,dt\Big]e\Big(\frac{-cy}{m}\Big)\,dy.$$

Anticipating the use of the Chinese remainder theorem, we let $c' = Dc$. So (3-11) equals

$$\tag{3-12} \frac{4\pi^2\tau(\chi)}{D}\sum_{m=1}\frac{1}{Dm}\sum_{\substack{c'\in\mathbb{Z},c'\equiv 0(D)\\ c'\neq 0,(c',m)=1}}\overline{\chi(m)}\tau(\overline{\chi})e\Big(\frac{l\overline{c'/D}}{Dm}\Big) \\ \times\int_{-\infty}^{\infty}J_{k-1}\Big(\frac{4\pi\sqrt{lDy}}{Dm}\Big)\Big[\int_0^{\infty}g(t)J_{k-1}\Big(\frac{4\pi\sqrt{ty}}{\sqrt{D}}\Big)\,dt\Big]e\Big(\frac{-c'y}{Dm}\Big)\,dy.$$

We focus on the arithmetic inside the c'-sum. We note, using $(m, D) = 1$ and the Chinese remainder theorem, that

$$\tag{3-13} \overline{\chi(m)}\tau(\overline{\chi})e\Big(\frac{l\overline{c'/D}}{Dm}\Big) = \Big[\sum_{a(D)}\overline{\chi}(a)e\Big(\frac{\overline{m}a}{D}\Big)\Big]\Big[\sum_{b(m)^*,\,\bar{b}l\equiv\frac{c'}{D}(m)}e\Big(\frac{b}{m}\Big)\Big] \\ = \sum_{\substack{x(Dm)\\ D\bar{x}l\equiv c'(Dm)}}\overline{\chi(x)}e\Big(\frac{x}{Dm}\Big).$$

Using (3-3) again the above equals

$$\sum_{\substack{x(Dm)\\ D\bar{x}l\equiv c'(Dm)}}\overline{\chi(x)}e\Big(\frac{x}{Dm}\Big) = \frac{1}{Dm}\sum_{x(Dm)}\overline{\chi(x)}e\Big(\frac{x}{Dm}\Big)\sum_{k(Dm)}e\Big(\frac{k(Dl\bar{x}-c')}{Dm}\Big).$$

Incorporating the above line and a rearrangement of the exponential sums, we have

$$\tag{3-14} \frac{4\pi^2\tau(\chi)}{D}\sum_{m=1}\frac{1}{(Dm)^2}\sum_{c'\in\mathbb{Z}}\sum_{x(Dm)}\overline{\chi(x)}e\Big(\frac{x}{Dm}\Big)\sum_{k(Dm)}e\Big(\frac{kDl\bar{x}}{Dm}\Big) \\ \times\int_0^{\infty}J_{k-1}\Big(\frac{4\pi\sqrt{lDy}}{Dm}\Big)\Big[\int_0^{\infty}g(t)J_{k-1}\Big(\frac{4\pi\sqrt{ty}}{\sqrt{D}}\Big)dt\Big]e\Big(\frac{-c'(y+k)}{Dm}\Big)\,dy.$$

We note the c'-sum has the restriction $c' \equiv 0(D)$ removed by the k-sum. With a change of variables $y \to y-k$, followed by $y \to Dmy$, we get

$$\frac{4\pi^2\tau(\chi)}{D}\sum_{m=1}\frac{1}{Dm}\sum_{x(Dm)^*}\overline{\chi(x)}e\Big(\frac{x}{Dm}\Big)\sum_{k(Dm)}e\Big(\frac{kDl\bar{x}}{Dm}\Big) \\ +\sum_{c'\in\mathbb{Z}}\int_0^\infty J_{k-1}\Big(\frac{4\pi\sqrt{lD(Dmy-k)}}{Dm}\Big) \\ \times\bigg[\int_0^\infty g(t)J_{k-1}\bigg(\frac{4\pi\sqrt{t(Dmy-k)}}{\sqrt{D}}\bigg)dt\bigg]e(-c'y)\,dy. \tag{3-15}$$

The c'-sum now clearly came from a Poisson summation, namely,

$$\sum_{c'\in\mathbb{Z}}\int_0^\infty J_{k-1}\Big(\frac{4\pi\sqrt{lD(Dmy-k)}}{Dm}\Big) \\ \times\bigg[\int_0^\infty g(t)J_{k-1}\bigg(\frac{4\pi\sqrt{t(Dmy-k)}}{\sqrt{D}}\bigg)dt\bigg]e(-c'y)\,dy \\ =\sum_{c\in\mathbb{Z}}J_{k-1}\Big(\frac{4\pi\sqrt{lD(Dmc'-k)}}{Dm}\Big) \\ \times\int_0^\infty g(t)J_{k-1}\bigg(\frac{4\pi\sqrt{t(Dmc-k)}}{\sqrt{D}}\bigg)dt. \tag{3-16}$$

In order to check that

$$F(w)=J_{k-1}\bigg(\frac{4\pi\sqrt{lD(Dmw-k)}}{Dm}\bigg)\int_0^\infty g(t)J_{k-1}\bigg(\frac{4\pi\sqrt{t(Dmw-k)}}{D}\bigg)dt$$

satisfies the conditions for Poisson summation, we use the following lemma of [Kowalski et al. 2002]:

Lemma 3.3. *Let $h(x)$ be a smooth function supported on $[M, 2M]$ that satisfies $|x^j h^{(j)}(x)| \ll (1+|\log x|)$ for all $i \geq 0$, $x > 0$. For ν complex and $j \geq 0$ we have*

$$\int_0^\infty J_\nu(x)h(x)\,dx \ll_{\nu,j} \frac{(1+|\log M|)}{M^{j-1}}\frac{M^{\Re\nu+j+1}}{(1+M)^{\Re\nu+j+1/2}}.$$

We apply this to the integral in $F(w)$ with

$$h(t)=\frac{D^2}{16\pi^2(Dmw-k)^2}tg\Big(\frac{D^2t^2}{16\pi^2(Dmw-k)^2}\Big).$$

It is easy, but tedious, to check that the assumptions of the lemma are fulfilled by using the assumption on g that $|x^j g^{(j)}(x)| \ll (1+|\log x|)$ (from the hypothesis of Theorem 3.1). The lemma then gives $F(w) \ll \min(w^{k-1}, 1/w^j)$ for any $j > 0$ for $w \in [0, \infty)$. So certainly Poisson summation holds in this case.

Defining $Dmc-k=-j$, (3-15) again by regrouping equals

$$\begin{aligned}\frac{4\pi^2\tau(\chi)}{D}\sum_{m=1}\frac{1}{Dm}\sum_{j\in\mathbb{Z}}\sum_{x(Dm)^*}\overline{\chi(x)}e\Big(\frac{x}{Dm}\Big)e\Big(\frac{jDl\bar{x}}{Dm}\Big)J_{k-1}\Big(\frac{4\pi\sqrt{ljD}}{Dm}\Big)&\\ \times\int_0^\infty g(t)J_{k-1}\left(\frac{4\pi\sqrt{tj}}{\sqrt{D}}\right)dt&\\ =\frac{4\pi^2\tau(\chi)}{D}\sum_{m=1}\frac{1}{Dm}\sum_{j\in\mathbb{Z}}\overline{\chi(j)}S_\chi(Dl,j,Dm)J_{k-1}\Big(\frac{4\pi\sqrt{lDj}}{Dm}\Big)&\\ \times\int_0^\infty g(t)J_{k-1}\left(\frac{4\pi\sqrt{tj}}{\sqrt{D}}\right)dt&\\ =\frac{4\pi^2\tau(\chi)}{D}\sum_{j\in\mathbb{Z}}\overline{\chi(j)}\Bigg[\sum_{m\equiv 0(D)}\frac{S_\chi(Dl,j,m)}{m}J_{k-1}\left(\frac{4\pi\sqrt{Dlj}}{m}\right)\Bigg]&\\ \times\int_0^\infty g(t)J_{k-1}\left(\frac{4\pi\sqrt{tj}}{\sqrt{D}}\right)dt.&\end{aligned}$$

Now recall we ignored $g(l)$ from (3-2), so (3-1) equals

$$g(l)+\frac{2\pi i^k\tau(\chi)}{D}\sum_j\overline{\chi(j)}\Bigg[2\pi i^{-k}\sum_{m\equiv 0(D)}\frac{S_\chi(Dl,j,m)}{m}J_{k-1}\Big(\frac{4\pi\sqrt{Dlj}}{m}\Big)\Bigg]\times\int_0^\infty g(t)J_{k-1}\left(\frac{4\pi\sqrt{tj}}{\sqrt{D}}\right)dt. \tag{3-17}$$

The $g(l)$ term is again the diagonal term for the geometric side of the trace formula that comes from the term

$$\sum_{f_D}a_{lD}(f_D)\overline{a_{lD}(f_D)}. \tag{3-18}$$

This is again using the fact that $|a_D(f_D)|=1$.

Now as D is squarefree and χ is primitive, the space $B_k(N,\chi)$ is spanned by newforms, which implies the Fourier coefficients are multiplicative in all the primes (including the bad primes) and $|c_D(f)|=1$. So using Petersson's formula again we get

$$\begin{aligned}\sum_{f\in B_k}\overline{a_l(f)}\left[\frac{2\pi i^k\tau(\chi)}{Da_D(f)}\sum_j\overline{\chi(j)}a_j(f)\int_0^\infty g(x)J_{k-1}\left(\frac{4\pi\sqrt{jx}}{\sqrt{D}}\right)dx\right]&\\ =\sum_{f\in B_k}\overline{a_l(f)}\left[\frac{2\pi i^k\eta(f)}{\sqrt{D}}\sum_j\overline{\chi(j)}a_j(f)\int_0^\infty g(x)J_{k-1}\left(\frac{4\pi\sqrt{jx}}{\sqrt{D}}\right)dx\right]&\\ =\sum_{f_D\in B_k}\overline{a_l(f_D)}\left[\frac{2\pi i^k\eta(f)}{\sqrt{D}}\sum_{(j,D)=1}a_j(f_D)\int_0^\infty g(x)J_{k-1}\left(\frac{4\pi\sqrt{jx}}{\sqrt{D}}\right)dx\right].&\end{aligned} \tag{3-19}$$

Note that to show the connection to Voronoi summation from Theorem 2.1, we need also the coefficients $a_j(f_D)$ with $(j, D) > 1$. We state a lemma:

Lemma 3.4. *For* $(l, D) = 1$,

$$\sum_{f_D \in B_k} \overline{a_l(f_D)} \left[\frac{2\pi i^k \eta(f)}{\sqrt{D}} \sum_{(j,D)>1} a_j(f_D) \int_0^\infty g(x) J_{k-1}\left(\frac{4\pi\sqrt{jx}}{\sqrt{D}} \right) dx \right] = 0. \tag{3-20}$$

Assuming the lemma for now, we get that (3-19) equals

$$\sum_{f_D \in B_k} \overline{a_l(f_D)} \left[\frac{2\pi i^k \eta(f)}{\sqrt{D}} \sum_{j=1}^{\infty} a_j(f_D) \int_0^\infty g(x) J_{k-1}\left(\frac{4\pi\sqrt{jx}}{\sqrt{D}} \right) dx \right],$$

which proves Theorem 3.1. □

Proof of Lemma 3.4. First we write $j = D^k j'$, $(j', D) = 1$. Using the definition of the coefficients $f_D(n)$ in Theorem 2.1, the left hand side of (3-20) equals

$$\frac{2\pi i^k}{\sqrt{D}} \sum_{k=1}^{\infty} \sum_{(j,D)=1} \overline{\chi(j)} \sum_{f \in B_k} \overline{a_{lD^{k+1}}(f)}\, a_j(f) \int_0^\infty g(x) J_{k-1}\left(\frac{4\pi\sqrt{jD^k x}}{\sqrt{D}} \right) dx.$$

Fix a k, and following the same argument as we made previously, we apply Petersson's formula to get

$$2\pi i^{-k} \sum_j \overline{\chi(j)} \sum_{c=1}^{\infty} \frac{S_\chi(j, lD^{k+1}, Dc)}{Dc} J_{k-1}\left(\frac{4\pi\sqrt{nlD^{k+1}}}{Dc} \right) \\ \times \int_0^\infty g(x) J_{k-1}\left(\frac{4\pi\sqrt{jD^k x}}{\sqrt{D}} \right) dx.$$

That we can apply Petersson's formula in this case follows from using the estimates of Lemma 3.3. With a change of variable in the Kloosterman sum this equals

$$2\pi i^{-k} \sum_j \sum_{c=1}^{\infty} \frac{S_\chi(1, jlD^{k+1}, Dc)}{Dc} J_{k-1}\left(\frac{4\pi\sqrt{nlD^{k+1}}}{Dc} \right) \\ \times \int_0^\infty g(x) J_{k-1}\left(\frac{4\pi\sqrt{jD^k x}}{\sqrt{D}} \right) dx.$$

Interchanging the j- and c-sums, justified by a similar Bessel function analysis as above, we apply Poisson summation to the j-sum modulo Dc. The crucial arithmetic sums, analogous to the ones in obtaining (3-4), are

$$\sum_{x(Dc)^*} \overline{\chi(x)} e\left(\frac{x}{Dc} \right) \sum_{a(Dc)} e\left(\frac{\bar{x} a D^{k+1} l}{Dc} \right) e\left(\frac{-am}{Dc} \right), \tag{3-21}$$

where m is the variable for Poisson summation. The inner sum is nonzero only when $D^{k+1}l \equiv mx(Dc)$. If $(m, c) = 1$, then it is easy to check (3-21) is zero. As well if $D^h|m$ then $D^{h-1}|c$ as $(xl, D) = 1$ for $h \le k+1$. So for a nonzero contribution we must have $D^{k+1}|m$ and $D^k|c$. Writing $c = D^k c'$ and $m = D^{k+1}m'$, x must satisfy $l \equiv m'x(c')$. We can write these solutions as $x \equiv \overline{m}'l + c'b(Dc)$, where $b(D^{k+1})$. So (3-21) equals

$$Dc \sum_{b(D^{k+1})} \overline{\chi}(\overline{m}'l + c'b) e\Big(\frac{\overline{m}'l+c'b}{Dc}\Big) = e\Big(\frac{\overline{m}'l}{Dc}\Big) Dc \sum_{b(D^{k+1})} \overline{\chi}(\overline{m}'l + c'b) e\Big(\frac{b}{D^{k+1}}\Big).$$

With a change of variables $b \to \bar{c}b$, $b \to b - \overline{m}'l$, the inner Gauss sum is

$$\sum_{b(D^{k+1})} \overline{\chi}(b) e\Big(\frac{\bar{c}'b}{D^{k+1}}\Big) = \overline{\chi(c')} \sum_{b(D^{k+1})} \overline{\chi}(b) e\Big(\frac{b}{D^{k+1}}\Big).$$

This last Gauss sum is zero as χ is a primitive character modulo D and $k+1 \ge 2$. $\square$

Remark. There is nothing special about the test function we used in the lemma, and by a similar argument it is easy to show that for a "nice" test function $V(x)$ and $k \ge 2$,

$$\sum_{j=1}^{\infty} \overline{\chi(j)} V(j) \sum_{f \in B_k} \overline{a_{D^k}(f)} a_j(f) = 0.$$

4. Application of Hecke theory

Now to prove Corollary 3.2. One can rewrite Theorem 3.1 as

$$\text{(4-1)} \quad \frac{1}{2\pi i} \int_{(\sigma)} G(s) \Bigg[\sum_{f \in B_k} \overline{a_l(f)} \Big(L(f, s) - \frac{i^k \gamma(f, 1-s) L(f, 1-s)}{\gamma(f, s)} \Big) \Bigg] ds = 0,$$

using that the Voronoi summation we take into consideration is equivalent to the functional equation. Since (4-1) holds for any g in $C_0^\infty(\mathbb{R}^+)$, and in fact holds with slightly more care for the transform of any Schwarz function, by completeness, it must hold that

$$\sum_{f \in B_k} \overline{a_l(f)} \Big(L(f, s) - \frac{i^k \gamma(f, 1-s) L(f, 1-s)}{\gamma(f, s)} \Big) = 0.$$

Fix a form $f^\circ \in B_k$. Now as l was arbitrary and the space of forms $f \in B_k$ is finite dimensional, using the relation

$$a_n(f) a_l(f) = \sum_{r|(n,l)} \chi(r) a_{nl/r^2}(f)$$

for $(nl, D) = 1$, one can build a polynomial in the Hecke coefficients, call it $F(a_{q_1}(f), a_{q_2}(f), \dots, a_{q_N}(f))$, such that

$$\sum_{f\in B_k} F(a_{q_1}(f), a_{q_2}(f), \dots, a_{q_N}(f))\left(L(f,s) - \frac{i^k\gamma(f,1-s)L(f,1-s)}{\gamma(f,s)}\right) = 0,$$

where F equals 1 for $f = f^\circ$, and equals 0 for $f \neq f^\circ$, following [Herman 2010]. So we get a pointwise equality

$$L(f,s) - \frac{i^k\gamma(f,1-s)L(f,1-s)}{\gamma(f,s)} = 0,$$

which proves the corollary. □

Appendix

We replicate Sarnak's argument [2001] from his letter to Langlands. In order to do so, we use the Kuznetsov trace formula for the entire GL_2 spectrum. We refer to [Herman 2011] for the details. Let $H(D,\chi)$ denote the GL_2 spectrum with level D and nebentypus χ.

Theorem A.1. *Let $g, V \in C_0^\infty(\mathbb{R}^+)$ with $|x^j g^{(j)}(x)| \ll (1 + |\log x|)$, X a large fixed real number, and D and χ as above. Then for any integer $A > 0$,*

$$\sum_{n\le X}\sum_{f\in H(D,\chi)} h(t_f, V)\overline{a_l(f)}a_n(f)g(n/X) = O(X^{-A}). \tag{A-1}$$

Proof. We apply the Kuznetsov trace formula and Poisson summation, similarly as we did in obtaining (3-4) for just the holomorphic forms, to get

$$\sum_{c=1}^{\infty}\frac{1}{Dc}\sum_{\substack{m\neq 0\in\mathbb{Z}\\ (m,Dc)=1}}\overline{\chi(m)}e\left(\frac{-l\overline{m}}{Dc}\right)\int_{-\infty}^{\infty} g\left(\frac{t}{X}\right)V\left(\frac{4\pi\sqrt{tl}}{Dc}\right)e\left(\frac{-mt}{Dc}\right)dt. \tag{A-2}$$

Essentially, the argument only depends on showing the integral is bounded by $O(X^{-A})$. Note that as V and g are compactly supported, the c-sum is restricted to size $a\sqrt{X} \le c \le b\sqrt{X}$, for some absolute constants $a, b \in \mathbb{R}^+$, notated $c \sim \sqrt{X}$. Note that $g^{(k)}(Dct/X) \ll 1/X^{k/2}$ and $V^{(h)}(4\pi\sqrt{tl}/\sqrt{Dc}) \ll 1/X^{h/2}$ for $h, k \ge 0$. Also the size of the integral is $X/c \sim \sqrt{X}$. Using these estimates and integrating by parts j-times, after a change of variables $t \to Dct$, it easy to check that

$$Dc\int_{-\infty}^{\infty} g\left(\frac{Dct}{X}\right)V\left(\frac{4\pi\sqrt{tl}}{\sqrt{Dc}}\right)e(-mt)\,dt \ll \frac{Dc}{(\sqrt{X})^{j-1}m^j} \ll \frac{1}{(\sqrt{X})^{j-2}m^j}. \tag{A-3}$$

So including the c- and m-sums we have

$$\ll \frac{1}{(\sqrt{X})^{j-2}} \sum_{c\sim\sqrt{X}} \sum_{m} \frac{1}{m^j} \ll \frac{1}{(\sqrt{X})^{j-3}}.$$

Obviously, this implies the theorem by taking $(j-3/2) > A$. □

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I want to dedicate this paper to my advisor, Jon Rogawski. I could not have asked for a better advisor than Jon as a graduate student at UCLA. His unfaltering patience and calm resolve balanced my personality, which was the opposite of patient in those days. When I would fail to understand an aspect of automorphic forms or the trace formula, Jon would dismiss my frustration and clarify the misunderstanding in a way that only Jon could with sympathetic composure and the knowledge of a veteran in the field. While he was always collected when he addressed the challenges that I faced, he was a passionate person and was ardent when he spoke about math. During times of stagnation, I would go to Jon to reinvigorate me. After talking to Jon, I always felt more inspired and confident.

From winter to the early part of summer of 2011, I was at the American Institute of Mathematics in Palo Alto and would visit Jon in Los Angeles every few months. I remember fondly going to coffee shops or to his home to discuss my new ideas as he shared his own. It was in one of these gatherings that he suggested how to isolate a single Hecke eigenform (found in Section 4). I want to point this out because even today, Jon continues to inspire me. I am honored to have been one of his students and am also saddened that I was his last.

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P. EDWARD HERMAN
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CHICAGO
5734 S. UNIVERSITY AVENUE
CHICAGO, IL 60637
UNITED STATES

peherman@math.uchicago.edu

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A CORRECTION TO *CONDUCTEUR DES REPRÉSENTATIONS DU GROUPE LINÉAIRE*

HERVÉ JACQUET

We give a correct proof for the existence of the essential vector of an irreducible admissible generic representation of the general linear group over a $\mathfrak{p}$-adic field.

Nadir Matringe has indicated to me that the paper "Conducteur des représentations du groupe linéaire" [Jacquet et al. 1981a; 1981b] contains an error. Since the result therein has applications (see [Jacquet and Shalika 1985] for instance), it may be useful to correct the error. In any case, the correct proof is actually simpler than the erroneous proof. Separately, Matringe [2011] has given a different proof, which is of independent interest.

First, I recall the result in question. Let F be a non-Archimedean local field. We denote by α or $|\cdot|$ the absolute value of F, by q the cardinality of the residual field and finally by v the valuation function on F. Thus, $\alpha(x) = |x| = q^{-v(x)}$. Let ψ be an additive character of F whose conductor is the ring of integers $\mathbb{O}_F$. Let G_r be the group $GL(r)$ regarded as an algebraic group. We denote by w_r the permutation matrix whose antidiagonal entries are 1. For instance,

$$w_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

We denote by dg the Haar measure of $G_r(F)$ for which the compact group $G_r(\mathbb{O}_F)$ has volume 1. Let N_r be the subgroup of upper triangular matrices with unit diagonal and A_r the group of diagonal matrices. We define a character

$$\theta_{r,\psi} : N_r(F) \to \mathbb{C}^\times$$

by the formula

$$\theta_{r,\psi}(u) = \psi\Bigg(\sum_{1 \le i \le r-1} u_{i,i+1}\Bigg).$$

We denote by du the Haar measure on $N_r(F)$ for which $N_r(\mathbb{O}_F)$ has measure 1. We have then an invariant quotient measure on $N_r(F)\backslash G_r(F)$.

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Let S_r be the algebra of symmetric polynomials in

$$(X_1, X_1^{-1}, X_2, X_2^{-1}, \dots, X_r, X_r^{-1}).$$

Let H_r be the Hecke algebra of $G_r(F)$, that is, the convolution algebra of compactly supported, complex-valued functions that are bi-invariant under the maximal compact group $G_r(\mathbb{O}_F)$. Let $\mathcal{S}_r : H_r \to S_r$ be the Satake isomorphism. Thus, for any r-tuple of nonzero complex numbers $(x_1, x_2, \dots, x_r)$ we have a homomorphism of algebras $\mathcal{S}_r(x_1, x_2, \dots, x_r) : H_r \to \mathbb{C}$, defined by

$$\mathcal{S}_r(x_1, x_2, \dots, x_r) : \phi \mapsto \mathcal{S}_r(\phi)(x_1, x_2, \dots, x_r)\,.$$

Concretely, it is defined in the following way. Let $t = (t_1, t_2, \dots, t_r)$ be a tuple of complex numbers such that $x_i = q^{-t_i}$ for each i. We denote by $\pi(t_1, t_2, \dots, t_r)$ the corresponding principal series representation of $G_{r-1}(F)$. It is the representation induced by the character

$$a = (a_1, a_2, \dots, a_r) \mapsto |a_1|^{t_1} |a_2|^{t_2} \cdots |a_r|^{t_r}$$

of $A_r(F)$. Its space $I(t_1, t_2, \dots, t_r)$ is the space of smooth functions $\phi : G_r(F) \to \mathbb{C}$ such that

$$\phi\left[\begin{pmatrix} a_1 & * & \cdots & \cdots & * \\ 0 & a_2 & \cdots & \cdots & * \\ 0 & 0 & \cdots & \cdots & a_r \end{pmatrix} g\right] = |a_1|^{t_1 + \frac{r-1}{2}} |a_2|^{t_2 + \frac{r-1}{2} - 1} \cdots |a_r|^{t_r - \frac{r-1}{2}} \phi(g)\,.$$

The space $I(t_1, t_2, \dots, t_r)$ contains a unique vector ϕ_0 equal to 1 on $G_r(\mathbb{O}_F)$ and thus invariant under $G_r(\mathbb{O}_F)$. Under convolution, it is an eigenfunction of H_r with eigenvalue $\mathcal{S}_r(x_1, x_2, \dots, x_r)$, that is,

$$\int_{G_r(F)} \phi_0(gh)\, \phi(h)\, dh = \mathcal{S}_r(\phi)(x_1, x_2, \dots, x_r)\, \phi_0(g)$$

for every ϕ in H_r.

There is a unique function $W : G_r(F) \to \mathbb{C}$ satisfying the following properties:

- $W(gk) = W(g)$ for $k \in G_r(\mathbb{O}_F)$,
- $W(ug) = \theta_\psi(u)\, W(g)$ for $u \in N_r(F)$,
- for all $(x_1, x_2, \dots, x_r)$ and all $\phi \in H_r$,

$$\int_{G_r(F)} W(gh)\, \phi(h)\, dh = \mathcal{S}_r(\phi)(x_1, x_2, \dots, x_r)\, W(g),$$

- $W(e) = 1$.

Thus, W is an eigenfunction of H_r with eigenvalue $S_r(x_1, x_2, \dots, x_r)$. We will denote this function by $W(x_1, x_2, \dots, x_r; \psi)$ and its value at g by

$$W(g; x_1, x_2, \dots x_r; \psi)\,.$$

Let (π, V) be an irreducible admissible representation of $G_r(F)$. We assume that π is *generic*, that is, there is a nonzero linear form $\lambda : V \to \mathbb{C}$ such that

$$\lambda(\pi(u)v) = \theta_{r,\psi}(u)\,\lambda(v)$$

for all $u \in N_r(F)$ and all $v \in V$. Recall that such a form is unique within a scalar factor. We denote by $\mathcal{W}(\pi; \psi)$ the space of functions of the form

$$g \mapsto \lambda(\pi(g)v)$$

with $v \in V$. It is the *Whittaker model* of π. On the other hand, we have the L-factor $L(s, \pi)$ [Godement and Jacquet 1972]. We denote by $P_\pi(X)$ the polynomial defined by $L(s, \pi) = P_\pi(q^{-s})^{-1}$. The main result of [Jacquet et al. 1981a] is the following theorem:

Theorem 1. *There is an element $W \in \mathcal{W}(\pi; \psi)$ such that, for any $(r-1)$-tuple of nonzero complex numbers $(x_1, x_2, \dots, x_{r-1})$,*

$$\int_{N_{r-1}(F)\backslash G_{r-1}(F)} W\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} W(g; x_1, x_2, \dots, x_{r-1}; \overline{\psi})|\det g|^{s-1/2}\, dg \\ = \prod_{1 \le i \le r-1} P_\pi(q^{-s}x_i)^{-1}.$$

In [Jacquet et al. 1981a] it is shown that if we impose the extra condition

$$W\begin{pmatrix} gh & 0 \\ 0 & 1 \end{pmatrix} = W\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}$$

for all $h \in G_{r-1}(\mathcal{O}_F)$ and $g \in G_{r-1}(F)$, then W is unique. The vector W is then called the *essential vector* of π, and further properties of this vector are obtained in [Jacquet et al. 1981a].

The proof of this theorem is incorrect in that paper. We give a correct proof here.

1. Review of the properties of the L-factor

Let $r \ge 2$ be an integer. Let $t = (t_1, t_2, \dots, t_{r-1})$ be an $(r-1)$-tuple of complex numbers. We assume that

$$\mathrm{Re}(t_1) \ge \mathrm{Re}(t_2) \ge \cdots \ge \mathrm{Re}(t_{r-1}).$$

Again, we consider the representation $\pi(t_1, t_2, \dots, t_{r-1})$ that acts on the space $I(t_1, t_2, \dots, t_{r-1})$. As before, let ϕ_0 be the unique vector of that space that is equal to 1 on $G_{r-1}(\mathbb{O}_F)$. Recall it is invariant under $G_{r-1}(\mathbb{O}_F)$. We recall a standard result.

Lemma 1. *For each tuple t satisfying the above inequalities the vector ϕ_0 is a cyclic vector for the representation $\pi(t_1, t_2, \ldots, t_{r-1})$.*

Proof. Indeed, if $\mathrm{Re}(t_1) = \mathrm{Re}(t_2) = \cdots = \mathrm{Re}(t_{r-1})$, the representation is irreducible and our assertion is trivial. If not, we use Langlands' construction [Silberger 1978]. For each root α of A_{r-1} we denote by N_α the corresponding subgroup of N_{r-1} or $\overline{N}_{r-1}$ and by $\check{\alpha}$ the corresponding co-root. Thus, if α is a positive root, we have

$$\alpha(a_1, a_2, \ldots, a_{r-1}) = a_i/a_j$$

with $i < j$ and

$$\langle t, \check{\alpha}\rangle = t_i - t_j\ .$$

Let $P(t)$ be the set of positive roots α such that $\mathrm{Re}\langle t, \check{\alpha}\rangle > 0$. Let U be the unipotent group generated by the subgroups $N_{-\alpha}$ with $\alpha \in P(t)$. The intertwining operator

$$N\phi(g) = \int_{U(F)} \phi(ug)\, du$$

is defined by a convergent integral, and its kernel is a maximal invariant subspace. The formula of [Gindikin and Karpelevič 1966; Gindikin 1961] gives

$$N\phi_0(e) = \prod_{\alpha\in P(t)} \frac{1 - q^{-\langle t,\check{\alpha}\rangle - 1}}{1 - q^{-\langle t,\check{\alpha}\rangle}}\ .$$

Thus, $N\phi_0 \neq 0$, and our assertion follows. □

The representation $I(t_1, t_2, \ldots, t_{r-1})$ admits a nonzero linear form λ such that, for $u \in N_{r-1}(F)$ and ϕ in the space of the representation,

$$\lambda(\pi(u)\phi) = \theta_{r-1,\overline{\psi}}(u)\, \lambda(\phi).$$

We denote by $\mathcal{W}(t_1, t_2, \ldots, t_{r-1}; \overline{\psi})$ the space spanned by the functions of the form

$$g \mapsto W_\phi(g), \quad W_\phi(g) = \lambda(\pi(t_1, t_2, \ldots, t_{r-1})(g)\phi)$$

with $\phi \in I(t_1, t_2, \ldots, t_{r-1})$. We recall the following result:

Lemma 2 [Jacquet and Shalika 1983]. *The map $\phi \mapsto W_\phi$ is injective.*

It follows that the image W_0 of ϕ_0 is a cyclic vector in $\mathcal{W}(t_1, t_2, \ldots, t_{r-1}; \overline{\psi})$. Up to a multiplicative constant, the function W_0 is equal to the function

$$W_0 = W(x_1, x_2, \ldots, x_{r-1}; \overline{\psi}).$$

Now let π be an irreducible generic representation of $G_r(F)$. For $W \in \mathcal{W}(\pi, \psi)$ and $W' \in \mathcal{W}(t_1, t_2, \ldots, t_{r-1}; \overline{\psi})$ we consider the integral

$$\Psi(s, W, W') = \int_{N_{r-1}\backslash G_{r-1}} W\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} W'(g) |\det g|^{s-1/2}\, dg.$$

The integral converges absolutely if $\mathrm{Re}(s) \gg 0$ and extends to a meromorphic function of s. In any case, it has a meaning as a formal Laurent series in the variable q^{-s} (see below). We recall a result from [Jacquet et al. 1983].

Lemma 3. *There are functions $W_j \in \mathcal{W}(\pi; \psi)$ and $W'_j \in \mathcal{W}(t_1, t_2, \ldots, t_{r-1}; \overline{\psi})$, $1 \le j \le k$, such that*

$$\sum_{1\le j\le k} \Psi(s, W_j, W'_j) = \prod_{1\le i\le r-1} L(s+t_i, \pi).$$

Since W_0 is a cyclic vector, after a change of notations, we see that there are functions $W_j \in \mathcal{W}(\pi; \psi)$ and integers n_j, $1 \le j \le k$, such that

$$\sum_j q^{-n_i s} \Psi(s, W_j, W(x_1, x_2, \ldots, x_{r-1}; \overline{\psi})) = \prod_{1\le i\le r-1} L(s+t_i, \pi).$$

In our discussion $|x_1| \le |x_2| \le \cdots \le |x_{r-1}|$. However, the functions

$$W(x_1, x_2, \ldots, x_{r-1}; \overline{\psi})$$

are symmetric in the variables x_i. Thus, we have the following result:

Lemma 4. *Given an $(r-1)$-tuple of nonzero complex numbers $(x_1, x_2, \ldots, x_{r-1})$ there are functions $W_j \in \mathcal{W}(\pi; \psi)$ and integers n_j, $1 \le j \le k$, such that*

$$\sum_j q^{-n_j s} \Psi(s, W_j, W(x_1, x_2, \ldots, x_{r-1}; \overline{\psi})) = \prod_{1\le i\le r-1} P_\pi(q^{-s} x_i)^{-1}.$$

2. The ideal I_π

We review the construction of [Jacquet et al. 1981a], adding a little more detail to some formal computations. First, we introduce a function

$$W(X_1, X_2, \ldots, X_{r-1}; \overline{\psi}) : G_{r-1}(F) \to S_{r-1}$$

whose value at a point $g \in G_{r-1}(F)$ is denoted $W(g; X_1, X_2, \ldots, X_{r-1}; \overline{\psi})$. It is defined by the following property: for every $(r-1)$-tuple $(x_1, x_2, \ldots, x_{r-1})$ and every g, the scalar $W(g; x_1, x_2, \ldots, x_{r-1}; \overline{\psi})$ is the value of the polynomial

$$W(g; X_1, X_2, \ldots, X_{r-1}; \overline{\psi})$$

at the point $(x_1, x_2, \dots, x_{r-1})$. For g in a set compact modulo $N_{r-1}(F)$, the polynomials $W(g; X_1, X_2, \dots, X_{r-1}; \overline{\psi})$ remain in a finite dimensional vector subspace of S_{r-1}. We have the relation

$$|\det g|^s W(g; x_1, x_2, \dots, x_{r-1}; \overline{\psi}) = W(g; q^{-s}x_1, q^{-s}x_2, \dots, q^{-s}x_{r-1}; \overline{\psi}) .$$

It follows that if $|\det g| = q^{-n}$, then the polynomial

$$W(g; X_1, X_2, \dots, X_{r-1}; \overline{\psi})$$

is homogeneous of degree n, that is,

$$W(g; XX_1, XX_2, \dots, XX_{r-1}; \overline{\psi}) = X^n W(g; X_1, X_2, \dots, X_{r-1}; \overline{\psi}).$$

For each integer n, we now define the integral

$$\Psi_n(W; X_1, X_2, \dots, X_{r-1}; \psi) \\ := \int_{|\det g| = q^{-n}} W \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} W(g, X_1, X_2, \dots, X_{r-1}; \overline{\psi}) |\det g|^{-1/2}\, dg.$$

The support of the integrand is contained in a set compact modulo $N_{r-1}(F)$, which depends on W. In addition, there is an integer $N(W)$ (depending on W) such that the support of the integrand is empty if $n < N(W)$. The polynomial

$$\Psi_n(W; X_1, X_2, \dots, X_{r-1}; \psi)$$

is homogeneous of degree n. We consider the following formal Laurent series with coefficients in S_{r-1}:

$$\Psi(X; W; X_1, X_2, \dots, X_{r-1}; \psi) = \sum_n X^n \Psi_n(W; X_1, X_2, \dots, X_{r-1}; \psi).$$

Hence, in fact

$$\Psi(X; W; X_1, X_2, \dots, X_{r-1}; \psi) = \sum_{n \geq N(W)} X^n \Psi_n(W; X_1, X_2, \dots, X_{r-1}; \psi).$$

If we multiply this Laurent series by $\prod_{1 \leq i \leq r-1} P_\pi(XX_i)$, we obtain a new Laurent series with coefficients in S_{r-1}, namely,

$$\Psi(X; W, X_1, X_2, \dots, X_{r-1}; \psi) \prod_{1 \leq i \leq r-1} P_\pi(XX_i) \\ = \sum_{n \geq N_1(W)} X^n a_n(X_1, X_2, \dots, X_{r-1}; \psi),$$

where $N_1(W)$ is another integer (depending on W) and $a_n \in S_{r-1}$. Each a_n is homogeneous of degree n. We can replace π by the contragredient representation $\widetilde{\pi}$,

ψ by $\overline{\psi}$ and the function W by the function $\widetilde{W}$ defined by

$$\widetilde{W}(g) = W(w_r {}^t g^{-1}).$$

The function $\widetilde{W}$ belongs to $\mathcal{W}(\widetilde{\pi}, \overline{\psi})$. We define similarly

$$\Psi(\widetilde{W}; X_1, X_2, \ldots, X_{r-1}; \overline{\psi}).$$

We have then the following functional equation [Jacquet et al. 1983]:

$$\Psi(q^{-1}X^{-1}; \widetilde{W}; X_1^{-1}, X_2^{-1}, \ldots, X_{r-1}^{-1}; \overline{\psi}) \prod_{i=1}^{r-1} P_{\widetilde{\pi}}(q^{-1}X^{-1}X_i^{-1})$$
$$= c_\pi \prod_{i=1}^{r-1} \epsilon_\pi(XX_i, \psi)\, \Psi(X; W, X_1, X_2, \ldots, X_{r-1}; \psi) \prod_{i=1}^{r-1} P_\pi(XX_i).$$

The ϵ factors are monomials and $c_\pi = \pm 1$. Thus, there is another integer $N_2(W)$ such that in fact

$$\Psi(X; W; X_1, X_2, \ldots, X_{r-1}; \psi) \prod_{1 \le i \le r-1} P_\pi(XX_i)$$
$$= \sum_{N_2(W) \ge n \ge N_1(W)} X^n a_n(X_1, X_2, \ldots, X_{r-1}).$$

From now on we drop the dependence on ψ from the notation.

From the above considerations it follows that the product

$$\Psi(X; W; X_1, X_2, \ldots, X_{r-1}) \prod_{1 \le i \le r-1} P_\pi(XX_i)$$

is in fact a polynomial in X with coefficients in S_{r-1}. Moreover, because the a_n are homogeneous of degree n, there is a polynomial $\Xi(W; X_1, X_2, \ldots, X_{r-1})$ in S_{r-1} such that

$$\Psi(X; W; X_1, X_2, \ldots, X_{r-1}) \prod_{1 \le i \le r-1} P_\pi(XX_i) = \Xi(W; XX_1, XX_2, \ldots, XX_{r-1}).$$

In a precise way, let us write

$$\prod_{1 \le i \le r-1} P_\pi(X_i) = \sum_{m=0}^{R} P_m(X_1, X_2, \ldots, X_{r-1}),$$

where each P_m is homogeneous of degree m. Then

$$\Psi(X; W; X_1, X_2, \ldots, X_{r-1}) \prod_{1 \le i \le r-1} P_\pi(XX_i)$$
$$= \sum_n X^n \sum_{m=0}^{R} \Psi_{n-m}(W; X_1, X_2, \ldots, X_{r-1})\, P_m(X_1, X_2, \ldots, X_{r-1}).$$

The polynomial $\Xi(W; X_1, X_2, \dots, X_{r-1})$ is then determined by the condition that its homogeneous component of degree n noted $\Xi_n(W; X_1, X_2, \dots, X_{r-1})$ be given by

$$\Xi_n(W; X_1, X_2, \dots, X_{r-1}) = \sum_{m=0}^{R} \Psi_{n-m}(W; X_1, X_2, \dots, X_{r-1}) P_m(X_1, X_2, \dots, X_{r-1}).$$

The theorem amounts to saying there is a W such that $\Xi(W; X_1, X_2, \dots, X_{r-1})$ equals 1.

Let I_π be the subvector space of S_{r-1} spanned by the polynomials

$$\Xi(W; X_1, X_2, \dots, X_{r-1}).$$

Lemma 5. *In fact I_π is an ideal of the algebra S_{r-1}.*

Proof. Let Q be an element of S_{r-1}. Let ϕ be the corresponding element of H_{r-1}. Then

$$\int W(gh; X_1, X_2, \dots, X_{r-1}) \phi(h)\, dh = W(g; X_1, X_2, \dots, X_{r-1}) Q(X_1, X_2, \dots, X_{r-1}).$$

Let W be an element of $\mathcal{W}(\pi, \psi)$. Define another element W_1 of $\mathcal{W}(\pi, \psi)$ by

$$W_1(g) = \int_{G_{r-1}} W\left[g \begin{pmatrix} h^{-1} & 0 \\ 0 & 1 \end{pmatrix} \right] \phi(h) |\det h|^{1/2}\, dh.$$

We claim that

$$\Xi(W_1; X_1, X_2, \dots, X_{r-1}) = \Xi(W; X_1, X_2, \dots, X_{r-1}) Q(X_1, X_2, \dots, X_{r-1}).$$

This will imply the Lemma.

By linearity, it suffices to prove our claim when Q is homogeneous of degree t. Then ϕ is supported on the set of h such that $|\det h| = q^{-t}$. We have then, for every n,

$$\begin{aligned}
\Psi_n(W_1; X_1, \dots, X_{r-1})
&= \int_{|\det g| = q^{-n}} W_1 \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} W(g; X_1, \dots, X_{r-1}) |\det g|^{-1/2}\, dg \\
&= \int_{|\det g| = q^{-n}} \int W \begin{pmatrix} gh^{-1} & 0 \\ 0 & 1 \end{pmatrix} W(g; X_1, \dots, X_{r-1})\, \phi(h) |\det h|^{1/2}\, dh \\
&\qquad \times |\det g|^{-1/2}\, dg \\
&= \int_{|\det g| = q^{-n+t}} W \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \int W(gh; X_1, \dots, X_{r-1})\, \phi(h)\, dh\, |\det g|^{-1/2}\, dg
\end{aligned}$$

$$= \int_{|\det g|=q^{-n+t}} W\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} W(g; X_1, \ldots, X_{r-1}) |\det g|^{-1/2}\, dg\; Q(X_1, \ldots, X_{r-1})$$

$$= \Psi_{n-t}(W; X_1, \ldots, X_{r-1})\; Q(X_1, \ldots, X_{r-1}).$$

Hence,

$$\begin{aligned}
&\Xi_n(W_1; X_1, \ldots, X_{r-1}) \\
&\qquad = \sum_{m=0}^{R} \Psi_{n-m}(W_1; X_1, \ldots, X_{r-1})\; P_m(X_1, \ldots, X_{r-1}) \\
&\qquad = \sum_{m=0}^{R} \Psi_{n-m-t}(W; X_1, \ldots, X_{r-1})\; P_m(X_1, \ldots, X_{r-1})\; Q(X_1, \ldots, X_{r-1}) \\
&\qquad = \Xi_{n-t}(W; X_1, \ldots, X_{r-1})\; Q(X_1, \ldots, X_{r-1}).
\end{aligned}$$

Since Q is homogeneous of degree t our assertion follows. □

3. Proof of the theorem

Proof. Given an $(r-1)$-tuple of nonzero complex numbers $(x_1, x_2, \ldots, x_{r-1})$, Lemma 4 shows that we can find W_j and integers n_j such that, for all s,

$$\sum_{1 \le j \le k} (q^{-s})^{n_j}\, \Xi(W_j, q^{-s}x_1, q^{-s}x_2, \ldots, q^{-s}x_{r-1}) = 1.$$

In particular,

$$\sum_{1 \le j \le k} \Xi(W_j, x_1, x_2, \ldots, x_{r-1}) = 1.$$

Thus, the element

$$\sum_{1 \le j \le k} \Xi(W_j; X_1, X_2, \ldots, X_{r-1})$$

of I_π does not vanish at $(x_1, x_2, \ldots, x_{r-1})$. By the theorem of zeros of Hilbert we have then $I_\pi = S_{r-1}$. In particular, there is W such that

$$\Xi(W; X_1, X_2, \ldots, X_{r-1}) = 1\,.$$

This implies the theorem. □

Remark 1. The proof in [Jacquet et al. 1981a] is correct if $L(s, \pi)$ is identically 1. In general, the proof there only shows that the polynomials in I_π cannot all vanish on a coordinate hyperplane $X_i = x$.

Remark 2. Consider an induced representation π of the form

$$\pi = I(\sigma_1 \otimes \alpha^{s_1}, \sigma_2 \otimes \alpha^{s_2}, \dots, \sigma_k \otimes \alpha^{s_k}),$$

where the representations $\sigma_1, \sigma_2, \dots, \sigma_k$ are tempered and $s_1, s_2, \dots, s_k$ are real numbers such that

$$s_1 > s_2 > \cdots > s_k \,.$$

The representation π may fail to be irreducible. But, in any case, it has a Whittaker model [Jacquet and Shalika 1983], and Theorem 1 is valid for the Whittaker model of π.

Remark 3. The proof of Matringe uses the theory of derivatives of a representation. The present proof appears simple only because we use Lemma 3, the proof of which is quite elaborate (and can be obtained from the theory of derivatives as in [Cogdell and Piatetski-Shapiro 2011]).

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HERVÉ JACQUET
DEPARTMENT OF MATHEMATICS
COLUMBIA UNIVERSITY
MC 4408
NEW YORK, NY 10027
UNITED STATES
hj@math.columbia.edu

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MODULAR L-VALUES OF CUBIC LEVEL

ANDREW KNIGHTLY AND CHARLES LI

In memory of Jonathan Rogawski

Using a simple relative trace formula, we compute averages of twisted modular L-values for newforms of cubic level. In the case of Maass forms, we obtain an exact formula. For holomorphic forms of weight $k > 2$, we obtain an asymptotic formula, which agrees with the estimate predicted by the Lindelöf hypothesis in the weight and level aspects.

1. Introduction

A *simple* trace formula is one in which a local discrete series matrix coefficient is used, thereby annihilating the contribution of the continuous spectrum (see Lecture V of [Gelbart 1996] for a general overview). By choosing the matrix coefficient appropriately, one can also project onto a particular local new vector. For example, using the matrix coefficient attached to a lowest weight vector for the weight k discrete series of $\mathrm{GL}_2(\mathbb{R})$, one isolates the space of holomorphic cusp forms of weight k from the rest of the automorphic spectrum. In essence, this was the method used by Selberg [1956, §4] in his formula for the trace of a Hecke operator.

In this paper we give a nonarchimedean illustration of this technique, using matrix coefficients attached to certain supercuspidal representations of $\mathrm{GL}_2(\mathbb{Q}_p)$. We work with a relative trace formula to compute averages of the form

$$\sum_{u\in\mathcal{F}} \frac{\lambda_n(u)\overline{a_r(u)}\Lambda(s,u,\chi)}{\|u\|^2}\, B_r(u),$$

where u ranges over the set of newforms of weight k and level N^3 for N squarefree and $k > 2$ or $k = 0$, $\lambda_n(u)$ is the associated eigenvalue of the Hecke operator T_n, $a_r(u)$ is the r-th Fourier coefficient, $\Lambda(s,u,\chi)$ is the completed L-function, twisted by a fixed primitive character χ of conductor D prime to N, and $B_r(u)$ is a function

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of the spectral parameter of u with sufficient decay, which we take to be 1 in the case of holomorphic forms (that is, when $k > 2$).

We have two main results, one for Maass newforms and one for holomorphic newforms. Each is an explicit version of the relative trace formula introduced by Jacquet [1987]. In broad terms, we start with a kernel function attached to the Hecke operator T_n, and integrate it (against a character) over the group $N \times M$, where N is unipotent and M is diagonal. The unipotent integral gives the Fourier coefficient $\overline{a_r(u)}$, and the diagonal integral gives the L-function. The geometric side reduces to the calculation of numerous local orbital integrals.

The result for Maass forms is given in Theorem 5.4 below. A special case of it is the following exact expression for a weighted average of Maass newform L-values:

Theorem 1.1. *Let χ be a primitive Dirichlet character with modulus D. Let $h(iz)$ be any even Paley–Wiener function, and let $h_1(s)$ be the $e^{-2\pi i x}$-twisted spherical transform of the inverse Selberg transform of h; cf.* (5-5). *Then there exists a constant $C \geq 1$ depending only on h, such that for all squarefree integers $N > C$ prime to D and all complex numbers s,*

$$\sum_{u_j \in \mathcal{F}_+^{\text{new}}(N^3)} \frac{\Lambda(s, u_j, \chi)}{\psi(N^3)\|u_j\|^2} h(t_j) K_{it_j}(2\pi) = 2h_1(s) \prod_{p \mid N} \Big(1 - \frac{1}{p}\Big). \tag{1-1}$$

Here, $\mathcal{F}_+^{\text{new}}(N^3)$ denotes the set of even Maass newforms on $\Gamma_0(N^3)$ of weight 0 and trivial central character, normalized with first Fourier coefficient $a_1(u) = 1$, t_j is the spectral parameter of u_j, $K_\nu(x)$ is the Bessel function, and $\psi(N^3)$ denotes the index $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N^3)]$.

Remarks. (1) It is interesting to note that the right-hand side of (1-1) (and hence also the left-hand side for N sufficiently large) is independent of χ.

(2) Given any $s \in \mathbb{C}$, we can choose h so that $h_1(s)$ is nonzero. Therefore an immediate consequence is the existence of a Maass newform of level N^3 with nonvanishing twisted L-value at s.

(3) We normalize the Petersson norm on page 530 so that it is independent of the choice of level and coincides with the adelic L^2-norm. Many people write $\|u\|^2$ where we have written $\psi(N^3)\|u\|^2$.

The analogous result for holomorphic cusp forms is stated in Theorem 4.1. In that case, we no longer have an exact formula because the archimedean discrete series matrix coefficient is not compactly supported. But the resulting asymptotic formula still gives nonvanishing, as well as a bound for the sum of the central L-values that is as strong as that predicted by the Lindelöf hypothesis in the weight and level aspects; see Corollary 4.3. In Corollary 4.4, we compare the contribution of newforms and oldforms in the analogous sum for the full space of cusp forms of level N^3. When N is prime, the contribution of oldforms becomes negligible as $N \to \infty$, but in the other extreme, if N is the product of the first m primes, the contribution of newforms becomes negligible as $m \to \infty$.

In both of our main results, we project onto the newforms of cubic level by using the simple supercuspidal representations defined by Gross and Reeder [2010]. Matrix coefficients for these representations have previously been used in the trace formula by Gross [2011], where, for a simple group over a totally real number field, he computed the multiplicities of cuspidal representations with certain prescribed local behavior in terms of values of modified Artin L-functions at negative integers. The local test vector used by Gross has a very simple matrix coefficient and is ideally suited for counting representations. However, it is not a new vector so it cannot be used for our purpose here.

In [KL 2012], we defined simple supercuspidal representations for the group $\mathrm{GL}_n(\mathbb{Q}_p)$, showing that they have conductor p^{n+1} and exhibiting the new vector. We then gave an explicit formula for the matrix coefficient attached to the new vector in the case where $n = 2$. Lastly, we showed that every irreducible admissible representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ with conductor p^3 is a simple supercuspidal representation, assuming that its central character is unramified or tamely ramified. In the present paper, at each place $p \mid N$ we sum the new vector matrix coefficients attached to the $2(p-1)$ distinct simple supercuspidal representations to obtain a test function which projects onto the newforms of level N^3 and annihilates the continuous spectrum.

We restrict to the field $\mathbb{Q}$ throughout for simplicity, but since all of the computations are local, there would be no serious obstruction to working over an arbitrary totally real number field.

2. Preliminaries

2A. *Orthogonality of matrix coefficients.* The proposition below, which has been attributed to Langlands, will be a key ingredient in what follows.

Proposition 2.1. *Let G be a unimodular locally compact group with center Z. Let (π, V) be an irreducible unitary square integrable representation of G with formal degree d_π. Let $w \in V$ be a unit vector, and suppose that the function $f(g) = d_\pi \overline{\langle \pi(g)w, w\rangle}$ is absolutely integrable over $\overline{G} = G/Z$. Then for any irreducible unitary representation (ρ, W) of G with the same central character as π (but not necessarily square integrable), the operator $\rho(f)$ is identically zero on W unless $\rho \cong \pi$. Furthermore, $\pi(f)$ is the orthogonal projection operator from V onto $\mathbb{C}w$.*

Remark. The formal degree d_π depends on a choice of Haar measure on $\overline{G}$, as does the operator $\pi(f)$. We must assume that these measures are the same.

Proof. See Corollary 10.29 of [KL 2006]. □

2B. ***Notation and measure.*** Given a prime number p and an integer x, we write $x_p = \mathrm{ord}_p(x)$, so that $x = \prod_p p^{x_p}$.

Let $\mathbb{A}$, $\mathbb{A}_{\mathrm{fin}}$ be the adeles and finite adeles of $\mathbb{Q}$, and henceforth let $G = \mathrm{GL}(2)$. Write $\overline{G} = G/Z$, where Z is the center of G. We let $Z_p = Z(\mathbb{Q}_p)$ and $Z_\infty = Z(\mathbb{R})$ be the respective centers of $G(\mathbb{Q}_p)$ and $G(\mathbb{R})$. We also set $K_\infty = \mathrm{SO}(2)$ and $K_p = \mathrm{GL}_2(\mathbb{Z}_p)$.

We take Lebesgue measure dx on $\mathbb{R}$, and we use the measure $d^*y = dy/|y|$ on $\mathbb{R}^*$. On $\mathbb{Q}_p$ and $\mathbb{Q}_p^*$ we normalize the Haar measures so that $\mathrm{meas}(\mathbb{Z}_p) = 1$ and $\mathrm{meas}(\mathbb{Z}_p^*) = 1$, respectively. With these choices, the product measure on $\mathbb{A}$ has the property that $\mathrm{meas}(\mathbb{Q}\backslash\mathbb{A}) = 1$. In $\mathbb{A}_{\mathrm{fin}}^*$ we have $\mathrm{meas}(\widehat{\mathbb{Z}}^*) = 1$. We normalize Haar measure on $G(\mathbb{Q}_p)$ by taking $\mathrm{meas}(K_p) = 1$. Likewise in $\overline{G}(\mathbb{Q}_p)$ we take $\mathrm{meas}(\overline{K_p}) = 1$. On $\overline{G}(\mathbb{A}_{\mathrm{fin}})$, we give $\overline{G}(\widehat{\mathbb{Z}})$ the measure 1. We normalize Haar measure on $\overline{G}(\mathbb{A})$ so that $\mathrm{meas}(\overline{G}(\mathbb{Q})\backslash\overline{G}(\mathbb{A})) = \pi/3$. See [KL 2006] for further details about this normalization.

We let $\theta : \mathbb{A} \to \mathbb{C}^*$ be the nontrivial character given locally by

$$\theta_p(x) = \begin{cases} e^{-2\pi i x} & \text{if } p = \infty \ (x \in \mathbb{R}), \\ e^{2\pi i r_p(x)} & \text{if } p < \infty \ (x \in \mathbb{Q}_p), \end{cases} \tag{2-1}$$

where $r_p(x) \in \mathbb{Q}$ is the p-principal part of x, a number with p-power denominator characterized up to $\mathbb{Z}$ by $x \in r_p(x) + \mathbb{Z}_p$. The kernel of θ_p is $\mathbb{Z}_p$, and θ is trivial on $\mathbb{Q} \subset \mathbb{A}$.

2C. ***Cusp forms.*** Let k be a nonnegative integer. Eventually we will assume further that $k \neq 1, 2$. Let N be a positive integer, and let ω' be a Dirichlet character modulo N satisfying $\omega'(-1) = (-1)^k$. Define the Hecke congruence subgroups

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \;\middle|\; c \in N\mathbb{Z} \right\}, \quad \Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \;\middle|\; d \in 1 + N\mathbb{Z} \right\},$$

and let

$$\psi(N) = [\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)] = N \prod_{p \mid N} \Big(1 + \frac{1}{p}\Big). \tag{2-2}$$

Consider the space of measurable complex-valued functions u on the complex upper half-plane $\mathbb{H}$ that have the following properties:

(1) For all $z \in \mathbb{H}$ and all
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N),$$

$$u\Big(\frac{az+b}{cz+d}\Big) = \overline{\omega'(d)}(cz+d)^k u(z). \tag{2-3}$$

(2) u has finite Petersson norm:

$$\|u\|^2 = \frac{1}{\psi(N)} \int_{\Gamma_0(N)\backslash\mathbb{H}} |u(x+iy)|^2 y^k \frac{dx\,dy}{y^2} < \infty.$$

(3) u is holomorphic if $k > 0$.

(4) u is cuspidal: at each cusp of $\Gamma_1(N)$ it has a constant term which vanishes almost everywhere (see for example [KL 2013, §4.1] for a detailed definition).

We denote this space by $S_k(N, \omega')$ if $k > 0$, and by $L^2_0(N, \omega')$ if $k = 0$. The latter space is infinite-dimensional if nonzero, but it has a basis consisting of Maass forms, that is, those elements which are eigenfunctions of the Laplacian $\Delta = -y^2((\partial^2/\partial x^2) + (\partial^2/\partial y^2))$. We write the Laplace eigenvalue as

$$\Delta u = (\tfrac{1}{4} + t^2)u,$$

and refer to t as the *spectral parameter* of u. We know that $t \in \mathbb{R}^* \cup i(-\frac{1}{2}, \frac{1}{2})$, with the number of u with *exceptional* (nonreal) parameter being finite.

If u is continuous, condition (1) implies u has a Fourier expansion of the form

$$u(x+iy) = \sum_{n\neq 0} a_n(u, y)\, e^{2\pi i n x}.$$

The coefficient $a_n(u, y)$ has the well-known form

$$a_n(u, y) = \begin{cases} a_n(u)e^{-2\pi n y} & \text{if } n, k > 0, \\ 0 & \text{if } k > 0,\ n < 0, \\ a_n(u) y^{1/2} K_{it}(2\pi|n|y) & \text{if } k = 0, \end{cases} \tag{2-4}$$

where K_{it} is the Bessel function and t is the spectral parameter of u.

The weight k Hecke operator T_n is defined by

$$T_n u(z) = n^{\alpha(k)} \sum_{\substack{ad=n \\ a>0}} \sum_{b=0}^{d-1} \overline{\omega'(a)} d^{-k} u\Big(\frac{az+r}{d}\Big),$$

where $\alpha(k) = k-1$ if $k > 0$ and $\alpha(k) = -\frac{1}{2}$ if $k = 0$. If u is a Hecke eigenform, we denote the eigenvalues by $T_n u = \lambda_n(u)u$. We say that u is a *newform* if its Hecke eigenvalue packet $\{\lambda_p(u)\}_{p\nmid N}$ has an eigenspace that is exactly one-dimensional. In this case, $a_1(u) \neq 0$, and we will normalize so that $a_1(u) = 1$. Under this normalization,

$$a_n(u) = \lambda_n(u) \tag{2-5}$$

for all n. We let

$$\mathcal{F}_k^{\mathrm{new}}(N, \omega') = \{\text{newforms } u, \text{ with } a_1(u) = 1\}.$$

We also define $T_{-1}u(x+iy) = u(-x+iy)$. A Maass cusp form is *even* or *odd* if $T_{-1}u = u$ or $T_{-1}u = -u$, respectively. If u is even, then in (2-4) we have $a_{-n}(u) = a_n(u)$, while if u is odd, $a_n(u) = -a_{-n}(u)$. It is a basic fact that $L^2_0(N,\omega')$ has an orthogonal basis consisting of Maass eigenforms which are also eigenfunctions of T_{-1}. We let

$$\mathcal{F}^{\text{new}}_+(N,\omega') = \{u \in \mathcal{F}^{\text{new}}_0(N,\omega') \mid u \text{ is even}\}.$$

We define the L-function of u by

$$L(s,u) = \sum_{n=1}^{\infty} a_n(u) n^{-s}.$$

This converges absolutely when $\mathrm{Re}(s)$ is sufficiently large. We define the completed L-function by

$$\Lambda(s,u) = \begin{cases} (2\pi)^{-s}\Gamma(s)L(s,u) & k>0, \\ \pi^{-s}\Gamma\left(\frac{s+\varepsilon+it}{2}\right)\Gamma\left(\frac{s+\varepsilon-it}{2}\right)L(s,u) & k=0, \end{cases} \tag{2-6}$$

where $\varepsilon = 0$ or 1 according to whether u is even or odd. It has an analytic continuation, which satisfies a functional equation relating values at s and $1-s$ when $k=0$, and at s and $k-s$ when $k>0$.

2D. ***Adelic cusp forms.*** Let ω be the Hecke character attached to ω' by

$$\omega : \mathbb{A}^* = \mathbb{Q}^*(\mathbb{R}^+ \times \widehat{\mathbb{Z}}^*) \to \widehat{\mathbb{Z}}^* \to (\mathbb{Z}/N\mathbb{Z})^* \to \mathbb{C}^*, \tag{2-7}$$

where the first two arrows are the canonical projections, and the last arrow is ω'. For $q>0$, let $L^q(\omega) = L^q(\overline{G}(\mathbb{Q})\backslash\overline{G}(\mathbb{A}),\omega)$ denote the space of measurable $G(\mathbb{Q})$-invariant functions $\phi : G(\mathbb{A}) \to \mathbb{C}$ that transform under the center by ω, and satisfy $\int_{\overline{G}(\mathbb{Q})\backslash\overline{G}(\mathbb{A})} |\phi(g)|^q\, dg < \infty$. When $q=2$, we let $L^2_0(\omega) \subset L^2(\omega)$ denote the subspace of cuspidal functions.

Letting

$$K_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G(\widehat{\mathbb{Z}}) \;\middle|\; c, d-1 \in N\widehat{\mathbb{Z}} \right\},$$

we embed $S_k(N,\omega')$ and $L^2_0(N,\omega')$ (taking $k=0$ in the latter case) isometrically into $L^2_0(\omega)$ by defining

$$\phi_u(\gamma(g_\infty \times g_{\text{fin}})) = j(g_\infty, i)^{-k} u(g_\infty(i)) \tag{2-8}$$

for $\gamma(g_\infty \times g_{\text{fin}}) \in G(\mathbb{Q})(G(\mathbb{R})^+ \times K_1(N)) = G(\mathbb{A})$, and

$$j\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z\right) = (ad-bc)^{-\frac{1}{2}}(cz+d).$$

In the $k=0$ case, the map $u \mapsto \phi_u$ is a surjective linear isometry from $L_0^2(N,\omega')$ to $L_0^2(\omega)^{K_\infty \times K_1(N)}$ (the $K_\infty \times K_1(N)$-invariant vectors); see [KL 2013, Proposition 4.5].

Lemma 2.2. *Let u be a holomorphic Hecke eigenform* ($k>0$) *or a Maass eigenform with spectral parameter t* ($k=0$). *Then for $r \in \mathbb{Q}$,*

$$\int_{\mathbb{Q}\backslash\mathbb{A}} \phi_u\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)\theta(rx)\,dx = \begin{cases} a_r(u)K_{it}(2\pi|r|) & \text{if } r \in \mathbb{Z}, k=0, \\ e^{-2\pi r}a_r(u) & \text{if } r \in \mathbb{Z}^+, k>0, \\ 0 & \text{otherwise}, \end{cases}$$

where θ is the character defined in (2-1). *For all $s \in \mathbb{C}$,*

$$\int_{\mathbb{Q}^*\backslash\mathbb{A}^*} \phi_u\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix}\right)|y|^{s-\frac{k'}{2}}\,d^*y = \begin{cases} \frac{1}{2}\Lambda(s,u) & \text{if } k=0, u \text{ is even}, \\ 0 & \text{if } k=0, u \text{ is odd}, \\ \Lambda(s,u) & \text{if } k>0, \end{cases}$$

where $\Lambda(s,u)$ is the completed L-function defined in (2-6) *and*

$$k' = \begin{cases} k & k>2, \\ 1 & k=0. \end{cases}$$

Each of the above integrals is absolutely convergent.

Proof. For a proof of the first statement, see [KL 2006, Corollary 12.4; 2013, Lemma 7.1]. For the second, suppose $k=0$. Using the fundamental domain $\mathbb{R}^+ \times \widehat{\mathbb{Z}}^*$ for $\mathbb{Q}^*\backslash\mathbb{A}^*$, we have

$$\int_{\mathbb{Q}^*\backslash\mathbb{A}^*} \phi_u\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix}\right)|y|^{s-\frac{1}{2}}\,d^*y = \int_0^\infty u(iy)y^{s-\frac{1}{2}}\,\frac{dy}{y}.$$

The result then follows by a well-known classical computation using the Fourier expansion; see [Goldfeld 2006, p. 86]. The proof when $k>0$ is similar; see, e.g., [KL 2010, Lemma 3.1]. □

2E. *Newforms.* Here we will define a space of adelic newforms, and realize the orthogonal projection onto it as an integral operator.

We wish to study newforms with certain local behavior. Let N be an integer multiple of the conductor of ω with the property that $N_p \geq 2$ for all $p \mid N$. For each $p \mid N$, let σ_p be a fixed supercuspidal representation of $G(\mathbb{Q}_p)$ with central character ω_p and conductor p^{N_p}. Let $\widehat{\sigma}$ denote the tuple $\{\sigma_p\}_{p \mid N}$.

Under the action of $G(\mathbb{A})$ on $L_0^2(\omega)$ by right translation, the space decomposes as a direct sum of irreducible cuspidal representations π. Given a nonnegative integer $k \neq 1, 2$ (that is, $k \in \{0,3,4,5,\dots\}$), we define the subspace

$$H_k(\widehat{\sigma},\omega) = \bigoplus \pi \subset L_0^2(\omega), \tag{2-9}$$

where π ranges through the irreducible cuspidal representations for which:

(1) $\pi_p = \sigma_p$ for all $p \mid N$.

(2) π_p is unramified for all finite $p \nmid N$.

(3) π_∞ is a spherical principal series representation of $G(\mathbb{R})$ with trivial central character if $k = 0$.

(4) π_∞ is the weight k discrete series representation π_k of $G(\mathbb{R})$ with central character

$$\begin{pmatrix} z & \\ & z \end{pmatrix} \mapsto \operatorname{sgn}(z)^k \quad \text{if } k > 2.$$

For each such $\pi = \bigotimes' \pi_p$, define a vector (the "newform") $w_\pi = \bigotimes w_{\pi_p}$ in the space of π by taking

$$w_{\pi_p} = \begin{cases} \text{unit new vector [Casselman 1973]} & \text{if } p \mid N, \\ \text{unit unramified vector} & \text{if } p \nmid N\infty, \\ \text{unit spherical vector} & \text{if } p = \infty,\ k = 0, \\ \text{unit lowest weight vector} & \text{if } p = \infty,\ k > 2, \end{cases}$$

where, in almost every unramified case, the unit vector is the one predetermined by the restricted tensor product. In each case, the vector w_{π_p} is unique up to unitary scaling. Let

$$A_k(\widehat{\sigma}, \omega) = \bigoplus_\pi \mathbb{C} w_\pi \subset H_k(\widehat{\sigma}, \omega). \tag{2-10}$$

This corresponds to a classical space of newforms of level N on the upper half-plane. Letting $\phi_\pi \in L_0^2(\omega)$ denote the function defined by ω_π, the associated cusp form on $\mathbb{H}$ is given by

$$u(x+iy) = y^{-\frac{k}{2}} \phi_\pi\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}_\infty \times 1_{\text{fin}} \right), \qquad y > 0. \tag{2-11}$$

This is the inverse of the association (2-8), that is, $\phi_u = \phi_\pi$.

For $p \mid N$, define a function $f_p : G(\mathbb{Q}_p) \to \mathbb{C}$ by

$$f_p(g) = d_p \overline{\langle \sigma_p(g) w_{\sigma_p}, w_{\sigma_p} \rangle}, \quad p \mid N, \tag{2-12}$$

where d_p is the formal degree of the supercuspidal representation σ_p relative to our choice of Haar measure on $\overline{G}(\mathbb{Q}_p)$, and the inner product is $G(\mathbb{Q}_p)$-invariant. Likewise, if $p = \infty$ and $k > 2$ we take

$$f_\infty(g) = d_k \overline{\langle \pi_\infty(g) w_{\pi_\infty}, w_{\pi_\infty} \rangle}, \quad k > 2, \tag{2-13}$$

where d_k is the formal degree of the discrete series representation $\pi_\infty = \pi_k$. The latter function is supported on the subgroup

$$G(\mathbb{R})^+ = \{g \in G(\mathbb{R}) \mid \det(g) > 0\}.$$

(We rule out $k = 2$ because the function (2-13) is integrable precisely when $k > 2$, and integrability is required by Proposition 2.1.)

For $p \nmid N\infty$, we assume that f_p is a bi-K_p-invariant function on $G(\mathbb{Q}_p)$ with compact support modulo the center, and that for all but finitely many such p, $f_p = \phi_p$ is the function supported on $Z_p K_p$ given by

$$\phi_p(z\kappa) = \overline{\omega_p(z)}, \quad z \in Z_p,\ \kappa \in K_p. \tag{2-14}$$

Likewise if $p = \infty$ and $k = 0$, we take

$$f_\infty \in C_c^\infty(G(\mathbb{R})^+//K_\infty), \quad k = 0. \tag{2-15}$$

The latter is the space of smooth functions on $G(\mathbb{R})^+$ which are biinvariant under $Z(\mathbb{R})K_\infty$ and have compact support modulo $Z(\mathbb{R})$. Such a function enables us to project onto the K_∞-invariant space of $L^2(\omega)$, which contains the Maass forms of weight $k = 0$.

Proposition 2.3. *With local functions as above, let $f = \prod f_p$ be the associated function on $G(\mathbb{A})$. Let $R(f)$ be the operator on $L^2(\omega)$ defined by*

$$R(f)\phi(x) = \int_{\overline{G}(\mathbb{A})} f(g)\phi(xg)\,dg.$$

Then $R(f)$ annihilates $L_0^2(\omega)^\perp$. In fact, it factors through the orthogonal projection of $L^2(\omega)$ onto $A_k(\widehat{\sigma}, \omega)$, and acts diagonally on the latter space, the vectors w_π being eigenvectors.

Proof. For a proof of the first statement, see [Rogawski 1983, Proposition 1.1]. Now suppose $v \in L_0^2(\omega)$. Since the latter space is a direct sum of cuspidal representations, we may assume that $v \in V_\pi$ for some $\pi = \bigotimes' \pi_p$. Likewise, we may assume that $v = \bigotimes' v_p$ is a pure tensor. For the purposes of this proof, let G' denote the restricted direct product $G' = \prod'_{p \nmid N} G(\mathbb{Q}_p)$. Decompose π as

$$\pi = \pi_\infty \otimes \pi' \otimes \bigotimes_{p \mid N} \pi_p,$$

where π' is a representation of G', and write

$$v = v_\infty \otimes v' \otimes \bigotimes_{p \mid N} v_p$$

accordingly. Then (by [KL 2006, Proposition 13.17], for instance)

$$R(f)v = \pi_\infty(f_\infty)v_\infty \otimes \pi'(f')v' \otimes \bigotimes_{p \mid N} \pi_p(f_p)v_p.$$

If $p \mid N$, or $p = \infty$ and $k > 2$, then by Proposition 2.1, the above vanishes unless $\pi_p = \sigma_p$ or, respectively, $\pi_p = \pi_k$, and in these cases $\pi_p(f_p)$ is the orthogonal projection onto $\mathbb{C}w_{\pi_p}$. Because f' is biinvariant under $\prod_{p\nmid N} K_p$, $\pi'(f')$ has its image in the space

$$\mathbb{C}w' = \bigotimes_{p\nmid N}{}' \mathbb{C}w_{\pi_p} \subset V_{\pi'},$$

and it annihilates the orthogonal complement of this subspace (see for instance [KL 2013, Lemma 3.10]). The analogous statement holds for $\pi_\infty(f_\infty)$ if $k = 0$ for the same reasons. It follows that $R(f)$ annihilates $A_k(\widehat{\sigma}, \omega)^\perp$, and acts by scalars on the vectors $w_\pi \in A_k(\widehat{\sigma}, \omega)$. □

2F. *Twisting.* Let D be a positive integer with $\gcd(D, N) = 1$, and let χ be a primitive Dirichlet character modulo D. Given a cusp form

$$u(z) = \sum_{n\neq 0} a_n(u, y) e^{2\pi i n x}$$

in $S_k(N, \omega')$ or $L^2_0(N, \omega')$, its twist by χ is the form

$$u_\chi(z) = \sum_{n\neq 0} \chi(n) a_n(u, y) e^{2\pi i n x},$$

which belongs to $S_k(D^2N, \overline{\chi}^2\omega')$ or $L^2_0(D^2N, \overline{\chi}^2\omega')$. If u is a Maass form with spectral parameter t, then so is u_χ. In this section we will define a function f^χ on $G(\mathbb{A}_{\text{fin}})$ for which $R(f^\chi)$ encodes the twisting operation adelically. See §3 of [Jackson and Knightly 2012] (henceforth referred to as [JK 2012]) for more detail. Beware that the nebentypus ψ in that paper plays the role of $\overline{\omega'}$ here, since we have a complex conjugate in (2-3) which is not present in [JK 2012].

We let $\chi^* : \mathbb{A}^* \to \mathbb{C}^*$ be the Hecke character attached to χ as in (2-7) (but using D in place of N).[1] We let χ_p be the local component of χ^*. It is a character of $\mathbb{Q}_p^*$, and when $p \mid D$ it can be viewed as a primitive character of the group $(\mathbb{Z}/p^{D_p}\mathbb{Z})^*$. The Gauss sum attached to χ is

$$\tau(\chi) = \sum_{m\in(\mathbb{Z}/D\mathbb{Z})^*} \chi(m) e^{\frac{2\pi i m}{D}}.$$

If we set

$$\tau(\chi)_p = \chi_p\Big(\frac{D}{p^{D_p}}\Big)\tau(\chi_p), \tag{2-16}$$

then $\tau(\chi) = \prod_{p\mid D} \tau(\chi)_p$; see [JK 2012], (3.10).

[1] Thus we use two sets of notation: ω' and χ are Dirichlet characters and ω, χ^* are the associated Hecke characters. This was done in order to conform to notation in papers we reference.

For each prime $p \mid D$, we define a local test function $f_p^\chi : G(\mathbb{Q}_p) \to \mathbb{C}$ by

$$f_p^\chi(x) = \begin{cases} \dfrac{\overline{\omega_p(z)}\ \overline{\chi_p(m)}}{\tau(\overline{\chi})_p} & \text{if } x = zg \text{ for } z \in Z_p \\ & \quad \text{and } g \in \begin{pmatrix} 1 & -m/D \\ 0 & 1 \end{pmatrix} K_p \text{ for } m \in (\mathbb{Z}_p/D\mathbb{Z}_p)^*, \\ 0 & \text{otherwise.} \end{cases}$$

For the primes $p \mid N$, we take f_p^χ to be the function supported on $Z_p K_1(N)_p$ given by

$$f_p^\chi(z\kappa) = \frac{\overline{\omega_p(z)}}{\operatorname{meas}(\overline{K_1(N)_p})} = \frac{\psi_p(N)}{\omega_p(z)}, \tag{2-17}$$

where $\psi_p(N) = [K_p : K_1(N)_p] = p^{N_p}(1+(1/p))$. Lastly, for $p \nmid DN$, we take f_p^χ to be the function defined in (2-14). Now let $f^\chi = \prod_{p<\infty} f_p^\chi$, and define the operator

$$R(f^\chi)\phi(x) = \int_{\overline{G}(\mathbb{A}_{\text{fin}})} f^\chi(g)\phi(xg)\,dg, \quad \phi \in L^2(\omega). \tag{2-18}$$

We call this the *twisting operator of level N* attached to χ.

Proposition 2.4. *For $y \in \mathbb{R}^+ \times \widehat{\mathbb{Z}}^* \cong \mathbb{Q}^*\backslash\mathbb{A}^*$ and u a holomorphic or Maass cusp form of level N and nebentypus ω',*

$$R(f^\chi)\phi_u\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix}\right) = \chi^*(y)\phi_{u_\chi}\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix}\right).$$

Proof. See Proposition 3.2 of [JK 2012]. That result is stated for holomorphic cusp forms, but the proof carries over verbatim to the case of Maass forms. ☐

Given two functions $f_1, f_2 \in L^1(\overline{G}(\mathbb{A}_{\text{fin}}), \overline{\omega})$, we define their convolution by

$$f_1 * f_2(x) = \int_{\overline{G}(\mathbb{A}_{\text{fin}})} f_1(g) f_2(g^{-1}x)\,dg = \int_{\overline{G}(\mathbb{A}_{\text{fin}})} f_1(xg^{-1}) f_2(g)\,dg.$$

Then $f_1 * f_2 \in L^1(\overline{G}(\mathbb{A}_{\text{fin}}), \overline{\omega})$. It is straightforward to show $R(f_1 * f_2) = R(f_1)R(f_2)$ as operators on $L^2(\omega)$.

Proposition 2.5. *Let $f = f_\infty \times f_{\text{fin}}$ be a function on $G(\mathbb{A})$ of the type defined in Section 2E, with the property that for all $p \mid D$, f_p is the function* (2-14). *Then*

$$R(f_\infty \times (f^\chi * f_{\text{fin}})) = R(f^\chi)R(f). \tag{2-19}$$

As a result, the above operator factors through the orthogonal projection of $L^2(\omega)$ onto $A_k(\widehat{\sigma}, \omega)$ by Proposition 2.3.

Proof. As mentioned above, $R(f^\chi * f_{\text{fin}}) = R(f^\chi)R(f_{\text{fin}})$. The local components of the convolution are given as follows:

$$(f^\chi * f_{\text{fin}})_p = f_p^\chi * f_p = \begin{cases} f_p^\chi & \text{if } p \mid D, \\ f_p & \text{if } p \nmid D. \end{cases} \tag{2-20}$$

Indeed, if $p \nmid DN$, then the assertion is immediate because f_p is bi-K_p-invariant and f_p^χ is the identity element of the local Hecke algebra of bi-K_p-invariant functions. Similarly, the case $p \mid D$ follows easily by the right K_p-invariance of f_p^χ and our assumption that f_p is given by (2-14). If $p \mid N$, then for $\kappa \in K_1(N)_p$, by (2-12) we have

$$f_p(\kappa^{-1}x) = d_p \overline{\langle \sigma_p(x) w_{\sigma_p}, \sigma_p(\kappa) w_{\sigma_p} \rangle} = f_p(x),$$

since w_{σ_p} is fixed by $K_1(N)_p$. Thus by (2-17),

$$f_p^\chi * f_p(x) = \int_{\overline{K_1(N)_p}} f_p^\chi(\kappa) f_p(\kappa^{-1}x)\, d\kappa = f_p(x) \int_{\overline{K_1(N)_p}} f_p^\chi(\kappa)\, d\kappa = f_p(x),$$

as claimed.

In view of (2-20), we may apply Proposition 2.3 to both sides of the proposed equality (2-19) to see that they each vanish on $L_0^2(\omega)^\perp$. Therefore it suffices to show that they agree on $L_0^2(\omega)$. Let (π, V_π) be a cuspidal representation in $L_0^2(\omega)$. Given $v = v_\infty \otimes v_{\text{fin}} \in V_\pi$, by (2-18) we have

$$\begin{aligned} R(f^\chi)v &= \int_{\overline{G}(\mathbb{A}_{\text{fin}})} f^\chi(g)\pi(1_\infty \times g)v\, dg \\ &= \int_{\overline{G}(\mathbb{A}_{\text{fin}})} v_\infty \otimes f^\chi(g)\pi_{\text{fin}}(g)v_{\text{fin}}\, dg = v_\infty \otimes \pi_{\text{fin}}(f^\chi)v_{\text{fin}}. \end{aligned}$$

For details justifying the movement of the tensor outside the integral, see [KL 2006, Lemma 13.16]. Applying the above identity with $R(f)v$ in place of v, the result follows:

$$\begin{aligned} R(f^\chi)R(f)v &= \pi_\infty(f_\infty)v_\infty \otimes \pi_{\text{fin}}(f^\chi)\pi_{\text{fin}}(f_{\text{fin}})v_{\text{fin}} \\ &= \pi_\infty(f_\infty)v_\infty \otimes \pi_{\text{fin}}(f^\chi * f_{\text{fin}})v_{\text{fin}} = R(f_\infty \times (f^\chi * f_{\text{fin}}))v. \end{aligned}$$

For a justification of the last step, see, e.g., [ibid., Proposition 13.17]. □

2G. *A particular choice of function.* The above discussion is rather general, and we will now define a very specific function f as in Section 2E, designed to project onto the newforms of cubic level and then act as a Hecke operator. For our main test function in the trace formula, we will then take $F = f_\infty \times (f^\chi * f_{\text{fin}})$, with f^χ a twisting operator defined as above.

Henceforth we take $N > 1$ to be a squarefree integer. We make the following assumption in all that follows:

(**) ω' is a Dirichlet character of modulus N^3 whose conductor divides N.

As before, we let ω be the associated Hecke character.

For each $p \mid N$, the conductor of ω_p divides p. Therefore by Proposition 7.2 of [KL 2012], there are exactly $2(p-1)$ irreducible admissible representations of $G(\mathbb{Q}_p)$ of conductor p^3 and central character ω_p, up to isomorphism. These are the simple supercuspidal representations, which are parametrized naturally by the pairs (t, ζ) with $t \in (\mathbb{Z}/p\mathbb{Z})^*$ and $\zeta \in \mathbb{C}$ satisfying $\zeta^2 = \omega_p(tp)$. The construction depends on the choice of a nontrivial character of $\mathbb{Z}/p\mathbb{Z}$, which we fix to be $x \mapsto \theta_p(x/p)$. Let $\sigma = \sigma_{t,\zeta}$ be the supercuspidal representation indexed by (t, ζ). It is defined precisely in [KL 2012], but all that we need here is the formula for its matrix coefficient

$$f^\sigma = d_\sigma \overline{\langle \sigma(g) w_\sigma, w_\sigma \rangle},$$

where the formal degree d_σ is taken relative to the Haar measure for which $\mathrm{meas}(\overline{K}_p)$ is 1, and w_σ is a unit new vector as before. By Theorem 7.1 of the same reference,

$$f^\sigma = f_1^\sigma + f_2^\sigma, \tag{2-21}$$

where f_1^σ and f_2^σ have disjoint support, and for $z \in Z_p$,

$$f_1^\sigma(zg) = \frac{p+1}{2\omega_p(z)} \sum_{w \in (\mathbb{Z}/p\mathbb{Z})^*} \theta_p\Big(\frac{-bw - tc(aw)^{-1}}{p}\Big) \tag{2-22}$$
$$\text{for } g = \begin{pmatrix} a & bp^{-1} \\ cp^2 & d \end{pmatrix} \in \begin{pmatrix} \mathbb{Z}_p^* & \frac{1}{p}\mathbb{Z}_p \\ p^2\mathbb{Z}_p & 1 + p\mathbb{Z}_p \end{pmatrix}$$

(a Kloosterman sum), and

$$f_2^\sigma(zg) = \frac{(p+1)\zeta}{2\omega_p(z)\omega_p(t)} \theta_p\Big(\frac{-b - tc(ad)^{-1}}{p}\Big) \tag{2-23}$$
$$\text{for } g = \begin{pmatrix} c & dp^{-2} \\ ap & b \end{pmatrix} \in \begin{pmatrix} \mathbb{Z}_p & \frac{1}{p^2}\mathbb{Z}_p^* \\ p\mathbb{Z}_p^* & \mathbb{Z}_p \end{pmatrix}.$$

The function f^σ vanishes outside the set

$$Z_p \cdot \begin{pmatrix} \mathbb{Z}_p^* & p^{-1}\mathbb{Z}_p \\ p^2\mathbb{Z}_p & 1 + p\mathbb{Z}_p \end{pmatrix} \bigcup Z_p \cdot \begin{pmatrix} \mathbb{Z}_p & p^{-2}\mathbb{Z}_p^* \\ p\mathbb{Z}_p^* & \mathbb{Z}_p \end{pmatrix}.$$

Fix an integer $n > 0$ with $\gcd(n, DN) = 1$. Let

$$M(n)_p = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}_p) \;\middle|\; ad - bc \in n\mathbb{Z}_p^* \right\}.$$

Define, for $p \mid n$, the local Hecke operator $f_p^n : G(\mathbb{Q}_p) \to \mathbb{C}$, supported on $Z_p M(n)_p$:

$$f_p^n(zg) = \omega_p(z), \quad z \in Z_p,\, g \in M(n)_p. \tag{2-24}$$

This plays the role of the classical Hecke operator $T_{p^{n_p}}$.

Finally, we let f_∞ be the matrix coefficient (2-13) if $k > 2$, or a spherical function as in (2-15) if $k = 0$.

With these choices, we define the global function $f : G(\mathbb{A}) \to \mathbb{C}$ by

$$f = f_\infty \times \prod_{p \mid N} \Bigl(\sum_{(t,\zeta)} f^{\sigma_{t,\zeta}} \Bigr) \prod_{p \mid n} f_p^n \prod_{p \nmid nN} \phi_p,$$

where, in the case $p \nmid nN$, ϕ_p is the unramified function supported on $Z_p K_p$ defined in (2-14). We remark that at the places $p \mid N$,

$$\sum_{(t,\zeta)} f^{\sigma_{t,\zeta}} = \sum_{(t,\zeta)} f_1^{\sigma_{t,\zeta}},$$

since from the definition (2-23) it follows easily that for each t, $f_2^{\sigma_{t,\zeta}} + f_2^{\sigma_{t,-\zeta}} = 0$. Nevertheless, we will compute the contribution of f_2^σ to the local orbital integrals in the trace formula that follows, since these do not vanish individually and may be of interest in other applications.

The function f defined above is a finite sum of functions of the type considered in Section 2E. Thus any new vector w_π belonging to the space

$$A_k(N^3, \omega) \overset{\text{def}}{=} \bigoplus_{\widehat{\sigma}} A_k(\widehat{\sigma}, \omega) \tag{2-25}$$

is an eigenvector of $R(f)$. Here, $\widehat{\sigma}$ runs through all tuples $\{\sigma_p\}_{p \mid N}$ of simple supercuspidal representations ($\sigma_p = \sigma_{t,\zeta}$) with central character ω_p, and $A_k(\widehat{\sigma}, \omega)$ is the space defined in (2-10).

Proposition 2.6. *Given a new vector $w_\pi \in A_k(N^3, \omega)$, let u be the associated newform. Then $R(f) w_\pi = \lambda_f(u) w_\pi$, for*

$$\lambda_f(u) = \begin{cases} n^{1-k/2} \lambda_n(u) & \text{if } k > 2, \\ n^{1/2} h(t) \lambda_n(u) & \text{if } k = 0, \end{cases} \tag{2-26}$$

where $\lambda_n(u)$ is the eigenvalue of the classical Hecke operator T_n acting on u, and in the $k = 0$ case, t is the spectral parameter of u and $h(t)$ is the Selberg transform of f_∞ (defined in (5-3) *below).*

Proof. We may write

$$R(f) w_\pi = \pi_\infty(f_\infty) w_\infty \otimes \pi'(f') w' \bigotimes_{p \mid N} \pi_p(f_p) w_p \bigotimes_{p \mid n} \pi_p(f_p) w_p,$$

where the $'$ indicates the contribution of the primes $p \nmid Nn\infty$ as in the proof of Proposition 2.3. If $p \mid N$, then

$$\pi_p(f_p)w_p = \sum_{\sigma} \pi_p(f^{\sigma})w_p = w_p$$

by Proposition 2.1, since exactly one of the representations σ is isomorphic to π_p. Likewise if $p \nmid nN$, then $\pi_p(f_p)w_p = \pi_p(\phi_p)w_p = w_p$ by the definition of ϕ_p. Hence

$$R(f)w_\pi = \pi_\infty(f_\infty)w_\infty \otimes w' \bigotimes_{p \mid N} w_p \bigotimes_{p \mid n} \pi_p(f_p)w_p.$$

Now

$$\pi_\infty(f_\infty)w_\infty = \begin{cases} w_\infty & \text{if } k > 2 \text{ (by Proposition 2.1)}, \\ h(t)w_\infty & \text{if } k = 0 \text{ (by [KL 2013, Proposition 3.9])}. \end{cases}$$

From the product over the places $p \mid n$, we get the scalar $\sqrt{n}\,\lambda_n(u)$ if $k = 0$ [ibid., Lemma 4.6], and $n^{1-k/2}\lambda_n(u)$ if $k > 2$ [KL 2006, Proposition 13.6]. $\square$

To incorporate twisting, we consider the function

$$F = f_\infty \times (f^\chi * f_{\text{fin}}) = f_\infty \times \prod_{p \mid N} \Bigl(\sum_{(t,\zeta)} f^{\sigma_{t,\zeta}}\Bigr) \prod_{p \mid D} f_p^\chi \prod_{p \mid n} f_p^n \prod_{p \nmid nDN} \phi_p, \tag{2-27}$$

where f^χ is the twisting operator of level N^3 as defined in Section 2F (where the level was denoted by N rather than N^3 used here). The second equality in (2-27) follows from (2-20). We will use the above as our test function in the trace formula. The kernel of the operator $R(F)$ is

$$K(x, y) = \sum_{\gamma \in \overline{G}(\mathbb{Q})} F(x^{-1}\gamma y). \tag{2-28}$$

Proposition 2.7. *Let $\mathcal{F}_k^{\text{new}}(N^3, \omega')$ be the set of newforms of weight k, level N^3 and central character ω'. Then the kernel function above has the spectral form*

$$K(x, y) = \sum_{u \in \mathcal{F}_k^{\text{new}}(N^3, \omega')} \frac{\lambda_f(u)R(f^\chi)\phi_u(x)\overline{\phi_u(y)}}{\|u\|^2} \tag{2-29}$$

for $\lambda_f(u)$ as in (2-26). The kernel function is continuous on $G(\mathbb{A}) \times G(\mathbb{A})$ and the above equality is valid for all points (x, y).

Proof. First, note that by Propositions 2.5 and 2.6, $R(F)\phi_u = \lambda_f(u)R(f^\chi)\phi_u$. Therefore the right-hand side of (2-29) is the same as

$$\sum_{\phi} \frac{R(F)\phi(x)\overline{\phi(y)}}{\|\phi\|^2}, \tag{2-30}$$

where ϕ ranges through an orthogonal basis for $A_k(N^3,\omega)$ (defined in (2-25)). In fact we may even allow ϕ to range over an orthogonal basis for the whole space $L^2(\omega)$ since $R(F)$ annihilates $A_k(N^3,\omega)^\perp$. The restriction of $R(F)$ to the cuspidal subspace is well-known to be Hilbert–Schmidt, and hence $R(F)$ is itself Hilbert–Schmidt since it vanishes on $L^2_0(\omega)^\perp$. (In fact, $R(F)$ has finite rank when $k>2$, but not when $k=0$.) Hence its kernel is equal to (2-30), proving that (2-29) holds almost everywhere.

The continuity of (2-28) is trivial when $k=0$, since in that case the defining sum is locally finite, F having compact support modulo the center and $\overline{G}(\mathbb{Q})$ being discrete and closed in $\overline{G}(\mathbb{A})$. When $k>2$, f_∞ is not compactly supported, so the continuity is not trivial, but a proof is given in [KL 2006, Proposition 18.4].

On the other hand, the continuity of the right-hand side of (2-29) is trivial when $k>2$ since in that case it is a finite sum of continuous functions. When $k=0$, a proof is given in [KL 2013, Corollary 6.12]. In all cases, it follows that (2-29) is valid everywhere. □

3. A relative trace formula

Our goal is to compute the relative trace formula given by the integral

$$\int_{\mathbb{Q}^*\backslash\mathbb{A}^*}\int_{\mathbb{Q}\backslash\mathbb{A}} K\left(\begin{pmatrix} y & \\ & 1\end{pmatrix},\begin{pmatrix} 1 & x\\ 0 & 1\end{pmatrix}\right)\overline{\theta(rx)}\,\overline{\chi^*(y)}|y|^{s-\frac{k'}{2}}\,dx\,d^*y, \tag{3-1}$$

where $k'=k$ if $k>2$ and $k'=1$ if $k=0$.

On the spectral side we evaluate the double integral using (2-29).

Proposition 3.1. *The integral*

$$\int_{\mathbb{Q}^*\backslash\mathbb{A}^*}\int_{\mathbb{Q}\backslash\mathbb{A}} \sum_{u\in\mathcal{F}_k^{\mathrm{new}}(N^3,\omega')}\left|\frac{\lambda_f(u)R(f^\chi)\phi_u\left(\left(\begin{smallmatrix} y & \\ & 1\end{smallmatrix}\right)\right)\overline{\phi_u\left(\left(\begin{smallmatrix} 1 & x\\ 0 & 1\end{smallmatrix}\right)\right)}}{\|u\|^2}\overline{\theta(rx)}\;\overline{\chi^*(y)}|y|^{s-k'/2}\right|\,dx\,d^*y$$

is convergent for all $s\in\mathbb{C}$ when $k>2$, and in some right half-plane when $k=0$. Hence for such s, (3-1) *is equal to*

$$\sum_{u\in\mathcal{F}_k^{\mathrm{new}}(N^3,\omega')}\frac{\lambda_f(u)\overline{a_r(u)}\Lambda(s,u,\chi)}{\|u\|^2}P_r(u), \tag{3-2}$$

where $\lambda_f(u)$ is given in (2-26) *and*

$$P_r(u)=\begin{cases}\frac{1}{2}K_{it}(2\pi|r|) & \text{if } k=0 \text{ and } u \text{ is even},\\ 0 & \text{if } k=0 \text{ and } u \text{ is odd},\\ e^{-2\pi r} & \text{if } k>0.\end{cases}$$

Proof. By Proposition 2.4, we have

$$R(f^{\chi})\phi_u\left(\begin{pmatrix} y & \\ & 1\end{pmatrix}\right) = \chi^*(y)\phi_{u_\chi}\left(\begin{pmatrix} y & \\ & 1\end{pmatrix}\right)$$

for all $y \in \mathbb{R}^+ \times \widehat{\mathbb{Z}}^* \cong \mathbb{Q}^*\backslash\mathbb{A}^*$. Therefore, whenever the double integral in the statement of the proposition is convergent, (3-1) is equal to

$$\sum_{u\in\mathcal{F}_k^{\text{new}}(N^3,\omega')} \frac{\lambda_f(u)}{\|u\|^2}\int_{\mathbb{Q}^*\backslash\mathbb{A}^*}\phi_{u_\chi}\left(\begin{pmatrix} y & \\ & 1\end{pmatrix}\right)|y|^{s-\frac{k'}{2}}\,d^*y\int_{\mathbb{Q}\backslash\mathbb{A}}\overline{\phi_u\left(\begin{pmatrix} 1 & x\\ 0 & 1\end{pmatrix}\right)\theta(rx)}\,dx,$$

which is equal to (3-2) by Lemma 2.2. Each of the above integrals is absolutely convergent, so the first assertion of the proposition is immediate when $k > 2$ since the sum over u is finite in that case. For the nonholomorphic case, we refer to Proposition 5.1 below. □

For the geometric side, we use the expression (2-28) and formally unfold (3-1) to obtain

$$\sum_{\delta}\int_{\mathbb{A}^*}\int_{\mathbb{A}} F\left(\begin{pmatrix} y^{-1} & \\ & 1\end{pmatrix}\delta\begin{pmatrix} 1 & x\\ 0 & 1\end{pmatrix}\right)\overline{\theta(rx)}\,\overline{\chi^*(y)}|y|^{s-\frac{k'}{2}}\,dx\,d^*y, \tag{3-3}$$

where δ ranges over $\overline{M}(\mathbb{Q})\backslash\overline{G}(\mathbb{Q})/N(\mathbb{Q})$. (See §7 of [JK 2012] for details.) By the Bruhat decomposition, the elements

$$1,\quad \begin{pmatrix} 0 & -1\\ 1 & 0\end{pmatrix},\quad \left\{\begin{pmatrix} a & -1\\ 1 & 0\end{pmatrix}\,\middle|\, a\in\mathbb{Q}^*\right\},$$

form a set of representatives for these double cosets.

Proposition 3.2. *The convergence*

$$\sum_{\delta}\int_{\mathbb{A}^*}\int_{\mathbb{A}}\left| F\left(\begin{pmatrix} y^{-1} & \\ & 1\end{pmatrix}\delta\begin{pmatrix} 1 & x\\ 0 & 1\end{pmatrix}\right)\overline{\theta(rx)}\,\overline{\chi^*(y)}|y|^{s-k'/2}\right|\,dx\,d^*y < \infty$$

is valid for all s when $k = 0$, and for $1 < \mathrm{Re}(s) < k - 1$ when $k > 2$. Hence the spectral side (3-2) *is equal to the geometric side* (3-3) *when $1 < \mathrm{Re}(s) < k-1$ if $k > 2$, and when $\mathrm{Re}(s)$ is sufficiently large if $k = 0$.*

Proof. We will show in the proof of Proposition 5.3 below that when $k = 0$, the integrand vanishes identically for all but finitely many δ. Since F also has compact support modulo the center in this case, the remaining integrals are absolutely convergent. When $k > 2$, the proof is essentially identical to that of Proposition 7.1 of [JK 2012], in view of the proof of Proposition 4.5 below. □

We let $I_\delta(s)$ denote the double integral attached to δ in (3-3). This orbital integral can be computed locally. The archimedean integral in the case $k = 0$ will be considered in Section 5 below. In the holomorphic case $k > 2$, the archimedean orbital integral was computed in [KL 2010] and [JK 2012]. The nonarchimedean local calculations at places $p \nmid N$ were carried out in [JK 2012]. Thus it remains here to compute the local integrals at places $p \mid N$. The results will be given in (3-5), (3-7) and (3-10) below.

3A. ***Orbital integrals for p | N.*** To simplify notation in this section, we will write k rather than k'. Suppose $p \mid N$, and let $\sigma = \sigma_{t,\zeta}$ be a supercuspidal representation of conductor p^3 and central character ω_p. Define

$$J_\delta(s, f^\sigma) = J_\delta(s, f_1^\sigma) + J_\delta(s, f_2^\sigma),$$

as in (2-21), where for $i = 1, 2$,

$$J_\delta(s, f_i^\sigma) = \int_{\mathbb{Q}_p^*} \int_{\mathbb{Q}_p} f_i^\sigma \left(\begin{pmatrix} y & \\ & 1 \end{pmatrix} \delta \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right) \overline{\theta_p(rx)} \chi_p(y) |y|_p^{\frac{k}{2}-s} \, dx \, d^*y.$$

Then replacing y by y^{-1} in (3-3), we see that

$$I_\delta(s)_p = \sum_\sigma J_\delta^\sigma(s).$$

Proposition 3.3. *Let* $\delta = 1$, *so that*

$$J_1(s, f^\sigma) = \int_{\mathbb{Q}_p^*} \int_{\mathbb{Q}_p} f^\sigma \left(\begin{pmatrix} y & xy \\ 0 & 1 \end{pmatrix} \right) \overline{\theta_p(rx)} \, dx \, \chi_p(y) |y|_p^{k/2-s} \, d^*y.$$

Then

$$J_1(s, f^\sigma) = J_1(s, f_1^\sigma) = \begin{cases} \frac{p(p+1)}{2} & \text{if } p \nmid r, \\ 0 & \text{if } p \mid r, \end{cases} \tag{3-4}$$

$$I_1(s)_p = \begin{cases} p^3 - p & \text{if } p \nmid r, \\ 0 & \text{otherwise.} \end{cases} \tag{3-5}$$

Proof. By (2-23), the matrix

$$\begin{pmatrix} y & xy \\ 0 & 1 \end{pmatrix}$$

never belongs to $\mathrm{Supp}(f_2^\sigma)$, so $J_1(s, f^\sigma) = J_1(s, f_1^\sigma)$. Note that

$$\begin{pmatrix} y & xy \\ 0 & 1 \end{pmatrix} \in \mathrm{Supp}(f_1^\sigma) = Z_p \begin{pmatrix} \mathbb{Z}_p^* & p^{-1}\mathbb{Z}_p \\ 0 & 1 \end{pmatrix}$$

if and only if $y \in \mathbb{Z}_p^*$ and $x \in p^{-1}\mathbb{Z}_p$. We substitute $u = y \in \mathbb{Z}_p^*$, and replace yx by $p^{-1}x$, so now $x \in \mathbb{Z}_p$. Then dx becomes $p\,dx$, and

$$\begin{aligned} J_1(s, f_1^\sigma) &= p \int_{\mathbb{Z}_p^*}\int_{\mathbb{Z}_p} f_1^\sigma\left(\begin{pmatrix} u & p^{-1}x \\ 0 & 1\end{pmatrix}\right)\overline{\theta_p\left(\frac{ru^{-1}x}{p}\right)}\,dx\,d^*u \\ &= \frac{p(p+1)}{2}\sum_{w\in(\mathbb{Z}/p\mathbb{Z})^*}\int_{\mathbb{Z}_p^*}\int_{\mathbb{Z}_p}\theta_p\left(\frac{-xw}{p}\right)\theta_p\left(\frac{-ru^{-1}x}{p}\right)dx\,d^*u. \end{aligned}$$

The integral over $\mathbb{Z}_p$ is equal to

$$\int_{\mathbb{Z}_p}\theta_p\left(\frac{(-w-ru^{-1})x}{p}\right)dx = \begin{cases} 1 & \text{if } u \equiv -rw^{-1} \bmod p, \\ 0 & \text{otherwise.}\end{cases}$$

In particular, this vanishes if $p \mid r$. Assuming $p \nmid r$,

$$J_1(s, f_1^\sigma) = \frac{p(p+1)}{2}\sum_{w\in(\mathbb{Z}/p\mathbb{Z})^*}\int_{-rw^{-1}+p\mathbb{Z}_p} d^*u = \frac{p(p+1)}{2}\operatorname{meas}(\mathbb{Z}_p^*),$$

which proves (3-4). The number of pairs (t, ζ) is $2(p-1)$. Since (3-4) is independent of σ (the parameters (t, ζ) not appearing in (3-4)),

$$I_1(s)_p = 2(p-1)J_1(s, f^\sigma),$$

and (3-5) follows. □

Proposition 3.4. *Let*

$$\delta = \begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix},$$

so that

$$J_\delta(s, f^\sigma) = \int_{\mathbb{Q}_p^*}\int_{\mathbb{Q}_p} f^\sigma\left(\begin{pmatrix} 0 & -y \\ 1 & x\end{pmatrix}\right)\overline{\theta_p(rx)}\,dx\,\chi_p(y)|y|_p^{\frac{k}{2}-s}\,d^*y.$$

Then

$$J_\delta(s, f^\sigma) = J_\delta(s, f_2^\sigma) = \begin{cases} \dfrac{(p^3)^{\frac{k}{2}-s}p(p+1)\omega_p(p)^2}{2\zeta\chi_p(p^3)} & \textit{if } r \equiv -1 \textit{ mod } p, \\ 0 & \textit{otherwise,}\end{cases} \tag{3-6}$$

$$I_\delta(s)_p = 0. \tag{3-7}$$

Proof. By (2-22), the matrix

$$\begin{pmatrix} 0 & -y \\ 1 & x\end{pmatrix}$$

never belongs to $\mathrm{Supp}(f_1^\sigma)$, so $J_\delta(s, f^\sigma) = J_\delta(s, f_2^\sigma)$. Note that

$$\begin{pmatrix} 0 & -y \\ 1 & x \end{pmatrix} \in \mathrm{Supp}(f_2^\sigma)$$

if and only if

$$\begin{pmatrix} 0 & -py \\ p & px \end{pmatrix} \in \begin{pmatrix} \mathbb{Z}_p & p^{-2}\mathbb{Z}_p^* \\ p\mathbb{Z}_p^* & \mathbb{Z}_p \end{pmatrix}.$$

In this case, we may write $y = p^{-3}u$ for $u \in \mathbb{Z}_p^*$, and $x' = px \in \mathbb{Z}_p$. Then $dx' = p^{-1}\,dx$, and dropping the $'$ from the notation, we have

$$\begin{aligned} &J_\delta(s, f_2^\sigma) \\ &= p\int_{\mathbb{Z}_p^*}\int_{\mathbb{Z}_p} f_2^\sigma\left(\begin{pmatrix} p^{-1} & \\ & p^{-1} \end{pmatrix}\begin{pmatrix} 0 & -p^{-2}u \\ p & x \end{pmatrix}\right)\theta_p\left(\frac{-rx}{p}\right) dx\, \chi_p(p^{-3})(p^3)^{\frac{k}{2}-s}\, d^*u \\ &\qquad = \frac{(p^3)^{k/2-s}\, p(p+1)\zeta\omega_p(p)}{2\chi_p(p^3)\omega_p(t)} \int_{\mathbb{Z}_p^*}\int_{\mathbb{Z}_p} \theta_p\left(\frac{-x}{p}\right)\theta_p\left(\frac{-rx}{p}\right) dx\, d^*u. \end{aligned}$$

The integral over $\mathbb{Z}_p$ is equal to

$$\int_{\mathbb{Z}_p} \theta_p\left(-\frac{(1+r)x}{p}\right) dx = \begin{cases} 1 & \text{if } r \equiv -1 \bmod p, \\ 0 & \text{otherwise.} \end{cases}$$

Equality (3-6) now follows, using the fact that

$$\frac{\zeta\omega_p(p)}{\omega_p(t)} = \frac{\zeta\omega_p(p)^2}{\omega_p(pt)} = \frac{\omega_p(p)^2}{\zeta}.$$

For fixed t, if we sum (3-6) over $\pm\zeta$, we get 0. It follows that

$$I_\delta(s)_p = \sum_\sigma J_\delta(s, f^\sigma) = 0. \qquad \square$$

Proposition 3.5. *For $a \in \mathbb{Q}^*$, let*

$$\delta_a = \begin{pmatrix} a & -1 \\ 1 & 0 \end{pmatrix},$$

so that

$$J_{\delta_a}(s, f^\sigma) = \int_{\mathbb{Q}_p^*}\int_{\mathbb{Q}_p} f^\sigma\left(\begin{pmatrix} ya & y(xa-1) \\ 1 & x \end{pmatrix}\right)\overline{\theta_p(rx)}\, dx\, \chi_p(y)|y|_p^{\frac{k}{2}-s}\, d^*y.$$

Then $J_{\delta_a}(s, f_1^\sigma)$ vanishes unless $a \in p^2\mathbb{Z}_p$ and $p \nmid r$. In this case, writing $a = p^{a_p}a_0$ for $a_0 \in \mathbb{Z}_p^* \cap \mathbb{Q}^*$, we have

$$J_{\delta_a}(s, f_1^\sigma) = \begin{cases} \dfrac{|a|_p^{2s-k} p(p+1)\omega_p(p^{a_p})}{2\chi_p(a^2)} \theta_p\Big(\dfrac{ta}{rp^3} - \dfrac{r}{a}\Big) & \text{if } a_0 \equiv 1 \bmod p, \\ 0 & \text{otherwise.} \end{cases} \tag{3-8}$$

The integral $J_{\delta_a}(s, f_2^\sigma)$ vanishes unless $a \in p^2\mathbb{Z}_p$. For such a,

$$J_{\delta_a}(s, f_2^\sigma) = \begin{cases} \dfrac{(p^3)^{k/2-s} p(p+1)\omega_p(p^2)}{2\chi_p(p^3)\zeta} \theta_p\Big(\dfrac{ta}{p^3}\Big) & \text{if } r \equiv -1 \bmod p, \\ 0 & \text{otherwise.} \end{cases} \tag{3-9}$$

Finally, $I_{\delta_a}(s)_p$ vanishes unless $p \nmid r$ and $a = p^{a_p}a_0$ for $a_p \geq 2$ and $a_0 \equiv 1 \bmod p\mathbb{Z}_p$. If these conditions are satisfied, then

$$I_{\delta_a}(s)_p = \frac{|a|_p^{2s-k} p(p+1)\omega_p(p^{a_p})\theta_p\big(-\frac{r}{a}\big)}{\chi_p(a^2)} \Delta_p(a), \tag{3-10}$$
$$\text{for } \Delta_p(a) = \begin{cases} p-1, & a_p > 2 \\ -1, & a_p = 2. \end{cases}$$

Proof. We start by computing $J_{\delta_a}(s, f_1^\sigma)$. From (2-22) we see that the determinant of any matrix in the support of f_1^σ is of the form $(p^m)^2u$ for some $m \in \mathbb{Z}$ and $u \in \mathbb{Z}_p^*$ (the square factor coming from the center). Since

$$\det\begin{pmatrix} y & \\ & 1 \end{pmatrix} \delta_a \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = y,$$

it follows that we may assume $y = u/p^{2\ell}$ for some $\ell \in \mathbb{Z}$ and $u \in \mathbb{Z}_p^*$, and that

$$\begin{pmatrix} p^\ell & \\ & p^\ell \end{pmatrix} \begin{pmatrix} ya & y(xa-1) \\ 1 & x \end{pmatrix} = \begin{pmatrix} p^{-\ell}au & p^{-\ell}u(xa-1) \\ p^\ell & p^\ell x \end{pmatrix} \in \begin{pmatrix} \mathbb{Z}_p^* & p^{-1}\mathbb{Z}_p \\ p^2\mathbb{Z}_p & 1+p\mathbb{Z}_p \end{pmatrix}.$$

This implies $a_p = \ell \geq 2$, and that $p^\ell x = 1 + px'$ for some $x' \in \mathbb{Z}_p$. Then $p^{-\ell}\, dx = p^{-1}\, dx'$. Making this substitution, we find that $J_{\delta_a}(s, f_1^\sigma)$ is equal to

$$\frac{(p^{\ell-1})\omega_p(p^\ell)(p^{2\ell})^{\frac{k}{2}-s}}{\chi_p(p^{2\ell})} \times \int_{\mathbb{Z}_p^*}\int_{\mathbb{Z}_p} f_1^\sigma\left(\begin{pmatrix} a_0u & \frac{1}{p^\ell}u((1+px')a_0-1) \\ p^\ell & 1+px' \end{pmatrix}\right) \overline{\theta_p\Big(\frac{r(1+px')}{p^\ell}\Big)}\, dx'\, d^*u.$$

In order that the integrand be nonzero, we need $p^{-\ell}((1+px')a_0-1) \in p^{-1}\mathbb{Z}_p$, that is, $1+px' \equiv a_0^{-1} \bmod p^{\ell-1}\mathbb{Z}_p$. This is only possible if $a_0 \equiv 1 \bmod p$. Assuming the latter condition holds, we set $1 + px' = a_0^{-1} + p^{\ell-1}x''$, so $p^{-1}\, dx' = p^{1-\ell}\, dx''$.

Then, writing x in place of x'', the double integral becomes

$$\frac{p\omega_p(p^\ell)(p^{2\ell})^{\frac{k}{2}-s}}{\chi_p(p^{2\ell})}\int_{\mathbb{Z}_p^*}\int_{\mathbb{Z}_p} f_1^\sigma\left(\begin{pmatrix} a_0u & p^{-1}xua_0 \\ p^\ell & a_0^{-1}+p^{\ell-1}x\end{pmatrix}\right)\overline{\theta_p\Big(\frac{r(a_0^{-1}+p^{\ell-1}x)}{p^\ell}\Big)}\,dx\,d^*u.$$

After replacing u by ua_0^{-1}, this becomes

$$\frac{(p^{2\ell})^{\frac{k}{2}-s}p(p+1)\omega_p(p^\ell)}{2\chi_p(p^{2\ell})}\theta_p\Big(\frac{-r}{p^\ell a_0}\Big)$$
$$\times\sum_{w\in(\mathbb{Z}/p\mathbb{Z})^*}\int_{\mathbb{Z}_p^*}\int_{\mathbb{Z}_p}\theta_p\Big(\frac{-xuw-tp^{\ell-2}(uw)^{-1}}{p}\Big)\theta_p\Big(\frac{-rx}{p}\Big)dx\,d^*u.$$

Replacing u by $-uw^{-1}$, we eliminate w, and the sum contributes a factor of $(p-1)$. The integral over x is then

$$\int_{\mathbb{Z}_p}\theta_p\Big(\frac{(u-r)x}{p}\Big)\,dx=\begin{cases}1 & \text{if } u\in r+p\mathbb{Z}_p,\\ 0 & \text{otherwise.}\end{cases}$$

In particular, it vanishes if $p\mid r$. Assuming $p\nmid r$, the sum over w of the double integral thus becomes

$$(p-1)\int_{r+p\mathbb{Z}_p}\theta_p\Big(\frac{tp^{\ell-2}u^{-1}}{p}\Big)\,d^*u=\theta_p\Big(\frac{tp^{\ell-2}}{rp}\Big).$$

Hence

$$J_{\delta_a}(s,f_1^\sigma)=\frac{(p^{2\ell})^{k/2-s}p(p+1)\omega_p(p^\ell)}{2\chi_p(p^{2\ell})}\theta_p\Big(\frac{tp^{\ell-2}}{rp}-\frac{r}{a}\Big),$$

which establishes (3-8).

Now consider

$$J_{\delta_a}(s,f_2^\sigma)=\int_{\mathbb{Q}_p^*}\int_{\mathbb{Q}_p} f_2^\sigma\left(\begin{pmatrix} ya & y(xa-1)\\ 1 & x\end{pmatrix}\right)\overline{\theta_p(rx)}\,dx\,\chi_p(y)|y|_p^{\frac{k}{2}-s}\,d^*y.$$

By (2-23), the integrand is nonzero precisely when

$$\begin{pmatrix} p & \\ & p\end{pmatrix}\begin{pmatrix} ya & y(xa-1)\\ 1 & x\end{pmatrix}=\begin{pmatrix} pya & py(xa-1)\\ p & px\end{pmatrix}\in\begin{pmatrix}\mathbb{Z}_p & p^{-2}\mathbb{Z}_p^*\\ p\mathbb{Z}_p^* & \mathbb{Z}_p\end{pmatrix}.$$

Taking the determinant, this says in particular that $p^2y\in p^{-1}\mathbb{Z}_p^*$, so we may write $y=u/p^3$ for $u\in\mathbb{Z}_p^*$. Writing $px=x'$, we have $p^{-1}\,dx=dx'$, and the double integral above equals

$$\frac{(p^3)^{\frac{k}{2}-s}p\omega_p(p)}{\chi_p(p^3)}\int_{\mathbb{Z}_p^*}\int_{\mathbb{Z}_p} f_2^\sigma\left(\begin{pmatrix}\frac{ua}{p^2} & \frac{u}{p^2}(\frac{xa}{p}-1)\\ p & x\end{pmatrix}\right)\theta_p\Big(\frac{-rx}{p}\Big)\,dx\,d^*u.$$

From the upper left entry, the integrand is nonzero only if $a_p \geq 2$. Assuming the latter, we also have $(xa/p) - 1 \in \mathbb{Z}_p^*$, so the upper right entry belongs to $p^{-2}\mathbb{Z}_p^*$ as required. Hence the above equals

$$\frac{(p^3)^{k/2-s}\, p(p+1)\zeta\omega_p(p)}{2\omega_p(t)\chi_p(p^3)} \times \int_{\mathbb{Z}_p^*}\int_{\mathbb{Z}_p} \theta_p\Big(\frac{-x - tp^{-2}ua\big(u\big(\frac{xa}{p}-1\big)\big)^{-1}}{p}\Big)\theta_p\Big(\frac{-rx}{p}\Big)\,dx\,d^*u.$$

Note that u disappears, and that $((xa/p) - 1)^{-1} \equiv -1 \bmod p$. Hence the above equals

$$\frac{(p^3)^{\frac{k}{2}-s}\, p(p+1)\zeta\omega_p(p)}{2\omega_p(t)\chi_p(p^3)}\,\theta_p\Big(\frac{ta}{p^3}\Big)\int_{\mathbb{Z}_p}\theta_p\Big(\frac{-(r+1)x}{p}\Big)\,dx.$$

Equation (3-9) follows upon using $\zeta\omega_p(p)/\omega_p(t) = \omega_p(p)^2/\zeta$.

Since $J_{\delta_a}(s, f_2^{\sigma_{t,\zeta}}) + J_{\delta_a}(s, f_2^{\sigma_{t,-\zeta}}) = 0$, we see that

$$I_{\delta_a}(s)_p = \sum_{(t,\zeta)} J_{\delta_a}(s, f_1^{\sigma}) = \frac{|a|_p^{2s-k}\, p(p+1)\omega_p(p)^{a_p}}{\chi_p(a^2)}\,\theta_p\Big(-\frac{r}{a}\Big)\Delta_p(a),$$

assuming $p \nmid r$, $a \in p^2\mathbb{Z}_p$, and $a_0 \equiv 1 \bmod p$, where

$$\Delta_p(a) = \sum_{t\in(\mathbb{Z}/p\mathbb{Z})^*}\theta_p\Big(\frac{tp^{a_p-2}}{rp}\Big) = \begin{cases} p-1 & \text{if } a_p > 2,\\ -1 & \text{if } a_p = 2.\end{cases}$$

This concludes the proof. □

3B. ***Summary of local results.*** We summarize here the contribution of $\delta = 1$, which turns out to be the main term. By (7.7) and (7.8) of [JK 2012], and (3-5) above, we have

(3-11)
$$I_1(s)_p = \begin{cases} \chi_p(r) & \text{if } p \mid D,\\ p(p+1)(p-1) & \text{if } p \mid N,\\ (p^{n_p})^{\frac{k'}{2}-s}\displaystyle\sum_{d_p=0}^{\min(r_p,n_p)}(p^{d_p})^{2s-k'+1}\,\overline{\omega_p\Big(\frac{p^{d_p}}{p^{n_p}}\Big)}\chi_p\Big(\frac{p^{2d_p}}{p^{n_p}}\Big) & \text{if } p \mid n,\\ \dfrac{2^{k-1}(2\pi r)^{k-s-1}}{(k-2)!\,e^{2\pi r}}\,\Gamma(s) & \text{if } p = \infty,\ k > 2,\\ 1 & \text{if } p \nmid NDn\infty.\end{cases}$$

The local integrals for $\delta = \left(\begin{smallmatrix} 0 & -1\\ 1 & 0\end{smallmatrix}\right)$ are irrelevant, since those at places dividing N vanish by (3-7).

We will discuss the local integrals for $\delta = \left(\begin{smallmatrix} a & -1\\ 1 & 0\end{smallmatrix}\right)$ in Section 4A.

4. Results for holomorphic cusp forms

In this section, we will prove the following:

Theorem 4.1. *Let $r, n, D, N, k \in \mathbb{Z}^+$ with N squarefree, $k > 2$, $(rn, ND) = 1$, and $(D, N) = 1$. Let ω' be a Dirichlet character modulo N, with $\omega'(-1) = (-1)^k$, and let χ be a primitive Dirichlet character modulo D. Then for all $s = \sigma + i\tau$ in the strip $1 < \sigma < k-1$,*

$$\sum_{u \in \mathcal{F}_k^{\mathrm{new}}(N^3,\omega')} \frac{\lambda_n(u)\overline{a_r(u)}\Lambda(s,u,\chi)}{\psi(N^3)\|u\|^2} = F + E, \tag{4-1}$$

where

$$F = \frac{2^{k-1}(2\pi rn)^{k-s-1}}{(k-2)!}\Gamma(s)\prod_{p \mid N}\Big(1-\frac{1}{p}\Big)\sum_{d \mid (n,r)} d^{2s-k+1}\overline{\omega'\Big(\frac{n}{d}\Big)\chi\Big(\frac{rn}{d^2}\Big)}$$

is the main term, and the error term E (an infinite series involving confluent hypergeometric functions) satisfies

$$|E| \le \frac{(4\pi rn)^{k-1}\varphi(D)\gcd(r,n)B(\sigma,k-\sigma)\prod_{p \mid N}\big(1-\frac{1}{p}\big)}{N^{2\sigma}D^{\sigma-k+\frac{1}{2}}(k-2)!} 2\cosh\frac{\pi\tau}{2}\,\zeta(k-\sigma)\zeta(\sigma)$$

for the beta function $B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x+y) \le 1$ and Euler's φ-function.

Remark. If $(r, N) > 1$, then (4-1) vanishes. This is a consequence of the fact that the local representations at $p \mid N$ are supercuspidal, which implies that $a_p(u) = 0$ (see, e.g., the proof of Corollary 45 of [Goldfeld et al. 2010]) and hence $a_r(u) = 0$. This is reflected on the geometric side in (3-5).

The L-functions in Theorem 4.1 are normalized so that the central point is $k/2$. In order to free s from dependence on k we shift the variable so that the critical strip becomes [0, 1] in the following:

Corollary 4.2. *Suppose for simplicity that $(n, r) = 1$. Then for any point s in the critical strip $0 < \mathrm{Re}(s) < 1$, the sum*

$$\sum_{u \in \mathcal{F}_k^{\mathrm{new}}(N^3,\omega')} \frac{\lambda_n(u)\overline{a_r(u)}\Lambda(s+\frac{k-1}{2},u,\chi)}{\psi(N^3)\|u\|^2}$$

is nonzero as long as $N + k$ is sufficiently large.

Proof. See the proof of Corollary 4.4 below. □

As another corollary, we can show that the central values $L(k/2, h, \chi)$ satisfy the Lindelöf hypothesis on average as $k + N \to \infty$ when χ is real and ω' is trivial. (Under these conditions, the central value is a nonnegative real number [Guo 1996].)

Corollary 4.3. *Suppose that ω' is trivial and χ is real. Then for $k > 2$,*

$$\sum_{u\in\mathcal{F}_k^{\text{new}}(N^3)} L(k/2, u, \chi) \ll_D (kN^3)^{1+\varepsilon}.$$

Proof. The proof is identical to that of Corollary 1.3 of [JK 2012]. □

Remark. This is the same bound we would get by assuming the Lindelöf hypothesis $L(k/2, u, \chi) \ll (D^2kN^3)^{\varepsilon}$, in view of the fact that

$$|\mathcal{F}_k^{\text{new}}(N^3)| \sim \frac{k-1}{12}\psi^{\text{new}}(N^3),$$

where $(N^3)^{1-\varepsilon} \ll \psi^{\text{new}}(N^3) \le N^3$ [Serre 1997, (60) on p. 86].

Returning to the case of general ω' and χ, let $\langle\Lambda_n(s, \chi), a_r\rangle^{\text{new}}$ denote the sum in (4-1). We can regard this as an inner product of elements of the dual space of $S_k(N^3, \omega')^{\text{new}}$. One can also define $\langle\Lambda_n(s, \chi), a_r\rangle$ in the same way, but where the sum is taken over an orthogonal basis for the full space $S_k(N^3, \omega')$. It is interesting to compare the two.

Corollary 4.4. *Let s belong to the critical strip $(k-1)/2 < \mathrm{Re}(s) < (k+1)/2$, and suppose that $(n, r) = 1$. Then with notation as above, we have*

$$\frac{\langle\Lambda_n(s, \chi), a_r\rangle^{\text{new}}}{\langle\Lambda_n(s, \chi), a_r\rangle} \sim \prod_{p\,|\,N}\Bigl(1-\frac{1}{p}\Bigr) \quad \textit{as } N+k\to\infty.$$

Remark. From this we can observe two extremes in behavior. If $N = p$ is prime, and $p\to\infty$, the above tends to 1, so the contribution of oldforms becomes negligible. This agrees with a prediction of Ellenberg [2004, Remark 3.11]. On the other hand, if we take N to be the product of the first ℓ primes not dividing D and let $\ell\to\infty$, the above goes to 0 and the contribution of the newforms becomes negligible.

Proof. In Theorem 1.1 of [JK 2012], it is proven that

$$\langle\Lambda_n(s, \chi), a_r\rangle = F' + E', \quad \text{where } F' = \frac{F}{\prod_{p\,|\,N}\bigl(1-\frac{1}{p}\bigr)},$$

and $|E'|$ satisfies a bound similar to the one given for $|E|$ in the above theorem, but without the factor of $\prod_{p\,|\,N}(1-\frac{1}{p})$, and with $N^{3\sigma}$ in the denominator rather than $N^{2\sigma}$. In the last line of [JK 2012], it is shown using Stirling's approximation that for $s = (k/2)+\delta+i\tau$ with $|\delta| < \frac{1}{2}$,

$$\Bigl|\frac{E'}{F'}\Bigr| \ll \frac{(4D\pi rne)^{\frac{k}{2}}}{(N^3)^{\frac{k-1}{2}}k^{\frac{k}{2}-1}},$$

where the implied constant depends on δ, D, R, n, τ. Clearly the above goes to 0 as $N+k\to\infty$. The same bound holds for $|E/F|$, but with N^2 in place of N^3, since the extra factor of $\prod\bigl(1-\frac{1}{p}\bigr)$ in the numerator and denominator cancels out. Likewise,

$$\left|\frac{E}{F'}\right| = \left|\frac{E}{F}\right| \prod_{p\,|\,N}\Bigl(1-\frac{1}{p}\Bigr) \le \left|\frac{E}{F}\right| \ll \frac{(4D\pi rne)^{\frac{k}{2}}}{(N^2)^{\frac{k-1}{2}} k^{\frac{k}{2}-1}}.$$

Now consider the quotient

$$\frac{\langle \Lambda_n(s,\chi), a_r\rangle^{\text{new}}}{\langle \Lambda_n(s,\chi), a_r\rangle} = \frac{F+E}{F'+E'} = \frac{\frac{F}{F'}+\frac{E}{F'}}{1+\frac{E'}{F'}} = \frac{\prod_{p\,|\,N}\bigl(1-\frac{1}{p}\bigr) + O\left(\dfrac{(4D\pi rne)^{\frac{k}{2}}}{(N^2)^{\frac{k-1}{2}} k^{\frac{k}{2}-1}}\right)}{1+O\left(\dfrac{(4D\pi rne)^{\frac{k}{2}}}{(N^3)^{\frac{k-1}{2}} k^{\frac{k}{2}-1}}\right)}$$

The corollary now follows easily. □

4A. ***Proof of Theorem 4.1.*** In the holomorphic case, the spectral side (3-2) becomes

$$\frac{n^{1-\frac{k}{2}}}{e^{2\pi r}} \sum_{u\in\mathcal{F}_k^{\text{new}}(N^3,\omega')} \frac{\lambda_n(u)\overline{a_r(u)}}{\|u\|^2}\Lambda(s,u,\chi).$$

By the local calculation (3-7), the geometric side has the form

$$I_1(s) + \sum_{a\in\mathbb{Q}^*} I_{\delta_a}(s).$$

As is typical, the identity term $I_1(s)$ is the dominant term as $N+k\to\infty$. Multiplying the local results (3-11) together, when $k>2$ (so $k'=k$) we obtain:

$$\begin{aligned}\frac{e^{2\pi r} n^{\frac{k}{2}-1}}{\psi(N^3)} & I_1(s) \\ &= \frac{2^{k-1}(2\pi rn)^{k-s-1}}{(k-2)!}\Gamma(s)\prod_{p\,|\,N}\frac{p(p+1)(p-1)}{p^2(p+1)} \sum_{d\,|\,(r,n)} d^{2s-k+1}\,\overline{\omega'\Bigl(\frac{n}{d}\Bigr)}\chi\Bigl(\frac{rn}{d^2}\Bigr).\end{aligned}$$

This is the leading term of (4-1).

Theorem 4.1 now follows immediately from the following proposition involving the remaining orbital integrals.

Proposition 4.5. *For $\delta_a = \bigl(\begin{smallmatrix} a & -1 \\ 1 & 0\end{smallmatrix}\bigr)$, with $a\in\mathbb{Q}^*$, the orbital integral $I_{\delta_a}(s)$ is absolutely convergent on the strip $0<\sigma<k$. It vanishes unless $a=(N^2b)/(nD)$ for $b\in\mathbb{Z}-\{0\}$. When $s=\sigma+i\tau$ for $1<\sigma<k-1$, the sum $\sum_{a\in\mathbb{Q}^*} I_{\delta_a}(s)$ is absolutely*

convergent, *and*

$$\frac{e^{2\pi r}n^{\frac{k}{2}-1}}{\psi(N^3)}\sum_{a\in\mathbb{Q}^*}|I_{\delta_a}(s)| \\ \le \frac{(4\pi rn)^{k-1}\varphi(D)\gcd(r,n)B(\sigma,k-\sigma)\prod_{p\mid N}\left(1-\frac{1}{p}\right)}{N^{2\sigma}D^{\sigma-k+\frac{1}{2}}(k-2)!}2\cosh\left(\frac{\pi\tau}{2}\right)\zeta(k-\sigma)\zeta(\sigma).$$

Using the results of [JK 2012] and (3-10) above, one can actually give a rather explicit formula for the sum of the $I_{\delta_a}(s)$ as an infinite series. However, this involves a lot of bookkeeping and seems of limited value, so we will just present the bound. First, by (3-10), we note that for $p\mid N$, $I_{\delta_a}(s)_p$ vanishes unless $a=(N^2b)/(nD)\in N^2\mathbb{Z}_p$, and

(4-2)
$$|I_{\delta_a}(s)_p|\le|N^2b|_p^{2\sigma-k}p(p+1)(p-1)\le|N^2|_p^{2\sigma-k}p(p+1)(p-1)\sum_{d_p=0}^{b_p}|p^{d_p}|_p^{2\sigma-k}.$$

Now suppose $p\nmid N\infty$. Then the value of $I_{\delta_a}(s)_p$ is not quite stated explicitly in [JK 2012], but a closely related integral is given. Start with

$$I_{\delta_a}(s)_p=\int_{\mathbb{Q}_p^*}\int_{\mathbb{Q}_p}f_p\left(\begin{pmatrix}ya & y(xa-1)\\ 1 & x\end{pmatrix}\right)\overline{\theta_p(rx)}\,dx\,\chi_p(y)|y|_p^{\frac{k}{2}-s}\,d^*y.$$

A matrix belongs to the support of f_p only if its determinant is of the form $(p^m)^2nu$ for some $m\in\mathbb{Z}$ and $u\in\mathbb{Z}_p^*$. This can be seen from (2-24) (for $p\mid n$), from the expression for f_p^χ three lines above (2-17) if $p\mid D$, and from (2-14) if $p\nmid nDN\infty$. (In the latter two cases, n is a unit.) The determinant of the matrix in the above integral is y, so the integrand vanishes unless $y\in p^{-2d_p}n\mathbb{Z}_p^*$ for some $d_p\in\mathbb{Z}$. Write $a=N^2b/(nD)$, where (for now) $b\in\mathbb{Q}^*$. It will be convenient to write $y=nu/(N^2d)^2$ for $d=p^{d_p}$ and $u\in\mathbb{Z}_p^*$. Then the above becomes

$$\sum_{d_p\in\mathbb{Z}}\frac{\chi_p\left(\frac{n}{N^4d^2}\right)}{\left|\frac{n}{d^2}\right|_p^{s-\frac{k}{2}}} \\ \times\int_{\mathbb{Z}_p^*}\int_{\mathbb{Q}_p}f_p\left(\begin{pmatrix}N^2d & \\ & N^2d\end{pmatrix}^{-1}\begin{pmatrix}\frac{nu}{N^2d}\frac{N^2b}{nD} & \frac{nu}{N^2d}\left(\frac{xN^2b}{nD}-1\right)\\ N^2d & dN^2x\end{pmatrix}\right)\overline{\theta_p(rx)}\,dx\,\chi_p(u)\,d^*u.$$

Since the second matrix has determinant $un\in n\mathbb{Z}_p^*$, all entries must lie in $\mathbb{Z}_p$ for the integrand to be nonzero. In particular, $d_p\ge 0$. We may substitute $x'=dN^2x$, so that $dx'=|dN^2|_p\,dx=|d|_p\,dx$. Using the fact that the central character ω_p is unramified, we obtain

$$\sum_{d_p \geq 0} \frac{\chi_p\bigl(\frac{n}{N^4 d^2}\bigr)\omega_p(d)}{\bigl|\frac{n}{d^2}\bigr|_p^{s-\frac{k}{2}}} |d|_p^{-1} \times \int_{\mathbb{Z}_p^*} \int_{\mathbb{Z}_p} f_p\left(\begin{pmatrix} \frac{ub}{dD} & \frac{ubx}{N^2 d^2 D} - \frac{nu}{N^2 d} \\ N^2 d & x \end{pmatrix}\right) \overline{\theta_p\Bigl(\frac{rx}{N^2 d}\Bigr)}\, dx\, \chi_p(u)\, d^*u.$$

The latter double integral coincides with (8.4) of [JK 2012], but with N where we have N^2. It is computed explicitly in Sections 8.1 and 8.2 of that reference. In particular, it vanishes unless $0 \leq d_p \leq b_p$, proving the assertion in Proposition 4.5 that the global integral vanishes unless $a = N^2 b/(nD)$ for nonzero $b \in \mathbb{Z}$. Multiplying the coefficient by the double integral, whose value is given in (8.7), (8.8), and (8.12) of the same work, we find

$$|I_{\delta_a}(s)_p| \leq \frac{\varphi(p^{D_p})}{|\tau(\overline{\chi})_p|} \sum_{d_p=0}^{b_p} |p^{d_p}|_p^{2\sigma-k} \qquad \text{if } (p \mid D), \tag{4-3}$$

$$|I_{\delta_a}(s)_p| \leq \frac{|n|_p^{\frac{k}{2}-\sigma}}{|\gcd(r,n)|_p} \sum_{d_p=0}^{b_p} |p^{d_p}|_p^{2\sigma-k} \qquad \text{if } (p \nmid DN\infty). \tag{4-4}$$

(For the latter, we have used the fact that $\gcd(\frac{b}{d}, Nd) \mid \gcd(r, Dn)$ in [JK 2012].) Multiplying the local bounds (4-2)–(4-4) together, we have

$$|I_{\delta_a}(s)_{\mathrm{fin}}| \leq \frac{n^{\sigma-\frac{k}{2}} \varphi(D) \gcd(r,n)}{(N^2)^{2\sigma-k} |\tau(\overline{\chi})|} \Bigl(\prod_{p \mid N} p(p+1)(p-1)\Bigr) \sum_{d \mid b} \frac{1}{d^{2\sigma-k}}. \tag{4-5}$$

For the archimedean part, we have from [ibid., (8.15)]

$$|I_{\delta_a}(s)_\infty| = \left| \frac{(4\pi r)^{k-1} (N^2)^{\sigma-k} b^{s-k} e^{\frac{i\pi s}{2}}}{(k-2)!\,(nD)^{\sigma-k} e^{2\pi r}} {}_1 f_1\Bigl(s; k; -\frac{2\pi i r n D}{N^2 b}\Bigr) \right|, \tag{4-6}$$

where $b^s = e^{-i\pi s}|a|^s$ if $b < 0$, and for $\mathrm{Re}(k) > \mathrm{Re}(s) > 0$,

$${}_1 f_1(s, k; w) = B(s, k-s)\, {}_1F_1(s; k; w) = \int_0^1 e^{wx} x^{s-1} (1-x)^{k-s-1}\, dx$$

[Slater 1960, §3.1]. Noting that

$$\left| {}_1 f_1\Bigl(s; k; -\frac{2\pi i r n D}{N^2 b}\Bigr) \right| \leq \int_0^1 x^{\sigma-1} (1-x)^{k-\sigma-1}\, dx = B(\sigma, k-\sigma),$$

and that

$$\frac{\prod_{p\mid N} p(p+1)(p-1)}{\psi(N^3)} = \prod_{p\mid N} \frac{p(p+1)(p-1)}{p^2(p+1)} = \prod_{p\mid N}\Big(1-\frac{1}{p}\Big),$$

we multiply (4-5) by (4-6) to get

$$\frac{e^{2\pi r} n^{\frac{k}{2}-1}}{\psi(N^3)} |I_{\delta_a}(s)| \\ \leq \frac{(4\pi r n)^{k-1}\varphi(D)e^{-\pi\frac{\tau}{2}}}{N^{2\sigma}D^{\sigma-k+\frac{1}{2}}(k-2)!}\bigg(\prod_{p\mid N}\Big(1-\frac{1}{p}\Big)\bigg)\big|b^{s-k}\big|B(\sigma,k-\sigma)\sum_{d\mid b}\frac{\gcd(r,n)}{d^{2\sigma-k}}.$$

Now we need to bound the sum over $b \in \mathbb{Z}-\{0\}$. Write $b = \pm cd$ for $c, d > 0$, and group the c, $-c$ terms together, so that

$$|c^{s-k}| + |(-c)^{s-k}| = c^{\sigma-k} + |(e^{-i\pi})^{s-k}c^{s-k}| = c^{\sigma-k}(1+e^{\pi\tau}).$$

Noting that $e^{-\pi\tau/2}(1+e^{\pi\tau}) = 2\cosh(\pi\tau/2)$, we obtain

$$e^{\frac{-\pi\tau}{2}}\sum_{b\neq 0}|b^{s-k}|\sum_{d\mid b}\frac{1}{d^{2\sigma-k}} \\ = 2\cosh\Big(\frac{\pi\tau}{2}\Big)\sum_{c,d>0} c^{\sigma-k}d^{-\sigma} = 2\cosh\Big(\frac{\pi\tau}{2}\Big)\zeta(k-\sigma)\zeta(\sigma).$$

Proposition 4.5 now follows immediately.

5. The case of nonholomorphic cusp forms

5A. ***Integral transforms.*** Here we define various integral transforms involving spherical functions. We refer to §3 of [KL 2013] for further detail.

Let $f_\infty \in C_c^\infty(G(\mathbb{R})^+//K_\infty)$ as in (2-15). The *Harish-Chandra transform* of f_∞ is the function on $\mathbb{R}^+$ defined by

$$\mathcal{H} f_\infty(y) = y^{\frac{-1}{2}}\int_{-\infty}^{\infty} f_\infty\bigg(\begin{pmatrix}1&x\\0&1\end{pmatrix}\begin{pmatrix}y&0\\0&1\end{pmatrix}\bigg)\,dx.$$

We will also encounter the twisted variant

$$\mathcal{H}_r f_\infty(y) = y^{-\frac{1}{2}}\int_{-\infty}^{\infty} f_\infty\bigg(\begin{pmatrix}1&x\\0&1\end{pmatrix}\begin{pmatrix}y&\\&1\end{pmatrix}\bigg)e^{-2\pi i r x}\,dx \tag{5-1}$$

for $r \in \mathbb{R}$, and a twisted variant in the big Bruhat cell

$$\mathcal{H}_{r,\alpha} f_\infty(y) = y^{-\frac{1}{2}}\int_{-\infty}^{\infty} f_\infty\bigg(\begin{pmatrix}1&x\\0&1\end{pmatrix}\begin{pmatrix}0&1\\-1&\alpha\end{pmatrix}\begin{pmatrix}y&\\&1\end{pmatrix}\bigg)e^{-2\pi i r x}\,dx \tag{5-2}$$

for $\alpha \in \mathbb{R}$. Each of the above functions is smooth with compact support in $\mathbb{R}^+$.

For $\phi \in C_c^\infty(\mathbb{R}^+)$, the Mellin transform is denoted

$$\mathcal{M}_s\phi = \int_0^\infty \phi(y) y^s \, d^*y.$$

Composing with the Harish-Chandra transform, we obtain the *spherical transform*

$$\mathcal{S} f_\infty(s) = \mathcal{M}_s \mathcal{H} f_\infty.$$

The *Selberg transform* of f_∞ is defined by

$$h(t) = \mathcal{S} f_\infty(it) = \mathcal{M}_{it} \mathcal{H} f_\infty. \tag{5-3}$$

Then $h(it)$ is an even Paley–Wiener function. This means that it is holomorphic and there exists a real number $C \geq 1$ depending only on h such that for any integer $M > 0$, we have

$$h(a+ib) \ll_{M,h} \frac{C^{|b|}}{(1+|a|)^M}. \tag{5-4}$$

Using (5-1) we also define a *twisted spherical transform* of f_∞ by

$$h_r(s) = \mathcal{M}_s \mathcal{H}_r f_\infty, \tag{5-5}$$

and a twisted variant in the big Bruhat cell

$$h_{r,\alpha}(s) = \mathcal{M}_s \mathcal{H}_{r,\alpha} f_\infty, \tag{5-6}$$

for $\alpha \in \mathbb{R}$, as in (5-2). These functions likewise are holomorphic and satisfy (5-4), though they are not even in general. Note that $h_0(s) = h(s)$.

5B. *Nonholomorphic case: spectral side.* When $k = 0$, the spectral side (3-2) of the relative trace formula becomes

$$I = \frac{\sqrt{n}}{2} \sum_{u_j \in \mathcal{F}_+^{\text{new}}(N^3,\omega')} \frac{\lambda_n(u_j)\overline{a_r(u_j)}\Lambda(s,u_j,\chi)}{\|u_j\|^2} h(t_j) K_{it_j}(2\pi|r|). \tag{5-7}$$

Proposition 5.1. *Let $k = 0$ and $r \in \mathbb{Q}$. If $\sigma = \mathrm{Re}(s)$ is sufficiently large, then the integral*

$$\int_{\mathbb{Q}^*\backslash\mathbb{A}^*} \int_{\mathbb{Q}\backslash\mathbb{A}} \sum_{u \in \mathcal{F}_0^{\text{new}}(N^3,\omega')} \left| \frac{\lambda_f(u) R(f^\chi)\phi_u\left(\left(\begin{smallmatrix} y & \\ & 1\end{smallmatrix}\right)\right)\overline{\phi_u\left(\left(\begin{smallmatrix} 1 & x \\ 0 & 1\end{smallmatrix}\right)\right)}}{\|u\|^2} |y|^{s-\frac{1}{2}} \right| dx\, d^*y$$

is absolutely convergent. Hence, as in Proposition 3.1, the integral (3-1) *is equal to* (5-7) *for such s. The sum* (5-7) *converges absolutely for all $s \in \mathbb{C}$, and defines an entire function.*

Proof. As in the proof of Proposition 3.1, by the fact there are at most finitely many u with exceptional spectral parameters, it suffices to sum over the set $\mathcal{F}'$ of newforms for which t is real. Thus we need to show that

$$\sum_{u\in\mathcal{F}'} \frac{|\lambda_f(u)|}{\|u\|^2} \int_{\mathbb{Q}^*\backslash\mathbb{A}^*} \left|\phi_{u_\chi}\left(\begin{pmatrix} y & \\ & 1\end{pmatrix}\right)\right| y^{\sigma-\frac{1}{2}}\, d^*y \int_{\mathbb{Q}\backslash\mathbb{A}} \left|\phi_u\left(\begin{pmatrix} 1 & x\\ 0 & 1\end{pmatrix}\right)\right| dx \tag{5-8}$$

is finite. The second integral is bounded by an absolute constant:

$$\begin{aligned}\int_{\mathbb{Q}\backslash\mathbb{A}} \left|\phi_u\left(\begin{pmatrix} 1 & x\\ 0 & 1\end{pmatrix}\right)\right| dx = \int_0^1 |u(i+x)|\, dx \le \int_0^1 \sum_{m\neq 0} \left|a_m(u)K_{it}(2\pi|m|)e^{2\pi imx}\right| dx\\ = \sum_{m\neq 0}\left|a_m(u)K_{it}(2\pi|m|)\right| \ll \sum_{m\neq 0} |m|^{\frac{1}{2}+\varepsilon} K_0(2\pi|m|) < \infty.\end{aligned}$$

Here, we have used the fact that since t is real,

$$|K_{it}(y)| = \left|\tfrac{1}{2}\int_0^\infty e^{-\frac{y(w+w^{-1})}{2}} w^{it}\, d^*w\right| \le \tfrac{1}{2}\int_0^\infty e^{\frac{-y(w+w^{-1})}{2}}\, d^*w = K_0(y), \tag{5-9}$$

and also the bound $|a_m(u)| \ll |m|^{1/2+\varepsilon}$ [Iwaniec and Kowalski 2004, (5.92)]. The strongest known bound of this nature is that of Kim and Sarnak [2003]:

$$|a_m(u)| \le \tau(|m|)|m|^{\frac{7}{64}}, \tag{5-10}$$

where τ is the divisor function. This, together with (2-26), gives $|\lambda_f(u)| \ll_n |h(t)|$. By the above observations, and following the proof of Lemma 2.2, we see that (5-8) is

$$\ll \sum_{u\in\mathcal{F}'} \frac{|h(t)|}{\|u\|^2} \int_0^\infty |u_\chi(iy)|\, y^{\sigma-\frac{1}{2}}\, d^*y. \tag{5-11}$$

Using the Fourier expansion of u_χ, the above integral is bounded by

$$\begin{aligned}\int_0^\infty \sum_{m\neq 0} \left|\chi(m)a_m(u)K_{it}(2\pi|m|y)\right| y^\sigma\, d^*y\\ \le (2\pi)^{-\sigma} \sum_{m\neq 0} \frac{|a_m(u)|}{m^\sigma} \int_0^\infty |K_{it}(y)| y^\sigma\, d^*y.\end{aligned}$$

Once again invoking (5-9), we have

$$\int_0^\infty |K_{it}(y)| y^\sigma\, d^*y \le \int_0^\infty K_0(y) y^\sigma\, d^*y = 2^{\sigma-2}\Gamma\left(\frac{\sigma}{2}\right)^2,$$

by a well-known identity for $\sigma > 0$ (cf. [Gradshteyn and Ryzhik 2007, 6.561.16]). Using (5-10), we see that the sum over m is bounded by an absolute constant when $\sigma \ge 3$. This shows that the integral of u_χ in (5-11) is bounded by a constant independent of u and depending continuously on $\sigma \geq 3$. As shown by Goldfeld, Hoffstein and Lieman (see [Hoffstein and Lockhart 1994, Appendix]),

$$\frac{1}{\|u\|^2} \ll_\varepsilon N^\varepsilon (1+|t|)^\varepsilon \tag{5-12}$$

for an absolute (ineffective) implied constant. Thus we reduce to proving that

$$\sum_{u_j \in \mathcal{F}'} |h(t_j)|(1+|t_j|)^\varepsilon < \infty. \tag{5-13}$$

This follows from (5-4) and the fact that $|t_j| \to \infty$ (for details, see the end of the proof of Proposition 7.5 of [KL 2013]).

Now we prove that the sum (5-7) is absolutely convergent for all $s \in \mathbb{C}$. Once again, it suffices to sum over $u \in \mathcal{F}'$. Thus (using (5-10) to bound $a_r(u)$ and $\lambda_n(u)$), we need to show

$$\sum_{u_j \in \mathcal{F}'} \frac{\left|\Gamma\left(\frac{s+it_j}{2}\right)\Gamma\left(\frac{s-it_j}{2}\right)L(s,u_j,\chi)h(t_j)K_{it_j}(2\pi|r|)\right|}{\|u_j\|^2} < \infty. \tag{5-14}$$

By Stirling's formula [Abramowitz and Stegun 1964, 6.1.39], for real $t \neq 0$ (taking $\arg(it) = \pm\frac{\pi}{2}$) we have

$$\begin{aligned}\left|\Gamma\left(\frac{s+it}{2}\right)\Gamma\left(\frac{s-it}{2}\right)\right| f \sim 2\pi\left|\left(\frac{it}{2}\right)^{\frac{s+it-1}{2}}\left(-\frac{it}{2}\right)^{\frac{s-it-1}{2}}\right| \\ = 2\pi\left(\frac{|t|}{2}\right)^{\sigma-1} e^{-\pi\frac{|t|}{2}}\end{aligned} \tag{5-15}$$

as $|t| \to \infty$. Similarly, because t is real, we have

$$K_{it}(2\pi|r|) \ll e^{-\pi|t|/2} \tag{5-16}$$

as $|t| \to \infty$; see (19) on p. 88 of [Erdélyi et al. 1953].

To bound the L-functions, by the functional equation we can assume without losing generality that $\sigma \geq \frac{1}{2}$. For such s, we have the uniform convexity bound

$$L(s,u_j,\chi) \ll_\varepsilon (D^2N^3)^{\frac{1}{4}+\varepsilon}(|s|+3)^{\frac{1}{2}+\varepsilon}(|t_j|+3)^{\frac{1}{2}+\varepsilon}$$

[Iwaniec and Kowalski 2004, Theorem 5.41 and (5.8)]. Here the implied constant is independent of u_j.

Using (5-12), we now find that the left-hand side of (5-14) is

$$\ll (|s|+3)^{\frac{1}{2}+\varepsilon} \sum_{u_j} (|t_j|+3)^{\frac{1}{2}+\varepsilon}\left(\frac{|t_j|}{2}\right)^{\sigma-1}|h(t_j)|e^{-\pi|t_j|}.$$

The finiteness of the above sum follows as for (5-13). It is clear as well that the convergence is uniform for s in compact sets, giving an entire function. □

5C. ***Nonholomorphic case: geometric side.*** By Proposition 3.2 and (3-7), the geometric side is equal to

$$I_1(s)+\sum_{a\in\mathbb{Q}^*} I_{\delta_a}(s)$$

for $\delta=\left(\begin{smallmatrix} a & -1\\ 1 & 0\end{smallmatrix}\right)$. The only changes from the holomorphic case discussed earlier are archimedean. For example, it remains true here that $I_{\delta_a}(s)\neq 0$ only if $a=N^2b/(nD)$ for some nonzero integer b. The local orbital integrals at ∞ are now given as general integral transforms of $f_\infty\in C_c^\infty(G(\mathbb{R})^+//K_\infty)$. Using the fact that f_∞ has compact support modulo Z_∞, we will see that all but finitely many of the geometric terms vanish, and indeed if N is sufficiently large, the only nonzero term is the main term.

For the main term we have, upon replacing y by y^{-1} in (3-3),

$$I_1(s)_\infty=\int_{\mathbb{R}^*}\int_{\mathbb{R}} f_\infty\left(\begin{pmatrix} y & xy\\ 0 & 1\end{pmatrix}\right)e^{2\pi irx}\,dx\,\chi_\infty(y)|y|^{\frac{1}{2}-s}\,d^*y.$$

Since f_∞ is supported on $G(\mathbb{R})^+$, the first integral can be taken over $\mathbb{R}^+$, where χ_∞ is trivial. Furthermore, since f_∞ is biinvariant under $Z_\infty K_\infty$, it follows easily (using the Cartan decomposition [KL 2013, §3.1]) that

$$f_\infty(g)=f_\infty(g^{-1}). \tag{5-17}$$

Therefore

$$\begin{aligned} I_1(s)_\infty &= \int_0^\infty\int_{-\infty}^\infty f_\infty\left(\begin{pmatrix} y^{-1} & -x\\ 0 & 1\end{pmatrix}\right)e^{2\pi irx}\,dx\,y^{\frac{1}{2}-s}\,d^*y \\ &= \int_0^\infty\left[y^{-1/2}\int_{-\infty}^\infty f_\infty\left(\begin{pmatrix} y & x\\ 0 & 1\end{pmatrix}\right)e^{-2\pi irx}\,dx\right]y^s\,d^*y \\ &= \mathcal{M}_s\mathcal{H}_r f_\infty = h_r(s), \end{aligned} \tag{5-18}$$

as in (5-5).

Multiplying the above by the local nonarchimedean values given in (3-11) (taking $k'=1$), we obtain the following:

Proposition 5.2. *The global integral $I_1(s)$ is nonzero only if* $\gcd(r,N)=1$. *In this case*,

$$I_1(s)=\frac{h_r(s)}{n^{s-\frac{1}{2}}}N\prod_{p\mid N}(p^2-1)\sum_{d\mid \gcd(n,r)} d^{2s}\,\overline{\omega'\left(\frac{n}{d}\right)\chi\left(\frac{rn}{d^2}\right)}. \tag{5-19}$$

For $\delta=\left(\begin{smallmatrix} a & -1\\ 1 & 0\end{smallmatrix}\right)$, the archimedean orbital integral is

$$\begin{aligned}
I_{\delta_a}(s)_\infty &= \int_{\mathbb{R}^*}\int_{\mathbb{R}} f_\infty\left(\begin{pmatrix} y & \\ & 1\end{pmatrix}\begin{pmatrix} a & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix} 1 & x\\ 0 & 1\end{pmatrix}\right)e^{2\pi i r x}\,dx\;\chi_\infty(y)|y|^{\frac{1}{2}-s}\,d^*y\\
&= \int_0^\infty\int_{\mathbb{R}} f_\infty\left(\begin{pmatrix} 1 & -x\\ 0 & 1\end{pmatrix}\begin{pmatrix} 0 & 1\\ -1 & a\end{pmatrix}\begin{pmatrix} y^{-1} & \\ & 1\end{pmatrix}\right)e^{2\pi i r x}\,dx\;y^{\frac{1}{2}-s}\,d^*y\\
&= \int_0^\infty\int_{\mathbb{R}} f_\infty\left(\begin{pmatrix} 1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix} 0 & 1\\ -1 & a\end{pmatrix}\begin{pmatrix} y & \\ & 1\end{pmatrix}\right)e^{-2\pi i r x}\,dx\;y^{s-\frac{1}{2}}\,d^*y\\
&= \mathcal{M}_s(\mathcal{H}_{r,a}f_\infty) = h_{r,a}(s),
\end{aligned}$$

as in (5-6).

Proposition 5.3. *For any choice of* $f_\infty \in C_c^\infty(G(\mathbb{R})^+//K_\infty)$, $I_\delta(s) = 0$ *for all but finitely many* δ. *Indeed, there exists a constant* C, *depending on* f_∞ *and* n, *such that*

$$\sum_\delta I_\delta(s) = I_1(s) = \frac{h_r(s)}{n^{s-\frac{1}{2}}} N \prod_{p\mid N}(p^2-1) \sum_{d\mid \gcd(n,r)} d^{2s}\overline{\omega'\Big(\frac{n}{d}\Big)}\chi\Big(\frac{rn}{d^2}\Big) \tag{5-20}$$

whenever $N > C$.

Proof. The function

$$\begin{pmatrix} \alpha & \beta\\ \gamma & \delta\end{pmatrix} \mapsto \frac{\alpha\gamma}{\alpha\delta-\beta\gamma}$$

is well-defined in $\overline{G}(\mathbb{R})$. Hence it is bounded on the compact set $\mathrm{Supp}(f_\infty)/Z_\infty$. Taking

$$\begin{pmatrix} \alpha & \beta\\ \gamma & \delta\end{pmatrix} = \begin{pmatrix} y & \\ & 1\end{pmatrix}\begin{pmatrix} a & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix} 1 & x\\ & 1\end{pmatrix} = \begin{pmatrix} ya & y(xa-1)\\ 1 & x\end{pmatrix},$$

we have $\alpha\gamma/(\alpha\delta-\beta\gamma) = a$. This shows that if $|a|$ is sufficiently large, the above matrix lies outside the support of f_∞ for all x, y, and hence $I_{\delta_a}(s) = 0$. Furthermore, because

$$|a| = \frac{N^2}{n}|b| \geq \frac{N^2}{n} \to \infty$$

as $N\to\infty$, when N is sufficiently large the only nonzero term is $I_1(s)$. □

Putting everything together, we now arrive at the main result for Maass forms.

Theorem 5.4. *Let* r, n, N *be positive integers with* N *squarefree and* $(rn, N) = 1$. *Let* χ *be a primitive Dirichlet character of modulus* D, *where* $(D, rnN) = 1$. *Let* $h(iz)$ *be an even Paley–Wiener function. When the squarefree integer* N *is sufficiently large, we have for all* $s \in \mathbb{C}$,

$$\begin{aligned}
&\sum_{u_j\in\mathcal{F}_+^{\mathrm{new}}(N^3,\omega')} \frac{\lambda_n(u_j)\,\overline{a_r(u_j)}\,\Lambda(s,u_j,\chi)}{\psi(N^3)\|u_j\|^2} h(t_j)\,K_{it_j}(2\pi|r|)\\
&\qquad = \frac{2}{n^s}h_r(s)\prod_{p\mid N}\Big(1-\frac{1}{p}\Big)\sum_{d\mid\gcd(n,r)} d^{2s}\,\overline{\omega'(\tfrac{n}{d})}\,\chi\Big(\frac{rn}{d^2}\Big)
\end{aligned} \tag{5-21}$$

for $h_r(s)$ *as in* (5-5).

Remarks. (1) An immediate corollary (at least when $\gcd(n, r) = 1$) is the existence of a Maass newform of level N^3 for which $\lambda_n(u)$, $a_r(u)$, and $\Lambda(s, u, \chi)$ are simultaneously nonzero.

(2) When $\gcd(r, n) = 1$, the sum on the right becomes $\overline{\omega'(n)}\chi(rn)$. If $r = n = 1$, then the right-hand side is independent of χ. This is the case stated as Theorem 1.1.

(3) Both sides vanish when $(r, N) > 1$. See the remark after Theorem 4.1.

(4) One can weaken the hypotheses somewhat. It is sufficient for $h(iz)$ to be Paley–Wiener of order $m \geq 8$; cf. [KL 2013, Corollary 6.12 and (3.14)].

Proof. As a consequence of Proposition 3.2 and the above discussion, the equality between the spectral side (5-7) and the geometric side (5-20) has been established in some right half-plane $\mathrm{Re}(s) \geq \alpha$ for α sufficiently large. Multiplying both sides of this relative trace formula by $2/(\sqrt{n}\psi(N^3))$, we obtain (5-21) for such s. On the other hand, by Proposition 5.1, each side of (5-21) is an entire function of s. Hence the equality is valid for all complex s. □

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ANDREW KNIGHTLY
DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MAINE
5752 NEVILLE HALL, RM 333
ORONO ME 04469-5752
UNITED STATES
knightly@math.umaine.edu

CHARLES LI
DEPARTMENT OF MATHEMATICS
THE CHINESE UNIVERSITY OF HONG KONG
ROOM 218
LADY SHAW BUILDING
SHATIN
HONG KONG
charles@charlesli.org

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ON OCCULT PERIOD MAPS

STEPHEN KUDLA AND MICHAEL RAPOPORT

In memoriam Jonathan Rogawski

We interpret the "occult" period maps of Allcock, Carlson, and Toledo (2002; 2011), of Looijenga and Swierstra (2007; 2008), and of Kondō (2000; 2002) in moduli theoretic terms, as a construction of certain families of polarized abelian varieties of Picard type. We show that these period maps are morphisms defined over their natural field of definition.

1. Introduction

In papers of Allcock, Carlson, and Toledo [Allcock et al. 2002; 2011], of Looijenga and Swierstra [2007; 2008], and of Kondō [2000; 2002], "hidden" period maps are constructed in certain cases. The target spaces of these maps are certain *arithmetic quotients of complex unit balls*. The basic observation which is the starting point of this paper is that these arithmetic quotients can be interpreted as the complex points of certain *moduli spaces of abelian varieties of Picard type*, of the kind considered in [Kudla and Rapoport 2009]. Consequently, the purpose in this paper is to interpret these hidden period maps in moduli-theoretic terms. The payoff of this exercise is that we can raise and partially answer some *descent problems* which seem natural from our viewpoint, and which are related to a similar descent problem addressed by Deligne [1972] in his theory of *complete intersections of Hodge level one*.

Why do we speak of "hidden" or "occult" period maps in this context? This is done in order to make the distinction with the usual period maps which associate to a family of smooth projective complex varieties (over some base scheme S) the (polarized) Hodge structures of its fibers, which then induces a map from S to a quotient by a discrete group of a period domain. Let us recall three examples of classical period maps:

(1) **Case of quartic surfaces.** In this case, the period map is a holomorphic map of orbifolds

$$\varphi : \mathrm{Quartics}^{\circ}_{2,\mathbb{C}} \to \big[\Gamma \backslash V(2, 19)\big].$$

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Here $\mathrm{Quartics}^{\circ}_{2,\mathbb{C}}$ denotes the stack parametrizing smooth quartic surfaces up to projective equivalence,

$$\mathrm{Quartics}^{\circ}_{2,\mathbb{C}} = \big[\mathrm{PGL}_4 \backslash \mathbb{P}\mathrm{Sym}^4(\mathbb{C}^4)^{\circ}\big]$$

(stack quotient in the orbifold sense). The target space is the orbifold quotient of the space of oriented positive 2-planes in a quadratic space V of signature (2, 19) by the automorphism group Γ of a lattice in V.

(2) **Case of cubic threefolds.** In this case, the period map is a holomorphic map of orbifolds

$$\varphi : \mathrm{Cubics}^{\circ}_{3,\mathbb{C}} \to \big[\Gamma \backslash \mathfrak{H}_5\big].$$

Here $\mathrm{Cubics}^{\circ}_{3,\mathbb{C}}$ denotes the stack parametrizing smooth cubic threefolds up to projective equivalence. The target space is the orbifold quotient of the Siegel upper half-space of genus 5 by the Siegel group $\Gamma = \mathrm{Sp}_5(\mathbb{Z})$.

(3) **Case of cubic fourfolds.** In this case, the period map is a holomorphic map of orbifolds

$$\varphi : \mathrm{Cubics}^{\circ}_{4,\mathbb{C}} \to \big[\Gamma \backslash V(2, 20)\big].$$

Here $\mathrm{Cubics}^{\circ}_{4,\mathbb{C}}$ denotes the stack parametrizing smooth cubic fourfolds up to projective equivalence. The target space is the orbifold quotient of the space of oriented positive 2-planes in a quadratic space V of signature $(2, 20)$ by the automorphism group Γ of a lattice in V.

In the first case, by the Torelli theorem of Piatetski-Shapiro and Shafarevich, the induced map $|\varphi|$ on coarse moduli spaces is an open embedding. In the second case, by the Torelli theorem of Clemens and Griffiths, the map $|\varphi|$ is a locally closed embedding (it is not an open embedding since the source of φ has dimension 10, and the target has dimension 15). In the third case, by the Torelli theorem of Voisin, the map $|\varphi|$ is an open embedding.

The construction of the occult period maps is quite different, although it does use the classical period maps indirectly. For instance, the construction of Allcock, Carlson, and Toledo attaches a certain Hodge structure to any smooth cubic surface which allows one to distinguish between nonisomorphic ones, even though the natural Hodge structures on the cohomology in the middle dimension of all cubic surfaces are isomorphic. Also, in one dimension higher, their construction allows them to define an *open embedding* of the space of cubic threefolds into an arithmetic quotient of the complex unit ball of dimension 10.

Our second aim in this paper is to identify the complements of the images of occult period maps with *special divisors* considered in [Kudla and Rapoport 2009].

The layout of the paper is as follows. In Sections 2, 3 and 4, we recall some of the theory and notation of [Kudla and Rapoport 2009]. In Sections 5, 6, 7 and 8, respectively, we explain in turn the cases of cubic surfaces, cubic threefolds, curves of genus 3, and curves of genus 4. In Section 9, we explain the descent problem and solve it in zero characteristic. In the final section, we make a few supplementary remarks.

We stress that the proofs of our statements are all contained in the papers mentioned above, and that our work only consists in interpreting these results.

2. Moduli spaces of Picard type

Let $\boldsymbol{k} = \mathbb{Q}(\sqrt{\Delta})$ be an imaginary-quadratic field with discriminant Δ, ring of integers $O_{\boldsymbol{k}}$, and a fixed complex embedding. We write $a \mapsto a^\sigma$ for the nontrivial automorphism of $O_{\boldsymbol{k}}$.

For integers $n \geq 1$ and r, $0 \leq r \leq n$, we consider the groupoid $\mathcal{M} = \mathcal{M}(n-r, r) = \mathcal{M}(\boldsymbol{k}; n-r, r)$ fibered over $(\mathrm{Sch}/O_{\boldsymbol{k}})$ which associates to an $O_{\boldsymbol{k}}$-scheme S the groupoid of triples (A, ι, λ). Here A is an abelian scheme over S, λ is a principal polarization, and $\iota : O_{\boldsymbol{k}} \to \mathrm{End}(A)$ is a homomorphism such that

$$\iota(a)^* = \iota(a^\sigma),$$

for the Rosati involution $*$ corresponding to λ. In addition, the following signature condition is imposed:

$$\text{(2-1)} \qquad \mathrm{char}\big(T, \iota(a) \mid \mathrm{Lie}\, A\big) = (T - i(a))^{n-r} \cdot (T - i(a^\sigma))^r, \quad \text{for all } a \in O_{\boldsymbol{k}},$$

where $i : O_{\boldsymbol{k}} \to \mathbb{O}_S$ is the structure map.

We will mostly consider the complex fiber $\mathcal{M}_{\mathbb{C}} = \mathcal{M} \times_{\mathrm{Spec}\, O_{\boldsymbol{k}}} \mathrm{Spec}\, \mathbb{C}$ of $\mathcal{M}$. In any case, $\mathcal{M}$ is a Deligne–Mumford stack and $\mathcal{M}_{\mathbb{C}}$ is smooth. We denote by $|\mathcal{M}_{\mathbb{C}}|$ the coarse moduli scheme.

We will also have to consider the following variant, defined by modifying the requirement above that the polarization λ be principal. Let $d > 1$ be a square-free divisor of $|\Delta|$. Then $\mathcal{M}(\boldsymbol{k}, d; n-r, r)^* = \mathcal{M}(\boldsymbol{k}; n-r, r)^*$ parametrizes triples (A, ι, λ) as in the case of $\mathcal{M}(\boldsymbol{k}; n-r, r)$, except that we impose the following condition on λ. We require first of all that $\ker \lambda \subset A[d]$, so that $O_{\boldsymbol{k}}/(d)$ acts on $\ker \lambda$. In addition, we require that this action factor through the quotient ring $\prod_{p|d} \mathbb{F}_p$ of $O_{\boldsymbol{k}}/(d)$, and that λ be of degree d^{n-1} if n is odd and of degree d^{n-2} if n is even. In the notation introduced in Section 13 of [Kudla and Rapoport 2009], we have $\mathcal{M}(\boldsymbol{k}, d; n-r, r)^* = \mathcal{M}(\boldsymbol{k}, \mathbf{t}; n-r, r)^{*,\mathrm{naive}}$, where the function $\mathbf{t}$ on the set of primes p with $p \mid \Delta$ assigns to p the integer $2[(n-1)/2]$ if $p \mid d$, and 0 if $p \nmid d$. Note that if $\boldsymbol{k}$ is the Gaussian field $\boldsymbol{k} = \mathbb{Q}(\sqrt{-1})$, then necessarily $d = 2$; if $\boldsymbol{k}$ is the Eisenstein field $\boldsymbol{k} = \mathbb{Q}(\sqrt{-3})$, then $d = 3$. We denote by $|\mathcal{M}^*_{\mathbb{C}}|$ the corresponding coarse moduli scheme.

3. Complex uniformization

Let us recall from [Kudla and Rapoport 2009] the complex uniformization of $\mathcal{M}(\boldsymbol{k}; n-1, 1)(\mathbb{C})$ in the special case that $\boldsymbol{k}$ has class number one. For $n > 2$, let $(V, (\ ,\ ))$ be a hermitian vector space over $\boldsymbol{k}$ of signature $(n-1, 1)$ which contains a self-dual $O_{\boldsymbol{k}}$-lattice L. By the class number hypothesis, V is unique up to isomorphism. When n is odd, or when n is even and Δ is odd, the lattice L is also unique up to isomorphism. We assume that one of these conditions is satisfied. Let $\mathfrak{D}$ be the space of negative lines in the $\mathbb{C}$-vector space $(V_{\mathbb{R}}, \mathbb{I}_0)$, where the complex structure $\mathbb{I}_0$ is defined in terms of the discriminant of $\boldsymbol{k}$, as $\mathbb{I}_0 = \sqrt{\Delta}/|\sqrt{\Delta}|$. Let Γ be the isometry group of L. Then the complex uniformization is the isomorphism of orbifolds,

$$\mathcal{M}(\boldsymbol{k}; n-1, 1)(\mathbb{C}) \simeq [\Gamma\backslash\mathfrak{D}].$$

There is an obvious $*$-variant of this uniformization, which gives

$$\mathcal{M}(\boldsymbol{k}; n-1, 1)^*(\mathbb{C}) \simeq [\Gamma^*\backslash\mathfrak{D}],$$

where Γ^* is the automorphism group of the (*parahoric*) lattice L^* corresponding to the $*$-moduli problem. The lattice L^* is uniquely determined up to isomorphism by the condition that there is a chain of inclusions of $O_{\boldsymbol{k}}$-lattices $L^* \subset (L^*)^\vee \subset (\sqrt{d})^{-1}L^*$, with quotient $(L^*)^\vee/L^*$ of dimension $n-1$ if n is odd and $n-2$ if n is even, when localized at any prime ideal $\mathfrak{p}$ dividing d. Here, for an $O_{\boldsymbol{k}}$-lattice M in V, we write

$$M^\vee = \{x \in V \mid h(x, L) \subset O_{\boldsymbol{k}}\}$$

for the dual lattice.

4. Special cycles (KM-cycles)

We continue to assume that the class number of $\boldsymbol{k}$ is one, and recall from [Kudla and Rapoport 2009] the definition of special cycles over $\mathbb{C}$. Let (E, ι_0) be an elliptic curve with CM by $O_{\boldsymbol{k}}$ over $\mathbb{C}$, which we fix in what follows. Note that, due to our class number hypothesis, (E, ι_0) is unique up to isomorphism. We denote its canonical principal polarization by λ_0. For any connected $\mathbb{C}$-scheme S and any $(A, \iota, \lambda) \in \mathcal{M}(\boldsymbol{k}; n-1, 1)(S)$, let

$$V'(A, E) = \mathrm{Hom}_{O_{\boldsymbol{k}}}(E_S, A),$$

where $E_S = E \times_{\mathbb{C}} S$ is the constant elliptic scheme over S defined by E. Then $V'(A, E)$ is a projective $O_{\boldsymbol{k}}$-module of finite rank with a positive definite $O_{\boldsymbol{k}}$-valued hermitian form given by

$$h'(x, y) = \lambda_0^{-1} \circ y^\vee \circ \lambda \circ x \in \mathrm{End}_{O_{\boldsymbol{k}}}(E_S) = O_{\boldsymbol{k}}.$$

For a positive integer t, we define the DM-stack[1] $\mathcal{Z}(t)$ by

$$\mathcal{Z}(t)(S)=\big\{(A,\iota,\lambda;x)\mid(A,\iota,\lambda)\in\mathcal{M}(\boldsymbol{k};n-1,1)(S),\ x\in V'(A,E),\ h'(x,x)=t\big\}.$$

Then $\mathcal{Z}(t)$ maps by a finite unramified morphism to $\mathcal{M}(\boldsymbol{k};n-1,1)_{\mathbb{C}}$, and its image is a divisor in the sense that, locally for the étale topology, it is defined by a nonzero equation.

The cycles $\mathcal{Z}(t)$ also admit a complex uniformization. More precisely, under the assumption of the triviality of the class group of $\boldsymbol{k}$, we have

$$\mathcal{Z}(t)(\mathbb{C})\simeq\Big[\Gamma\backslash\Big(\coprod_{\substack{x\in L\\ h(x,x)=t}}\mathcal{D}_x\Big)\Big],$$

where $\mathcal{D}_x$ is the set of lines in $\mathcal{D}$ which are perpendicular to x.

Again, there is a $*$-variant of these definitions and a corresponding DM-stack $\mathcal{Z}(t)^*$ above $\mathcal{M}(\boldsymbol{k};n-1,1)^*$.

5. Cubic surfaces

In this paper we consider four occult period mappings. We start with the case of cubic surfaces, following [Allcock et al. 2002]; compare [Beauville 2009]. As explained in the introduction, in these sources, the results are formulated in terms of arithmetic ball quotients; here we use the complex uniformization of the previous two sections to express these results in terms of moduli spaces of Picard type.

Let $S\subset\mathbb{P}^3$ be a smooth cubic surface. Let V be a cyclic covering of degree 3 of $\mathbb{P}^3$, ramified along S. Explicitly, if S is defined by the homogeneous equation of degree 3 in 4 variables

$$F(X_0,\dots,X_3)=0,$$

then V is defined by the homogeneous equation of degree 3 in 5 variables,

$$X_4^3-F(X_0,\dots,X_3)=0.$$

Let $\boldsymbol{k}=\mathbb{Q}(\omega)$, $\omega=e^{2\pi i/3}$. Then the obvious μ_3-action on V determines an action of $O_{\boldsymbol{k}}=\mathbb{Z}[\omega]$ on $H^3(V,\mathbb{Z})$. For the (alternating) cup product pairing $\langle\ ,\ \rangle$,

$$\langle\omega x,\omega y\rangle=\langle x,y\rangle,$$

which implies that

$$\langle ax,y\rangle=\langle x,a^\sigma y\rangle,\quad\text{for all } a\in O_{\boldsymbol{k}}.$$

[1] This notation differs from that in [Kudla and Rapoport 2009], in that here the special cycles are defined over $\mathbb{C}$, and are considered as lying over $\mathcal{M}(\boldsymbol{k};n-1,1)_{\mathbb{C}}$.

Hence there is a unique $O_{\boldsymbol{k}}$-valued hermitian form h on $H^3(V,\mathbb{Z})$ such that

$$\langle x, y\rangle = \operatorname{tr}\Big(\frac{1}{\sqrt{\Delta}} h(x,y)\Big), \tag{5-1}$$

where the discriminant Δ of $\boldsymbol{k}$ is equal to -3 in the case at hand. Explicitly,

$$h(x,y) = \tfrac{1}{2}\big(\langle\sqrt{\Delta}x, y\rangle + \langle x, y\rangle\sqrt{\Delta}\big). \tag{5-2}$$

Furthermore, an $O_{\boldsymbol{k}}$-lattice is self-dual with respect to $\langle\ ,\ \rangle$ if and only if it is self-dual with respect to $h(\ ,\ )$.

Fact. *$H^3(V,\mathbb{Z})$ is a self-dual hermitian $O_{\boldsymbol{k}}$-module of signature* (4, 1).

As noted above, such a lattice is unique up to isomorphism.

Let

$$A = A(V) = H^3(V,\mathbb{Z})\backslash H^3(V,\mathbb{C})/H^{2,1}(V)$$

be the intermediate Jacobian of V. Then A is an abelian variety of dimension 5 which is principally polarized by the intersection form. Since the association $V \mapsto (A(V), \lambda)$ is functorial, we obtain an action ι of $O_{\boldsymbol{k}}$ on $A(V)$.

Theorem 5.1. (i) *The object (A,ι,λ) lies in $\mathcal{M}(\boldsymbol{k}; 4, 1)(\mathbb{C})$.*

(ii) *This construction is functorial and compatible with families, and defines a morphism of DM-stacks,*

$$\varphi : \mathrm{Cubics}^{\circ}_{2,\mathbb{C}} \to \mathcal{M}(\boldsymbol{k}; 4, 1)_{\mathbb{C}}.$$

Here $\mathrm{Cubics}^{\circ}_{2,\mathbb{C}}$ *denotes the stack parametrizing smooth cubic surfaces up to projective equivalence,*

$$\mathrm{Cubics}^{\circ}_{2,\mathbb{C}} = [\mathrm{PGL}_4\backslash\mathbb{P}\mathrm{Sym}^3(\mathbb{C}^4)^{\circ}]$$

(*stack quotient in the orbifold sense*).

(iii) *The induced morphism on coarse moduli spaces*

$$|\varphi| : |\mathrm{Cubics}^{\circ}_{2,\mathbb{C}}| \to |\mathcal{M}(\boldsymbol{k}; 4, 1)_{\mathbb{C}}|$$

is an open embedding. Its image is the complement of the image of the KM-cycle $\mathcal{Z}(1)$ in $|\mathcal{M}(\boldsymbol{k}; 4, 1)_{\mathbb{C}}|$.

Proof. We only comment on the assertions in (ii) and (iii). In (ii), the compatibility with families is always true of Griffiths' intermediate jacobians (which however are abelian varieties only when the Hodge structure is of type $(m+1,m)+(m,m+1)$). This constructs φ as a complex-analytic morphism. The algebraicity of φ then follows from Borel's theorem [1972] that any analytic family of abelian varieties over a $\mathbb{C}$-scheme is automatically algebraic. The fact that the image is contained in the complement of $\mathcal{Z}(1)$ is true because, by the Clemens–Griffiths theory, intermediate

Jacobians of cubic threefolds are simple as polarized abelian varieties, whereas over $\mathscr{X}(1)$, the polarized abelian varieties split off an elliptic curve. However, the fact that $\mathscr{X}(1)$ makes up the whole complement is surprising and results from the fact that the morphism φ extends to an isomorphism from a partial compactification $|\mathrm{Cubics}^{s}_{2,\mathbb{C}}|$ of $|\mathrm{Cubics}^{\circ}_{2,\mathbb{C}}|$ (obtained by adding *stable* cubics) to $|\mathcal{M}(\boldsymbol{k}; 4, 1)_{\mathbb{C}}|$, such that the complement of $|\mathrm{Cubics}^{\circ}_{2,\mathbb{C}}|$ in $|\mathrm{Cubics}^{s}_{2,\mathbb{C}}|$ is an irreducible divisor; see [Beauville 2009, Propositions 6.7 and 8.2]. $\square$

Remark 5.2. Let us comment on the stacks aspect of Theorem 5.1. Any automorphism of S is induced by an automorphism of $\mathbb{P}^3$, which in turn induces an automorphism of V. We therefore obtain a homomorphism $\mathrm{Aut}(S) \to \mathrm{Aut}(A(V), \iota, \lambda)$. The statement of [Allcock et al. 2002, Theorem 2.20] implies that this homomorphism induces an isomorphism

$$\mathrm{Aut}(S) \xrightarrow{\sim} \mathrm{Aut}(A(V), \iota, \lambda)/O_{\boldsymbol{k}}^{\times}, \tag{5-3}$$

where the units $O_{\boldsymbol{k}}^{\times} \simeq \mu_6$ act via ι on $A(V)$. Indeed, in [Allcock et al. 2002] it is asserted that φ is an open immersion of orbifolds $\mathrm{Cubics}^{\circ}_{2,\mathbb{C}} \to [P\Gamma\backslash\mathscr{D}]$, where $P\Gamma = \Gamma/O_{\boldsymbol{k}}^{\times}$; however, we were not able to follow the argument. Note that the orbifold $[P\Gamma\backslash\mathscr{D}]$ is different from $[\Gamma\backslash\mathscr{D}]$, which occurs in Section 3.

6. Cubic threefolds

Our next example concerns cubic threefolds, following Allcock et al. [2011] and Looijenga and Swierstra [2007].

Let $T \subset \mathbb{P}^4$ be a cubic threefold. Let V be the cyclic covering of degree 3 of $\mathbb{P}^4$, ramified in T. Then V is a cubic hypersurface in $\mathbb{P}^5$ and we define the primitive cohomology as

$$L = H^4_0(V, \mathbb{Z}) = \{x \in H^4(V, \mathbb{Z}) \mid (x, \rho) = 0\}, \tag{6-1}$$

where ρ is the square of the hyperplane section class. Note that $\mathrm{rk}_{\mathbb{Z}} L = 22$. Again, let $\boldsymbol{k} = \mathbb{Q}(\omega)$, with $\omega = e^{2\pi i/3}$, so that L becomes an $O_{\boldsymbol{k}}$-module. Now the cup product $(\ ,\ )$ on $H^4(V, \mathbb{Z})$ is a perfect *symmetric* pairing satisfying $(ax, y) = (x, a^{\sigma} y)$ for $a \in O_{\boldsymbol{k}}$. It induces on L a symmetric bilinear form $(\ ,\ )$ of discriminant 3. We wish to define an *alternating* pairing $\langle\ ,\ \rangle$ on L satisfying $\langle ax, y\rangle = \langle x, a^{\sigma} y\rangle$ for $a \in O_{\boldsymbol{k}}$. We do this by giving the associated $O_{\boldsymbol{k}}$-valued hermitian pairing $h(\ ,\ )$, in the sense of (5-1), defined by

$$h(x, y) = \tfrac{3}{2}\Big((x, y) + (x, \sqrt{\Delta} y)\frac{1}{\sqrt{\Delta}}\Big). \tag{6-2}$$

Here the factor $\frac{3}{2}$ is used instead of $\frac{1}{2}$ to have better integrality properties. Set $\pi = \sqrt{\Delta}$.

Fact. *For the pairing* (6-2), $L^\vee$ *contains* $\pi^{-1}L$ *with* $L^\vee/\pi^{-1}L \simeq \mathbb{Z}/3\mathbb{Z}$.

For this result, see [Allcock et al. 2011, Theorem 2.6 and its proof], as well as [Looijenga and Swierstra 2007, the passage below (2.1)].

Now consider the eigenspace decomposition of $H_0^4(V, \mathbb{C})$ under $\boldsymbol{k} \otimes \mathbb{C} = \mathbb{C} \oplus \mathbb{C}$.

Fact. *The Hodge structure of* $H_0^4(V, \mathbb{R})$ *is of type*

$$H_0^4(V, \mathbb{C}) = H^{3,1} \oplus H_0^{2,2} \oplus H^{1,3},$$

with $\dim H^{3,1} = \dim H^{1,3} = 1$. *Furthermore, the only nontrivial eigenspaces of the generator* ω *of* μ_3 *are*

$$H_0^4(V, \mathbb{C})_\omega = H^{3,1} \oplus (H_0^{2,2})_\omega, \quad \textit{with } \dim(H_0^{2,2})_\omega = 10,$$
$$H_0^4(V, \mathbb{C})_{\overline{\omega}} = (H_0^{2,2})_{\overline{\omega}} \oplus H^{1,3}, \quad \textit{with } \dim(H_0^{2,2})_{\overline{\omega}} = 10.$$

(See [Allcock et al. 2011, §2] and [Looijenga and Swierstra 2007, §4], respectively.)

Now set $\Lambda = \pi L^\vee$. Then we have the chain of inclusions of $O_{\boldsymbol{k}}$-lattices

$$\Lambda \subset \Lambda^\vee \subset \pi^{-1}\Lambda,$$

where the quotient $\Lambda^\vee/\Lambda$ is isomorphic to $(\mathbb{Z}/3\mathbb{Z})^{10}$, and where $\pi^{-1}\Lambda/\Lambda^\vee$ is isomorphic to $\mathbb{Z}/3\mathbb{Z}$. Let

$$A = \Lambda \backslash H_0^4(V, \mathbb{C})/H^-,$$

where

$$H^- = H^{3,1} \oplus (H_0^{2,2})_{\overline{\omega}}.$$

Note that the map $\Lambda \to H_0^4(V, \mathbb{C})/H^-$ is an $O_{\boldsymbol{k}}$-linear injection; hence A is a complex torus. In fact, the hermitian form h and its associated alternating form $\langle\ ,\ \rangle$ define a polarization λ on A. Hence A is an abelian variety of dimension 11, with an action of $O_{\boldsymbol{k}}$ and a polarization of degree 3^{10}. In fact, we obtain in this way an object (A, ι, λ) of $\mathcal{M}(\boldsymbol{k}; 10, 1)^*(\mathbb{C})$ (see Section 2 for the definition of the $*$-variants of our moduli stacks).

Theorem 6.1. (i) *The construction which associates to a smooth cubic* T *in* $\mathbb{P}^4$ *the object* (A, ι, λ) *of* $\mathcal{M}(\boldsymbol{k}; 10, 1)^*(\mathbb{C})$ *is functorial and compatible with families, and defines a morphism of DM-stacks*

$$\varphi : \mathrm{Cubics}^\circ_{3,\mathbb{C}} \to \mathcal{M}(\boldsymbol{k}; 10, 1)^*_{\mathbb{C}}.$$

(ii) *The induced morphism on coarse moduli spaces*

$$|\varphi| : |\,\mathrm{Cubics}^\circ_{3,\mathbb{C}}\,| \to |\mathcal{M}(\boldsymbol{k}; 10, 1)^*_{\mathbb{C}}|$$

is an open embedding. Its image is the complement of the image of the KM-cycle $\mathfrak{Z}(3)^*$ *in* $|\mathcal{M}(\boldsymbol{k}; 10, 1)^*_{\mathbb{C}}|$.

Proof. The compatibility with families is due to the fact that the eigenspaces for the μ_3-action and the Hodge filtration both vary in a holomorphic way. Point (ii) follows from [Allcock et al. 2011, Theorem 1.1] or [Looijenga and Swierstra 2007, Theorem 3.1]. □

Remark 6.2. The stack aspect is not treated in these sources. However, it seems reasonable to conjecture that the analogue of (5-3) is also true in this case, that is, that there is an isomorphism

$$\mathrm{Aut}(T) \xrightarrow{\sim} \mathrm{Aut}(A, \iota, \lambda)/O_{\boldsymbol{k}}^{\times}, \tag{6-3}$$

where (A, ι, λ) is the object of $\mathcal{M}(\boldsymbol{k}; 10, 1)^*_{\mathbb{C}}$ attached to T.

Remark 6.3. The construction of the rational Hodge structure $H^1(A, \mathbb{Q})$ from $H^4_0(V, \mathbb{Q})$ is a very special case of a general construction due to van Geemen [2001]. More precisely, it arises (up to Tate twist) as the *inverse half-twist* in the sense of [van Geemen 2001] of the Hodge structure $H^4_0(V, \mathbb{Q})$ with complex multiplication by $\boldsymbol{k}$. The *half-twist* construction attaches to a rational Hodge structure V of weight w with complex multiplication by a CM-field $\boldsymbol{k}$ a rational Hodge structure of weight $w+1$. More precisely, if Σ is a fixed half-system of complex embeddings of $\boldsymbol{k}$, then van Geemen defines a new Hodge structure on V by setting

$$V^{r,s}_{\mathrm{new}} = V^{r-1,s}_{\Sigma} \oplus V^{r,s-1}_{\overline{\Sigma}},$$

where V_Σ (resp. $V_{\overline{\Sigma}}$) denotes the sum of the eigenspaces for the $\boldsymbol{k}$-action corresponding to the complex embeddings in Σ (resp. in $\overline{\Sigma}$).

7. Curves of genus 3

Our third example concerns the moduli space of curves of genus 3 following Kondō [2000].

Let C be a non-hyperelliptic smooth projective curve of genus 3. The canonical system embeds C as a quartic curve in $\mathbb{P}^2$. Let $X(C)$ be the μ_4-covering of $\mathbb{P}^2$ ramified in C. Then the quartic $X(C) \subset \mathbb{P}^3$ is a K3-surface with an automorphism τ of order 4 and hence an action of μ_4. Let

$$L = \{x \in H^2(X(C), \mathbb{Z}) \mid \tau^2(x) = -x\}.$$

Let $\boldsymbol{k} = \mathbb{Q}(i)$ be the Gaussian field.

Fact. *L is a free $\mathbb{Z}$-module of rank* 14. *The restriction* $(\ ,\ )$ *of the symmetric cup product pairing to L has discriminant* 2^8; *more precisely, for the dual lattice L^* for the symmetric pairing,*

$$L^*/L \cong (\mathbb{Z}/2)^8.$$

(See [Kondō 2000, top of p. 222].)

Now consider the eigenspace decomposition of $L_{\mathbb{C}} = L \otimes \mathbb{C}$ under $\boldsymbol{k} \otimes \mathbb{C} = \mathbb{C} \oplus \mathbb{C}$, where $i \otimes 1$ acts via τ.

Fact. *The induced Hodge structure on* $L_{\mathbb{C}}$ *is of type*

$$L_{\mathbb{C}} = L^{2,0} \oplus L^{1,1} \oplus L^{0,2},$$

with $\dim L^{2,0} = \dim L^{0,2} = 1$. *Furthermore, the only nontrivial eigenspaces of* τ *are*

$$(L_{\mathbb{C}})_i = L^{2,0} \oplus (L^{1,1})_i, \quad \textit{with } \dim(L^{1,1})_i = 6,$$
$$(L_{\mathbb{C}})_{-i} = (L^{1,1})_{-i} \oplus L^{0,2}, \quad \textit{with } \dim(L^{1,1})_{-i} = 6.$$

We define an $O_{\boldsymbol{k}}$-valued hermitian pairing h on $L_{\mathbb{Q}}$ by setting

$$h(x, y) = (x, y) + (x, \tau y)i. \tag{7-1}$$

Then it is easy to see that the dual lattice $L^{\vee}$ of L for the hermitian form h is the same as the dual lattice L^* for the symmetric form.

Now set $\Lambda = \pi L^{\vee}$, where $\pi = 1 + i$. Then we obtain a chain of inclusions of $O_{\boldsymbol{k}}$-lattices

$$\Lambda \subset \Lambda^{\vee} \subset \pi^{-1}\Lambda,$$

where the quotient $\Lambda^{\vee}/\Lambda$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^6$, and where $\pi^{-1}\Lambda/\Lambda^{\vee}$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

Let

$$A = \Lambda \backslash L_{\mathbb{C}}/L^{-},$$

where

$$L^{-} = L^{2,0} \oplus (L^{1,1})_{-i}.$$

Note that the map $\Lambda \to L_{\mathbb{C}}/L^{-}$ is an $O_{\boldsymbol{k}}$-linear injection; hence A is a complex torus. In fact, the hermitian form h and its associated alternating form $\langle\ ,\ \rangle$ define a polarization λ on A. Hence A is an abelian variety of dimension 7, with an action of $O_{\boldsymbol{k}}$ and a polarization of degree 2^6. In fact, we obtain in this way an object (A, ι, λ) of $\mathcal{M}(\boldsymbol{k}; 6, 1)^*(\mathbb{C})$. Now [Kondō 2000, Theorem 2.5] implies the following theorem.

Theorem 7.1. (i) *The construction which associates to a non-hyperelliptic curve of genus* 3 *the object* (A, ι, λ) *of* $\mathcal{M}(\boldsymbol{k}; 6, 1)^*(\mathbb{C})$ *is functorial and compatible with families, and defines a morphism of DM-stacks*

$$\varphi : \mathcal{N}^{\circ}_{3,\mathbb{C}} \to \mathcal{M}(\boldsymbol{k}; 6, 1)^*_{\mathbb{C}}.$$

Here $\mathcal{N}^{\circ}_{3,\mathbb{C}}$ *denotes the stack of smooth non-hyperelliptic curves of genus* 3, *that is, of smooth non-hyperelliptic quartics in* $\mathbb{P}^2$ *up to projective equivalence.*

(ii) *The induced morphism on coarse moduli schemes* $|\varphi| : |\mathcal{N}^\circ_{3,\mathbb{C}}| \to |\mathcal{M}(\boldsymbol{k}; 6, 1)^*_{\mathbb{C}}|$ *is an open embedding. Its image is the complement of the image of the KM-cycle* $\mathfrak{X}(2)^*$ *in* $|\mathcal{M}(\boldsymbol{k}; 6, 1)^*_{\mathbb{C}}|$. □

Remark 7.2. Again, the stack aspect is not treated in [Kondō 2000]. It seems reasonable to conjecture that the analogue of (5-3) is also true in this case, that is, that there is an isomorphism

$$\mathrm{Aut}(C) \xrightarrow{\sim} \mathrm{Aut}(A, \iota, \lambda)/O_{\boldsymbol{k}}^{\times}, \tag{7-2}$$

where (A, ι, λ) is the object of $\mathcal{M}(\boldsymbol{k}; 6, 1)^*_{\mathbb{C}}$ attached to C, and where $O_{\boldsymbol{k}}^{\times} = \mu_4$.

8. Curves of genus 4

Our final example concerns the moduli space of curves of genus 4 and is also due to Kondō [2002].

Let C be a non-hyperelliptic curve of genus 4. The canonical system embeds C into $\mathbb{P}^3$. More precisely, C is the intersection of a smooth cubic surface S and a quartic Q which is either smooth or a quadratic cone. Furthermore, Q is uniquely determined by C. Let X be a cyclic cover of degree 3 over Q branched along C (if Q is singular, we take the minimal resolution of the singularities; see [Kondō 2002]). Then X is a K3-surface with an action of μ_3. Let

$$L = (H^2(X, \mathbb{Z})^{\mu_3})^{\perp}$$

be the orthogonal complement of the invariants of this action in $H^2(X, \mathbb{Z})$, equipped with the symmetric form (,) obtained by restriction.

Fact. *L is a free $\mathbb{Z}$-module of rank* 20, *with dual L^* for the symmetric form satisfying*

$$L^*/L \simeq (\mathbb{Z}/3\mathbb{Z})^2.$$

(See [Kondō 2002, top of p. 386].)

For $\boldsymbol{k} = \mathbb{Q}(\omega)$, $\omega = e^{2\pi i/3}$, we again define an alternating form $\langle\ ,\ \rangle$ through its associated $O_{\boldsymbol{k}}$-valued hermitian form h. Using the action of $O_{\boldsymbol{k}}$ on L, we set

$$h(x, y) = \tfrac{3}{2}\Big((x, y) + (x, \sqrt{\Delta}y)\frac{1}{\sqrt{\Delta}}\Big). \tag{8-1}$$

Set $\pi = \sqrt{\Delta}$.

Fact. *For the hermitian pairing* (8-1), *$L^\vee$ is an over-lattice of $\pi^{-1}L$ with*

$$L^\vee/\pi^{-1}L \simeq (\mathbb{Z}/3\mathbb{Z})^2.$$

Now consider the eigenspace decomposition of $L \otimes \mathbb{C}$ under $\boldsymbol{k} \otimes \mathbb{C} = \mathbb{C} \oplus \mathbb{C}$.

Fact. *The induced Hodge structure on $L_{\mathbb{C}}$ is of type*

$$L_{\mathbb{C}} = L^{2,0} \oplus L^{1,1} \oplus L^{0,2},$$

with $\dim L^{2,0} = \dim L^{0,2} = 1$. *Furthermore, the only nontrivial eigenspaces of μ_3 are*

$$(L_{\mathbb{C}})_{\omega} = L^{2,0} \oplus (L^{1,1})_{\omega}, \quad \textit{with } \dim(L^{1,1})_{\omega} = 9,$$
$$(L_{\mathbb{C}})_{\overline{\omega}} = (L^{1,1})_{\overline{\omega}} \oplus L^{0,2}, \quad \textit{with } \dim(L^{1,1})_{\overline{\omega}} = 9.$$

Now set $\Lambda = \pi L^{\vee}$. Then we have the chain of inclusions of $O_{\boldsymbol{k}}$-lattices

$$\Lambda \subset \Lambda^{\vee} \subset \pi^{-1}\Lambda,$$

where the quotient $\Lambda^{\vee}/\Lambda$ is isomorphic to $(\mathbb{Z}/3\mathbb{Z})^8$, and where $\pi^{-1}\Lambda/\Lambda^{\vee}$ is isomorphic to $(\mathbb{Z}/3\mathbb{Z})^2$.

Let

$$A = \Lambda \backslash L_{\mathbb{C}}/L^{-},$$

where

$$L^{-} = L^{2,0} \oplus (L^{1,1})_{\overline{\omega}}.$$

Then the map $\Lambda \to L_{\mathbb{C}}/L^{-}$ is an $O_{\boldsymbol{k}}$-linear injection; hence A is a complex torus. In fact, the hermitian form h and its associated alternating form $\langle\ ,\ \rangle$ define a polarization λ on A. Hence A is an abelian variety of dimension 10, with an action of $O_{\boldsymbol{k}}$ and a polarization of degree 3^8. In fact, we obtain in this way an object (A, ι, λ) of $\mathcal{M}(\boldsymbol{k}; 9, 1)^*(\mathbb{C})$.

Theorem 8.1. (i) *The construction which associates to a non-hyperelliptic curve of genus* 4 *the object* (A, ι, λ) *of* $\mathcal{M}(\boldsymbol{k}; 9, 1)^*(\mathbb{C})$ *is functorial and compatible with families, and defines a morphism of DM-stacks*

$$\varphi : \mathcal{N}^{\circ}_{4,\mathbb{C}} \to \mathcal{M}(\boldsymbol{k}; 9, 1)^*_{\mathbb{C}}.$$

Here $\mathcal{N}^{\circ}_{4,\mathbb{C}}$ *denotes the stack of smooth non-hyperelliptic curves of genus* 4.

(ii) *The induced morphism on coarse moduli schemes* $|\varphi| : |\mathcal{N}^{\circ}_{4,\mathbb{C}}| \to |\mathcal{M}(\boldsymbol{k}; 9, 1)^*_{\mathbb{C}}|$ *is an open embedding. Its image is the complement of the image of the KM-cycle* $\mathcal{Z}(2)^*$ *in* $|\mathcal{M}(\boldsymbol{k}; 9, 1)^*_{\mathbb{C}}|$. □

Remark 8.2. Again, the stack aspect is not treated in [Kondō 2002]. It seems reasonable to conjecture that the analogue of (5-3) is also true in this case, that is, that there is an isomorphism

$$\mathrm{Aut}(C) \xrightarrow{\sim} \mathrm{Aut}(A, \iota, \lambda)/O_{\boldsymbol{k}}^{\times}, \tag{8-2}$$

where (A, ι, λ) is the object of $\mathcal{M}(\boldsymbol{k}; 9, 1)^*_{\mathbb{C}}$ attached to C, and where $O_{\boldsymbol{k}}^{\times} = \mu_6$.

9. Descent

In all four cases discussed above, we obtain morphisms over $\mathbb{C}$ between DM-stacks defined over $\boldsymbol{k}$. These morphisms are constructed using transcendental methods. In this section we will show that these morphisms are in fact defined over $\boldsymbol{k}$. The argument is modeled on Deligne's solution [1972] of the analogous problem for complete intersections of Hodge level one, where he shows that the corresponding family of intermediate jacobians is an abelian scheme over the moduli scheme over $\mathbb{Q}$ of complete intersections of given multidegree.

In our discussion below, to simplify notations, we will deal with the case of cubic threefolds, as explained in Section 6; the other cases are completely analogous. Below we will shorten the notation Cubics_3° to $\mathscr{C}$, and consider this as a DM-stack over $\operatorname{Spec}\boldsymbol{k}$. Let $v : V \to \mathscr{C}$ be the universal family of cubic threefolds, and let $a : A \to \mathscr{C}_{\mathbb{C}}$ be the polarized family of abelian varieties constructed from V in Section 6. Hence A is the pullback of the universal abelian scheme over $\mathscr{M}(\boldsymbol{k}; 10, 1)^*_{\mathbb{C}}$ under the morphism $\varphi : \mathscr{C}_{\mathbb{C}} \to \mathscr{M}(\boldsymbol{k}; 10, 1)^*_{\mathbb{C}}$.

Lemma 9.1. *Let $b : B \to \mathscr{C}_{\mathbb{C}}$ be a polarized abelian scheme with $O_{\boldsymbol{k}}$-action, which is the pullback under a morphism $\psi : \mathscr{C}_{\mathbb{C}} \to \mathscr{M}(\boldsymbol{k}; 10, 1)^*_{\mathbb{C}}$ of the universal abelian scheme, and such that there exists ℓ and an $O_{\boldsymbol{k}}$-linear isomorphism of lisse ℓ-adic sheaves on $\mathscr{C}_{\mathbb{C}}$,*

$$\alpha_\ell : R^1 a_* \mathbb{Z}_\ell \simeq R^1 b_* \mathbb{Z}_\ell,$$

compatible with the Riemann forms on source and target. Then there exists a unique isomorphism $\alpha : A \to B$ that induces α_ℓ. This isomorphism is compatible with polarizations.

To prove this, we are going to use the following lemma. In it, we denote by Λ the hermitian $O_{\boldsymbol{k}}$-module $H^1(A_s, \mathbb{Z})$, for $s \in \mathscr{C}_{\mathbb{C}}$ a fixed base point. Recall from Section 6 that there is a chain of inclusions $\Lambda \subset \Lambda^\vee \subset \pi^{-1}\Lambda$, where $\pi = \sqrt{-3}$ is a generator of the unique prime ideal of $O_{\boldsymbol{k}}$ dividing 3.

Lemma 9.2. *Let $s \in \mathscr{C}_{\mathbb{C}}$ be the chosen base point.*

(i) *The monodromy representation $\rho_A : \pi_1(\mathscr{C}_{\mathbb{C}}, s) \to \mathrm{GL}_{\boldsymbol{k}}(\Lambda \otimes_{O_k} \boldsymbol{k})$ is absolutely irreducible.*

(ii) *For every prime ideal $\mathfrak{p}$ prime to* 3, *the monodromy representation*

$$\pi_1(\mathscr{C}_{\mathbb{C}}, s) \to \mathrm{GL}_{\kappa(\mathfrak{p})}(\Lambda/\mathfrak{p}\Lambda)$$

is absolutely irreducible.

(iii) *For the unique prime ideal $\mathfrak{p} = (\pi)$ lying over* 3, *the monodromy representation $\pi_1(\mathscr{C}_{\mathbb{C}}, s) \to \mathrm{GL}_{\kappa(\mathfrak{p})}(\Lambda/\mathfrak{p}\Lambda)$ is not absolutely irreducible, but there is a unique* *nontrivial stable subspace, namely, the* 10*-dimensional image of* $\pi\Lambda^\vee$ *in* $\Lambda/\pi\Lambda$.

Proof. The monodromy representations in question are induced by the composition of homomorphisms

$$\pi_1(\mathscr{C}_{\mathbb{C}}, s) \longrightarrow \pi_1\big(\mathcal{M}(\boldsymbol{k}; 10, 1)^*_{\mathbb{C}}, \varphi(s)\big) \longrightarrow \mathrm{GL}_{O_{\boldsymbol{k}}}(H^1(A_s, \mathbb{Z})). \tag{9-1}$$

Here by Theorem 6.1, and using complex uniformization (see Section 3), the first homomorphism is induced by the inclusion of connected spaces

$$\iota : \mathscr{D}\setminus\Big(\bigcup_{\substack{x\in L\\ h(x,x)=3}} \mathscr{D}_x\Big) \hookrightarrow \mathscr{D},$$

followed by quotienting out by the free action of Γ^*. Since $\mathscr{D}$ is simply connected, it follows that $\pi_1\big(\mathcal{M}(\boldsymbol{k}; 10, 1)^*_{\mathbb{C}}, \varphi(s)\big) = \Gamma^*$ and that the first homomorphism in (9-1) is surjective. Now, Γ^* can be identified with the group of unitary automorphisms of the *parahoric* lattice Λ, and it is elementary that the representations of Γ^* on $\Lambda \otimes_{O_{\boldsymbol{k}}} \boldsymbol{k}$ and on $\Lambda/\mathfrak{p}\Lambda$ for $\mathfrak{p}$ prime to 3 are absolutely irreducible (the latter since $\Lambda^\vee \otimes \mathbb{Z}_\ell = \Lambda \otimes \mathbb{Z}_\ell$ for $\ell \neq 3$). The statement (iii) is proved in the same way. □

Proof of Lemma 9.1. Let us compare the monodromy representations

$$\begin{aligned} \rho_A &: \pi_1(\mathscr{C}_{\mathbb{C}}, s) \to \mathrm{GL}_{O_{\boldsymbol{k}}}(H^1(A_s, \mathbb{Z})),\\ \rho_B &: \pi_1(\mathscr{C}_{\mathbb{C}}, s) \to \mathrm{GL}_{O_{\boldsymbol{k}}}(H^1(B_s, \mathbb{Z})). \end{aligned} \tag{9-2}$$

By hypothesis, these representations are isomorphic after tensoring with $\mathbb{Z}_\ell$. Hence, they are also isomorphic after tensoring with $\boldsymbol{k}$. Hence there exists a $\pi_1(\mathscr{C}_{\mathbb{C}}, s)$-equivariant $\boldsymbol{k}$-linear isomorphism

$$\beta : H^1(A_s, \mathbb{Q}) \simeq H^1(B_s, \mathbb{Q}).$$

By the irreducibility of the representation of $\pi_1(\mathscr{C}_{\mathbb{C}}, s)$ in $H^1(A_s, \mathbb{Q})$, β is unique up to a scalar in $\boldsymbol{k}^\times$. Let us compare the $O_{\boldsymbol{k}}$-lattices $\beta^{-1}(H^1(B_s, \mathbb{Z}))$ and $H^1(A_s, \mathbb{Z})$. Since we are assuming that $O_{\boldsymbol{k}}$ is a PID, after replacing β by a multiple $\beta_{\mathbb{O}} = c\beta$, we may assume that $L_B = \beta_{\mathbb{O}}^{-1}(H^1(B_s, \mathbb{Z}))$ is a primitive $O_{\boldsymbol{k}}$-sublattice in $\Lambda = H^1(A_s, \mathbb{Z})$. Let $\mathfrak{p}$ be a prime ideal in $O_{\boldsymbol{k}}$, and let us consider the image of L_B in $\Lambda/\mathfrak{p}\Lambda$. Since L_B is primitive in Λ, this image is nonzero. If $\mathfrak{p}$ is prime to 3, the irreducibility statement in (ii) of Lemma 9.2 implies that this image is everything, and hence $L_B \otimes O_{\boldsymbol{k},\mathfrak{p}} = \Lambda \otimes O_{\boldsymbol{k},\mathfrak{p}}$ in this case.

To handle the prime ideal $\mathfrak{p} = (\pi)$ over 3, we use the polarizations. By the irreducibility statement in (i) of Lemma 9.2, the polarization forms on $H^1(A_s, \mathbb{Q})$ and on $H^1(B_s, \mathbb{Q})$ differ by a scalar in $\mathbb{Q}^\times$ under the isomorphism $\beta_{\mathbb{O}}$. Now, by hypothesis on B, with respect to the polarization form on $H^1(B_s, \mathbb{Q})$, we have a chain of inclusions $L_B \subset L_B^\vee \subset \pi^{-1}L_B$ with respective quotients of dimension 10 and 1 over $\mathbb{F}_p$, just as for Λ. Since the two polarization forms differ by a scalar, this excludes the possibility that the image of L_B in $\Lambda/\pi\Lambda$ be nontrivial. It follows that $L_B = \Lambda$.

Furthermore, the isomorphism $\beta_{\mathbb{C}}$ is unique up to a unit in $O_{\boldsymbol{k}}^{\times}$, and it is an isometry with respect to both polarization forms. Now, by [Deligne 1971, 4.4.11 and 4.4.12], $\beta_{\mathbb{C}}$ is induced by an isomorphism of polarized abelian schemes. Finally, $\beta_{\mathbb{C}} \otimes_{\mathbb{Z}} \mathbb{Z}_\ell = \alpha_\ell$ up to a unit, since these homomorphisms differ by a scalar and both preserve the Riemann forms.

The uniqueness of α follows from Serre's Lemma. □

Now Lemma 9.1 implies that over any field extension k' of $\boldsymbol{k}$ inside $\mathbb{C}$, there exists at most one polarized abelian variety $b : B \to \mathscr{C}_{k'}$ obtained by pull-back from the universal abelian variety over $\mathcal{M}(\boldsymbol{k}; 10, 1)^*$, equipped with an $O_{\boldsymbol{k}}$-linear isomorphism of lisse ℓ-adic sheaves over $\mathscr{C}_{\mathbb{C}}$

$$R^1 a_* \mathbb{Z}_\ell \simeq R^1 b_{\mathbb{C}*} \mathbb{Z}_\ell,$$

preserving the Riemann forms. By the argument in [Deligne 1972, 2.2], this implies that, in fact, B exists (since it does for $k' = \mathbb{C}$). Hence the morphism φ is defined over $\boldsymbol{k}$. Put otherwise, for any $\boldsymbol{k}$-automorphism τ of $\mathbb{C}$, the conjugate embedding φ^τ, which corresponds to the conjugate $(A, \iota, \lambda)^\tau$, is equal to φ; hence φ is defined over $\boldsymbol{k}$.

Conjecture 9.3. *In all four cases above, the morphisms φ can be extended over $O_{\boldsymbol{k}}[\Delta^{-1}]$.*

Since we circulated a first version of our paper, this has been proved by J. Achter [2012] in the case of cubic surfaces.

10. Concluding remarks

Remark 10.1. In all four cases, the complement of $\mathrm{Im}(|\varphi|)$ is identified with a certain KM-divisor. In fact, for other KM-divisors, the intersection with $\mathrm{Im}(|\varphi|)$ sometimes has a geometric interpretation. For example, in the case of cubic surfaces, the intersection of $\mathrm{Im}(|\varphi|)$ with the image of the KM-divisor $\mathscr{Z}(2)$ in $|\mathcal{M}(\boldsymbol{k}; 4, 1)_{\mathbb{C}}|$ can be identified with the locus of cubic surfaces admitting Eckardt points; see [Dolgachev et al. 2005, Theorem 8.10]. Similarly, in the case of curves of genus 3, the intersection of $\mathrm{Im}(|\varphi|)$ with the image of $\mathscr{Z}(t)^*$ in $|\mathcal{M}(\boldsymbol{k}; 6, 1)^*_{\mathbb{C}}|$ can be identified with the locus of curves C where the K3-surface $X(C)$ admits a "splitting curve" of a certain degree depending on t; see [Artebani 2008, Theorem 4.6].

Remark 10.2. In [Dolgachev and Kondō 2007; Dolgachev et al. 2005; Matsumoto et al. 1992], occult period morphisms are often set in comparison with the Deligne–Mostow theory, which establishes a relation between configuration spaces (for example, of points in the projective line) and quotients of the complex unit ball by complex reflection groups, via monodromy groups of hypergeometric equations. This aspect of these examples has been suppressed entirely here. Also, it should be mentioned that there are other ways of constructing the period map for cubic surfaces; see, e.g., [Dolgachev and Kondō 2007; Dolgachev et al. 2005].

Remark 10.3. Let us return to Section 3. There we had fixed a hermitian vector space $(V, (\ ,\ ))$ over $\boldsymbol{k}$ of signature $(n-1, 1)$. Let V_0 be the underlying $\mathbb{Q}$-vector space, with the symmetric pairing defined by

$$s(x, y) = \mathrm{tr}(h(x, y)).$$

Then s has signature $(2(n-1), 2)$, and we obtain an embedding of $\mathrm{U}(V)$ into $\mathrm{O}(V_0)$. This also induces an embedding of symmetric spaces,

$$\mathscr{D} \hookrightarrow \mathscr{D}_{\mathrm{O}}, \tag{10-1}$$

where, as before, $\mathscr{D}$ is the space of negative (complex) lines in $(V_{\mathbb{R}}, \mathbb{I}_0)$, and where $\mathscr{D}_{\mathrm{O}}$ is the space of oriented negative 2-planes in $V_{\mathbb{R}}$. The image of (10-1) is precisely the set of negative 2-planes that are stable by $\mathbb{I}_0$. In the cases of the Gauss field and the Eisenstein field, this invariance is equivalent to being stable under the action of μ_4 or μ_6, respectively. Hence in these two cases, the image of (10-1) can also be identified with the fixed point locus of μ_4 or μ_6, respectively, in $\mathscr{D}_{\mathrm{O}}$.

Remark 10.4. By going through the tables in [Rapoport 1972, §2], one sees that there is no further example of an occult period map of the type above which embeds the moduli stack of *hypersurfaces* of suitable degree and dimension into a Picard type moduli stack of abelian varieties. Note, however, that, in the case of curves of genus 4, the source of the hidden period morphism is a moduli stack of *complete intersections* of a certain multidegree of dimension one, and there may be more examples of this type.

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STEPHEN KUDLA
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF TORONTO
40 ST. GEORGE ST., BA6290
TORONTO, ON M5S 2E4
CANADA
skudla@math.toronto.edu

MICHAEL RAPOPORT
MATHEMATISCHES INSTITUT
UNIVERSITÄT BONN
ENDENICHER ALLEE 60
53115 BONN
GERMANY
rapoport@math.uni-bonn.de

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A PROLOGUE TO "FUNCTORIALITY AND RECIPROCITY" PART I

ROBERT LANGLANDS

In memoriam — Jonathan Rogawski

Contents of Part I

Contents of Part II

1. Introduction

The first of my students who struggled seriously with the problems raised by the letter to André Weil of 1967 was Diana Shelstad in 1970–74, at Yale University and at the Institute for Advanced Study, who studied what we later called, at her suggestion and with the advice of Avner Ash, endoscopy, but for real groups. Although endoscopy for reductive groups over nonarchimedean fields was an issue from 1970 onwards, especially for SL(2), it took a decade to arrive at a clear and confident statement of one central issue, the fundamental lemma. This was given in my lectures at the École normale supérieure des jeunes filles in Paris in the summer

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of 1980, although not in the form finally proved by Ngô, which is a statement that results from a sequence of reductions of the original statement. The theory of endoscopy is a theory for certain pairs (H, G) of reductive groups. A central property of these pairs is a homomorphism $\phi_G^H : f^G \mapsto f^H$ of the Hecke algebra for $G(F)$ to the Hecke algebra for $H(F)$, F a nonarchimedean local field, and the fundamental lemma was an equality between certain orbital integrals for f^G and stable orbital integrals for f^H.

Clozel observed at an early stage that, with the help of the trace formula, it was sufficient to treat the case that f^H and f^G were both the unit element in the respective Hecke algebras. Other more difficult reductions came later, but in the seventies it was the fundamental lemma in a raw form, but for specific groups, that I proposed as a problem to a number of students who worked with me at the Institute for Advanced Study, although their formal advisors were elsewhere because the IAS had no graduate program: first Robert Kottwitz in 1976–77, Tate's student at Harvard; then Jonathan Rogawski, who received his degree in 1980; and later Thomas Hales, in the mid-eighties. Nicholas Katz was the formal advisor of both Rogawski and Hales. The experience was perhaps not entirely a happy one for at least two of the three students, but all survived to thrive as mathematicians. Jon left us too soon.

He had come to me on arriving at Princeton from Yale thanks to the advice of Serge Lang. It was Jon's ambition to become a number theorist, an ambition he ultimately realized, but the fundamental lemma for SU(2, 1) looked to him, with reason, to be far from real number theory. I think he would rather have proved the lemma wrong for SU(2, 1), abandoned the whole project, and gone on to something where elliptic curves figured more prominently. Fortunately, in my view, he never found a semisimple element for which the desired equality was false, proved the lemma for this group, and went on to write an extremely useful treatise on SU(2, 1), *Automorphic representations of unitary groups in three variables*, with very instructive examples of endoscopy, and then spent a good part of the remainder of his life with automorphic forms as an expression of the theory of numbers. Unfortunately, I never had an opportunity to discuss with him the very sophisticated, and very difficult, subsequent development of the fundamental lemma as a central element in the analytic theory of automorphic forms at the hands of Kottwitz, Waldspurger, Ngô and many others.

Indeed, we lived on opposite coasts of North America, and met only rarely, so that we never had an occasion to share our views on the changing face of the theory of automorphic forms in the years after late sixties when functoriality first appeared, together with some indications of reciprocity, or after the seventies, when the trace formula began to be used more systematically in the study of automorphic forms and of Shimura varieties. We were together for a conference on Picard modular surfaces in Montreal in the late eighties, at which his book was a central reference, but we were both too busy to have much time for conversation. Moreover, the subject was changing around both of us: the geometrical theory was growing at an astronomical rate; the genuinely arithmetical applications, such as Fermat's theorem were utterly unexpected; and the trace formula was being developed by Arthur not merely as an occasional tool but as an elaborate theory crying out for applications. In spite of many remaining points that are both obscure and difficult, many more, and more important, applications are in the offing.

That there are common threads running through this material and many later contributions, often referred to in the bulk as the Langlands program, was generally accepted, but, as I found when attempting — to some extent to indulge my vanity, because of the label, and to some extent for sentimental reasons, for it is also related to a number of topics that appealed to me in my early years as a mathematician but that I had never actively investigated — to acquire some understanding of the scope of the program at present, there is a great deal of confusion: the central issues are not always distinguished from the peripheral; partial results obtained by methods that are almost certainly dead-ends are offered with a frequently misplaced satisfaction; many suggestions are facile and, in my eyes, more than doubtful. Some of these shortcomings reflect the failings of our current mathematical culture; others may be inevitable in any cooperative intellectual effort. They are nonetheless troubling and, for the incautious, misleading. Some coherent reflection on the topic, its goals, its limitations at present, and achievements so far, is necessary. It is also difficult.

To write at this point a synopsis of the subject would be premature. Too much is left to do and my command of the material is inadequate. Nevertheless, I am trying to describe the goals of the theory and the methods with which they might be achieved — for my own satisfaction first of all, but secondly because the subject of automorphic representations and their applications appears to me central. As I attempted to explain in the essay "Is there beauty in mathematical theories?" [ND], it is the natural issue of several major currents in pure mathematics of the past two centuries: algebraic number theory; algebraic geometry; group representations — as created by Frobenius, Weyl and Harish-Chandra; and even a dollop of topological ideas, such as perverse sheaves. There is a speculative element in this attempt, and I try to be clear about it when the occasion arises. Nevertheless, the intention is to offer, when I can, possibilities that are not, in my view, impasses and that will lead to a theory at the level of its historical origins. If some results, even, or especially, much-acclaimed or important results, are not mentioned, it may because I see them as leading ultimately nowhere, not as an absolute conviction — absolute convictions are seldom useful — but as a suspicion; but it may also be because they refer to issues like endoscopy or the fundamental lemma, which are basic and important, but for reasons that are tactical more than strategic. Unfortunately, inadequate as it will be, there was no question of completing this description in time for it to appear in the present collection; there are far too many questions and difficulties on which I have hardly begun to reflect. At my age the future offers an uncertain quantity of time, so that whatever success I have will certainly be limited. Nevertheless, this memorial volume is an opportunity to describe and explain in a provisional and, at this time, necessarily incomplete form not only what I mean, in the context of the Langlands program — even in that part of it that owes little or nothing to me — by the two words *functoriality* and *reciprocity* — concepts that are maturing only slowly and in whose development Jon participated — but also how I expect them to be given a clear mathematical content. I apologize, once and for all, for the large tentative element that still remains not only in this prologue but also in the longer, more substantial text "Functoriality and Reciprocity" that it anticipates.

It is best to begin with a rough description of some basic concepts, concepts which it would be idle at this point to formulate too precisely, but which help in appreciating the structure of the theory we are attempting to construct. To introduce the notions of functoriality and reciprocity we need a crude notion of a mock Tannakian category: a generalization of the notion of the category of representations of a group.

Take, as an introductory example, G to be a group, for example, to be as simple as possible, a finite group. Suppose we have a family $\mathcal{G}^{\mathrm{range}}$ of groups and homomorphisms between then, for example the family $\{\mathrm{GL}(n, K) \mid n = 1, 2, \dots\}$, where K is a field, say the field of complex numbers. Consider the collection of homomorphisms $\varphi : G \to \mathrm{GL}(n, K)$ from G to an element of $\mathcal{G}^{\mathrm{range}}$. This is a Tannakian category in the sense of, for example, [M], a very simple one. The morphisms are given by $\varphi \to \phi \circ \varphi$, where ϕ is an algebraic homomorphism from $\mathrm{GL}(n, K)$ to $\mathrm{GL}(n', K)$.

A more sophisticated choice for $\mathcal{G}^{\mathrm{range}}$ would be the collection ${}^L\mathcal{G}(K/F)$ of all L-groups ${}^LG$ defined for a given extension K/F (see [BC]). The possible choices for these Galois extensions will be described later. The objects of the category will be pairs, the first element of which is an L-group ${}^LG$ in ${}^L\mathcal{G}(K/F)$ and the second an object whose nature depends on F. What is important is that these objects behave functorially: given a pair with first element ${}^LG$ and a homomorphism $\phi : G \to G'$ in ${}^L\mathcal{G}(K/F)$, thus a homomorphism for which the diagram

$$\begin{tikzcd} {}^LG \arrow[rr, "\phi"] \arrow[dr] & & {}^LG' \arrow[dl] \\ & \mathrm{Gal}(K/F) & \end{tikzcd} \tag{1.1}$$

is commutative, there is an image — of the given pair with first element ${}^LG$ — whose first element is ${}^LG'$. We have, for the moment, to be coy about the second element of the pairs because its nature depends on the nature of the field F, whether it is local or global, a field of algebraic numbers or the function field of an algebraic curve.

If K/F is given, the principal property of a mock Tannakian category — and the word “mock” is there to allow a certain latitude and a certain imprecision — is that there is a group $\mathfrak{G}$, usually not a group in ${}^L\mathcal{G}$ although it will have to be provided with a homomorphism $\mathfrak{G} \to \mathrm{Gal}(K/F)$, such that for any ${}^LG \in {}^L\mathcal{G}(K/F)$ the set of pairs with first element ${}^LG$ may be identified with — or, better, parametrized by — the homomorphisms $\varphi : \mathfrak{G} \to {}^LG$ for which the diagram

$$\begin{tikzcd} \mathfrak{G} \arrow[rr, "\varphi"] \arrow[dr] & & {}^LG' \arrow[dl] \\ & \mathrm{Gal}(K/F) & \end{tikzcd} \tag{1.2}$$

is commutative. The existence of this group is closely related to functoriality, usually by no means evident, and in many cases of great interest it is not yet established, although it is expected that it can, in the simpler local contexts, be identified with groups familiar from mid-twentieth century algebraic number theory or from geometry. For global fields, the group cannot be known except through the category it describes. We are striving for a notion that is, in one way, more general than that of a Tannakian category formulated in [M] and, in another, less broad. It is certainly at the moment much less precise.

Functoriality in the L-group appeared first, although not with that name and not so clearly circumscribed as at present, in my letter to Weil of 1967 [LW]. There is a reciprocity — of which, even today, only a small part has been realized — already implicit in functoriality, but a general form of reciprocity only appeared in connection with Shimura varieties, first with the theorem of Eichler–Shimura, but later in a bolder form, once the relation between the cohomology of general Shimura varieties and the discrete series of Harish-Chandra was clarified. The appropriate expression of its general form is one of the central issues of the program, an issue that, as will be intimated later in this prologue, not yet fully resolved, even conjecturally. Indeed in spite of the spectacular success of Wiles with the conjecture for elliptic curves to which the names of Taniyama and Shimura are attached, it is hardly broached. That the initial expression of reciprocity was the Artin reciprocity law for abelian characters of the Galois group of number fields, which itself had its historic source in the quadratic reciprocity law is, however, clear. In the letter to Weil, it was global functoriality that manifested itself as a possible strategy for the analytic continuation of automorphic L-functions. It can still be regarded as the only genuinely promising method for attacking this problem. Local functoriality only manifested itself later, as the consequences and the possibility of more precise formulations of global functoriality began to appear.

In a general context, in which all avatars of the original "program" are to be embraced not by an absolutely uniform collection of definitions and theorems, but by structures which do bear a family resemblance to each other, the origins are a less useful source of understanding than a few general concepts. There are three different theories that find a place in the program, each with a global and a local aspect. By and large, the local theory is a prerequisite for the global. Each of the three is attached to a different type of field. Globally these are (i) algebraic number fields of finite degree over $\mathbb{Q}$; (ii) function fields of algebraic curves over a finite field; (iii) function field of a complete nonsingular curve over $\mathbb{C}$. The corresponding local fields are (i) real, complex and p-adic local fields; (ii) fields of Laurent series over a finite field; (iii) fields of Laurent series over $\mathbb{C}$. The second type of field, a kind of poor relative, usually ignored, shares properties with both the first and the third, themselves quite different from each other in the details although with a common structure.

Let, for example, $F = F_v$ be a local field, for the moment the completion of a number field, and G a reductive group over F_v. Consider the collection of irreducible representations of $G(F_v)$. These are usually infinite-dimensional. The theory of irreducible representations of $G(F_v)$ is a theory that began with Dedekind, Frobenius, and Schur, and whose current structure, the structure with which we shall be dealing, owes an enormous debt to Élie Cartan, Weyl, and Harish-Chandra, but it is nonetheless a theory that is far from complete. We know more for $F = \mathbb{R}$, $\mathbb{C}$ than for a nonarchimedean F, but the theory appears to be similar for all.

An important observation is that the theory of which we speak is, for a given F_v, not unique. There are several possibilities. First of all, the representations being infinite-dimensional, there are technical constraints, discussed in all the standard texts: they are to be admissible. The notion of equivalent representation has also to be formulated carefully. That demands a good deal of understanding of the structure of the group, its subgroups, and its Lie algebra, all of which I take for granted. Secondly the theories for different groups should be fused. There are distinguished reductive groups: the quasisplit groups, even perhaps, in a pinch, the quasisplit simply connected groups. The reduction of the representation theory for general reductive groups to the theory for quasisplit groups is a part of endoscopy, for which the famous fundamental lemma is necessary and which is absolutely essential for the representation theory of reductive groups over local fields, those of the first two types and probably those of the third type as well. This reduction I do not emphasize; I take it for granted, simply confining myself to quasisplit groups. Moreover, even for quasisplit groups there is another consequence of the yet only very incompletely developed endoscopy that we accept: classes of irreducible representations and their characters are not the objects to which functoriality and reciprocity apply. It is stable classes of representations and, at least for fields of the first or second kinds, their characters that are pertinent. The local theory for function fields over $\mathbb{C}$ is only available in a nascent form, and it is by no means certain that characters have a role to play. It is hard — at the moment — to imagine that their equivalent will not appear. For local fields of the first two types, a stable class π^{st} consists of finitely many equivalence classes of representations and the character of a stable class is a sum

$$\chi_{\pi^{\mathrm{st}}} = \sum_{\pi \in \pi^{\mathrm{st}}} \alpha_\pi \chi_\pi ,$$

where the coefficients are often, perhaps always, integers and these stable characters are not merely class functions, which is what we normally expect from characters, but functions on stable conjugacy classes, stable conjugacy meaning — essentially — conjugacy in $G(F^{\mathrm{sep}})$, F^{sep} being the separable algebraic closure of F, of two elements in $G(F)$. In their full generality both functoriality and reciprocity are predicated on complete theories of endoscopy. Although we are far from possessing such theories, there are many questions related to the two notions on which we can reflect at present.

In order to describe the mock Tannakian categories that are of concern to us, we need to fix a field, either global or local. The first element of a pair is then the L-group ${}^LG$ associated to a reductive group G over F or to the quasisplit group associated to it. They are the same. The second, about which we have been until now reticent, is a stable conjugacy class π^{st} of irreducible representations of $G(F)$, if F is local, or of automorphic representations $G(\mathbb{A}_F)$, if F is global. As already observed, it is best to take G itself quasisplit, referring the rest to endoscopy.

The second element introduces new subtleties. Suppose, for example, that F is a local field and that we are dealing the first of the three types, so that F is the completion of a number field, even $\mathbb{R}$ or $\mathbb{C}$. Then there are several categories of irreducible admissible representation that can — and must — be distinguished: all; unitary; tempered; the Arthur class. For each of these classes, in so far as it is in the present context of any interest at all, there will be a mock Tannakian category, each a slight modification of the others. If we are dealing with all representations then in the semidirect product defining the group ${}^LG = \hat{G} \rtimes \mathrm{Gal}(K/F)$, the connected component $\hat{G}$ is the group of complex points of a reductive group. It is not clear that it is appropriate to consider the category of unitary representations in the context of functoriality. They are, as a class, recalcitrant, and it is very likely that only the Arthur class, of which tempered representations, which are unitary, form an important subclass, is needed. So it may be best to exclude the class of unitary representations as such. For tempered representations, $\hat{G}$ is taken to be the unitary form of $\hat{G}$ — with no change in notation. For the parametrization of the Arthur class, ${}^LG$ is presumably replaced by $\mathrm{SL}(2,\mathbb{C}) \times {}^LG$, but here again it is best to impose some growth conditions on the characters and some unitary condition on the parameters, thus, as it turns out, some growth conditions, not yet understood or formulated, on the matrix coefficients of the representation. The class of all representations has obscure aspects that remain unsettled. We can classify its elements, so that we have a notion of L-packet for them, but so far as I know, there is no stable theory available, even over $\mathbb{R}$; there are L-packets but, at this moment, no stable characters. They may not exist outside the Arthur class. Over nonarchimedean local fields, our ignorance is even more thorough. When the groups ${}^LG$ are replaced by their unitary form, the conditions on the homomorphisms between them are modified as well. For example, for tempered L-packets, homomorphisms from $\hat{H} \rtimes \mathrm{Gal}(K/F)$ to $\hat{G} \rtimes \mathrm{Gal}(K/F)$ restricted to $\hat{H}$ are homomorphisms from a compact group to a compact group.

In the simple example we gave of a Tannakian category, each morphism $\varphi : G \to \mathrm{GL}(n,K)$ represented something, namely itself, a linear representation of G. Composed with $\phi : \mathrm{GL}(n,K) \to \mathrm{GL}(m,K)$ it represents a second linear representation, this time m-dimensional. In the more general mock Tannakian categories, like those associated to ${}^L\mathcal{G}(K/F)$, and we may as well restrict our attention to it, each object is a pair, the first element of which is the L-group ${}^LG$, which determines and is uniquely determined by the corresponding quasisplit group G over F. There is also a second element, a stable conjugacy class π^{st} of irreducible representations $G(F)$, whose type must, as observed, be specified, whether all, tempered, or in the Arthur class. The geometric theory, as described in [CFT], is still immature, so that the possibilities for this second element are even less clear. We shall come to the geometric theory, with its many unresolved questions, later. For now it is best to concentrate on the arithmetic theory.

In order to be able to discuss reciprocity we need, whether at a local or global level, a group $\mathfrak{A}$ such that for a given group G or ${}^LG$ the stable conjugacy classes of irreducible or, globally, irreducible automorphic representations are represented by homomorphism of $\mathfrak{A}$ to ${}^LG$. This possibility was already mentioned, and it was intimated that to prove the existence of $\mathfrak{A}$, it was necessary to prove first that to any homomorphism $\phi : {}^LG_1 \to {}^LG_2$, there was associated a map $\Pi_\phi : \pi^{\mathrm{st}} \to \pi_\phi^{\mathrm{st}}$ of L-packets for G_1 to L-packets for G_2. This possibility I refer to as functoriality or, at more length, functoriality in the L-group.

Once functoriality in the L-group is proved, we shall be on the road to the proof of the existence of $\mathfrak{A}$, locally or globally, and for each kind of field. We have, as will be explained, to envisage different kinds of proof for the various types of fields. Before attempting to describe the possible nature of these proofs, I comment on the second principal topic of this prologue and of the essay to follow it: reciprocity.

We have suggested that the existence of groups $\mathfrak{A} = \mathfrak{A}_F = \mathfrak{A}_{K/F}$, one associated to each field F of the six various types of field, and, to be pedantic, to each sufficiently large Galois extension K of F, was the appropriate classification of representations either locally or globally. Indeed there are other constraints that have to be taken into account: first of all, whether the representations considered are tempered, of Arthur type, or, globally, of Ramanujan type, which entails, even for the same field, the introduction of more than one $\mathfrak{A}_F$; secondly, and this is important only in order not to be forced to pass to senselessly large inverse limits, we should consider the stable classes of representations generated by a finite set. This provides us with one ingredient of reciprocity. The other has been provided, at least partially, by two mathematicians: Galois in the early nineteenth century and Grothendieck in the late twentieth century. Galois groups and their importance are well understood; Grothendieck's notion of motivic Galois group is not well understood and not yet even in a satisfactory form. One task for mathematicians in the coming decades is to discover a better form. Whatever else, these are groups $\mathfrak{M} = \mathfrak{M}_F = \mathfrak{M}_{K/F}$ attached to the various fields. It can be said once again that they are only known through the objects they describe. Over local fields these groups are familiar, especially those for the fields of the first two types, and are known as Weil(–Shafarevich) groups. Globally, however, they are not and the function of reciprocity is to provide some understanding of them. It will be expressed as a homomorphism from $\mathfrak{A}_{K/F}$ to $\mathfrak{M}_{K/F}$, so that it attaches a representation of the former group to one of the latter. Reciprocity is of course the most abstruse, the most profound, and the most difficult of the topics discussed in this prologue and in the essay to follow. I do not expect to have much useful to write. So far as I can tell, we do not understand motives, not even hypothetically, and any real understanding of them requires the solution of major problems in algebraic and diophantine geometry. It would be presumptuous for me even to attempt to describe them at this moment.

I am not certain how it is best to refer to the various groups $\mathfrak{A}$ and $\mathfrak{M}$, in either their local or global forms. For lack of anything better, I shall refer to automorphic and motivic galoisian groups, the adjective *galoisian* indicating that the group describes some other kind of algebraic structure or is defined by it. It may be useful to observe immediately that, in the arithmetic theory, the relation between $\mathfrak{A}$ and $\mathfrak{M}$ is inevitably reflected in an important analytic object associated to irreducible representations and automorphic representations on one hand and motives on the other: L-functions. A homomorphism from $\mathfrak{A}$ to $\mathfrak{M}$ entails a mapping from *complex* representations of $\mathfrak{M}$ to *complex* representations of $\mathfrak{A}$. The definitions on the motivic side are delicate because of the intervention of ℓ-adic-representations. An ℓ-adic representation is not, at least not without further ado, a complex representation. Useful and important as ℓ-adic representations are — they are indeed indispensable —

some reflection is necessary before incorporating them into statements of reciprocity. I find that this preliminary reflection is often missing.

Further discussion of these questions will appear in the essay itself. What is important at present, especially for a reader who may not appreciate the need for the development of sound general concepts, is some understanding of how the general concepts are incorporated into the search for proofs. I begin with a brief list of the necessary steps, employing sometimes notions that have yet to be introduced.

(i) *The local theory over the real field.* What is needed is, first of all, to complete the theory for real groups developed by Harish-Chandra. This means, first of all, a theory of the Arthur class, and secondly a theory of stable transfer.

(ii) *The local theory over nonarchimedean fields.* It is again a matter of completing the theory created by Harish-Chandra, but, as he knew, he left the theory for p-adic fields in a form in which much that he had established over $\mathbb{R}$ was not yet available. Not only is there no theory for the Arthur class and no theory of stable transfer over p-adic fields, there is also no adequate tempered theory.

(iii) *The global theory for algebraic number fields.* In my view, which may not be unanimously shared, the only possibility is to pursue the suggestions of [FLN; BE; ST]. This is no easy in matter. It requires the local theories of (i) and (ii). Globally, it demands a completion of the analytic beginnings of [ST], thus some way of transforming the limits that appear in [ST] into a useful form. Efforts with some promise are being made, although not by me. I am keeping my fingers crossed that they succeed. These will be, at first, results only for $G = \mathrm{P\,GL}(2)$, but it is possible that they will substantially strengthen our confidence in the trace formula as the route to global functoriality. Moreover the creation of the theories of (i) and (ii) will make it possible to pursue the general global theory effectively. For $\mathrm{P\,GL}(2)$, there are two bench marks: (a) the second symmetric power and dihedral representations; (b) the fourth symmetric power and quaternionic representations. The second of these bench marks, if reached, would, I believe, encourage the search for concrete methods of counting fields with specific properties in a way that can be compared with the results reached analytically with the trace formula. This may more closely resemble the class field theory of the first half of the twentieth century than of the second.

(iv) *The local geometric theory.* This is the local theory for the field of Laurent series over $\mathbb{C}$. The fascination of contemporary mathematicians with sheaves has, on the one hand, encouraged the development of the local and global theories, but only in the context of spherical functions, which were also of considerable importance in the early years of representation theory for semisimple groups. It has, at the same time, inhibited the development of a theory with ramification, although not entirely [FG]. If this were remedied, the theory would be richer. The structure of a complete local theory is by no means evident, although there are some intimations of the form to be taken by reciprocity, or, better, of the form of the galoisian group $\mathfrak{A}_{\text{geom}}$.

(v) *The global geometric theory.* This is a theory strongly related to the theory of abelian integrals on one hand and the theory of ordinary differential equations with singularities on the other. As with the local theory, the contributions of algebraic geometers, among them Drinfeld, and of mathematical physicists, among them Witten, to the theory have greatly enriched it, but we do prefer a mathematical theory that includes ramification. The best I will be able to do in this prologue are some, with any luck instructive, observations not about reciprocity in a geometric context, where it may not exist, but about the new features that its relation to field theories reveal. I hope that, before coming to this part of the essay itself, I shall have acquired more familiarity and more understanding of the contributions of the mathematical physicists and the geometers.

(vi) *The p-adic theory and diophantine geometry.* These, or rather reciprocity, which can be considered the link between them and the theory of automorphic forms, have to be postponed to the second half of the prologue. It is not clear that, even with the longer period of time available to me for its preparation, I shall be able to write anything useful about these topics. I do hope, at least, to make the stakes clear.

I have no doubt that a lot of reflection will be necessary before the problems presented by (vi) can be broached in any serious way. Deep, quiet reflection over many years may be an indispensable preliminary. My thesis in this prologue and in the essay is that we have, nevertheless, enough tactical understanding to attack the other five problems successfully on a broad front now. Immediate victory is unlikely, but steady advances are not.

2. The local theory over the real field

For reasons connected with the Ramanujan conjecture and its generalizations and with the theory of Eisenstein series, the tempered irreducible representations of $G(F_v)$, F_v a local field, in particular, $F_v = \mathbb{R}$, are not adequate for the modern theory of automorphic forms. There is a larger class of irreducible representations needed that we have already introduced as the Arthur class. The simplest such representation is the trivial one-dimensional representation, which is present for every G and an important factor in the global analytic theory. We have also observed that the local group $\mathfrak{A}_{\mathbb{R}}$ for tempered representations is known to be the Weil group $W_{\mathbb{C}/\mathbb{R}}$, of which we then admit only representations with relatively compact image. The local group for Arthur classes over $\mathbb{R}$, at least for the analogues of tempered representations in the context of Arthur classes, is the group $\mathrm{SL}(2, \mathbb{C}) \times W_{\mathbb{C}/\mathbb{R}}$, or, perhaps better $\mathrm{SU}(2) \times W_{\mathbb{C}/\mathbb{R}}$, but they give equivalent results, and it is better to use the first, for which the Jacobson–Morozov theorem is more easily stated. It is also more concise, if less precise, to use the notation $W_{\mathbb{R}}$ rather than $W_{\mathbb{C}/\mathbb{R}}$ for the Weil group of $\mathbb{R}$.

When attempting to formulate the missing spectral theory for the Arthur packets, we will need to be aware of the need when applying the trace formula for a stable transfer of L-packets. Some very simple cases of this transfer were examined in [ST], but no general theory is available even over $\mathbb{R}$. It is closely related to the stable character for Arthur packets for a (quasisplit) group G that Arthur introduced with the packets in [A1] and their transfers from one group to another are presumably functorial with respect to homomorphisms from ${}^LH$ to ${}^LG$. There is, by the way, no need to introduce any kind of unitary constraint on these homomorphisms: if the image of φ is relatively compact, then so is the image of $\psi \circ \varphi$.

Our focus at the moment is the theory for the real field, which implicitly includes the theory for the complex field. Harish-Chandra's theory for tempered representations, which is the special case of

$$\varphi = \sigma \times \psi : \mathrm{SL}(2, \mathbb{C}) \times W_{\mathbb{R}} \to {}^LG, \qquad W_{\mathbb{R}} = W_{\mathbb{C}/\mathbb{R}}, \tag{2.1}$$

for which σ is trivial, will be in so far as possible the model. It will certainly be used. It is a spectral theory, thus an analytic theory, but it differs from the usual spectral theory. The space of functions to be decomposed is $L^2(G(\mathbb{R}))$, but, as on the line, it is really a more subtle space that is to be decomposed, a Schwartz space. The eigenfunctions or eigendistributions to be employed are invariant under conjugation, thus characters, which are tempered distributions on the Schwartz space. So there is a passage in the theory, not present in, for example, Fourier analysis on the line. From functions on the group, through orbital integrals, to functions on the semisimple conjugacy classes, which for a reductive group is itself a space easily enough described in terms of Cartan subgroups. There is also a passage backwards from distributions on the center to conjugation-invariant distributions on the group and then, by integration on parameters and convolution with functions on the maximal compact subgroup, to functions in the Schwartz space. This means, incidentally, that at this stage it is best not to work with stable packets but with the appropriate classes of irreducible representations, referred to by Harish-Chandra as tempered, those whose matrix-coefficients lie in the Schwartz space. All characters satisfy differential equations, differential equations whose solutions can be concretely described in terms of exponential functions, growth conditions, and jump conditions. Harish-Chandra recognized this. He recognized also, after many years of reflection, that this was all he needed to construct a complete spectral theory for tempered characters. For a more detailed description of Harish-Chandra's representation theory for real groups, I refer to Varadarajan's introduction to his collected works [HC].

In the homomorphism (2.1) σ has a different character than ψ. Only its conjugacy class under interior automorphisms of $\hat{G}$ is pertinent and these are finite in number and correspond to conjugacy classes of unipotent elements in $\hat{G}$ or nilpotent elements in its Lie algebra $\hat{\mathfrak{g}}$. It is usual to study the homomorphisms with a fixed σ and the associated class of representations $\Pi_\sigma(G)$ as a unit, it being understand that the image of ψ is relatively compact. For example, if σ is trivial, we are dealing with the class of tempered representations. To a pair consisting of a homomorphism

$$\phi = \sigma \times \psi : \mathrm{SL}(2, \mathbb{C}) \times {}^L H \to {}^L G$$

and a homomorphism

$$\varphi_H = \sigma_H \times \psi_H : \mathrm{SL}(2, \mathbb{C}) \times W_{\mathbb{C}/\mathbb{R}} \to {}^L H,$$

in which ψ has relatively compact image, we can associate

$$\varphi_G = \sigma_G \times \psi_G : \mathrm{SL}(2, \mathbb{C}) \to {}^L G,$$

in which σ_G is the composite of $\sigma_G \times \sigma_H \circ \phi_H$ with the diagonal imbedding $\mathrm{SL}(2, \mathbb{C}) \to \mathrm{SL}(2, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C})$ and $\psi_G = \psi \circ \psi_H$.

If we had the theory of stable characters for each Arthur class envisaged in [A1], then we would have the mapping that assigns to the stable character of $\pi^{\mathrm{st}}_{\varphi_H}$ on $H(\mathbb{R})$ the stable character of $\pi^{\mathrm{st}}_{\varphi_G}$ on $G(\mathbb{R})$. A grave question, or rather a question central for the trace formula, arises! Is there, for a given ϕ a dual mapping — or, better, correspondence because it will not be single-valued — from smooth functions f^G with compact support on $G(\mathbb{R})$ to smooth functions f^H with compact support on H, thus $f^G \to f^H$, such that

$$\int_{H(\mathbb{R})} f^H(h) \pi^{\mathrm{st}}_{\varphi_H}(h)\, dh = \int_{G(\mathbb{R})} f^G(g) \pi^{\mathrm{st}}_{\varphi_G}(g)\, dg,$$

for all ψ? This question was broached for a very special case in [ST]. It would be premature to attempt to discuss it further here. It is necessary to understand the transfer of stable characters. For this, the first step is to ask what must be done to establish the existence of $\pi^{\mathrm{st}}_{\varphi_H}$.

The stable character will be an eigendistribution, and thus, by an important theorem, an eigenfunction of the center of the universal enveloping algebra with eigenvalues that are given because the definition of Arthur prescribes one representation in it — or rather one stable packet in the sense of the Langlands classification of all irreducible representations, namely, the packet π^{st} associated, as in [A1], to the homomorphism

$$\phi_w = \sigma_G\left(\begin{pmatrix} |w|^{1/2} & 0 \\ 0 & |w|^{-1/2} \end{pmatrix}\right) \cdot \psi_G(w) \tag{2.2}$$

of $W_{\mathbb{R}}$ into ${}^LG$. The infinitesimal character of a representation with Langlands parameter given can be readily calculated from the parameter. So we know the infinitesimal character corresponding to π^{st} and thus that corresponding to the associated — conjecturally — Arthur packet. We can safely assume that all representations in it have the same infinitesimal character, for otherwise we will have no theory. We can study the papers of Harish-Chandra to learn how to calculate all possible eigenfunctions χ associated to this infinitesimal character. On each connected component of the regular elements in each Cartan subgroup T they have the form

$$\chi(t) = \frac{\sum_{w\in\Omega} a_w \exp(w\lambda X)}{|\Delta(t)|}, \tag{2.3}$$

where Ω is the Weyl group, X lies in the Lie algebra of $\mathfrak{t}$ of T, $t = \exp X$, λ is a complex linear form on $\mathfrak{t}$ and the a_w are complex constants. The function $|\Delta(t)|$ is defined as usual by a product of differences of roots. There are constraints attached to χ by the parameter φ_G, constraints studied carefully by Harish-Chandra when σ_G is trivial, thus when the packet is tempered. The constraints, basically on the growth of the function (2.3) and on the propagation of the constants across the subvarieties of singular elements, can be studied as for the tempered characters, although there will be complications that I am in no position to anticipate. They will have to be determined by experience and by the study of some low-dimensional cases.

The existence of the transfer $f^G \to f^H$ and its properties is not the only local problem raised in [ST] in connection with the combined use of the trace formula for G and Poisson summation formula on the Steinberg–Hitchin base. It was also necessary to understand the singularities of θ^G. Both questions are related to the asymptotic behavior of orbital integrals and stable orbital integrals $\mathrm{Orb}(\gamma_{\mathrm{st}}, f^G)$. I have not tried to reflect on them in any serious way.

3. The local theory over nonarchimedean fields

The problems are the same as for the real field; the difficulties are different. I — and, I suspect, many other people — find ourselves here face-to-face with our own ignorance, not just in one domain, but in several. Over both fields we are dealing with problems for characters. Over the real numbers, characters are solutions of a system of holonomic differential equations. Such systems are intimately related to perverse sheaves. In particular, for the complex numbers, the relation between functions and perverse sheaves is mediated by differential equations and belongs to a system of reference familiar to all mathematicians. For representation theory, the real field is more important, but we can overlook that for the sake of the analogy. Over nonarchimedean fields, characters are functions, but there is, as yet, no convenient characterization of them. We have to appeal to the original definition of Dedekind–Frobenius. There are perverse sheaves over these fields and, apparently, perverse sheaves on varieties over finite fields yield functions through the trace of Frobenius — or of the inverse Frobenius. So, if we are willing to overlook the difference between p-adic fields and finite fields, we have parallel constructions for the real field and for nonarchimedean fields

$$\begin{array}{ccccc} \text{perverse sheaves} & \longrightarrow & \text{differential equations and functions} & \longrightarrow & \text{characters} \\ \text{perverse sheaves} & \longrightarrow & \text{trace of Frobenius and functions} & \longrightarrow & \text{characters} \end{array}$$

The trick will be to discover how the perverse sheaves on the second line are to be defined and how they are to be calculated

In the theory of Harish-Chandra ([HC]), whether over an archimedean or over a nonarchimedean field, at least one of characteristic zero, the characters are distributions on $G(F)$ given by functions, or, more precisely, by the product of invariant, but singular, functions with the Haar measure. Over the real or complex field, these singular functions, as distributions, satisfy differential equations, which are — in some sense — holonomic. Since the distributions are invariant, the functions can be considered as functions on the (regular, semisimple) conjugacy classes, and the problem faced and solved by Harish-Chandra was to translate the differential equations satisfied by the characters into jump-conditions for these singular functions. For nonarchimedean fields, there will presumably be similar problems, but I am still uncertain of their nature and certainly in no position to attempt to solve them.

I content myself with a few remarks, influenced, but in no very precise way, by [Wa]. I have no grounds for taking them very seriously, nor do I have any genuine understanding of the necessary algebraic geometry. My goal is to complete the Harish-Chandra theory by finding a handle on the explicit forms of the characters over nonarchimedean fields for tempered representations and, more generally, for characters of representations in the Arthur class; my immediate question is whether, with this precise goal in mind, it is worthwhile to learn the theory of perverse ℓ-adic sheaves. We shall need sheaves on the Cartan subgroups of G and the functions are to be obtained by the traces of the Frobenius on the ℓ-adic cohomology of the fibers.

We also need to convert varieties over a nonarchimedean local field F_v of characteristic 0, or rather schemes over $\mathbb{O}_v$ with residue field κ_v. Let q be the number of elements in κ_v. A preliminary study of [Ha] suggests that Witt vectors are the appropriate instrument for this. The elements of $\mathbb{O}_v$ or, more generally, of the analogous ring $\bar{\mathbb{O}}_v$ in the maximal unramified extension $\bar{F}_v$ of F_v, can be written as series $(x_0, x_1, \dots)$ with coefficients in κ_v or $\bar{\kappa}_v$. This applies to the equations defining any scheme being considered. If the scheme A lies in an n-dimensional space $(X_1, \dots, X_m)$, and if we truncate the coefficients of the equations and of the variables after the mth variable, we obtain equations in $(m+1)n$ variables and schemes A_m, $m=0, 1, 2, \dots$. There is a morphism $A_{m+1}\to A_m$, $m=0, 1, \dots$. We might guess that for large m this will usually be smooth with fiber the n-dimensional affine space.

If there is a perverse ℓ-adic sheaf on the scheme being considered, we can think of restricting it to A_{m+1} and to the fibers of the morphism. If this restriction is a constant sheaf, just the pull-back of the restriction to the base point of the fiber, then the restriction has cohomology with compact support only in dimension 0. So the summation over the points in the fiber of the trace of the Frobenius is q^n. There is, however, something to remember. Although the character is a function, it always appears multiplied by a measure, either the Haar measure on the group or, if we pass to an integral not against a function f on $G(F)$, but against the orbital integral of F a measure on the Steinberg–Hitchin base or on a Cartan subgroup. The two are locally equivalent. A is either this base or the Cartan subgroup — give or take some singular subvarieties. The measure of a point on A_{m+1} is, up to a constant, $1/q^n$ times the measure of its image, so that the factors q^n and $1/q^n$ cancel each other. When passing to the Steinberg–Hitchin base, we multiply the character by $|\Delta(t)|$ and the measure is the measure on the Cartan subgroup, for which conventions have been established in [FLN]. The remaining factor in the measure is implicit in the orbital integrals.

I am tempted to think that the road to follow is already blazed in the literature. The theory over $\mathbb{R}$, with the Borel–Weil–Bott theorem, the homological realization of the discrete series verified by W. Schmid, Harish-Chandra's analytic construction of tempered representations from the discrete series, and the proof by Deligne–Lusztig of a conjecture of Macdonald, all point in the same direction: first introduce the characters of tori in a form adapted to the use of perverse sheaves, then combine it with some twisted form of parabolic induction — which can be formulated I suppose, in the context of perverse sheaves. This is not a matter of an effort lasting a few days or a few weeks, but unless the basic idea of using truncated Witt vectors is misguided, a careful study of the works mentioned should allow some progress.

I confess that I have never attempted, even in a modest experimental way, to examine the possibilities or to understand the initial difficulties when attempting to transpose the constructions in [DL] to a nonarchimedean context using truncated Witt vectors. To begin would be easy enough, as the only difficulty is to find the time, but the possible virtues of these secondary constructions was not apparent to me until I began to write this prologue and the essay on functoriality and reciprocity to follow, both a continuation of the reflections begun in [FLN; BE; ST]. In the following section, I simply take the existence of the necessary local theory as established. A good deal of it, not necessarily in the most suggestive form, is certainly available for $G = \mathrm{SL}(2)$ for which the global analytic and arithmetic problems are already daunting and well worth investigating.

4. The global theory for algebraic number fields

There are two aspects to the continuing reflections on the methods suggested in [BE]: the formal or structural aspects and the analytic aspects. The latter are extremely difficult. Ali Altuğ has been reflecting on them and I leave it to him to present, when it is appropriate, his conclusions. I concentrate on the former. The principal goal, indeed the overriding goal, is to establish functoriality and its consequences with the help of the trace formula and Poisson summation. The objects studied are the automorphic L-functions $L(s,\pi,\rho)$ associated to an automorphic representation π or, better, a stable class of automorphic representations π^{st} that contains π. It is their analytic properties that need to be studied, especially near $s=1$ or in the half-plane $\operatorname{Re} s>1$.

There are two possibilities: examine $L(s,\pi,\rho)$ itself or examine its logarithmic derivative. Although the logarithmic derivative contains in clearer form the pertinent information, the function $L(s,\pi,\rho)$ is the more accessible. So there is a difficult passage, as with the prime number theorem, from its study to that of $-L'(s,\pi,\rho)/L(s,\pi,\rho)$. This I leave, at least for the moment, to others and concentrate on the properties of $-L'(s,\pi,\rho)/L(s,\pi,\rho)$ that one hopes can be established and that lead to functoriality.

We anticipate a complete form of endoscopy, which is certainly available in some simple and instructive cases. With an appropriate choice of test functions, the stable trace formula leads to sums

$$\sum_{\pi^{\mathrm{st}}} m_{\pi^{\mathrm{st}}}\left\{\prod_{v\in S}\operatorname{tr}\pi_v^{\mathrm{st}}(f_v)\left\{\sum_{v\notin S}\ln L_v(s,\pi_v,\rho)\right\}\right\}. \tag{4.1}$$

Here S is an arbitrary finite collection of places, containing the infinite places, Each f_v is a smooth function with compact support, taken otherwise arbitrary, and ρ is an algebraic representation of ${}^LG$. There are loose ends, some terms missing, and some imprecision in the formula (4.1), but none of this is a serious issue for us here. The sum itself is over stable classes of representations unramified outside of S. So it is likely that only H that are unramified outside of S are pertinent.

The first serious issue is related to the generalized form of Ramanujan's conjecture and Arthur L-packets. The global L-packets are expected to be related to homomorphisms

$$\phi=\sigma\times\psi:\mathrm{SL}(2,\mathbb{C})\times{}^{\lambda}H\to{}^LG, \tag{4.2}$$

where, for the present purposes, we can quite comfortably write $\mathrm{SL}(2,\mathbb{C})\times{}^LH$, the necessity of modifying the L-groups ${}^LH$ slightly to ${}^{\lambda}H$ for technical homological reasons being one of the minor nuisances that plague the subject. Our principal purpose is to establish that the stable classes of automorphic representations can be written as a sum over the functorial transfers associated to (4.2) of the stable tempered automorphic representations of $H(\mathbb{A}_F)$. There will be ambiguities to be clarified.

One stable class π_G for G may be the image of several π_H. This is why we appeal in the essay to the notion of a hadronic or thick class introduced in [LSP]. We use only the transfers associated to classes that are hadronic. It is then to be expected that each class π_G^{st} is associated to a unique pair $\mathrm{SL}(2,\mathbb{C}) \times {}^L H$, although we will have to allow different ψ, for the reasons that can be inferred from [LP], and perhaps even different σ, although this is unlikely.

In the discussion of local packets and local transfers, we made it clear that the transfers associated to (4.2) are of tempered representations of H to representations of G that are tempered if and only if σ is trivial. The σ-factor is otherwise a measure of the extent to which the local images $\pi_{G,v}$ are not tempered. This is measured by the eigenvalues of

$$\rho\left(\sigma\left(\begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix}\right)\right)$$

in the various representations ρ of ${}^L G$. We want to sort the representations appearing in (4.1) according to type. This means we take the sum over pairs $\mathrm{SL}(2,\mathbb{C}) \times {}^L H$ and over conjugacy classes of ϕ but only include for a given such pair — if we include ϕ, such triple — hadronic π_H. Such a sum introduces multiplicities. It is natural to assume, and the evidence, such as it is, supports the assumption, that they are accommodated by the multiplicity with which various representations of $G(\mathbb{A}_F)$ occur in the space of automorphic forms.

So (4.1) should be equal to a sum, implicitly over triples $(\phi, \mathrm{SL}(2,\mathbb{C}) \times {}^L H)$,

$$\sum\Big\{\sum\nolimits_H^{\mathrm{temp}}\Big\{\prod_{v\in S} \operatorname{tr}\pi_v^{\mathrm{st}}\big(f_v^H\big)\Big\{\sum_{v\notin S}\ln L_v(s,\pi_v,\rho)\Big\}\Big\}\Big\} \tag{4.3}$$

It is understood that at each place f_v^H is the transfer in the sense of the previous sections of f^G. I have left out any reference to multiplicities on the assumption, made largely for the purposes of simplicity, that the multiplicities are largely caused by multiple homomorphisms ϕ. Any necessary corrections can be made when proofs have been found. What is important at the moment is to be clear about the structure proposed for the proof. For (4.1) there is a formula, the trace formula. In (4.3), the outer sum is over triples, the first inner sum, $\sum_H^{\mathrm{temp}}$ is over the stable tempered automorphic representations of $H(\mathbb{A}_F)$. We can assume by induction that for each triple, except the triple with $H = G$, thus with ϕ trivial on SL(2) and the identity on G itself, we have a formula for the inner sum $\sum_H^{\mathrm{temp}}$. This would yield a formula for the remaining term of the inner sum,

$$\sum\nolimits_{G}^{\mathrm{temp}}\left\{\prod_{v\in S}\operatorname{tr}\pi_v^{\mathrm{st}}\big(f_v^G\big)\left\{\sum_{v\notin S}\ln L_v(s,\pi_v,\rho)\right\}\right\}, \tag{4.4}$$

except that we would not know that the only automorphic representations that are not the image of a hadronic tempered automorphic representation with respect to some ψ with $H\neq G$ are themselves hadronic and tempered. However, in the terms of the sum (4.4) the first factor $\prod_{v\in S}\operatorname{tr}\pi_v^{\mathrm{st}}(f_v^G)$ is essentially arbitrary and serves to distinguish one π from another. So an understanding of (4.3) is essentially an understanding of the logarithmic derivative

$$\frac{d}{ds}\ln L_v(s,\pi_v,\rho)=\frac{L_v'(s,\pi_v,\rho)}{L_v(s,\pi_v,\rho)}. \tag{4.5}$$

The analytic problem is to show, with the aid of the formula for (4.4) just described, that it is holomorphic to the right of $\operatorname{Re}s=1$ for every ρ. This implies not only that the representation is tempered but that it is hadronic. This problem is central, very serious, and, in my view, it will be a matter of decades before it is solved in any generality. The method suggested in [FLN; ST] was to use the Poisson summation formula on the Steinberg–Hitchin base, but the hard questions were not broached.

Although it is premature to make too much of a fuss of the notion of hadronic representation, one observation is in order. If π_G^{st} is the image of π_H^{st} under the functorial transfer associated to ϕ in (4.2). Then

$$L(s,\phi_G^{\mathrm{st}},\rho)=L(s,\phi_H^{\mathrm{st}},\rho\circ\psi). \tag{4.6}$$

The representation $\rho\circ\phi$ of $\mathrm{SL}(2,\mathbb{C})\times{}^LH$ decomposes into a direct sum $\bigoplus_n\tau_n\otimes\rho_n$, where τ_n is the irreducible representation of $\mathrm{SL}(2,\mathbb{C})$ of degree n, which can be any positive integer. The L-function (4.6) is then given by

$$\prod_{n=1}^{\infty}\prod_{j=0}^{n}L(s-2j+n,\pi_H^{\mathrm{st}},\rho_n).$$

Each representation ρ_n is a direct sum of irreducible representations $\bigoplus_{i=1}^{m_n}\rho_{n,i}$. To show that (4.4) is holomorphic for $\operatorname{Re}s>1$ for every choice of S and every choice of the functions f_v, $v\in S$ is to show that $\dim\rho_n=0$ for $n>0$ and that for all ρ none of the representations $\rho_{0,i}$ is trivial. It may be appropriate to remind ourselves at this point that the L-groups that appear are defined with respect to any extension K/F, which can be arbitrarily large. Since H itself may be the group $\{1\}$, we will be dealing, in particular, with those representations that are attached to homomorphisms of the Galois group into ${}^LG$.

The solution of these analytic problems, even for very specific low-dimensional questions, for example, the existence of automorphic representations associated to quaternionic representations, can entail at least partial answers to the arithmetic questions raised in §2 and §3 and to their global forms. I, myself, find that these questions and their answers add considerably to the appeal of the algebraic theory of numbers [D; JY]. The Dedekind paper [D], which we shall review in the next section, is particularly charming. The arithmetic problems to be confronted and solved in the course of establishing global functoriality are nevertheless every bit as formidable, if not more so, than the analytic problems.

5. Classical algebraic number theory

There are two very different aspects of the construction of global functoriality proposed in the previous section: analytic and arithmetic. The analysis does not end with the introduction of the Steinberg–Hitchin base and the use of the Poisson summation formula; as in [FLN; ST], considerably more is needed. I hope that this will be explained in Altuğ's forthcoming thesis. As just intimated, there will also be arithmetic problems. In §4 it was explained that we expect, for a given G and each representation ρ of its L-group, thanks in part to the trace formula and Poisson summation, to be able to express the sum of the logarithmic derivatives of the L-functions $L(s,\pi,\rho)$ as a sum over imbeddings $\phi : \mathrm{SL}(2,\mathbb{C}) \times {}^L H \to {}^L G$, and in particular, with this in hand, to examine the asymptotic behavior of this sum as $s \to 1$. This will be complicated, because, for example, the L-groups ${}^L H = \hat{H} \rtimes \mathrm{Gal}(K/F)$ and ϕ can reduce to an imbedding of a Galois group in ${}^L G$. As a result the proposed analysis entails an understanding of such imbeddings. For abelian class field theory, this becomes an understanding of, say, the cyclic extensions of a given degree of the base field F. For the group GL(2) or PGL(2), it becomes an understanding of the imbeddings of Galois groups in GL(2) or SL(2). If ρ is the fourth symmetric power of the defining representation of SL(2), the most interesting possibility is that $\mathrm{Gal}(K/F)$ is imbedded as the quaternion group. Such extensions were studied, as observed in the previous section, not so long ago by Jensen and Yui, to whose paper my attention was drawn by Anthony Pulido. They were influenced by an earlier paper of Reichardt ([Re], see also [Ri]). There is a much earlier, more concrete paper by Dedekind ([D]), that it is worthwhile to review briefly, because, or so it seems to me, algebraic number theory in the, often concrete, style of Dedekind was abandoned after the Second World War, with the mounting enthusiasm in the USA and elsewhere for the more formal, more abstract styles of Artin and Chevalley. It may be that a successful attack in the spirit of §4 will demand a return to Dedekind.

The focus in Dedekind's paper is on quaternion extensions of $F = \mathbb{Q}$. Following his notation, let the quaternion group be formed by 1, a central element ϵ, $\epsilon^2 = 1$, and elements $\alpha, \beta, \gamma, \epsilon\alpha, \epsilon\beta, \epsilon\gamma$, $\alpha^2 = \beta^2 = \gamma^2 = \epsilon$, $\alpha\beta = \gamma = \epsilon\beta\alpha$. Any such extension contains a biquadratic extension, the fixed field H of ϵ. This is a field of the form $\mathbb{Q}(\sqrt{a}, \sqrt{b}, \sqrt{c})$, $c = ab$. Dedekind takes $a = 2$, $b = 3$, which pretty much leads to the minimal ramification of the field, which, as it turns out, has to be real. This is very convenient in connection with the trace formula. The field H is the only biquadratic field unramified outside of $\{2, 3\}$ and, using spherical functions outside of $\{2, 3\}$, in particular at the infinite place, we exclude all representations π with ramification outside this set. So our comparison will be very focussed. The field Ω whose Galois group is the quaternion group will be of the form $\Omega = H(\omega)$, $\omega^2 = \mu \in H$ and the problem is to determine those ω that lead to an Ω also unramified outside of $\{2, 3\}$. A helpful feature that simplifies the constructions, but that is not present in general, is that the class number of H is one. We shall verify this later.

We begin with some other considerations, more generally valid. After some hesitation, I chose to follow Dedekind’s convention of writing the action of the Galois group on the right. This is, otherwise, inconsistent with the notation of the paper, but without it the comparison with Dedekind’s paper is awkward. The elements $\omega\alpha$, $\omega\beta$, $\omega\gamma$ all lie in Ω and their squares all lie in H. Since they themselves do not lie in H, they all lie in $H\omega$. As a result, we obtain,

$$\omega\alpha = u\omega, \quad \omega\beta = v\omega, \quad \omega\gamma = w\omega, \qquad u, v, w \in H. \tag{5.1}$$

Moreover, $\omega\epsilon = -\omega$. Applying α to the first of the equations (5.1), we obtain $-\omega = \omega\epsilon = u\alpha u\omega$ or, as Dedekind writes, $u\alpha = -u^{-1}$. There is a collection of similar equations, verified in the same way, that Dedekind writes as

$$\begin{array}{lll} u\alpha = -u^{-1}, & u\beta = wv^{-1}, & u\gamma = -vw^{-1}, \\ v\alpha = -wu^{-1}, & v\beta = -v^{-1}, & v\gamma = uw^{-1}, \\ w\alpha = vu^{-1}, & w\beta = -uv^{-1}, & w\gamma = -w^{-1}, \\ \mu\alpha = \mu u^2, & \mu\beta = \mu v^2, & \mu\gamma = \mu w^2. \end{array} \tag{5.2}$$

If μ is replaced by $\mu\nu^2$, the extension does not change and u, v, w can be replaced by $u\nu\alpha/\nu$, $v\nu\beta/\nu$, $w\nu\gamma/\nu$. This allows simplifications. For example, Dedekind observes that if the class number of H is one, then we can assume that μ is integral and not divisible by the square of any prime ideal. This is then also true of its conjugates $\mu\alpha$, $\mu\beta$, and $\mu\gamma$, so that, thanks to the last line in (5.2), u, v, w must all be units. As a consequence, if μ is divisible by any prime ideal π, it is divisible by all the conjugates of that ideal. If, therefore, p is the prime number in $\mathbb{Q}$ that π divides and if p does not divide the discriminant, then p divides μ. Dedekind concludes that μ must be the product of a natural number m, a unit, and perhaps powers of the generators of the prime divisors of the discriminant. The pertinent information for our particular H is (i) its discriminant is $48^2 = 2^8 3^2$; (ii) the ideal

$(3) = (\sqrt{3})^2$ and $\sqrt{3}$ is a prime in H; (iii) the ideal (2) is the fourth power of the ideal $(1+\eta)$, $\eta = (1+\sqrt{3})/\sqrt{2}$, with $\eta^2 = 2+\sqrt{3}$, $\eta^{-2} = 2-\sqrt{3}$; (iv) the fundamental units in H are $a = 1+\sqrt{2}$, η, $\tau = \sqrt{2}+\sqrt{3}$, with inverses, $\sqrt{2}-1$, $(\sqrt{3}-1)/\sqrt{2}$, $\sqrt{3}-\sqrt{2}$. The possibilities for μ are therefore

$$\mu = \pm m a^{e_1} \eta^{e_2} \tau^{e_3} (1+\eta)^{e_4} (\sqrt{3})^{e_5}, \tag{5.3}$$

in which each e_i, $i = 1, \dots, 5$, is 0 or 1 and m is a natural number prime to 6 and a product of primes. Not all possible values of the exponents are admissible. Examining (5.3) on the basis of (i)–(iv), Dedekind arrives at the conclusion that $e_1 = e_2 = 1,\ e_3 = e_4 = 0,\ e_5 = 1$. As a consequence

$$\mu = \pm m a \eta \sqrt{3}. \tag{5.4}$$

I repeat his calculations. It is necessary to calculate $\mu\alpha/\mu$, $\mu\beta/\mu$, $\mu\gamma/\mu$ and to demand that they all be squares. For this, following Dedekind, we compute the Galois action on each factor of (5.3). We repeat that

$$\begin{aligned} (\sqrt{2}, \sqrt{3}, \sqrt{6}, \omega)\alpha &= (\sqrt{2}, -\sqrt{3}, -\sqrt{6}, u\omega), \\ (\sqrt{2}, \sqrt{3}, \sqrt{6}, \omega)\beta &= (-\sqrt{2}, \sqrt{3}, -\sqrt{6}, v\omega), \qquad \omega\epsilon = -\omega, \quad \mu\epsilon = \mu, \\ (\sqrt{2}, \sqrt{3}, \sqrt{6}, \omega)\gamma &= (-\sqrt{2}, -\sqrt{3}, \sqrt{6}, w\omega). \end{aligned}$$

The Galois action on the units is given by

$$\begin{array}{lll} a\alpha = a, & a\beta = -a^{-1}, & a\gamma = -a^{-1}, \\ \eta\alpha = -\eta^{-1}, & \eta\beta = -\eta, & \eta\gamma = \eta^{-1}, \\ \tau\alpha = -\tau^{-1}, & \tau\beta = \tau^{-1}, & \tau\gamma = -\tau. \end{array} \tag{5.5}$$

The first line follows from $(1+\sqrt{2})(1-\sqrt{2}) = -1$. The action of the Galois group takes η to $\pm(1 \pm \sqrt{3})/\sqrt{2}$ and

$$\frac{1+\sqrt{3}}{\sqrt{2}} \cdot \frac{1-\sqrt{3}}{\sqrt{2}} = -1,$$

thus η to $\pm\eta^{\pm 1}$. This is the second line. Since $(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2}) = 1$, the Galois group also takes τ to $\pm\tau^{\pm 1}$. This is the third line.

The action of the Galois group on $\sqrt{3}$ is given by $(\sqrt{3})\alpha = -\sqrt{3}$; $(\sqrt{3})\beta = \sqrt{3}$; $(\sqrt{3})\gamma = -\sqrt{3}$. The second line yields immediately a first form for the Galois action on the supplementary prime $1+\eta$ that divides 2,

$$(1+\eta)\alpha = -\eta^{-1}(1-\eta), \quad (1+\eta)\beta = (1-\eta), \quad (1+\eta)\gamma = \eta^{-1}(1+\eta). \tag{5.6}$$

Each of these numbers are units. It only remains to express them as products of powers of the fundamental units times $1+\eta$.

$$\frac{1-\eta}{1+\eta}=\frac{\sqrt{2}-1-\sqrt{3}}{\sqrt{2}+1+\sqrt{3}}=\frac{(\sqrt{2}-1-\sqrt{3})^2}{2-(1+\sqrt{3})^2}=-\frac{2-2\sqrt{2}(1+\sqrt{3})+(1+\sqrt{3})^2}{2+2\sqrt{3}}.$$

Multiplying numerator and denominator by $1-\sqrt{3}$, we obtain

$$-\frac{2(1-\sqrt{3})-2\sqrt{2}(1-3)-2(1+\sqrt{3})}{2(1-3)}=-(\sqrt{2}-\sqrt{3})=-\tau^{-1}.$$

Thus the three numbers (5.6) are $1+\eta$ times, respectively,

$$\eta^{-1}\tau^{-1},\quad -\tau^{-1},\quad \eta^{-1}.$$

From these relations, we conclude with Dedekind that

$$\mu\alpha=\pm ma^{e_1}(-\eta)^{-e_2-e_4}(-\tau)^{-e_3-e_4}(1+\eta)^{e_4}(-\sqrt{3})^{e_5},$$

the sign being the same as in (5.3), and that

$$u^2=\frac{\mu\alpha}{\mu}=(-1)^{e_2+e_3+2e_4+e_5}\eta^{-2e_2-e_4}\tau^{-2e_3-e_4}. \tag{5.7}$$

For this to be a square it is necessary and sufficient that e_4 be 0 and $e_2+e_3+e_5\equiv 0 \pmod 2$. Further conditions are given by $\mu\beta/\mu$. Since

$$\mu\beta=\pm m(-a)^{-e_1}(-\eta)^{e_2}(\tau)^{-e_3-e_4}(1+\eta)^{e_4}$$

and $e_4=0$, the quotient $v^2=\mu\beta/\mu$ is

$$(-1)^{e_1+e_2}a^{-2e_1}\tau^{-2e_3}. \tag{5.8}$$

For this to be a square $e_1+e_2\equiv 0 \pmod 2$. Thus $e_1=e_2$.

The first two of the equations in the last line of (5.2) imply the third. They imply as well that Ω is a quadratic extension of H, Galois over $\mathbb{Q}$. They do not imply that Ω is a quaternion extension of $\mathbb{Q}$. For that we need the earlier lines, which assure us that this is so. Dedekind uses the first two of the three diagonal equations, which must imply all nine equations because the first completely defines the action of α on Ω and the second that of β. Consider the first diagonal equation. The number u is the square root of (5.7). The information at our disposition yields

$$u=(\pm)'\eta^{-e_1}\tau^{-e_3},$$

where, following Dedekind, we have explicitly indicated with a prime that the sign appearing here is not the sign in (5.3). The first diagonal equation yields

$$\begin{aligned}u\alpha=(\pm)'(-\eta^{-1})^{-e_1}(-\tau^{-1})^{-e_3}&=(\pm)'(-1)e_1+e_3\eta^{e_1}\tau^{e_3}\\&=-u^{-1}=-(\pm)'\eta^{e_1}\tau^{e_3},\end{aligned}$$

from which we conclude that $e_1 + e_3 \equiv 1 \pmod 2$. This implies that $e_5 = 1$ and that (e_1, e_3) is either $(1, 0)$ or $(0, 1)$. Dedekind settles the matter with the second diagonal equation.

The element v is the square root of (5.8), $v = (\pm)'' a^{-e_1}\tau^{-e_3}$ and, thanks to (5.5),

$$\begin{aligned} v\beta = (\pm)''(-a)^{e_1}\tau^{e_3} &= (\pm)''(-1)^{e_1}a^{e_1}\tau^{e_3} \\ &= -v^{-1} = -(\pm)''a^{e_1}\tau^{e_3}. \end{aligned}$$

We infer that $e_1 = 1$, and therefore that $e_3 = 0$, arriving finally at Dedekind's conclusion (5.4).

Dedekind does not offer any hints for the verification that the class number is one. So we apply the standard theorems. Since there are a number of other points about the field H to be verified, we postpone this until the end of the section and explain first the pertinence of the quaternionic fields to the study of the trace formula and its applications.

There are two tests that may be undertaken to persuade oneself of the validity of the strategy proposed in §4 and of Altuğ's analytic development of the necessary analysis. He, himself, has begun to reflect on them. The two tests are: the application to dihedral automorphic representations and the possible application to quaternionic representations. The interest is less in the results and more in the conviction to be obtained that the methods proposed, although difficult, are sound. As explained in §4, the method, as so often with L-functions, is focussed on the behavior of $-dL(s, \pi, \rho)/ds$ as $s \searrow 1$, or, rather, assuming for simplicity that $F = \mathbb{Q}$, on that of

$$\sum_{\pi}\left\{\operatorname{tr}\pi(f_\infty^\rho) + \sum_p \sum_{n=1}^{\infty} \frac{n \ln p}{p^{ns}} \operatorname{tr}\pi_v(f_v^n)\right\}, \tag{5.9}$$

where outside of a finite set S of places v, the functions f_v^n are chosen to be spherical functions such that

$$\operatorname{tr}\pi_v^n = \rho(A^n(\pi_v)),$$

where $A(\pi_v)$ is the Frobenius–Hecke class attached to π_v. The representation ρ is a representation of ${}^LG$. The development of the stable trace formula described in §4 allows for the removal of all nontempered π from (5.9), thus of all stable π whose parameter contains a nontrivial $\mathrm{SL}(2)$ component. It is understood that these have been removed, so that the remaining sum has no singularities to the right of $\operatorname{Re} s = 1$. It will be part of the analysis to show this! It is moreover expected, and will have to be shown, that only those π associated to a homomorphism $\phi : {}^\lambda H \to {}^LG$ whose image is a proper subgroup of ${}^LG$ will contribute to the pole at $s = 1$. The representations of $G(\mathbb{A}_F)$, thus of $H(\mathbb{A}_F)$, are to be understood inductively. For the two tests, we take $G = \mathrm{PGL}(2)$ or, but that would be slightly more elaborate, GL(2). The L-groups are SL(2) or GL(2) or, better because we must consider all possible H, SL(2) $\times$ Gal($K/\mathbb{Q}$) or GL(2) $\times$ Gal($K/\mathbb{Q}$). We consider only the first.

If $\pi = \pi_G$ is the image of π_H, then the contribution of π to the sum (5.9) is $1/(s-1)$ times the multiplicity of the trivial representation of ${}^\lambda H$ in $\rho \circ \phi$. To avoid redundancy, we always suppose π_H is hadronic. In particular, if ρ is irreducible and nontrivial, as we may as well suppose, there is no contribution from any hadronic π_G. For P GL(2) or GL(2), say GL(2) because this allows a simpler notation, this means any one of the following three possibilities. First ${}^\lambda H = {}^L H$, $H = \mathrm{GL}(1) \times \mathrm{GL}(1)$. For the second there is a quadratic extension E of $\mathbb{Q}$, H is the two-dimensional torus obtained from GL(1) by restriction of scalars from E to $\mathbb{Q}$, ${}^\lambda H = {}^L H$, and

$$\begin{aligned} &\phi : (a, b) \rtimes 1 \mapsto \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \quad a, b \in \mathbb{C},\ 1 \in \mathrm{Gal}(E/\mathbb{Q}), \\ &\phi : (1, 1) \rtimes \sigma \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \end{aligned} \tag{5.10}$$

where $\mathrm{Gal}(E/\mathbb{Q}) = \{1, \sigma\}$. The representation π_H is attached to a homomorphism φ of the global Weil group

$$\{1\} \to E^\times \backslash I_E \to W_{E/F} \to \mathrm{Gal}(E/F) \to \{1\}$$

into ${}^L H$ and this homomorphism is defined by a character χ of $E^\times \backslash I_E$. If w is a fixed element in $W_{E/F}$, $w \notin I_E$, then

$$\varphi : \begin{cases} \alpha \in I_E \mapsto (\chi(\alpha), \chi(\sigma\alpha)) \rtimes 1, \\ \qquad w \mapsto (\chi(w^2), 1) \rtimes \sigma. \end{cases} \tag{5.11}$$

The third possibility is that H=$\{1\}$. There is overlapping of all three cases.

The first case leads to noncuspidal representations and is thus understood. The third case is most interesting when we take ${}^\lambda H$ to be a Galois group, especially when this group is tetrahedral, octahedral or icosahedral. We have not reached the stage where they can be treated by the methods under discussion. The overlapping occurs when the Galois group is a finite dihedral group and, in particular, when it is a quaternion group.

Consider first the case (ii) and let ρ be the $2n$-th symmetric power of the defining representation of GL(2). Then the L-function $L(s, \pi_G, \rho) = L(s, \pi_H, \rho \circ \phi)$ will be the product of the L-function of $\mathbb{Q}$ associated to the character $\chi|I_{\mathbb{Q}}$ and the L-functions of the field E associated to the characters $\chi^2, \dots, \chi^n$. The first of these functions has a pole of order 1 at $s = 1$ if and only if $\chi|I_{\mathbb{Q}}$ is trivial. So for any natural number n, these functions will contribute a pole at $s = 1$ to the sum (4.3), in particular for $n = 1$. For the results achieved by the method of §4 for $n = 1$, it will be best to refer to Altuğ’s thesis. Since they concern functoriality only for the group GL(2) and a torus H, a case that can be treated in the context of the Hecke theory, they may not convince the sceptical, however interesting they may be for those whose concern is with functoriality and its consequences in general. It is only for exceptional n and very exceptional H and ϕ that further poles appear.

One possibility is that H is associated to a quadratic extension, χ is exactly of order 4, $\chi(\alpha)$ is not identically equal to $\chi(\sigma\alpha)$, and $\chi(w^2) = -1$. Of course, π_H is then not hadronic, but π_G is also associated to another group H, the group $H = \{1\}$, and extensions $K/\mathbb{Q}$ with $\mathrm{Gal}(K/\mathbb{Q})$ isomorphic to the image of the original ${}^LH$ under $\phi \circ \varphi$. This image is a group of order 8, isomorphic to the group

$$\left\{\begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix}\right\} \cup \left\{\begin{pmatrix} 0 & a \\ -\bar{a} & 0 \end{pmatrix}\right\}, \quad a^4 = 1.$$

This is the quaternion group imbedded in SL(2). In addition to the irreducible representation this yields, the group has four one-dimensional representations, the trivial representation and the three nontrivial characters of the group divided by its center, which is, of course, $\pm I$. The even symmetric powers of the two-dimensional representation are clearly the direct sums of characters. Since the quaternion group has a group of outer automorphisms of order three, $\alpha \to \beta \to \gamma \to \alpha$, all three nontrivial characters appear with the same multiplicity μ and the trivial representation then appears with multiplicity $\nu = 2n + 1 - 3\mu$. For $n = 1$, $\mu = 1$, $\nu = 0$; for $n = 2$, $\mu = 1$, $\nu = 2$. This means that if ρ is the fourth symmetric power and π_G is the image of the trivial representation of π_H, then $L(s, \pi_G, \rho)$ has a pole of order two at $s = 1$. This will not be so for hadronic π_G, nor for other dihedral π_G, nor for tetrahedral, octahedral, or icosahedral π_G. For these, as observed in [BE], the exceptional poles at $s = 1$ begin only with higher values of n.

Thus the method of §4, if it is to work at all, must detect the quaternionic representations — and only the quaternionic representations — by the extra pole for $n = 4$. Although this leads to no new number-theoretical conclusions, it would be a very important indication of the promise of the method. It would also be a sign that the investigations of Dedekind or Jensen–Yui and the other authors have to be pursued, perhaps along the lines suggested in [ST], perhaps in other ways.

When applying the method, we usually fix a bound for the ramification of the representations π that we wish to consider. This is done by choosing $f = \prod_v f_v$ to be a spherical function outside of a finite set S of places and then choosing the f_v, $v \in S$, with appropriate restrictions. Their precise description is limited by one's understanding of the local harmonic analysis and arithmetic. In the present case, we might want to take $m = 1$ in (5.4), which restricts the ramification to 2 and 3, where it can be the minimum that permits the quaternion group to appear. Ramification for nonabelian representations of the Galois group and for representations of a general reductive group over a local field demands, of course, a more sophisticated, more technical examination, than necessary for abelian Galois representations or for the group GL(1).

To complete our report on Dedekind's paper, we have still to deal with some details of the structure of H, which can be obtained in two steps: (i) by the adjunction of $\sqrt{3}$ to obtain $H_1 = \mathbb{Q}(\sqrt{3})$; (ii) by the subsequent adjunction of $\sqrt{2}$, $H = H_1(\sqrt{2})$. Unfortunately, I am not so familiar with such calculations as Dedekind.

The discriminant of $H_1/\mathbb{Q}$ is $2^2 3$ and the ideal $\sqrt{3}$ is clearly unramified in H_2 where it does not split. So, by the usual formulas for the differents and discriminants of fields obtained by repeated extensions, the contribution of 3 to the discriminant of $H/\mathbb{Q}$ is 3^2. The two numbers a and τ are clearly integral. Moreover $\eta^2 = 2+\sqrt{3}$, so that η is also integral. It follows from (5.5) that all three of these numbers are units and from (5.6) that

$$N_{H/\mathbb{Q}}(1+\eta) = -\eta^{-2}(1-\eta^2)^2 = -\frac{(1+\sqrt{3})^2}{2+\sqrt{3}} = -2$$

is a unit times the fourth power of $1+\eta$.

The only other prime dividing the discriminant is 2. Let Z_2 be the 2-adic integers. Since the powers $(1+\eta)^j$, $j = 0, 1, 2, 3$ form an integral basis over $\mathbb{Z}_2$ of $H \otimes \mathbb{Z}_2$, we can calculate the power of 2 in the discriminant as $\prod_{i\neq j}(\eta_i - \eta_j)$, where η_i, $i = 1, 2, 3, 4$ are the conjugates of η, namely $\eta, -\eta, \eta^{-1}, -\eta^{-1}$. The result is $\pm(\eta^2 - \eta^{-2})^4$ and

$$16(\eta^2 - \eta^{-2})^4 = ((2+\sqrt{3}) - (2-\sqrt{3})^4 = 16^2 \cdot 3^2 = 2^8 \cdot 3^2.$$

This gives the correct result not only for 2 but also for 3. As a consequence, $\{1, 1+\eta, (1+\eta)^2, (1+\eta)^3\}$ or $\{1, \eta, \eta^2, \eta^3\}$ is an integral basis for the ring of integers in H.

Dedekind observes — discretely and without comment — that $\eta^2 = 2+\sqrt{3}$ and $\tau^2 = 5+2\sqrt{6}$. This is very useful information. The three quadratic subfields of H are $E_1 = \mathbb{Q}(\sqrt{2})$, $E_2 = \mathbb{Q}(\sqrt{3})$, $E_3 = \mathbb{Q}(\sqrt{6})$. For the units in E_1, the two basic hyperbolas are $x^2 - 2y^2 = \pm 1$. The units of positive norm are contained in $x^2 - 2y^2 = 1$ and generated, up to sign, by $3 \pm 2\sqrt{2}$, themselves the square of $1 \pm \sqrt{2}$. So a is a fundamental unit of E_1. For E_2, the corresponding hyperbolas are $x^2 - 3y^2 = \pm 1$. The units of positive norm are generated, again up to sign, by $2 \pm \sqrt{3} = \eta^2$. There are none with negative norm. For E_3 the hyperbolas are $x^2 - 6y^2 = \pm 1$ with points $5 \pm 2\sqrt{6}$, thus τ^2, $(\tau\alpha)^2$. They generate the units of positive norm. There are again no units of negative norm.

Consider a unit $x = x_1$ and its conjugates, $x_2 = x\alpha$, $x_3 = x\beta$, $x_4 = x\gamma$. Thus $|x_1| \cdot |x_2| \cdot |x_3| \cdot |x_4| = 1$, $x_1x_2x_3x_4 = \pm 1$. Since x_1x_3 is a unit in E_2, it is up to sign an even power of η. Thus, dividing x by an appropriate power of η we can conclude that $x_1x_3 = \pm 1$. Then, of course, $x_2x_4 = \pm 1$ as well. Now we divide by a power of

τ to obtain $x_1x_4 = \pm 1$, but without affecting the value of x_1x_3. As a result

$$\pm 1 = x_1x_2x_3x_4 = \pm\frac{x_2}{x_1} \quad \text{and} \quad \frac{x_1}{x_2} = \pm 1,$$

so that $x_1\alpha = \pm x_1$. If $x_1\alpha = x_1$, then x_1 is in E_1 and up to sign a power of a. Otherwise, $x_1 = y\sqrt{3}$, $y \in E_2$. Since 3 remains prime in E_2 and x_1 is a unit, this is impossible. We conclude that, as affirmed by Dedekind, a, η, and τ generate, up to sign, the group of units of H. Dedekind's example is marvelously simple!

Unfortunately, I am not familiar enough with Dedekind's style to know how he would have established that the class number of H is one. It follows readily enough from standard theorems. Dedekind's argument would have been more elegant. According to a familiar theorem [He, Satz 96], if there is a prime ideal in H that is not principal, there is one with norm less than or equal to the square root of the discriminant of H, thus $2^4 \cdot 3 = 48$.

The field H is a composite of two quadratic fields the class field associated to the group of ideles multiplicatively congruent to 1 or 7 modulo 8, 1 modulo 3, and positive. So there are four classes of primes different from 2 and 3. According to the law of quadratic reciprocity, they are distinguished by their residues modulo 3 and 8. First of all, in the field $\mathbb{Q}(\sqrt{2})$ the decomposition is:

(i) If $p \equiv 1, 7 \pmod 8$ then p splits.

(ii) If $p \equiv 3, 5 \pmod 8$ then p does not split.

In the field $\mathbb{Q}(\sqrt{3})$:

(i) If $p \equiv 1 \pmod 3$ and $p \equiv 1 \pmod 4$ or $p \equiv 2 \pmod 3$ and $p \equiv 3 \pmod 4$ then p splits.

(ii) If $p \equiv 2 \pmod 3$ and $p \equiv 1 \pmod 4$ or $p \equiv 1 \pmod 3$ and $p \equiv 3 \pmod 4$ then p does not split.

In the field $\mathbb{Q}(\sqrt{6})$, if $p \equiv 1 \pmod 8$ and $p \equiv 1 \pmod 3$, if $p \equiv 7 \pmod 8$ and $p \equiv 2 \pmod 3$, if $p \equiv 3 \pmod 8$ and $p \equiv 1 \pmod 3$, or if $p \equiv 5 \pmod 8$ and $p \equiv 2 \pmod 3$ then p splits, otherwise it does not. From this, we determine immediately the nature of the decomposition in H, whether a prime different from 2, 3 splits into 1, 2 or 4 primes. It splits into four if and only if it splits into two in the three intermediate fields.

According to the theorem cited, all we need do is show, first of all, that every prime ideal $\mathfrak{p}$ of norm p in one of the three fields E dividing a prime $p > 3$ in $\mathbb{Q}$ and with $\mathrm{N}\,\mathfrak{p}$ less than or equal to the square root of the discriminant of E is principal and, secondly, that every prime ideal of norm p in H dividing a prime $p > 3$ and with $\mathrm{N}\,\mathfrak{p}$ less than or equal to the discriminant of H is also principal. For the first type this is hardly necessary, but the results are as follows.

(1) For $\mathbb{Q}(\sqrt{2})$, the discriminant is 8 and there are no such primes.

(2) For $\mathbb{Q}(\sqrt{3})$, the discriminant is 12 and the only pertinent prime seems to be 11. Since $\mathrm{N}(1+2\sqrt{3}) = -11$, $\mathfrak{p} = (1+2\sqrt{3})$ is a prime of norm 11.

(3) For $\mathbb{Q}(\sqrt{6})$, the discriminant is 24. Of the primes 5, 7, 11, 13, 17, 19, 23, only 5, 19, 23 seem to satisfy the necessary conditions. We have $N(1+\sqrt{6}) = -5$, $N(5+\sqrt{6}) = 19$, $\mathrm{N}(1+2\sqrt{6}) = -23$.

(4) There are many primes less than or equal to 48, namely

$$5, 7, 11, 13, 17, 19, 23, 31, 37, 41, 43, 47,$$

but very few with the correct congruence properties, namely that split completely in H. For this we need either $p \equiv 1 \pmod 8$, $p \equiv 1 \pmod 3$, or $p \equiv 7 \pmod 8$, $p \equiv 2 \pmod 3$. Thus $p = 23$, 47 seem to be the only possibilities. We have to show that each of them factors in H into the product of four distinct principal prime ideals. It is enough to show that each of them is the norm of an element in H.

We can factor each of them in the three quadratic fields.

$$\mathrm{N}(5-\sqrt{2}) = 23; \quad \mathrm{N}(2-3\sqrt{3}) = -23; \quad \mathrm{N}(1-2\sqrt{6}) = -23,$$
$$\mathrm{N}(7-\sqrt{2}) = 47; \quad \mathrm{N}(1-4\sqrt{3}) = -47; \quad \mathrm{N}(7-4\sqrt{6}) = -47.$$

Because we have so much information about η, it is convenient — and sufficient — to establish that the central element in each of these rows is, up to a unit, the norm in $E = \mathbb{Q}(\sqrt{3})$ of an element u in H, which is $E(\eta)$, because $\eta^2 = 2+\sqrt{3}$. The field E is the fixed field of β. Thus, if we can find one $u = a + b\eta$ such that

$$\mathrm{N}_{H/E}(u) = u \cdot u\beta = a^2 - b^2(2+\sqrt{3}), \qquad a, b \in E. \tag{5.12}$$

differs from $2-3\sqrt{3}$ by a unit in E and another such that it differs from $1-4\sqrt{3}$ by another unit, then our task will be complete. So ran my first reflections.

I thought it would be necessary to attack the problem systematically, by a careful analysis that would determine where the numbers whose norm was ± 23 or ± 47 were to be found. The field H seemed to be singularly adapted to the necessary calculation. Consider the absolute values of the numbers a, η, τ and of their conjugates in the order: the number itself, then its conjugate under α, β, γ in that vertical order. The first column is supplementary, $x > 0$, $x \neq 1$.

$$\begin{array}{cccc} 1 & a & \eta & \tau \\ x & |a| & |\eta| & |\tau| \\ x & |a| & |\eta|^{-1} & |\tau|^{-1} \\ x & |a|^{-1} & |\eta| & |\tau|^{-1} \\ x & |a|^{-1} & |\eta|^{-1} & |\tau|. \end{array} \tag{5.13}$$

Taking the logarithms, we obtain a matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} \ln x & 0 & 0 & 0 \\ 0 & \ln|a| & 0 & 0 \\ 0 & 0 & \ln|\eta| & 0 \\ 0 & 0 & 0 & \ln|\tau| \end{pmatrix} \tag{5.14}$$

The first matrix is up to a factor an orthogonal matrix with inverse,

$$\frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

I wrote this down, looked at it, thought of the effort that a further systematic analysis would require, and decided I would just resort to Mathematica and calculate the norms of a few numbers in H in order to have a better feel for the sizes entailed. To my surprise and delight, because I was growing very fatigued, among the first ten norms generated appeared both -23 and -47.

$$\mathrm{N}(1-\eta^2-\eta^3) = -23; \quad \mathrm{N}(1-\eta-\eta^3) = -47.$$

6. Reciprocity

The meaning of reciprocity, as it appears in this prologue, is somewhat uncertain and variable. This appears to be inevitable. Although I have attempted to confine it to a relation between a group of $\mathfrak{A}$ and a group $\mathfrak{M}$, it sometimes appears to be simply a description of a group, either a motivic group or, more often, an automorphic group. This is, to a large extent, because the traditional Weil group already incorporates both aspects: (i) the multiplicative group of the field or of the idele classes as a carrier of characters; (ii) the Galois group as a description of finite extensions of the base field F, thus as a description of motives of dimension 0. Moreover, although reciprocity has a certain universality, it appears under more than form and this form adapts itself to the circumstances, local or global, geometric or arithmetic, and is, as a consequence, somewhat protean.

The Ramanujan conjecture in its general form — if properly interpreted, even in its classical form — is a statement about the local factors of automorphic representations $\pi = \otimes\pi_v$ and their Arthur parameters. We have not had occasion to comment on the local form of the group $\mathfrak{A}$ in an Arthurian context over nonarchimedean fields F_v. It appears to be $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C}) \times W_{F_v}$. As for archimedean fields, the first factor determines whether the representation is tempered or not and, if it is not, the nature of its failure to be tempered: determined by the asymptotic behavior of characters or matrix-coefficients. The second $\mathrm{SL}(2,\mathbb{C})$ does not appear for archimedean fields and is present to accommodate the needs of local reciprocity, which some authors satisfy by introducing the Weil–Deligne group WD.

We can introduce into the global arithmetic theory a formal but suggestive diagram:

$$\text{(6.1)}\quad \ell\text{-adic representations} \xleftarrow{\otimes\bar{\mathbb{Q}}_\ell} \text{motives}/F \xrightarrow{\otimes\mathbb{C}} \text{automorphic representations},$$

The Weil–Deligne group is introduced in the context of ℓ-adic representations, thus on the left; the motivic groups, at least those introduced by Grothendieck can be considered, for the present purposes, as being defined over $\mathbb{Q}$; the group $\mathfrak{A}$ is defined over $\mathbb{C}$. So an arrow from the extreme left to the extreme right is, without further explanations, not immediately at hand. The further explanations necessary are not, given the theorems and conjectures currently at hand, particularly difficult. The Weil–Deligne group has two disadvantages: (i) it introduces isomorphisms between fields that have no natural relation to each other, namely ℓ-adic fields and the complex field; (ii) it introduces classes of representations that are not semisimple. Neither of these is overwhelming, but both are unnecessary and clumsy. It is best to introduce a second SL(2) factor, either in the local $\mathfrak{A}$ or in the local $\mathfrak{M}$. This second factor is not present over $\mathbb{R}$ or $\mathbb{C}$.

Our immediate task, however, is to introduce the appropriate local structures on the left-hand side, for which all we have at hand are the ℓ-adic representations. We begin with them in their local form, taking the necessary material from [T]. Suppose that the local field $F = F_v$ is nonarchimedean with residue characteristic p and $\ell \neq p$. The theory of p-adic representations is more difficult and certainly pertinent, but not for this article.

In [T] a Frobenius element Φ is an element of the Galois group such that $\Phi^{-1}x = x^q$ on the residue field. I follow this convention. The elements of the Galois group that concern us are those that can be written as a product $\Phi^n\iota$, where ι lies in the inertia group. They form a dense subgroup, to be identified with the Weil group, of the Galois group. There is a homomorphism of the inertia group onto $\prod_{p\neq\ell}\mathbb{Z}_\ell$. Let it send ι to $\prod_{\ell\neq p}t_\ell(\iota)$. In [T] the notion of an ℓ-adic WD-representation or a representation of the Weil–Deligne group on a finite-dimensional ℓ-adic vector space is introduced. A “representation” of this group is not a true representation, it is a pair (r, N), where N is a nilpotent transformation of a finite-dimensional vector space V over $\bar{\mathbb{Q}}_\ell$ and r a representation of the Weil group on the same space. The representation r is to be continuous, so that its kernel is open in W. Moreover

$$\text{(6.2)}\qquad r(\Phi)Nr(\Phi^{-1}) = q^{-1}N,$$

a condition imposed for every choice of Φ. Thus N commutes with the inertia group.

There are supplementary conditions that can be imposed. One, that the Zariski closure G_r of the image of r is reductive, seems especially important. We impose it. As a consequence the image $r(\Phi^n\iota)$ of any element of the Weil group is semisimple. Other conditions of a topological or analytical nature are of lesser importance and we leave them aside for the moment. There is then a second representation associated to the pair (r, N),

$$\rho : \Phi^n\iota \mapsto r(\Phi^n\iota)\exp(t_l(\iota)N).$$

Clearly ρ determines r and N, but it is the pair to which we attach here the most importance, not the ℓ-adic representation ρ.

The restriction of r to the inertia group is defined by a representation of a finite quotient of this group. There is, consequently, an integer $m \neq 0$ such that $r(\Phi^m\iota) = r(\iota\Phi^m)$ for all ι in the inertia group. If $r(\Phi)$ is equal to $\Phi_{\mathrm{ss}}\Phi_{\mathrm{un}} = \Phi_{\mathrm{un}}\Phi_{\mathrm{ss}}$, with Φ_{ss} semisimple and Φ_{un} unipotent, then

$$\Phi_{\mathrm{ss}}^m r(\iota) = r(\iota)\Phi_{\mathrm{ss}}^m, \quad \Phi_{\mathrm{un}}^m r(\iota) = r(\iota)\Phi_{\mathrm{un}}^m,$$

for all ι in the inertia group. The second equation implies that $r(\iota)\Phi_{\mathrm{un}} = \Phi_{\mathrm{un}} r(\iota)$ for all ι. Consequently we can introduce a new representation r_{ss} such that $r_{\mathrm{ss}}(\iota) = r(\iota)$ for all ι, while $r_{\mathrm{ss}}(\Phi) = \Phi_{\mathrm{ss}}$. The representation r_{ss} is a canonical semisimplification of r. This is a representation of the Weil group.

It may seem idle, but we want to be able to replace the homomorphism r by a homomorphism of the WD-group to any L-group ${}^LG$, taken not over $\mathbb{C}$ but over $\bar{\mathbb{Q}}_\ell$. This is easily done. The definitions are identical to those just given for ${}^LG = \mathrm{GL}(n, \bar{\mathbb{Q}}_\ell)$. We shall continue in this vein for it allows us to consider homomorphisms of the categories we construct into the category of (algebraic) representations of ${}^LG(\bar{Q}_\ell)$. We retain the assumption that the Zariski closure of the image of the Weil group in ${}^LG$ is reductive, not forgetting the necessary compatibility with projections to finite Galois groups.

The construction of the WD-group is, unfortunately, clumsy and misleading, because it permits a passage to quotients by kernels of the transformation N. In order to avoid this possibility, we appeal to the Jacobson–Morozov lemma as formulated in [K], but we use it not over $\mathbb{C}$, rather over $\bar{\mathbb{Q}}_\ell$. Let N be a nilpotent element in the Lie algebra ${}^L\mathfrak{g}$ of a reductive group ${}^LG$. The superscript on ${}^L\mathfrak{g}$ serves a largely mnemonic function. There exists an $X \in \operatorname{ad} N({}^L\mathfrak{g})$ such that $[X, N] = 2N$. In addition, for each such X there exists a unique N' such that $[X, N'] = 2N'$, $[N, N'] = X$. The algebra $\mathfrak{s} = \{N', X, N\}$ is therefore isomorphic to the algebra $\mathfrak{sl}(2)$. Let σ be the isomorphism

$$\sigma : \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mapsto N, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mapsto X, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mapsto N'$$

of the algebra $\mathfrak{sl}(2)$ with $\mathfrak{s}$.

For us, N is the element in the equation (6.2). Corresponding to this equation, there is a character of the Galois group Gal_F, such that

$$r(\iota)Nr(\iota)^{-1} = \chi(\iota)N, \quad \forall g \in \mathrm{Gal}_F .$$

On the inertial group $\chi(\iota) = 1$.

Let $\mathfrak{h}$ be the centralizer of the image of the inertial group in the algebra ${}^L\mathfrak{g}$. The algebra $\mathfrak{h}$ is reductive because, by hypothesis, the image of r is reductive. We apply the Jacobson–Morozov theorem to the algebra $\mathfrak{h}$ and the element $N \in \mathfrak{h}$. Let H be the connected component of the identity in the centralizer of the inertia group in ${}^LG$ and S the connected subgroup of H corresponding to $\mathfrak{s}$. The group S has a unique Cartan subgroup, isomorphic to the multiplicative group of the field $\bar{\mathbb{Q}}_\ell$, whose Lie algebra contains X and this subgroup contains an element P such that $\mathrm{Ad}(P)(N) = q^{-1}N$, $P = q^{-X/2}$. Define ψ by the relation

$$\psi(\Phi^n\iota) = P^{-n}r(\Phi^n\iota). \tag{6.3}$$

The set of products $\Phi^n\iota$, $n \in \mathbb{Z}$, is of course the Weil group and ψ is a representation of it. Since $\psi(\iota) = r(\iota)$,

$$\mathrm{Ad}(\psi(\iota))N = N, \quad \mathrm{Ad}(\psi(\iota))X = X, \quad \mathrm{Ad}(\psi(\iota))N' = N'. \tag{6.4a}$$

Moreover,

$$\mathrm{Ad}(\psi(\Phi))N = N, \quad \mathrm{Ad}(\psi(\Phi))X = X. \tag{6.4b}$$

Consequently, $\mathrm{Ad}(\psi(\Phi))N'$ satisfies the conditions of the theorem of Jacobson–Morozov, so that $\mathrm{Ad}(\psi(\Phi))N' = N'$. As a consequence, rather than a representation of the WD-group in the sense given to it in [T] and other sources, we may use the representation (σ, ψ) of the thickened Weil group $\mathbb{W}$. I prefer this. Of course, σ has to be interpreted as a representation of the group, rather than of the algebra, and we have to replace $\mathbb{C}$ by $\bar{\mathbb{Q}}_\ell$. There is nothing to be done about this. It can be effected by an imbedding $\bar{\mathbb{Q}}_\ell \hookrightarrow \mathbb{C}$, disturbing but in the nature of things. We can, if we prefer, rather take the thickened Weil group not over $\mathbb{C}$ but over $\bar{\mathbb{Q}}_\ell$.

This possibility raises many questions. Since we do not yet have a complete theory of the representations of reductive groups over nonarchimedean local fields, we do not have a parametrization of the various classes, tempered, arbitrary, or the class introduced by Arthur. Moreover, even over archimedean fields there is, so far as I know, no clear indication, even at the speculative level, that there will be a stable theory for arbitrary irreducible representations. On the other hand, the classification of tempered representations, over $\mathbb{R}$ or, presumably, any other local field, will certainly demand a constraint of relative compactness on the image of ψ in, for example, (2.1), and this condition is not one that is invariant under an imbedding of $\bar{\mathbb{Q}}_\ell$ in $\mathbb{C}$. So there is room for reflexion on the local form of (6.1).

We take the imbeddings $\bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_\ell$ as given; so the algebraic closures of $\mathbb{Q}$ in the two fields are identified. The considerations that follow would lead to definitions that are independent of this identification.

There is a distinguished subgroup of $\mathbb{Q}$, the group S_1 of algebraic numbers β all of whose conjugates have absolute value 1 in $\mathbb{C}$. We introduce, at a local level, the set of parameters ψ, or (σ, ψ), or $(\sigma_1, \sigma_2, \psi)$ such that there is a homomorphism

$$\xi : \mathrm{GL}(1) \to {}^L G \tag{6.5}$$

for which:

(i) The image of ξ commutes with the image of ψ and, if appropriate, the image of σ or $\sigma_1 \times \sigma_2$.

(ii) For every Frobenius element Φ, every element ι of the group of inertia, every integer m, and for every (algebraic) representation ρ of ${}^L G$ all the eigenvalues of $\rho(\xi(|w|^{-m/2})\psi(\Phi^m \iota))$ lie in S_1.

If ξ exists it is determined by r so that when there is no danger of misunderstanding it need not be explicitly given. The distinction between ψ and r is somewhat pedantic. In (6.4a) and (6.4b), r is given and ψ depends on the choice of a square root of q; the representation r does not. In (ii) we are, in effect, making a further modification of ψ, leading to a further dependence on the choice of the square root $q^{1/2}$. The second condition does not, however, depend on the choice of the square root $q^{1/2}$. We must, nevertheless, take care that no implicit dependence on this choice occurs in other definitions, for example, in the L-functions associated to ℓ-adic representations. This would be a different dependence than that entailed by the simultaneous imbeddings of $\bar{\mathbb{Q}}$ in $\bar{\mathbb{C}}$ and $\bar{\mathbb{Q}}_\ell$. In a global context, ξ would first be given and then the various conditions would be satisfied by the local restrictions and this fixed ξ.

For the field $\bar{\mathbb{Q}}_\ell$ — and for ${}^L G = \mathrm{GL}(n)$ — these are the parameters that, because of the last Weil conjecture, yield the ℓ-adic representations with which we are principally concerned. According to the yet to be established local parametrization — for tempered and nontempered representations — they would correspond not only to tempered representations — for ξ trivial — but often to nontempered representations. Thus, it is entirely appropriate to introduce *weighted* parameters locally as well as the attendant global modifications. At a nonarchimedean place the local parametrization consist of pairs $\{\sigma \times \psi, \xi\}$ that satisfy the conditions described, or, for Arthur parameters, pairs $\{\sigma_1 \times \sigma_2 \times \psi, \xi\}$. At an archimedean place, they would just be pairs $\{\psi, \xi\}$ or $\{\sigma \times \psi, \xi\}$, the first condition remaining unchanged, but the second being replaced by the condition that the eigenvalues of $\rho\left(\xi(|w|^{-1})\psi(w)\right)$ have absolute value 1. The local form of the relation (6.1) would no longer be mediated by motives. It would be

(6.6) weighted ℓ-adic representations $\longrightarrow$ (automorphic) representations.

The arrow is now independent of the imbedding $\bar{\mathbb{Q}}_\ell \hookrightarrow \mathbb{C}$, at least in so far as it is compatible with the identification of $\bar{\mathbb{Q}} \subset \mathbb{C}$ and $\bar{\mathbb{Q}} \subset \bar{\mathbb{Q}}_\ell$. All this, in view of the scarcity of general results, has a very pedantic air, but, for my own sake, I find it useful to have a clear notion of the goals. They are not always clearly understood or formulated. Locally, what is often wanted — apart from careful, appropriate definitions — is simply an independent description of $\mathfrak{A}$ or $\mathfrak{M}$, in terms of familiar objects: the Weil group, the Galois group, or differentials. Globally, at least for arithmetical fields, it is a matter of proving the existence of both $\mathfrak{A}$ and $\mathfrak{M}$, deciding what their relation is and proving it. All three are major problems. For the global geometric theory, it is not clear to me at the moment, whether it is a description of $\mathfrak{A}$ in classical terms that is wanted, or whether there is a motivic group $\mathfrak{M}$ to be introduced and a relation of $\mathfrak{A}$ and $\mathfrak{M}$ to be discovered. The following two sections suggest that there is no $\mathfrak{M}$ in the geometric theory, but they are hardly conclusive.

I add that the supplementary Arthur parameters may not play a role in the correspondence (6.6). It appears to me that the image is likely to consist of objects whose supplementary Arthur parameter is trivial, so that the homomorphism of groups, which is from the right-hand side to the left, will be trivial on the SL(2) component of the global automorphic $\mathfrak{A}$.

There are many relatively simple examples of the various parameters that it might be appropriate to introduce here: (i) for the Arthur parameters, the conjecture of Jacquet for $\mathrm{SL}(n)$ proved by Mœglin and Waldspurger; (ii) for the second SL(2) parameter, the reciprocity for elliptic curves with nonintegral j-invariant, a very important and very early example in the development of a general reciprocity. Although they are well-known, they belong in any introduction to the theory. This is none the less only the prologue to an introduction. So I omit them.

7. The geometric theory for the group GL(1)

For the local arithmetic theory, we can identify the group $\mathfrak{A}_F$ as the Weil group or — for Arthur packets or if the local field is nonarchimedean — as a modified form of the Weil group, but we are not yet able to supply the necessary proofs. For the local geometric theory the abelian quotient, thus the group appropriate for $G = \mathrm{GL}(1)$, of the local group $\mathfrak{A}_F$ is readily identified, although the definitions are somewhat forced. The description of this quotient for the global geometric theory can be deduced, as we shall describe, from the classical theory of abelian integrals on a Riemann surface. The description of the abelian quotient of $\mathfrak{A}_F$ suggests, both locally and globally, a definition of $\mathfrak{A}_F$ itself, but as I discovered, one is faced almost immediately with the need for theories that have yet to be developed. I stress that, although the concepts emphasized here differ in some ways from those preferred by Edward Frenkel and have been influenced as well by the proof of a theorem of Weil, the initial impulse has been taken from his writings.

I should perhaps confess as well that, although the references [CFT; CLG; GT], from which I profited considerably, were, together with a letter from their author, my introduction to the geometric theory, my impulses, aesthetic and mathematical, are more analytic, less formal, perhaps less geometric, than those of their author. Even though I have not yet succeeded in exploiting the analytic possibilities of the theory, I do want to draw them to the reader's attention.

For the geometric theory, the local field at a point x is the field F_x of formal Laurent series

$$f(z)=\sum_{n=k}^{\infty} a_n z^n, \quad k\in\mathbb{Z}.$$

In the present context reciprocity — not the correct word in this context, in which our goal is simply the description of the local automorphic galoisian group $\mathfrak{A}_{F_v}$ — is, at least at first, simply a matter of expressing the characters of $F_x^\times$, or rather the group formed by these characters, in some appealing arithmetic or geometric manner. We must of course fix the choice of characters — unitary, nonunitary, holomorphic, whatever.

The local group $F_x^\times$ is abelian with two particularly important subgroups,

$$\begin{aligned}\mathbb{O}_x^\times &= \{a+bz+cz^2+\cdots \mid a\neq 0\},\\ \mathbb{O}_x^+ &= \{1+bz+cz^2+\cdots\},\end{aligned} \tag{7.1}$$

and $\mathbb{O}_x^\times=\mathbb{C}^\times\mathbb{O}_x^+$, $\mathbb{O}_x^+\backslash\mathbb{O}_x^\times\simeq\mathbb{C}^\times$, $\mathbb{O}_x^\times\backslash F_x^\times\simeq\mathbb{Z}$. The elements of the group $\mathbb{O}_x^+$ are best written in exponential form

$$\exp(\alpha_1 z+\alpha_2 z^2+\cdots). \tag{7.2}$$

The characters of $\mathbb{O}_x^\times$ are, for our purposes, most conveniently given by the residues of the differential forms defined by the product of the logarithm of

$$f=\alpha_0\exp(\alpha_1 z+\alpha_2 z^2+\cdots), \quad \alpha_0\neq 0,$$

and a given local differential form

$$\omega=\frac{\beta_{-k+1}}{z^k}+\frac{\beta_{-k+2}}{z^{k-1}}+\cdots+\frac{\beta_{-1}}{z^2}+\frac{\beta_0}{z}+\sum_{j=1}^{\infty}\beta_j z^{j-1}, \quad \beta_j\in\mathbb{C}\;\forall j,\quad \beta_0\in\mathbb{Z}, \tag{7.3}$$

although the coefficients $\beta_j,\ j > 0$, which are redundant and present only in anticipation of the global theory, do not affect the pairing.

$$(7.4)\quad (\omega, f) = (\omega, f)_x = \exp(i\,\mathrm{Re}(\mathrm{res}\,\omega \ln f)),$$
$$\mathrm{res}\,\omega \ln f = \beta_0 \ln \alpha_0 + \sum_{i=1}^{k-1} \alpha_i \beta_{-i}.$$

There is a second pairing implicit in (7.4), obtained on replacing $\mathrm{Re}(\mathrm{res}\,\omega \ln f)$ by the real linear form $\mathrm{Im}(\mathrm{res}\,\omega \ln f)$. It is understood that both are to be used, alone and in products. If $\beta_0 = 0$ one of them is redundant, since $\mathrm{Im}(\mathrm{res}\,\omega \ln f) = \mathrm{Re}(\mathrm{res}(-i\omega \ln f))$. If $\beta_0 \neq 0$, $i\omega$ is not admissible, because $i\beta_0 \notin \mathbb{Z}$. These characters are unitary. There is another possibility,

$$f \mapsto \exp(\mathrm{Re}(\omega \ln f)).$$

These characters are not unitary, but are pertinent in a more geometric theory, like that of [CFT], if ramification is admitted. We keep them in mind, because the two theories, analytic and geometric, are conceived as parallel to each other.

The group $\mathbb{O}_x^+$ is an infinite-dimensional complex vector space, the inverse limit of finite-dimensional vector spaces. Its dual space is taken to be a direct limit not of the complex dual spaces of the distinguished finite-dimensional spaces defined by the inverse limit, but of the distinguished real linear forms defined by the real and imaginary parts of the complex forms. Since $\mathrm{Im}(\omega, f) = \mathrm{Re}(-i\omega, f)$, this leads to a real vector space of dimension twice — and not four times — the complex spaces from which they arise. The dual space of $\mathbb{C}^\times = \mathbb{O}^+\backslash\mathbb{O}^\times$ is taken to be $\mathbb{R} \times \mathbb{Z}$, $\alpha \to \alpha^m \bar{\alpha}^n$, $m + n \in \mathbb{R}$, $m - n \in \mathbb{Z}$. Here, however, we need to use both the real and the imaginary parts of (ω, f), thus $\beta_0 \,\mathrm{Re} \ln \alpha_0$ and $\beta_0 \,\mathrm{Im} \ln \bar{\alpha}_0$, because β_0 is constrained to be integral, in particular, real. The pairing (ω, f) is linear in ω and multiplicative in f.

There seems to be no natural or unique way to extend this identification of the space Ω_x of local differential forms ω_x at x, implicitly taken modulo their regular parts and modulo the identification described, with the character group of $\mathbb{O}_x^\times$ to a concrete identification of $F_x^\times$, thus no way to incorporate naturally the dual of $\mathbb{O}_x^\times\backslash F_x^\times \simeq \mathbb{Z}$. This dual can be taken to be $\mathbb{C}^\times$. There will be a commutative diagram

$$(7.5)\quad \begin{array}{ccccccccc}
\{1\} & \longrightarrow & \mathbb{C}^\times & \longrightarrow & \widetilde{\Omega}_x & \longrightarrow & \Omega_x & \longrightarrow & \{1\} \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle\tilde{\eta}} & & \downarrow{\scriptstyle\eta} & & \downarrow \\
\{1\} & \longrightarrow & \mathbb{C}^\times & \longrightarrow & \mathrm{Char}(F_x^\times) & \longrightarrow & \mathrm{Char}\,\mathbb{O}_x & \longrightarrow & \{1\},
\end{array}$$

in which the kernel of $\tilde{\eta}$ is equal to the kernel of η, but a natural precise description is not available. To split the extension, a local parameter must be chosen.

This said and the necessary precisions kept in mind, we have defined the local group $\mathfrak{A}$ for the abelian geometric theory in terms of local differentials with singularities. For the global theory on a Riemann surface X, there are not only global differential forms, there are a number of supplementary objects, whose purpose took me some time to recognize. I brought with me from the arithmetic theory a notion of an automorphic form as a function on $G(F)\backslash G(\mathbb{A}_F)$. I shall return to the notion for a general group in §8. For now, I recall it for GL(1). It is the quotient I_F of the restricted districted product $\mathbb{I}_F = \prod_{x\in X} F_x^\times$ by the diagonally imbedded $F^\times$. There is a filtration

$$\{1\} \subset I_F^{\mathrm{tr}} \subset I_F^{\mathrm{unr}} \subset I_F^0 \subset I_F$$

of the group of ideles $I_F = F^\times\backslash\mathbb{I}_F$ of the group of idele classes, with

$$I_F^{\mathrm{tr}} = \prod_{x\in X} \mathbb{O}_x^+, \quad I_F^{\mathrm{unr}} = \mathbb{C}^\times\backslash\prod_x \mathbb{O}^\times,$$

$$I_F^0 = F^\times\backslash\mathbb{I}_F^0 = \Big\{x = \prod f_x \in \mathbb{I}_F \;\Big|\; \sum_x \mathrm{ord}_x(f_x) = 0\Big\}.$$

The quotients are

$$I_F^{\mathrm{tr}}\backslash I_F^{\mathrm{unr}} = \mathbb{C}^\times\backslash\prod_x \mathbb{C}^\times,$$

where $\mathbb{C}^\times$ is diagonally imbedded,

$$I_F^{\mathrm{unr}}\backslash I_F^0 = \Big\{(n_x) \in \bigoplus_x \mathbb{Z} \;\Big|\; \sum_x n_x x = \mathrm{div}(f),\ f \in F^\times\Big\} \backslash \Big\{(n_x) \in \bigoplus_x \mathbb{Z} \;\Big|\; \sum_x n_x = 0\Big\},$$

thus the group of divisors of degree 0 modulo principal divisors, and $I_F^0\backslash I_F = \mathbb{Z}$. The idele-class characters in the geometric theory are continuous functions on I_F equal to 1 on a subgroup $\prod_{x\notin S} \mathbb{O}_x^\times$, S a finite set of points in X, and on $\prod_{x\in S} F_x^\times$ to a product of the local characters already introduced.

These characters, or these automorphic forms, certainly need to be considered, but the geometric theory takes a broader view that it took me a good deal of time to appreciate and to reconcile with my simple ideas. The pertinent clue lies in the statement of Theorem 3 of §3.8 of [CFT].

Assertion. *For each irreducible rank n local system E on X there exists a perverse sheaf* Aut_E *on* Bun_n *which is a Hecke eigensheaf with respect to E. Moreover,* Aut_E *is irreducible on each connected component* Bun_n^d.

For the moment, I take $G = \mathrm{GL}(1)$, thus G to be not a general reductive group, and not $\mathrm{GL}(n)$, with n arbitrary as in the assertion, but with $n = 1$, and try to understand the meaning of this assertion. Among other things, it will be important to be clear, as soon as the initial explanations are concluded, about the nature of the difference between automorphic forms in the naive, but legitimate sense taken from the arithmetic theory, even those that are eigenfunctions of the Hecke operators, and a Hecke eigensheaf, at first when there is no ramification. It is an immediate result of Diagram I that for the group GL(1) a local system is just a coset in $H^1(X,\mathbb{Z})\backslash H^1(X,\mathbb{C})$. This is not emphasized in [CFT] or even mentioned, perhaps because the emphasis is on nonabelian groups. Handicapped by my inexperience, I am often, in the theory of algebraic curves, at a loss to distinguish theorems from definitions. Our first task will be to acquire some concrete understanding of the Assertion for $n = 1$ and to introduce its geometric counterpart. I do find it convenient, when reflecting on the Assertion and its analytic counterpart, to fix in mind the dimensions that appear. We shall see, for example, that the line bundles on X are parametrized by a $2g$-dimensional torus and the local systems attached to a given line bundle by a $2g$-dimensional real vector space, so that in the Assertion the possibilities are parametrized by the quotient of a $4g$-dimensional vector lattice by a $2g$-dimensional lattice. Such information assures a failing memory that nothing has been forgotten and nothing counted twice.

The quotient of I_F by $\prod_x \mathbb{O}_x^\times$ is the group of divisors on the nonsingular algebraic curve X for which the global theory is to be developed, taken modulo linear equivalence; it can be given the structure of an algebraic variety. The connected component of this variety, formed by the divisors of degree 0, is then the jacobian of X, which could be identified with the moduli space $\mathcal{P}^0$ of line bundles of degree 0 on X, but we do not do so. The full group is the Picard variety $\mathcal{P}$ itself, which can be identified with the quotient $F^\times \prod_x \mathbb{O}_x^\times\backslash\mathbb{I}_F$, but once again it is convenient to distinguish them.

We can be more precise. Let g be the genus of X. We introduce a complex vector space Ξ of dimension g, the dual space of the space of differential forms of the first kind on X and a lattice Δ in Ξ, given by the complex linear forms

$$\omega \mapsto \int_\delta \omega, \tag{7.6a}$$

$\delta \in H_1(X,\mathbb{Z})$, thus, more informally, but more instructively, δ being a closed curve on X. We introduce as well the real dual space $\hat{\Xi}$ of Ξ (sometimes identified with the space of conjugate linear complex-valued forms, but often with the space of complex linear forms) on Ξ by sending the conjugate linear form μ, $\mu(\alpha x) = \bar{\alpha}\mu(x)$, to $\mathrm{Re}(\mu)$ and the lattice $\hat{\Delta}$ defined by $\hat{\delta} \in \hat{\Delta}$ if and only if $\mathrm{Re}(\hat{\delta}(\delta)) \in 2\pi\mathbb{Z}$ for all $\delta \in \Delta$. It is difficult to distinguish Ξ and $\hat{\Xi}$ or Δ and $\hat{\Delta}$, but $\Delta = H_1(X,\mathbb{Z})$ and $\hat{\Xi}$ may, of course, be identified with the space of differential forms of the first kind on X. Then, thanks to the Abel–Jacobi theory, the map that assigns to the divisor $p_1 + \cdots + p_n - q_1 - \cdots - q_n$ the linear form

$$\omega \mapsto \sum_{i=1}^{n} \int_{q_i}^{p_i} \omega$$

defines an isomorphism — for both the group structure and the holomorphic structure — of $\Delta\backslash\Xi$ with the jacobian of X. The group and $\hat{\Delta}\backslash\hat{\Xi}$ the group (or moduli space) $\mathcal{P}^0$ of line bundles of degree 0, thus the connected component of the Picard group. Each of these line bundles admits a connection and the family of connections is given by a coset of $\hat{\Delta}$ in $\hat{\Xi}$. More precisely, as is explained on page 313 of [GH] in the context of complex tori, but the explanation is also valid here, the exact sequence of sheaves

$$H^1(X,\mathbb{Z}) \to H^1(X,\mathcal{O}) \to H^1(X,\mathcal{O}^\times) \to H^2(X,\mathbb{Z}), \tag{7.7}$$

in which there is a factor 2π in the first arrow that must not be forgotten, leads to an identification

$$\mathcal{P}^0 = H^1(X,\mathbb{Z})\backslash H^1(X,\mathcal{O}), \quad \hat{\Xi} = H^1(X,\mathcal{O}) = H^{0,1}(X), \quad \hat{\Delta} = H^1(X,\mathbb{Z}), \tag{7.8}$$

the notation $H^{0,1}(X)$ being taken from Hodge theory, where the space $H^{1,0}(X)$ is the space of differentials of the first kind. Unfortunately, I have difficulty remembering which is which because of the reversal of the order of the 0 and the 1 in the relations $H^1(X,\mathcal{O}) = H^1(X,\Omega^0) \simeq H^{0,1}(X)$, $H^0(X,\Omega^1) \simeq H^{1,0}(X)$.

It is undoubtedly best that I be as precise as I can because my experience with differentials and Hodge theory, even on curves, is limited. For example, $H^1(X,\mathcal{O}) = H^{0,1}(X)$ is the complex conjugate of $H^{1,0}(X)$, the space of differential forms of the first kind, this identification being given by the Hodge $\star$-operator ([GH, page 82]). There are two isomorphisms of $H^{1,0}(X)$ as a vector space over $\mathbb{R}$ to $\hat{\Xi}$; they are given by the real and imaginary parts of the periods (7.6a). To continue, I return to an enlarged form of the diagram (7.7), suppressing the explicit reference to X from the notation.

$$\begin{array}{ccccccccccc}
\{0\} & \longrightarrow & H^1(\mathbb{Z}) & \longrightarrow & H^1(\mathbb{C}) & \longrightarrow & H^1(\mathbb{C}^\times) & \longrightarrow & H^2(\mathbb{Z}) & \longrightarrow & H^2(\mathbb{C}) = \mathbb{C} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\{0\} & \longrightarrow & H^1(\mathbb{Z}) & \longrightarrow & H^1(\mathcal{O}) & \longrightarrow & H^1(\mathcal{O}^\times) & \longrightarrow & H^2(\mathbb{Z}) & \longrightarrow & H^2(\mathcal{O}) = \{0\}
\end{array}$$

Diagram I

The central square of the diagram is summarized in [GT, §2], although with reference to a general group G, not just GL(1): *"A flat connection has two components. The* (0, 1) *component, with respect to the complex structure on X, defines holomorphic structure, and the* (1, 0) *component defines a holomorphic connection."* According to the Hodge theory an element of $H^1(\mathbb{C}) = H^{1,0}(X) \oplus H^{0,1}(X)$ is realized uniquely as a sum of a holomorphic form and an antiholomorphic form. In the diagram, the third vertical arrow, $H^1(\mathbb{C}) \to H^1(\mathcal{O})$, is the projection on the second factor. Since the last arrow in the first line is an injection, the kernel of $H^1(\mathbb{C}^\times) \to H^1(\mathcal{O}^\times)$ is a complex vector space of dimension g, isomorphic to $H^{1,0}(X)$.

A flat connection is a connection in which there is a local notion of constant section; these are obviously given by $H^1(\mathbb{C}^\times)$, while $H^1(\mathbb{O}^\times)$ parametrizes line bundles. Since the Chern class, given by the degree of a line bundle, is the image of its parameter in $H^2(\mathbb{Z})$, we see that the collection of connections on a given line bundle is parametrized by $H^{1,0}(X)$. Thus the collection of flat connections on a given line bundle, which are parametrized by the inverse image of an element of $H^1(\mathbb{O}^\times)$ form an affine space over $H^{1,0}(X)$. A line bundle of degree 0, thus of Chern class 0, is an element of $H^1(\mathbb{O}^\times)$, thus an element of $H^1(\mathbb{Z})\backslash H^1(\mathbb{O})$ or a coset of $H^1(\mathbb{Z})$ in $H^1(\mathbb{O}) = H^{0,1}(X)$ or, as seems to be demanded by the formalism, by its complex (thus $\star$-)conjugate in $H^{1,0}(X)$. The comment of [GT] cited is the observation that an element of $H^1(\mathbb{C}^\times)$ is the image of an element of $H^1(\mathbb{C}) = H^{0,1} \oplus H^{1,0}$. Mapping this element to $H^1(\mathbb{O})$ amounts to projection on its first component. The horizontal arrow then yields a line bundle. So the bundle determines the first component. (As a test for my orientation: this is $H^1(\Omega^0)$ appearing in the Hodge theory as $H^0(\overline{\Omega})$, thus as an antiholomorphic differential, whose orthogonal complement lies in the holomorphic direction $\partial/\partial z$.) To determine the image of the upper horizontal arrow, we need to know both the first and the second component. So the supplementary information needed to determine the second component of an element of $H^1(\mathbb{C})$ is contained in its image in $H^1(\mathbb{C}^\times)$, thus in the connection.

It will be worthwhile to return to the geometric theory at the end of this section, just to understand better what the Assertion means for $n = 1$ and how it can be proved, but our principal goal is to introduce an analytic form of it that will allow us to introduce a candidate for the abelian quotient $\mathfrak{A}_{\mathrm{ab}}$ of $\mathfrak{A}$. The analytic form has quite a different flavor.

We have already defined $\hat{\Xi}$ as $H^{1,0}(X)$ and defined the periods of an element of $H^{1,0}(X)$, thus of a differential form of the first kind, by (7.6a). Then the conjugate space $\hat{\Xi}^{\mathrm{conj}}$ is $H^{0,1}(X)$ and their sum can be identified with $H^1(X,\mathbb{C})$, a $2g$-dimensional space, represented by holomorphic differential forms with arbitrary periods. We conclude that the span of the periods, either the real periods or the complex periods, for both are not simultaneously necessary,

$$\delta \mapsto \mathrm{Re}\int_\delta \omega, \quad \delta \mapsto \mathrm{Im}\int_\delta \omega, \quad \omega \in H^{1,0}(X),$$

is just Ξ, treated as a real vector space, thus the real dual of $\hat{\Xi}$. If we express the surface X in the usual way as a disc with boundary

$$\delta_1\delta_{g+1}\delta_1^{-1}\delta_{g+1}^{-1}\cdots\delta_g\delta_{2g}\delta_g^{-1}\delta_{2g}^{-1},$$

the various segments on the boundary being identified as indicated by the subscripts, then any additive mapping of this sort is determined by its values on $\delta_1,\ldots,\delta_{2g}$.

Observe that, because there is a multiplication by $2\pi i$, $z \to 2\pi i z$, the image of $H^1(X, \mathbb{Z})$ is characterized by the conditions that the real parts of the periods are 0 and the imaginary parts lie in $2\pi\mathbb{Z}$.

If $\omega \in \hat{\Delta}$ and p_0 is an arbitrary but fixed point on X, then

$$p \mapsto \exp\left(i \operatorname{Im} \int_{p_0}^{p} \omega\right) \tag{7.6b}$$

is a continuous character of the jacobian (or of the Picard variety $\mathcal{P}^0$) and, as ω varies over $\hat{\Delta}$, we obtain in this way a family of $\mathbb{Z}^{2g}$ characters of I_F^0, which can, of course, be extended to I_F, but this is a secondary matter, the choice of a nonzero constant. The character is defined by its differential equation,

$$\chi_R^{-1} d\chi_R = i \operatorname{Re} \omega = i\omega_R \quad \text{ or } \quad \chi_I^{-1} d\chi_I = i \operatorname{Im} \omega = i\omega_I, \tag{7.9a}$$

either of which defines in some sense a holonomic system or a *perverse* sheaf, but in a real context. The usual holonomic system would be given by the complex equation

$$\chi^{-1} d\chi = \omega, \tag{7.9b}$$

which may also be treated as two real equations. So it has more boundary conditions, thus conditions of periodicity. If the local coordinate is $z = x + iy$, $\omega = (\mu + i\nu)(dx + i dy)$, and if $\chi = \exp(\alpha(x, y) + i\beta(x, y))$, then (7.9b) amounts to

$$\left(\frac{\partial\alpha}{\partial x}, \frac{\partial\beta}{\partial x}\right) = (\mu, \nu), \qquad \left(\frac{\partial\alpha}{\partial y}, \frac{\partial\beta}{\partial y}\right) = (-\nu, \mu).$$

For the second equation, that in (7.9b), periodic conditions are not appropriate; for one or the other of the first, they are. For the second, boundary conditions would be to combine both conditions of (7.9a). So they are again usually impossible to satisfy. In the analytic or arithmetic theory, it is the second of equations (7.9a) that is pertinent. In the context of perverse sheaves, thus in the context of the Assertion, the issue of a global solution of the differential equation is inappropriate. I was troubled and confused by this difference for some time. Its source has become clearer. One thinks of the exponential function $\exp \lambda z$ on the interval $[0, 1]$, with 0 and 1 identified. If one wants functions, one needs $\lambda \in 2\pi i\mathbb{Z}$; if one accepts sheaves, thus the differential equation

$$\frac{dh}{dz} = \text{constant}$$

is acceptable. One reflects an analytic impulse, my dominant impulse; the other a geometric impulse, by which [CFT] is guided. The notion of a Hecke eigensheaf that appears there is, as we shall see, a clever way of admitting this greater generality. As already observed, it can also be incorporated into the analytic theory mediated by characters of the fundamental group The two possibilities could be examined separately, but as my purpose here was to adumbrate an analytic theory that would not lag behind the geometric theory, I have preferred to incorporate some to-and-fro in the exposition, as well as some redundancy.

To add to it, we reflect just a minute on the equation

$$\frac{1}{h}\frac{dh}{dz} = \lambda$$

on the circle, realized as the real line modulo $2\pi z$. On the line it defines a flat connection on the trivial bundle because the quotient of any two solutions $c_1 \exp(\lambda z)$, $c_2 \exp(\lambda z)$ differ by a multiplicative constant. It also defines a flat connection on the circle because $c_1 \exp(\lambda z)$, $c_1 \exp(\lambda(z + 2\pi))$ also differs by a multiplicative constant. It does not, however, define a section of the trivial bundle as a bundle on $\mathbb{Z}\backslash\mathbb{R}$. Trivial as the difference is, I find it, as the reader will discover, hard to fix in my mind. The integral of a constant function becomes linear and after passing to the exponential even more difficult to recognize. This becomes even worse with a curve and its jacobian. The jacobian is a quotient of a linear space on which a differential of the first kind is just a constant element of the dual; on the curve itself, it is hardly linear. The danger of confusing the intuition is even more severe for differentials with values in a vector bundle or in a Lie algebra. Another feature that leads to confusion is that the equations (7.9a) and (7.9b) describe the development of a complex line, thus of a real plane, or better a local section of a $\mathbb{U}(1) \subset \mathrm{GL}(1)$ bundle over X, even of a local system of X, because for it there is a local notion of constant section. I have only increased the possibility of confusion by referring to boundary conditions; at best, we are dealing with boundary conditions on a rectangle.

In the analytic theory, we are dealing with characters, thus with functions with values in the group $\mathbb{U}(1)$ of complex numbers of absolute value 1. So we are dealing with one or the other of the equations (7.9a), say the first, or, in other words, with $\mathbb{U}(1)$-bundles. The sequence

$$\{1\} \to \mathbb{Z} \to \mathbb{R} \to \mathbb{U}(1) \to \{1\}$$

yields an analogue of Diagram I, in which the vertical arrow $H^1(\mathbb{R}) \to H^1(\mathbb{O})$ is an isomorphism.

$$\begin{array}{ccccccccccc}
\{0\} & \longrightarrow & H^1(\mathbb{Z}) & \longrightarrow & H^1(\mathbb{R}) & \longrightarrow & H^1(\mathbb{U}(1)) & \longrightarrow & H^2(\mathbb{Z}) & \longrightarrow & H^2(\mathbb{R}) = \mathbb{R} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\{0\} & \longrightarrow & H^1(\mathbb{Z}) & \longrightarrow & H^1(\mathbb{O}) & \longrightarrow & H^1(\mathbb{O}^\times) & \longrightarrow & H^2(\mathbb{Z}) & \longrightarrow & H^2(\mathbb{O}) = \{0\}.
\end{array}$$

Diagram II

The significance of the diagram is that every holomorphic line bundle is realized as a $\mathbb{U}(1)$-bundle, that each of them carries a unique local system in the real sense, thus with constant transition functions in $\mathbb{U}(1)$. All this is simple, but it has taken me some time to appreciate the consequence: the analytic theory is closely related to the holomorphic theory but different from it. In the analytic theory, the local systems have automorphisms: sections of the associated $\mathbb{U}(1)$-bundle. These are automorphic forms, but this may not be pertinent.

It is worthwhile to explain this further. In the theory of algebraic curves, there is a great deal of structure crammed into a very small space and it is difficult to describe it in an orderly fashion. Starting with an element η in $H^1(\mathbb{O}) = H^{0,1}$, we add its image $\star\eta$, which lies in $H^0(\Omega^1) = H^{1,0}$, to it and divide by 2 to obtain a form $\operatorname{Re}\omega$. Then the element of $H^1(\mathbb{U}(1))$, thus a flat sections of the bundle is given by

$$\exp\left(2\pi i \int_{p_0}^{p} \operatorname{Re}\omega\right).$$

Notice that, because of the presence of $H^1(\mathbb{Z})$ at the beginning of each line and because of the factor $2\pi i$ that appears in the passage from $\mathbb{R}$ to $\mathbb{U}(1)$, $\operatorname{Re}\omega$ is determined only up to a form with integral periods.

When the geometric theory is treated as an offshoot of the arithmetic theory, the restriction to unramified representations or forms is unnatural. It is also unnecessary. For the global theory on a nonsingular algebraic curve X, the space of differentials of the first kind is replaced by the space $\Omega = \Omega_X$ of global meromorphic differentials ω with local expressions ω_x. There is one condition on the forms ω considered that one might be tempted to impose: the residue at each point must be integral. I omit it for a brief moment, because it took me sometime to understand the significance of such a condition, but I shall very quickly impose it. Even if it is unnecessary, there are already enough other complications to master. One consequence is that the solutions of the differential equation $df/f = \omega$ are single-valued in a neighborhood of each point, so that no singularities are introduced locally into the sheaf of solutions. So the distinction is — perhaps — between a *sheaf*, thus by the differential equation, that is by ω, which is well defined in all cases, and, up to a constant, a *single-valued* function, its solution, which is not! The analytic impulse, as well as the arithmetic, is to emphasize the function; the geometric impulse is to emphasize the sheaf. The consequences of the two points of view have already revealed themselves. If the condition of integrality is imposed, the periods of ω, thus its integrals over one-cycles, are well-defined modulo $2\pi i\mathbb{Z}$. In particular the real and imaginary parts are defined modulo $2\pi\mathbb{Z}$. The periods as such are not defined because an integral of ω even over a cycle homologous to zero is given by the sum of the residues in the 1-chain that it bounds.

There are two difficulties in the Assertion, apart from understanding how it is verified. First of all, it is an assertion only in an unramified context. Coming from the arithmetic theory of automorphic forms or representations, I find this an unacceptable restriction. We shall remove it. Secondly, there is no immediate link to a theorem of Weil that we shall recall later and that offers a partial solution to the problem of identifying the geometric galoisian group $\mathfrak{A}$, a kind of self-duality similar to that of class field theory.

I recall the structure of the vector space of meromorphic differentials on X. First of all the space of differentials without singularities, thus differentials of the first kind, has dimension equal to g, the genus of the curve. Secondly, the singularities may be assigned almost arbitrarily. There is only one constraint: the sum of the residues must be 0. This is a consequence of, for example, the theorem of Riemann–Roch, which can be cited in the form given in [Sp]. Take a finite set of points $x_1, \dots, x_n$ on the curve X and integers $d_1, \dots, d_n$. Then the space of possible singularities concentrated on this finite set and of degree at most d_i at x_i is of dimension $\sum_i d_i$. If a given singularity can be realized by a meromorphic differential ω then any other realization is of the form $\omega + \omega'$, where ω' lies in the g-dimensional space of holomorphic differentials. So to prove that differentials can be assigned arbitrarily, we need only verify that for all choices of $x_1, \dots, x_n$ and of $d_1, \dots, d_n$ with

$$d_1 + \cdots + d_n > 0, \tag{7.10}$$

the space of differentials with the singularities allowed by these choices is of dimension $g - 1 + \sum_i d_i$. The condition (7.10) takes account of the constraint that the sum of the residues is 0. Take the divisor $\mathfrak{a}$ on page 264 of Springer's book [Sp] to be $-\sum_i d_i x_i$. Then, according to the form of the Riemann–Roch theorem given there, the dimension of the space of possible differentials is

$$i(\mathfrak{a}) = g - 1 + \sum d_i,$$

because, in the notation of [Sp], $d(\mathfrak{a}) = -\sum d_i$ and $r(-\mathfrak{a}) = 0$.

It is convenient to introduce an increasing sequence of differential forms: the forms with no singularities, thus the forms of the first kind; the forms whose only singularities are simple poles; finally, the forms with arbitrary singularities. We can then add the supplementary condition, already introduced, that the residues be integral. If $\delta_1, \delta_2, \dots, \delta_{2g}$ is the base of the integral cycles, then $\omega \to \operatorname{Re} \int_{\delta_i} \omega$ (or $\operatorname{Im} \int_{\delta_i} \omega$) defines $2g$ linear forms linearly independent over $\mathbb{R}$ on the g-dimensional complex space of forms of the first kind. Moreover, as ω varies, the $2g$-dimensional vectors in $\mathbb{R}^{2n}$ or, perhaps better, in $\hat{\Xi}$,

$$\begin{aligned}\varpi_R &= \{\varpi_{R,i}\} = \bigl\{\operatorname{Re}\textstyle\int_{\delta_i}\omega \bigm| i = 1, \dots, 2g\bigr\},\\ \varpi_I &= \{\varpi_{I,i}\} = \bigl\{\operatorname{Im}\textstyle\int_{\delta_i}\omega \bigm| i = 1, \dots, 2g\bigr\},\end{aligned} \tag{7.11}$$

are arbitrary, but not independently arbitrary. Without the condition on the residues, they are both path-dependent. Even if the condition on the residues is imposed, the second is only defined modulo $2\pi\mathbb{Z}$, but, as already observed, that is all we need because we use $\exp(i \operatorname{Im} \int \omega)$. The condition on the residue prevents us from multiplying all ω by i or $-i$, so that the set of ϖ_I and ϖ_R may be different.

For the moment, we are dealing with line bundles, so that $n = 1$. My impulse was to look for a theorem in which irreducible, thus one-dimensional, automorphic representations of the geometric form of the group of idele classes appear. If there is no ramification — and if we admit as automorphic representations only continuous functions in the parameter $x \in X$, thus only continuous characters of the group of divisors modulo those linearly equivalent to 0, a group whose connected component is the jacobian, thus a complex variety of a dimension g that, as a group, may be identified with $\mathbb{U}^{2g}$, $\mathbb{U} = \{z \in \mathbb{C}^\times \mid |z| = 1\}$ — its group of unitary characters, thus the set of irreducible unramified automorphic representations, is isomorphic to $\mathbb{Z}^{2g}$, or to the group of differentials ω of the first kind for which ϖ_I lies in $(2\pi\mathbb{Z})^{2g}$. It is easy enough to make the isomorphism explicit in terms of $F^\times\backslash\mathbb{I}_F^0$ — rather than in terms of the jacobian — by applying the method of [GH] for proving a theorem of Weil, and we shall do so.

What then is the purpose of the remaining ω, either the remaining ω of the first kind or, more generally, the differentials with singularities? For the comparison with the Assertion, which is implicitly stated in an unramified context, it is the differentials of the first kind that are relevant, but for the description of the global group $\mathfrak{A}$ in the geometric context, it will be necessary to admit differentials with singularities, thus with negative powers in their local Laurent expansions. For this prologue, however, it is best to consider only those with integral residue, since a nonintegral residue introduces ramification in the line bundles themselves — local sections at some points behave like z^α, $\alpha \in \mathbb{C}$. That would, at this stage, be one complication too many.

For the moment, we remain with differentials with no singularities. We count — once again — the parameters available. I refer to Diagram I. It is clear from the lower line of the diagram that line bundles are parametrized by $\mathbb{Z}^{2g}\backslash\mathbb{R}^{2g}$. The upper line then shows that the possible local systems on a given line bundle are parametrized by $H^{1,0}(X) = H^0(X, \Omega^1)$, thus by $\mathbb{R}^{2g}$. So all in all, we need $\mathbb{Z}^{2g}\backslash\mathbb{R}^{4g}$ to specify a local system. On the other hand, in the analytic/arithmetic context the set of unramified automorphic forms is given by the $\mathbb{Z}^{2g}$ characters of the jacobian, or rather by their extensions to characters of $F^\times\backslash\mathbb{I}_F$, parametrized by $\mathbb{U}$ or by characters of $\mathbb{I}_F^0\backslash\mathbb{I}_F$. These extensions are incidental to the central issue. So the puzzling matter is the presence in the geometric theory of supplementary parameters in $\mathbb{Z}^{4g}\backslash\mathbb{R}^{4g}$. We shall introduce them artificially. I was, initially, made more than a little uneasy by the artifice.

We shall return to this point, but only after broaching the question of attaching, in the geometric theory, an idele-class character to a differential ω, perhaps singular but with integral residues. It turns out that this entails an enlargement of the notion of idele class. We take the product of $\mathbb{I}_F$ with $4g$ copies of $\mathbb{Z}$, thus with two copies $\mathfrak{Z}_R$ and $\mathfrak{Z}_I$ of $\mathbb{Z}^{2g}$, so that the dual of the modified group is the group of characters of $\mathbb{I}_F$ multiplied by two copies of the $2g$-fold product of $\mathbb{Z}\backslash\mathbb{R} = \mathbb{U}(1)$ with itself. We introduce an imbedding of $F^\times$ in $\mathfrak{I}_F = \mathfrak{Z}_R \times \mathfrak{Z}_I \times \mathbb{I}_F$ by

$$f \mapsto \prod_{i=1}^{2g} \frac{\int_{\delta_i} d\ln f}{2\pi i} \times \prod_{i=1}^{2g} \frac{\int_{\delta_i} d\ln f}{2\pi i} \times f. \tag{7.12}$$

There will be a finite number of points $q_1, q_2, \dots$ at which ω has a singularity and, for any given idele f, a finite number of points $p_1, p_2, \dots$ at which $f = \prod_x f_x$ has a zero or pole. If the sets $D_\omega = \{q_1, q_2, \dots\}$ and $D_f = \{p_1, p_2, \dots\}$ are disjoint and if $f \in F^\times$ is a principal idele we may introduce λ_R as the difference of

$$\mathrm{Re}\Big\{\sum_j \mathrm{res}_{q_j}(\omega \ln f) - \sum_i \mathrm{ord}_{p_i}(f) \int_p^{p_i} \omega - \gamma \sum_i \mathrm{ord}_{p_i}(f)\Big\} \tag{7.13a}$$

and

$$\frac{1}{2\pi i} \sum_{k=1}^{g} \Big\{\varpi_{R,g+k} \cdot \int_{\delta_k} d\ln f - \varpi_{R,k} \cdot \int_{\delta_{g+k}} d\ln f\Big\}, \tag{7.13b}$$

where γ is a supplementary complex parameter, p is a supplementary point, and a choice of path from p to p_i that avoids the singularities of ω is implicit for each i. It modifies the value of λ_R only by an additive constant in $2\pi\mathbb{Z}$. We want to introduce a pairing $(\omega, f)_R = \exp(i\lambda_R)$ defined for all ideles f. The expression (7.13a) is certainly defined; the expression (7.13b) is not, but it is defined if we replace f by an element $\tilde{f}$ of $\mathfrak{Z}_R \times \mathfrak{Z}_I \times \mathbb{I}_F$ and $\int_{\delta_k} d\ln f$ by 2π times the appropriate coordinate of the $\mathfrak{Z}_R$ component of $\tilde{f}$. This defines $(\omega, \tilde{f})_R$ in general. We define $(\omega, \tilde{f})_I$ in the same manner. It is simpler to abbreviate $\tilde{f}$ to f, and I do so in the following discussion, inserting the tilde if its omission would lead to confusion or as a reminder.

The parameter γ only affects the pairing at those f whose total degree $\sum_i \mathrm{ord}_{p_i} f$ does not vanish and two pairs (ω, γ), (ω', γ') yield the same pairing if $\gamma' - \gamma = \int_p^{p'} \omega$. As we did for the local parameters, we shall have to use both pairings $(\omega, f)_R$

and $(\omega, f)_I$ unless all the residues β_0 of ω are zero. So the parametrization of characters might best be expressed in terms of pairs (ϖ_R, ϖ_I) with an appropriate equivalence relation, but this would be too fastidious for a prologue and, in any case, obvious.

The key to the global definition of $(\omega, f)_R$ or $(\omega, f)_I$, whose properties have yet to be discussed, is a generalization of a theorem attributed in [GH] to Weil. I formulate the generalization as a lemma that implies that, for each ω, $f \mapsto (\omega, f)_R$, $f \mapsto (\omega, f)_I$ define idele-class characters.

Lemma 7.1. *If f is a principal idele and D_ω and D_f are disjoint, then $(\omega, f)_R = (\omega, f)_I = 1$.*

The theorem of Weil affirms that if f and g are meromorphic functions on the compact Riemann surface X such that the set of zeros and poles of f is disjoint from the set of zeros and poles of G then

$$\prod_p f(p)^{\mathrm{ord}_p(g)} = \prod_p g(p)^{\mathrm{ord}_p(f)}. \tag{7.14}$$

For the simplest example, the projective line $\mathbb{P}^1$, the theorem is elementary and easy to prove. Suppose, for example that $f = (x - a_1)/(x - b_1)$, $g = (x - a_2)/(x - b_2)$. Then

$$(f, g) = \frac{g(a_1)}{g(b_1)} \frac{f(b_2)}{f(a_2)} = \frac{a_1 - a_2}{a_1 - b_2} \frac{b_1 - b_2}{b_1 - a_2} \frac{b_2 - a_1}{b_2 - b_1} \frac{a_2 - b_1}{a_2 - a_1} = 1$$

In general, the theorem is a consequence of a relation like that of the lemma, but for $\omega = dg/g$. The idele f is still to be principal. The relation becomes

$$\sum_j \mathrm{res}_{q_j}(\omega \ln f) - \sum_i \mathrm{ord}_{p_i}(f) \int_p^{p_i} \omega = \lambda \in 2\pi i \mathbb{Z}. \tag{7.15}$$

The new relation is stronger, or, rather, more compact, because the periods of ω themselves now lie in $2\pi i\mathbb{Z}$. This is not just a condition on the real or imaginary parts. We recall the proof given on page 229 and on pages 242–243 of [GH], following, so far as possible, the notation of that book. We have already followed it with the usual description of the basic cycles $\delta_1, \ldots, \delta_g, \delta_{g+1}, \ldots, \delta_{2g}$ that display the surface as a planar polygon Δ with sides identified. We have a function f with poles and zeros at p_i and a form $\omega = dg/g$ with first-order poles at q_j. The sets $\{p_i\}$ and $\{q_j\}$ are taken to be disjoint. The p_i and the q_j are to lie in the interior of the planar region and we join each p_i to a common point p on the boundary by a curve α_i that avoids the q_j, thus introducing incisions that reduce Δ to a region Δ' and add several curves to its boundary, the curve α_i and the curve in the inverse direction.

Since $\sum_i \operatorname{ord}_{p_i}(f) = 0$ and $\phi(p_i) = \ln\int_p^{p_i}\omega$ is well-determined up to a constant independent of p_i, the possible ambiguities, for example in the choice of the base point p, have no affect on the relation (7.15).

As in [GH], we integrate the form $\varphi = \omega \ln f$ over the boundary of Δ'. By the residue theorem, this integral is given by

$$\int_{\partial\Delta'}\varphi = 2\pi i\sum_{q_j}\operatorname{res}_{q_j}\varphi = 2\pi i\sum_{q_j}\operatorname{res}_{q_j}(\omega\ln f). \tag{7.16}$$

We collect terms as in [GH]. First of all, for identified pairs p, p' on the arc δ_i and on the inverse arc δ_i^{-1},

$$\ln f(p') = \ln f(p) + \int_{\delta_{g+i}} d\ln f, \tag{7.17}$$

so that

$$\int_{\delta_i+\delta_i^{-1}}\varphi = \Big(\int_{\delta_i}\omega\Big)\Big(-\int_{\delta_{g+i}} d\ln f\Big). \tag{7.18}$$

In the same way,

$$\int_{\delta_{g+i}+\delta_{g+i}^{-1}}\varphi = \Big(\int_{\delta_{g+i}}\omega\Big)\Big(-\int_{\delta_i} d\ln f\Big). \tag{7.19}$$

Moreover for identified points $p \in \alpha_i$ and $p' \in \alpha_i^{-1}$,

$$\ln f(p') - \ln f(p) = -2\pi i \operatorname{ord}_{p_i}(f). \tag{7.20}$$

so that[1]

$$\int_{\alpha_i+\alpha_i^{-1}}\varphi = 2\pi i \operatorname{ord}_{p_i}(f)\int_p^{p_i}\omega.$$

As in [GH], the conclusion is that

$$2\pi i\Big\{\sum_j \operatorname{res}_{q_j}(\omega\ln f) - \sum_i \operatorname{ord}_{p_i}(f)\int_p^{p_i}\omega\Big\}$$

is equal to

$$\sum_{k=1}^{g}\{\int_{\delta_k} d\ln f\cdot\int_{\delta_{g+k}}\omega - \int_{\delta_k}\omega\cdot\int_{\delta_{g+k}} d\ln f\} \tag{7.21$'$}$$

[1] In the diagram of [GH], δ_0 is meant to be s_0, an arbitrarily chosen point on the boundary of the planar region. I have denoted it above by p.

or

$$\sum_{k=1}^{g}\left\{\int_{\delta_k} d\ln f\cdot\int_{\delta_{g+k}} d\ln g-\int_{\delta_k} d\ln g\cdot\int_{\delta_{g+k}} d\ln f\right\}. \tag{7.21$''$}$$

In (7.21″), all four integrals are of functions all of whose residues are integral and all integrals are over closed curves. The conclusion is, as in [GH], that the sum is an integral multiple of $(2\pi i)^2$. The relation (7.14) follows.

To prove the lemma itself, we deal with $(\omega,\cdot)_R$ and, implicitly, $(\omega,\cdot)_I$ with essentially the same sequence of formulas. Since f is now a principal idele, the term in (7.13a) that contains γ is 0, and (7.13a) itself is reduced to the real part of

$$\lambda=\sum_j \operatorname{res}_{q_j}\{\omega\ln f(q_j)\}-\sum_i \operatorname{ord}_{p_i}(f)\int_p^{p_i}\omega \tag{7.22}$$

and the assertion is that the difference between the real part of (7.22) and (7.13b) lies in $2\pi\mathbb{Z}$. The proof is the same as before; we deal with (7.22) as we dealt with (7.15), collecting terms in the same way:

$$\int_{\delta_i+\delta_i^{-1}}\varphi=\left(\int_{\delta_i}\omega\right)\left(-\int_{\delta_{g+i}} d\ln f\right);\quad \int_{\delta_{g+i}+\delta_{g+i}^{-1}}\varphi=\left(\int_{\delta_{g+i}}\omega\right)\left(-\int_{\delta_i} d\ln f\right).$$

The conclusion is that

$$2\pi i\sum_j\left(\operatorname{res}_{q_j}(\omega)\ln f(q_j)-\sum_i \operatorname{ord}_{p_i}(f)\int_p^{p_i}\omega\right)$$

is equal to

$$\sum_{k=1}^{g}\left(\int_{\delta_k} d\ln f\cdot\int_{\delta_{g+k}}\omega-\int_{\delta_k}\omega\cdot\int_{\delta_{g+k}} d\ln f\right). \tag{7.23}$$

To calculate λ, we take the imaginary part of this, divide by 2π, and subtract (7.13b). This yields

$$\sum_{k=1}^{g}\left(\int_{\delta_k} d\ln f\cdot\left(\operatorname{Re}\int_{\delta_{g+k}}\omega-\varpi_{R,g+k}\right)-\left(\operatorname{Re}\int_{\delta_k}\omega-\varpi_{R,k}\right)\cdot\int_{\delta_{g+k}} d\ln f\right). \tag{7.24}$$

The periods of $d\ln f$ are all multiples of $2\pi i$ and the numbers $\operatorname{Re}\int_{\delta_k}\omega-\omega_{R,k}$, $k=1,\dots,2g$, are also all integral multiples of 2π. Indeed they are 0, but that is not the point here. This proves the lemma!

There is a difficulty with the pairings $(\omega,f)_R$ and $(\omega,f)_I$ that is resolved by the lemma. For a given ω, it is not defined for all ideles f, or, to be precise, $\tilde{f}$, only for those for which D_f and D_ω are disjoint. We can extend it to all ideles by setting any given idele f equal to f_1f_2, where f_2 is principal and f_1 is an idele whose set of zeros and poles is disjoint from the set of singularities of ω. Then we set $(\varpi_R, f) = (\varpi_R, f_1)$, $(\varpi_I, f) = (\varpi_I, f_1)$. Thanks to the lemma, the result will be independent of the choice of the factorization of f.

There is a second difficulty, not resolved by the lemma, at least not without closer examination. What do we do if the function f or the differential ω has a singularity at a point x on the boundary of $\triangle$, say in δ_l and thus in δ_l^{-1}. So it can be approached in two ways from within $\triangle$, one through a half-neighborhood of a subinterval of δ_l, the other through a half-neighborhood of δ_l^{-1}. If the limiting results for the differences of (7.13a) and (7.13b) are the same modulo 2π, there is no problem. We just deform δ_l a little around the offending point and the choice of the deformation, whether we deform a little to the left in the sense of δ_l or in the sense of δ_l^{-1} to make the calculation does not matter. Since the singularities of ω are assumed not to fall on the singularities of f, we can treat the two independently.

The contribution of a singularity of ω to the first term of (7.13a) does not depend on the relation of its position to the curve δ_l. On the other hand, the second term is affected as are the factors $\varpi_{R,k}$. The first is affected because the integral, inside $\triangle$, from p to p_i as a point on δ_l^{-1} is replaced by an integral over a path inside $\triangle$ from p to p_i as a point on δ_l. The difference is a multiple of $2\pi i$ and is multiplied by $\mathrm{ord}_{p_i}(f)$. So it causes no problem. The factor $\varpi_{R,l}$ is deformed but the result is an additive modification by $2\pi i$ times the residue of ω, which is assumed to be integral.

The singularities of f appear in both (7.13a) and (7.13b). Since it is easier, we consider first the effect on (7.13b). The path δ_l first passes to the right of the point and then to the left. So the modification in $\int_{\delta_l} d\ln f$ is $2\pi i\,\mathrm{ord}_{p_i}(f)$, and in (7.13b) $\pm\varpi_{R,g+l'}\,\mathrm{ord}_{p_i} f$, where l' is l or $l-g$ according as $l \le g$ or $g < l \le 2g$. It is evident that something similar will happen with (7.13a). The factor $\mathrm{ord}_{p_i} f$ is already in evidence. The modification is therefore given by the negative of the integral over the path from the point p_0 to p_i on δ_l followed by the inverse path from p_i on δ_l^{-1} to p_0. The two together, with sign, yield a closed path within $\triangle$ from p_i on δ_l^{-1} to p_i on δ_l. Since we can deform the path inside the contour at the cost of adding an integral multiple of 2π, we might as well move directly along the boundary. The integrals along δ_l and δ_l^{-1} cancel and we are left with the integral along δ_{l+g} if $l \le g$ and along the inverse of δ_{l-g} if $l > g$. So up to an additive factor that is an integral multiple of 2π, the difference does not change. I apologize to the more skillful reader for the clumsy argument. I hope it is correct!

The conclusion is that we have attached to ω two characters $\tilde{f} \mapsto (\omega, \tilde{f})_R$ and $\tilde{f} \mapsto (\omega, \tilde{f})_I$ of $F^\times\backslash\mathfrak{I}_F$. In order to persuade ourselves that we indeed have, in a useful way, identified all idele class characters, but also to understand what we have in hand, we remind ourselves of the structure of the group of ideles, or rather of $\mathfrak{Z}_F$, and of its character group, and then of the structure of the group of characters constructed from the admissible differentials.

I observe first of all, to make the task easier, that $\mathfrak{Z}_R \times \mathfrak{Z}_I$ is a subgroup of $F^\times\backslash\mathfrak{I}_F$ and that the characters $(\omega, \cdot\,)_R$ and $(\omega, \cdot\,)_I$, certainly yield, upon restriction, all characters of this subgroup. The restrictions are trivial if

$$\varpi_{R,i} \equiv 0 \pmod{2\pi}, \quad \varpi_{I,i} \equiv 0 \pmod{2\pi}, \qquad 1 \le i \le 2g. \tag{7.25}$$

So the issue is whether we obtain all characters of $F^\times\backslash I_F$ from forms ω satisfying one or the other of the two conditions.

We are dealing with a great deal of structure in a very small space. We begin with the curve X, an intuitively difficult object. Then we pass to its jacobian jac_X, the quotient of a vector space Ξ, which is a vector space over $\mathbb{C}$ and thus over $\mathbb{R}$ as well by a distinguished lattice Δ. The jacobian carries not only the structure of a complex manifold, but also the structure of an algebraic variety, and of a group. There are also algebraic mappings of $X \times X$, $(x, y) \mapsto x - y$ of $X \times X$ to jac_X. Analytically — and if we exclude all ramification — the functions of immediate interest are functions on a subgroup of the group of idele-classes, namely on the group $I_F^{\mathrm{unr}}\backslash I_F^0 = F^\times \mathbb{I}^{\mathrm{unr}}\backslash\mathbb{I}_F^0$, indeed they are characters of this group. Such characters are determined by their values on the elements represented by $f_{u,v} = \prod_x f_x$, where $f_x = 1$, except for $x = u, v$ and $f_u = z_u^{-1}$, $f_v = z_v$, z_u and z_v being local parameters at u and v. Such functions are obtained by taking characters χ of $\Delta\backslash\Xi$ and pulling them back to functions χ' on $I_F^{\mathrm{unr}}\backslash I_F^0$ by setting $\chi'(fu, v) = \chi(u - v)$, $u - v$ being the image of (u, v) in the jacobian. This does not function in the geometric context because the functions χ are not holomorphic. It does function in the geometric context if we take χ as a holomorphic character of Ξ, thus a function $\exp(\lambda(\,\cdot\,))$ where λ lies in the dual of Ξ as a complex space. This appears to be the expedient found by the geometers. It suggests that analysts, too, not demand that χ be a character of $\Delta\backslash\Xi$, only that it be given by a real linear form λ on Ξ, $\chi(\,\cdot\,) = \exp(i\,\mathrm{Re}\,\lambda(\,\cdot\,))$. This is effectively what we have done.

Each element of the parameters that we propose for the characters of I_F is determined by two elements $\mathrm{Re}\,\omega$, $\mathrm{Im}\,\omega'$ — the first element satisfying the first set of conditions (7.25), the second the second set — because we allow products of $(\omega, \cdot\,)_R$ and $(\omega', \cdot\,)_I$, where ω' may or may not be equal to ω, and by a constant γ that may be taken as real and is only pertinent modulo $2\pi\mathbb{Z}$. It is clear that with the duality proposed, the function of γ is to generate the characters of $I_F^0\backslash I_F$. It is the characters determined by $\mathrm{Re}\,\omega$ and $\mathrm{Im}\,\omega'$ that matter. We pass to them, thus implicitly passing to the quotient by the subgroup of characters generated by the γ. It has already been observed that there is a classical filtration: forms of the first kind (with no singularities) are a subset of forms of the third kind (with at most simple poles, where for our purposes the residues must be integral), and these are in turn a subset of the forms with singularities of arbitrarily high order (but always with integral residues.)

If ω is a form of the first kind, the form $i\omega$ satisfies the condition on the residue of integrality because the residues are all 0. It is therefore unnecessary to include the second $\omega(=\omega')$ or, rather, the contribution $(\omega',\cdot)_I$. More precisely, we have to divide by pairs (ω,ω') of differentials of the first kind for which $\omega=i\omega'$, but this is a fastidious point of the parametrization. Since we know that γ accounts for all characters of $I^0\backslash I$, to establish the desired duality we need only examine the restriction of the remaining characters to I^0. The p that appears in (7.22) is a matter of indifference. The differential forms of the first kind can be regarded as complex linear forms on the complex vector space defining the jacobian. For the exponential $\exp(i \operatorname{Re}\int_0^p \omega)$ to define a character of the jacobian, the real parts of the $2g$ periods of ω must lie in $2\pi\mathbb{Z}$. This is the real part of the condition (7.25). The imaginary part is not relevant here. It clearly defines a lattice in the g-dimensional complex dual of the space defining the jacobian. Thus the characters defined by the ω chosen give exactly the continuous characters of $I^{\mathrm{unr}}\backslash I^0$, which by the classical theory may be identified with the jacobian. This is a repetition — and not the first — of previous reflections. I should probably observe as well that with the conditions (7.25), the formula for λ_R given by the difference between (7.13a) and (7.13b) reduces when $\gamma=0$ to

$$\operatorname{Re}\left\{\sum_j \operatorname{res}_{q_j}(\omega\ln f)-\sum_i \operatorname{ord}_{p_i}(f)\int_p^{p_i}\omega\right\} \tag{7.13c}$$

Since the differential forms of the first kind give, what may be regarded as a complete set of characters on the quotient $I^{\mathrm{unr}}\backslash I$, all we have to do is assure ourselves that differentials with arbitrary singularities, but otherwise satisfying our conditions, give a complete set of characters on I^{unr}, where, of course, the characters defined by the differential forms of the first kind give 1. We must now employ both ω and ω'. On the other hand, we need no longer concern ourselves with the behavior outside of I^{tr}. If we can match, at least on I^{tr}, a given continuous character χ with one χ_1 given by a differential, then we can complete the matching by identifying $\chi\chi_1^{-1}$ with a character associated to a differential form of the first kind, perhaps multiplied, in addition, by the character associated to one of the supplementary parameters $\gamma\in\mathbb{R}$. We first consider forms of the third kind, or rather their real and imaginary parts, treating the two separately. They define characters of $I^{\mathrm{tr}}\backslash I^{\mathrm{unr}}$.

It is clear from (8.6) that for a form ω of the third kind and an idele in I^{unr} the value of $(\omega, f)_R$ is

$$\prod_x (f_x\bar{f}_x)^{in_x/2}=\prod_x \exp(in_xa_x)$$

where n_x is the residue of ω at x, the only constraint being $\sum_x n_x=0$, and where

$$f_x=\exp(a_x+ib_x).$$

For the imaginary part we obtain

$$\prod_x \left(\frac{f_x}{\bar{f}_x}\right)^{n_x/2} = \prod_x \exp(i n_x b_x).$$

These together yield a complete set of characters of $I^{\mathrm{tr}}\backslash I^{\mathrm{unr}}$. The differentials of the first kind yield of course the trivial character.

All that is left to show is that the real and the imaginary parts of all differentials yield all characters of I^{tr}, the differentials of the third kind yielding the trivial character. This is clear from formula (7.4)

One point of view, the analytic, has been explained. Although it is not the immediate issue in this prologue, it is important to explain how the geometric theory and the notion of Hecke eigensheaf accommodate the same — or similar — structures. It seems to me that with some of these matters, whether geometric or analytic, one is walking a fine line between the manipulation of definitions and genuine theorems. So there is reason to be uneasy. One goal, here and in the following section, is to offer, at least conjecturally, a precise description of the group $\mathfrak{A}$ in the global geometrical theory. For its abelian quotient this will be, almost inevitably, a reformulation of classical results for abelian integrals, well understood by specialists and, to some extent, familiar to all. We have just rehearsed those necessary for the analytic theory. I found that there was a kaleidoscopic variability in the way these results presented themselves. I hope I have finally arrived at a stable configuration of the constitutive elements. I now describe briefly the geometric theory, but without attempting to include ramification. In the analytic theory, the parametrization by $\mathcal{P}_X = \mathrm{Bun}_1(X)$ is optional; it seems, on the other hand, to be intrinsic to the Assertion.

The Hecke eigensheaves are supported, according to the definitions of [CFT] on $\mathrm{Bun}\, G$, thus in the context of $G = \mathrm{GL}(1)$ on $\mathrm{Bun} = \mathrm{Bun}_1$. This is also a double coset space of $G(\mathbb{A}_F)$, namely

$$\mathcal{B} = G(F)\backslash G(\mathbb{A}_F)/K,$$

where F is the field of algebraic functions on F, $K = \prod_{x\in X} K_x$, where $K_x = G(\mathcal{O}_x)$ for almost all x but for a finite number of places, thus for $x \in S$, K_x lies between $G(\mathcal{O}_x)$ and a congruence subgroup $\{g \in G(F_x) \mid g \equiv I \pmod{z_x^n}\}$, $n \in \mathbb{N}$. Of course, $G(\mathbb{A}_F) = \prod G(F_x)$. In [CFT] — for $\mathrm{Bun}\, G$ itself — the set S is taken to be empty, but this can scarcely be necessary, and it must be possible, with just a little care, to incorporate the congruence conditions into the discussion. They may even simplify matters, because the introduction of a level structure can remove, I suppose, the vexing complications introduced by stacks.

Hecke eigensheaves accommodate many possibilities because they are sheaves, namely perverse sheaves, but for our purposes here, which is just to make the connection between the geometric theory and the analytic theory, we can take these perverse sheaves to be a local systems of dimension 1, thus line bundles with a connection or locally distinguished constant sections. According to my innocent reading of the notion of perverse sheaf, these are the simplest possibilities. A possibility at a higher level would be the flat structure given not by differentials of the first kind, but by differentials with singularities. Whether they have to be singularities with integral residue, so that the sheaves are single valued locally, I am not yet certain. Perverse sheaves with support are outside my range of experience, as is the extension of a local system over the complement of a proper subvariety to a perverse sheaf over the whole variety. For a first explanation of the notion of a Hecke eigensheaf and its relation with geometric automorphic forms — in the more general form envisaged as functions on $F^\times\backslash\mathfrak{I}_F$ — differentials of the first kind are adequate. The rest the reader can discover on his own. We shall incorporate ramification into the discussion only in so far as necessary to make the ideas clear, perhaps not at all. It is important to understand that the complexities introduced by ramification are an essential feature of the theoretical structure even in the geometric theory, but that the notion of a Hecke eigensheaf as such is of interest in itself and that its extension to the ramified context offers only a very modest addition to one's intuitive understanding.

In the context of line bundles, we consider the Picard group $\mathcal{P}$, which is the moduli space for line bundles. Given a line bundle $\mathcal{L}$ on X, thus a point in $\mathcal{P}$, and a point $x \in X$, we can create a bundle $\mathcal{L}_x$ on X by modifying the notion of a section of L in a neighborhood of x. If the local coordinate near x on X is taken to be z, $z(x) = 0$, then the sections of the modified bundle $\mathcal{L}_x$ are the sections of $\mathcal{L}$ divided by z. As a part of the construction of $\mathcal{P}$ as an algebraic variety, which is, of course, a core element of the theory of algebraic curves, the map h from $X \times \mathcal{P}$ to itself given by $x \times \mathcal{L} \to x \times \mathcal{L}_x$ is algebraic or, if one prefers, holomorphic, with a holomorphic inverse. A perverse sheaf $\mathcal{K}$ on $\mathcal{P}$ can be pulled back to $X \times \mathcal{P}$ and then transferred by h to one on the same space. For our purposes at present, this perverse sheaf need be nothing more than a line bundle provided with a local notion of a constant section, thus a local system, but it is best to be aware of the possibilities. I denote the new sheaf by $h_*\mathcal{K}$. The sheaf K is called a Hecke eigensheaf with respect to a local system E on X if

$$h_*\mathcal{K} = E \otimes \mathcal{K}. \tag{7.26}$$

For those who, like me, are not fully at ease with contemporary mathematics, I recall that a local system is also a perverse sheaf. For $n = 1$, the Assertion is that, given E, we can find a $\mathcal{K}$ that satisfies this equation.

The intuition is clear. Translating within $\mathcal{P}$ by the action of $x \in X$, we modify K, but not in a way that can be detected locally, not even locally over X, although it can be detected globally over X. It is difficult, however, not to become entangled in the various strands of the geometry. The connected component $\mathcal{P}^0$ of the Picard variety parametrizes bundles of degree 0 and differs only slightly from the full variety, but it differs in an important way. The homology and cohomology groups of $\mathcal{P}^0$ over $\mathbb{Z}$ and $\mathbb{C}$ are the same as those of X in degrees 0 and 1. So, in the following form the first part of Diagram I applies to both X and $\mathcal{P}^0$,

$$\begin{array}{ccccccc} \{0\} & \longrightarrow & H^1(\mathbb{Z}) & \longrightarrow & H^1(\mathbb{C}) & \longrightarrow & H^1(\mathbb{C}^\times) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \{0\} & \longrightarrow & H^1(\mathbb{Z}) & \longrightarrow & H^1(\mathcal{O}) & \longrightarrow & H^1(\mathcal{O}^\times)_{c_1=0}, \end{array} \tag{7.27}$$

where in the lower right-hand corner only those elements with Chern class equal to 0 are allowed, thus line bundles of degree 0.

Consequently, in the case of $\mathcal{P}^0$, we may continue to consider local systems on $\mathcal{P}^0$ as line bundles together with a differential form of the first kind. Local systems on $\mathcal{P}$ — the only kind of perverse sheaf that I want to consider here — are just pieced together from local systems on its various components. Different components are linked by (7.26), which appears in [CFT] as Equation (3.9). Let $\mathcal{P}^n$ be the elements of $\mathcal{P}$ of degree n, $n \in \mathbb{Z}$. The comparison (7.26) effectively compares a sheaf on the connected component $\mathcal{P}^n$ on the right with the same sheaf but over $\mathcal{P}^{n+1}$ on the left, but on both sides there is an extra parameter, one of which, that on the left, is modifying the sheaf, while the other does not. So if we apply the equality twice, once in one sense, once in the other, and take the varying parameters into account, we see that we are imposing a condition on $\mathcal{K}$, a condition that is described by E. All we need to do is ensure that the condition is satisfied as we pass from 0 to 1 and then, back again, from 1 to 0. That takes care of the necessary equality at the level 0, and then (7.26) routinely takes us through the other integers $n = \pm 1, \pm 2, \ldots$.

From the identity of (7.27) for X and $\mathcal{P}^0$, we may identify a line bundle with Chern class 0 on X and with one on $\mathcal{P}^0$ and a flat connection on the first with one on the second. How does this function? We denote the construction in which rather than admitting a pole of order 1 at x, we add a zero, passing from $\mathcal{L}$ to $\mathcal{L}_{-x}$ and introduce the corresponding map from $X \times \mathcal{P}$ to $\mathcal{P}$ by $h' : x \times \mathcal{L} \to x \times \mathcal{L}_{-x}$. Then $y \times x \times \mathcal{L} \to y \times x \times \mathcal{L}_{x-y}$ takes $X \times X \times \mathcal{P}^0$ to $\mathcal{P}^0$ and (7.26) is replaced by an equation for the restriction $\mathcal{K}^0$ of $\mathcal{K}$ to $\mathcal{P}^0$,

$$h'_* h_* \mathcal{K}^0 = E^{-1} \otimes E \otimes \mathcal{K}^0 \tag{7.28}$$

on $X \times X \times \mathcal{P}^0$. The notation $\mathcal{L}_{x-y}$ is simply a more elegant, and perhaps more suggestive, way of writing $(\mathcal{L}_x)_{-y}$.

To prove (7.28) we need to know:

($\star$) The jacobian, thus the group of divisors of degree 0, is identical with the elements of degree 0 in $\mathcal{P}$, this identification being given by mapping the divisor $\delta = \sum_i \pm x_i$, $x_i \in X$ to the line bundle $\mathcal{L}_\delta$ whose sections are functions f with $\operatorname{div} f + \delta \geq 0$.

($\star\star$) The isomorphism between the various cohomology groups appearing in Diagram I on the one hand and (7.27) on the other can be obtained by pull-back from $x \to \mathcal{L}_{x-y}$ with a fixed y and a fixed $\mathcal{L}$.

So (7.28) is simply the assertion that E is the pull-back of $\mathcal{K}$. It seems to be much ado about nothing, but that would be, I suspect, a view that failed to appreciate the marvels of the theory created by Abel, Jacobi and others.

This discussion suggests that, at least for GL(1), one neither wins nor loses by working with the arithmetic/analytic structures rather than the geometric, but it does not suggest to me a direct equivalence. The space $\mathrm{Bun}_1(X) = \mathcal{P}$ is implicated in an essential way in the statement of the (geometric) Assertion. In the analytic theory $\mathrm{Bun}_1(X)$, or rather its connected component, appears as an optional enlargement of the group of characters. There is one respect in which the analytic theory appears to offer an advantage: the description of the group $\mathfrak{A}$. This description, which shall be formulated and verified for GL(1) in this section, and for general quasisplit G in the following section, but only as a conjecture that will not be entirely precise, has to serve as my apology for an irritatingly lengthy rehearsal of familiar classical material and the modern geometrical viewpoint.

For the local theory, an analytic theory, the group to be parametrized is formed by the characters of $F_x^\times$. Apart from the ambiguities in the extension of diagram (7.5), the parametrization is given by differentials. So, to be as precise as possible, because we are (almost!) dealing with definitions rather than theorems, just as the characters of $\mathrm{GL}(1, F_x)$ are identified with homomorphisms of the Weil group into GL(1) in the local arithmetic, so characters of $\mathrm{GL}(1, F_x)$ (or, at first, $\mathrm{GL}(1, \mathcal{O}_x)$) are associated with differentials ω with values in the Lie algebra of GL(1) over F_x, or rather with their principal parts. These form a group and should be regarded as the abelian form $\mathfrak{A}_x$ of the local Weil group in the geometric context, with multiplication given by addition of differentials, except that the extension of Ω_x to $\tilde{\Omega}_x$ of diagram (7.5) is needed to complete the construction.

Globally, we have introduced a similar relation between differentials and characters, except that there is no longer a question of discarding the regular parts of the differentials. Moreover, the characters are not characters of idele classes I_F but of an enlarged group $F^\times \backslash \mathfrak{I}_F$. Multiplication of characters becomes addition of differentials. It is this group, or rather an extension of it by the group of characters of the group $\mathcal{P}^0 \backslash \mathcal{P} \simeq \mathbb{Z}$, that functions as the abelianized form $\mathfrak{A}_{\mathrm{ab}}$ of the group

$\mathfrak{A}_F$. So it is an analogue of the abelianized Weil group with multiplication given by addition of differentials,

$$\frac{df}{f} = \omega_1 + \omega_2. \tag{7.29}$$

I add that class field theory has accustomed us to identify, in the arithmetic theory, the abelianized form of the Weil group with I_F and the Weil group itself with a subgroup of the Galois group. There is a merging of definition and theorems that, if we are not careful, obscures for us the accomplishments of the past.

Before turning to the theory for a general group, I remark that I may have found partial answers to two questions while struggling not with proofs, but just with the formulation of conjectures and assertions in the geometric theory: (i) what are the respective merits of the geometric and analytic standpoint? (ii) what is the interest of the geometric theory in itself, thus what are the principal theorems or conjectures, independently of any relation to quantum field theory? The response to the second question is best left to §9. The response to the first question is tentative, especially as there are a number of clumsy aspects to the analytic theory for a general group and even for $\mathrm{GL}(1)$. The difficulty with the geometric theory is that there are so many possibilities that they are never exhausted. In the theory of Fourier transforms there are many possibilities: the spectral theory for square-integrable functions; Paley–Wiener theorems; theorems related to Schwartz distributions of various sorts; the Laplace transform. I am inclined to take the spectral theory as central. For the geometric theory, there is a similar difficulty. What is the core problem? My hope for a spectral theory is that one could formulate a clearly defined spectral problem, thus an L^2-problem — differential operators with boundary conditions — whose solutions on Bun_G could be regarded in at least some respects as a definitive formulation of the existence problem for Hecke eigensheaves: an eigensheaf (or eigenfunction) $\mathcal{K} = \mathrm{Aut}_E$ on Bun_G with eigenvalue a ${}^LG$ local system E on X is a pair characterized by a certain set of conditions on E and by the relation between E and $\mathcal{K}$.

The eigenvalue — in a sense like that of the geometric theory — is $\exp(i\,\mathrm{Re}\,\omega)$. It is $\mathrm{Re}\,\omega$ (or $\mathrm{Im}\,\omega$) that is characterized by a differential equation, as the real part of an analytic function it is harmonic outside of the singularities and with circumscribed behavior at the singularities, for the residue is integral. Notice, in passing, that we can recover $\mathrm{Im}\,\omega$ or ω — up to an unimportant constant from $\mathrm{Re}\,\omega$ — and the Cauchy–Riemann equations. The function $\mathrm{Re}\,\omega$ is moreover implicitly subject to a boundary condition. We have made the boundary condition more flexible, even removed it, by introducing $\mathfrak{Z}_R$ (or $\mathfrak{Z}_I$), but that was necessary only to keep up with the geometers. The boundary condition is a condition not on $\mathrm{Re}\,\omega$ as a function on X but on the function (sheaf for the geometers) associated to it on $\mathcal{K}$. Boundary conditions on ω itself would double their number and yield an overdetermined eigenvalue problem.

When we allow singularities, $\mathcal{H}$ is replaced by a quotient $\mathbb{I}_F/\prod_x K_x$, where $K_x = \mathbb{O}_x^\times$ for almost all x, but for a finite number of x it is the set of $f_x \in \mathbb{O}^\times$ that are congruent to 1 modulo some positive power $z_x^{n_x}$ of z_x. It may not be immediately apparent, but the definitions (7.13a), (7.13b), and the formulas for $(\omega,\cdot)_R$, $(\omega,\cdot)_I$ are an expression of the condition (7.26), although that condition refers more to the characters defined by $\omega = \operatorname{Re}\omega + i \operatorname{Im}\omega$. In the geometric theory it is only the local conditions on ω that matter, not the boundary conditions or conditions of periodicity. As a consequence, or so it seems to me, there is for a general group some difficulty in formulating the problem of existence.

It is difficult to recognize the equation (7.26) in the conditions (7.13a) and (7.13b) for at least three reasons: (i) the condition (7.13b) is a matter of conditions of periodicity; (ii) the final term of (7.13a) is constant on connected components of Bun_1, so that in the geometric or sheaf-theoretic context it has no meaning; (iii) the first term is not present when ω has no singularities. Although the geometers are well aware of the possibility of singularities [FG], they are not studied in [CLG]. Even when singularities are present, (7.26) is likely to remain, in that form, the telling geometric condition.

One of the purposes of the next section is to begin the search, for a general G, for a construction of automorphic representations analogous to that given for $\mathrm{GL}(1)$ by $(\omega,\cdot)_R$ or $(\omega,\cdot)_I$. Since these are unitary characters, we have to expect unitary characters (representations) for a general group as well. The middle term of expression 7.12.a controls the unramified contribution. The first term controls the character on each $\mathbb{O}_x^\times$. The condition that the residue n_x of ω at each point x be integral implies that $(\omega,\cdot)_R$ and $(\omega,\cdot)_I$ yield respectively r^{in} or e^{imn}, where $n = \operatorname{res}_{q_j}\omega$, $f(q_j) = re^{im}$.

8.a. The geometric theory for a general group (provisional)

Such a theory is not yet available even in embryonic form, although some reflections are suggested by the previous constructions. As I observed in the previous section, these constructions are perhaps not merely my interpretation of those explained in [CFT], but are implicit in the proof of Weil's theorem. The relation between Lemma 7.1 and the calculation that yields it differ on the face of things from formula (7.28) and its proof. Their interpretations are also informed by a different impulse: sheaves are replaced by differential equations. For regular holonomic systems, there is presumably an equivalence available [HTT]. After the admission of differentials with more general singularities, this may no longer be so, although that is unlikely. My impulse arises, however, from other sources: from a greater familiarity and perhaps even greater ease with differential equations than with sheaves, or perhaps from a suspicion that, important, powerful, and fascinating as they are, in recent decades an excessive, sometimes inappropriate, appeal to sheaves has, inadvertently, had an unhealthy influence on some parts of mathematics or, rather, on some of its practitioners; and from an attachment, already expressed, to representation theory, as introduced, in a remarkable, but little read, sequence of papers by Frobenius, in response to a suggestion of Dedekind and then developed by several major mathematicians of the twentieth century.

For a general group G, even if it is split, as I suppose in this text, there is not only no global geometric theory yet available, there is also no local theory. Moreover, there is an extra question. What is the relation between, on the one hand, the functoriality of the geometric theory, the identification of the group $\mathfrak{A} = \mathfrak{A}_{\text{geom}}$, and a description of its properties and, on the other, the Langlands duality featured in gauge theory? Are they one and the same, or are they different? That they are different, occurred to me on reading a brief, but instructive and suggestive letter, that I received from David Nadler in March of 2011. Nadler writes: "The 6-dimensional theory Z depends not on a group G but only on the combinatorics of G in a way that is unbiased towards G and its dual group $G^\vee$." This is not so for the theory to whose preliminary exploration this section is devoted: G and ${}^LG$ (or $G^\vee$) do not play symmetric roles! Moreover, there is no 6-dimensional field theory in sight. So there is a great deal left for me, and perhaps not for me alone, to understand. It will be best not to broach this question until Section 9, yet to be written. It requires a good deal of supplementary reflection, informed by some knowledge of field theory.

Indeed, even my attempt to broach the purely mathematical questions turned out to be premature. One of the principal mathematical problems of the geometric theory, perhaps *the* principal one, is the identification of the geometric galoisian group $\mathfrak{A}_{\text{geom}}$ in terms of differentials, thus the general form of the identification of its abelian quotient in the previous section. This is by no means a simple matter, for it demands a serious understanding, not merely a formal understanding, of moduli spaces for vector bundles and G-bundles, of the differential geometry of these bundles as in [Si], and of the relation between $\mathfrak{D}$-modules and perverse sheaves. These are all very rich subjects, of which I could not hope to acquire an adequate understanding before the deadline imposed by the editor of this volume, if ever. So I was forced to content myself with some provisional suggestions just to intimate to the reader what I have in mind. As will be almost immediately evident, there are major unresolved difficulties left open.

The identification of $\mathfrak{A}_{\text{geom}}$ entails functoriality for the geometric theory. If there is some form of reciprocity — different as our title implies from functoriality — in the geometric theory, I do not know how to formulate it. The geometric Langlands program as envisioned in [KW] contains, I suspect, much, much more than the identification of the geometric galoisian group $\mathfrak{A}_{\text{geom}}$ in terms of differentials envisaged in this section. It may, indeed, have little relation to it. It does contain a kind of *duality*, but it may be best to distinguish this duality from the *reciprocity* in the arithmetic theory and from any concrete identification of $\mathfrak{A}_{\text{geom}}$, although it is clearly related to this. It had been my intention to begin, in a ninth section, the attempt to understand [KW] and, more generally, the many and various contributions to the geometric theory and its relation to quantum field theory, but that, as I have already confessed, is matter for an even more distant and more uncertain future.

In Section 7, the emphasis was on functions on $\text{Bun}_G = \mathcal{P}_X$, $G = \text{GL}(1)$; sheaves were not emphasized. There is, indeed, a major difference. The forms ω were parametrized by a local system and then by a second parameter in $H^1(\mathbb{Z})$. Two elements of $H^1(\mathbb{R})$ in the first line of Diagram I that differ by an element of $H^1(\mathbb{Z})$, define isomorphic local systems or, viewed from another angle, an automorphism of a given line bundle replacing one local system by another, thus, in terms of one of the local systems multiplication of the flat connection by the character $\exp(i\,\text{Re}\,\omega)$, where the exponent is constrained to be the imaginary part of a holomorphic function. So there is a mixing of a real (unitary) theory and a complex (holomorphic) theory. This brings with it advantages but also difficulties. One of the difficulties for me is that — as is clear from [Si] and the works there cited — the mixing for a general group demands very serious differential-geometric preparation, not merely the Cauchy–Riemann equations. One advantage, already explained, is that, in the analytic theory, we can hope to formulate the problems in the context of a spectral theory in an L^2-space.

I expected, on first reflecting on the matter, that, as for GL(1), the group $\mathfrak{A}_{\text{geom}}$ will be given by a kind of inverse limit of differentials with values in ${}^L\mathfrak{g}$, the inverse limit being taken over $\omega \to \omega'$, where ω' is the image of ω under a homomorphism ${}^LG \to {}^LG'$ in the sense of L-groups. So the inverse limit is over the group, the direct limit over the differentials. In order to deal with all automorphic representations, we would have to admit, as for GL(1), differentials with singularities. Whether there should be restrictions on the residues similar to those we described for GL(1) can be left moot. Even for those without singularities, there is a great deal of theory to understand.

As an aside, I mention that, following [CFT], I shall take the group action on G-bundles and on ${}^LG$-bundles to be on the right. So, once we have fixed a local trivialization, the differentials generate along curves a function with values in ${}^LG$ according to the differential equation $dg \cdot g^{-1} = \omega$.

My first expectations were perhaps, in the light of our understanding of the geometric theory, too naive, too influenced by the construction of the geometric $\mathfrak{A}_{\text{ab}}$. If there is homomorphism ${}^LH \to {}^LG$, then the differential with values in $\hat{\mathfrak{h}}$ transfers to $\hat{\mathfrak{g}}$, so that if a parametrization of automorphic representations or forms in the geometric theory is established, functoriality will be an immediate consequence. In the arithmetic theory the definition of the galoisian group $\mathfrak{A}_{\text{arith}}$ is based on functoriality and on the notion of a hadronic representation, itself based on the properties of $L(s, \pi, \rho)$, $\rho : {}^L G \to \mathrm{GL}(n)$. This assumes, in particular, that in establishing functoriality we have also completely understood the nature of the Ramanujan conjecture and the Arthur parameters. Although I have alluded to these in the arithmetic context, I have not attempted any, even conjectural, definition in a geometric context. This would be reckless without more experience with the classification of bundles for specific groups on specific curves, with the parameters, and with the corresponding automorphic representations. We need more concrete assurance that the parametrization proposed here is correct and some insight into its specific consequences. There may be surprises. This is one of the many reasons that this section is provisional. In one way or another, the parameter obtained from ω on transfer to $\mathrm{GL}(n)$ under an irreducible ρ will be a direct sum of irreducible parameters, ω_i, for $\mathrm{GL}(n_i)$, $\sum_i n_i = n$,

$$\omega \to \bigoplus_i \omega_i. \tag{8.1}$$

The initial parameter ω would be hadronic if there were no i for which $n_i = 1$ and ω_i is trivial.

There are also many other many other matters to consider. We have somehow to reconcile the unitary and holomorphic (or meromorphic) forms of this equation. I am not yet in a position to do so and am uneasy about suggesting definitions that I do not understand. For the moment, the definition of $\mathfrak{A}_{\text{geom}}$ remains, at best, imprecise. To make it clear that anyone who, like me, has little or no differential geometrical experience has much to learn, I quote one of the first paragraphs in [Si], which treats $\mathrm{GL}(n)$, which can for us be regarded as typical. Although our concern is with complete nonsingular curves, the statement in [Si] refers more generally to smooth, projective X. "A *harmonic bundle* on X is a C^∞ vector bundle E with differential operators δ and $\bar{\delta}$ and algebraic operators θ and $\bar{\theta}$ (operators from E to one-forms with coefficients in E), such that the following hold. There exists a metric K so that $\partial + \bar{\partial}$ is a unitary connection and $\theta + \bar{\theta}$ is self-adjoint. And if we set $D = \partial + \bar{\partial} + \theta + \bar{\theta}$ and $D'' = \bar{\delta} + \theta$, then $D^2 = 0$ and $(D'')^2 = 0$. With these conditions, (E, D) is a vector bundle with flat connection, and $(E, \bar{\partial})$ is a *Higgs bundle*: a holomorphic vector bundle with holomorphic section θ such that $\theta \wedge \theta = 0$." Of course, for a surface some of the assertions are superfluous. This statement is followed by a theorem whose first sentence I repeat. "There is a natural equivalence between the categories of harmonic bundles on X and semisimple flat bundles (or representations of $\pi_1(X)$)." I observe that this statement does not take into account a possibility that we encountered for $\mathrm{GL}(1)$: automorphisms of the unitary bundles and associated automorphic forms.

There are so many questions in the geometric theory, both local and global, that have never been touched, that I am more than a little uncertain of the similarities and differences between it and the arithmetic theory. Perhaps we should begin by stating clearly the difference between the objects in the analytic theory and the objects in the geometric theory. They are all constructed from the curve X, a set of points, but also a Riemann surface and the "set" $\mathrm{Bun}_G(X)$ of G-bundles on X, with or without the extra structure that allows the introduction of ramified automorphic forms. The set Bun_G may be identified with $G(F)\backslash G(\mathbb{A}_F)/\prod_{x\in X} G(\mathbb{O}_x)$ (see [CLG]) or, if there is extra structure, with $G(F)\backslash G(\mathbb{A}_F)/\prod_{x\in X} K_x$, where K_x is equal to $G(\mathbb{O}_x)$ for almost all x, say $x \notin S$, S finite, and, for example, equal to $G_x^{n_x}$ for $x \in S$, where n_x is a nonnegative integer and $G_x^{n_x}$ the set of elements in $G(\mathbb{O}_x)$ congruent to 1 modulo $\mathfrak{m}_x^{n_x}$, where $\mathfrak{m}_x$ is the maximal ideal in $\mathbb{O}_x$, but it may also be constructed geometrically.

Although these two descriptions yield the same two sets, they yield functionally dissimilar objects. As identified with $G(F)\backslash G(\mathbb{A}_F)/\prod_{x\in X} G(\mathbb{O}_x)$, $\mathrm{Bun}_G(X)$ is a set in the familiar sense; constructed as a *stack* it has, essentially, the structure of an algebraic variety, which can be recovered from that on the quotients $G(F_x)/G(\mathbb{O}_x)$, a matter to which we shall return but not to examine it in depth. In the one context, the appropriate objects are functions; in the others, sheaves. As we have seen, we can expect — or hope — that for functions more precision is possible because more structure is possible.

It is not clear where it is best to begin, perhaps by reviewing the very little we know about the local theory, or, if one prefers, what we clearly do not know. Like the theory over a number field or one of its completions, the analytic theory over the function field and over its completions F_x is a theory about representations, usually infinite-dimensional. The group $G(F_x)$ has a sequence of decreasing subgroups: $G(F_x) \supset G_x^{\mathrm{unr}} \supset G_x^{\mathrm{tr}}$, where $G_x^{\mathrm{unr}} = G(\mathbb{O}_x)$, $G_x^{\mathrm{tr}} = G_x^1$ is the set of g in G_x^{unr} whose power series expansion begins with the identity. These subgroups are of course not normal, but we can consider, as a first, coarse classification locally, irreducible representations whose restriction to one or the other of the subgroups contains this or that irreducible representation. The most important are those whose restriction to G_x^{unr} contains the trivial representations. Their theory is the theory of spherical functions and characters of the Hecke algebra. We can admit all characters; we can admit "tempered" characters; we can admit those characters that correspond over a local arithmetic field to Arthur parameters, although Arthur parameters in the geometric context are certainly not a topic to broach in this prologue. The theory of spherical functions, a generalized form of the theory of elementary divisors, will be, in many respects, the same for the geometric theory as for the arithmetic theory. So unramified characters will be parametrized by a conjugacy class t in ${}^LG(\mathbb{C})$, or, even, if we assume, as we have done, that G is split, by a semisimple class in $\hat{G}(\mathbb{C})$. We can take this class to be arbitrary and this corresponds to a geometric theory or we can take it to lie in a compact form $\hat{U}(\mathbb{R})$ of $\hat{G}(\mathbb{C})$ and this corresponds to the tempered analytic theory, the adjectives "geometric" and "analytic" having for the moment only the imprecise meaning suggested by various remarks in the previous section.

In contrast to the arithmetic theory, the representations of the group $G_x^{\mathrm{unr}}/G_x^{\mathrm{tr}}$ may be infinite-dimensional. There are two possibilities. We can consider either representations of $G(\mathbb{C})$ or the representations of its compact form $U(\mathbb{R})$. As for GL(1), when dealing with a higher-dimensional G we have to choose between a holomorphic theory and an analytic theory. I choose, because of preferences already acknowledged and for reasons already explained, the analytic theory. It is not entirely clear what this implies even locally. The abelian theory suggests that it is the representations of $G(\mathbb{C})$ that we need. Although this is a noncompact reductive group and the distinctions of §2 — the class of tempered representations, the Arthur class, general representations whether unitary or not — may be pertinent, it may be the finite-dimensional representations, these suggested because the trivial representation certainly appears in the unramified theory, to which we should pay the most attention. They can be holomorphic or antiholomorphic or some mixture of the two. The usual (Langlands) classification parametrizes the tempered representations by conjugacy classes of homomorphisms of $W_{\mathbb{C}} = \mathbb{C}^\times$ into ${}^LG$, thus by $z = re^{i\theta} \to r^\lambda e^{i\mu(\theta)}$ where λ is a real linear combination of weights of G and μ a weight of G, the pair (λ, μ) being given up to conjugation up to the action of the Weyl group. The holomorphic finite-dimensional representations correspond to unitary representations of the unitary form of G and these correspond to homomorphisms of $U(1)$ into ${}^LG$, thus to $\lambda = 0$. The similarity of the parametrizations of the characters of the Hecke algebra (related to unramified representations) and of certain representations of $G_x^{\mathrm{unr}}/G_x^{\mathrm{tr}}$ (related to tamely ramified representations) is curious and gives pause for reflection.

It is suggested by the theory for GL(1) — and confirmed by various reflections, although by no means certain — that a central role will be played by differentials with values in the Lie algebra $\hat{\mathfrak{g}} = {}^L\mathfrak{g}$ and their real parts, taken in an appropriate sense, which I hesitate to attempt to make precise without a better understanding of the differential geometric theory [Si]. They will define the local system E of the Assertion of §7 or of the related Conjecture that follows in this section. What are the restrictions on these differentials and what do we mean by their real parts? The first question to be answered is what the nature of their residues must be, for — as I suppose — the residue at a point controls, when their is no higher order singularity, the representation of $G_x^{\mathrm{unr}}/G_x^{\mathrm{tr}}$. So far as I know, the representations of G_x^{unr}/G_x^n, $n > 1$, have been little studied, nor, of course, have those of $G(F_x)/G_x^n$, $n \geq 1$. The discussion in this section is predicated on the hypothesis that they are controlled by the singular part of a Laurent expansion of a differential ω with values in $\hat{\mathfrak{g}}$. The residue, thus the coefficient of $1/z$, will be an element of $\hat{\mathfrak{g}}$. The demand that it be integral is compatible with our discussion of the tamely ramified parameter μ in the preceding paragraph.

The group G_x^{tr} is an infinite-dimensional Lie group. We shall only be concerned with representations of the finite-dimensional quotients G_x^{tr}/G_x^n. It is a finite-dimensional simply connected nilpotent Lie group and its irreducible unitary representations are classified by the method of coadjoint orbits [VE], thus by conjugacy classes in the dual of the Lie algebra over $\mathbb{C}$ of G_x^1/G_x^n. Thanks to the sequence $G_x^1 \supset G_x^2 \supset \cdots$ these coadjoint orbits form an increasing sequence of sets. This can be interpreted to state that they are parametrized by the singular parts of local differentials ω at x with residue 0 and values in $\hat{\mathfrak{g}}$.

These facts together suggest, but hardly prove, that the local parametrization for a general group is very much like that of the diagram (7.5) for GL(1), although I do not yet know how to define in general the patching of conjugacy classes of GL($\mathbb{C}$) that appears in its upper line. This suggestion will be taken as an hypothesis for the remainder of this section. I have made no attempt to prove it. The local theory is only a part of the unresolved difficulties, and this prologue, even the essay *Functoriality and Reciprocity* that I hope will follow it, is intended to be no more than a first exploration of possibilities.

As with the arithmetic theory, the major issues in the geometric theory will be global. They may not be so difficult as for the arithmetic theory, but the theory of vector bundles or of G-bundles on curves over $\mathbb{C}$ is very rich and for me largely unfamiliar, so that I could very easily overstep the limits of my knowledge, which are severe. It would certainly be presumptuous for me to say too much at this stage, but I do want to sketch the possibilities. Although the problem of describing the global geometrical galoisian $\mathfrak{A}_{\mathrm{geom}}$ may be more accessible than that of describing $\mathfrak{A}_{\mathrm{arith}}$, we can expect it to be difficult and to require a good deal of experience and technical skill. It is my hope that the arguments for GL(1), especially the proof of Lemma 7.1 in which the calculus of residues is applied, will serve as a model.

The principal issue is to understand the unramified theory or, better, the theory at the unramified places. For the unramified theory, the basic object is

$$\text{(8.2a)}\quad \begin{gathered} G(F)\Big\backslash \prod_{x\in X} G(F_x)\Big/\prod_x G(\mathbb{O}_x) = \varinjlim\nolimits_T G_T(F)\Big\backslash \prod_{x\in T} G(F_x) \prod_{x\notin T} G(\mathbb{O}_x)\Big/\prod_x G(\mathbb{O}_x), \\ G_T(F) = G(F)\cap\Big\{\prod_{x\in T} G(F_x) \prod_{x\notin T} G(\mathbb{O}_x)\Big\}, \end{gathered}$$

where T can be taken as large as appropriate. If there is ramification, the first line will be replaced by

$$\text{(8.2b)}\quad G(F)\backslash \prod_{x\in X} G(F_x)\Big/\prod_x K_x = \varinjlim_T G_T(F)\backslash \prod_{x\in T} G(F_x)\prod_{x\notin T} G(\mathbb{O}_x)\Big/\prod_x K_x,$$

where K_x is open in $G(\mathbb{O}_x)$ for all x and $K_x = G(\mathbb{O}_x)$ for $x \notin S \subset T$.

The set (8.2a) is Bun_G and the set (8.2b) is Bun_G with frills. There are explanations to be given because Bun_G is, whether as a variety, as an injective limit of varieties, or as a stack, an algebro-geometric object. For us, however, who want to make the connection with the geometric theory of automorphic forms, it is also simply a set. Whether as a variety or as a set it is a quotient. We begin in the unramified context with the trivial G-bundle, which we modify by an element of $\prod_{x\in T} g_x$, $g_x \in G(F)$. If we take $g_x = 1$, $x \in T' - T$, $T' \supset T$, this can be regarded as an element of $\prod_{x\in T'} G(F_x)/\prod_{x\in T'} G(\mathbb{O}_x)$, so that we have an injective family of sets. Since each g_x is defined in a neigborhood of x and regular in this neighborhood except at x and since we can enlarge T to T', we can suppose these neighborhoods cover X, so that the collection $\{g_x\}$ defines a G-bundle. We can even suppose that $g_x \in G(F)$ because the set $g_x G(\mathbb{O}_x) \cap G(F)$ will not be empty, and we can replace g_x by an element of this set. The choice does not affect the bundle. The conclusion is that any choice of $g \in \prod G(F_x)$ defines a G-bundle on X and that all G-bundles are obtained in this way.

Thus Bun_G is constructed as a limit of the quotient of

$$\prod_{x\in T} G(F_x)/G(\mathbb{O}_x)$$

by $G_T(F)$. Each point of $G(F_x)/G(\mathbb{O}_x)$ represents a modification of Bun_G at the point x; it is an extremely complicated variety, the direct limit of finite-dimensional subvarieties. Thus, starting from single point of $\mathrm{Bun}\, G$, the trivial bundle, and repeatedly modifying the bundle, each modification at perhaps a different point, we can reach any bundle. So there are two sources of complexity in the construction of Bun_G. They are the modifications and the divisions by $G_T(F)$. It is well to give some examples for vector bundles, thus for the groups $\mathrm{SL}(n)$ and $\mathrm{GL}(n)$, to see how the parameters in $G(F_x)/G(\mathbb{O}_x)$ and the parameter x together lead to very complex modifications that may, because of the division by $G_T(F)$, yield a bundle isomorphic to that with which we began. The topological or geometrical structure is a combination of the double parametrization: by the parameter of $x \in X$ and by the coordinates on $G(F_x)/G(\mathbb{O}_x)$. For $\mathrm{GL}(1)$ this double parametrization is simple because $G(F_x)/G(\mathbb{O}_x) \simeq \mathbb{Z}$. Thus, as in the second term on the left of (7.15), the only relevant parameters are p_i, basically a point in a neigborhood of p, and the integer $\mathrm{ord}_{p_i}(f)$. For groups of higher dimension, the parameters are far more complex.

Consider $G = \mathrm{GL}(n)$ and, first of all, the structure of $\mathrm{Bun}_x = G(F_x)/G(\mathbb{O}_x)$ as a space or variety, whose dimension is infinite, on which $G(\mathbb{O}_x)$ acts to the left. If z is the local coordinate at x, the representatives of the double cosets in $G(\mathbb{O}_x)\backslash G(F_x)/G(\mathbb{O}_x)$ are the matrices

$$t(m_1,\ldots,m_n)=\begin{pmatrix} z^{m_1} & 0 & 0 & \ldots & 0\\ 0 & z^{m_2} & 0 & \ldots & 0\\ 0 & 0 & z^{m_3} & \ldots & 0\\ \vdots & \vdots & \vdots & & \vdots\\ 0 & 0 & 0 & \ldots & z^{m_n}\end{pmatrix},$$

where the m_i are integers, implicitly subject to the condition $m_1 \geq m_2 \geq \cdots \geq m_n$. The variety Bun_x is a union of the connected components $\mathrm{Bun}_{x,m}$ defined by the condition $\sum_i m_i = m$. Multiplying by a scalar matrix, we replace $\mathrm{Bun}_{x,m}$ by $\mathrm{Bun}_{x,m+kn}$, $k \in \mathbb{Z}$. We shall consider some examples, taking small values of m and n.

For $m = 0$ and all n, there is a distinguished point,

$$q_0 = G(\mathbb{O}_x)t(0,0,\ldots,0)G(\mathbb{O}_x)/G(\mathbb{O}_x).$$

If $n = 1$, it is the only point in $\mathrm{Bun}_{x,0}$. In general,

(8.3)

$$\mathrm{Bun}_{x,m_1,\ldots,m_n} = G(\mathbb{O}_x)t(m_1,\ldots,m_n)G(\mathbb{O}_x)/G(\mathbb{O}_x) \simeq G(\mathbb{O}_x)/P(m_1,m_2,\ldots,m_n),$$

where

$$\begin{aligned} P(m_1,m_2,\ldots,m_n) &= G(\mathbb{O}_x)\cap(tG(\mathbb{O}_x)t^{-1}\cap G(\mathbb{O}_x))\\ &= \{(a_{i,j}) \ni a_{i,j} \equiv 0 \pmod{z^{m_i-m_j}}\}, \end{aligned}$$

with $t = t(m_1,\ldots,m_n)$.

As in [CFT], we can try to grasp the full space $G(F_x)/G(\mathbb{O}_x)$ by writing the elements of $G(F_x)$ as products nak, where n is unipotent and upper-triangular, thus $n \in N(F_x)$, a is a diagonal matrix $T = t(m_1,\ldots,m_n) = \mathrm{diag}(t^{m_1},\ldots,t^{m_n})$, and $k \in G(\mathbb{O}_x)$. The connected components are then given as algebraic varieties by

$$N(F_x)/(N(F_x)\cap TG(\mathbb{O}_x)T^{-1}) \times G(\mathbb{O}_x), \tag{8.4}$$

which is closed in the full variety. The structure of the first factor has to be explained, but it is intuitively clear. For example, if $n = 2$ and $m = m_1 - m_2$, then a full set of representatives for the quotient in (8.4) is given by

$$n(p) = \begin{pmatrix} 1 & p(t)\\ 0 & 1\end{pmatrix}, \tag{8.5}$$

where p is a finite Laurent series with an indefinite number of nonzero terms of negative degree, $p(t) = \sum_{k<m} a_k t^k$. If $p(t)$ is identically 0, then $n(p)T$ lies in the

double coset with parameter (m_1', m_2'), where $\{m_1', m_2'\} = \{m_1, m_2\}$ and $m_1' \geq m_2'$. Otherwise, let l be the least k for which $a_k \neq 0$. Then

$$n(p)T = \begin{pmatrix} t^{m_1} & t^{m_2+l}\alpha(t) \\ 0 & t^{m_2} \end{pmatrix}, \tag{8.6}$$

where $\alpha(t)$ is a polynomial with nonzero constant term. If $m_2 + l \geq \min\{m_1, m_2\}$, this lies in the same double coset as when $p(t) = 0$, otherwise it lies in the double coset with parameter $\{m_1', m_2'\} = \{m_1 - l, m_2 + l\}$ and $m_1 - l \geq m_2 + l$. Since we can choose l to lie as far to the left as we like and then let all the coefficients of $\alpha(t)$ approach 0, we conclude that one coset can lie in the closure of many others.

On the other hand, some of the double cosets (8.3) are closed. If $m_1 \geq m_2$, the relation between the parameters at the end of the preceding paragraph is $m_1 - l \geq m_1 \geq m_2 \geq m_2 + l$. If $(m_1 - l) - (m_2 + l) = 1$, this is out of the question. If $m_2 \geq m_1$, the relation is $m_1 - l \geq m_2 \geq m_1 \geq m_2 + l$ and there is the same difficulty. So the set $\mathrm{Bun}_{x,1,0}$ is closed.

There are clear relations of containment between the various groups $P(m_1, m_2, \ldots, m_n)$, that yield mappings between the various sets $\mathrm{Bun}_{x,m_1,\ldots,m_n}$ or, more generally, between the analogous varieties for a general G. They are usually referred to as a blowing-up or a blowing-down, or as Hecke correspondences, or as modifications. It is certainly appealing and useful to keep the geometric language and the geometric context in mind, but we shall not always do so. A partial order on the weights of the usual kind and, for example, arguments along the lines of the discussion of the previous paragraph provide a partial order by inclusion on the set of closures $\overline{\mathrm{Bun}}_{x,\tau}$ of the varieties $\mathrm{Bun}_{x,\tau} = G(\mathbb{O}_x)\tau/\operatorname{Gal} O_x)$ and these closures are complete. The element τ is a matrix $t(m_1, \ldots, m_n)$ for $\mathrm{GL}(n)$ and an element in, say, a split torus for a general (split) G. The varieties themselves are open in their closure. It is possible — it is so already for $\mathrm{GL}(1)$ — that we cannot find a cofinal set of $\overline{\mathrm{Bun}}_{x,\tau}$, but we can find a cofinal family of finite unions $\bigcup_i \overline{\mathrm{Bun}}_{x,\tau_i}$. So we can introduce the union of these varieties to obtain a variety $\mathrm{Bun}_x = G(F_x)/G(\mathbb{O}_x)$, infinite-dimensional but the union of closed, finite-dimensional subvarieties.

This is very likely all familiar. We use it to construct the global Bun_X. Let T be a finite set in X and $\{\tau_x, x \in T\}$ a collection of τ. Consider

$$\prod_{x\in T} \overline{\mathrm{Bun}}_{x,\tau_x} \subset \prod \overline{\mathrm{Bun}}_{x,\tau_x}, \tag{8.7}$$

where for the imbedding of the left side in the right, it is understood that $\tau_x = 1$, $x \notin T$. In principle, we can fix T and take a union to arrive at

$$\prod_{x\in T} G(F_x)/G(\mathbb{O}_x), \tag{8.8a}$$

which we can divide by

$$G_T(F) = G(F) \cap \left\{ \prod_{x\in T} G(F_x) \prod_{x\notin T} G(\mathbb{O}_x) \right\}, \tag{8.8b}$$

but this will lead to a discrete object. There is a second, more important limit implicit in (8.8a). We can first fix the number of elements in $T = \{x_1, \dots, x_n\} \in X \times \dots X = X^{(n)}$, so that we introduce n supplementary parameters, as in the theory of the Picard variety, although repetitions are not necessary. They are already at hand in (8.8a). It is presumably better to take the limits in the order opposite to that suggested in (8.2a) with a finite number of double cosets at first but with all possible T. Then, as for GL(1) and the Picard variety, we may reach the limit before exhausting the possibilities offered by (8.8a). This construction poses problems of various kinds, with stability, stacks and with other matters. I am in no position to deal with them at the moment and prefer to pass on to another issue, the central question of this section. So I simply take them as solved or, at least, solvable. In essence, however, we arrive at the algebro-geometric form of the set appearing as a limit in the modified form of (8.2a),

$$\lim_n \bigcup_{|T|=n} G_T(F)\backslash \prod_{x\in T} G(F_x)/G(\mathbb{O}_x). \tag{8.8c}$$

It appears that, in spite of its formidable appearance, the algebro-geometric result is finite-dimensional. Indeed, for some curves X, it is, I find, strangely simple [At; Le].

The central question for us here is whether the method used in the previous section to construct a character from the differential form ω can function for nonabelian groups. There are three issues raised by formula (7.13a) and (7.13b): the periods that appear in (7.13b) and whose existence was accommodated by the introduction of $\mathfrak{Z}_R$ and $\mathfrak{Z}_I$; the contributions of the singularities of ω; the contributions of the singularities of f. It is clear that ω in both its holomorphic (in general, meromorphic) and unitary form will yield a homomorphism of the fundamental group into $G(\mathbb{C})$ or into its unitary form, thus a nonabelian form of the periods. Any study of this will have to wait until I better understand the issues arising from a study of [Si].

We do not yet understand the local ramified theory. So we have to exclude, at least provisionally, all ramification. One possibility is to assume that ω itself has no singularities. Another possibility to keep in mind is that we can fix a finite set $S \subset X$, which T is always supposed to contain, and, for $x \in S$, fix $g_x \in G(F_x)$ and a nonnegative integer n_x and work not on (8.8a) but on

$$\prod_{x\in S} g_x G^{n_x}(\mathbb{O}_x) \prod_{x\in T/S} G(F_x)/G(\mathbb{O}_x) \tag{8.9}$$

and to divide not by (8.8b) but by the subset of this set consisting of g for which $gg_xG^{n_x} \subseteq g_xG^{n_x}$, for all $x \in S$. If we work on the set (8.9), we are excluding the effect of the singularities of ω, thus effectively imposing the condition of Lemma 7.1 that the singularities of f and ω be disjoint. So we are left with the middle term of (7.13a).

What is the issue? For GL(1), the form ω leads not immediately but in connection with the Abel–Jacobi theory to a Hecke parameter at every place and the identity of Lemma 7.1 obtained from the residue theorem shows that the character constructed as a product of the local parameters is an idele-class character. Apart from the ambiguity already noted, which has to be resolved, the form ω should also give a local parameter everywhere and thus local spherical functions ϕ_x (normalized, say, to be 1 at the identity). The problem is to show that the local parameters together yield an admissible global parameter. In other words, there is a compatible family of functions f_T (associated in some way to a perverse sheaf) on the varieties of (8.2a), each f_T being, first of all, a linear combination (perhaps in some general sense — a direct image of some perverse sheaf) of left-translations of $\prod_x \phi_x$ and, secondly, invariant under the group $G_T(F)$. The function f_T once chosen — in whatever way imagination suggests — integration around the outside of $\triangle'$ and the residue, can with any luck, be used to show that it is invariant under $G_T(F)$. There is, as will be apparent, a gulf here, maybe two, that I make, for the moment, no proposal for bridging.

I had initially hoped that even if I was unable in this, the first part of the prologue, to reach the relation of the geometric theory to quantum field theory, I would be able to make a convincing suggestion about the construction of the mathematical theory, thus about the construction of the group $\mathfrak{A}_{\text{geom}}$. The possibility of constructing it in terms of differentials with values in ${}^L\mathfrak{g}$ is suggested by the abelian theory and I had hoped — and still hope — that one could prove the appropriate theorem with the help of the residue theorem as for Lemma 7.1. There are encouraging signs, but there are, as I have just explained, also obstacles: for example the full determination from the differentials of the parameters of the spherical functions at each unramified place. On the other hand, the example of elliptic curves [At] suggests that the moduli spaces for G-bundles may be simpler, at least in some respects, than one fears. Although I still had a few weeks grace until the deadline for submitting the paper, I concluded in the face of this and other formidable obstacles that it would be best to stop at the point I had reached, where an uncertain optimism was still possible, and to give myself the leisure — more than a few weeks — to understand better not only the spaces Bun_G and their differential-geometric and algebro-geometric properties but also the quotients $G(F_v)/G(\mathbb{O}_v)$ and the geometrical spherical "functions".

Certainly my limited understanding of the construction of Bun_G is a handicap. There are two puzzles that I have already mentioned. The first is the construction of the local parameters $\mu_x \in {}^LG$ for the spherical functions, defined only at the unramified points. I have suggested that for the abelian theory they are to be defined by the integral of the differential form as $\exp \int_{p_0}^{p} \omega = \mu_p \mu_{p_0}^{-1}$, but without being myself sufficiently clear of how μ_{p_0} was defined. In fact, for $G = \mathrm{GL}(1)$, $\mathrm{Bun}_X = \mathrm{Pic}_X = \bigcup_{n \in \mathbb{Z}} \mathcal{P}^n$ and $p_0 \in X$ is to be interpreted as a point in $\mathcal{P}^1$, the bundle attached to it being the trivial bundle modified by permitting a pole at p_0. The value μ_{p_0} is given by γ in formula (7.13a), thus by a supplementary parameter, so that the parameter of the automorphic representations contains, in addition, to the differential ω a complex number $e^{i\gamma}$ of absolute value 1. Until I understand better the nature of differentials on X with values in ${}^LG$ and the structure of Bun_G, it is idle to make suggestions about the form of $\mathfrak{A}_{\mathrm{geom}}$ that are more precise than that already made at the beginning of this section.

Although I prefer to fix my attention on the geometrical theory as a theory of automorphic forms, thus on functions on Bun_G, it is still necessary to reckon with algebro-geometric aspects of the problem. Geometrically, the Hecke algebra has, it appears, to be defined geometrically. The double-coset space

$$G(\mathbb{O}_v)\backslash G(F_v)/G(\mathbb{O}_v)$$

may be discrete, but $G(\mathbb{O}_x)$ and the spaces $G(F_v)/G(\mathbb{O}_v)$ are algebraic varieties, so that convolution of two elements in the Hecke algebra, formally $\int f_1(gh^{-1}) f_2(h)dh$, has to be defined — so far as I can seen — not as an integral but in terms of direct images of (perverse) sheaves under the mapping of $(g_1, g_2) \mapsto g = g_1 g_2$. I would suppose that, if convolution is to be defined, the spherical functions will also have to be interpreted as sheaves, again perverse, and as sheaves will not have support in a compact subvariety of $G(F_v)$ or $G(F_v)/G(\mathbb{O}_v)$. I am not absolutely certain that we will need a formula for these sheaves, but suspect we will; I am also not certain what form it will take. I take the existence of the formula, in some form, for granted below. The necessary formula could very well be discovered and proved by taking the theory over p-adic fields as a model. So far as I know this has not yet been done. What we have to do, at least for the unramified theory, is first establish, or at least surmise, what the local parameters are. Each of them is supposed to be a conjugacy class in ${}^LG(\mathbb{C})$ or even, if a form of the Ramanujan conjecture is valid, in the unitary form ${}^LU$ of this group, although my interpretation of the results of [At] suggests that the possibility of Arthur parameters intervening has to be kept in mind. It may be that some authors have reflected on the presence of the distinctions familiar from the arithmetic context — tempered or of Arthur type — in the geometric context.

The potential local parameters for a given differential form on hand, one has to show that among the "linear combinations" of left translates of the associated spherical function on (8.8a), a product — over T, a set that has to be allowed to grow larger and larger — of the spherical functions on the individual factors, there is one invariant under the group (8.8b). I should think that to establish this it would be a help to have an explicit formula for the spherical functions.

With the reader's permission, I introduce an intuitive fashion of thinking, based on our experience with spherical functions over archimedean and nonarchimedean local fields. In the representation of $G(F_v)$ (resp. $\prod_T G(F_x)$) on $G(F_x)/G(\mathbb{O}_x)$ (resp. $\prod_T G(F_v)/G(\mathbb{O}_v)$) each representation that occurs, occurs with multiplicity one. We consider the representation with parameter μ_x or rather, at this stage, with parameter $\prod_T \mu_x$. We need to establish that it contains a vector fixed by all elements of $G_T(F)$. For GL(1), this vector was essentially unique and could be determined by integrating the differential. For a general group, we can expect, because of endoscopy and multiplicity, both familiar from the arithmetic theory, the unicity to fail.

We have, at the same time, to contend with something more serious. Functions are not sheaves; sheaves are not functions. Rather they are not, even with the Riemann–Hilbert correspondence, uniquely functions. As I have already made clear, it seems to me that completeness theorems, to assure that we have in hand all pertinent objects of some given sort, require something less ethereal than sheaves, even than perverse sheaves. Nevertheless, for the sake of the argument I confound briefly, in the following observations, functions and sheaves.

The object $\mathrm{Bun}_G(X)$ as defined in terms of (8.8a) and (8.8b) appears far too large, far too coarse, to admit any analysis, but the goal of the theory of moduli spaces as expounded in [Le] is to show that they are, in essence, algebraic varieties, on which differential equations of various sorts can be introduced and studied, so that the exclusive use of sheaves, as in [CFT] is not obligatory, not, in my view, even to be recommended! The general theory does, however, differ from the abelian theory in that a given parameter does not correspond to a single function — up to a scalar factor — but to an infinite-dimensional space of functions, but many of us are already familiar with this from the arithmetic theory.

So what are we to do, keeping in mind that we are working — initially — with the group G_T? The function/sheaf for which we are searching will be a product of spherical functions $\prod_{x\notin T} \phi_x$ times some linear combination of left-translates of $\prod_{x\in T} \phi_x$, presumably by elements in $G_F(T)$. We cannot at first take an average because $G_F(T)$ is infinite, but also because, so far as I know, we cannot take the average of sheaves. On the other hand, we might be able to calculate the change in the function imposed by the translations by $g \in G_T(F)$ by adding up the local modifications as an integral around the boundary of $\triangle'$ finding either that the total change is 0 or that it permits an averaging. This is, at the moment, where the problem stands. Nothing is certain, but there is a great deal on which to reflect!

At this point, I cannot be very much clearer about this proposal for constructing $\mathfrak{A}_{\text{geom}}$. A few observations are, however, in order. It is useful, first of all, to compare it with a conjecture in §6.1 of [CFT], although this conjecture is formulated only in an unramified context.

Conjecture. *Let E be an irreducible ${}^LG$-local system on X. Then there exists a nonzero Hecke eigensheaf* Aut_E *on* Bun_G *with the eigenvalue E whose restriction to each connected component of* Bun_G *is an irreducible perverse sheaf.*

The earlier Assertion is this conjecture for $\mathrm{GL}(n)$. Let me try to explain the general form and its relation with our tentative proposal.

As will be obvious, I have been strongly influenced when composing this section by the geometric and sheaf-theoretic formalism for the Hecke theory with which this conjecture is expressed. This formalism is very elegant, but I did not understand the intuition that informs it. Perhaps I still do not. Nonetheless, if I had not struggled to interpret it in a perhaps more mundane but also more concrete analytic context, it may never have meant anything to me at all. Implicit in the conjecture there are conventions and conceptions — familiar in some circles, less so in those to which I belong — of which we remind ourselves before explaining the relation between it and our goal. Locally the trivial bundle is $G(\mathbb{O}_x)$, a set on which $G(\mathbb{O}_x)$ acts from the right. If $\gamma \in G(F_x)$ then the action of $G(\mathbb{O}_x)$ to the right on $G(\mathbb{O}_x)\gamma G(\mathbb{O}_x)$ defines a G-bundle on $G(\mathbb{O}_x)\gamma G(\mathbb{O}_x)/G(\mathbb{O}_x)$, an operation of blowing-up or modification that we can, if desired, repeat, passing to $G(\mathbb{O}_x)\gamma' G(\mathbb{O}_x)\gamma G(\mathbb{O}_x)/G(\mathbb{O}_x)$, and so on, or just blowing up a given point of $G(\mathbb{O}_x)\gamma G(\mathbb{O}_x)/G(\mathbb{O}_x)$. At all events, this operation allows us to introduce a G-bundle structure on

$$\bigcup_T \prod_{x \in T} G(F_x)/G(\mathbb{O}_x)$$

and then, passing to the limit over $T = \{x_1, \ldots, x_n\}$ as before, first over all possibilities for the set T with a given n and then over n, we arrive, at Bun_G and the G-bundle over it.

It is bundles and modifications that are the preferred form of expression in [CFT]. A central diagram is found in §6.1 of those lectures.

(8.10)
$$\begin{tikzcd} & \mathit{Hecke} \arrow[dl, "\mathfrak{h}^{\leftarrow}"'] \arrow[dr, "\mathfrak{h}^{\rightarrow}"] & \\ \mathrm{Bun}_G & & X \times \mathrm{Bun}_G \end{tikzcd}$$

I do not find the definition of *Hecke* in [CFT, §6.1] perfectly transparent, but I think it safe to take it to be the union over increasing T of the union (or sum) over $x \in T$

of the quotient of[2]

$$\{x \times G(F_x) \times G(F_x)/G(\mathbb{O}_x)\} \times \prod_{\substack{y \in T \\ y \neq x}} G(F_y)/G(\mathbb{O}_y) \tag{8.11}$$

by $G_T(F)$, whose action on $G(F_x) \times G(F_x)/G(\mathbb{O}_x)$ is through the first factor alone, a definition compatible — I hope and, indeed, believe — with that of [CFT]. The arrow on the left of (8.10) takes $x \times h_x \times g_x \times \prod_{y \in T, y \neq x} g_y$ to $x \times h_x \times \prod_{y \in T, y \neq x} g_y$; the arrow on the right takes it to $x \times h_x g_x \times \prod_{y \in T, y \neq x} g_y$. This seems equivalent to the assertion in [CFT], which I have difficulty understanding, and I, myself, see no reasonable alternative to (8.11). Informally, the object *Hecke* consists of quadruples $(\mathcal{M}, \mathcal{M}', x, \beta)$, where $\mathcal{M}$ is a G-bundle on X, $\mathcal{M}'$ is a modification at a single point x, and β is an expression of the identity of $\mathcal{M}$ and $\mathcal{M}'$ outside of x. Of course T grows to include more and more points. Observe that it defines a correspondence that commutes with the action of $G_T(F)$.

In the Conjecture, the initial object is the local system. Our initial object is more, it is the differential. The difference is somewhat difficult to describe, but its source is clear. It is the difference between a local system and a local system with isomorphism. For example, in the analytic context, there is, on the curve X or on its jacobian, the trivial bundle itself, but there is also the trivial bundle plus a section, $\exp(i \operatorname{Re} \omega)$, where ω is a holomorphic differential with real periods in $2\pi\mathbb{Z}$, a set parametrized by $\mathbb{Z}^{2g}$. Here we distinguish between them. In the Conjecture and in the earlier Assertion, both taken from [CFT], they are confounded. The advantage of the local systems with isomorphism is that it refers to the set of solutions of a precise analytic problem, an eigenvalue problem for the Laplacian, so that we can treat the set without having to exhibit its individual elements. This is what, I hope, the differentials — with whatever supplementary data are necessary — will offer in general.

Having affirmed, for the second time, that there is a difference between the local system and the differential, I now retract and explain that, when trying to understand the meaning of the conjecture, I discovered that this supposed difference was the result either of my careless reading of [CFT] or of the author's careless writing. The author speaks of local systems, local systems for vector bundles and "local systems" for ${}^LG$-bundles — the latter seem to be no more than ${}^LG$-bundles — for they are what allow the definition of the vector bundles V_λ^E, which are local systems, defined

[2]The pertinent phrase from [CFT] is, "Note that the fiber of *Hecke* is the moduli space of pairs (M, β), where $\mathcal{M}$ is a G-bundle on X and $\beta : \mathcal{M}'|_{X \setminus x}$. It is known that this moduli space is isomorphic to a twist of $\mathrm{Gr}_x = G(F_x)/G(\mathbb{O}_x)$ by the $G(\mathbb{O}_x)$-torsor $\mathcal{M}'(\mathbb{O}_x)$ of sections of $\mathcal{M}'$ over $\operatorname{Spec}\mathbb{O}_x$:

$$(h^{\rightarrow})^{-1}(x, \mathcal{M}') = \mathcal{M}'(\mathbb{O}_x) \times_{G(\mathbb{O}_x)} \mathrm{Gr}_x\text{."}$$

I hope it means what I suggest.

by the constant sections of an ${}^LG$-bundle. For groups with no center, the distinction between the two notions of local system is barely perceptible, and this may be the source of the confusion. The emphasis in [CFT] is often on semisimple groups. The group GL(1) that we were examining in §7 is, however, all center!

This confusion, all being well, clarified, let us try to understand the conjecture. Certainly, whatever we manage to establish, we want it to imply the conjecture! What does it mean for the sheaf $\mathcal{F}$ on Bun_G to be an eigensheaf with eigenvalue E? The condition is formulated sheaf-theoretically as equations (6.1) and (6.2) of [CFT].

$$\text{(CFT-6.1)} \qquad H_\lambda(\mathcal{F}) = \mathfrak{h}_*^{\rightarrow}(\mathfrak{h}^{*\leftarrow}\mathcal{F} \otimes \mathrm{IC}_\lambda);$$

$$\text{(CFT-6.2)} \qquad \iota_\lambda : H_\lambda(\mathcal{F}) \simeq V_\lambda^E \boxtimes \mathcal{F}, \qquad \lambda \in P_+.$$

The first line is a definition. In the second line λ is a dominant weight of ${}^LG$ or a double coset $G(\mathbb{O}_x)\tau_\lambda G(\mathbb{O}_x)$, ι_λ is an isomorphism and V_λ^E is the vector bundle $E \times_{{}^LG} V_\lambda$. The sheaf IC_λ is a perverse sheaf associated to the subvariety $G(\mathbb{O}_x)\tau_\lambda/G(\mathbb{O}_F)$ of $\mathrm{Bun}_{x,\tau_\lambda}$, the Goresky–MacPherson or intersection cohomology sheaf described in [CFT] and many other places. It appears to be the cohomological representative of this subvariety in the context of perverse sheaves, thus the cohomological representative of a spherical function, the characteristic function of a double coset. In any case, the second line is the condition that $\mathcal{F}$ has eigenvalue E. The almost imperceptible mixing of G-bundles and ${}^LG$-bundles is striking!

We replace the ${}^LG$ local system by the differential ω or rather by the set of parameters $\{\mu_x \mid x \in X\}$ associated to it, without troubling ourselves by the imprecisions that this entails at this stage. Let us try to understand the situation in the context of group representations, but only in a grossly informal manner. At all but a finite set S of points in X, we have a representation $\pi_x = \pi(\mu_x)$ of $G(F_x)$, a representation that contains a nonzero vector fixed by $G(\mathbb{O}_F)$. It occurs in the space of functions on $G(F_x)/G(\mathbb{O}_x)$ and, as we infer — for the sake of the argument — from the usual theories of spherical functions, with multiplicity one. So we have a clearly defined space of functions on $\prod_{x\in T} G(F_x)/G(\mathbb{O}_x)$. On each $G(F_x)$, we take a left translate of the spherical function with parameter μ_x. Then we take a tensor product of such functions over $x \in T$ and then linear combinations, perhaps in a topological sense — for example, by convolution with a function, a measure, or a distribution. The group $\prod_{x\in T} G(F_x)$ acts on this space and we assume that there is a nonzero vector Φ in it invariant under $G_T(F)$. That would be our solution of the problem. Does it offer a sheaf $\mathcal{F} = \mathrm{Aut}_E$ satisfying the conditions of the conjecture? The question, at the moment, is not in what sense it might be a sheaf, or in what sense a function, but whether and why we can expect the equation (CFT-6.2) to be valid.

The appropriate construction works entirely from the right, so that the invariance under $G_T(F)$ on the left plays no role in the arguments. It only assures, because the constructions are undertaken from the right, that the result continues to be invariant under $G_T(F)$, so that it can be transferred to Bun_G. In other words, we replace the diagram (8.10) with

$$\begin{tikzcd}
 & H \arrow[dl, "\mathfrak{h}^{\leftarrow}"'] \arrow[dr, "\mathfrak{h}^{\rightarrow}"] & \\
\prod_{x\in T} G(F_x)/G(\mathbb{O}_x) & & T \times \prod_{y\in T} G(F_y)/G(\mathbb{O}_y)
\end{tikzcd} \tag{8.12}$$

The diagram defines H; it is given by (8.11), but there is now no division by $G_T(F)$, neither of H nor of $\prod G(F_y)/G(\mathbb{O}_y)$

If $\mathfrak{h}^{\leftarrow}$ and $\mathfrak{h}^{\rightarrow}$ can be interpreted as actions on the right, then (8.12) may be interpreted as a covering of (8.10). A typical element of H is $(x, h_x, g_x, \prod_{y\neq x} g_y)$. The maps $\mathfrak{h}^{\leftarrow}$ and $\mathfrak{h}^{\rightarrow}$ are defined independently on the various summands and on the various factors, in particular:

$$\mathfrak{h}_x^{\leftarrow} : (h_x, g_x) \mapsto h_x G(\mathbb{O}_x); \qquad \mathfrak{h}_x^{\rightarrow} : x \times (h_x, g_x) \mapsto x \times h_x g_x G(\mathbb{O}_x). \tag{8.13}$$

All these morphisms commute with the action of $G_T(F)$. The fiber of $\mathfrak{h}^{\rightarrow}$ over $(x, g_x G(\mathbb{O}_x))$ is, if I am not mistaken, the set $\{(x, h_x, h_x^{-1} g_x)\}$, thus $G(F_x)$. It is perhaps important to stress as well that the local factors of IC_λ are sheaves on $G(F_x)/G(\mathbb{O}_x)$, so that the global product is a sheaf

The intersection cohomology sheaves IC_λ are defined in [CFT] locally, one at each point of X. We have agreed that a provisional section of a prologue is not the place to describe them precisely. They are, as suggested, the intersection-cohomological representatives of the subvarieties

$$G(\mathbb{O}_x)\tau_\lambda G(\mathbb{O}_x)/G(\mathbb{O}_x)$$

of $G(F_x)/G(\mathbb{O}_x)$. It is plausible that, whatever the precise definition is, we can, as in [CFT] extend it from a local construct to a global construct. Indeed, from the point of view adopted in this prologue, we just define it on (8.11) by pulling back the local IC_λ through the projection on $g_x \in G(F_x)/G(\mathbb{O}_x)$, the third coordinate in (8.11). The result is not invariant under $G_T(F)$. Moreover, there are implicit parameters with an algebraic or function-theoretical significance that are being kept in reserve, the points x in T. One might want to verify that the constructions were compatible with this aspect of the construction — but not now.

I am a tyro here and have by no means understood in any genuine sense intersection cohomology. So I am reduced to guessing what the relations (CFT-6.1) and (CFT-6.2) mean not only in that context, but in the context of functions, if they have an interpretation there. In (CFT-6.1) the sheaf $\mathcal{F}$ or our function Φ depends on the first coordinates h_x alone; the sheaf $\Psi = \mathrm{IC}_\lambda$ depends on the coordinate g_x

alone. The direct image $H_*^{\rightarrow}$ is an integral, in this case,

$$\int \Phi(hg)\Psi(g^{-1})dg = \prod_x \int \Phi_x(h_x g_x)\Psi_x(g_x^{-1})\, dg_x \tag{8.14}$$

thus convolution on the right by Ψ, which does not destroy the invariance under $G_T(F)$.

The function Φ has moreover been obtained as a limit of linear combination of left translates of a product of spherical functions $\otimes_x \phi_x$, whose eigenvalues we know. If we have a formula for these functions, we can calculate

$$\prod_x \int \Phi_x(h_x g_x)\Psi_x(g_x^{-1})\, dg_x,$$

explicitly. This achieved, it should not be too difficult to deduce the relation CFT-6.2)!

On closer examination, there are several troubling aspects to these reflections. It appears to me, as already explained, that the theory of spherical functions in the geometric context necessarily entails the use of sheaves because there is no G-invariant measure with respect to which convolutions of spherical functions on $G(F_x)$ can be defined. The integral in (8.14) is fictional. A graver flaw is that we have not succeeded in introducing into our discussion the essential ingredient of the proof of Weil’s identity and Lemma 7.1, namely the residue theorem. To a large extent, although not entirely, this is because I am working around my ignorance of the theory of Bun_G. The conviction that we can deal, even in a geometric context, with automorphic forms on Bun_G as functions is because the spaces defined by (8.2a), or at least large pieces of them, are finite-dimensional algebraic varieties. Their construction as such is difficult and technical and it is fatuous to attempt, as I have been doing, to discuss the geometric theory without having understood it and its results.

The form ω defines a ${}^LG$-bundle with singularities, the bundle E of (CFT - 6.2) Then each irreducible representation $\rho = \rho_\lambda$ of ${}^LG$, λ being the highest weight of ρ, defines a local system, the local system V_λ^E of that formula. If, as before, μ_x are the parameters defined by integrating ω, $\alpha \in V_\lambda$, $\alpha^\star \in V_\lambda^*$, its dual space, then $\alpha^*(\mu_x\alpha)$ is a function on Bun_G. The question is how to combine it with a rational function f on X with values in G so that the result can be integrated over the boundary of $\triangle'$ as in the proof of Lemma 7.1. Although there is a duality between G and ${}^LG$ or between $\mathfrak{g}$ and ${}^L\mathfrak{g}$, it is coarse and I cannot see, at the moment, how it can be used. I am handicapped not only by an aging brain but also by a lack of facility with all the pertinent notions. In addition, the meddle of promising clues and doubtful juxtapositions is daunting to all but the very determined. Nevertheless, although

I have less confidence in the suggestions of this section than in those of the first six, I think there is something to be done.

For example, it is troubling that, as I have observed, the pairing between G and ${}^LG$ or, perhaps better, $\mathfrak{g}$ and ${}^L\mathfrak{g}$ is very coarse, apparently at most a pairing at the level of conjugacy classes, but that may be just as well, because what we sum are residues or products of residues with factors defined by ω with values in ${}^L\mathfrak{g}$. The residues themselves are logarithmic derivatives $df \cdot f^{-1}$. The following relations are clear.

(i) If $f_1 = uf$,

$$df_1 \cdot f_1^{-1} = du \cdot u^{-1} + udf \cdot f^{-1}u^{-1},$$

and the first term has no residue at x if $u \in G(\mathbb{O}_x)$. The conjugacy class of the second term is that of $df \cdot f^{-1}$

(ii) There is a similar relation for $f_1 = fu$,

$$df_1 \cdot f_1^{-1} = df \cdot f^{-1} + fdu \cdot u^{-1}f^{-1}.$$

If $u \in G(\mathbb{O}_x)$, the conjugacy class of the second term is regular at x.

(iii) If $f_1 = ufu^{-1}$,

$$df_1 \cdot f_1^{-1} = du \cdot u^{-1} + udf \cdot f^{-1}u^{-1} - uf^{-1}(u^{-1}du)fu^{-1}.$$

Thus any linear function of the residue at x of $df \cdot f^{-1}$ that is invariant under conjugation does not change on passing from f to f_1. This linear function should be the substitute for the right-hand side of (7.20). It will have to be matched with a substitute for $\int \omega$ as in the proof of the lemma.

Consider for example the group $\mathrm{GL}(n)$, then if z is a local parameter at x, the matrix-valued function f may be written locally as u_1Tu_2, where u_1, u_2 lie in $G(\mathbb{O}_x)$ and

$$T = t(-m_1, -m_2, \dots, -m_n), \quad m_1 \geq m_2 \cdots \geq m_n,\ m_i \in \mathbb{Z},$$

so that the residue of $f^{-1}df$ is the matrix

$$-\begin{pmatrix} m_1 & 0 & \dots & 0 \\ 0 & m_2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & m_n \end{pmatrix},$$

which can be considered a parameter for the double cosets $G(\mathbb{O}_x)\backslash G(F_x)/G(\mathbb{O}_x)$, or as a highest weight for $\hat{G}$ or ${}^LG$. We of course have a pairing of it with the Lie algebra of ${}^LG$ or with the group ${}^LG$ itself, through the trace of the corresponding representation, thus with ω or $\int \omega$.

So there are a good number of clues that could lead to a nonabelian theory similar to the abelian theory of §7. I do not have a clear notion of how to follow them. I hesitate moreover to search for the theory so long as I have not mastered the techniques for constructing moduli spaces described in [Le]. The moduli space as described in (8.8a) and (8.8b) is convenient in some respects, but it is analytically awkward and, as we found when discussing the conjecture, it encourages us to work not with functions, thus not with solutions of partial differential equations, but with sheaves, for which convolution is possible, at least in a topological sense. I tried in the essay to pass from one to the other by sleight-of-hand, but was not, as even a casual examination reveals, successful. The usual convolution is not defined because $G(F_x)/G(\mathbb{O}_x)$ does not carry an invariant measure. On the other hand, the moduli spaces — or at least large parts of them — are finite-dimensional, even compact, as with the jacobians, algebraic varieties and we might expect to define the eigensheaves as functions satisfying differential equations. I do not yet know what these might be. Moreover, whatever form the final theory takes, I certainly hope it embraces all possibilities: sheaf-theoretic, analytic, and geometric.

It seems best to leave all these questions aside until I acquire a more intimate understanding not only of the nature, both algebro-geometric and differential-geometric, of the moduli spaces, but also of the contributions of the mathematical physicists to what they refer to as the geometric Langlands program.

Contrary to my hopes, which were, in part, unreasonable, the last two sections of this first half of the prologue have turned out to differ sharply from the first six. Although the first six are speculative, they are informed by years of reflection, which has sufficiently matured that I have considerable confidence not only in the correctness of the theory suggested but also in the soundness of the methods proposed for arriving at it. This is not so for the last section, for which the penultimate section was preparation. The last section is only provisional. I hope that on returning to the material in §8.b I can do better!

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ROBERT LANGLANDS
SCHOOL OF MATHEMATICS
INSTITUTE FOR ADVANCED STUDY
1 EINSTEIN DRIVE
PRINCETON, NJ 08540
UNITED STATES
rpl@ias.edu

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TRUNCATION OF EISENSTEIN SERIES

EREZ LAPID AND KEITH OUELLETTE

To the memory of Jonathan Rogawski

We give a generalization of the Maass–Selberg relations for general Eisenstein series, providing a different approach to Arthur's asymptotic inner product formula.

1. Introduction

In this short note we study truncation of Eisenstein series. The truncation operator was introduced by Arthur [1980]. It plays a ubiquitous role in the trace formula. In the case of a cuspidal Eisenstein series (that is, one induced from a cuspidal representation) one can write its truncation as a modified Eisenstein series (previously introduced by Langlands). From this, one obtains the Maass–Selberg relations for the inner product of truncated Eisenstein series [Arthur 1980, §4] (see also Section 3). In the case of Eisenstein series induced from the discrete spectrum, Arthur [1982] obtained an asymptotic formula for the inner product above. His method was rather indirect and in particular, it required Langlands' description of the discrete spectrum in terms of residues of Eisenstein series. A different approach which avoids this description was taken in [Lapid 2011]. It uses the regularized integral developed in [Jacquet et al. 1999]. While the approach of [Lapid 2011] is reasonably conceptual, one still encounters some unpleasant technical difficulties. The purpose of this short paper is to rederive Arthur's asymptotic result more directly

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by writing down explicitly the truncation of a general Eisenstein series. This is a pleasant combinatorial exercise in truncation. As explained in [Lapid 2011], the asymptotic formula can be used to compute the inner product of Eisenstein integrals, a key fact in Langlands spectral theory.

We cannot close the introduction without recalling our deep appreciation to our teacher Jonathan Rogawski. His unlimited encouragement and keen interest in mathematics, even in difficult times, will not be forgotten. We miss him greatly.

2. Notation and conventions

Let F be a number field and $\mathbb{A}$ its ring of adeles. Throughout, we denote by boldface letters, such as $\boldsymbol{Y}$, algebraic varieties over F and we write $Y = \boldsymbol{Y}(F)$, $Y_{\mathbb{A}} = \boldsymbol{Y}(\mathbb{A})$. Sometimes we will not distinguish between $\boldsymbol{Y}$ and Y. Let $\boldsymbol{G}$ be a reductive group over a number field F. (Henceforth, all the algebraic subgroups of $\boldsymbol{G}$ that we consider are implicitly assumed to be defined over F.) We fix a maximal F-split torus $\boldsymbol{T}_0$ and a minimal parabolic subgroup $\boldsymbol{P}_0$ containing $\boldsymbol{T}_0$. We have a Levi decomposition $\boldsymbol{P}_0 = \boldsymbol{M}_0 \ltimes \boldsymbol{U}_0$ where $\boldsymbol{M}_0 = C_G(\boldsymbol{T}_0)$. Let $\mathfrak{a}_0^*$ be the $\mathbb{R}$-vector space spanned by the lattice $X^*(\boldsymbol{T}_0)$ of F-rational characters of $\boldsymbol{T}_0$ (or alternatively, by the commensurable lattice $X^*(\boldsymbol{M}_0)$ of F-rational characters of $\boldsymbol{M}_0$). The dimension of $\mathfrak{a}_0^*$ is the split rank of $\boldsymbol{G}$. The dual space $\mathfrak{a}_0$ of $\mathfrak{a}_0^*$ is the $\mathbb{R}$-vector space spanned by the lattice of cocharacters $X_*(\boldsymbol{T}_0)$ of $\boldsymbol{T}_0$. We write $\mathfrak{a}_{0,\mathbb{C}}$ for the complexification of $\mathfrak{a}_0$. We denote by $\Delta_0 \subseteq X^*(\boldsymbol{T}_0)$ the set of simple roots of $\boldsymbol{T}_0$ on $\operatorname{Lie} \boldsymbol{U}_0$ and by $\Delta_0^\vee \subseteq X_*(\boldsymbol{T}_0)$ the set of simple coroots.

We write $H^g = gHg^{-1}$ for any subgroup $H \subseteq G$ and an element $g \in G$.

For any algebraic group $\boldsymbol{Y}$, we write δ_Y for the modulus function on $Y_{\mathbb{A}}$. We also write $Y_{\mathbb{A}}^1 = \bigcap \operatorname{Ker}|\chi|$ where χ ranges over the lattice of F-rational characters of $\boldsymbol{Y}$ and $|\chi|(y) = \prod_v |\chi_v(y_v)|_v$ for $y = (y_v) \in Y_{\mathbb{A}}$.

Let $P = M \ltimes U$ be a standard parabolic subgroup of G defined over F, with $M \supset M_0$. Let $\Delta_0^M \subseteq \Delta_0$ be the set of simple roots of T_0 in $\operatorname{Lie}(U_0 \cap M)$ and denote the span of Δ_0^M by $(\mathfrak{a}_0^M)^*$. Let $\boldsymbol{T}_M$ be the identity component of the split part of the center of M — a subtorus of T_0. We identify $\mathfrak{a}_M^* = X^*(T_M) \otimes \mathbb{R} = X^*(M) \otimes \mathbb{R}$ with a subspace of $\mathfrak{a}_0^*$. Occasionally we also write $\mathfrak{a}_P = \mathfrak{a}_M$. In particular, $\mathfrak{a}_{P_0} = \mathfrak{a}_{M_0} = \mathfrak{a}_0$. We write $r(P) = r(M) = \dim \mathfrak{a}_M$. We have $\mathfrak{a}_0 = \mathfrak{a}_M \oplus \mathfrak{a}_0^M$ and similarly for $\mathfrak{a}_0^*$. Denote by $\Delta_M = \Delta_P \subseteq X^*(\boldsymbol{T}_M)$ the simple roots of $\boldsymbol{T}_M$ on $\operatorname{Lie} \boldsymbol{U}$ — these are the projections of $\Delta_0 \setminus \Delta_0^M$ to $\mathfrak{a}_M^*$. For any $\alpha \in \Delta_P$ we have the corresponding coroot $\alpha^\vee \in X_*(\boldsymbol{T}_M)$.

We reserve the letters $P = MU$ and $Q = LV$ (possibly appended with primes or subscripts) for standard parabolic subgroups of G with their standard Levi decomposition. Since M and P determine each other, we often use them interchangeably as subscripts or superscripts in various notation. Occasionally we will use R and S

to denote auxiliary standard parabolic subgroups. We write M_R for the standard Levi subgroup of R and N_R for its unipotent radical.

For any $Q \subseteq P$, we write $\Delta_L^M = \Delta_Q^P \subseteq \Delta_Q$ for the simple roots of T_L on $\mathrm{Lie}(V \cap M)$. We have $\mathfrak{a}_L = \mathfrak{a}_L^M \oplus \mathfrak{a}_M$ where $\mathfrak{a}_L^M = \mathfrak{a}_Q^P$ is the span of

$$(\Delta_Q^P)^\vee = (\Delta_L^M)^\vee = \{\alpha^\vee : \alpha \in \Delta_Q^P\}.$$

Consequently, $\mathfrak{a}_0 = \mathfrak{a}_0^L \oplus \mathfrak{a}_L^M \oplus \mathfrak{a}_M$. The dual basis of $(\Delta_L^M)^\vee$ in $(\mathfrak{a}_L^M)^*$ will be denoted by $\hat{\Delta}_L^M$. We write X_Q^P or X_L^M for the image of $X \in \mathfrak{a}_0$ under the projection from $\mathfrak{a}_0$ to $\mathfrak{a}_L^M$.

We write $[P, Q]$ for the set of parabolic subgroups of Q containing P. Thus, $[P_0, G]$ is the set of all standard parabolic subgroups of G.

Denote by $W = W_G$ the Weyl group $N_G(T_0)/M_0$ of G. For any M, we identify the cosets $W^M \backslash W$ (resp. W/W^M) with the set of left- (resp. right-) W^M reduced elements of W, that is, those for which $w^{-1}\alpha > 0$ (resp. $w\alpha > 0$) for all $\alpha \in \Delta_0^M$.

Now let M and L be standard Levi subgroups. We identify $W^M \backslash W / W^L$ with the set of left-W^M and right-W^L reduced elements of W. Define subsets

$$W(L, M) \subseteq W(L; M) \subseteq W^M \backslash W / W^L$$

by

$$W(L, M) = \{w \in W^M \backslash W : L^w = M\} = \{w \in W^M \backslash W : w\Delta_0^L = \Delta_0^M\}$$

and

$$W(L; M) = \{w \in W^M \backslash W : L^w \subseteq M\} = \{w \in W^M \backslash W : w\Delta_0^L \subseteq \Delta_0^M\}.$$

Note that if $L' \subseteq L$ then $W(L; M) \subseteq W(L'; M)$.

We write $\mathscr{C}_{0,-}$ for the closed negative obtuse Weyl chamber

$$\mathscr{C}_{0,-} = \Big\{ \sum_{\alpha \in \Delta_0} x_\alpha \alpha^\vee : x_\alpha \leq 0 \text{ for all } \alpha \Big\}.$$

More generally, for any $Q \subseteq P$ we write

$$\mathscr{C}_{Q,-}^P = \Big\{ \sum_{\alpha \in \Delta_Q^P} x_\alpha \alpha^\vee : x_\alpha \leq 0 \text{ for all } \alpha \Big\}.$$

We fix a positive definite W-invariant scalar product, and hence a norm $\|\cdot\|$ on $\mathfrak{a}_0$. This defines a measure on any subspace of $\mathfrak{a}_0$.

We fix a "good" maximal compact subgroup K of $G_{\mathbb{A}}$. Using the Iwasawa decomposition, we define $H : G_{\mathbb{A}} \to \mathfrak{a}_0$ to be the left-$U_{0,\mathbb{A}}$ right-K invariant function such that

$$e^{\langle \chi, H(m) \rangle} = \prod_v |\chi_v(m_v)|_v$$

for any $\chi \in X^*(M)$ where $m = (m_v)_v$ and χ_v is the extension of χ to $M(F_v)$.

Let A_0 be the identity component of $\boldsymbol{T}_0(\mathbb{R}) \subseteq T_{0,\mathbb{A}}$ where $\mathbb{R}$ is embedded in $\mathbb{A}$ diagonally at the archimedean places. The map H gives rise to an isomorphism $A_0 \to \mathfrak{a}_0$. We denote by $X \mapsto e^X$ the inverse map. More generally, for any M let $A_M = A_0 \cap T_M$. The map H restricts to an isomorphism $A_M \to \mathfrak{a}_M$.

Let $\mathfrak{a}_{0,+}$ be the positive Weyl chamber

$$\mathfrak{a}_{0,+} = \{X \in \mathfrak{a}_0 : \langle \alpha, X\rangle > 0 \text{ for all } \alpha \in \Delta_0\}.$$

Similarly, we write for any P

$$\mathfrak{a}^*_{M,+} = \{\lambda \in \mathfrak{a}^*_M : \langle \lambda, \alpha^\vee\rangle > 0 \text{ for all } \alpha \in \Delta_P\}.$$

Let $\mathscr{A}_P$ be the space of automorphic forms on $PU_{\mathbb{A}}\backslash G_{\mathbb{A}}$, that is, the smooth, K-finite, and $\mathfrak{z}$-finite functions of moderate growth where as usual $\mathfrak{z}$ is the center of the universal enveloping algebra of the complexified Lie algebra of $\boldsymbol{G}(\mathbb{R})$. We write $\mathscr{A}^n_P$ for those $\varphi \in \mathscr{A}_P$ such that $\varphi(ag) = \delta_P(a)^{\frac{1}{2}}\varphi(g)$ for all $a \in A_M$, $g \in G$. We denote by $\mathscr{A}^2_P$ the subspace of $\mathscr{A}^n_P$ consisting of the functions such that

$$\langle \varphi, \varphi\rangle_{A_M U_{\mathbb{A}} M\backslash G_{\mathbb{A}}} = \|\varphi\|_2^2 = \int_{A_M U_{\mathbb{A}} M\backslash G_{\mathbb{A}}} |\varphi(g)|^2\, dg < \infty$$

and by $\mathscr{A}^{\mathrm{cusp}}_P$ the subspace consisting of the cuspidal automorphic forms.

For any $\varphi \in \mathscr{A}_P$ and $\lambda \in \mathfrak{a}^*_M$ let

$$\varphi_\lambda(g) = e^{\langle \lambda, H_P(g)\rangle}\varphi(g), \quad g \in G_{\mathbb{A}}.$$

For any $Q \supset P$ the Eisenstein series is defined by

$$E^Q_P(g, \varphi, \lambda) = \sum_{\gamma \in P\backslash Q} \varphi_\lambda(\gamma g).$$

(If $Q = G$ we omit it from the notation.) The series converges absolutely for $\operatorname{Re}\lambda \in \mathfrak{a}^*_{P,+}$ sufficiently regular. We will *assume* that $E(\,\cdot\,, \varphi, \lambda)$ admits meromorphic continuation with hyperplane singularities. This is proved in [Langlands 1976] (cf. [Mœglin and Waldspurger 1994]) first for $\varphi \in \mathscr{A}^{\mathrm{cusp}}_P$ and then for $\varphi \in \mathscr{A}^2_P$ as a consequence of the description of the discrete spectrum in terms of residues of Eisenstein series. An argument of Bernstein gives such a result (for any $\varphi \in \mathscr{A}_P$) without appealing to Langlands' description of the discrete spectrum. Unfortunately, this argument is still unpublished. However, for our purposes we will simply admit it.

Alongside, we have the intertwining operators

$$M(w, \lambda) : \mathscr{A}_P \to \mathscr{A}_{P'}$$

for any $w \in W(M, M')$ given by

$$(M(w,\lambda)\varphi)_{w\lambda}(g) = \int_{(U'_{\mathbb{A}} \cap U^w_{\mathbb{A}})\backslash U'_{\mathbb{A}}} \varphi_\lambda(w^{-1}ug)\, du.$$

Once again, the integral converges absolutely provided that $\mathrm{Re}\langle\lambda, \alpha^\vee\rangle \gg 0$ for all roots α of T_M on $\mathrm{Lie}(U)$ such that $w\alpha < 0$. We *admit* the meromorphic continuation of $M(w, \lambda)$ and the functional equations

$$M(w_1w_2, \lambda) = M(w_1, w_2\lambda)M(w_2, \lambda).$$

for any $w_1 \in W(M', L)$ and $w_2 \in W(M, M')$. In particular,

$$M(w, \lambda)^{-1} = M(w^{-1}, w\lambda).$$

We also have

$$M(w, \lambda)^* = M(w^{-1}, -w\bar{\lambda})$$

on $\mathcal{A}^2_{P'}$. Thus, $M(w, \lambda)$ is unitary (and in particular, holomorphic) on $\mathcal{A}^2_P$ for $\lambda \in \mathrm{i}\mathfrak{a}^*_M$.

For any $\varphi \in \mathcal{A}_P$ and $Q \subseteq P$, we write φ_Q for the constant term along Q, namely

$$\varphi_Q(g) = \int_{V\backslash V_{\mathbb{A}}} \varphi(vg)\, dv = \int_{(V\cap M)\backslash(V_{\mathbb{A}}\cap M_{\mathbb{A}})} \varphi(vg)\, dv.$$

Occasionally we also write φ_V or φ_L for φ_Q.

For any $w \in W^M\backslash W/W^L$ let $P_w \subseteq P$ be the parabolic subgroup with Levi $M_w = M \cap L^w$ and let Q_w be the parabolic subgroup with Levi $L_w = L \cap M^{w^{-1}}$. Note that $w \in W(L_w, M_w)$. The constant term of the Eisenstein series $E_P(\varphi, \lambda)$ along Q is given by

$$\sum_{w\in W^M\backslash W/W^L} E^Q_{Q_w}(M(w^{-1}, \lambda)\varphi_{P_w}, w^{-1}\lambda). \tag{1}$$

This is proved in [Mœglin and Waldspurger 1994, II.1.7] in the case $\varphi \in \mathcal{A}^{\mathrm{cusp}}_P$, in which only the terms involving w such that $L^w \supset M$ (that is, $M_w = M$) contribute. The proof easily extends to the general case — there are simply more contributions. Note that (1) is an identity of meromorphic functions on $\mathfrak{a}^*_{M,\mathbb{C}}$; the terms in (1) are absolutely convergent for $\mathrm{Re}\,\lambda \in \mathfrak{a}^*_{P,+}$ sufficiently regular.

It will also be useful to introduce the following notation for any $\varphi \in \mathcal{A}_P$, $w \in W(L; M)$, and $\lambda \in \mathfrak{a}^*_{M,\mathbb{C}}$:

$$B_Q(g, \varphi, w, \lambda) = (M(w^{-1}, \lambda)\varphi_{L^w})_{w^{-1}\lambda}(g)$$

so that $B_Q(\varphi, w, \lambda) \in \mathcal{A}_Q$. The following result is standard. For completeness we include the proof.

Lemma 1. *Suppose that $w \in W(L'; M)$ and $L \subseteq L'$. Then the constant term of $B_{Q'}(\varphi, w, \lambda)$ along Q is $B_Q(\varphi, w, \lambda)$.*

Proof. Let $Q = LV$ (resp. $Q' = L'V'$, R, R') be the parabolic subgroups with Levi parts L (resp. L', L^w, L'^w). Since $w \in W(L'; M)$ we have

$$(V \cap L')^w = N_R \cap L'^w.$$

The constant term of $B_{Q'}(\varphi, w, \lambda)$ along Q is

$$\int_{(V\cap L')\backslash(V_{\mathbb{A}}\cap L'_{\mathbb{A}})} \int_{V'_{\mathbb{A}}\cap N^{w^{-1}}_{R',\mathbb{A}}\backslash V'_{\mathbb{A}}} (\varphi_{L'^w})_\lambda(wuv\,\cdot)\,du\,dv.$$

Since $V \cap L'$ normalizes both $V'_{\mathbb{A}}$ and $N^{w^{-1}}_{R',\mathbb{A}}$ we can change variables in u to get

$$\int_{(V\cap L')\backslash(V_{\mathbb{A}}\cap L'_{\mathbb{A}})} \int_{V'_{\mathbb{A}}\cap N^{w^{-1}}_{R',\mathbb{A}}\backslash V'_{\mathbb{A}}} (\varphi_{L'^w})_\lambda(wvu\,\cdot)\,du\,dv$$

or

$$\int_{(N_R\cap L'^w)\backslash(N_{R,\mathbb{A}}\cap L'^w_{\mathbb{A}})} \int_{V'_{\mathbb{A}}\cap N^{w^{-1}}_{R',\mathbb{A}}\backslash V'_{\mathbb{A}}} (\varphi_{L'^w})_\lambda(vwu\,\cdot)\,du\,dv$$
$$= \int_{V'_{\mathbb{A}}\cap N^{w^{-1}}_{R',\mathbb{A}}\backslash V'_{\mathbb{A}}} (\varphi_{L^w})_\lambda(wu\,\cdot)\,du.$$

We have $V = V' \rtimes (V \cap L')$ and $N_R = N_{R'} \rtimes (N_R \cap L'^w)$. Therefore $N_R^{w^{-1}} = N_{R'}^{w^{-1}} \rtimes (V \cap L')$, and we can rewrite the integral above as

$$\int_{V_{\mathbb{A}}\cap N^{w^{-1}}_{R,\mathbb{A}}\backslash V_{\mathbb{A}}} (\varphi_{L^w})_\lambda(wu\,\cdot)\,du = (M(w^{-1}, \lambda)\varphi_{L^w})_{w^{-1}\lambda}$$

as required. □

2.1. *Truncation.* For convenience we recall a few facts about Arthur's truncation operator Λ^T [Arthur 1980]. For any $P \subseteq Q$, let τ_P^Q be the characteristic function of the Weyl chamber

$$(\mathfrak{a}_P^Q)_+ = \{X \in \mathfrak{a}_P^Q : \langle \alpha, X\rangle > 0 \text{ for all } \alpha \in \Delta_P^Q\}$$

and let $\hat{\tau}_P^Q$ be the characteristic function of the obtuse Weyl chamber

$$\Big\{ \sum_{\alpha\in\Delta_P^Q} x_\alpha \alpha^\vee : x_\alpha > 0 \text{ for all } \alpha \Big\}.$$

We extend τ_P^Q and $\hat{\tau}_P^Q$ to $\mathfrak{a}_0$ by letting $\tau_P^Q(X) = \tau_P^Q(X_P^Q)$ and $\hat{\tau}_P^Q(X) = \hat{\tau}_P^Q(X_P^Q)$.

For T sufficiently regular in $\mathfrak{a}_0^+$, the truncation operator is given by

$$\Lambda^T\varphi(g) = \sum_{P\supset P_0} (-1)^{r(P)-r(G)} \sum_{\gamma\in P\backslash G} \varphi_P(\gamma g)\hat{\tau}_P(H(\gamma g) - T)$$

for any locally bounded measurable function φ on $G\backslash G^1_{\mathbb{A}}$. It defines an orthogonal projection on $L^2(G\backslash G^1_{\mathbb{A}})$. If φ is of uniform moderate growth, then $\Lambda^T\varphi$ is rapidly decreasing.

More generally, for any Q, one defines the relative truncation with respect to Q by

$$\Lambda^{T,Q}\varphi(g) = \sum_{P\in[P_0,Q]} (-1)^{r(P)-r(Q)} \sum_{\gamma\in P\backslash Q} \varphi_P(\gamma g)\hat{\tau}_P^Q(H(\gamma g)-T).$$

By the Langlands combinatorial lemma, we have the inversion formula

$$\varphi_P(g) = \sum_{Q\subseteq P}\sum_{\gamma\in Q\backslash P} \Lambda^{T,Q}\varphi_Q(\gamma g)\tau_Q^P(H(\gamma g)-T) \tag{2}$$

[Arthur 1980, Lemma 1.5].

For any $\varphi\in\mathcal{A}_P$ and $Q\subseteq P$, we write $\mathcal{E}_Q(\varphi)=\mathcal{E}_Q(\varphi_Q)\subseteq\mathfrak{a}^*_{Q,\mathbb{C}}$ for the multiset of cuspidal exponents of φ along Q — see [Mœglin and Waldspurger 1994, I.3.4]. We also write $\mathcal{E}_{\subseteq P}(\varphi)=\bigcup_{Q\in[P_0,P]}\mathcal{E}_Q(\varphi)$. In the case $P=G$ we simply write $\mathcal{E}(\varphi)$ for $\mathcal{E}_{\subseteq G}(\varphi)$.

For a multiset $A=\{\lambda_1,\dots,\lambda_m\}\subseteq\mathfrak{a}^*_{0,\mathbb{C}}$ (including multiplicities) we write $\mathcal{PE}(A)$ for the space of polynomial exponential functions on $\mathfrak{a}_0$ with exponents $\subseteq A$. This means that any $f\in\mathcal{PE}(A)$ has the form

$$f(X)=\sum_{\lambda\in A}P_\lambda(X)e^{\langle\lambda,X\rangle},$$

where for any $\lambda\in A$, P_λ is a polynomial in $\mathfrak{a}_0$ whose degree is smaller than the multiplicity of λ in A. Equivalently, $f\in\mathcal{PE}(A)$ if and only if for any $v_1,\dots,v_m\in\mathfrak{a}_0$, f is annihilated by the differential operator

$$\prod_{i=1}^m(D_{v_i}-\langle\lambda_i,v_i\rangle),$$

where D_v denotes taking the partial derivative along $v\in\mathfrak{a}_0$. We also write $\mathcal{PE}_-=\mathcal{PE}(\mathcal{C}_{0,-}\setminus\{0\})$, where we limit the exponents λ to $\mathcal{C}_{0,-}\setminus\{0\}$, but we do not limit the degree of P_λ.

The following lemma is a simple consequence of the properties of truncation.

Lemma 2 [Lapid and Rogawski 2003, Proposition 8.4.1]. *For any automorphic forms $\varphi_i\in\mathcal{A}^n_G$, $i=1,2$, we have*

$$\langle\varphi_1,\Lambda^T\varphi_2\rangle_{G\backslash G^1_{\mathbb{A}}}\in\mathcal{PE}(\mathcal{E}(\varphi_1)+\overline{\mathcal{E}(\varphi_2)}).$$

Moreover, if $\varphi_1,\varphi_2\in\mathcal{A}^2_G$ then

$$\langle\varphi_1,\Lambda^T\varphi_2\rangle_{G\backslash G^1_{\mathbb{A}}}-\langle\varphi_1,\varphi_2\rangle_{G\backslash G^1_{\mathbb{A}}}\in\mathcal{PE}_-.$$

We also recall the following elementary fact.

Lemma 3. *Let* $\mathscr{C} = \left\{\sum_{i=1}^m a_i v_i : a_1, \dots, a_m \ge 0\right\}$ *be a salient*[1] *polyhedral cone in a finite dimensional vector space* V *over* $\mathbb{R}$ *(for some* $v_1, \dots, v_m \in V \setminus \{0\}$*). Then for any* $A \subseteq V^*$ *and* $f \in \mathscr{PE}(A)$ *the function*

$$\int_V \mathbf{1}_{\mathscr{C}}(X-T) f(X) e^{\langle \lambda, X\rangle}\, dX$$

converges for $\{\lambda \in V_{\mathbb{C}}^* : \operatorname{Re}\langle \lambda, v_i\rangle \ll 0, i = 1, \dots, m\}$ *and extends to a meromorphic function on* $V_{\mathbb{C}}^*$ *with hyperplane singularities. As a function of* T*, it belongs to* $\mathscr{PE}(A+\lambda)$.

This is a straightforward computation if $\mathscr{C}$ is simplicial. Otherwise, it follows by subdivision of $\mathscr{C}$ into simplicial subcones.

3. Cuspidal Eisenstein series

For the convenience of the reader we recall the results of Langlands and Arthur for cuspidal Eisenstein series.[2]

For any $w \in W(L, M)$ let ϕ_L^w be the function on $\mathfrak{a}_L^G$ given by

$$\phi_L^w\Big(\sum_{\alpha\in\Delta_Q} x_\alpha \alpha^\vee\Big) = \begin{cases} (-1)^{\#\{\alpha\in\Delta_Q : x_\alpha > 0\}} & \text{if } \{\alpha\in\Delta_Q : x_\alpha > 0\} = \{\alpha \in \Delta_Q : w\alpha < 0\}, \\ 0 & \text{otherwise.}\end{cases}$$

The Laplace transform of ϕ_L^w is given by

$$\int_{\mathfrak{a}_L^G} e^{\langle\lambda, X\rangle} \phi_L^w(X)\, dX = \frac{\operatorname{vol}(\mathfrak{a}_L^G/\mathbb{Z}\Delta_Q^\vee)}{\prod_{\alpha\in\Delta_Q}\langle\lambda, \alpha^\vee\rangle}, \quad \lambda \in (\mathfrak{a}_L^G)_{\mathbb{C}}^*, \quad \operatorname{Re} w\lambda \in \mathfrak{a}_{P,+}^*. \tag{3}$$

By [Arthur 1980, Lemma 4.1], for $\operatorname{Re}\lambda \in \mathfrak{a}_{P,+}^*$ sufficiently regular we have

$$\begin{aligned}\Lambda^T E_P(\varphi,\lambda) \\ = \sum_{Q\in[P_0,G]} \sum_{w\in W(L,M)} \sum_{\gamma\in Q\backslash G} (M(w^{-1},\lambda)\varphi)_{w^{-1}\lambda}(\gamma g)\phi_L^w(H(\gamma g) - T).\end{aligned} \tag{4}$$

Suppose that $\varphi_j \in \mathscr{A}_{P_j}^{\text{cusp}}$, $j = 1, 2$. Set $f_i = E_{P_i}(\varphi_i, \lambda_i)$, $i = 1, 2$. Using (4) we write $\langle f_1, \Lambda^T f_2\rangle_{G\backslash G_{\mathbb{A}}^1}$ as the sum over Q and $w_2 \in W(L, M_2)$ of

$$\Big\langle f_1, \sum_{\gamma\in Q\backslash G} (M(w_2^{-1},\lambda_2)\varphi_2)_{w_2^{-1}\lambda_2}(\gamma g)\phi_L^{w_2}(H(\gamma g)-T)\Big\rangle_{G\backslash G_{\mathbb{A}}^1}$$

[1] That is, such that $\mathscr{C} \cap -\mathscr{C} = \{0\}$.

[2] A similar argument to the one below was given by Labesse in the 1983 Friday morning seminar on the twisted trace formula. See lecture 12 in http://www.math.ubc.ca/~cass/Langlands/friday/friday.html and [Labesse and Waldspurger 2012, §5.4].

provided $\operatorname{Re}\lambda_2 \in \mathfrak{a}^*_{M_2,+}$ is sufficiently regular. (This will be justified in Lemma 14 below.) Each summand is equal to

$$\langle (f_1)_Q, (M(w_2^{-1}, \lambda_2)\varphi_2)_{w_2^{-1}\lambda_2}\phi_L^{w_2}(H(\cdot) - T)\rangle_{Q\backslash G^1_{\mathbb{A}}}.$$

Using the formula for the constant term, we get

$$\begin{aligned}(5)\quad & (E_{P_1}(\varphi_1, \lambda_1), \Lambda^T E_{P_2}(\varphi_2, \lambda_2))_{G\backslash G^1_{\mathbb{A}}} \\ &= \sum_Q \sum_{w_1 \in W(L,M_1)} \sum_{w_2 \in W(L,M_2)} \big\langle (M(w_1^{-1}, \lambda_1)\varphi_1)_{w_1^{-1}\lambda_1}, \\ &\qquad (M(w_2^{-1}, \lambda_2)\varphi_2)_{w_2^{-1}\lambda_2}\phi_L^{w_2}(H(\cdot) - T)\big\rangle_{Q\backslash G^1_{\mathbb{A}}}.\end{aligned}$$

Finally, using (3) we get

$$(E_{P_1}(\varphi_1, \lambda_1), \Lambda^T E_{P_2}(\varphi_2, \lambda_2))_{G\backslash G^1_{\mathbb{A}}} = \mathfrak{M}^T(\varphi_1, \lambda_1, \varphi_2, \lambda_2),$$

where

$$\begin{aligned}(6)\quad & \mathfrak{M}^T(\varphi_1, \lambda_1, \varphi_2, \lambda_2) \\ &= \sum_Q \sum_{w_1 \in W(L,M_1)} \sum_{w_2 \in W(L,M_2)} \frac{e^{\langle w_1^{-1}\lambda_1 + w_2^{-1}\bar{\lambda}_2, T\rangle}}{\prod_{\alpha\in\Delta_Q}\langle w_1^{-1}\lambda_1 + w_2^{-1}\bar{\lambda}_2, \alpha^\vee\rangle} \\ &\qquad \operatorname{vol}(\mathfrak{a}_L^G/\mathbb{Z}\Delta_Q^\vee)\langle M(w_1^{-1}, \lambda_1)(\varphi_1), M(w_2^{-1}, \lambda_2)(\varphi_2)\rangle_{A_L V_{\mathbb{A}} L\backslash G_{\mathbb{A}}}.\end{aligned}$$

These are the usual Maass–Selberg relations proved in [Arthur 1980, §4]. Note that the intricate residue calculus of [loc. cit.] is unnecessary.

4. Some combinatorial lemmas

In order to analyze the truncation of Eisenstein series and the Maass–Selberg relations in the general case we will need a few combinatorial definitions and lemmas in the spirit of [Arthur 1978, §6].

Let L' and M be standard Levi subgroups and let $w \in W(L'; M)$ and $Q \supset Q'$. For any $X \in \mathfrak{a}_0$ with $X_{Q'}^Q = \sum_{\alpha\in\Delta_{Q'}^Q} x_\alpha\alpha^\vee \in \mathfrak{a}_{Q'}^Q$ we set

$$D_{Q',+}^Q(X) = \{\alpha \in \Delta_{Q'}^Q : x_\alpha > 0\} \subseteq \Delta_{Q'}^Q.$$

Observe that for any $Q_2 \supset Q_1$, $D_{Q_2,+}^Q(X)$ consists of the nonzero projections of the elements of $D_{Q_1,+}^Q(X)$.

Also set

$$\phi_{L',M,w}^Q(X) = \begin{cases} (-1)^{|D_{Q',+}^Q(X)|} & \text{if } D_{Q',+}^Q(X) = \{\alpha\in\Delta_{Q'}^Q : w\alpha < 0 \text{ or } w\alpha \in \Delta_{(L')^w}^M\}, \\ 0 & \text{otherwise.}\end{cases}$$

Note that the condition $w\alpha \in \Delta_{(L')^w}^M$ is equivalent to $(w\alpha)_M = 0$.

As usual, we suppress the superscript if $Q = G$.

Note that if $w \in W(L, M)$ then $\phi_{L,M,w}$ is the function denoted by ϕ_L^w in the previous section. In particular, in this case

$$\int_{\mathfrak{a}_L^G} e^{\langle\lambda, X\rangle}\phi_{L,M,w}(X)\,dX = \frac{\operatorname{vol}(\mathfrak{a}_L^G/\mathbb{Z}\Delta_Q^\vee)}{\prod_{\alpha\in\Delta_Q}\langle\lambda, \alpha^\vee\rangle}, \quad \lambda \in (\mathfrak{a}_L^G)^*_{\mathbb{C}}, \quad \operatorname{Re} w\lambda \in \mathfrak{a}^*_{P,+}. \tag{7}$$

Lemma 4. *Suppose that $R \subseteq S \subseteq Q$ and $w \in W(M_S; M)$. Then*

$$\sum_{Q'\in[R,S]} \phi^Q_{L',M,w}(X) = \begin{cases} \phi^Q_{S,M,w}(X) & \text{if } D^Q_{R,+}(X) \cap \Delta^S_R = \varnothing, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We observe that for any $Q' \in [R, S]$ we have $\phi^Q_{L',M,w}(X) \neq 0$ if and only if $\phi^Q_{M_S,M,w}(X) \neq 0$ and $D^Q_{Q',+}(X) \supset \Delta^S_{Q'}$. In this case,

$$\phi^Q_{L',M,w}(X) = (-1)^{r(Q')-r(S)}\phi^Q_{M_S,M,w}(X).$$

The lemma follows from [Arthur 1978, Proposition 1.1]. □

We also recall the following version of Langlands' combinatorial lemma.

Lemma 5 (Arthur). *Let $w \in W(L'; M)$ and $Q \supset Q'$. Then we have*

$$\sum_{R\in[Q',Q]} \phi^R_{L',M,w}(X)\tau^Q_R(X) = \begin{cases} 1 & \text{if } w\alpha > 0 \text{ and } w\alpha \notin \Delta^M_{(L')^w} \text{ for all } \alpha \in \Delta^Q_{Q'}, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, taking $Q' = P$ and $w = 1$, for any $X \in \mathfrak{a}_P$ there exists a unique $Q \in [P, G]$ such that $\tau_Q(X) = 1$ and $X^Q \in \mathscr{C}^Q_{P,-}$. Moreover, $\langle\alpha, X\rangle > 0$ for any $\alpha \in \Delta_P \setminus \Delta^Q_P$ and $D_{P,+}(X) \supset \Delta_P \setminus \Delta^Q_P$.

This follows from [Arthur 1978, Lemma 6.3] by taking $\Lambda = -w^{-1}\Lambda_0$ where $\Lambda_0 \in \mathfrak{a}^*_{M,+}$.

For nonnegative quantities A and B (depending on parameters) we will write $A \ll B$ if there exists a constant $c > 0$ (independent of the parameters) such that $A \le cB$.

Lemma 6. *Suppose that $P \in [R, Q]$, $X \in \mathfrak{a}^Q_R$, and*

$$D^Q_{R,+}(X) \cap \Delta^P_R = \{\alpha \in \Delta^P_R : \langle\alpha, X\rangle \le 0\}.$$

Then $\|X\| \ll \|X_M\|$.

Proof. Write $X = \sum_{\alpha\in\Delta_R} x_\alpha\alpha^\vee$ as $X_1 + X_2$ where

$$X_1 = \sum_{\alpha\in\Delta^P_R} x_\alpha\alpha^\vee \quad \text{and} \quad X_2 = \sum_{\alpha\in\Delta_R\setminus\Delta^P_R} x_\alpha\alpha^\vee.$$

We have to show that under the conditions of the lemma we have $\|X_1\| \le C\|X_2\|$ for some constant C which is independent of X. Let $S(X)$ be such that

$$\Delta_R^{S(X)} = \{\alpha \in \Delta_R^P : \langle \alpha, X\rangle > 0\} = \Delta_R^P \setminus D_{R,+}^Q(X).$$

Fix $\lambda \in (\mathfrak{a}_R^{S(X)})_+^*$. Since the coefficients of λ in the basis $\Delta_R^{S(X)}$ are positive, we have

$$0 \le \langle \lambda, X\rangle = \langle \lambda, X_1\rangle + \langle \lambda, X_2\rangle.$$

On the other hand, we have $\lambda = \sum_{\varpi \in \hat{\Delta}_R^Q} \lambda_\varpi \varpi$ where $\lambda_\varpi > 0$ for $\varpi \in \hat{\Delta}_R^Q \setminus \hat{\Delta}_{S(X)}^Q$ and $\lambda_\varpi \le 0$ for $\varpi \in \hat{\Delta}_{S(X)}^Q$. Thus,

$$\sum_{\alpha \in \Delta_R^{S(X)}} |x_\alpha| \ll -\langle \lambda, X_1\rangle.$$

(There are of course only finitely many possibilities for $S(X)$, so the dependence of the implied constant on λ is immaterial.)

Similarly, fix $\mu \in (\mathfrak{a}_R^{S'(X)})_+^*$ where

$$\Delta_R^{S'(X)} = \Delta_R^P \setminus \Delta_R^{S(X)} = \Delta_R^P \cap D_{R,+}^Q(X).$$

Then

$$\langle \mu, X_1\rangle \le -\langle \mu, X_2\rangle$$

and

$$\sum_{\alpha \in \Delta_R^{S'(X)}} |x_\alpha| \ll \langle \mu, X_1\rangle.$$

Thus, $\langle \mu - \lambda, X_1\rangle \le \langle \lambda - \mu, X_2\rangle$ while $\|X_1\| \ll \langle \mu - \lambda, X_1\rangle$. The claim follows. $\square$

As before, fix P and Q. For any $R \subseteq Q$ and $w \in W(M_R; M)$ define

$$\chi_{M_R,M,w}^Q(X) = \sum_{Q' \in [R,Q] : w \in W(L';M)} \tau_R^{Q'}(X)\phi_{L',M,w}^Q(X). \tag{8}$$

Lemma 7. *We have*

$$\chi_{M_R,M,w}^Q(X) = \begin{cases} (-1)^{|D_{R,+}^Q(X)|} & \text{if } D_{R,+}^Q(X) = \{\alpha \in \Delta_R^Q : w\alpha < 0 \text{ or} \\ & \qquad (w\alpha \in \Delta_{M_R^w}^M \text{ and } \langle \alpha, X\rangle \le 0)\}, \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, if $\chi_{M_R,M,w}^Q(X) \ne 0$ *and* $X \in \mathfrak{a}_R^Q$ *then* $wX \in \mathscr{C}_{M_R^w,-}$ *and* $\|X\| \ll \|(wX)_M\|$.

Proof. Let $R_1 \in [R, Q]$ be the parabolic subgroup such that

$$\Delta_R^{R_1} = \{\alpha \in \Delta_R^Q : w\alpha \in \Delta_{M_R^w}^M\}$$

so that $w \in W(L'; M)$ if and only if $Q' \in [R, R_1]$. Let $R_2 \in [R, Q]$ be such that

$$\Delta_R^{R_2} = \{\alpha \in \Delta_R^Q : \langle \alpha, X\rangle > 0\}$$

so that $\tau_R^{Q'}(X) = 1$ if and only if $Q' \subseteq R_2$. Let $S = R_1 \cap R_2$. Then

$$\chi^Q_{M_R,M,w}(X) = \sum_{Q' \in [R,S]} \phi^Q_{L',M,w}(X).$$

The first part now follows from Lemma 4.

In order to prove the second part, let $X \in \mathfrak{a}_R^Q$ and write $X = \sum_{\alpha \in \Delta_R^Q} x_\alpha \alpha^\vee$. Suppose that $\chi^Q_{M_R,M,w}(X) \neq 0$ and let

$$A = \{\alpha \in \Delta_R^Q : w\alpha \in \Delta^M_{M_R^w} \text{ and } \langle \alpha, X\rangle \le 0\}.$$

Let Q_1 be the parabolic subgroup with Levi M_R^w. We write

$$wX = \sum_{\alpha \in A} x_\alpha w\alpha^\vee + \sum_{\alpha \notin A} x_\alpha w\alpha^\vee$$

and observe that the first sum is a linear combination of roots in $wA \subseteq \Delta^M_{Q_1}$ with positive coefficients, while the second sum lies in $\mathscr{C}_{Q_1,-}$. Thus, $D_{Q_1,+}(wX) \subseteq wA$. Using Lemma 5, let $R_1 \in [Q_1, G]$ be such that $\tau_{R_1}(wX) = 1$ and $(wX)^{R_1} \in \mathscr{C}^{R_1}_{Q_1,-}$. Then $\langle \alpha, wX\rangle > 0$ for all $\alpha \in \Delta_{Q_1} \setminus \Delta^{R_1}_{Q_1}$ and $D_{Q_1,+}(wX) \supset \Delta_{Q_1} \setminus \Delta^{R_1}_{Q_1}$. In particular, $\Delta_{Q_1} \setminus \Delta^{R_1}_{Q_1} \subseteq wA$. On the other hand, from the definition of A, we have $\langle w\alpha, wX\rangle = \langle \alpha, X\rangle \le 0$ for any $\alpha \in A$. It follows that $R_1 = G$, that is, $wX \in \mathscr{C}_{Q_1,-}$ as required.

It remains to show that $\|X\| \ll \|(wX)_M\|$ if $X \in \mathfrak{a}_R^Q$ and $\chi^Q_{M_R,M,w}(X) \neq 0$. Write $X = X_1 + X_2$ where

$$X_1 = \sum_{\alpha \in \Delta_R^Q : w\alpha \in \Delta^M_{M_R^w}} x_\alpha \alpha^\vee \quad \text{and} \quad X_2 = \sum_{\alpha \in \Delta_R^Q : w\alpha \notin \Delta^M_{M_R^w}} x_\alpha \alpha^\vee.$$

We can apply Lemma 6 (with $L \cap M^{w^{-1}}$ instead of M) to infer that $\|X_1\| \ll \|X_2\|$. On the other hand, since

$$(wX_2)_M = \sum_{\alpha \in \Delta_R^Q : w\alpha \notin \Delta^M_{M_R^w}} x_\alpha (w\alpha^\vee)_M$$

and each $w\alpha^\vee$ has the opposite sign of x_α, we conclude that $\|X_2\| \ll \|(wX_2)_M\|$. Our claim follows. □

Corollary 8. *For any k, we have*

$$\int_{\mathfrak{a}_R^G} \chi_{M_R,M,w}(X) e^{k\|X\| + \langle w^{-1}\lambda, X\rangle}\, dX < \infty$$

for any $\lambda \in \mathfrak{a}^*_{M,+}$ sufficiently regular (depending on k).

For the rest of the section, suppose that we are given Q, M_i, and $w_i \in W(L; M_i)$, $i = 1, 2$.

Corollary 9. *For any k and $Q \subseteq Q_2$ we have*

$$\int_{\mathfrak{a}_Q^G} \chi_{L,M_1,w_1}^{Q_2}(X)\chi_{L_2,M_2,w_2}(X)e^{k\|X\|+\langle w_1^{-1}\lambda_1+w_2^{-1}\lambda_2,X\rangle}\,dX < \infty$$

provided that $\lambda_1 \in \mathfrak{a}_{M_1,+}^*$ is sufficiently regular (depending on k) and $\lambda_2 \in \mathfrak{a}_{M_2,+}^*$ is sufficiently regular (depending on λ_1 and k).

Proof. It follows from Lemma 7 that for any $C > 0$ we have

$$-\langle \lambda_2, w_2 X\rangle \geq C\|X_{Q_2}\|$$

if $\chi_{L_2,M_2,w_2}(X) \neq 0$ provided that $\lambda_2 \in \mathfrak{a}_{M_2,+}^*$ is sufficiently regular (depending on C, but not on X). Similarly, for any $C > 0$ we have

$$-\langle \lambda_1, w_1 X\rangle = -\langle \lambda_1, w_1 X^{Q_2}\rangle - \langle \lambda_1, w_1 X_{Q_2}\rangle \geq C\|X^{Q_2}\| - C_2\|X_{Q_2}\|$$

if $\chi_{L,M_1,w_1}^{Q_2}(X) \neq 0$ provided that $\lambda_1 \in \mathfrak{a}_{M_1,+}^*$ is sufficiently regular, depending on C, but not on X, and with C_2 depending only on λ_1. Thus for any C, we have

$$-\langle w_1^{-1}\lambda_1 + w_2^{-1}\lambda_2, X\rangle \geq C\|X\|$$

if $\chi_{L,M_1,w_1}^{Q_2}(X)\chi_{L_2,M_2,w_2}(X) \neq 0$, provided that $\lambda_1 \in \mathfrak{a}_{M_1,+}^*$ is sufficiently regular (depending on C) and $\lambda_2 \in \mathfrak{a}_{M_2,+}^*$ is sufficiently regular (depending on λ_1 and C). The corollary follows. □

We define

$$\Psi_{L,M_1,w_1,M_2,w_2}(X) = \sum_{Q_2 \supset Q\,:\,w_2 \in W(L_2;\,M_2)} \chi_{L,M_1,w_1}^{Q_2}(X)\chi_{L_2,M_2,w_2}(X).$$

We can explicate the function Ψ_{L,M_1,w_1,M_2,w_2} as follows.

Proposition 10. *Let R_i, $i = 1, 2$, be such that*

$$\Delta_Q^{R_i} = \{\alpha \in \Delta_Q : w_i\alpha \in \Delta_{L^{w_i}}^{M_i}\}$$

and let R_1' be such that

$$\Delta_Q^{R_1'} = \{\alpha \in \Delta_Q : w_1\alpha > 0 \text{ and } w_1\alpha \notin \Delta_{L^{w_1}}^{M_1}\}.$$

Then

$$\Psi_{L,M_1,w_1,M_2,w_2}(X) = \begin{cases} (-1)^{|D_{Q,+}(X)|} & \text{if } D_{Q,+}(X) = \{\alpha \in \Delta_Q : w_2\alpha < 0\} \\ & \qquad \cup\,(\Delta_Q^{R_2} \setminus (\{\alpha \in \Delta_Q^{R_1} : \langle \alpha, X\rangle > 0\} \cup \Delta_Q^{R_1'})), \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\Psi_{L,M_1,w_1,M_2,w_2} \not\equiv \Psi_{L,M_2,w_2,M_1,w_1}$.

Proof. Note that for $L_i \supset L$, we have $w_i \in W(L_i; M_i)$ if and only if $Q_i \subseteq R_i$, $i = 1, 2$. Thus, upon substituting (8) for χ, we get that $\Psi_{L,M_1,w_1,M_2,w_2}(X)$ is equal to

$$\sum_{Q_2 \in [Q, R_2]} \sum_{Q_1 \in [Q, Q_2 \cap R_1]} \tau_Q^{Q_1}(X)\phi^{Q_2}_{Q_1,M_1,w_1}(X) \sum_{Q_2' \in [Q_2, R_2]} \tau_{Q_2}^{Q_2'}(X)\phi_{Q_2',M_2,w_2}(X).$$

We write this differently as

$$\sum_{Q_1 \in [Q, R_1]} \tau_Q^{Q_1}(X) \sum_{Q_2' \in [Q_1, R_2]} \phi_{Q_2',M_2,w_2}(X) \sum_{Q_2 \in [Q_1, Q_2']} \phi^{Q_2}_{Q_1,M_1,w_1}(X)\tau_{Q_2}^{Q_2'}(X).$$

By Lemma 5, we get

$$\Psi_{L,M_1,w_1,M_2,w_2}(X) = \sum_{Q_1 \in [Q, R_1]} \tau_Q^{Q_1}(X) \sum_{Q_2' \in [Q_1, R_2 \cap Q_1^\sharp]} \phi_{Q_2',M_2,w_2}(X),$$

where $Q_1^\sharp$ is such that

$$\Delta_{Q_1}^{Q_1^\sharp} = \{\alpha \in \Delta_{Q_1} : w_1\alpha > 0 \text{ and } w_1\alpha \notin \Delta_{L_1^{w_1}}^{M_1}\}.$$

Observe that $\Delta_{Q_1}^{Q_1^\sharp}$ consists of the projections of $\Delta_Q^{R_1'}$, that is,

$$\Delta_Q^{Q_1^\sharp} = \Delta_Q^{Q_1} \cup \Delta_Q^{R_1'}$$

(disjoint union). In particular, $Q^\sharp = R_1'$. Thus, by Lemma 4, we get

$$\Psi_{L,M_1,w_1,M_2,w_2}(X) = \sum \tau_Q^{Q_1}(X)\phi_{R_2 \cap Q_1^\sharp,M_2,w_2}(X),$$

where the sum is over $Q_1 \in [Q, R_1 \cap R_2]$ such that $D_{Q_1,+}(X) \cap \Delta_{Q_1}^{R_2 \cap Q_1^\sharp} = \varnothing$, or equivalently, $Q_1 \in [S_1(X), R_1 \cap R_2]$ where

$$\Delta_Q^{S_1(X)} = \Delta_Q^{R_2 \cap R_1'} \cap D_{Q,+}(X).$$

On the other hand, let $S_2(X)$ be such that

$$\Delta_Q^{S_2(X)} = \{\alpha \in \Delta_Q : \langle \alpha, X \rangle > 0\}.$$

Then $\tau_Q^{Q_1}(X) = 1$ if and only if $Q_1 \subseteq S_2(X)$. All in all, we get

$$\sum_{Q_1 \in [S_1(X), R_1 \cap R_2 \cap S_2(X)]} \phi_{R_2 \cap Q_1^\sharp,M_2,w_2}(X).$$

Note that since $R_1 \cap R_1' = Q$ and $S_1(X) \subseteq R_1'$, we have $S_1(X) \subseteq R_1 \cap R_2 \cap S_2(X)$ if and only if $S_1(X) = Q$. In this case, the map $Q_1 \mapsto R_2 \cap Q_1^\sharp$ is a bijection between

$[S_1(X), R_1 \cap R_2 \cap S_2(X)]$ and $[R_2 \cap R_1', S_2'(X)]$ where $S_2'(X) = R_2 \cap (R_1 \cap R_2 \cap S_2(X))^\sharp$. We thus get (assuming $S_1(X) = Q$)

$$\sum_{Q_1' \in [R_2 \cap R_1', S_2'(X)]} \phi_{Q_1', M_2, w_2}(X).$$

Invoking Lemma 4 once again, we get that

$$\Psi_{L,M_1,w_1,M_2,w_2}(X) = \phi_{S_2'(X),M_2,w_2}(X)$$

if $S_1(X) = Q$ and

$$D_{R_2 \cap R_1',+}(X) \cap \Delta_{R_2 \cap R_1'}^{S_2'(X)} = \varnothing. \tag{9}$$

Otherwise, $\Psi_{L,M_1,w_1,M_2,w_2}(X) = 0$. We can rewrite condition (9) equivalently as

$$D_{Q,+}(X) \cap \Delta_Q^{S_2'(X)} \subseteq \Delta_Q^{R_2 \cap R_1'}.$$

Once again, since $R_1 \cap R_1' = Q$, this becomes

$$\Delta_Q^{R_1 \cap R_2 \cap S_2(X)} \cap D_{Q,+}(X) = \varnothing.$$

The proposition follows. □

Corollary 11. *For any k we have*

$$\int_{\mathfrak{a}_Q^G} \Psi_{L,M_1,w_1,M_2,w_2}(X) e^{k\|X\| + \langle w_1^{-1}\lambda_1 + w_2^{-1}\lambda_2, X\rangle}\, dX < \infty$$

provided that $\lambda_1 \in \mathfrak{a}_{M_1,+}^*$ is sufficiently regular (depending on k) and $\lambda_2 \in \mathfrak{a}_{M_2,+}^*$ is sufficiently regular (depending on λ_1 and k). Moreover, for any $f_i \in \mathcal{PE}(A_i)$, $i = 1, 2$,

$$\int_{\mathfrak{a}_Q^G} \Psi_{L,M_1,w_1,M_2,w_2}(X - T) f_1(w_1 X) f_2(w_2 X) e^{\langle w_1^{-1}\lambda_1 + w_2^{-1}\lambda_2, X\rangle}\, dX$$

has meromorphic continuation for $\lambda_i \in \mathfrak{a}_{M_i,\mathbb{C}}^*$, $i = 1, 2$, with hyperplane singularities, and as a function of T, it belongs to $\mathcal{PE}(w_1^{-1}A_1 + w_2^{-1}A_2 + w_1^{-1}\lambda_1 + w_2^{-1}\lambda_2)$.

Proof. The first part follows from Corollary 9 and the defining expression for Ψ_{L,M_1,w_1,M_2,w_2}. Alternatively, we can deduce it from Proposition 10 as follows. Suppose that

$$X = \sum_{\alpha \in \Delta_Q} x_\alpha \alpha^\vee$$

and $\Psi_{L,M_1,w_1,M_2,w_2}(X) \neq 0$. Write $X = X_1 + X_2 + X_3$ where

$$X_1 = \sum_{\alpha \in \Delta_Q \setminus \Delta_Q^{R_2}} x_\alpha \alpha^\vee, \quad X_2 = \sum_{\alpha \in \Delta_Q^{R_2} \setminus \Delta_Q^{R_1}} x_\alpha \alpha^\vee, \quad \text{and} \quad X_3 = \sum_{\alpha \in \Delta_Q^{R_1 \cap R_2}} x_\alpha \alpha^\vee.$$

By Proposition 10, the coefficients x_α in X_1 are positive precisely when $w_2\alpha < 0$, the coefficients in X_2 are positive precisely when $w_1\alpha < 0$, and the coefficients in X_3 are positive precisely when $\langle \alpha, X\rangle \le 0$. Then $w_2 X = w_2 X_1 + w_2(X_2 + X_3)$ where $w_2 X_1 \in \mathscr{C}_{0,-}$ and $w_2(X_2 + X_3) \in w_2 \mathfrak{a}_Q^{R_2} \subseteq \mathfrak{a}_0^{M_2}$. Thus, $\langle \lambda_2, w_2 X\rangle = \langle \lambda_2, w_2 X_1\rangle$. Note that the kernel of the map $X \mapsto (w_2 X)_{M_2}$ is $\mathfrak{a}_Q^{R_2}$. Therefore, for any $C_1 > 0$, we have

$$-\langle w_2^{-1}\lambda_2, X\rangle = -\langle \lambda_2, w_2 X\rangle \ge C_1 \|X_1\|$$

provided that $\lambda_2 \in \mathfrak{a}^*_{M_2,+}$ is sufficiently regular (depending on C_1, but not on X).

We also have $w_1 X_2 \in \mathscr{C}_{0,-}$ and $\langle \lambda_1, w_1 X\rangle = \langle \lambda_1, w_1 X_1\rangle + \langle \lambda_1, w_1 X_2\rangle$. By the same reasoning, we infer that for any $C_2 > 0$, we have

$$-\langle w_1^{-1}\lambda_1, X\rangle = -\langle \lambda_1, w_1 X\rangle \ge C_2\|X_2\| - C_3\|X_1\|$$

for all $\lambda_1 \in \mathfrak{a}^*_{M_1,+}$ sufficiently regular (depending on C_2 but not on X) where C_3 depends on λ_1 but not on X.

Thus for any $C > 0$ and for $\lambda_1 \in \mathfrak{a}^*_{M_1,+}$ sufficiently regular (depending on C) and $\lambda_2 \in \mathfrak{a}^*_{M_2,+}$ sufficiently regular (depending on C and λ_1), we have

$$-\langle \lambda_1, w_1 X\rangle - \langle \lambda_2, w_2 X\rangle \ge C\|X_1 + X_2\|. \tag{10}$$

On the other hand, by Lemma 6, it follows that $\|X_3\| \ll \|X_1 + X_2\|$ on the support of Ψ_{L,M_1,w_1,M_2,w_2}. Thus we can replace the right-hand side of (10) by $C\|X\|$. The first part of the corollary follows.

The second part follows from Lemma 3. □

5. Truncation of a general Eisenstein series

We will use the notation of the previous sections.

We have the following generalization of (4).

Lemma 12. *For* $\operatorname{Re}\lambda \in \mathfrak{a}^*_{P,+}$ *sufficiently regular we have*

$$\begin{aligned} \Lambda^{T,Q} E_P(g, \varphi, \lambda) \\ = \sum_{Q' \in [P_0, Q],\, w \in W(L';M)} \sum_{\gamma \in Q'\backslash Q} B_{Q'}(\gamma g, \varphi, w, \lambda)\phi^Q_{L',M,w}(H(\gamma g) - T). \end{aligned} \tag{11}$$

Proof. Let $E = E_P(\varphi, \lambda)$. Then

$$\begin{aligned} &\Lambda^{T,Q} E_P(g, \varphi, \lambda) \\ &= \sum_{P' \in [P_0, Q]} (-1)^{r(P')-r(Q)} \sum_{\gamma \in P'\backslash Q} (E_P(\varphi, \lambda))_{P'}(\gamma g)\hat{\tau}^Q_{P'}(H(\gamma g) - T). \end{aligned}$$

Using (1) for the constant term of Eisenstein series we get

$$\sum_{P'\in[P_0,Q]}(-1)^{r(P')-r(Q)}\sum_{\gamma\in P'\backslash Q}\sum_{w\in W^M\backslash W/W^{M'}}E^{P'}_{P'_w}(\gamma g, M(w^{-1},\lambda)\varphi_{P_w}, w^{-1}\lambda)\\ \times\hat\tau^Q_{P'}(H(\gamma g)-T),$$

where P_w (resp. P'_w) is the standard parabolic with Levi part $M_w = M\cap M'^w$ (resp. $M'_w = M'\cap M^{w^{-1}}$). Unfolding the Eisenstein series, we get

$$\sum_{P'\in[P_0,Q]}(-1)^{r(P')-r(Q)}\sum_{w\in W^M\backslash W/W^{M'}}\sum_{\gamma\in P'_w\backslash Q}B_{P_w}(\gamma g,\varphi,w,\lambda)\hat\tau^Q_{P'}(H(\gamma g)-T).$$

The sum is absolutely convergent by the assumption on λ. For any $w\in W^M\backslash W/W^{M'}$, we have $w\in W(M'_w;M)$. Therefore, we may rearrange the sums differently as

$$\sum_{Q'\in[P_0,Q]}\sum_{w\in W(L';M)}\sum_{\gamma\in Q'\backslash Q}B_{Q'}(\gamma g,\varphi,w,\lambda)\\ \sum_{P'\in[P_0,Q]:w\in W/W^{M'},P'_w=Q'}(-1)^{r(P')-r(Q)}\hat\tau^Q_{P'}(H(\gamma g)-T). \tag{12}$$

It remains to analyze the inner sum. Fix Q', $w\in W(L';M)$, $\gamma\in Q'\backslash Q$, and $g\in G_{\mathbb{A}}$. Let $X = H(\gamma g)-T$ and let $R\in[Q',Q]$ be the parabolic subgroup such that $\hat\Delta^Q_R = \{\varpi\in\hat\Delta^Q_{Q'}:\langle\varpi,X\rangle>0\}$. Note that $\hat\tau^Q_{P'}(H(\gamma g)-T)=1$ if and only if $P'\in[R,Q]$. On the other hand, we can rewrite the conditions $w\in W/W^{M'}$ and $P'_w=Q'$ as $P'\in[Q',S]$ where $S\in[Q',Q]$ is such that

$$\Delta^S_{Q'} = \{\alpha\in\Delta^Q_{Q'}: w\alpha>0 \text{ but } w\alpha\notin\Delta^M_{L'^w}\}.$$

We infer that the inner sum of (12) is nonzero only if $S=R$ and this happens exactly when $\phi^Q_{L',M,w}(X)\neq 0$. In this case, $\phi^Q_{L',M,w}(X) = (-1)^{r(S)-r(Q)}$. The lemma follows. $\square$

The lemma just proved is not so useful as it stands, for in practice, it may be difficult to work analytically with the right-hand side of (11) since the constant terms of φ are not rapidly decreasing in general. We seek a similar expression where $B_{Q'}$ is replaced by a function which is rapidly decreasing on $L'\backslash L'^1_{\mathbb{A}}$. To that end we will use the inversion formula (2) to rewrite the right-hand side of (11) as

$$\sum_{Q'\in[P_0,Q]}\sum_{w\in W(L';M)}\sum_{\gamma\in Q'\backslash Q}\sum_{R\subseteq Q'}\sum_{\delta\in R\backslash Q'}\Lambda^{T,R}B_{Q'}(\delta\gamma g,\varphi,w,\lambda)\\ \tau^{Q'}_R(H(\delta\gamma g)-T)\phi^Q_{L',w,M}(H(\gamma g)-T).$$

Applying Lemma 1 (with R instead of Q) and combining the sums over γ and δ, we get:

Proposition 13. *With χ given by Lemma 7*,

$$\Lambda^{T,Q}E_P(\varphi,\lambda) = \sum_{R\in[P_0,Q]}\ \sum_{w\in W(M_R;M)}\ \sum_{\gamma\in R\backslash Q}\Lambda^{T,R}B_R(\gamma g,\varphi,w,\lambda)\chi^Q_{M_R,M,w}(H(\gamma g)-T).$$

6. Maass–Selberg relations

We will use Proposition 13 to obtain the Maass–Selberg relations in this context. First we need a lemma.

Lemma 14. *Let f be a function of moderate growth on $G\backslash G^1_{\mathbb{A}}$ and let φ be a function of moderate growth on $QV_{\mathbb{A}}\backslash G_{\mathbb{A}}$ which is rapidly decreasing in $L\backslash L^1_{\mathbb{A}}\times K$. Then for any $w\in W(L;M)$ and for* $\operatorname{Re}\lambda\in\mathfrak{a}^*_{M,+}$ *sufficiently regular*, *we have*

$$\Big\langle f,\sum_{\gamma\in Q\backslash G}\varphi_{w^{-1}\lambda}(\gamma g)\chi_{L,M,w}(H(\gamma g)-T)\Big\rangle_{G\backslash G^1_{\mathbb{A}}} = \langle f_Q,\varphi_{w^{-1}\lambda}\chi_{L,M,w}(H(\cdot)-T)\rangle_{Q\backslash G^1_{\mathbb{A}}}.$$

Proof. This is the usual unfolding. In order to justify it, we need to show the convergence of

$$\int_{Q\backslash G^1_{\mathbb{A}}}|f|_Q(g)|\varphi_{w^{-1}\lambda}(g)\chi_{L,M,w}(H(g)-T)|\,dg.$$

We use Iwasawa decomposition to write this as

$$\int_K\int_{\mathfrak{a}^G_L}\int_{L\backslash L^1_{\mathbb{A}}}|f|_Q(e^Xlk)\,|\varphi(e^Xlk)|\,\delta_Q(e^X)^{-1}e^{\operatorname{Re}\langle w^{-1}\lambda,X\rangle}|\chi_{L,M,w}(X-T)|\,dl\,dX\,dk.$$

By the moderate growth of f and φ, there exist c and N such that

$$|f|_Q(e^Xlk)|\varphi(e^Xlk)|\le c(e^{\|X\|}\|l\|)^N,\quad X\in\mathfrak{a}^G_L,\ l\in L_{\mathbb{A}},\ k\in K.$$

The convergence follows from the rapid decay of φ in $L\backslash L^1_{\mathbb{A}}$ and Corollary 8. □

Proposition 15. *We have the identity* (*in the sense of meromorphic continuation*)

$$\langle E_{P_1}(\varphi_1,\lambda_1),\Lambda^TE_{P_2}(\varphi_2,\lambda_2)\rangle_{G\backslash G^1_{\mathbb{A}}} = \sum_Q\ \sum_{w_1\in W(L;M_1)}\ \sum_{w_2\in W(L;M_2)} \langle\Lambda^{T,Q}B_Q(\varphi_1,w_1,\lambda_1),B_Q(\varphi_2,w_2,\lambda_2)\Psi_{L,M_1,w_1,M_2,w_2}(H(\cdot)-T)\rangle_{Q\backslash G^1_{\mathbb{A}}},\tag{13}$$

where each summand converges for $\operatorname{Re}\lambda_1\in\mathfrak{a}^*_{M_1,+}$ *sufficiently regular and* $\operatorname{Re}\lambda_2\in\mathfrak{a}^*_{M_2,+}$ *sufficiently regular* (*depending on* $\operatorname{Re}\lambda_1$) *and as a function of T belongs to*

$$\mathcal{PE}(\mathcal{E}_{\subseteq L^{w_1}}(\varphi_1)+\overline{\mathcal{E}_{\subseteq L^{w_2}}(\varphi_2)}+w_1^{-1}\lambda_1+w_2^{-1}\bar{\lambda}_2).$$

Interestingly, because of the asymmetry of Ψ, the individual terms on the right-hand side are *not* invariant (up to complex conjugation) under interchanging φ_i, w_i, and M_i.

Proof. Set $f_i = E_{P_i}(\varphi_i, \lambda_i)$, $i = 1, 2$. Using Proposition 13 we write $\langle f_1, \Lambda^T f_2\rangle_{G\backslash G^1_{\mathbb{A}}}$ as the sum over Q_2 and $w_2 \in W(L_2; M_2)$ of

$$\Big\langle f_1, \sum_{\gamma \in Q_2\backslash G} \Lambda^{T,Q_2} B_{Q_2}(\gamma g, \varphi_2, w_2, \lambda_2)\chi_{L_2,M_2,w_2}(H(\gamma g) - T)\Big\rangle_{G\backslash G^1_{\mathbb{A}}}$$

provided that each term is defined. By Lemma 14, this is indeed the case for $\operatorname{Re}\lambda_2 \in \mathfrak{a}^*_{M_2,+}$ sufficiently regular and each summand is equal to

$$\langle (f_1)_{Q_2}, \Lambda^{T,Q_2} B_{Q_2}(\varphi_2, w_2, \lambda_2)\chi_{L_2,M_2,w_2}(H(\cdot) - T)\rangle_{Q_2\backslash G^1_{\mathbb{A}}}.$$

This is equal to

$$\langle \Lambda^{T,Q_2} f_1, B_{Q_2}(\varphi_2, w_2, \lambda_2)\chi_{L_2,M_2,w_2}(H(\cdot) - T)\rangle_{Q_2\backslash G^1_{\mathbb{A}}}.$$

Using Proposition 13 once more, we obtain the sum over $Q_1 \in [P_0, Q_2]$ and $w_1 \in W(L_1; M_1)$ of

$$\Big\langle \sum_{\gamma \in Q_1\backslash Q_2} \Lambda^{T,Q_1} B_{Q_1}(\gamma g, \varphi_1, w_1, \lambda_1)\chi^{Q_2}_{L_1,M_1,w_1}(H(\gamma g) - T), \\ B_{Q_2}(\varphi_2, w_2, \lambda_2)\chi_{L_2,M_2,w_2}(H(\cdot) - T)\Big\rangle_{Q_2\backslash G^1_{\mathbb{A}}}.$$

Using the argument of Lemma 14 together with Corollary 9 and applying Lemma 1 we get

$$\sum_{Q_1 \subseteq Q_2} \sum_{w_1 \in W(L_1; M_1)} \sum_{w_2 \in W(L_2; M_2)} \langle \Lambda^{T,Q_1} B_{Q_1}(\varphi_1, w_1, \lambda_1)\chi^{Q_2}_{L_1,M_1,w_1}(H(\cdot) - T), \\ B_{Q_1}(\varphi_2, w_2, \lambda_2)\chi_{L_2,M_2,w_2}(H(\cdot) - T)\rangle_{Q_1\backslash G^1_{\mathbb{A}}}.$$

Upon rewriting, we obtain (13) from the definition of Ψ_{L,M_1,w_1,M_2,w_2}. The last part follows from Corollary 11 and Lemma 2. □

Remark 16. The careful reader would have noticed that the exact description of Ψ_{L,M_1,w_1,M_2,w_2} provided by Proposition 10 was not really used in the argument above. It will be of interest to describe the Laplace transform of Ψ_{L,M_1,w_1,M_2,w_2} explicitly, thereby explicating further the Maass–Selberg relations above. We will not go in this direction in this paper. We mention, however, the following special case: the volume olume of the truncated fundamental domain, namely $\langle 1, \Lambda^T 1\rangle_{G\backslash G^1_A}$, was computed explicitly in [Kim and Weng 2007].

If $\varphi_j \in \mathcal{A}^{\mathrm{cusp}}_{P_j}$, the identity (13) reduces to (5), which is equal to the expression $\mathfrak{M}^T(\varphi_1, \lambda_1, \varphi_2, \lambda_2)$ defined in (6).

In the case where $\varphi_j \in \mathcal{A}^2_{P_j}$ we recover Arthur's asymptotic result.

Proposition 17 [Arthur 1982]. *Suppose $\varphi_j \in \mathcal{A}^2_{P_j}$ and $\lambda_j \in \mathrm{i}\mathfrak{a}^*_{M_j}$, $j = 1, 2$. Then*

$$\langle E_{P_1}(\varphi_1, \lambda_1), \Lambda^T E_{P_2}(\varphi_2, \lambda_2)\rangle_{G\backslash G^1_{\mathbb{A}}} = \mathfrak{M}^T(\varphi_1, \lambda_1, \varphi_2, \lambda_2) + \mathfrak{E}^T(\varphi_1, \lambda_1, \varphi_2, \lambda_2),$$

where

$$\mathfrak{E}^T(\varphi_1, \lambda_1, \varphi_2, \lambda_2) \in \mathcal{PE}_-.$$

Proof. Consider the right-hand side of (13). Each summand belongs to $\mathcal{PE}_-$ unless $w_1 \in W(L, M_1)$ and $w_2 \in W(L, M_2)$. In this case, the summand is equal to

$$\langle (\Lambda^{T,Q}(M(w_1^{-1}, \lambda_1)\varphi_1)_{w_1^{-1}\lambda_1}, (M(w_2^{-1}, \lambda_2)\varphi_2)_{w_2^{-1}\lambda_2}\phi_{Q,M_2,w_2}(H(\cdot) - T)\rangle_{Q\backslash G^1_{\mathbb{A}}}.$$

The proposition follows from Lemma 2 applied with L instead of G, using the Iwasawa decomposition and (7) □

As in [Lapid 2011, §8] one can infer from Proposition 17 the holomorphy of $E(\varphi, \lambda)$ on $\lambda \in \mathrm{i}\mathfrak{a}^*_M$ for any $\varphi \in \mathcal{A}^2_P$. Moreover, for any smooth compactly supported function $\varphi : \mathrm{i}\mathfrak{a}^*_M \to \mathcal{A}^2_P$ with values in a finite-dimensional subspace of $\mathcal{A}^2_P$, define the Eisenstein integral

$$\Theta_{P,\varphi} = \int_{\mathrm{i}\mathfrak{a}^*_M} E(\varphi(\lambda), \lambda)\, d\lambda.$$

Then $\Theta_{P,\varphi} \in L^2(G\backslash G^1_{\mathbb{A}})$ and

$$(14)\quad \langle \Theta_{P,\varphi}, \Theta_{P',\varphi'}\rangle_{G\backslash G^1_{\mathbb{A}}} = \int_{\mathrm{i}\mathfrak{a}^*_M} \sum_{w \in W(M,M')} \langle M(w, \lambda)\varphi(\lambda), \varphi'(w\lambda)\rangle_{A_{M'}U'_{\mathbb{A}}M'\backslash G_{\mathbb{A}}}\, d\lambda.$$

We note that the argument in [Lapid 2011, §8] depends on the second half of [ibid., §7] (which is elementary), but is otherwise self-contained.

We can write (14) more symmetrically as follows. For any parabolic subgroup R, write

$$\varphi^{P,R}_{\#}(\lambda) = \sum_{w \in W(P,R)} M(w, w^{-1}\lambda)\varphi(w^{-1}\lambda), \quad \lambda \in \mathrm{i}\mathfrak{a}^*_R.$$

By the properties of the intertwining operators, we have

$$\varphi^{P,Q}_{\#}(s\lambda, g) = M(s, \lambda)\varphi^{P,R}_{\#}(\lambda, g)$$

for any $s \in W(R, Q)$. Therefore, for any Q and $s \in W(P, Q)$ we can write the right-hand side of (14) as

$$\int_{\mathrm{i}\mathfrak{a}^*_Q} \langle M(s, s^{-1}\mu)\varphi(s^{-1}\mu), \varphi'^{P',Q}_{\#}(\mu)\rangle_{A_L V_{\mathbb{A}} L\backslash G_{\mathbb{A}}}\, d\mu.$$

Averaging over Q and s we get

$$\langle \Theta_{P,\varphi}, \Theta_{P',\varphi'}\rangle_{G\backslash G^1_{\mathbb{A}}} = n(\mathfrak{a}_P)^{-1} \sum_Q \int_{\mathrm{i}\mathfrak{a}^*_Q} \langle \varphi^{P,Q}_{\#}(\mu), \varphi'^{P',Q}_{\#}(\mu)\rangle_{A_L V_{\mathbb{A}} L\backslash G_{\mathbb{A}}}\, d\mu,$$

where $n(\mathfrak{a}_P) = \sum_Q |W(P, Q)|$ is the number of chambers for $\mathfrak{a}_P$ [Arthur 1978, p. 919].

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EREZ LAPID
INSTITUTE OF MATHEMATICS
THE HEBREW UNIVERSITY OF JERUSALEM
91904 JERUSALEM
ISRAEL
erezla@math.huji.ac.il

KEITH OUELLETTE
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA
LOS ANGELES, CA 90095-1555
UNITED STATES
kouellet@math.ucla.edu

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SOME COMMENTS ON WEYL'S COMPLETE REDUCIBILITY THEOREM

JONATHAN ROGAWSKI AND V. S. VARADARAJAN

In memoriam: Jonathan Rogawski

In this note we discuss a purely algebraic proof of Weyl's theorem that all finite-dimensional representations of a complex semisimple Lie algebra are completely reducible. We give a simple and direct proof which is elementary in the sense that it does not use cohomology, and which is a synthesis of the older proofs of Casimir – van der Waerden and of Brauer.

1. Introduction

In the theory of semisimple Lie algebras, one of the central results is the theorem of Weyl that says that all finite-dimensional representations of a complex semisimple Lie algebra $\mathfrak{g}$ are completely reducible, that is, direct sums of the irreducible ones. In this note we present a purely algebraic proof of this result. Our aim is pedagogical, and so we have made an effort to explain in detail facts that are usually taken for granted in expositions, so as to make this accessible to graduate students and advanced undergraduates. The result is valid over any field of characteristic 0; but this generalization can be deduced from the complex case by standard arguments. Hence we restrict ourselves to working over $\mathbb{C}$.

The complete reducibility theorem was first proved by Hermann Weyl [1968] in his great series of papers on the theory of representations of semisimple Lie groups. Although Elie Cartan had already obtained a complete description of the *irreducible* representations, he did not go seriously into the issue of how an arbitrary representation could be built out of irreducible representations.

Weyl's proof remains one of the most beautiful in the entire theory of representations. The basis of his proof is the following: there is a real form $\mathfrak{u}$ of $\mathfrak{g}$, the so-called *compact form*, with the property that the simply connected group U corresponding to $\mathfrak{u}$ is compact; here we recall that to say that $\mathfrak{u}$ is a real form of $\mathfrak{g}$ is to require that $\mathfrak{u}$ is a real Lie subalgebra of $\mathfrak{g}$ whose real dimension is the complex dimension of $\mathfrak{g}$.

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Once the existence of $\mathfrak{u}$ is assumed, the proof is straightforward. The representations of $\mathfrak{g}$ correspond naturally to the representations of $\mathfrak{u}$; in one direction one restricts from $\mathfrak{g}$ to $\mathfrak{u}$ and in the other direction one extends from $\mathfrak{u}$ to $\mathfrak{g}$ by complexification. On the other hand, there is a natural correspondence between the representations of $\mathfrak{u}$ and the continuous representations of U. Now the compactness of U means that any continuous representation of U is unitarizable, that is, is equivalent to a unitary representation, namely a continuous homomorphism of U into the group of unitary operators on a finite-dimensional complex Hilbert space. So, assuming the representation is unitary, its complete reducibility is immediate, because for any U-invariant subspace M, its orthogonal complement $M^{\perp}$ is also U-invariant and is moreover complementary to M; the complete irreducibility is then clear by induction on the dimension of the representation. The unitarizability of any representation of U means that if we start with any representation, we can find a U-invariant scalar product in the representation space. To do this, we start with an arbitrary scalar product in the representation space and then introduce the scalar product which is the group average of its transforms by elements of U; this latter scalar product is invariant under U so that U is unitary with respect to it (see [Varadarajan 1984, Chapter 4, §4.11]; see also [Hawkins 2000, pp. 465–484]).

This proof is essentially transcendental: it uses invariant integration on the compact group and also the topological fact that the universal covering group of the adjoint group of the compact form is still compact. And neither of these is elementary. The reduction to the compact form $\mathfrak{u}$ and the compact group U was named the *unitarian trick* by Weyl.

It must be mentioned that the idea of averaging on special compact groups (such as the orthogonal group) goes back to Hurwitz and Schur. For Hurwitz, the goal was to prove the finite generation of the invariants for actions of $\mathrm{SL}(n, \mathbb{C})$ and $\mathrm{SO}(n, \mathbb{C})$. Although these groups are not compact, they contain the compact groups $\mathrm{SU}(n)$ and $\mathrm{SO}(n, \mathbb{R})$, and invariance with respect to these will imply invariance with respect to the complex groups they are contained in (this is the first instance of the unitarian trick); this is because a holomorphic function on the complex groups which vanishes on the relevant compact subgroup is identically 0. For Schur, the goal was the representation theory of the compact groups $\mathrm{SU}(n)$ and $\mathrm{SO}(n, \mathbb{R})$, and he used integration over the group to determine all the irreducible representations, their characters, and their dimensions. The work of Hurwitz and Schur triggered Weyl's imagination, and after he got the results for all semisimple compact groups, he described them in a letter to Schur (see [Borel 1986] for references to the work of Hurwitz and Schur and to Weyl's letter to Schur).

So in the years after Weyl's proof appeared, the question of a purely algebraic proof became a natural issue. The first such algebraic proof was given by Casimir and van der Waerden [1935].[1] They give a beautiful purely algebraic argument which is elementary in the sense that it relies only on the known structure of irreducible modules as highest-weight modules, and the Casimir operator. A key element of their proof is a delicate calculation in $\mathfrak{sl}(2)$.

After the original proof, additional algebraic proofs appeared in [Brauer 1936] and [Rashevski 1953]. There was also a proof that was cohomological: here it is a question of first establishing that $H^1(\mathfrak{g}, M) = 0$ for any semisimple Lie algebra $\mathfrak{g}$ and any finite-dimensional $\mathfrak{g}$-module M, from which the complete reducibility will follow via standard arguments (see [Varadarajan 1984, Chapter 3, §§3.12–3.13]).

In this note we shall present a proof which is in some sense a synthesis of the earlier proofs of [Casimir and van der Waerden 1935; Brauer 1936]. It is very short and very direct. It uses the Casimir operator in an essential manner (as do all other proofs), the fact that any irreducible module is the one with highest weight equal to some λ, and the fact that it is enough to lift invariant vectors from the quotient of a module to the module itself. All modules considered from now on are finite-dimensional.

For a detailed historical account of Weyl's theorem and the various proofs of it, see [Borel 1998].

2. The Casimir operator and the lifting of invariant vectors

Let $\mathfrak{g}$ be a complex semisimple Lie algebra. If (X_i) is a basis for $\mathfrak{g}$ and (X^i) is the dual basis with respect to the Cartan–Killing form $(\cdot\,,\cdot)$ (that is, $(X_i, X^j) = \delta_{ij}$), then the Casimir operator is

$$\omega = \sum_i X_i X^i;$$

it is independent of the choice of the basis and lies in the center of the enveloping algebra of $\mathfrak{g}$. It commutes with the action of $\mathfrak{g}$ and hence it goes into a scalar in any irreducible module for $\mathfrak{g}$. We write g_λ for the value of this scalar when the module is V_λ, the irreducible module of highest weight λ. Here we have chosen a Cartan subalgebra $\mathfrak{h}$, and a positive system of roots, so that $\lambda \in \mathfrak{h}^*$. The isomorphism between $\mathfrak{h}$ and $\mathfrak{h}^*$ carries the form on $\mathfrak{h}$ to one on $\mathfrak{h}^*$ which is denoted by the same symbol. It is known that $(\cdot\,,\cdot)$ is real and positive definite on the real span of the roots, which includes all the highest weights. If λ is a highest weight, $(\lambda, \alpha) \geq 0$ for all roots $\alpha > 0$.

The first lemma is the calculation of g_λ. As usual, let

$$\rho = \tfrac{1}{2} \sum_{\alpha>0} \alpha.$$

[1] V. S. V. was fortunate to hear an exposition by Casimir himself of this proof in a conference in Utrecht in 1985. See the Appendix.

Lemma 1 [Casimir and van der Waerden 1935]. *The scalar g_λ is given by*

$$g_\lambda = (\lambda+\rho, \lambda+\rho) - (\rho, \rho).$$

Moreover,

$$g_\lambda = 0 \iff \lambda = 0.$$

In other words, an irreducible module where ω is 0 *is necessarily the trivial module.*

Proof. We select an ON basis (K_i) for $\mathfrak{h}$ and root vectors X_α with $(X_\alpha, X_{-\alpha}) = 1$ for all roots α. Then

$$\omega = \sum_i K_i^2 + \sum_{\alpha>0} (X_\alpha X_{-\alpha} + X_{-\alpha} X_\alpha) = \sum_i K_i^2 + \sum_{\alpha>0} H_\alpha + 2 \sum_{\alpha>0} X_{-\alpha} X_\alpha.$$

If m_λ is a highest-weight vector for M_λ, then

$$\omega m_\lambda = g_\lambda m_\lambda.$$

Since the $X_\alpha (\alpha > 0)$ annihilate m_λ and $H m_\lambda = \lambda(H) m_\lambda$ for $H \in \mathfrak{h}$, it follows that

$$\begin{aligned} g_\lambda &= \sum_i \lambda(K_i)^2 + \sum_{\alpha>0} \lambda(H_\alpha) = (\lambda, \lambda) + \sum_{\alpha>0} (\lambda, H_\alpha) \\ &= (\lambda, \lambda) + 2(\lambda, \rho) = (\lambda+\rho, \lambda+\rho) - (\rho, \rho). \end{aligned}$$

If $\lambda = 0$, then $g_\lambda = 0$. Conversely, suppose that $g_\lambda = 0$. Then $(\lambda, \lambda) + 2(\lambda, \rho) = 0$. As both (λ, λ) and (λ, ρ) are ≥ 0, we must have $(\lambda, \lambda) = 0$, so that $\lambda = 0$. □

The second lemma is that invariant vectors can be lifted. A vector v in a $\mathfrak{g}$-module is invariant if $\mathfrak{g} v = 0$.

Lemma 2. *Let*

$$M \to P \to 0$$

be an exact sequence of $\mathfrak{g}$-modules and let $p \in P$ be an invariant vector. Then we can lift p to an invariant vector in M, namely, find an invariant $u \in M$ such that u maps to p.

Proof. Replacing M by the preimage of the line $\mathbb{C}p$, we may assume that $P = \mathbb{C}p$. Now M is the direct sum of the generalized subspaces M_r of the Casimir operator, M_r being the largest subspace where ω has the single eigenvalue r. The M_r are stable under $\mathfrak{g}$, and as $\omega p = 0$, it follows that all the M_r for $r \neq 0$ map to 0 in P. Since M maps onto P, this means that $M_0 \neq 0$ and maps onto P. In other words, we may assume that $M = M_0$. But then, by Lemma 1, M has a Jordan composition series consisting only of trivial modules. Let $H_i, X_i, Y_i (1 \leq i \leq r = \text{rank of } \mathfrak{g})$ be the usual Chevalley generators for $\mathfrak{g}$. Then, for each i, H_i has the sole eigenvalue 0 in M. If $M[H_i : a]$ is the generalized eigensubspace of H_i for the eigenvalue a, it is standard that X_i (resp. Y_i) maps $M[H_i : a]$ into $M[H_i : a+2]$ (resp. $M[H_i : a-2]$).

But as 0 is the only eigenvalue for H_i in M, it follows that X_i and Y_i act as 0 on M. But then $H_i = [X_i, Y_i]$ also acts as 0, so that $\mathfrak{g}$ acts trivially on M. This means that any vector u of M above p satisfies our requirements. $\square$

3. Proof of Weyl's theorem

Weyl's theorem can now be obtained by a standard general argument.

Weyl's theorem. *All $\mathfrak{g}$-modules are completely reducible.*

Proof. Suppose M is a $\mathfrak{g}$-module and N a proper submodule. It is a question of finding a submodule Q such that $M = N \oplus Q$. If $P = M/N$, we have an exact sequence of $\mathfrak{g}$-modules

$$0 \to N \to M \to P \to 0.$$

So we get the exact sequence $\mathfrak{g}$-modules

$$0 \to \mathrm{Hom}(P, N) \to \mathrm{Hom}(P, M) \to \mathrm{Hom}(P, P) \to 0.$$

Now the identity I in $\mathrm{Hom}(P, P)$ is an invariant element, and so, by Lemma 2, can be lifted to an invariant element t in $\mathrm{Hom}(P, M)$. Since t maps to I, we see that t splits the map $M \to P$. Since t is invariant, the range Q of t is a submodule of M complementary to N. $\square$

Remark. The reduction to trivial modules goes back to [Brauer 1936]. It was resurrected by Chevalley [1955] in his proof of Weyl's theorem. Later on, when Mumford [1965] needed a characteristic p version of Lemma 2, he formulated it as follows: if G is a connected semisimple group over an algebraically closed field K of characteristic $p > 0$, and V, W are G-modules with $V \to W \to 0$ exact, and if $w \in W$ is an invariant vector, there are an integer $d > 0$ and an invariant vector $v \in S_d(V)$ which maps to w^d (here S_d refers to the component of degree d in the symmetric algebra). It is known that the smallest value of d is a power p^m of p.

Appendix: A historical note on Casimir and his operator

Hendrik Brugt Gerhard Casimir (1909–2000) was a physicist whose Leiden thesis [1933] on the theory of diatomic molecules introduced the operator now known as the Casimir operator, as an element of the enveloping algebra of the rotation Lie algebra $\mathfrak{o}$. It commutes with all the elements of $\mathfrak{o}$ and so is a scalar in any irreducible representation of $\mathfrak{o}$. Shortly afterwards, he discovered the analogue of this for any semisimple Lie algebra [1931]. The corresponding operator in any representation of the Lie algebra eventually became known as the *Casimir operator*. It plays a central role in harmonic analysis and representation theory, even though for groups of higher dimension its value on an irreducible representation no longer determines the representation, unlike what happens for $\mathfrak{sl}(2)$. In [Casimir 1931], he himself pioneered the idea that the Casimir operator should be viewed as a second-order differential operator on the group manifold, and that the matrix elements of irreducible representations of the group are eigenfunctions for this operator. The connection between representation theory and differential equations on the group manifold introduced by Casimir found its full force and scope only with the work of Harish-Chandra.

Casimir was one of the great Dutch physicists of the twentieth century who made significant contributions to both experimental and theoretical physics as well as to pure mathematics. In addition, he had a big influence on industry as the head of the research division of Philips. His mathematical contributions include the discovery and use of the Casimir operator as described above. In experimental physics, he predicted what is now known as the *Casimir effect*, which is a quantum mechanical attraction between conducting plates. His theoretical contributions are quite well known, such as his work on Lars Onsager's microscopic reversibility. One of us (V. S. V.) was present at a conference on semisimple Lie groups in Utrecht in 1985 when Casimir gave a talk on the history of the proof of Weyl's complete reducibility theorem and presented a brief sketch of the algebraic proof given in [Casimir and van der Waerden 1935]. His autobiography [Casimir 1983] is a wonderful document of great interest and humanity.

From the modern perspective, the Casimir operator is an element in the center of the universal enveloping algebra of a semisimple Lie algebra. For simple Lie algebras of dimension greater than 3, the Casimir element does not generate the center of the enveloping algebra. Some of the additional elements of the center were written down by Giulio Racah (1909–1965), an Israeli physicist and mathematician. Racah [1965] determined in some implicit manner the full center of the enveloping algebra of an arbitrary semisimple Lie algebra. The generators of the center discovered by Racah are known to physicists as *generalized Casimir operators*. For many in the 1950's and early 1960's (including V. S. V.), the Racah notes [1965] were almost the only sources of information on the structure and representations of semisimple Lie algebras till the appearance of [Blanchard et al. 1955] and [Jacobson 1962]. From the mathematical side, the center of the enveloping algebra was first investigated by Harish-Chandra (1923–1983). Harish-Chandra constructed what is now known as the *Harish-Chandra isomorphism* of the center of the universal enveloping algebra of a general semisimple Lie algebra $\mathfrak{g}$ with the algebra of Weyl group invariants of the algebra of polynomials on a Cartan subalgebra of $\mathfrak{g}$. The center is very closely related to the algebra of polynomial invariants on the Lie algebra, and this was determined by Claude Chevalley (1909–1984), who proved that it is isomorphic, via restriction to a Cartan subalgebra, to the Weyl group invariants of the algebra of polynomials on the Cartan subalgebra. The Harish-Chandra isomorphism, and its p-adic twin, the *Satake isomorphism*, play a fundamental role in harmonic analysis on semisimple groups.

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This paper had its origins in discussions between Jonathan Rogawski and myself. At some point in 2008, although we had a draft in existence, we decided to put it on hold, because of his medical commitments. After Jonathan's untimely passing away in 2011, Don Blasius suggested to me that perhaps this manuscript should see the light of day as an exposition, pedagogical in scope, of a beautiful aspect of semisimple Lie algebras. It is essentially the same as the earlier draft. I thank Don not only for his suggestion to write up these remarks, but also for reading the manuscript and suggesting a number of improvements.

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JONATHAN ROGAWSKI
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA, LOS ANGELES
LOS ANGELES, CA 90095
UNITED STATES

jonr@math.ucla.edu

V. S. VARADARAJAN
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA, LOS ANGELES
LOS ANGELES, CA 90095
UNITED STATES

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ON EQUALITY OF ARITHMETIC AND ANALYTIC FACTORS THROUGH LOCAL LANGLANDS CORRESPONDENCE

FREYDOON SHAHIDI

To the memory of Jonathan Rogawski

In this article we pursue the problem of equality of Artin factors with those defined on the representation theoretic (analytic) side by the local Langlands correspondence. We propose a set of axioms for the factors on the analytic side which allows us to prove the equality of the factors. In the case of L-functions the equality can be proved in a number of cases appearing in the Langlands–Shahidi method since one of the axioms, stability under highly ramified twists, is already available for the L-functions coming from this method.

Introduction

The local Langlands correspondence (LLC) for $\mathrm{GL}(n)$ is formulated through the equality of the Artin factors attached to tensor products on the Galois side with the factors defined on the representation theoretic side, namely those of Rankin–Selberg product L-functions for $\mathrm{GL}(n) \times \mathrm{GL}(m)$ [Jacquet et al. 1983; Shahidi 1984]. The LLC, which was proved for $\mathrm{GL}(n)$ in [Harris and Taylor 2001; Henniart 2000], also suggests that other Artin (or arithmetic) factors should be equal to their representation theoretic (or analytic) counterparts, if they exist. In fact, one important fact about analytic objects is that they always correspond to a global theory and thus are of automorphic significance. On the other hand so long as the problem of global parametrization or the global Langlands correspondence, a problem whose formulation is still unavailable [Langlands 2012], is not settled, one cannot expect to produce a global theory of L-functions from those defined by local Artin factors. The problem is thus to show the equality of Artin factors with the corresponding analytic ones whenever LLC is available.

The purpose of this article is to formulate a set of axioms to be satisfied by the objects on the analytic side attached to every representation r of the L-group so

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as to imply the equality of arithmetic (Artin) factors with analytic (representation theoretic or automorphic) factors through LLC (Theorem 2.1). This formalizes and generalizes some ideas of Harris [1998] as pursued later by Henniart [2010].

While the equality of γ-functions requires the validity of stability (Axiom 2), our Theorem 3.1 proves the equality of L-functions in certain special cases coming from Langlands–Shahidi method [Shahidi 1990; 2010] through LLC with no assumptions. They include the cases of twisted exterior square L-functions for $\mathrm{GL}(n)$ as well as twisted exterior cube for $\mathrm{GL}(6)$. This equality can be used to prove special cases of the generic Arthur packet conjecture [Arthur 1984; Shahidi 2011] as we explain in Section 3. Finally in Section 4 we address the issue of stability of γ-factors within our method and discuss the progress made on it and some of its consequences.

1. Axiomatic *r*-theory

Let G be a connected reductive algebraic group over a local field F of characteristic zero. Denote by ${}^LG$ its L-group. Let W'_F be the Weil–Deligne group of F. Let $\rho : W'_F \to {}^LG$ be an admissible homomorphism (see [Arthur 1984; Shahidi 2011]). Let r be an irreducible complex representation of ${}^LG$ on a finite dimensional complex vector space V, i.e., $r : {}^LG \to \mathrm{GL}(V)$ is an analytic homomorphism. Then $r \cdot \rho : W'_F \to \mathrm{GL}(V)$ defines a representation of W'_F, which we assume to be Frobenius-semisimple.

Let us now assume we have a theory of L-functions attached to r. More precisely, assume that for each irreducible admissible representation π of $G(F)$, there are defined an L-function $L(s, \pi, r)$ and an ε-factor $\varepsilon(s, \pi, r, \psi_F)$, where $s \in \mathbb{C}$ and ψ_F is a nontrivial additive character of F, satisfying (1) multiplicativity (additivity), (2) stability under highly ramified character twists, (3) a global functional equation whenever π becomes a local component of a global cusp form, and (4) archimedean matching, each of which we shall now explain. It is best to formulate them in terms of γ-functions

$$\gamma(s, \pi, r, \psi_F) = \varepsilon(s, \pi, r, \psi_F) L(1-s, \pi, \tilde{r}) / L(s, \pi, r).$$

1) ***Multiplicativity.*** This basically expresses γ-functions of a particular constituent of an induced representation as a product of γ-functions for the inducing data. One special and important case of it is that of Langlands quotients [Langlands 1989; Silberger 1978]. If π is an irreducible admissible representation of $G(F)$, then Langlands classification determines a standard parabolic subgroup P with a Levi decomposition $P = MN$ and a quasitempered representation σ of $M(F)$, in the "positive Weyl chamber", such that $\pi = J(P, \sigma)$. Here $J(P, \sigma)$ is the unique irreducible quotient of $I(P, \sigma)$, which is the representation of $G(k)$ induced by σ. Note that fixing the minimal parabolic subgroup $P_0 \subset P$, making P standard,

automatically determines the unique positive Weyl chamber. Now, let

$$\iota : {}^LM \hookrightarrow {}^LG$$

be the natural embedding. Let $\rho_M : W'_F \to {}^LM$ be the parameter defining σ (or its L-packet), if known. Then $\rho = \iota \cdot \rho_M$ will be the parameter for π. Let r be a finite dimensional irreducible complex representation of ${}^LG$ as before. Decompose

$$r \cdot \iota = \bigoplus_j r_j^M \tag{1-1}$$

into its irreducible constituents. Multiplicativity in this case simply requires

$$\gamma(s, \pi, r, \psi_F) = \prod_j \gamma(s, \sigma, r_j^M, \psi_F), \tag{1-2}$$

$$L(s, \pi, r) = \prod_j L(s, \sigma, r_j^M), \tag{1-3}$$

$$\varepsilon(s, \pi, r, \psi_F) = \prod_j \varepsilon(s, \sigma, r_j^M, \psi_F). \tag{1-4}$$

In fact, this is how these factors are defined: One first defines the factors for quasitempered but unitary data and then extends the unitary complex parameters to all of the complex dual of the complex Lie algebra of the split component of the center of M [Langlands 1989; Shahidi 1990]. When F is an archimedean field, LLC was established by Langlands [1989] and the L-functions were defined to be those of Artin attached to the parameter. They satisfy Equations (1-2)–(1-4).

When one restricts oneself to those representations r that appear in constant terms of Eisenstein series (Langlands–Shahidi method [Langlands 1971a; 1976; Shahidi 2010]), in which case G will be assumed to be quasisplit, then these formulas play a central role. In fact, what is defined with no reservations is the γ-function $\gamma(s, \pi, r'_i, \psi_F)$, where r'_i is any irreducible constituent of the adjoint action of ${}^LM'$ on ${}^L\mathfrak{n}'$, the Lie algebra of the complex Lie group ${}^LN'$ [Langlands 1971a; Shahidi 1990; 2010]. The representation π is any irreducible admissible ψ_F-generic representation of $M'(F)$, where $P' = M'N'$ is the defining parabolic subgroup for the Eisenstein series which we may assume to be maximal. Here F is a completion of the number field defining the Eisenstein series. As explained in [Shahidi 1990; 2010], the knowledge of γ-factors immediately defines the L-functions and ε-factors if π is also tempered. The extension to any irreducible admissible representation (not necessarily generic) π of $M'(F)$ is given by Langlands classification and Equations (1-3) and (1-4) [Shahidi 1990]. In this case multiplicativity is valid when π is the unique ψ_F-generic constituent of $\mathrm{Ind}_{M(F)N(F)}^{M'(F)} \sigma \otimes \mathbf{1}$, where P is any standard parabolic subgroup of M' defined over F and σ is any irreducible admissible ψ_F-generic representation of $M(F)$, $P = MN$. One then has the appropriate version of (1-2) for each γ-function $\gamma(s, \pi, r_i', \psi_F)$, where r_i' is an irreducible constituent of the adjoint action of ${}^LM'$ on ${}^L\mathfrak{n}'$ [Shahidi 1990; 2010].

Example. Assume $G = \mathrm{GL}(n_1 + n_2)$ and $M = \mathrm{GL}(n_1) \times \mathrm{GL}(n_2)$. Let r_N be Λ^2, the exterior square representation of $\mathrm{GL}(N, \mathbb{C})$ for any positive integer N. Then one has

$$ {}^LM = \mathrm{GL}(n_1, \mathbb{C}) \times \mathrm{GL}(n_2, \mathbb{C}) $$

and

$$ r_{n_1+n_2}|{}^LM = r_{n_1} \oplus r_{n_2} \oplus (\rho_{n_1} \otimes \rho_{n_2}), \tag{1-5} $$

where ρ_N is the standard representation of $\mathrm{GL}(N, \mathbb{C})$. If π is the Langlands quotient or the unique irreducible generic constituent of $\mathrm{Ind}_{M(F)N(F)}^{G(F)} \sigma_1 \otimes \sigma_2 \otimes \mathbf{1}$, where σ_i, $i = 1, 2$, is an irreducible generic representation of $\mathrm{GL}_{n_i}(F)$, which we will assume to be quasitempered in the positive Weyl chamber if π is the Langlands quotient, then

$$ \gamma(s, \pi, \Lambda^2, \psi_F) = \gamma(s, \sigma_1, \Lambda^2, \psi_F)\gamma(s, \sigma_2, \Lambda^2, \psi_F)\gamma(s, \sigma_1 \times \sigma_2, \psi_F). $$

Here $\gamma(s, \sigma_1 \times \sigma_2, \psi_F)$ is the Rankin–Selberg product γ-function defined in [Jacquet et al. 1983]. It is also obtained from the Langlands–Shahidi method if we consider $M' = \mathrm{GL}(n_1) \times \mathrm{GL}(n_2)$ inside $G = \mathrm{GL}(n_1 + n_2)$; see [Shahidi 1984].

One simple way of seeing the branching rule (1-5) is to consider $M' = \mathrm{GL}_{n_1+n_2}$ as the Siegel Levi subgroup of $G = \mathrm{SO}(2n_1 + 2n_2)$. Here one gets only one irreducible representation r_1' of ${}^LM' = \mathrm{GL}(n_1 + n_2, \mathbb{C})$ in ${}^L\mathfrak{n}'$, $r_1' = \Lambda^2_{n_1+n_2}$. One can then immediately see the restriction decomposition (branching rule) (1-5) if one considers the adjoint action of ${}^LM = \mathrm{GL}_{n_1}(\mathbb{C}) \times \mathrm{GL}_{n_2}(\mathbb{C})$ on ${}^L\mathfrak{n}'$ which is isomorphic to (the second diagonal) skew-symmetric elements of complex matrices of size $n_1 + n_2$.

Finally we remark that if one knows LLC and lets ρ be the parameter of π, and further assume the equality

$$ \gamma(s, \pi, r, \psi_F) = \gamma(s, r \cdot \rho, \psi_F), \tag{1-6} $$

where the factor on the right is that of Artin attached to the representation $r \cdot \rho$, then one immediately has

$$ \begin{aligned} \gamma(s, \pi, r, \psi_F) &= \gamma(s, r \cdot \rho, \psi_F) \\ &= \gamma(s, r \cdot \iota \cdot \rho_M, \psi_F) \\ &= \gamma(s, \oplus r_j^M \cdot \rho_M, \psi_F) \\ &= \prod_j \gamma(s, r_j^M \cdot \rho_M, \psi_F) = \prod_j \gamma(s, \sigma, r_j^M, \psi_F), \end{aligned} $$

where σ is a member of the L-packet attached to ρ_M. This immediately implies (1-2). The point is that even if one knows LLC, one would know the equality (1-6) only for certain r [Langlands 1971a; Shahidi 1990; 2010] and not necessarily for the family of L-functions attached to a given r. In practice one would need to know multiplicativity for γ-functions $\gamma(s, \pi, r, \psi_F)$ on the representation theoretic side in order to prove (1-6) for a given r.

2) ***Stability.*** This is again a local statement. Moreover, F will need to be assumed to be nonarchimedean. We also need to assume $X(G)_F \neq \{\mathbf{1}\}$, i.e., that G has a nontrivial F-rational character. This clearly rules out G being semisimple. Choose and fix $\mathbf{1} \neq \nu \in X(G)_F$. Note that $\nu(G(F)) \subset F^*$ is of finite index and thus open. Let χ be a highly ramified character of F^*. Then $\chi \cdot \nu$ is what we call a *highly ramified character* of $G(F)$.

Let π_1 and π_2 be two irreducible admissible representations of $G(F)$. Let ω_{π_i} denote the central character of π_i, $i = 1, 2$. Stability requires:

Assume $\omega_{\pi_1} = \omega_{\pi_2} = \omega$. Then for every sufficiently highly ramified character χ of $G(F)$ with the level of ramification depending on π_1 and π_2, one has

$$\gamma(s, \pi_1 \otimes \chi, r, \psi_F) = \gamma(s, \pi_2 \otimes \chi, r, \psi_F), \tag{1-7}$$

$$L(s, \pi_1 \otimes \chi, r) = L(s, \pi_2 \otimes \chi, r) \equiv 1, \tag{1-8}$$

and thus

$$\varepsilon(s, \pi_1 \otimes \chi, r, \psi_F) = \varepsilon(s, \pi_2 \otimes \chi, r, \psi_F). \tag{1-9}$$

By virtue of [Deligne 1973], stability is valid for all the Artin factors, and as in multiplicativity, stability will also be true for our factors (see [Cogdell, Shahidi and Tsai $\geq$ 2012]) if LLC is valid and moreover our factors are equal to those of Artin. But again stability is a tool which is needed to prove this equality which is known in only a few cases.

At present this is the only result that needs to be established even in the context of L-functions that come from the Langlands–Shahidi method [Shahidi 2002; 2010], although special cases of it are available from either methods of L-functions. More precisely, stability is known for the Rankin product factors $\gamma(s, \pi_1 \times \pi_2, \psi_F)$, where π_1 and π_2 are irreducible admissible representations of $\mathrm{GL}(n_1, F)$ and $\mathrm{GL}(n_2, F)$, respectively [Jacquet and Shalika 1985], or of $\mathrm{GL}_1(F) = F^*$ and $G(F)$, whenever G is a group for which the derived group of ${}^L G^0$ is a classical group [Cogdell et al. 2001; 2004; 2005; 2008; Kim and Krishnamurthy 2005; Asgari and Shahidi 2006; 2011].

On the other hand, in the context of L-functions in [Langlands 1989; Shahidi 1990; 2010], a stability statement for L-functions to the effect that

$$L(s, \pi \otimes \chi, r_i) \equiv 1$$

for every L-function obtained from our method, and suitably highly ramified characters χ, was proved in [Shahidi 2000]. Thus it is the stability of γ-functions $\gamma(s, \pi, r_i, \psi_F)$ which needs to be proved in a given case. We will discuss this problem shortly.

We conclude by pointing out that in the case of $G \times \mathrm{GL}(1)$ discussed above stability has been an important tool to prove functorial transfers from the generic spectrum of $G(\mathbb{A}_k)$ to appropriate $\mathrm{GL}(N, \mathbb{A}_k)$ [Cogdell et al. 2001; 2004; Kim and Krishnamurthy 2005; Asgari and Shahidi 2006; 2011]. Here $\mathbb{A}_k$ is the ring of adeles of a number field k.

3) ***Functional equations.*** The main reason for introducing local Artin root numbers (ε-factors) in [Dwork 1956; Langlands 1970; 1971b; Deligne 1973] was to decompose Artin's global root numbers and ε-factors into products of local objects. Under the validity of LLC, these local Artin factors can be used to define local factors attached to irreducible admissible representations (L-packets) of groups over local fields. On the other hand if one considers cuspidal automorphic forms over a global number field, then for each r one expects global functional equations whose root numbers will have to be a product of local ones. One thus needs to define a collection of local ε-factors and L-functions within the same machinery that establishes the global functional equations [Jacquet et al. 1983; Cogdell and Piatetski-Shapiro 2004; Shahidi 1990; 2010]. It is thus by no means clear that these factors are equal to those defined by Artin factors through LLC, and the challenge is to show that they are in fact equal. This is done by using these global functional equations, but for a very special class of cusp forms, those attached to certain irreducible continuous representations of global Galois (or Weil) group. We now formulate this as follows.

Let k be a global field whose ring of adeles is $\mathbb{A}_k$ and let $\pi = \bigotimes_v \pi_v$ be an automorphic cuspidal representation of $G(\mathbb{A}_k)$, where G is a connected reductive group over k. Let r be an irreducible complex analytic representation (thus finite dimensional and conversely) of ${}^LG$. Let $\eta_v : {}^LG_v \to {}^LG$ be the natural map, where ${}^LG_v$ is the L-group of G as a group over k_v. Write $r_v = r \cdot \eta_v$. Let S be a finite set of places of k such that for all $v \notin S$ both the group G, as a group over k_v, and π_v are unramified. Fix a complex number s. Let $L(s, \pi_v, r_v)$ and $\varepsilon(s, \pi_v, r_v, \psi_v)$ be the local L-function and root number attached to this data from our theory, where $\psi = \bigotimes_v \psi_v$ is a nontrivial additive character of $\mathbb{A}/k$ with ψ_v unramified outside S. Set

$$L(s, \pi, r) = \prod_v L(s, \pi_v, r_v), \tag{1-10}$$

and

$$\varepsilon(s,\pi,r)=\prod_v \varepsilon(s,\pi_v,r_v,\psi_v), \tag{1-11}$$

where (1-10) *converges absolutely for* $\operatorname{Re} s \gg 0$ *while* (1-11) *is just a finite product. Then*

$$L(s,\pi,r)=\varepsilon(s,\pi,r)L(1-s,\pi,\tilde{r}). \tag{1-12}$$

Here $\tilde{r}$ *denotes the contragredient of* r. *In terms of* γ*-functions this can be written as*

$$L^S(s,\pi,r)=\prod_{v\in S}\gamma(s,\pi_v,r_v,\psi_v)L^S(1-s,\pi,\tilde{r}), \tag{1-13}$$

where

$$L^S(s,\pi,r)=\prod_{v\notin S}L(s,\pi_v,r_v). \tag{1-14}$$

Here by an unramified group we mean a quasisplit group to split over an unramified extension. It will then have a hyperspecial maximal compact subgroup with respect to which π_v *has an invariant* (*one dimensional*) *subspace if* π_v *is unramified.*

There are a good number of cases where these functional equations are proved. The most general results here are those in the Langlands–Shahidi method, using Eisenstein series [Langlands 1989; Shahidi 1990; 2010]. On the other hand, they are also proved using the method of integral representations in a number of cases, most notably and completely by Jacquet, Piatetski–Shapiro and Shalika for Rankin product L-functions for $\mathrm{GL}(n_1)\times\mathrm{GL}(n_2)$ as discussed earlier [Jacquet et al. 1983; Cogdell and Piatetski-Shapiro 2004]. We refer to [Soudry 2006] for a survey of the results obtained from the integral representations method for other groups.

4) ***Archimedean matching.*** When k is a number field one has the benefit of using the Langlands classification [Langlands 1989; Silberger 1978] and thus LLC for real groups to define local factors at archimedean primes to be those of Artin through LLC. The theory must then require:

Let F *be either* $\mathbb{R}$ *or* $\mathbb{C}$ *and, for each irreducible admissible representation* π *of* $G(F)$, *let* $\rho: W_F\to {}^LG$ *be the corresponding parameter. Then for each finite dimensional irreducible complex representation* r *of* ${}^LG$ *we have*

$$\gamma(s,r\cdot\rho,\psi_F)=\gamma(s,\pi,r,\psi_F).$$

We also have similar identities for root numbers and L*-functions.*

Again, the most general case of this is proved within the context of the Langlands–Shahidi method [Shahidi 1990; 2010]. The work is carried on in [Shahidi 1985] when "local coefficients" are expressed as Artin factors. We recall that the γ-factors within this method are defined inductively by these local coefficients.

We refer to [Jacquet and Shalika 1990; Cogdell and Piatetski-Shapiro 2004] for the archimedean work within Rankin–Selberg theory for $\mathrm{GL}(n)$.

In the case of function fields, where no distinguished archimedean place stands out, other techniques are needed to develop the theory. We refer to L. Lomelí's work in [Lomelí 2009; Henniart and Lomelí 2011], where the method is developed at least for classical groups.

Definition 1.1. Let F be a local field together with a nontrivial additive character ψ and let G be a connected reductive group over F. Fix a (finite dimensional) complex analytic representation r of ${}^LG$. We will say we have a *theory of L-functions attached to* r, or in short an r*-theory*, if there exist complex functions $L(s, \pi, r)$ and $\varepsilon(s, \pi, r, \psi_F)$ satisfying axioms 1–4.

2. Equality of Artin (arithmetic) and automorphic (analytic) factors

With notation as in the previous section, let

$$\theta : {}^LG \hookrightarrow \mathrm{GL}(N, \mathbb{C}) \times W'_F$$

be a minimal embedding. Let r be a finite dimensional complex representation

$$r : \mathrm{GL}(N, \mathbb{C}) \times W'_F \to \mathrm{Aut}\, V.$$

Let $\rho : W'_F \to {}^LG$ be an admissible homomorphism and let $\pi(\rho)$ be a fixed element in the L-packet attached to ρ. Then

$$\begin{aligned} \gamma(s, r \cdot \theta \cdot \rho, \psi) &= \gamma(s, \pi(\rho), r \cdot \theta, \psi) \\ &= \gamma(s, \pi(\theta \cdot \rho), r, \psi) \end{aligned}$$

if the middle factor $\gamma(s, \pi(\rho), r \cdot \theta, \psi)$ is defined. Here $\pi(\theta \cdot \rho)$ is the representation of $\mathrm{GL}(N, F)$ attached to $\theta \cdot \rho$ as in [Harris and Taylor 2001; Henniart 2000]. In particular, r-factors for $\mathrm{GL}(N, F)$ define $r \cdot \theta$-factors for $G(F)$. We may therefore, at least for $r \cdot \theta$-factors of the group G, appeal to r-factors of $\mathrm{GL}(N)$.

Let us therefore concentrate on $\mathrm{GL}(N)$, where LLC is already established [Harris and Taylor 2001; Henniart 2000]. Assume our theory of γ-factor axioms (1)–(4) of the previous section. We thus consider a parameter $\rho : W'_F \to \mathrm{GL}(N, \mathbb{C})$ and let $\pi(\rho)$ be the corresponding irreducible admissible representation of $\mathrm{GL}_N(F)$ through LLC.

If ρ_1 and ρ_2 are two homomorphisms (representations) of W'_F,

$$\rho_i : W'_F \to \mathrm{GL}(n_i, \mathbb{C}),$$

we let r be a representation of $\mathrm{GL}(n_1+n_2, \mathbb{C})$ and assume a branching rule of the form

$$r\cdot(\rho_1\oplus\rho_2) = r\cdot\rho_1\oplus r\cdot\rho_2\oplus R(\rho_1, \rho_2), \tag{2-1}$$

where $R(\rho_1, \rho_2)$ is a representation of $\mathrm{GL}(n_1, \mathbb{C})\times\mathrm{GL}(n_2, \mathbb{C})$, $n_i = \dim\rho_i$, $i = 1, 2$, in which $r\cdot\rho_1$ and $r\cdot\rho_2$ do not appear; or said in other terms, they appear in $r\cdot(\rho_1\oplus\rho_2)$ with multiplicity one. We can in fact write $R(\rho_1, \rho_2)$ as the composite of

$$R : \mathrm{GL}(n_1, \mathbb{C})\times\mathrm{GL}(n_2, \mathbb{C}) \to \mathrm{GL}(N, \mathbb{C}),$$

$N = \dim R$, and

$$\begin{aligned}(\rho_1, \rho_2) : W'_F &\to \mathrm{GL}(n_1, \mathbb{C})\times\mathrm{GL}(n_2, \mathbb{C})\\ w &\mapsto (\rho_1(w), \rho_2(w)).\end{aligned}$$

We note that

$$\rho_1\oplus\rho_2 : W'_F \to \mathrm{GL}(n_1, \mathbb{C})\times\mathrm{GL}(n_2, \mathbb{C}) \hookrightarrow \mathrm{GL}(n_1+n_2, \mathbb{C}),$$

to which r can be applied. Here are some examples. Let $r = \Lambda^2$, in which case $R(\rho_1, \rho_2) = \rho_1\otimes\rho_2$, or $r = \Lambda^3$, for which $R(\rho_1, \rho_2) = \Lambda^2\rho_1\otimes\rho_2\oplus\rho_1\otimes\Lambda^2\rho_2$. Similar examples can be given for Sym^3 or higher powers of both Λ and Sym [Fulton and Harris 1991]. We recall that exterior powers are irreducible representations of highest weight δ_i, fundamental weights of $\mathrm{SL}(N, \mathbb{C})$. We will then assume that we also have

$$\gamma(s, R\cdot(\rho_1, \rho_2), \psi_F) = \gamma(s, (\pi(\rho_1), \pi(\rho_2)), R, \psi_F), \tag{2-2}$$

which of course requires the validity of an R-theory for $\mathrm{GL}(n_1)\times\mathrm{GL}(n_2)$.

Tracing through the tables in [Langlands 1971a; Shahidi 1988; 2010], it can be seen that the existence of corresponding R-theories for Λ^3 may be available within the same machinery, at least for $n \leq 6$ as we explain in the next section.

Now, fix a representation r with an r-theory and assume one has an R-theory for the representation R appearing in (2-1). We will briefly sketch how to show:

Theorem 2.1. *Fix r satisfying branching rule* (2-1). *Assume the existence of an r-theory and the corresponding R-theory for R satisfying* (2-2). *Then*

$$\gamma(s, r\cdot\rho, \psi_F) = \gamma(s, \pi(\rho), r, \psi_F)$$

for every n-dimensional continuous complex Frobenius-semisimple representation ρ of W'_F, where $\pi(\rho)$ is the irreducible admissible representation of $\mathrm{GL}_n(F)$ *attached to ρ by LLC.*

Proof. We pursue the ideas presented in [Harris 1998; Henniart 2010]. By Brauer's theorem ρ is a $\mathbb{Z}$-linear combination of monomial representations. Thus monomial representations, i.e., those induced from characters of subgroups of finite index in W'_F, form a basis for the Grothendieck ring of W'_F. Starting with a local monomial representation ρ, one chooses a global monomial representation $\tilde{\rho}$ which has ρ as $\tilde{\rho}|W'_F$, where $F = k_v$ at one place of the global field k as in [Harris 1998; Henniart 2010; Cogdell, Shahidi and Tsai $\geq$ 2012]. For each place w of k, let $\tilde{\rho}_w = \tilde{\rho}|W'_{k_w}$, and consider $\pi(\tilde{\rho}) := \bigotimes_w \pi(\tilde{\rho}_w)$, where $\pi(\tilde{\rho}_w)$ is the representation of $\mathrm{GL}(n, k_w)$ attached to $\tilde{\rho}_w$ by LLC. (We remind the reader that there are serious restrictions present in the choices of k and ρ as explained in [Harris 1998; Henniart 2010].)

Then $\pi(\tilde{\rho})$ is an automorphic representation of $\mathrm{GL}_n(\mathbb{A}_k)$, given by an automorphic induction from a grössencharacter. We then twist $\pi(\tilde{\rho})$ by a grössencharacter $\tilde{\chi} = \bigotimes_w \tilde{\chi}_w$ that is highly ramified at all finite places where $\pi(\tilde{\rho}_w)$ is ramified except at v. By stability we get

$$\gamma(s, r_w \cdot (\tilde{\rho}_w \otimes \tilde{\chi}_w), \tilde{\psi}_w) = \gamma(s, \pi(\tilde{\rho}_w) \otimes \tilde{\chi}_w, r_w, \tilde{\psi}_w),$$

which can be seen by computing each side, using a principal series with the same central character as $\pi(\tilde{\rho}_w)$ on the representation theoretic side and [Deligne 1973] on the Artin side.

We will assume $\tilde{\chi}_v \equiv 1$. By archimedean matching the factors are equal whenever $w = \infty$. Comparing functional equations for $\tilde{\rho}$ and $\pi(\tilde{\rho})$, we get

$$\gamma(s, r \cdot \rho, \psi_F) = \gamma(s, \pi(\rho), r, \psi_F)$$

for every member of a basis for the Grothendieck ring of W'_F. Here $\psi_F = \psi_v$ for a global nontrivial character $\psi = \bigotimes_w \psi_w$ of $k\backslash\mathbb{A}_k$.

Next we appeal to our R-theory satisfying (2-2), and multiplicativity, to extend the equality to the full Grothendieck ring. This completes our sketch of the proof. □

3. Equality of L-functions through LLC

While the equality of γ-factors in Theorem 2.1 requires availability of stability for them, stability for L-functions, expressed as Equation (1-8), is a lot less subtle. In what follows, we will show the equality of L-functions defined by the Langlands–Shahidi method with those of Artin in a number of cases previously not available.

A result like this has an interesting application in proving the generic A-packet conjecture discussed in [Shahidi 2011]. This is a kind of converse to the tempered L-packet conjecture, which asserts that every tempered L-packet of a quasisplit group has a generic member [Shahidi 1990; Vogan 1978]. On the other hand, the generic A-packet conjecture states that if the L-packet attached to ϕ_ψ, the Langlands parameter attached to an Arthur parameter ψ, has a generic member, then ϕ_ψ is tempered. We note that the elements of ϕ_ψ are supposed to provide the main nontempered members of ψ (see [Arthur 1984]), i.e., those which have not already appeared in other A-packets. The proof given in [Shahidi 2011] is based on the matching of only L-functions for certain Levi factors through LLC.

The work of Y. Kim [2012], where he uses the matching for the twisted exterior and symmetric square L-functions for $\mathrm{GL}(n)$ [Henniart 2010] and those of certain Rankin product ones [Asgari and Shahidi 2006; 2011], has now established this for split GSpin groups, generalizing the work of Ban [2006] and Liu [2011] for classical groups. Moreover, the examples of Λ^3 discussed below should handle some cases of exceptional groups. More precisely, using [Shahidi 2011] the work in [Kim 2012] proves that if ψ is an Arthur packet for $\mathrm{GSpin}(F)$, where F is a p-adic field, then the Langlands packet ϕ_ψ attached to ψ has a generic member only if ϕ_ψ is tempered. This clearly gives a converse to the tempered (or generic) L-packet conjecture [Shahidi 1990; Vogan 1978]. For an archimedean field F this is proved in [Shahidi 2011] and follows from the equality of Artin factors with those defined by the Langlands–Shahidi method [Langlands 1989; Shahidi 1985]. Here is now the matching theorem for L-functions:

Theorem 3.1. *Let (G, M) be a pair of a quasisplit connected reductive group and one of its maximal Levi subgroups defined over a local field F. Assume there exists a homomorphism $\varphi : M \to \mathrm{GL}(n) \times \mathrm{GL}(1)$ that is an isomorphism on derived groups, i.e., $M_D \simeq \mathrm{SL}(n)$. Let $\pi = \pi_0 \otimes \eta$ be an irreducible admissible representation of $\mathrm{GL}(n, F) \times F^*$ and consider it as one of $M(F)$. Assume $\pi = \pi(\rho)$, $\rho : W'_F \to \mathrm{GL}(n, \mathbb{C}) \times \mathrm{GL}(1, \mathbb{C})$. Let r_i be an irreducible constituent of the adjoint action of ${}^LM$ on ${}^L\mathfrak{n}$, the Lie algebra of ${}^LN$. Using the dual map*

$$ {}^L\varphi : \mathrm{GL}(n, \mathbb{C}) \times \mathbb{C}^* \to {}^LM, \tag{3.1.1} $$

we then have

$$ L(s, \pi \cdot \varphi, r_i) = L(s, \pi, r_i \cdot {}^L\varphi). \tag{3.1.2} $$

Assume $r_i \cdot {}^L\varphi$ satisfies the branching rule (2-1). *Moreover, assume the equality* (2-2), *but only for L-functions, that is, the validity of*

$$ L(s, R \cdot (\rho_1, \rho_2)) = L(s, (\pi(\rho_1), \pi(\rho_2)), R). \tag{3.1.3} $$

Then

$$ L(s, r_i \cdot {}^L\varphi \cdot \rho) = L(s, \pi(\rho), r_i \cdot {}^L\varphi). \tag{3.1.4} $$

Remark. The extension from generic representations to any irreducible admissible one is rather routine as explained on page 322 of [Shahidi 1990].

Proof. We may assume F is p-adic. We again use Brauer's theorem and prove (3.1.4) for monomial representations as in Theorem 2.1. We choose k and $\tilde{\rho}$ such that $k_v = F$, $\tilde{\rho}|W'_F = \rho$ and consider $\pi(\tilde{\rho}) := \bigotimes_v \pi(\tilde{\rho}_w)$, where $\tilde{\rho}_w = \tilde{\rho}|W'_{k_w}$. We again twist $\pi(\tilde{\rho})$ by a grössencharacter $\tilde{\chi} = \bigotimes_w \tilde{\chi}_w$ that is highly ramified at all finite places where $\pi(\tilde{\rho}_w)$ is ramified except v, where we will assume $\tilde{\chi}_v \equiv 1$. Then for each finite ramified w, $w \neq v$, stability for L-functions, i.e., (1-8), implies

$$\gamma\big(s, \pi(\tilde{\rho}_w) \otimes \tilde{\chi}_w, r_{i,w} \cdot {}^L\varphi, \psi_F\big) = c_w q_w^{-n_w s}, \tag{3.1.5}$$

where $c_w \in \mathbb{C}^*$, $n_w \in \mathbb{Z}$ and q_w is the cardinality of the residue field of k at w. Using the equality at archimedean primes for γ-functions we thus have

$$\prod_{\substack{w \in S \\ w \neq v}} c_w q_w^{-n_w s} \gamma\big(s, \pi(\rho), r_i \cdot {}^L\varphi, \psi_F\big) = \prod_{\substack{w \in S \\ w \neq v}} c'_w q_w^{-n'_w s} \gamma\big(s, r_i \cdot {}^L\varphi \cdot \rho, \psi_F\big), \tag{3.1.6}$$

where c'_w and n'_w are the corresponding objects on the Artin side and S is the set of ramified finite primes, whenever ρ is monomial.

On the other hand, by equality (3.1.3) of L-functions for constituents of our branching rule, we get an equality like (3.1.6) for every pair $\gamma(s, R \cdot (\rho_1, \rho_2), \psi_F)$ and $\gamma(s, (\pi(\rho_1), \pi(\rho_2)), R, \psi_F)$. We can then extend (3.1.6) from monomial representations, that is, a $\mathbb{Z}$-basis for the Grothendieck ring of W'_F, to the full ring.

We now assume ρ is bounded so that $\pi(\rho)$ is tempered. We then have that $L(s, \pi(\rho), r_i \cdot {}^L\varphi)$ gives the zeros of $\gamma(s, \pi(\rho), r_i \cdot {}^L\varphi)$ [Shahidi 1990; 2010]. The same is true of $L(s, r_i \cdot {}^L\varphi \cdot \rho)$ and $\gamma(s, r_i \cdot {}^L\varphi \cdot \rho)$. By standard properties of L-functions, we then get the equality (3.1.4) for a bounded ρ. The case of arbitrary ρ and $\pi(\rho)$ now follows from Langlands classification upon which factors for $\pi(\rho)$ are defined [Langlands 1989; Shahidi 1990; 2010] as well as those of Artin. This completes the proof of Theorem 3.1. □

Remark 3.2. One may replace Equation (3.1.3) with the equality of γ-factors only up to a monomial in q^{-s}, which is a much weaker statement than (2-2).

Example 3.3 (twisted exterior and symmetric square L-functions for $\mathrm{GL}(n)$). The pair in this case is $G = \mathrm{GSpin}$ and M is generated by all simple roots but the last one, i.e., the Siegel parabolic of G. In the case of exterior squares the equality is

$$\begin{aligned} L(s, \Lambda^2\rho_0 \otimes \eta) &= L(s, \pi(\rho_0) \otimes \eta, \Lambda^2 \otimes St) \\ &= L(s, \pi_0, \Lambda^2 \otimes \eta), \end{aligned} \tag{3.3.1}$$

where the L-functions on the right are from [Shahidi 1990; 2010]. This was first proved in [Henniart 2010]. The case of twisted symmetric square is similar. Here St denotes the standard representation of $\mathrm{GL}_1(\mathbb{C})$.

Example 3.4 (twisted exterior cube for GL(6)). Here the pair is (E_6^{sc}, M^{α_4}), where M^{α_4} is the Levi subgroup generated by $\Delta - \{\alpha_4\}$, Δ being the set of simple roots. This is case (x) in [Langlands 1971a] or equally $(E_{6,ii})$ in [Shahidi 2010]. The map φ is defined in 2.5.3 of [Kim 2005]. With notation as in Theorem 3.1 here

$$r_i \cdot {}^L\varphi = r_1 \cdot {}^L\varphi = \Lambda^3 \otimes St, \tag{3.4.1}$$

and thus Theorem 3.1 should imply

$$\begin{aligned} L(s, \Lambda^3 \rho_0 \otimes \eta) &= L(s, r_1 \cdot {}^L\varphi \cdot (\rho_0 \otimes \eta)) \\ &= L(s, \pi(\rho), r_1 \cdot {}^L\varphi) \\ &= L(s, \pi(\rho_0) \otimes \eta, \Lambda^3 \otimes St) \\ &= L(s, \pi_0, \Lambda^3 \otimes \eta), \end{aligned} \tag{3.4.2}$$

where $\pi(\rho_0) = \pi_0$, $\pi = \pi_0 \otimes \eta$ and $\rho = \rho_0 \otimes \eta$, if we can show (2-2) and (3.1.3) hold. We remark that in this case $\dim r_2 = 1$ and there are no other constituents.

As discussed in Section 2, the branching rule (2-1) in this case reads

$$R(\rho_1, \rho_2) = \Lambda^2 \rho_1 \otimes \rho_2 \oplus \rho_1 \otimes \Lambda^2 \rho_2. \tag{3.4.3}$$

Dimensions $n_i = \dim \rho_i$, $i = 1, 2$, are a partition of 6, i.e., $n_1 + n_2 = 6$. By symmetry we need to know the validity of

$$L(s, \Lambda^2 \rho_1 \otimes \rho_2) = L(s, (\pi(\rho_1), \pi(\rho_2)), \Lambda^2 \otimes St), \tag{3.4.4}$$

$1 \leq n_1 \leq 5$, $n_1 + n_2 = 6$, where St denotes the standard representation of $\mathrm{GL}(n_2, \mathbb{C})$.

When $1 \leq n_1 \leq 3$, (3.4.3) is valid by [Harris and Taylor 2001; Henniart 2000]. For $n_1 = 5$ and thus $n_2 = 1$, (3.4.4) is Example 3.3. It remains to address the case $n_1 = 4$ and $n_2 = 2$. Equality (3.4.4) in this case follows from Kim's work on functoriality for $\Lambda^2 : \mathrm{GL}_4(\mathbb{C}) \to \mathrm{GL}_6(\mathbb{C})$. In fact, (3.4.4) is equivalent to

$$L(s, \Lambda^2 \rho_1 \otimes \rho_2) = L(s, \pi(\Lambda^2 \rho_1) \times \pi(\rho_2)), \tag{3.4.5}$$

by [Harris and Taylor 2001; Henniart 2000] in which $\Lambda^2 \rho_1$ is a six dimensional continuous representation of W_F'. What we need to verify is the equality

$$\begin{aligned} L(s, \Lambda^2 \rho_1 \otimes \rho_2) &= L(s, \Lambda^2(\pi(\rho_1)) \times \pi(\rho_2)) \\ &= L(s, (\pi(\rho_1), \pi(\rho_2)), \Lambda^2 \otimes St). \end{aligned} \tag{3.4.6}$$

This is proved by Kim [2003]. We collect this as:

Proposition 3.5. *Let $(\rho, \pi(\rho))$ be a pair with $\rho = \rho_0 \otimes \eta$ a representation of W'_F into* $\mathrm{GL}(6, \mathbb{C}) \times \mathrm{GL}(1, \mathbb{C})$. *Let $\pi_0 = \pi(\rho_0)$. Then*

$$\begin{aligned} L(s, \Lambda^3\rho_0 \otimes \eta) &= L(s, \pi_0, \Lambda^3 \otimes \eta) \\ &= L(s, \pi(\rho_0) \otimes \eta, \Lambda^3 \otimes St). \end{aligned} \tag{3.5.1}$$

Remark 3.6. The pairs (E_7^{sc}, M^{α_4}) and (E_8, M^{α_4}) give $L(s, \pi_0 \otimes \eta, \Lambda^3 \otimes St)$, where $\eta \in \widehat{F}^*$ and π_0 is an irreducible admissible representation of either $\mathrm{GL}(7, F)$ or $\mathrm{GL}(8, F)$, respectively [Langlands 1971a; Shahidi 2010]. To get equality (3.5.1) in these cases requires equality (3.4.6) for $n_1 = 5$ and 6, respectively, which unfortunately are not yet available.

4. Comments on stability of γ-functions

As explained in Section 1, it is the stability of γ-functions, condition (2) of our r-theory, which is not available in any generality, even within the Langlands–Shahidi method. On the other hand γ-functions within this method are defined inductively by means of "local coefficients" [Shahidi 1990; 2010]. These are complex functions defined by means of standard intertwining operators and Whittaker functionals for induced representations [Shahidi 2010]. Their definition clearly requires the representation π of $M(F)$ be generic. But γ-functions defined through the method can be extended even to cases where π is not generic. This is done by means of Langlands classification (page 322 of [Shahidi 1990]).

It is thus enough to show that each local coefficient is stable under twists by highly ramified characters. We shall now briefly explain how one expects to prove stability.

As before, we assume (G, M) is a pair of a quasisplit connected reductive group G and a Levi subgroup M of one of its maximal parabolics, $P = MN$, both defined over F which we will assume to be a p-adic field of characteristic zero. We let α denote the unique simple root in N. The method is now being developed for fields of positive characteristic mainly by Luis Lomelí with some collaboration by Guy Henniart (see [Lomelí 2009; Henniart and Lomelí 2011]).

With notation as in the previous section, we let ${}^LM$ act on ${}^L\mathfrak{n}$ and let r_i, $1 \le i \le m$, be its irreducible subrepresentations ordered as in [Shahidi 1990; 2010]. The γ-factors $\gamma(s, \pi, r_i, \psi_F)$, when π is an irreducible admissible generic representation of $M(F)$, satisfy

$$C(s, \pi) = C_{\psi_F}(s, \pi) = \lambda_G(\psi_F, w_0)^{-1} \prod_{i=1}^{m} \gamma(is, \pi, \tilde{r}_i, \overline{\psi}_F), \tag{4.1}$$

where $C(s, \pi)$ is the corresponding local coefficient. Here π is assumed to be generic with respect to the generic character of $U_M(F)$ defined by ψ_F and a fixed

F-splitting of G (and thus M). For simplicity we call π ψ_F-generic, not mentioning the splitting. The factor $\lambda(\psi_F, w_0)$ is a product of Langlands λ-functions, Hilbert symbols, and w_0 is the representative of the element $\tilde{w}_\ell \tilde{w}_{\ell,M}^{-1}$ of Weyl group $W(G, T)$. Here $B = TU$ is a fixed Borel subgroup over F, giving our splitting, $M \supset T$, $U \supset N$, $U_M = U \cap M$. We recall that fixing the splitting leads to a choice of a representative for any Weyl group element, w_0 representing that of $\tilde{w}_\ell \tilde{w}_{\rho,M}^{-1}$. We refer to Chapter 8 of [Shahidi 2010], specifically Remarks 8.2.1 and 8.2.2, for a complete discussion of these factors and their choices.

With notation as in Section 1, item 2 (stability), one can formulate stability for $C(s, \pi)$ as follows:

Conjecture 4.1. *Given a pair of irreducible admissible ψ_F-generic representations π_1 and π_2 of $M(F)$ with same central characters,*

$$C(s, \pi_1 \otimes \chi) = C(s, \pi_2 \otimes \chi),$$

where χ is a suitably highly ramified character of $M(F)$.

As experience has shown, at least in a number of important cases [Asgari and Shahidi 2006; 2011; Cogdell et al. 2004; 2005; 2008; $\geq$ 2012; Kim and Krishnamurthy 2005], this can be proved by expressing $C(s, \pi)$ as a Mellin transform of a Bessel function on $M(F)$. This was attained by establishing an integral representation for $C(s, \pi)^{-1}$ in [Shahidi 2002]. The formula is under the assumption that P is self-associate. This means that $\overline{N} = w_0 N w_0^{-1} = N^-$, where N^- is the unipotent subgroup opposed to N.

We first recall the partial Bessel function involved. Let ω_π be the central character of π and define $w_0(\omega_\pi)(z) = \omega_\pi(w_0^{-1} z w_0)$. Given $s \in \mathbb{C}$, set $\pi_s = \pi \otimes q^{\langle s\tilde{\alpha}, H_M(\cdot)\rangle}$ and define

$$\omega_{\pi_s}(z) = \omega_\pi(z) q^{\langle s\tilde{\alpha}, H_M(z)\rangle}. \tag{4.2}$$

We refer to [Shahidi 1988] for the definition of $\tilde{\alpha}$. Fix a sufficiently large open compact subgroup $\overline{N}_0 \subset \overline{N}$. Let φ denote its characteristic function.

For almost all $n \in N(F)$,

$$w_0^{-1} n = m n' \overline{n}, \tag{4.3}$$

$m \in M(F)$, $n' \in N(F)$, $\overline{n} \in \overline{N}(F)$. This sets up a densely defined map

$$n \mapsto (m, \overline{n})$$

from $N(F)$ into $M(F) \times \overline{N}(F)$. While $n \mapsto \overline{n}$ is a bijection, $n \mapsto m$ may not be one; see [Shahidi 2002].

Let W_v be a Whittaker function in the space $W(\pi_s)$ of π_s such that $W_v(e)=1$. Given $z\in Z_M(F)$, we define the partial Bessel function

$$j_{v,\varphi}(m,\overline{n},z):=\int_{U_{M,n}(F)\backslash U_M(F)} W_v(mu^{-1})\varphi(zu\overline{n}u^{-1}z^{-1})\psi_F(u)\,du. \tag{4.4}$$

Let α be the unique simple root of T in U generating N.

We may assume $H^1(F,Z_G)=1$, which we can attain by enlarging G without changing its derived group. It will not affect our results. Lemma 5.2 of [Shahidi 2002] then implies existence of a map $\alpha^\vee$ from F^* into $Z_M^0=Z_G(F)\backslash Z_M(F)$ such that $\alpha'(\alpha^\vee(t))=t$, $t\in F^*$, for any root α' of T that restricts to α.

We need to define a scalar x_α defined by $\overline{n}$. It is simply the α-coordinate of $w_0^{-1}\overline{n}w_0\in N$ by means of our fixed splitting.

Given $y\in F^*$, set

$$j_{v,\varphi}(m,\overline{n},y):=j_{v,\varphi}(m,\overline{n},\alpha^\vee(y^{-1}\cdot x_\alpha)), \tag{4.5}$$

whenever $x_\alpha\neq 0$.

We also let $Z_M^0U_M(F)$ act on $N(F)$ by conjugation and write $Z_M^0U_M(F)\backslash N(F)$ for the corresponding quotient space.

Theorem 4.2 [Shahidi 2002, Theorem 6.2, second part]. *Suppose $\omega_\pi(w_0\omega_\pi^{-1})$ is ramified. Fix $y_0\in F$ such that $\mathrm{ord}_F(y_0)=-d-f$, where d and f are conductors of ψ_F and $\omega_\pi^{-1}\cdot(w_0\omega_\pi)$, respectively. Then up to an abelian Tate γ-factor attached to $\omega_\pi\cdot(w_0\omega_\pi^{-1})$ and ψ_F,*

$$C(s,\pi)^{-1} \sim\int_{Z_M^0U_M(F)\backslash N(F)} j_{\tilde{v},\varphi}(m,\overline{n},y_0)\omega_{\pi_s}^{-1}(x_\alpha)(w_0\omega_{\pi_s})(x_\alpha)q^{\langle s\tilde{\alpha}+\rho,H_M(m)\rangle}d\dot{n}. \tag{4.6}$$

Here x_α is embedded in $Z_M(F)$ through $\alpha^\vee$ and $v=\tilde{v}\otimes q^{\langle s\tilde{\alpha},H_M(\,)\rangle}$. More precisely, $\tilde{v}$ is the vector in the space of π that goes to v in the space of π_s.

We refer to [Shahidi 2009] for some of the geometric issues in analyzing the integral in (4.6).

It is Equation (4.6) which has been the main tool in proving stability in a number of important cases, all of significance in establishing functoriality [Cogdell and Piatetski-Shapiro 1998; Cogdell et al. 2004; 2005; 2008; Kim and Krishnamurthy 2005; Asgari and Shahidi 2006; 2011].

What one has to do is to prove an asymptotic expansion for the partial Bessel function $j_{\tilde{v},\varphi}$. In fact, in the cases of classical or GSpin groups, one basically needs to deal with $M=\mathrm{GL}(1)\times G_1$, where G_1 is one of these groups, as a maximal Levi subgroup inside a larger group G of the same type.

The philosophy of expressing γ-functions as a Mellin transform of a partial Bessel function goes back to Cogdell and Piatetski-Shapiro [1998] who proved such a formula as well as the asymptotic expansion for the corresponding partial Bessel functions when $G_1 = \mathrm{SO}(2n+1)$. Using Equation (4.6), which was established in [Shahidi 2002], the corresponding stability for other cases were proved in [Cogdell et al. 2004; 2005; 2008; Kim and Krishnamurthy 2005; Asgari and Shahidi 2006].

In [Cogdell, Shahidi and Tsai $\geq$ 2012], the authors study the case $(G, M) = (\mathrm{GSp}(2n),\ \mathrm{GL}(n) \times \mathrm{GL}(1))$, where the γ-factor $\gamma(s, \pi, \Lambda^2, \psi_F)$ appears. Using a robust deformation argument which should apply more generally whenever LLC is available, the equality

$$\gamma(s, \Lambda^2 \cdot \rho, \psi_F) = \gamma(s, \pi(\rho), \Lambda^2, \psi_F) \tag{4.7}$$

is reduced to a proof of stability for only when ρ is irreducible and thus only when $\pi = \pi(\rho)$ is supercuspidal in [Cogdell, Shahidi and Tsai $\geq$ 2012]. A proof of stability in the supercuspidal case also seems to be within reach, using (4.6) and the asymptotics of the full Bessel functions for $\mathrm{GL}(n)$ proved by Jacquet and Ye [1996]. In particular, it is shown that the asymptotics of the partial Bessel function $j_{\tilde{v},\varphi}$ can still be deduced from those of full Bessel functions and thus germ expansions in [Jacquet and Ye 1996]. The case of symmetric squares

$$\gamma(s, \mathrm{Sym}^2 \cdot \rho, \psi_F) = \gamma(s, \pi(\rho), \mathrm{Sym}^2, \psi_F) \tag{4.8}$$

follows immediately from

$$\gamma(s, \pi \times \pi, \psi_F) = \gamma(s, \pi, \Lambda^2, \psi_F)\gamma(s, \pi, \mathrm{Sym}^2, \psi_F), \tag{4.9}$$

$$\gamma(s, \rho \otimes \rho, \psi_F) = \gamma(s, \Lambda^2 \cdot \rho, \psi_F)\gamma(s, \mathrm{Sym}^2 \cdot \rho, \psi_F), \tag{4.10}$$

and

$$\gamma(s, \rho \otimes \rho, \psi_F) = \gamma(s, \pi(\rho) \times \pi(\rho), \psi_F), \tag{4.11}$$

the last being part of LLC in [Harris and Taylor 2001; Henniart 2000]. The γ-factors $\gamma(s, \pi, \Lambda^2, \psi_F)$ and $\gamma(s, \pi, \mathrm{Sym}^2, \psi_F)$ are those defined by the Langlands–Shahidi method as special cases of the general definition given in [Shahidi 1990].

The case of Rankin product L-functions for $\mathrm{GL}(n) \times \mathrm{GL}(n)$ using this approach has been addressed in [Tsai 2011]. The cases of non-self-associate maximal parabolics are also being addressed, and an analogue of (4.6) for $\mathrm{GL}(n) \times \mathrm{GL}(m)$, $n \neq m$, seems to be in hand. This seems to be the most complicated among the cases to be considered.

For the record, we also refer to [Ramakrishnan 2000] and [Kim and Shahidi 2002], where the equality of certain triple product factors is proved, but using other techniques such as base change, combined with functoriality.

We should finally mention the possible application of (4.6), or rather its more general form (6.38) or its initial form (6.55), both of [Shahidi 2002], in establishing the local Langlands correspondence for GSp(4) over function fields through Deligne–Kazhdan philosophy of close fields. If successful the problem is then reduced to that of LLC for GSp(4) over number fields, already established in [Gan and Takeda 2011]. We refer to [Ganapathy 2012] for a discussion of this philosophy and the treatment of LLC for $\mathrm{GL}(n)$ through this approach.

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FREYDOON SHAHIDI
DEPARTMENT OF MATHEMATICS
PURDUE UNIVERSITY
WEST LAFAYETTE, IN 47907
UNITED STATES
shahidi@math.purdue.edu

CONTENTS

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