*Pacific Journal of Mathematics*

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Volume 260 No. 2 December 2012

PACIFIC JOURNAL OF MATHEMATICS
Vol. 260, No. 2, 2012
dx.doi.org/10.2140/pjm.2012.260.665

# TRUNCATION OF EISENSTEIN SERIES

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*To the memory of Jonathan Rogawski*

**We give a generalization of the Maass–Selberg relations for general Eisenstein series, providing a different approach to Arthur's asymptotic inner product formula.**



## 1. Introduction

In this short note we study truncation of Eisenstein series. The truncation operator was introduced by Arthur [1980]. It plays a ubiquitous role in the trace formula. In the case of a cuspidal Eisenstein series (that is, one induced from a cuspidal representation) one can write its truncation as a modified Eisenstein series (previously introduced by Langlands). From this, one obtains the Maass–Selberg relations for the inner product of truncated Eisenstein series [Arthur 1980, §4] (see also Section 3). In the case of Eisenstein series induced from the discrete spectrum, Arthur [1982] obtained an asymptotic formula for the inner product above. His method was rather indirect and in particular, it required Langlands' description of the discrete spectrum in terms of residues of Eisenstein series. A different approach which avoids this description was taken in [Lapid 2011]. It uses the regularized integral developed in [Jacquet et al. 1999]. While the approach of [Lapid 2011] is reasonably conceptual, one still encounters some unpleasant technical difficulties. The purpose of this short paper is to rederive Arthur's asymptotic result more directly

Lapid was partially supported by the Israel Science Foundation Center of Excellence, grant 1691/10. Ouellette would like to thank the Hebrew University of Jerusalem for its hospitality.
*MSC2010:* 11F70, 11F72.
*Keywords:* Eisenstein series, spectral theory.

by writing down explicitly the truncation of a general Eisenstein series. This is a pleasant combinatorial exercise in truncation. As explained in [Lapid 2011], the asymptotic formula can be used to compute the inner product of Eisenstein integrals, a key fact in Langlands spectral theory.

We cannot close the introduction without recalling our deep appreciation to our teacher Jonathan Rogawski. His unlimited encouragement and keen interest in mathematics, even in difficult times, will not be forgotten. We miss him greatly.

## 2. Notation and conventions

Let $F$ be a number field and $\mathbb{A}$ its ring of adeles. Throughout, we denote by boldface letters, such as $\boldsymbol{Y}$, algebraic varieties over $F$ and we write $Y = \boldsymbol{Y}(F)$, $Y_{\mathbb{A}} = \boldsymbol{Y}(\mathbb{A})$. Sometimes we will not distinguish between $\boldsymbol{Y}$ and $Y$. Let $\boldsymbol{G}$ be a reductive group over a number field $F$. (Henceforth, all the algebraic subgroups of $\boldsymbol{G}$ that we consider are implicitly assumed to be defined over $F$.) We fix a maximal $F$-split torus $\boldsymbol{T}_0$ and a minimal parabolic subgroup $\boldsymbol{P}_0$ containing $\boldsymbol{T}_0$. We have a Levi decomposition $\boldsymbol{P}_0 = \boldsymbol{M}_0 \ltimes \boldsymbol{U}_0$ where $\boldsymbol{M}_0 = C_{\boldsymbol{G}}(\boldsymbol{T}_0)$. Let $\mathfrak{a}_0^*$ be the $\mathbb{R}$-vector space spanned by the lattice $X^*(\boldsymbol{T}_0)$ of $F$-rational characters of $\boldsymbol{T}_0$ (or alternatively, by the commensurable lattice $X^*(\boldsymbol{M}_0)$ of $F$-rational characters of $\boldsymbol{M}_0$). The dimension of $\mathfrak{a}_0^*$ is the split rank of $\boldsymbol{G}$. The dual space $\mathfrak{a}_0$ of $\mathfrak{a}_0^*$ is the $\mathbb{R}$-vector space spanned by the lattice of cocharacters $X_*(\boldsymbol{T}_0)$ of $\boldsymbol{T}_0$. We write $\mathfrak{a}_{0,\mathbb{C}}$ for the complexification of $\mathfrak{a}_0$. We denote by $\Delta_0 \subseteq X^*(\boldsymbol{T}_0)$ the set of simple roots of $\boldsymbol{T}_0$ on $\operatorname{Lie} \boldsymbol{U}_0$ and by $\Delta_0^\vee \subseteq X_*(\boldsymbol{T}_0)$ the set of simple coroots.

We write $H^g = gHg^{-1}$ for any subgroup $H \subseteq G$ and an element $g \in G$.

For any algebraic group $\boldsymbol{Y}$, we write $\delta_Y$ for the modulus function on $Y_{\mathbb{A}}$. We also write $Y_{\mathbb{A}}^1 = \bigcap \operatorname{Ker}|\chi|$ where $\chi$ ranges over the lattice of $F$-rational characters of $\boldsymbol{Y}$ and $|\chi|(y) = \prod_v |\chi_v(y_v)|_v$ for $y = (y_v) \in Y_{\mathbb{A}}$.

Let $P = M \ltimes U$ be a standard parabolic subgroup of $G$ defined over $F$, with $M \supset M_0$. Let $\Delta_0^M \subseteq \Delta_0$ be the set of simple roots of $T_0$ in $\operatorname{Lie}(U_0 \cap M)$ and denote the span of $\Delta_0^M$ by $(\mathfrak{a}_0^M)^*$. Let $\boldsymbol{T}_M$ be the identity component of the split part of the center of $M$ — a subtorus of $T_0$. We identify $\mathfrak{a}_M^* = X^*(T_M) \otimes \mathbb{R} = X^*(M) \otimes \mathbb{R}$ with a subspace of $\mathfrak{a}_0^*$. Occasionally we also write $\mathfrak{a}_P = \mathfrak{a}_M$. In particular, $\mathfrak{a}_{P_0} = \mathfrak{a}_{M_0} = \mathfrak{a}_0$. We write $r(P) = r(M) = \dim \mathfrak{a}_M$. We have $\mathfrak{a}_0 = \mathfrak{a}_M \oplus \mathfrak{a}_0^M$ and similarly for $\mathfrak{a}_0^*$. Denote by $\Delta_M = \Delta_P \subseteq X^*(\boldsymbol{T}_M)$ the simple roots of $\boldsymbol{T}_M$ on $\operatorname{Lie} \boldsymbol{U}$ — these are the projections of $\Delta_0 \setminus \Delta_0^M$ to $\mathfrak{a}_M^*$. For any $\alpha \in \Delta_P$ we have the corresponding coroot $\alpha^\vee \in X_*(\boldsymbol{T}_M)$.

We reserve the letters $P = MU$ and $Q = LV$ (possibly appended with primes or subscripts) for standard parabolic subgroups of $G$ with their standard Levi decomposition. Since $M$ and $P$ determine each other, we often use them interchangeably as subscripts or superscripts in various notation. Occasionally we will use $R$ and $S$

to denote auxiliary standard parabolic subgroups. We write $M_R$ for the standard Levi subgroup of $R$ and $N_R$ for its unipotent radical.

For any $Q \subseteq P$, we write $\Delta_L^M = \Delta_Q^P \subseteq \Delta_Q$ for the simple roots of $T_L$ on $\mathrm{Lie}(V \cap M)$. We have $\mathfrak{a}_L = \mathfrak{a}_L^M \oplus \mathfrak{a}_M$ where $\mathfrak{a}_L^M = \mathfrak{a}_Q^P$ is the span of

$$(\Delta_Q^P)^\vee = (\Delta_L^M)^\vee = \{\alpha^\vee : \alpha \in \Delta_Q^P\}.$$

Consequently, $\mathfrak{a}_0 = \mathfrak{a}_0^L \oplus \mathfrak{a}_L^M \oplus \mathfrak{a}_M$. The dual basis of $(\Delta_L^M)^\vee$ in $(\mathfrak{a}_L^M)^*$ will be denoted by $\hat{\Delta}_L^M$. We write $X_Q^P$ or $X_L^M$ for the image of $X \in \mathfrak{a}_0$ under the projection from $\mathfrak{a}_0$ to $\mathfrak{a}_L^M$.

We write $[P, Q]$ for the set of parabolic subgroups of $Q$ containing $P$. Thus, $[P_0, G]$ is the set of all standard parabolic subgroups of $G$.

Denote by $W = W_G$ the Weyl group $N_G(T_0)/M_0$ of $G$. For any $M$, we identify the cosets $W^M \backslash W$ (resp. $W/W^M$) with the set of left- (resp. right-) $W^M$ reduced elements of $W$, that is, those for which $w^{-1}\alpha > 0$ (resp. $w\alpha > 0$) for all $\alpha \in \Delta_0^M$.

Now let $M$ and $L$ be standard Levi subgroups. We identify $W^M \backslash W / W^L$ with the set of left-$W^M$ and right-$W^L$ reduced elements of $W$. Define subsets

$$W(L, M) \subseteq W(L; M) \subseteq W^M \backslash W / W^L$$

by

$$W(L, M) = \{w \in W^M \backslash W : L^w = M\} = \{w \in W^M \backslash W : w\Delta_0^L = \Delta_0^M\}$$

and

$$W(L; M) = \{w \in W^M \backslash W : L^w \subseteq M\} = \{w \in W^M \backslash W : w\Delta_0^L \subseteq \Delta_0^M\}.$$

Note that if $L' \subseteq L$ then $W(L; M) \subseteq W(L'; M)$.

We write $\mathscr{C}_{0,-}$ for the closed negative obtuse Weyl chamber

$$\mathscr{C}_{0,-} = \Big\{ \sum_{\alpha \in \Delta_0} x_\alpha \alpha^\vee : x_\alpha \le 0 \text{ for all } \alpha \Big\}.$$

More generally, for any $Q \subseteq P$ we write

$$\mathscr{C}_{Q,-}^P = \Big\{ \sum_{\alpha \in \Delta_Q^P} x_\alpha \alpha^\vee : x_\alpha \le 0 \text{ for all } \alpha \Big\}.$$

We fix a positive definite $W$-invariant scalar product, and hence a norm $\|\cdot\|$ on $\mathfrak{a}_0$. This defines a measure on any subspace of $\mathfrak{a}_0$.

We fix a "good" maximal compact subgroup $K$ of $G_{\mathbb{A}}$. Using the Iwasawa decomposition, we define $H : G_{\mathbb{A}} \to \mathfrak{a}_0$ to be the left-$U_{0,\mathbb{A}}$ right-$K$ invariant function such that

$$e^{\langle \chi, H(m) \rangle} = \prod_v |\chi_v(m_v)|_v$$

for any $\chi \in X^*(M)$ where $m = (m_v)_v$ and $\chi_v$ is the extension of $\chi$ to $M(F_v)$.

Let $A_0$ be the identity component of $\boldsymbol{T}_0(\mathbb{R}) \subseteq T_{0,\mathbb{A}}$ where $\mathbb{R}$ is embedded in $\mathbb{A}$ diagonally at the archimedean places. The map $H$ gives rise to an isomorphism $A_0 \to \mathfrak{a}_0$. We denote by $X \mapsto e^X$ the inverse map. More generally, for any $M$ let $A_M = A_0 \cap T_M$. The map $H$ restricts to an isomorphism $A_M \to \mathfrak{a}_M$.

Let $\mathfrak{a}_{0,+}$ be the positive Weyl chamber

$$\mathfrak{a}_{0,+} = \{X \in \mathfrak{a}_0 : \langle \alpha, X \rangle > 0 \text{ for all } \alpha \in \Delta_0\}.$$

Similarly, we write for any $P$

$$\mathfrak{a}^*_{M,+} = \{\lambda \in \mathfrak{a}^*_M : \langle \lambda, \alpha^\vee \rangle > 0 \text{ for all } \alpha \in \Delta_P\}.$$

Let $\mathcal{A}_P$ be the space of automorphic forms on $PU_{\mathbb{A}} \backslash G_{\mathbb{A}}$, that is, the smooth, $K$-finite, and $\mathfrak{z}$-finite functions of moderate growth where as usual $\mathfrak{z}$ is the center of the universal enveloping algebra of the complexified Lie algebra of $\boldsymbol{G}(\mathbb{R})$. We write $\mathcal{A}^n_P$ for those $\varphi \in \mathcal{A}_P$ such that $\varphi(ag) = \delta_P(a)^{\frac{1}{2}} \varphi(g)$ for all $a \in A_M$, $g \in G$. We denote by $\mathcal{A}^2_P$ the subspace of $\mathcal{A}^n_P$ consisting of the functions such that

$$\langle \varphi, \varphi \rangle_{A_M U_{\mathbb{A}} M \backslash G_{\mathbb{A}}} = \|\varphi\|_2^2 = \int_{A_M U_{\mathbb{A}} M \backslash G_{\mathbb{A}}} |\varphi(g)|^2 \, dg < \infty$$

and by $\mathcal{A}^{\text{cusp}}_P$ the subspace consisting of the cuspidal automorphic forms.

For any $\varphi \in \mathcal{A}_P$ and $\lambda \in \mathfrak{a}^*_M$ let

$$\varphi_\lambda(g) = e^{\langle \lambda, H_P(g) \rangle} \varphi(g), \quad g \in G_{\mathbb{A}}.$$

For any $Q \supset P$ the Eisenstein series is defined by

$$E^Q_P(g, \varphi, \lambda) = \sum_{\gamma \in P \backslash Q} \varphi_\lambda(\gamma g).$$

(If $Q = G$ we omit it from the notation.) The series converges absolutely for $\operatorname{Re} \lambda \in \mathfrak{a}^*_{P,+}$ sufficiently regular. We will *assume* that $E(\cdot, \varphi, \lambda)$ admits meromorphic continuation with hyperplane singularities. This is proved in [Langlands 1976] (cf. [Mœglin and Waldspurger 1994]) first for $\varphi \in \mathcal{A}^{\text{cusp}}_P$ and then for $\varphi \in \mathcal{A}^2_P$ as a consequence of the description of the discrete spectrum in terms of residues of Eisenstein series. An argument of Bernstein gives such a result (for any $\varphi \in \mathcal{A}_P$) without appealing to Langlands' description of the discrete spectrum. Unfortunately, this argument is still unpublished. However, for our purposes we will simply admit it.

Alongside, we have the intertwining operators

$$M(w, \lambda) : \mathcal{A}_P \to \mathcal{A}_{P'}$$

for any $w \in W(M, M')$ given by

$$(M(w,\lambda)\varphi)_{w\lambda}(g) = \int_{(U'_{\mathbb{A}} \cap U^w_{\mathbb{A}})\backslash U'_{\mathbb{A}}} \varphi_\lambda(w^{-1}ug)\,du.$$

Once again, the integral converges absolutely provided that $\operatorname{Re}\langle\lambda, \alpha^\vee\rangle \gg 0$ for all roots $\alpha$ of $T_M$ on $\operatorname{Lie}(U)$ such that $w\alpha < 0$. We *admit* the meromorphic continuation of $M(w, \lambda)$ and the functional equations

$$M(w_1w_2, \lambda) = M(w_1, w_2\lambda)M(w_2, \lambda).$$

for any $w_1 \in W(M', L)$ and $w_2 \in W(M, M')$. In particular,

$$M(w, \lambda)^{-1} = M(w^{-1}, w\lambda).$$

We also have

$$M(w, \lambda)^* = M(w^{-1}, -w\bar{\lambda})$$

on $\mathcal{A}^2_{P'}$. Thus, $M(w, \lambda)$ is unitary (and in particular, holomorphic) on $\mathcal{A}^2_P$ for $\lambda \in \mathrm{i}\mathfrak{a}^*_M$.

For any $\varphi \in \mathcal{A}_P$ and $Q \subseteq P$, we write $\varphi_Q$ for the constant term along $Q$, namely

$$\varphi_Q(g) = \int_{V\backslash V_{\mathbb{A}}} \varphi(vg)\,dv = \int_{(V\cap M)\backslash(V_{\mathbb{A}}\cap M_{\mathbb{A}})} \varphi(vg)\,dv.$$

Occasionally we also write $\varphi_V$ or $\varphi_L$ for $\varphi_Q$.

For any $w \in W^M\backslash W/W^L$ let $P_w \subseteq P$ be the parabolic subgroup with Levi $M_w = M \cap L^w$ and let $Q_w$ be the parabolic subgroup with Levi $L_w = L \cap M^{w^{-1}}$. Note that $w \in W(L_w, M_w)$. The constant term of the Eisenstein series $E_P(\varphi, \lambda)$ along $Q$ is given by

$$\sum_{w \in W^M\backslash W/W^L} E^Q_{Q_w}(M(w^{-1}, \lambda)\varphi_{P_w}, w^{-1}\lambda). \tag{1}$$

This is proved in [Mœglin and Waldspurger 1994, II.1.7] in the case $\varphi \in \mathcal{A}^{\text{cusp}}_P$, in which only the terms involving $w$ such that $L^w \supset M$ (that is, $M_w = M$) contribute. The proof easily extends to the general case — there are simply more contributions. Note that (1) is an identity of meromorphic functions on $\mathfrak{a}^*_{M,\mathbb{C}}$; the terms in (1) are absolutely convergent for $\operatorname{Re}\lambda \in \mathfrak{a}^*_{P,+}$ sufficiently regular.

It will also be useful to introduce the following notation for any $\varphi \in \mathcal{A}_P$, $w \in W(L; M)$, and $\lambda \in \mathfrak{a}^*_{M,\mathbb{C}}$:

$$B_Q(g, \varphi, w, \lambda) = (M(w^{-1}, \lambda)\varphi_{L^w})_{w^{-1}\lambda}(g)$$

so that $B_Q(\varphi, w, \lambda) \in \mathcal{A}_Q$. The following result is standard. For completeness we include the proof.

**Lemma 1.** *Suppose that $w \in W(L'; M)$ and $L \subseteq L'$. Then the constant term of $B_{Q'}(\varphi, w, \lambda)$ along $Q$ is $B_Q(\varphi, w, \lambda)$.*

*Proof.* Let $Q = LV$ (resp. $Q' = L'V'$, $R$, $R'$) be the parabolic subgroups with Levi parts $L$ (resp. $L'$, $L^w$, $L'^w$). Since $w \in W(L'; M)$ we have

$$(V \cap L')^w = N_R \cap L'^w.$$

The constant term of $B_{Q'}(\varphi, w, \lambda)$ along $Q$ is

$$\int_{(V\cap L')\backslash(V_{\mathbb{A}}\cap L'_{\mathbb{A}})} \int_{V'_{\mathbb{A}}\cap N^{w^{-1}}_{R',\mathbb{A}}\backslash V'_{\mathbb{A}}} (\varphi_{L'^w})_\lambda(wuv\,\cdot)\,du\,dv.$$

Since $V \cap L'$ normalizes both $V'_{\mathbb{A}}$ and $N^{w^{-1}}_{R',\mathbb{A}}$ we can change variables in $u$ to get

$$\int_{(V\cap L')\backslash(V_{\mathbb{A}}\cap L'_{\mathbb{A}})} \int_{V'_{\mathbb{A}}\cap N^{w^{-1}}_{R',\mathbb{A}}\backslash V'_{\mathbb{A}}} (\varphi_{L'^w})_\lambda(wvu\,\cdot)\,du\,dv$$

or

$$\int_{(N_R\cap L'^w)\backslash(N_{R,\mathbb{A}}\cap L'^w_{\mathbb{A}})} \int_{V'_{\mathbb{A}}\cap N^{w^{-1}}_{R',\mathbb{A}}\backslash V'_{\mathbb{A}}} (\varphi_{L'^w})_\lambda(vwu\,\cdot)\,du\,dv$$
$$= \int_{V'_{\mathbb{A}}\cap N^{w^{-1}}_{R',\mathbb{A}}\backslash V'_{\mathbb{A}}} (\varphi_{L^w})_\lambda(wu\,\cdot)\,du.$$

We have $V = V' \rtimes (V \cap L')$ and $N_R = N_{R'} \rtimes (N_R \cap L'^w)$. Therefore $N_R^{w^{-1}} = N_{R'}^{w^{-1}} \rtimes (V \cap L')$, and we can rewrite the integral above as

$$\int_{V_{\mathbb{A}}\cap N^{w^{-1}}_{R,\mathbb{A}}\backslash V_{\mathbb{A}}} (\varphi_{L^w})_\lambda(wu\,\cdot)\,du = (M(w^{-1},\lambda)\varphi_{L^w})_{w^{-1}\lambda}$$

as required. □

**2.1.** ***Truncation.*** For convenience we recall a few facts about Arthur's truncation operator $\Lambda^T$ [Arthur 1980]. For any $P \subseteq Q$, let $\tau_P^Q$ be the characteristic function of the Weyl chamber

$$(\mathfrak{a}_P^Q)_+ = \{X \in \mathfrak{a}_P^Q : \langle\alpha, X\rangle > 0 \text{ for all } \alpha \in \Delta_P^Q\}$$

and let $\hat{\tau}_P^Q$ be the characteristic function of the obtuse Weyl chamber

$$\Big\{\sum_{\alpha\in\Delta_P^Q} x_\alpha \alpha^\vee : x_\alpha > 0 \text{ for all } \alpha\Big\}.$$

We extend $\tau_P^Q$ and $\hat{\tau}_P^Q$ to $\mathfrak{a}_0$ by letting $\tau_P^Q(X) = \tau_P^Q(X_P^Q)$ and $\hat{\tau}_P^Q(X) = \hat{\tau}_P^Q(X_P^Q)$.

For $T$ sufficiently regular in $\mathfrak{a}_0^+$, the truncation operator is given by

$$\Lambda^T\varphi(g) = \sum_{P\supset P_0} (-1)^{r(P)-r(G)} \sum_{\gamma\in P\backslash G} \varphi_P(\gamma g)\hat{\tau}_P(H(\gamma g)-T)$$

for any locally bounded measurable function $\varphi$ on $G\backslash G^1_{\mathbb{A}}$. It defines an orthogonal projection on $L^2(G\backslash G^1_{\mathbb{A}})$. If $\varphi$ is of uniform moderate growth, then $\Lambda^T\varphi$ is rapidly decreasing.

More generally, for any $Q$, one defines the relative truncation with respect to $Q$ by

$$\Lambda^{T,Q}\varphi(g) = \sum_{P\in[P_0,Q]} (-1)^{r(P)-r(Q)} \sum_{\gamma\in P\backslash Q} \varphi_P(\gamma g)\hat{\tau}_P^Q(H(\gamma g)-T).$$

By the Langlands combinatorial lemma, we have the inversion formula

(2) $$\varphi_P(g) = \sum_{Q\subseteq P} \sum_{\gamma\in Q\backslash P} \Lambda^{T,Q}\varphi_Q(\gamma g)\tau_Q^P(H(\gamma g)-T)$$

[Arthur 1980, Lemma 1.5].

For any $\varphi\in\mathcal{A}_P$ and $Q\subseteq P$, we write $\mathcal{E}_Q(\varphi)=\mathcal{E}_Q(\varphi_Q)\subseteq\mathfrak{a}^*_{Q,\mathbb{C}}$ for the multiset of cuspidal exponents of $\varphi$ along $Q$ — see [Mœglin and Waldspurger 1994, I.3.4]. We also write $\mathcal{E}_{\subseteq P}(\varphi)=\bigcup_{Q\in[P_0,P]}\mathcal{E}_Q(\varphi)$. In the case $P=G$ we simply write $\mathcal{E}(\varphi)$ for $\mathcal{E}_{\subseteq G}(\varphi)$.

For a multiset $A=\{\lambda_1,\dots,\lambda_m\}\subseteq\mathfrak{a}^*_{0,\mathbb{C}}$ (including multiplicities) we write $\mathcal{PE}(A)$ for the space of polynomial exponential functions on $\mathfrak{a}_0$ with exponents $\subseteq A$. This means that any $f\in\mathcal{PE}(A)$ has the form

$$f(X)=\sum_{\lambda\in A} P_\lambda(X)e^{\langle\lambda,X\rangle},$$

where for any $\lambda\in A$, $P_\lambda$ is a polynomial in $\mathfrak{a}_0$ whose degree is smaller than the multiplicity of $\lambda$ in $A$. Equivalently, $f\in\mathcal{PE}(A)$ if and only if for any $v_1,\dots,v_m\in\mathfrak{a}_0$, $f$ is annihilated by the differential operator

$$\prod_{i=1}^m (D_{v_i}-\langle\lambda_i,v_i\rangle),$$

where $D_v$ denotes taking the partial derivative along $v\in\mathfrak{a}_0$. We also write $\mathcal{PE}_-=\mathcal{PE}(\mathcal{C}_{0,-}\setminus\{0\})$, where we limit the exponents $\lambda$ to $\mathcal{C}_{0,-}\setminus\{0\}$, but we do not limit the degree of $P_\lambda$.

The following lemma is a simple consequence of the properties of truncation.

**Lemma 2** [Lapid and Rogawski 2003, Proposition 8.4.1]. *For any automorphic forms $\varphi_i\in\mathcal{A}^n_G$, $i=1,2$, we have*

$$\langle\varphi_1,\Lambda^T\varphi_2\rangle_{G\backslash G^1_{\mathbb{A}}}\in\mathcal{PE}(\mathcal{E}(\varphi_1)+\overline{\mathcal{E}(\varphi_2)}).$$

*Moreover, if* $\varphi_1, \varphi_2 \in \mathcal{A}^2_G$ *then*

$$\langle \varphi_1, \Lambda^T \varphi_2 \rangle_{G\backslash G^1_{\mathbb{A}}} - \langle \varphi_1, \varphi_2 \rangle_{G\backslash G^1_{\mathbb{A}}} \in \mathcal{PE}_-.$$

We also recall the following elementary fact.

**Lemma 3.** *Let* $\mathscr{C} = \left\{\sum_{i=1}^m a_i v_i : a_1, \dots, a_m \geq 0\right\}$ *be a salient*[1] *polyhedral cone in a finite dimensional vector space* $V$ *over* $\mathbb{R}$ *(for some* $v_1, \dots, v_m \in V \setminus \{0\}$*). Then for any* $A \subseteq V^*$ *and* $f \in \mathcal{PE}(A)$ *the function*

$$\int_V \mathbf{1}_{\mathscr{C}}(X - T) f(X) e^{\langle \lambda, X \rangle}\, dX$$

*converges for* $\{\lambda \in V^*_{\mathbb{C}} : \operatorname{Re}\langle \lambda, v_i \rangle \ll 0, i = 1, \dots, m\}$ *and extends to a meromorphic function on* $V^*_{\mathbb{C}}$ *with hyperplane singularities. As a function of* $T$*, it belongs to* $\mathcal{PE}(A + \lambda)$.

This is a straightforward computation if $\mathscr{C}$ is simplicial. Otherwise, it follows by subdivision of $\mathscr{C}$ into simplicial subcones.

## 3. Cuspidal Eisenstein series

For the convenience of the reader we recall the results of Langlands and Arthur for cuspidal Eisenstein series.[2]

For any $w \in W(L, M)$ let $\phi^w_L$ be the function on $\mathfrak{a}^G_L$ given by

$$\phi^w_L\Big(\sum_{\alpha \in \Delta_Q} x_\alpha \alpha^\vee\Big) = \begin{cases} (-1)^{\#\{\alpha \in \Delta_Q : x_\alpha > 0\}} & \text{if } \{\alpha \in \Delta_Q : x_\alpha > 0\} = \{\alpha \in \Delta_Q : w\alpha < 0\}, \\ 0 & \text{otherwise.} \end{cases}$$

The Laplace transform of $\phi^w_L$ is given by

$$\int_{\mathfrak{a}^G_L} e^{\langle \lambda, X \rangle} \phi^w_L(X)\, dX = \frac{\operatorname{vol}(\mathfrak{a}^G_L / \mathbb{Z}\Delta^\vee_Q)}{\prod_{\alpha \in \Delta_Q} \langle \lambda, \alpha^\vee \rangle}, \quad \lambda \in (\mathfrak{a}^G_L)^*_{\mathbb{C}}, \quad \operatorname{Re} w\lambda \in \mathfrak{a}^*_{P,+}. \tag{3}$$

By [Arthur 1980, Lemma 4.1], for $\operatorname{Re}\lambda \in \mathfrak{a}^*_{P,+}$ sufficiently regular we have

$$\Lambda^T E_P(\varphi, \lambda) = \sum_{Q \in [P_0, G]} \sum_{w \in W(L,M)} \sum_{\gamma \in Q\backslash G} (M(w^{-1}, \lambda)\varphi)_{w^{-1}\lambda}(\gamma g) \phi^w_L(H(\gamma g) - T). \tag{4}$$

[1] That is, such that $\mathscr{C} \cap -\mathscr{C} = \{0\}$.

[2] A similar argument to the one below was given by Labesse in the 1983 Friday morning seminar on the twisted trace formula. See lecture 12 in http://www.math.ubc.ca/~cass/Langlands/friday/friday.html and [Labesse and Waldspurger 2012, §5.4].

Suppose that $\varphi_j \in \mathcal{A}^{\mathrm{cusp}}_{P_j}$, $j = 1, 2$. Set $f_i = E_{P_i}(\varphi_i, \lambda_i)$, $i = 1, 2$. Using (4) we write $\langle f_1, \Lambda^T f_2\rangle_{G\backslash G^1_{\mathbb{A}}}$ as the sum over $Q$ and $w_2 \in W(L, M_2)$ of

$$\Big\langle f_1, \sum_{\gamma\in Q\backslash G} (M(w_2^{-1}, \lambda_2)\varphi_2)_{w_2^{-1}\lambda_2}(\gamma g)\phi_L^{w_2}(H(\gamma g) - T)\Big\rangle_{G\backslash G^1_{\mathbb{A}}}$$

provided $\operatorname{Re}\lambda_2 \in \mathfrak{a}^*_{M_2,+}$ is sufficiently regular. (This will be justified in Lemma 14 below.) Each summand is equal to

$$\langle (f_1)_Q, (M(w_2^{-1}, \lambda_2)\varphi_2)_{w_2^{-1}\lambda_2}\phi_L^{w_2}(H(\cdot) - T)\rangle_{Q\backslash G^1_{\mathbb{A}}}.$$

Using the formula for the constant term, we get

$$(5)\quad (E_{P_1}(\varphi_1, \lambda_1), \Lambda^T E_{P_2}(\varphi_2, \lambda_2))_{G\backslash G^1_{\mathbb{A}}} = \sum_Q \sum_{w_1\in W(L,M_1)} \sum_{w_2\in W(L,M_2)} \big\langle (M(w_1^{-1}, \lambda_1)\varphi_1)_{w_1^{-1}\lambda_1}, (M(w_2^{-1}, \lambda_2)\varphi_2)_{w_2^{-1}\lambda_2}\phi_L^{w_2}(H(\cdot) - T)\big\rangle_{Q\backslash G^1_{\mathbb{A}}}.$$

Finally, using (3) we get

$$(E_{P_1}(\varphi_1, \lambda_1), \Lambda^T E_{P_2}(\varphi_2, \lambda_2))_{G\backslash G^1_{\mathbb{A}}} = \mathfrak{M}^T(\varphi_1, \lambda_1, \varphi_2, \lambda_2),$$

where

$$(6)\quad \mathfrak{M}^T(\varphi_1, \lambda_1, \varphi_2, \lambda_2) = \sum_Q \sum_{w_1\in W(L,M_1)} \sum_{w_2\in W(L,M_2)} \frac{e^{\langle w_1^{-1}\lambda_1 + w_2^{-1}\bar{\lambda}_2, T\rangle}}{\prod_{\alpha\in\Delta_Q}\langle w_1^{-1}\lambda_1 + w_2^{-1}\bar{\lambda}_2, \alpha^\vee\rangle} \operatorname{vol}(\mathfrak{a}_L^G/\mathbb{Z}\Delta_Q^\vee)\langle M(w_1^{-1}, \lambda_1)(\varphi_1), M(w_2^{-1}, \lambda_2)(\varphi_2)\rangle_{A_L V_{\mathbb{A}} L\backslash G_{\mathbb{A}}}.$$

These are the usual Maass–Selberg relations proved in [Arthur 1980, §4]. Note that the intricate residue calculus of [loc. cit.] is unnecessary.

## 4. Some combinatorial lemmas

In order to analyze the truncation of Eisenstein series and the Maass–Selberg relations in the general case we will need a few combinatorial definitions and lemmas in the spirit of [Arthur 1978, §6].

Let $L'$ and $M$ be standard Levi subgroups and let $w \in W(L'; M)$ and $Q \supset Q'$. For any $X \in \mathfrak{a}_0$ with $X^Q_{Q'} = \sum_{\alpha\in\Delta^Q_{Q'}} x_\alpha\alpha^\vee \in \mathfrak{a}^Q_{Q'}$ we set

$$D^Q_{Q',+}(X) = \{\alpha \in \Delta^Q_{Q'} : x_\alpha > 0\} \subseteq \Delta^Q_{Q'}.$$

Observe that for any $Q_2 \supset Q_1$, $D^Q_{Q_2,+}(X)$ consists of the nonzero projections of the elements of $D^Q_{Q_1,+}(X)$.

Also set

$$\phi^Q_{L',M,w}(X)=\begin{cases}(-1)^{|D^Q_{Q',+}(X)|} & \text{if } D^Q_{Q',+}(X)=\{\alpha\in\Delta^Q_{Q'}:w\alpha<0 \text{ or } w\alpha\in\Delta^M_{(L')^w}\},\\ 0 & \text{otherwise.}\end{cases}$$

Note that the condition $w\alpha \in \Delta^M_{(L')^w}$ is equivalent to $(w\alpha)_M = 0$.

As usual, we suppress the superscript if $Q = G$.

Note that if $w \in W(L, M)$ then $\phi_{L,M,w}$ is the function denoted by $\phi^w_L$ in the previous section. In particular, in this case

$$(7)\quad \int_{\mathfrak{a}^G_L} e^{\langle\lambda,X\rangle}\phi_{L,M,w}(X)\,dX = \frac{\operatorname{vol}(\mathfrak{a}^G_L/\mathbb{Z}\Delta^\vee_Q)}{\prod_{\alpha\in\Delta_Q}\langle\lambda,\alpha^\vee\rangle},\qquad \lambda\in(\mathfrak{a}^G_L)^*_{\mathbb{C}},\qquad \operatorname{Re} w\lambda\in\mathfrak{a}^*_{P,+}.$$

**Lemma 4.** *Suppose that $R \subseteq S \subseteq Q$ and $w \in W(M_S; M)$. Then*

$$\sum_{Q'\in[R,S]}\phi^Q_{L',M,w}(X)=\begin{cases}\phi^Q_{S,M,w}(X) & \text{if } D^Q_{R,+}(X)\cap\Delta^S_R=\varnothing,\\ 0 & \text{otherwise.}\end{cases}$$

*Proof.* We observe that for any $Q' \in [R, S]$ we have $\phi^Q_{L',M,w}(X) \neq 0$ if and only if $\phi^Q_{M_S,M,w}(X) \neq 0$ and $D^Q_{Q',+}(X) \supset \Delta^S_{Q'}$. In this case,

$$\phi^Q_{L',M,w}(X) = (-1)^{r(Q')-r(S)}\phi^Q_{M_S,M,w}(X).$$

The lemma follows from [Arthur 1978, Proposition 1.1]. □

We also recall the following version of Langlands' combinatorial lemma.

**Lemma 5** (Arthur). *Let $w \in W(L'; M)$ and $Q \supset Q'$. Then we have*

$$\sum_{R\in[Q',Q]}\phi^R_{L',M,w}(X)\tau^Q_R(X)=\begin{cases}1 & \text{if } w\alpha>0 \text{ and } w\alpha\notin\Delta^M_{(L')^w} \text{ for all } \alpha\in\Delta^Q_{Q'},\\ 0 & \text{otherwise.}\end{cases}$$

*In particular, taking $Q' = P$ and $w = 1$, for any $X \in \mathfrak{a}_P$ there exists a unique $Q \in [P, G]$ such that $\tau_Q(X) = 1$ and $X^Q \in \mathscr{C}^Q_{P,-}$. Moreover, $\langle\alpha, X\rangle > 0$ for any $\alpha \in \Delta_P \setminus \Delta^Q_P$ and $D_{P,+}(X) \supset \Delta_P \setminus \Delta^Q_P$.*

This follows from [Arthur 1978, Lemma 6.3] by taking $\Lambda = -w^{-1}\Lambda_0$ where $\Lambda_0 \in \mathfrak{a}^*_{M,+}$.

For nonnegative quantities $A$ and $B$ (depending on parameters) we will write $A \ll B$ if there exists a constant $c > 0$ (independent of the parameters) such that $A \le cB$.

**Lemma 6.** *Suppose that $P \in [R, Q]$, $X \in \mathfrak{a}^Q_R$, and*

$$D^Q_{R,+}(X)\cap\Delta^P_R=\{\alpha\in\Delta^P_R:\langle\alpha,X\rangle\le 0\}.$$

*Then $\|X\| \ll \|X_M\|$.*

*Proof.* Write $X=\sum_{\alpha\in\Delta_R} x_\alpha\alpha^\vee$ as $X_1+X_2$ where

$$X_1=\sum_{\alpha\in\Delta_R^P} x_\alpha\alpha^\vee \quad\text{and}\quad X_2=\sum_{\alpha\in\Delta_R\setminus\Delta_R^P} x_\alpha\alpha^\vee.$$

We have to show that under the conditions of the lemma we have $\|X_1\|\le C\|X_2\|$ for some constant $C$ which is independent of $X$. Let $S(X)$ be such that

$$\Delta_R^{S(X)}=\{\alpha\in\Delta_R^P:\langle\alpha,X\rangle>0\}=\Delta_R^P\setminus D_{R,+}^Q(X).$$

Fix $\lambda\in(\mathfrak{a}_R^{S(X)})_+^*$. Since the coefficients of $\lambda$ in the basis $\Delta_R^{S(X)}$ are positive, we have

$$0\le\langle\lambda,X\rangle=\langle\lambda,X_1\rangle+\langle\lambda,X_2\rangle.$$

On the other hand, we have $\lambda=\sum_{\varpi\in\hat\Delta_R^Q}\lambda_\varpi\varpi$ where $\lambda_\varpi>0$ for $\varpi\in\hat\Delta_R^Q\setminus\hat\Delta_{S(X)}^Q$ and $\lambda_\varpi\le 0$ for $\varpi\in\hat\Delta_{S(X)}^Q$. Thus,

$$\sum_{\alpha\in\Delta_R^{S(X)}}|x_\alpha|\ll-\langle\lambda,X_1\rangle.$$

(There are of course only finitely many possibilities for $S(X)$, so the dependence of the implied constant on $\lambda$ is immaterial.)

Similarly, fix $\mu\in(\mathfrak{a}_R^{S'(X)})_+^*$ where

$$\Delta_R^{S'(X)}=\Delta_R^P\setminus\Delta_R^{S(X)}=\Delta_R^P\cap D_{R,+}^Q(X).$$

Then

$$\langle\mu,X_1\rangle\le-\langle\mu,X_2\rangle$$

and

$$\sum_{\alpha\in\Delta_R^{S'(X)}}|x_\alpha|\ll\langle\mu,X_1\rangle.$$

Thus, $\langle\mu-\lambda,X_1\rangle\le\langle\lambda-\mu,X_2\rangle$ while $\|X_1\|\ll\langle\mu-\lambda,X_1\rangle$. The claim follows. □

As before, fix $P$ and $Q$. For any $R\subseteq Q$ and $w\in W(M_R;M)$ define

$$\chi_{M_R,M,w}^Q(X)=\sum_{Q'\in[R,Q]:w\in W(L';M)}\tau_R^{Q'}(X)\phi_{L',M,w}^Q(X).\tag{8}$$

**Lemma 7.** *We have*

$$\chi_{M_R,M,w}^Q(X)=\begin{cases}(-1)^{|D_{R,+}^Q(X)|} & \text{if } D_{R,+}^Q(X)=\{\alpha\in\Delta_R^Q:w\alpha<0 \text{ or}\\ & \qquad (w\alpha\in\Delta_{M_R^w}^M \text{ and } \langle\alpha,X\rangle\le0)\},\\ 0 & \text{otherwise.}\end{cases}$$

*Furthermore, if* $\chi_{M_R,M,w}^Q(X)\neq0$ *and* $X\in\mathfrak{a}_R^Q$ *then* $wX\in\mathscr{C}_{M_R^w,-}$ *and* $\|X\|\ll\|(wX)_M\|$.

*Proof.* Let $R_1 \in [R, Q]$ be the parabolic subgroup such that

$$\Delta_R^{R_1} = \{\alpha \in \Delta_R^Q : w\alpha \in \Delta_{M_R^w}^M\}$$

so that $w \in W(L'; M)$ if and only if $Q' \in [R, R_1]$. Let $R_2 \in [R, Q]$ be such that

$$\Delta_R^{R_2} = \{\alpha \in \Delta_R^Q : \langle \alpha, X \rangle > 0\}$$

so that $\tau_R^{Q'}(X) = 1$ if and only if $Q' \subseteq R_2$. Let $S = R_1 \cap R_2$. Then

$$\chi_{M_R,M,w}^Q(X) = \sum_{Q' \in [R,S]} \phi_{L',M,w}^Q(X).$$

The first part now follows from Lemma 4.

In order to prove the second part, let $X \in \mathfrak{a}_R^Q$ and write $X = \sum_{\alpha \in \Delta_R^Q} x_\alpha \alpha^\vee$. Suppose that $\chi_{M_R,M,w}^Q(X) \neq 0$ and let

$$A = \{\alpha \in \Delta_R^Q : w\alpha \in \Delta_{M_R^w}^M \text{ and } \langle \alpha, X \rangle \le 0\}.$$

Let $Q_1$ be the parabolic subgroup with Levi $M_R^w$. We write

$$wX = \sum_{\alpha \in A} x_\alpha w\alpha^\vee + \sum_{\alpha \notin A} x_\alpha w\alpha^\vee$$

and observe that the first sum is a linear combination of roots in $wA \subseteq \Delta_{Q_1}^M$ with positive coefficients, while the second sum lies in $\mathscr{C}_{Q_1,-}$. Thus, $D_{Q_1,+}(wX) \subseteq wA$. Using Lemma 5, let $R_1 \in [Q_1, G]$ be such that $\tau_{R_1}(wX) = 1$ and $(wX)^{R_1} \in \mathscr{C}_{Q_1,-}^{R_1}$. Then $\langle \alpha, wX \rangle > 0$ for all $\alpha \in \Delta_{Q_1} \setminus \Delta_{Q_1}^{R_1}$ and $D_{Q_1,+}(wX) \supset \Delta_{Q_1} \setminus \Delta_{Q_1}^{R_1}$. In particular, $\Delta_{Q_1} \setminus \Delta_{Q_1}^{R_1} \subseteq wA$. On the other hand, from the definition of $A$, we have $\langle w\alpha, wX \rangle = \langle \alpha, X \rangle \le 0$ for any $\alpha \in A$. It follows that $R_1 = G$, that is, $wX \in \mathscr{C}_{Q_1,-}$ as required.

It remains to show that $\|X\| \ll \|(wX)_M\|$ if $X \in \mathfrak{a}_R^Q$ and $\chi_{M_R,M,w}^Q(X) \neq 0$. Write $X = X_1 + X_2$ where

$$X_1 = \sum_{\alpha \in \Delta_R^Q : w\alpha \in \Delta_{M_R^w}^M} x_\alpha \alpha^\vee \quad \text{and} \quad X_2 = \sum_{\alpha \in \Delta_R^Q : w\alpha \notin \Delta_{M_R^w}^M} x_\alpha \alpha^\vee.$$

We can apply Lemma 6 (with $L \cap M^{w^{-1}}$ instead of $M$) to infer that $\|X_1\| \ll \|X_2\|$. On the other hand, since

$$(wX_2)_M = \sum_{\alpha \in \Delta_R^Q : w\alpha \notin \Delta_{M_R^w}^M} x_\alpha (w\alpha^\vee)_M$$

and each $w\alpha^\vee$ has the opposite sign of $x_\alpha$, we conclude that $\|X_2\| \ll \|(wX_2)_M\|$. Our claim follows. □

**Corollary 8.** *For any* $k$, *we have*

$$\int_{\mathfrak{a}_R^G} \chi_{M_R,M,w}(X) e^{k\|X\|+\langle w^{-1}\lambda, X\rangle}\, dX < \infty$$

*for any* $\lambda \in \mathfrak{a}^*_{M,+}$ *sufficiently regular* (*depending on* $k$).

For the rest of the section, suppose that we are given $Q$, $M_i$, and $w_i \in W(L; M_i)$, $i = 1, 2$.

**Corollary 9.** *For any* $k$ *and* $Q \subseteq Q_2$ *we have*

$$\int_{\mathfrak{a}_Q^G} \chi^{Q_2}_{L,M_1,w_1}(X) \chi_{L_2,M_2,w_2}(X) e^{k\|X\|+\langle w_1^{-1}\lambda_1 + w_2^{-1}\lambda_2, X\rangle}\, dX < \infty$$

*provided that* $\lambda_1 \in \mathfrak{a}^*_{M_1,+}$ *is sufficiently regular* (*depending on* $k$) *and* $\lambda_2 \in \mathfrak{a}^*_{M_2,+}$ *is sufficiently regular* (*depending on* $\lambda_1$ *and* $k$).

*Proof.* It follows from Lemma 7 that for any $C > 0$ we have

$$-\langle \lambda_2, w_2 X\rangle \geq C\|X_{Q_2}\|$$

if $\chi_{L_2,M_2,w_2}(X) \neq 0$ provided that $\lambda_2 \in \mathfrak{a}^*_{M_2,+}$ is sufficiently regular (depending on $C$, but not on $X$). Similarly, for any $C > 0$ we have

$$-\langle \lambda_1, w_1 X\rangle = -\langle \lambda_1, w_1 X^{Q_2}\rangle - \langle \lambda_1, w_1 X_{Q_2}\rangle \geq C\|X^{Q_2}\| - C_2\|X_{Q_2}\|$$

if $\chi^{Q_2}_{L,M_1,w_1}(X) \neq 0$ provided that $\lambda_1 \in \mathfrak{a}^*_{M_1,+}$ is sufficiently regular, depending on $C$, but not on $X$, and with $C_2$ depending only on $\lambda_1$. Thus for any $C$, we have

$$-\langle w_1^{-1}\lambda_1 + w_2^{-1}\lambda_2, X\rangle \geq C\|X\|$$

if $\chi^{Q_2}_{L,M_1,w_1}(X)\chi_{L_2,M_2,w_2}(X) \neq 0$, provided that $\lambda_1 \in \mathfrak{a}^*_{M_1,+}$ is sufficiently regular (depending on $C$) and $\lambda_2 \in \mathfrak{a}^*_{M_2,+}$ is sufficiently regular (depending on $\lambda_1$ and $C$). The corollary follows. □

We define

$$\Psi_{L,M_1,w_1,M_2,w_2}(X) = \sum_{Q_2 \supset Q\,:\, w_2 \in W(L_2; M_2)} \chi^{Q_2}_{L,M_1,w_1}(X)\chi_{L_2,M_2,w_2}(X).$$

We can explicate the function $\Psi_{L,M_1,w_1,M_2,w_2}$ as follows.

**Proposition 10.** *Let* $R_i$, $i = 1, 2$, *be such that*

$$\Delta_Q^{R_i} = \{\alpha \in \Delta_Q : w_i\alpha \in \Delta^{M_i}_{L^{w_i}}\}$$

*and let* $R_1'$ *be such that*

$$\Delta_Q^{R_1'} = \{\alpha \in \Delta_Q : w_1\alpha > 0 \text{ and } w_1\alpha \notin \Delta^{M_1}_{L^{w_1}}\}.$$

*Then*

$$\Psi_{L,M_1,w_1,M_2,w_2}(X)=\begin{cases}(-1)^{|D_{Q,+}(X)|} & \text{if } D_{Q,+}(X)=\{\alpha\in\Delta_Q : w_2\alpha<0\}\\ & \quad\cup(\Delta_Q^{R_2}\setminus(\{\alpha\in\Delta_Q^{R_1}:\langle\alpha,X\rangle>0\}\cup\Delta_Q^{R_1'})),\\ 0 & \text{otherwise.}\end{cases}$$

Note that $\Psi_{L,M_1,w_1,M_2,w_2}\not\equiv\Psi_{L,M_2,w_2,M_1,w_1}$.

*Proof.* Note that for $L_i\supset L$, we have $w_i\in W(L_i;M_i)$ if and only if $Q_i\subseteq R_i$, $i=1,2$. Thus, upon substituting (8) for $\chi$, we get that $\Psi_{L,M_1,w_1,M_2,w_2}(X)$ is equal to

$$\sum_{Q_2\in[Q,R_2]}\ \sum_{Q_1\in[Q,Q_2\cap R_1]}\tau_Q^{Q_1}(X)\phi_{Q_1,M_1,w_1}^{Q_2}(X)\sum_{Q_2'\in[Q_2,R_2]}\tau_{Q_2}^{Q_2'}(X)\phi_{Q_2',M_2,w_2}(X).$$

We write this differently as

$$\sum_{Q_1\in[Q,R_1]}\tau_Q^{Q_1}(X)\sum_{Q_2'\in[Q_1,R_2]}\phi_{Q_2',M_2,w_2}(X)\sum_{Q_2\in[Q_1,Q_2']}\phi_{Q_1,M_1,w_1}^{Q_2}(X)\tau_{Q_2}^{Q_2'}(X).$$

By Lemma 5, we get

$$\Psi_{L,M_1,w_1,M_2,w_2}(X)=\sum_{Q_1\in[Q,R_1]}\tau_Q^{Q_1}(X)\sum_{Q_2'\in[Q_1,R_2\cap Q_1^\sharp]}\phi_{Q_2',M_2,w_2}(X),$$

where $Q_1^\sharp$ is such that

$$\Delta_{Q_1}^{Q_1^\sharp}=\{\alpha\in\Delta_{Q_1}:w_1\alpha>0\text{ and }w_1\alpha\notin\Delta_{L_1^{w_1}}^{M_1}\}.$$

Observe that $\Delta_{Q_1}^{Q_1^\sharp}$ consists of the projections of $\Delta_Q^{R_1'}$, that is,

$$\Delta_Q^{Q_1^\sharp}=\Delta_Q^{Q_1}\cup\Delta_Q^{R_1'}$$

(disjoint union). In particular, $Q^\sharp=R_1'$. Thus, by Lemma 4, we get

$$\Psi_{L,M_1,w_1,M_2,w_2}(X)=\sum\tau_Q^{Q_1}(X)\phi_{R_2\cap Q_1^\sharp,M_2,w_2}(X),$$

where the sum is over $Q_1\in[Q,R_1\cap R_2]$ such that $D_{Q_1,+}(X)\cap\Delta_{Q_1}^{R_2\cap Q_1^\sharp}=\varnothing$, or equivalently, $Q_1\in[S_1(X),R_1\cap R_2]$ where

$$\Delta_Q^{S_1(X)}=\Delta_Q^{R_2\cap R_1'}\cap D_{Q,+}(X).$$

On the other hand, let $S_2(X)$ be such that

$$\Delta_Q^{S_2(X)}=\{\alpha\in\Delta_Q:\langle\alpha,X\rangle>0\}.$$

Then $\tau_Q^{Q_1}(X)=1$ if and only if $Q_1\subseteq S_2(X)$. All in all, we get

$$\sum_{Q_1\in[S_1(X),R_1\cap R_2\cap S_2(X)]}\phi_{R_2\cap Q_1^\sharp,M_2,w_2}(X).$$

Note that since $R_1\cap R_1'=Q$ and $S_1(X)\subseteq R_1'$, we have $S_1(X)\subseteq R_1\cap R_2\cap S_2(X)$ if and only if $S_1(X)=Q$. In this case, the map $Q_1\mapsto R_2\cap Q_1^\sharp$ is a bijection between $[S_1(X),R_1\cap R_2\cap S_2(X)]$ and $[R_2\cap R_1', S_2'(X)]$ where $S_2'(X)=R_2\cap(R_1\cap R_2\cap S_2(X))^\sharp$. We thus get (assuming $S_1(X)=Q$)

$$\sum_{Q_1'\in[R_2\cap R_1',S_2'(X)]}\phi_{Q_1',M_2,w_2}(X).$$

Invoking Lemma 4 once again, we get that

$$\Psi_{L,M_1,w_1,M_2,w_2}(X)=\phi_{S_2'(X),M_2,w_2}(X)$$

if $S_1(X)=Q$ and

$$D_{R_2\cap R_1',+}(X)\cap\Delta_{R_2\cap R_1'}^{S_2'(X)}=\varnothing. \tag{9}$$

Otherwise, $\Psi_{L,M_1,w_1,M_2,w_2}(X)=0$. We can rewrite condition (9) equivalently as

$$D_{Q,+}(X)\cap\Delta_Q^{S_2'(X)}\subseteq\Delta_Q^{R_2\cap R_1'}.$$

Once again, since $R_1\cap R_1'=Q$, this becomes

$$\Delta_Q^{R_1\cap R_2\cap S_2(X)}\cap D_{Q,+}(X)=\varnothing.$$

The proposition follows. □

**Corollary 11.** *For any k we have*

$$\int_{\mathfrak{a}_Q^G}\Psi_{L,M_1,w_1,M_2,w_2}(X)e^{k\|X\|+\langle w_1^{-1}\lambda_1+w_2^{-1}\lambda_2,X\rangle}\,dX<\infty$$

*provided that* $\lambda_1\in\mathfrak{a}_{M_1,+}^*$ *is sufficiently regular* (*depending on* $k$) *and* $\lambda_2\in\mathfrak{a}_{M_2,+}^*$ *is sufficiently regular* (*depending on* $\lambda_1$ *and* $k$). *Moreover, for any* $f_i\in\mathcal{PE}(A_i)$, $i=1,2$,

$$\int_{\mathfrak{a}_Q^G}\Psi_{L,M_1,w_1,M_2,w_2}(X-T)f_1(w_1X)f_2(w_2X)e^{\langle w_1^{-1}\lambda_1+w_2^{-1}\lambda_2,X\rangle}\,dX$$

*has meromorphic continuation for* $\lambda_i\in\mathfrak{a}_{M_i,\mathbb{C}}^*$, $i=1,2$, *with hyperplane singularities, and as a function of* $T$, *it belongs to* $\mathcal{PE}(w_1^{-1}A_1+w_2^{-1}A_2+w_1^{-1}\lambda_1+w_2^{-1}\lambda_2)$.

*Proof.* The first part follows from Corollary 9 and the defining expression for $\Psi_{L,M_1,w_1,M_2,w_2}$. Alternatively, we can deduce it from Proposition 10 as follows. Suppose that

$$X = \sum_{\alpha \in \Delta_Q} x_\alpha \alpha^\vee$$

and $\Psi_{L,M_1,w_1,M_2,w_2}(X) \neq 0$. Write $X = X_1 + X_2 + X_3$ where

$$X_1 = \sum_{\alpha \in \Delta_Q \setminus \Delta_Q^{R_2}} x_\alpha \alpha^\vee, \quad X_2 = \sum_{\alpha \in \Delta_Q^{R_2} \setminus \Delta_Q^{R_1}} x_\alpha \alpha^\vee, \quad \text{and} \quad X_3 = \sum_{\alpha \in \Delta_Q^{R_1 \cap R_2}} x_\alpha \alpha^\vee.$$

By Proposition 10, the coefficients $x_\alpha$ in $X_1$ are positive precisely when $w_2\alpha < 0$, the coefficients in $X_2$ are positive precisely when $w_1\alpha < 0$, and the coefficients in $X_3$ are positive precisely when $\langle \alpha, X \rangle \leq 0$. Then $w_2 X = w_2 X_1 + w_2(X_2 + X_3)$ where $w_2 X_1 \in \mathscr{C}_{0,-}$ and $w_2(X_2 + X_3) \in w_2 \mathfrak{a}_Q^{R_2} \subseteq \mathfrak{a}_0^{M_2}$. Thus, $\langle \lambda_2, w_2 X \rangle = \langle \lambda_2, w_2 X_1 \rangle$. Note that the kernel of the map $X \mapsto (w_2 X)_{M_2}$ is $\mathfrak{a}_Q^{R_2}$. Therefore, for any $C_1 > 0$, we have

$$-\langle w_2^{-1} \lambda_2, X \rangle = -\langle \lambda_2, w_2 X \rangle \geq C_1 \|X_1\|$$

provided that $\lambda_2 \in \mathfrak{a}_{M_2,+}^*$ is sufficiently regular (depending on $C_1$, but not on $X$).

We also have $w_1 X_2 \in \mathscr{C}_{0,-}$ and $\langle \lambda_1, w_1 X \rangle = \langle \lambda_1, w_1 X_1 \rangle + \langle \lambda_1, w_1 X_2 \rangle$. By the same reasoning, we infer that for any $C_2 > 0$, we have

$$-\langle w_1^{-1} \lambda_1, X \rangle = -\langle \lambda_1, w_1 X \rangle \geq C_2 \|X_2\| - C_3 \|X_1\|$$

for all $\lambda_1 \in \mathfrak{a}_{M_1,+}^*$ sufficiently regular (depending on $C_2$ but not on $X$) where $C_3$ depends on $\lambda_1$ but not on $X$.

Thus for any $C > 0$ and for $\lambda_1 \in \mathfrak{a}_{M_1,+}^*$ sufficiently regular (depending on $C$) and $\lambda_2 \in \mathfrak{a}_{M_2,+}^*$ sufficiently regular (depending on $C$ and $\lambda_1$), we have

$$-\langle \lambda_1, w_1 X \rangle - \langle \lambda_2, w_2 X \rangle \geq C \|X_1 + X_2\|. \tag{10}$$

On the other hand, by Lemma 6, it follows that $\|X_3\| \ll \|X_1 + X_2\|$ on the support of $\Psi_{L,M_1,w_1,M_2,w_2}$. Thus we can replace the right-hand side of (10) by $C\|X\|$. The first part of the corollary follows.

The second part follows from Lemma 3. □

## 5. Truncation of a general Eisenstein series

We will use the notation of the previous sections.

We have the following generalization of (4).

**Lemma 12.** *For* $\operatorname{Re}\lambda \in \mathfrak{a}^*_{P,+}$ *sufficiently regular we have*

$$\Lambda^{T,Q} E_P(g,\varphi,\lambda) = \sum_{Q'\in[P_0,Q],w\in W(L';M)} \sum_{\gamma\in Q'\backslash Q} B_{Q'}(\gamma g,\varphi,w,\lambda)\phi^Q_{L',M,w}(H(\gamma g)-T). \tag{11}$$

*Proof.* Let $E = E_P(\varphi,\lambda)$. Then

$$\Lambda^{T,Q} E_P(g,\varphi,\lambda) = \sum_{P'\in[P_0,Q]} (-1)^{r(P')-r(Q)} \sum_{\gamma\in P'\backslash Q} (E_P(\varphi,\lambda))_{P'}(\gamma g)\hat\tau^Q_{P'}(H(\gamma g)-T).$$

Using (1) for the constant term of Eisenstein series we get

$$\sum_{P'\in[P_0,Q]} (-1)^{r(P')-r(Q)} \sum_{\gamma\in P'\backslash Q} \sum_{w\in W^M\backslash W/W^{M'}} E^{P'}_{P'_w}(\gamma g, M(w^{-1},\lambda)\varphi_{P_w}, w^{-1}\lambda) \times \hat\tau^Q_{P'}(H(\gamma g)-T),$$

where $P_w$ (resp. $P'_w$) is the standard parabolic with Levi part $M_w = M\cap M'^w$ (resp. $M'_w = M'\cap M^{w^{-1}}$). Unfolding the Eisenstein series, we get

$$\sum_{P'\in[P_0,Q]} (-1)^{r(P')-r(Q)} \sum_{w\in W^M\backslash W/W^{M'}} \sum_{\gamma\in P'_w\backslash Q} B_{P_w}(\gamma g,\varphi,w,\lambda)\hat\tau^Q_{P'}(H(\gamma g)-T).$$

The sum is absolutely convergent by the assumption on $\lambda$. For any $w\in W^M\backslash W/W^{M'}$, we have $w\in W(M'_w;M)$. Therefore, we may rearrange the sums differently as

$$\sum_{Q'\in[P_0,Q]} \sum_{w\in W(L';M)} \sum_{\gamma\in Q'\backslash Q} B_{Q'}(\gamma g,\varphi,w,\lambda) \sum_{P'\in[P_0,Q]:w\in W/W^{M'},P'_w=Q'} (-1)^{r(P')-r(Q)}\hat\tau^Q_{P'}(H(\gamma g)-T). \tag{12}$$

It remains to analyze the inner sum. Fix $Q'$, $w\in W(L';M)$, $\gamma\in Q'\backslash Q$, and $g\in G_{\mathbb{A}}$. Let $X = H(\gamma g)-T$ and let $R\in[Q',Q]$ be the parabolic subgroup such that $\hat\Delta^Q_R = \{\varpi\in\hat\Delta^Q_{Q'} : \langle\varpi,X\rangle > 0\}$. Note that $\hat\tau^Q_{P'}(H(\gamma g)-T) = 1$ if and only if $P'\in[R,Q]$. On the other hand, we can rewrite the conditions $w\in W/W^{M'}$ and $P'_w = Q'$ as $P'\in[Q',S]$ where $S\in[Q',Q]$ is such that

$$\Delta^S_{Q'} = \{\alpha\in\Delta^Q_{Q'} : w\alpha > 0 \text{ but } w\alpha\notin\Delta^M_{L'^w}\}.$$

We infer that the inner sum of (12) is nonzero only if $S = R$ and this happens exactly when $\phi^Q_{L',M,w}(X)\neq 0$. In this case, $\phi^Q_{L',M,w}(X) = (-1)^{r(S)-r(Q)}$. The lemma follows. □

The lemma just proved is not so useful as it stands, for in practice, it may be difficult to work analytically with the right-hand side of (11) since the constant terms of $\varphi$ are not rapidly decreasing in general. We seek a similar expression where $B_{Q'}$ is replaced by a function which is rapidly decreasing on $L'\backslash {L'}^1_{\mathbb{A}}$. To that end we will use the inversion formula (2) to rewrite the right-hand side of (11) as

$$\sum_{Q'\in[P_0,Q]} \sum_{w\in W(L';M)} \sum_{\gamma\in Q'\backslash Q} \sum_{R\subseteq Q'} \sum_{\delta\in R\backslash Q'} \Lambda^{T,R} B_{Q'}(\delta\gamma g,\varphi,w,\lambda) \\ \tau_R^{Q'}(H(\delta\gamma g)-T)\phi^Q_{L',w,M}(H(\gamma g)-T).$$

Applying Lemma 1 (with $R$ instead of $Q$) and combining the sums over $\gamma$ and $\delta$, we get:

**Proposition 13.** *With $\chi$ given by Lemma 7,*

$$\Lambda^{T,Q} E_P(\varphi,\lambda) \\ = \sum_{R\in[P_0,Q]} \sum_{w\in W(M_R;M)} \sum_{\gamma\in R\backslash Q} \Lambda^{T,R} B_R(\gamma g,\varphi,w,\lambda)\chi^Q_{M_R,M,w}(H(\gamma g)-T).$$

## 6. Maass–Selberg relations

We will use Proposition 13 to obtain the Maass–Selberg relations in this context. First we need a lemma.

**Lemma 14.** *Let $f$ be a function of moderate growth on $G\backslash G^1_{\mathbb{A}}$ and let $\varphi$ be a function of moderate growth on $QV_{\mathbb{A}}\backslash G_{\mathbb{A}}$ which is rapidly decreasing in $L\backslash L^1_{\mathbb{A}}\times K$. Then for any $w\in W(L;M)$ and for* $\operatorname{Re}\lambda\in\mathfrak{a}^*_{M,+}$ *sufficiently regular, we have*

$$\Big\langle f, \sum_{\gamma\in Q\backslash G} \varphi_{w^{-1}\lambda}(\gamma g)\chi_{L,M,w}(H(\gamma g)-T)\Big\rangle_{G\backslash G^1_{\mathbb{A}}} \\ = \langle f_Q, \varphi_{w^{-1}\lambda}\chi_{L,M,w}(H(\cdot)-T)\rangle_{Q\backslash G^1_{\mathbb{A}}}.$$

*Proof.* This is the usual unfolding. In order to justify it, we need to show the convergence of

$$\int_{Q\backslash G^1_{\mathbb{A}}} |f|_Q(g)|\varphi_{w^{-1}\lambda}(g)\chi_{L,M,w}(H(g)-T)|\,dg.$$

We use Iwasawa decomposition to write this as

$$\int_K\int_{\mathfrak{a}^G_L}\int_{L\backslash L^1_{\mathbb{A}}} |f|_Q(e^X lk)\,|\varphi(e^X lk)|\,\delta_Q(e^X)^{-1}e^{\operatorname{Re}\langle w^{-1}\lambda,X\rangle}|\chi_{L,M,w}(X-T)|\,dl\,dX\,dk.$$

By the moderate growth of $f$ and $\varphi$, there exist $c$ and $N$ such that

$$|f|_Q(e^X lk)|\varphi(e^X lk)|\le c(e^{\|X\|}\|l\|)^N,\quad X\in\mathfrak{a}^G_L,\ l\in L_{\mathbb{A}},\ k\in K.$$

The convergence follows from the rapid decay of $\varphi$ in $L\backslash L^1_{\mathbb{A}}$ and Corollary 8. □

**Proposition 15.** *We have the identity* (*in the sense of meromorphic continuation*)

$$(13)\quad \langle E_{P_1}(\varphi_1,\lambda_1),\Lambda^T E_{P_2}(\varphi_2,\lambda_2)\rangle_{G\backslash G^1_{\mathbb{A}}}=\sum_{Q}\sum_{w_1\in W(L;M_1)}\sum_{w_2\in W(L;M_2)}$$
$$\langle\Lambda^{T,Q}B_Q(\varphi_1,w_1,\lambda_1),B_Q(\varphi_2,w_2,\lambda_2)\Psi_{L,M_1,w_1,M_2,w_2}(H(\cdot)-T)\rangle_{Q\backslash G^1_{\mathbb{A}}},$$

*where each summand converges for* $\operatorname{Re}\lambda_1\in\mathfrak{a}^*_{M_1,+}$ *sufficiently regular and* $\operatorname{Re}\lambda_2\in\mathfrak{a}^*_{M_2,+}$ *sufficiently regular* (*depending on* $\operatorname{Re}\lambda_1$) *and as a function of* $T$ *belongs to*

$$\mathcal{PE}(\mathcal{E}_{\subseteq L^{w_1}}(\varphi_1)+\overline{\mathcal{E}_{\subseteq L^{w_2}}(\varphi_2)}+w_1^{-1}\lambda_1+w_2^{-1}\bar{\lambda}_2).$$

Interestingly, because of the asymmetry of $\Psi$, the individual terms on the right-hand side are *not* invariant (up to complex conjugation) under interchanging $\varphi_i$, $w_i$, and $M_i$.

*Proof.* Set $f_i=E_{P_i}(\varphi_i,\lambda_i)$, $i=1,2$. Using Proposition 13 we write $\langle f_1,\Lambda^T f_2\rangle_{G\backslash G^1_{\mathbb{A}}}$ as the sum over $Q_2$ and $w_2\in W(L_2;M_2)$ of

$$\Big\langle f_1,\sum_{\gamma\in Q_2\backslash G}\Lambda^{T,Q_2}B_{Q_2}(\gamma g,\varphi_2,w_2,\lambda_2)\chi_{L_2,M_2,w_2}(H(\gamma g)-T)\Big\rangle_{G\backslash G^1_{\mathbb{A}}}$$

provided that each term is defined. By Lemma 14, this is indeed the case for $\operatorname{Re}\lambda_2\in\mathfrak{a}^*_{M_2,+}$ sufficiently regular and each summand is equal to

$$\langle (f_1)_{Q_2},\Lambda^{T,Q_2}B_{Q_2}(\varphi_2,w_2,\lambda_2)\chi_{L_2,M_2,w_2}(H(\cdot)-T)\rangle_{Q_2\backslash G^1_{\mathbb{A}}}.$$

This is equal to

$$\langle\Lambda^{T,Q_2}f_1,B_{Q_2}(\varphi_2,w_2,\lambda_2)\chi_{L_2,M_2,w_2}(H(\cdot)-T)\rangle_{Q_2\backslash G^1_{\mathbb{A}}}.$$

Using Proposition 13 once more, we obtain the sum over $Q_1\in[P_0,Q_2]$ and $w_1\in W(L_1;M_1)$ of

$$\Big\langle\sum_{\gamma\in Q_1\backslash Q_2}\Lambda^{T,Q_1}B_{Q_1}(\gamma g,\varphi_1,w_1,\lambda_1)\chi^{Q_2}_{L_1,M_1,w_1}(H(\gamma g)-T),$$
$$B_{Q_2}(\varphi_2,w_2,\lambda_2)\chi_{L_2,M_2,w_2}(H(\cdot)-T)\Big\rangle_{Q_2\backslash G^1_{\mathbb{A}}}.$$

Using the argument of Lemma 14 together with Corollary 9 and applying Lemma 1 we get

$$\sum_{Q_1\subseteq Q_2}\sum_{w_1\in W(L_1;M_1)}\sum_{w_2\in W(L_2;M_2)}\langle\Lambda^{T,Q_1}B_{Q_1}(\varphi_1,w_1,\lambda_1)\chi^{Q_2}_{L_1,M_1,w_1}(H(\cdot)-T),$$
$$B_{Q_1}(\varphi_2,w_2,\lambda_2)\chi_{L_2,M_2,w_2}(H(\cdot)-T)\rangle_{Q_1\backslash G^1_{\mathbb{A}}}.$$

Upon rewriting, we obtain (13) from the definition of $\Psi_{L,M_1,w_1,M_2,w_2}$. The last part follows from Corollary 11 and Lemma 2. □

**Remark 16.** The careful reader would have noticed that the exact description of $\Psi_{L,M_1,w_1,M_2,w_2}$ provided by Proposition 10 was not really used in the argument above. It will be of interest to describe the Laplace transform of $\Psi_{L,M_1,w_1,M_2,w_2}$ explicitly, thereby explicating further the Maass–Selberg relations above. We will not go in this direction in this paper. We mention, however, the following special case: the volume olume of the truncated fundamental domain, namely $\langle 1, \Lambda^T 1\rangle_{G\backslash G_A^1}$, was computed explicitly in [Kim and Weng 2007].

If $\varphi_j \in \mathcal{A}_{P_j}^{\mathrm{cusp}}$, the identity (13) reduces to (5), which is equal to the expression $\mathfrak{M}^T(\varphi_1, \lambda_1, \varphi_2, \lambda_2)$ defined in (6).

In the case where $\varphi_j \in \mathcal{A}_{P_j}^2$ we recover Arthur's asymptotic result.

**Proposition 17** [Arthur 1982]. *Suppose $\varphi_j \in \mathcal{A}_{P_j}^2$ and $\lambda_j \in i\mathfrak{a}_{M_j}^*$, $j = 1, 2$. Then*

$$\langle E_{P_1}(\varphi_1, \lambda_1), \Lambda^T E_{P_2}(\varphi_2, \lambda_2)\rangle_{G\backslash G_{\mathbb{A}}^1} = \mathfrak{M}^T(\varphi_1, \lambda_1, \varphi_2, \lambda_2) + \mathfrak{E}^T(\varphi_1, \lambda_1, \varphi_2, \lambda_2),$$

*where*

$$\mathfrak{E}^T(\varphi_1, \lambda_1, \varphi_2, \lambda_2) \in \mathcal{PE}_-.$$

*Proof.* Consider the right-hand side of (13). Each summand belongs to $\mathcal{PE}_-$ unless $w_1 \in W(L, M_1)$ and $w_2 \in W(L, M_2)$. In this case, the summand is equal to

$$\langle (\Lambda^{T,Q}(M(w_1^{-1}, \lambda_1)\varphi_1)_{w_1^{-1}\lambda_1}, (M(w_2^{-1}, \lambda_2)\varphi_2)_{w_2^{-1}\lambda_2}\phi_{Q,M_2,w_2}(H(\cdot) - T)\rangle_{Q\backslash G_{\mathbb{A}}^1}.$$

The proposition follows from Lemma 2 applied with $L$ instead of $G$, using the Iwasawa decomposition and (7) □

As in [Lapid 2011, §8] one can infer from Proposition 17 the holomorphy of $E(\varphi, \lambda)$ on $\lambda \in i\mathfrak{a}_M^*$ for any $\varphi \in \mathcal{A}_P^2$. Moreover, for any smooth compactly supported function $\varphi : i\mathfrak{a}_M^* \to \mathcal{A}_P^2$ with values in a finite-dimensional subspace of $\mathcal{A}_P^2$, define the Eisenstein integral

$$\Theta_{P,\varphi} = \int_{i\mathfrak{a}_M^*} E(\varphi(\lambda), \lambda)\, d\lambda.$$

Then $\Theta_{P,\varphi} \in L^2(G\backslash G_{\mathbb{A}}^1)$ and

$$\text{(14)}\quad \langle \Theta_{P,\varphi}, \Theta_{P',\varphi'}\rangle_{G\backslash G_{\mathbb{A}}^1} = \int_{i\mathfrak{a}_M^*} \sum_{w \in W(M,M')} \langle M(w, \lambda)\varphi(\lambda), \varphi'(w\lambda)\rangle_{A_{M'}U'_{\mathbb{A}}M'\backslash G_{\mathbb{A}}}\, d\lambda.$$

We note that the argument in [Lapid 2011, §8] depends on the second half of [ibid., §7] (which is elementary), but is otherwise self-contained.

We can write (14) more symmetrically as follows. For any parabolic subgroup $R$, write

$$\varphi_{\#}^{P,R}(\lambda) = \sum_{w \in W(P,R)} M(w, w^{-1}\lambda)\varphi(w^{-1}\lambda), \quad \lambda \in i\mathfrak{a}_R^*.$$

By the properties of the intertwining operators, we have

$$\varphi_{\#}^{P,Q}(s\lambda, g) = M(s,\lambda)\varphi_{\#}^{P,R}(\lambda, g)$$

for any $s \in W(R, Q)$. Therefore, for any $Q$ and $s \in W(P, Q)$ we can write the right-hand side of (14) as

$$\int_{i\mathfrak{a}_Q^*} \langle M(s, s^{-1}\mu)\varphi(s^{-1}\mu), \varphi'^{P',Q}_{\#}(\mu)\rangle_{A_L V_{\mathbb{A}} L\backslash G_{\mathbb{A}}}\, d\mu.$$

Averaging over $Q$ and $s$ we get

$$\langle \Theta_{P,\varphi}, \Theta_{P',\varphi'}\rangle_{G\backslash G^1_{\mathbb{A}}} = n(\mathfrak{a}_P)^{-1} \sum_Q \int_{i\mathfrak{a}_Q^*} \langle \varphi_{\#}^{P,Q}(\mu), \varphi'^{P',Q}_{\#}(\mu)\rangle_{A_L V_{\mathbb{A}} L\backslash G_{\mathbb{A}}}\, d\mu,$$

where $n(\mathfrak{a}_P) = \sum_Q |W(P, Q)|$ is the number of chambers for $\mathfrak{a}_P$ [Arthur 1978, p. 919].

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Received May 25, 2012. Revised June 29, 2012.

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Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)



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The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 969 Evans Hall, Berkeley, CA 94720-3840, is published monthly except July and August. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFlow® from Mathematical Sciences Publishers.

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Typeset in LaTeX

# PACIFIC JOURNAL OF MATHEMATICS

Volume 260 No. 2 December 2012

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Special issue
devoted to the memory of Jonathan Rogawski