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ON EQUALITY OF ARITHMETIC AND ANALYTIC FACTORS THROUGH LOCAL LANGLANDS CORRESPONDENCE

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To the memory of Jonathan Rogawski

In this article we pursue the problem of equality of Artin factors with those defined on the representation theoretic (analytic) side by the local Langlands correspondence. We propose a set of axioms for the factors on the analytic side which allows us to prove the equality of the factors. In the case of L-functions the equality can be proved in a number of cases appearing in the Langlands–Shahidi method since one of the axioms, stability under highly ramified twists, is already available for the L-functions coming from this method.

Introduction

The local Langlands correspondence (LLC) for $\mathrm{GL}(n)$ is formulated through the equality of the Artin factors attached to tensor products on the Galois side with the factors defined on the representation theoretic side, namely those of Rankin–Selberg product L-functions for $\mathrm{GL}(n) \times \mathrm{GL}(m)$ [Jacquet et al. 1983; Shahidi 1984]. The LLC, which was proved for $\mathrm{GL}(n)$ in [Harris and Taylor 2001; Henniart 2000], also suggests that other Artin (or arithmetic) factors should be equal to their representation theoretic (or analytic) counterparts, if they exist. In fact, one important fact about analytic objects is that they always correspond to a global theory and thus are of automorphic significance. On the other hand so long as the problem of global parametrization or the global Langlands correspondence, a problem whose formulation is still unavailable [Langlands 2012], is not settled, one cannot expect to produce a global theory of L-functions from those defined by local Artin factors. The problem is thus to show the equality of Artin factors with the corresponding analytic ones whenever LLC is available.

The purpose of this article is to formulate a set of axioms to be satisfied by the objects on the analytic side attached to every representation r of the L-group so

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as to imply the equality of arithmetic (Artin) factors with analytic (representation theoretic or automorphic) factors through LLC (Theorem 2.1). This formalizes and generalizes some ideas of Harris [1998] as pursued later by Henniart [2010].

While the equality of γ-functions requires the validity of stability (Axiom 2), our Theorem 3.1 proves the equality of L-functions in certain special cases coming from Langlands–Shahidi method [Shahidi 1990; 2010] through LLC with no assumptions. They include the cases of twisted exterior square L-functions for $\mathrm{GL}(n)$ as well as twisted exterior cube for GL(6). This equality can be used to prove special cases of the generic Arthur packet conjecture [Arthur 1984; Shahidi 2011] as we explain in Section 3. Finally in Section 4 we address the issue of stability of γ-factors within our method and discuss the progress made on it and some of its consequences.

1. Axiomatic r-theory

Let G be a connected reductive algebraic group over a local field F of characteristic zero. Denote by ${}^LG$ its L-group. Let W'_F be the Weil–Deligne group of F. Let $\rho : W'_F \to {}^LG$ be an admissible homomorphism (see [Arthur 1984; Shahidi 2011]). Let r be an irreducible complex representation of ${}^LG$ on a finite dimensional complex vector space V, i.e., $r : {}^LG \to \mathrm{GL}(V)$ is an analytic homomorphism. Then $r \cdot \rho : W'_F \to \mathrm{GL}(V)$ defines a representation of W'_F, which we assume to be Frobenius-semisimple.

Let us now assume we have a theory of L-functions attached to r. More precisely, assume that for each irreducible admissible representation π of $G(F)$, there are defined an L-function $L(s, \pi, r)$ and an ε-factor $\varepsilon(s, \pi, r, \psi_F)$, where $s \in \mathbb{C}$ and ψ_F is a nontrivial additive character of F, satisfying (1) multiplicativity (additivity), (2) stability under highly ramified character twists, (3) a global functional equation whenever π becomes a local component of a global cusp form, and (4) archimedean matching, each of which we shall now explain. It is best to formulate them in terms of γ-functions

$$\gamma(s, \pi, r, \psi_F) = \varepsilon(s, \pi, r, \psi_F)L(1-s, \pi, \tilde{r})/L(s, \pi, r).$$

1) ***Multiplicativity.*** This basically expresses γ-functions of a particular constituent of an induced representation as a product of γ-functions for the inducing data. One special and important case of it is that of Langlands quotients [Langlands 1989; Silberger 1978]. If π is an irreducible admissible representation of $G(F)$, then Langlands classification determines a standard parabolic subgroup P with a Levi decomposition $P = MN$ and a quasitempered representation σ of $M(F)$, in the "positive Weyl chamber", such that $\pi = J(P, \sigma)$. Here $J(P, \sigma)$ is the unique irreducible quotient of $I(P, \sigma)$, which is the representation of $G(k)$ induced by σ. Note that fixing the minimal parabolic subgroup $P_0 \subset P$, making P standard,

automatically determines the unique positive Weyl chamber. Now, let

$$\iota : {}^L M \hookrightarrow {}^L G$$

be the natural embedding. Let $\rho_M : W_F' \to {}^L M$ be the parameter defining σ (or its L-packet), if known. Then $\rho = \iota \cdot \rho_M$ will be the parameter for π. Let r be a finite dimensional irreducible complex representation of ${}^L G$ as before. Decompose

$$r \cdot \iota = \bigoplus_j r_j^M \tag{1-1}$$

into its irreducible constituents. Multiplicativity in this case simply requires

$$\gamma(s, \pi, r, \psi_F) = \prod_j \gamma(s, \sigma, r_j^M, \psi_F), \tag{1-2}$$

$$L(s, \pi, r) = \prod_j L(s, \sigma, r_j^M), \tag{1-3}$$

$$\varepsilon(s, \pi, r, \psi_F) = \prod_j \varepsilon(s, \sigma, r_j^M, \psi_F). \tag{1-4}$$

In fact, this is how these factors are defined: One first defines the factors for quasitempered but unitary data and then extends the unitary complex parameters to all of the complex dual of the complex Lie algebra of the split component of the center of M [Langlands 1989; Shahidi 1990]. When F is an archimedean field, LLC was established by Langlands [1989] and the L-functions were defined to be those of Artin attached to the parameter. They satisfy Equations (1-2)–(1-4).

When one restricts oneself to those representations r that appear in constant terms of Eisenstein series (Langlands–Shahidi method [Langlands 1971a; 1976; Shahidi 2010]), in which case G will be assumed to be quasisplit, then these formulas play a central role. In fact, what is defined with no reservations is the γ-function $\gamma(s, \pi, r_i', \psi_F)$, where r_i' is any irreducible constituent of the adjoint action of ${}^L M'$ on ${}^L \mathfrak{n}'$, the Lie algebra of the complex Lie group ${}^L N'$ [Langlands 1971a; Shahidi 1990; 2010]. The representation π is any irreducible admissible ψ_F-generic representation of $M'(F)$, where $P' = M'N'$ is the defining parabolic subgroup for the Eisenstein series which we may assume to be maximal. Here F is a completion of the number field defining the Eisenstein series. As explained in [Shahidi 1990; 2010], the knowledge of γ-factors immediately defines the L-functions and ε-factors if π is also tempered. The extension to any irreducible admissible representation (not necessarily generic) π of $M'(F)$ is given by Langlands classification and Equations (1-3) and (1-4) [Shahidi 1990]. In this case multiplicativity is valid when π is the unique ψ_F-generic constituent of $\mathrm{Ind}_{M(F)N(F)}^{M'(F)} \sigma \otimes \mathbf{1}$, where P is any standard parabolic subgroup of M' defined over F and σ is any irreducible admissible ψ_F-generic representation of $M(F)$, $P = MN$. One then has the appropriate version of (1-2) for each γ-function $\gamma(s, \pi, r_i', \psi_F)$, where r_i' is an irreducible constituent of the adjoint action of ${}^LM'$ on ${}^L\mathfrak{n}'$ [Shahidi 1990; 2010].

Example. Assume $G = \mathrm{GL}(n_1 + n_2)$ and $M = \mathrm{GL}(n_1) \times \mathrm{GL}(n_2)$. Let r_N be Λ^2, the exterior square representation of $\mathrm{GL}(N, \mathbb{C})$ for any positive integer N. Then one has

$$ {}^LM = \mathrm{GL}(n_1, \mathbb{C}) \times \mathrm{GL}(n_2, \mathbb{C}) $$

and

$$ r_{n_1+n_2} | {}^LM = r_{n_1} \oplus r_{n_2} \oplus (\rho_{n_1} \otimes \rho_{n_2}), \tag{1-5} $$

where ρ_N is the standard representation of $\mathrm{GL}(N, \mathbb{C})$. If π is the Langlands quotient or the unique irreducible generic constituent of $\mathrm{Ind}_{M(F)N(F)}^{G(F)} \sigma_1 \otimes \sigma_2 \otimes \mathbf{1}$, where σ_i, $i = 1, 2$, is an irreducible generic representation of $\mathrm{GL}_{n_i}(F)$, which we will assume to be quasitempered in the positive Weyl chamber if π is the Langlands quotient, then

$$ \gamma(s, \pi, \Lambda^2, \psi_F) = \gamma(s, \sigma_1, \Lambda^2, \psi_F)\gamma(s, \sigma_2, \Lambda^2, \psi_F)\gamma(s, \sigma_1 \times \sigma_2, \psi_F). $$

Here $\gamma(s, \sigma_1 \times \sigma_2, \psi_F)$ is the Rankin–Selberg product γ-function defined in [Jacquet et al. 1983]. It is also obtained from the Langlands–Shahidi method if we consider $M' = \mathrm{GL}(n_1) \times \mathrm{GL}(n_2)$ inside $G = \mathrm{GL}(n_1 + n_2)$; see [Shahidi 1984].

One simple way of seeing the branching rule (1-5) is to consider $M' = \mathrm{GL}_{n_1+n_2}$ as the Siegel Levi subgroup of $G = \mathrm{SO}(2n_1 + 2n_2)$. Here one gets only one irreducible representation r_1' of ${}^LM' = \mathrm{GL}(n_1 + n_2, \mathbb{C})$ in ${}^L\mathfrak{n}'$, $r_1' = \Lambda^2_{n_1+n_2}$. One can then immediately see the restriction decomposition (branching rule) (1-5) if one considers the adjoint action of ${}^LM = \mathrm{GL}_{n_1}(\mathbb{C}) \times \mathrm{GL}_{n_2}(\mathbb{C})$ on ${}^L\mathfrak{n}'$ which is isomorphic to (the second diagonal) skew-symmetric elements of complex matrices of size $n_1 + n_2$.

Finally we remark that if one knows LLC and lets ρ be the parameter of π, and further assume the equality

$$ \gamma(s, \pi, r, \psi_F) = \gamma(s, r \cdot \rho, \psi_F), \tag{1-6} $$

where the factor on the right is that of Artin attached to the representation $r \cdot \rho$, then one immediately has

$$ \begin{aligned} \gamma(s, \pi, r, \psi_F) &= \gamma(s, r \cdot \rho, \psi_F) \\ &= \gamma(s, r \cdot \iota \cdot \rho_M, \psi_F) \\ &= \gamma(s, \oplus r_j^M \cdot \rho_M, \psi_F) \\ &= \prod_j \gamma(s, r_j^M \cdot \rho_M, \psi_F) = \prod_j \gamma(s, \sigma, r_j^M, \psi_F), \end{aligned} $$

where σ is a member of the L-packet attached to ρ_M. This immediately implies (1-2). The point is that even if one knows LLC, one would know the equality (1-6) only for certain r [Langlands 1971a; Shahidi 1990; 2010] and not necessarily for the family of L-functions attached to a given r. In practice one would need to know multiplicativity for γ-functions $\gamma(s, \pi, r, \psi_F)$ on the representation theoretic side in order to prove (1-6) for a given r.

2) ***Stability.*** This is again a local statement. Moreover, F will need to be assumed to be nonarchimedean. We also need to assume $X(G)_F \neq \{\mathbf{1}\}$, i.e., that G has a nontrivial F-rational character. This clearly rules out G being semisimple. Choose and fix $\mathbf{1} \neq \nu \in X(G)_F$. Note that $\nu(G(F)) \subset F^*$ is of finite index and thus open. Let χ be a highly ramified character of F^*. Then $\chi \cdot \nu$ is what we call a *highly ramified character* of $G(F)$.

Let π_1 and π_2 be two irreducible admissible representations of $G(F)$. Let ω_{π_i} denote the central character of π_i, $i = 1, 2$. Stability requires:

Assume $\omega_{\pi_1} = \omega_{\pi_2} = \omega$. Then for every sufficiently highly ramified character χ of $G(F)$ with the level of ramification depending on π_1 and π_2, one has

$$\gamma(s, \pi_1 \otimes \chi, r, \psi_F) = \gamma(s, \pi_2 \otimes \chi, r, \psi_F), \tag{1-7}$$

$$L(s, \pi_1 \otimes \chi, r) = L(s, \pi_2 \otimes \chi, r) \equiv 1, \tag{1-8}$$

and thus

$$\varepsilon(s, \pi_1 \otimes \chi, r, \psi_F) = \varepsilon(s, \pi_2 \otimes \chi, r, \psi_F). \tag{1-9}$$

By virtue of [Deligne 1973], stability is valid for all the Artin factors, and as in multiplicativity, stability will also be true for our factors (see [Cogdell, Shahidi and Tsai $\geq$ 2012]) if LLC is valid and moreover our factors are equal to those of Artin. But again stability is a tool which is needed to prove this equality which is known in only a few cases.

At present this is the only result that needs to be established even in the context of L-functions that come from the Langlands–Shahidi method [Shahidi 2002; 2010], although special cases of it are available from either methods of L-functions. More precisely, stability is known for the Rankin product factors $\gamma(s, \pi_1 \times \pi_2, \psi_F)$, where π_1 and π_2 are irreducible admissible representations of $\mathrm{GL}(n_1, F)$ and $\mathrm{GL}(n_2, F)$, respectively [Jacquet and Shalika 1985], or of $\mathrm{GL}_1(F) = F^*$ and $G(F)$, whenever G is a group for which the derived group of ${}^LG^0$ is a classical group [Cogdell et al. 2001; 2004; 2005; 2008; Kim and Krishnamurthy 2005; Asgari and Shahidi 2006; 2011].

On the other hand, in the context of L-functions in [Langlands 1989; Shahidi 1990; 2010], a stability statement for L-functions to the effect that

$$L(s, \pi \otimes \chi, r_i) \equiv 1$$

for every L-function obtained from our method, and suitably highly ramified characters χ, was proved in [Shahidi 2000]. Thus it is the stability of γ-functions $\gamma(s, \pi, r_i, \psi_F)$ which needs to be proved in a given case. We will discuss this problem shortly.

We conclude by pointing out that in the case of $G \times \mathrm{GL}(1)$ discussed above stability has been an important tool to prove functorial transfers from the generic spectrum of $G(\mathbb{A}_k)$ to appropriate $\mathrm{GL}(N, \mathbb{A}_k)$ [Cogdell et al. 2001; 2004; Kim and Krishnamurthy 2005; Asgari and Shahidi 2006; 2011]. Here $\mathbb{A}_k$ is the ring of adeles of a number field k.

3) ***Functional equations.*** The main reason for introducing local Artin root numbers (ε-factors) in [Dwork 1956; Langlands 1970; 1971b; Deligne 1973] was to decompose Artin's global root numbers and ε-factors into products of local objects. Under the validity of LLC, these local Artin factors can be used to define local factors attached to irreducible admissible representations (L-packets) of groups over local fields. On the other hand if one considers cuspidal automorphic forms over a global number field, then for each r one expects global functional equations whose root numbers will have to be a product of local ones. One thus needs to define a collection of local ε-factors and L-functions within the same machinery that establishes the global functional equations [Jacquet et al. 1983; Cogdell and Piatetski-Shapiro 2004; Shahidi 1990; 2010]. It is thus by no means clear that these factors are equal to those defined by Artin factors through LLC, and the challenge is to show that they are in fact equal. This is done by using these global functional equations, but for a very special class of cusp forms, those attached to certain irreducible continuous representations of global Galois (or Weil) group. We now formulate this as follows.

Let k be a global field whose ring of adeles is $\mathbb{A}_k$ and let $\pi = \bigotimes_v \pi_v$ be an automorphic cuspidal representation of $G(\mathbb{A}_k)$, where G is a connected reductive group over k. Let r be an irreducible complex analytic representation (*thus finite dimensional and conversely*) *of ${}^LG$. Let $\eta_v : {}^LG_v \to {}^LG$ be the natural map, where ${}^LG_v$ is the L-group of G as a group over k_v. Write $r_v = r \cdot \eta_v$. Let S be a finite set of places of k such that for all $v \notin S$ both the group G, as a group over k_v, and π_v are unramified. Fix a complex number s. Let $L(s, \pi_v, r_v)$ and $\varepsilon(s, \pi_v, r_v, \psi_v)$ be the local L-function and root number attached to this data from our theory, where $\psi = \bigotimes_v \psi_v$ is a nontrivial additive character of $\mathbb{A}/k$ with ψ_v unramified outside S. Set*

$$L(s, \pi, r) = \prod_v L(s, \pi_v, r_v), \tag{1-10}$$

and

$$\varepsilon(s,\pi,r)=\prod_v \varepsilon(s,\pi_v,r_v,\psi_v), \tag{1-11}$$

where (1-10) converges absolutely for $\mathrm{Re}\, s \gg 0$ *while (1-11) is just a finite product. Then*

$$L(s,\pi,r)=\varepsilon(s,\pi,r)L(1-s,\pi,\tilde{r}). \tag{1-12}$$

Here $\tilde{r}$ *denotes the contragredient of* r. *In terms of* γ*-functions this can be written as*

$$L^S(s,\pi,r)=\prod_{v\in S}\gamma(s,\pi_v,r_v,\psi_v)L^S(1-s,\pi,\tilde{r}), \tag{1-13}$$

where

$$L^S(s,\pi,r)=\prod_{v\notin S} L(s,\pi_v,r_v). \tag{1-14}$$

Here by an unramified group we mean a quasisplit group to split over an unramified extension. It will then have a hyperspecial maximal compact subgroup with respect to which π_v *has an invariant* (*one dimensional*) *subspace if* π_v *is unramified.*

There are a good number of cases where these functional equations are proved. The most general results here are those in the Langlands–Shahidi method, using Eisenstein series [Langlands 1989; Shahidi 1990; 2010]. On the other hand, they are also proved using the method of integral representations in a number of cases, most notably and completely by Jacquet, Piatetski–Shapiro and Shalika for Rankin product L-functions for $\mathrm{GL}(n_1)\times\mathrm{GL}(n_2)$ as discussed earlier [Jacquet et al. 1983; Cogdell and Piatetski-Shapiro 2004]. We refer to [Soudry 2006] for a survey of the results obtained from the integral representations method for other groups.

4) ***Archimedean matching.*** When k is a number field one has the benefit of using the Langlands classification [Langlands 1989; Silberger 1978] and thus LLC for real groups to define local factors at archimedean primes to be those of Artin through LLC. The theory must then require:

Let F *be either* $\mathbb{R}$ *or* $\mathbb{C}$ *and, for each irreducible admissible representation* π *of* $G(F)$, *let* $\rho : W_F \to {}^LG$ *be the corresponding parameter. Then for each finite dimensional irreducible complex representation* r *of* ${}^LG$ *we have*

$$\gamma(s,r\cdot\rho,\psi_F)=\gamma(s,\pi,r,\psi_F).$$

We also have similar identities for root numbers and L*-functions.*

Again, the most general case of this is proved within the context of the Langlands–Shahidi method [Shahidi 1990; 2010]. The work is carried on in [Shahidi 1985] when "local coefficients" are expressed as Artin factors. We recall that the γ-factors within this method are defined inductively by these local coefficients.

We refer to [Jacquet and Shalika 1990; Cogdell and Piatetski-Shapiro 2004] for the archimedean work within Rankin–Selberg theory for $\mathrm{GL}(n)$.

In the case of function fields, where no distinguished archimedean place stands out, other techniques are needed to develop the theory. We refer to L. Lomelí's work in [Lomelí 2009; Henniart and Lomelí 2011], where the method is developed at least for classical groups.

Definition 1.1. Let F be a local field together with a nontrivial additive character ψ and let G be a connected reductive group over F. Fix a (finite dimensional) complex analytic representation r of ${}^LG$. We will say we have a *theory of L-functions attached to r*, or in short an *r-theory*, if there exist complex functions $L(s,\pi,r)$ and $\varepsilon(s,\pi,r,\psi_F)$ satisfying axioms 1–4.

2. Equality of Artin (arithmetic) and automorphic (analytic) factors

With notation as in the previous section, let

$$\theta : {}^LG \hookrightarrow \mathrm{GL}(N,\mathbb{C}) \times W'_F$$

be a minimal embedding. Let r be a finite dimensional complex representation

$$r : \mathrm{GL}(N,\mathbb{C}) \times W'_F \to \mathrm{Aut}\, V.$$

Let $\rho : W'_F \to {}^LG$ be an admissible homomorphism and let $\pi(\rho)$ be a fixed element in the L-packet attached to ρ. Then

$$\begin{aligned}\gamma(s, r\cdot\theta\cdot\rho, \psi) &= \gamma(s, \pi(\rho), r\cdot\theta, \psi)\\ &= \gamma(s, \pi(\theta\cdot\rho), r, \psi)\end{aligned}$$

if the middle factor $\gamma(s,\pi(\rho), r\cdot\theta,\psi)$ is defined. Here $\pi(\theta\cdot\rho)$ is the representation of $\mathrm{GL}(N,F)$ attached to $\theta\cdot\rho$ as in [Harris and Taylor 2001; Henniart 2000]. In particular, r-factors for $\mathrm{GL}(N,F)$ define $r\cdot\theta$-factors for $G(F)$. We may therefore, at least for $r\cdot\theta$-factors of the group G, appeal to r-factors of $\mathrm{GL}(N)$.

Let us therefore concentrate on $\mathrm{GL}(N)$, where LLC is already established [Harris and Taylor 2001; Henniart 2000]. Assume our theory of γ-factor axioms (1)–(4) of the previous section. We thus consider a parameter $\rho : W'_F \to \mathrm{GL}(N,\mathbb{C})$ and let $\pi(\rho)$ be the corresponding irreducible admissible representation of $\mathrm{GL}_N(F)$ through LLC.

If ρ_1 and ρ_2 are two homomorphisms (representations) of W'_F,

$$\rho_i : W'_F \to \mathrm{GL}(n_i, \mathbb{C}),$$

we let r be a representation of $\mathrm{GL}(n_1+n_2, \mathbb{C})$ and assume a branching rule of the form

$$r \cdot (\rho_1 \oplus \rho_2) = r \cdot \rho_1 \oplus r \cdot \rho_2 \oplus R(\rho_1, \rho_2), \tag{2-1}$$

where $R(\rho_1, \rho_2)$ is a representation of $\mathrm{GL}(n_1, \mathbb{C}) \times \mathrm{GL}(n_2, \mathbb{C})$, $n_i = \dim \rho_i$, $i = 1, 2$, in which $r \cdot \rho_1$ and $r \cdot \rho_2$ do not appear; or said in other terms, they appear in $r \cdot (\rho_1 \oplus \rho_2)$ with multiplicity one. We can in fact write $R(\rho_1, \rho_2)$ as the composite of

$$R : \mathrm{GL}(n_1, \mathbb{C}) \times \mathrm{GL}(n_2, \mathbb{C}) \to \mathrm{GL}(N, \mathbb{C}),$$

$N = \dim R$, and

$$\begin{aligned}(\rho_1, \rho_2) : W'_F &\to \mathrm{GL}(n_1, \mathbb{C}) \times \mathrm{GL}(n_2, \mathbb{C})\\ w &\mapsto (\rho_1(w), \rho_2(w)).\end{aligned}$$

We note that

$$\rho_1 \oplus \rho_2 : W'_F \to \mathrm{GL}(n_1, \mathbb{C}) \times \mathrm{GL}(n_2, \mathbb{C}) \hookrightarrow \mathrm{GL}(n_1+n_2, \mathbb{C}),$$

to which r can be applied. Here are some examples. Let $r = \Lambda^2$, in which case $R(\rho_1, \rho_2) = \rho_1 \otimes \rho_2$, or $r = \Lambda^3$, for which $R(\rho_1, \rho_2) = \Lambda^2\rho_1 \otimes \rho_2 \oplus \rho_1 \otimes \Lambda^2\rho_2$. Similar examples can be given for Sym^3 or higher powers of both Λ and Sym [Fulton and Harris 1991]. We recall that exterior powers are irreducible representations of highest weight δ_i, fundamental weights of $\mathrm{SL}(N, \mathbb{C})$. We will then assume that we also have

$$\gamma(s, R \cdot (\rho_1, \rho_2), \psi_F) = \gamma(s, (\pi(\rho_1), \pi(\rho_2)), R, \psi_F), \tag{2-2}$$

which of course requires the validity of an R-theory for $\mathrm{GL}(n_1) \times \mathrm{GL}(n_2)$.

Tracing through the tables in [Langlands 1971a; Shahidi 1988; 2010], it can be seen that the existence of corresponding R-theories for Λ^3 may be available within the same machinery, at least for $n \le 6$ as we explain in the next section.

Now, fix a representation r with an r-theory and assume one has an R-theory for the representation R appearing in (2-1). We will briefly sketch how to show:

Theorem 2.1. *Fix r satisfying branching rule (2-1). Assume the existence of an r-theory and the corresponding R-theory for R satisfying (2-2). Then*

$$\gamma(s, r \cdot \rho, \psi_F) = \gamma(s, \pi(\rho), r, \psi_F)$$

for every n-dimensional continuous complex Frobenius-semisimple representation ρ of W'_F, where $\pi(\rho)$ is the irreducible admissible representation of $\mathrm{GL}_n(F)$ *attached to ρ by LLC.*

Proof. We pursue the ideas presented in [Harris 1998; Henniart 2010]. By Brauer's theorem ρ is a $\mathbb{Z}$-linear combination of monomial representations. Thus monomial representations, i.e., those induced from characters of subgroups of finite index in W'_F, form a basis for the Grothendieck ring of W'_F. Starting with a local monomial representation ρ, one chooses a global monomial representation $\tilde{\rho}$ which has ρ as $\tilde{\rho}|W'_F$, where $F = k_v$ at one place of the global field k as in [Harris 1998; Henniart 2010; Cogdell, Shahidi and Tsai $\geq$ 2012]. For each place w of k, let $\tilde{\rho}_w = \tilde{\rho}|W'_{k_w}$, and consider $\pi(\tilde{\rho}) := \bigotimes_w \pi(\tilde{\rho}_w)$, where $\pi(\tilde{\rho}_w)$ is the representation of $\mathrm{GL}(n, k_w)$ attached to $\tilde{\rho}_w$ by LLC. (We remind the reader that there are serious restrictions present in the choices of k and ρ as explained in [Harris 1998; Henniart 2010].)

Then $\pi(\tilde{\rho})$ is an automorphic representation of $\mathrm{GL}_n(\mathbb{A}_k)$, given by an automorphic induction from a grössencharacter. We then twist $\pi(\tilde{\rho})$ by a grössencharacter $\tilde{\chi} = \bigotimes_w \tilde{\chi}_w$ that is highly ramified at all finite places where $\pi(\tilde{\rho}_w)$ is ramified except at v. By stability we get

$$\gamma(s, r_w \cdot (\tilde{\rho}_w \otimes \tilde{\chi}_w), \tilde{\psi}_w) = \gamma(s, \pi(\tilde{\rho}_w) \otimes \tilde{\chi}_w, r_w, \tilde{\psi}_w),$$

which can be seen by computing each side, using a principal series with the same central character as $\pi(\tilde{\rho}_w)$ on the representation theoretic side and [Deligne 1973] on the Artin side.

We will assume $\tilde{\chi}_v \equiv 1$. By archimedean matching the factors are equal whenever $w = \infty$. Comparing functional equations for $\tilde{\rho}$ and $\pi(\tilde{\rho})$, we get

$$\gamma(s, r \cdot \rho, \psi_F) = \gamma(s, \pi(\rho), r, \psi_F)$$

for every member of a basis for the Grothendieck ring of W'_F. Here $\psi_F = \psi_v$ for a global nontrivial character $\psi = \bigotimes_w \psi_w$ of $k\backslash\mathbb{A}_k$.

Next we appeal to our R-theory satisfying (2-2), and multiplicativity, to extend the equality to the full Grothendieck ring. This completes our sketch of the proof. $\square$

3. Equality of L-functions through LLC

While the equality of γ-factors in Theorem 2.1 requires availability of stability for them, stability for L-functions, expressed as Equation (1-8), is a lot less subtle. In what follows, we will show the equality of L-functions defined by the Langlands–Shahidi method with those of Artin in a number of cases previously not available.

A result like this has an interesting application in proving the generic A-packet conjecture discussed in [Shahidi 2011]. This is a kind of converse to the tempered L-packet conjecture, which asserts that every tempered L-packet of a quasisplit group has a generic member [Shahidi 1990; Vogan 1978]. On the other hand, the generic A-packet conjecture states that if the L-packet attached to ϕ_ψ, the Langlands parameter attached to an Arthur parameter ψ, has a generic member, then ϕ_ψ is tempered. We note that the elements of ϕ_ψ are supposed to provide the main nontempered members of ψ (see [Arthur 1984]), i.e., those which have not already appeared in other A-packets. The proof given in [Shahidi 2011] is based on the matching of only L-functions for certain Levi factors through LLC.

The work of Y. Kim [2012], where he uses the matching for the twisted exterior and symmetric square L-functions for $\mathrm{GL}(n)$ [Henniart 2010] and those of certain Rankin product ones [Asgari and Shahidi 2006; 2011], has now established this for split GSpin groups, generalizing the work of Ban [2006] and Liu [2011] for classical groups. Moreover, the examples of Λ^3 discussed below should handle some cases of exceptional groups. More precisely, using [Shahidi 2011] the work in [Kim 2012] proves that if ψ is an Arthur packet for $\mathrm{GSpin}(F)$, where F is a p-adic field, then the Langlands packet ϕ_ψ attached to ψ has a generic member only if ϕ_ψ is tempered. This clearly gives a converse to the tempered (or generic) L-packet conjecture [Shahidi 1990; Vogan 1978]. For an archimedean field F this is proved in [Shahidi 2011] and follows from the equality of Artin factors with those defined by the Langlands–Shahidi method [Langlands 1989; Shahidi 1985]. Here is now the matching theorem for L-functions:

Theorem 3.1. *Let (G, M) be a pair of a quasisplit connected reductive group and one of its maximal Levi subgroups defined over a local field F. Assume there exists a homomorphism $\varphi : M \to \mathrm{GL}(n) \times \mathrm{GL}(1)$ that is an isomorphism on derived groups, i.e., $M_D \simeq \mathrm{SL}(n)$. Let $\pi = \pi_0 \otimes \eta$ be an irreducible admissible representation of $\mathrm{GL}(n, F) \times F^*$ and consider it as one of $M(F)$. Assume $\pi = \pi(\rho)$, $\rho : W_F' \to \mathrm{GL}(n, \mathbb{C}) \times \mathrm{GL}(1, \mathbb{C})$. Let r_i be an irreducible constituent of the adjoint action of ${}^LM$ on ${}^L\mathfrak{n}$, the Lie algebra of ${}^LN$. Using the dual map*

$$
{}^L\varphi : \mathrm{GL}(n, \mathbb{C}) \times \mathbb{C}^* \to {}^LM, \tag{3.1.1}
$$

we then have

$$
L(s, \pi \cdot \varphi, r_i) = L(s, \pi, r_i \cdot {}^L\varphi). \tag{3.1.2}
$$

Assume $r_i \cdot {}^L\varphi$ satisfies the branching rule (2-1). Moreover, assume the equality (2-2), but only for L-functions, that is, the validity of

$$
L(s, R \cdot (\rho_1, \rho_2)) = L(s, (\pi(\rho_1), \pi(\rho_2)), R). \tag{3.1.3}
$$

Then

$$
L(s, r_i \cdot {}^L\varphi \cdot \rho) = L(s, \pi(\rho), r_i \cdot {}^L\varphi). \tag{3.1.4}
$$

Remark. The extension from generic representations to any irreducible admissible one is rather routine as explained on page 322 of [Shahidi 1990].

Proof. We may assume F is p-adic. We again use Brauer's theorem and prove (3.1.4) for monomial representations as in Theorem 2.1. We choose k and $\tilde{\rho}$ such that $k_v = F$, $\tilde{\rho}|W'_F = \rho$ and consider $\pi(\tilde{\rho}) := \bigotimes_v \pi(\tilde{\rho}_w)$, where $\tilde{\rho}_w = \tilde{\rho}|W'_{k_w}$. We again twist $\pi(\tilde{\rho})$ by a grössencharacter $\tilde{\chi} = \bigotimes_w \tilde{\chi}_w$ that is highly ramified at all finite places where $\pi(\tilde{\rho}_w)$ is ramified except v, where we will assume $\tilde{\chi}_v \equiv 1$. Then for each finite ramified w, $w \neq v$, stability for L-functions, i.e., (1-8), implies

$$\gamma\big(s, \pi(\tilde{\rho}_w) \otimes \tilde{\chi}_w, r_{i,w} \cdot {}^L\varphi, \psi_F\big) = c_w q_w^{-n_w s}, \tag{3.1.5}$$

where $c_w \in \mathbb{C}^*$, $n_w \in \mathbb{Z}$ and q_w is the cardinality of the residue field of k at w. Using the equality at archimedean primes for γ-functions we thus have

$$\prod_{\substack{w \in S \\ w \neq v}} c_w q_w^{-n_w s} \gamma\big(s, \pi(\rho), r_i \cdot {}^L\varphi, \psi_F\big) = \prod_{\substack{w \in S \\ w \neq v}} c'_w q_w^{-n'_w s} \gamma\big(s, r_i \cdot {}^L\varphi \cdot \rho, \psi_F\big), \tag{3.1.6}$$

where c'_w and n'_w are the corresponding objects on the Artin side and S is the set of ramified finite primes, whenever ρ is monomial.

On the other hand, by equality (3.1.3) of L-functions for constituents of our branching rule, we get an equality like (3.1.6) for every pair $\gamma(s, R \cdot (\rho_1, \rho_2), \psi_F)$ and $\gamma(s, (\pi(\rho_1), \pi(\rho_2)), R, \psi_F)$. We can then extend (3.1.6) from monomial representations, that is, a $\mathbb{Z}$-basis for the Grothendieck ring of W'_F, to the full ring.

We now assume ρ is bounded so that $\pi(\rho)$ is tempered. We then have that $L(s, \pi(\rho), r_i \cdot {}^L\varphi)$ gives the zeros of $\gamma(s, \pi(\rho), r_i \cdot {}^L\varphi)$ [Shahidi 1990; 2010]. The same is true of $L(s, r_i \cdot {}^L\varphi \cdot \rho)$ and $\gamma(s, r_i \cdot {}^L\varphi \cdot \rho)$. By standard properties of L-functions, we then get the equality (3.1.4) for a bounded ρ. The case of arbitrary ρ and $\pi(\rho)$ now follows from Langlands classification upon which factors for $\pi(\rho)$ are defined [Langlands 1989; Shahidi 1990; 2010] as well as those of Artin. This completes the proof of Theorem 3.1. $\square$

Remark 3.2. One may replace Equation (3.1.3) with the equality of γ-factors only up to a monomial in q^{-s}, which is a much weaker statement than (2-2).

Example 3.3 (twisted exterior and symmetric square L-functions for $\mathrm{GL}(n)$). The pair in this case is $G = \mathrm{GSpin}$ and M is generated by all simple roots but the last one, i.e., the Siegel parabolic of G. In the case of exterior squares the equality is

$$\begin{aligned} L(s, \Lambda^2 \rho_0 \otimes \eta) &= L(s, \pi(\rho_0) \otimes \eta, \Lambda^2 \otimes St) \\ &= L(s, \pi_0, \Lambda^2 \otimes \eta), \end{aligned} \tag{3.3.1}$$

where the L-functions on the right are from [Shahidi 1990; 2010]. This was first proved in [Henniart 2010]. The case of twisted symmetric square is similar. Here St denotes the standard representation of $\mathrm{GL}_1(\mathbb{C})$.

Example 3.4 (twisted exterior cube for GL(6)). Here the pair is (E_6^{sc}, M^{α_4}), where M^{α_4} is the Levi subgroup generated by $\Delta - \{\alpha_4\}$, Δ being the set of simple roots. This is case (x) in [Langlands 1971a] or equally $(E_{6,ii})$ in [Shahidi 2010]. The map φ is defined in 2.5.3 of [Kim 2005]. With notation as in Theorem 3.1 here

$$r_i \cdot {}^L\varphi = r_1 \cdot {}^L\varphi = \Lambda^3 \otimes St, \tag{3.4.1}$$

and thus Theorem 3.1 should imply

$$\begin{aligned} L(s, \Lambda^3\rho_0 \otimes \eta) &= L(s, r_1 \cdot {}^L\varphi \cdot (\rho_0 \otimes \eta)) \\ &= L(s, \pi(\rho), r_1 \cdot {}^L\varphi) \\ &= L(s, \pi(\rho_0) \otimes \eta, \Lambda^3 \otimes St) \\ &= L(s, \pi_0, \Lambda^3 \otimes \eta), \end{aligned} \tag{3.4.2}$$

where $\pi(\rho_0) = \pi_0$, $\pi = \pi_0 \otimes \eta$ and $\rho = \rho_0 \otimes \eta$, if we can show (2-2) and (3.1.3) hold. We remark that in this case $\dim r_2 = 1$ and there are no other constituents.

As discussed in Section 2, the branching rule (2-1) in this case reads

$$R(\rho_1, \rho_2) = \Lambda^2\rho_1 \otimes \rho_2 \oplus \rho_1 \otimes \Lambda^2\rho_2. \tag{3.4.3}$$

Dimensions $n_i = \dim \rho_i$, $i = 1, 2$, are a partition of 6, i.e., $n_1 + n_2 = 6$. By symmetry we need to know the validity of

$$L(s, \Lambda^2\rho_1 \otimes \rho_2) = L(s, (\pi(\rho_1), \pi(\rho_2)), \Lambda^2 \otimes St), \tag{3.4.4}$$

$1 \le n_1 \le 5$, $n_1 + n_2 = 6$, where St denotes the standard representation of $\mathrm{GL}(n_2, \mathbb{C})$.

When $1 \le n_1 \le 3$, (3.4.3) is valid by [Harris and Taylor 2001; Henniart 2000]. For $n_1 = 5$ and thus $n_2 = 1$, (3.4.4) is Example 3.3. It remains to address the case $n_1 = 4$ and $n_2 = 2$. Equality (3.4.4) in this case follows from Kim's work on functoriality for $\Lambda^2 : \mathrm{GL}_4(\mathbb{C}) \to \mathrm{GL}_6(\mathbb{C})$. In fact, (3.4.4) is equivalent to

$$L(s, \Lambda^2\rho_1 \otimes \rho_2) = L(s, \pi(\Lambda^2\rho_1) \times \pi(\rho_2)), \tag{3.4.5}$$

by [Harris and Taylor 2001; Henniart 2000] in which $\Lambda^2\rho_1$ is a six dimensional continuous representation of W_F'. What we need to verify is the equality

$$\begin{aligned} L(s, \Lambda^2\rho_1 \otimes \rho_2) &= L(s, \Lambda^2(\pi(\rho_1)) \times \pi(\rho_2)) \\ &= L(s, (\pi(\rho_1), \pi(\rho_2)), \Lambda^2 \otimes St). \end{aligned} \tag{3.4.6}$$

This is proved by Kim [2003]. We collect this as:

Proposition 3.5. *Let $(\rho, \pi(\rho))$ be a pair with $\rho = \rho_0 \otimes \eta$ a representation of W'_F into* $\mathrm{GL}(6, \mathbb{C}) \times \mathrm{GL}(1, \mathbb{C})$. *Let $\pi_0 = \pi(\rho_0)$. Then*

$$L(s, \Lambda^3 \rho_0 \otimes \eta) = L(s, \pi_0, \Lambda^3 \otimes \eta) = L(s, \pi(\rho_0) \otimes \eta, \Lambda^3 \otimes St). \tag{3.5.1}$$

Remark 3.6. The pairs (E_7^{sc}, M^{α_4}) and (E_8, M^{α_4}) give $L(s, \pi_0 \otimes \eta, \Lambda^3 \otimes St)$, where $\eta \in \widehat{F}^*$ and π_0 is an irreducible admissible representation of either $\mathrm{GL}(7, F)$ or $\mathrm{GL}(8, F)$, respectively [Langlands 1971a; Shahidi 2010]. To get equality (3.5.1) in these cases requires equality (3.4.6) for $n_1 = 5$ and 6, respectively, which unfortunately are not yet available.

4. Comments on stability of γ-functions

As explained in Section 1, it is the stability of γ-functions, condition (2) of our r-theory, which is not available in any generality, even within the Langlands–Shahidi method. On the other hand γ-functions within this method are defined inductively by means of "local coefficients" [Shahidi 1990; 2010]. These are complex functions defined by means of standard intertwining operators and Whittaker functionals for induced representations [Shahidi 2010]. Their definition clearly requires the representation π of $M(F)$ be generic. But γ-functions defined through the method can be extended even to cases where π is not generic. This is done by means of Langlands classification (page 322 of [Shahidi 1990]).

It is thus enough to show that each local coefficient is stable under twists by highly ramified characters. We shall now briefly explain how one expects to prove stability.

As before, we assume (G, M) is a pair of a quasisplit connected reductive group G and a Levi subgroup M of one of its maximal parabolics, $P = MN$, both defined over F which we will assume to be a p-adic field of characteristic zero. We let α denote the unique simple root in N. The method is now being developed for fields of positive characteristic mainly by Luis Lomelí with some collaboration by Guy Henniart (see [Lomelí 2009; Henniart and Lomelí 2011]).

With notation as in the previous section, we let ${}^LM$ act on ${}^L\mathfrak{n}$ and let r_i, $1 \le i \le m$, be its irreducible subrepresentations ordered as in [Shahidi 1990; 2010]. The γ-factors $\gamma(s, \pi, r_i, \psi_F)$, when π is an irreducible admissible generic representation of $M(F)$, satisfy

$$C(s, \pi) = C_{\psi_F}(s, \pi) = \lambda_G(\psi_F, w_0)^{-1} \prod_{i=1}^{m} \gamma(is, \pi, \tilde{r}_i, \overline{\psi}_F), \tag{4.1}$$

where $C(s, \pi)$ is the corresponding local coefficient. Here π is assumed to be generic with respect to the generic character of $U_M(F)$ defined by ψ_F and a fixed

F-splitting of G (and thus M). For simplicity we call π ψ_F-generic, not mentioning the splitting. The factor $\lambda(\psi_F, w_0)$ is a product of Langlands λ-functions, Hilbert symbols, and w_0 is the representative of the element $\tilde{w}_\ell \tilde{w}_{\ell,M}^{-1}$ of Weyl group $W(G, T)$. Here $B = TU$ is a fixed Borel subgroup over F, giving our splitting, $M \supset T$, $U \supset N$, $U_M = U \cap M$. We recall that fixing the splitting leads to a choice of a representative for any Weyl group element, w_0 representing that of $\tilde{w}_\ell \tilde{w}_{\rho,M}^{-1}$. We refer to Chapter 8 of [Shahidi 2010], specifically Remarks 8.2.1 and 8.2.2, for a complete discussion of these factors and their choices.

With notation as in Section 1, item 2 (stability), one can formulate stability for $C(s, \pi)$ as follows:

Conjecture 4.1. *Given a pair of irreducible admissible ψ_F-generic representations π_1 and π_2 of $M(F)$ with same central characters,*

$$C(s, \pi_1 \otimes \chi) = C(s, \pi_2 \otimes \chi),$$

where χ is a suitably highly ramified character of $M(F)$.

As experience has shown, at least in a number of important cases [Asgari and Shahidi 2006; 2011; Cogdell et al. 2004; 2005; 2008; $\geq$ 2012; Kim and Krishnamurthy 2005], this can be proved by expressing $C(s, \pi)$ as a Mellin transform of a Bessel function on $M(F)$. This was attained by establishing an integral representation for $C(s, \pi)^{-1}$ in [Shahidi 2002]. The formula is under the assumption that P is self-associate. This means that $\overline{N} = w_0 N w_0^{-1} = N^-$, where N^- is the unipotent subgroup opposed to N.

We first recall the partial Bessel function involved. Let ω_π be the central character of π and define $w_0(\omega_\pi)(z) = \omega_\pi(w_0^{-1} z w_0)$. Given $s \in \mathbb{C}$, set $\pi_s = \pi \otimes q^{\langle s\tilde{\alpha}, H_M(\cdot)\rangle}$ and define

$$\omega_{\pi_s}(z) = \omega_\pi(z) q^{\langle s\tilde{\alpha}, H_M(z)\rangle}. \tag{4.2}$$

We refer to [Shahidi 1988] for the definition of $\tilde{\alpha}$. Fix a sufficiently large open compact subgroup $\overline{N}_0 \subset \overline{N}$. Let φ denote its characteristic function.

For almost all $n \in N(F)$,

$$w_0^{-1} n = m n' \overline{n}, \tag{4.3}$$

$m \in M(F)$, $n' \in N(F)$, $\overline{n} \in \overline{N}(F)$. This sets up a densely defined map

$$n \mapsto (m, \overline{n})$$

from $N(F)$ into $M(F) \times \overline{N}(F)$. While $n \mapsto \overline{n}$ is a bijection, $n \mapsto m$ may not be one; see [Shahidi 2002].

Let W_v be a Whittaker function in the space $W(\pi_s)$ of π_s such that $W_v(e) = 1$. Given $z \in Z_M(F)$, we define the partial Bessel function

$$j_{v,\varphi}(m, \overline{n}, z) := \int_{U_{M,n}(F)\backslash U_M(F)} W_v(mu^{-1})\varphi(zu\overline{n}u^{-1}z^{-1})\psi_F(u)\,du. \tag{4.4}$$

Let α be the unique simple root of T in U generating N.

We may assume $H^1(F, Z_G) = 1$, which we can attain by enlarging G without changing its derived group. It will not affect our results. Lemma 5.2 of [Shahidi 2002] then implies existence of a map $\alpha^\vee$ from F^* into $Z_M^0 = Z_G(F)\backslash Z_M(F)$ such that $\alpha'(\alpha^\vee(t)) = t$, $t \in F^*$, for any root α' of T that restricts to α.

We need to define a scalar x_α defined by $\overline{n}$. It is simply the α-coordinate of $w_0^{-1}\overline{n}w_0 \in N$ by means of our fixed splitting.

Given $y \in F^*$, set

$$j_{v,\varphi}(m, \overline{n}, y) := j_{v,\varphi}(m, \overline{n}, \alpha^\vee(y^{-1} \cdot x_\alpha)), \tag{4.5}$$

whenever $x_\alpha \neq 0$.

We also let $Z_M^0 U_M(F)$ act on $N(F)$ by conjugation and write $Z_M^0 U_M(F)\backslash N(F)$ for the corresponding quotient space.

Theorem 4.2 [Shahidi 2002, Theorem 6.2, second part]. *Suppose $\omega_\pi(w_0\omega_\pi^{-1})$ is ramified. Fix $y_0 \in F$ such that* $\mathrm{ord}_F(y_0) = -d - f$, *where d and f are conductors of ψ_F and $\omega_\pi^{-1} \cdot (w_0\omega_\pi)$, respectively. Then up to an abelian Tate γ-factor attached to $\omega_\pi \cdot (w_0\omega_\pi^{-1})$ and ψ_F,*

$$\begin{aligned} &C(s,\pi)^{-1} \\ &\qquad\sim \int_{Z_M^0 U_M(F)\backslash N(F)} j_{\tilde{v},\varphi}(m, \overline{n}, y_0)\omega_{\pi_s}^{-1}(x_\alpha)(w_0\omega_{\pi_s})(x_\alpha)q^{\langle s\tilde{\alpha}+\rho, H_M(m)\rangle}\,d\dot{n}. \end{aligned} \tag{4.6}$$

Here x_α is embedded in $Z_M(F)$ through $\alpha^\vee$ and $v = \tilde{v} \otimes q^{\langle s\tilde{\alpha}, H_M(\ )\rangle}$. More precisely, $\tilde{v}$ is the vector in the space of π that goes to v in the space of π_s.

We refer to [Shahidi 2009] for some of the geometric issues in analyzing the integral in (4.6).

It is Equation (4.6) which has been the main tool in proving stability in a number of important cases, all of significance in establishing functoriality [Cogdell and Piatetski-Shapiro 1998; Cogdell et al. 2004; 2005; 2008; Kim and Krishnamurthy 2005; Asgari and Shahidi 2006; 2011].

What one has to do is to prove an asymptotic expansion for the partial Bessel function $j_{\tilde{v},\varphi}$. In fact, in the cases of classical or GSpin groups, one basically needs to deal with $M = \mathrm{GL}(1) \times G_1$, where G_1 is one of these groups, as a maximal Levi subgroup inside a larger group G of the same type.

The philosophy of expressing γ-functions as a Mellin transform of a partial Bessel function goes back to Cogdell and Piatetski-Shapiro [1998] who proved such a formula as well as the asymptotic expansion for the corresponding partial Bessel functions when $G_1 = \mathrm{SO}(2n+1)$. Using Equation (4.6), which was established in [Shahidi 2002], the corresponding stability for other cases were proved in [Cogdell et al. 2004; 2005; 2008; Kim and Krishnamurthy 2005; Asgari and Shahidi 2006].

In [Cogdell, Shahidi and Tsai $\geq$ 2012], the authors study the case $(G, M) = (\mathrm{GSp}(2n),\ \mathrm{GL}(n)\times \mathrm{GL}(1))$, where the γ-factor $\gamma(s, \pi, \Lambda^2, \psi_F)$ appears. Using a robust deformation argument which should apply more generally whenever LLC is available, the equality

$$\gamma(s, \Lambda^2 \cdot \rho, \psi_F) = \gamma(s, \pi(\rho), \Lambda^2, \psi_F) \tag{4.7}$$

is reduced to a proof of stability for only when ρ is irreducible and thus only when $\pi = \pi(\rho)$ is supercuspidal in [Cogdell, Shahidi and Tsai $\geq$ 2012]. A proof of stability in the supercuspidal case also seems to be within reach, using (4.6) and the asymptotics of the full Bessel functions for $\mathrm{GL}(n)$ proved by Jacquet and Ye [1996]. In particular, it is shown that the asymptotics of the partial Bessel function $j_{\tilde{v},\varphi}$ can still be deduced from those of full Bessel functions and thus germ expansions in [Jacquet and Ye 1996]. The case of symmetric squares

$$\gamma(s, \mathrm{Sym}^2 \cdot \rho, \psi_F) = \gamma(s, \pi(\rho), \mathrm{Sym}^2, \psi_F) \tag{4.8}$$

follows immediately from

$$\gamma(s, \pi\times\pi, \psi_F) = \gamma(s, \pi, \Lambda^2, \psi_F)\gamma(s, \pi, \mathrm{Sym}^2, \psi_F), \tag{4.9}$$

$$\gamma(s, \rho\otimes\rho, \psi_F) = \gamma(s, \Lambda^2 \cdot \rho, \psi_F)\gamma(s, \mathrm{Sym}^2 \cdot\rho, \psi_F), \tag{4.10}$$

and

$$\gamma(s, \rho\otimes\rho, \psi_F) = \gamma(s, \pi(\rho)\times\pi(\rho), \psi_F), \tag{4.11}$$

the last being part of LLC in [Harris and Taylor 2001; Henniart 2000]. The γ-factors $\gamma(s, \pi, \Lambda^2, \psi_F)$ and $\gamma(s, \pi, \mathrm{Sym}^2, \psi_F)$ are those defined by the Langlands–Shahidi method as special cases of the general definition given in [Shahidi 1990].

The case of Rankin product L-functions for $\mathrm{GL}(n)\times\mathrm{GL}(n)$ using this approach has been addressed in [Tsai 2011]. The cases of non-self-associate maximal parabolics are also being addressed, and an analogue of (4.6) for $\mathrm{GL}(n)\times\mathrm{GL}(m)$, $n\neq m$, seems to be in hand. This seems to be the most complicated among the cases to be considered.

For the record, we also refer to [Ramakrishnan 2000] and [Kim and Shahidi 2002], where the equality of certain triple product factors is proved, but using other techniques such as base change, combined with functoriality.

We should finally mention the possible application of (4.6), or rather its more general form (6.38) or its initial form (6.55), both of [Shahidi 2002], in establishing the local Langlands correspondence for GSp(4) over function fields through Deligne–Kazhdan philosophy of close fields. If successful the problem is then reduced to that of LLC for GSp(4) over number fields, already established in [Gan and Takeda 2011]. We refer to [Ganapathy 2012] for a discussion of this philosophy and the treatment of LLC for GL(n) through this approach.

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