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ROBERT STEINBERG (1922–2014): IN MEMORIAM

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In memoriam

Robert Steinberg

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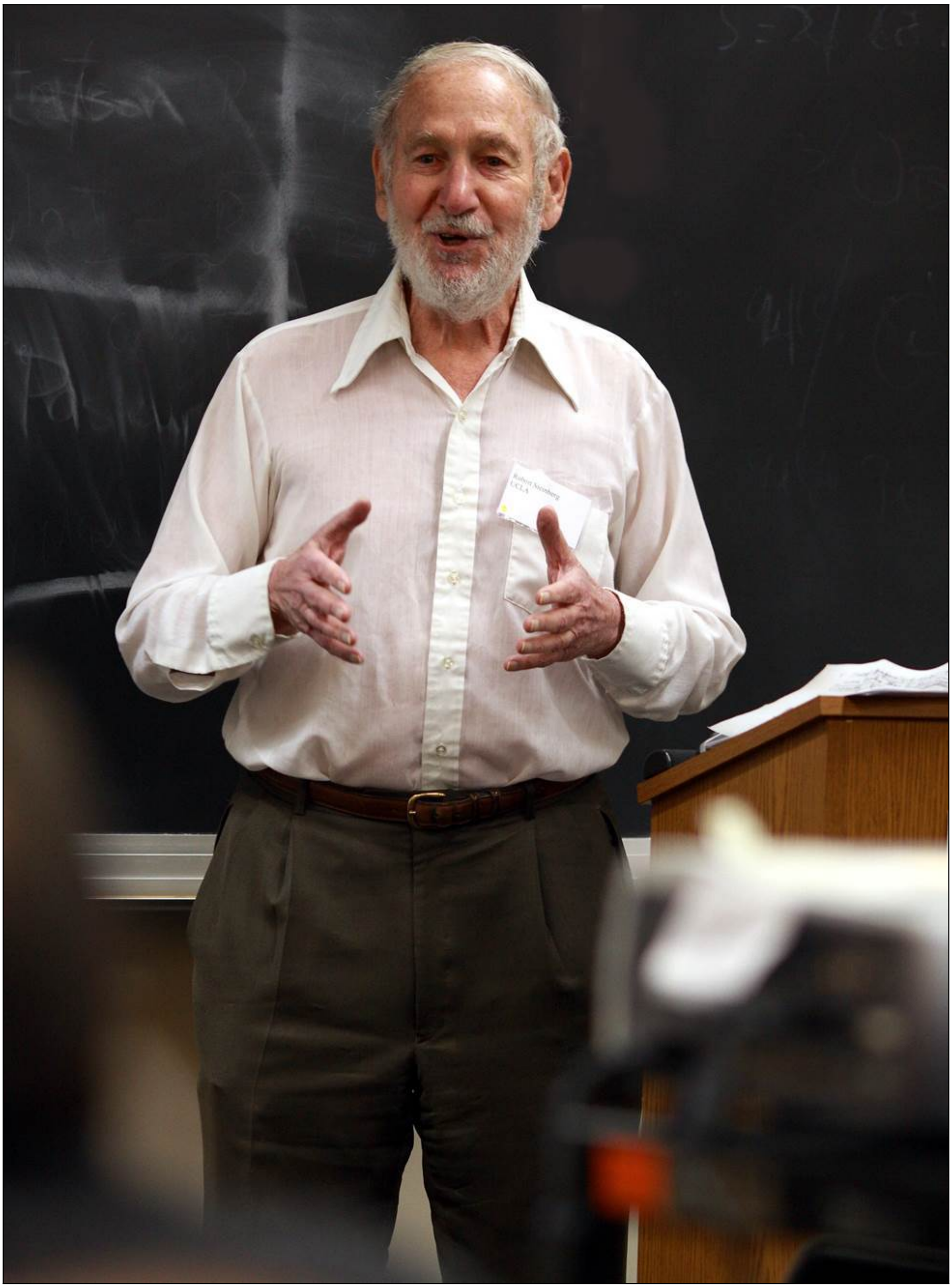

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ROBERT STEINBERG (1922–2014): IN MEMORIAM

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He touched nothing that he did not adorn.

The quotation above is by Samuel Johnson, writing about his friend Oliver Goldsmith. I think it is the most satisfying way to describe the work and legacy of Robert Steinberg, who passed away on May 25, 2014 on his 92nd birthday. His towering stature as one of the great masters of the theory of algebraic groups and finite groups, the vast scope, depth, and beauty of his papers (some key ones were published in the PJM), and his gigantic presence in the algebraic scene in Southern California, are the reasons that led the Board of Governors of the PJM to request that a special volume of the PJM be published in his memory. In this brief essay I shall try to sketch a portrait of a master who wore his mantle of greatness with unassuming simplicity and charm.

This is a melancholy task for me, to write about someone who was a good friend and role model for me for nearly fifty years. In these days of ever multiplying awards, million dollar grants, medals, and so on, it is refreshing, even humbling, to talk about a man who never sought the limelight, who worked quietly on the problems that appealed to him, and evolved into one of the great masters and innovators of the theory of semisimple algebraic groups. The problems he worked on and considered important became the central problems of the subject. His influence on the subject was enormous. Even after he retired he could surprise experts with new and easier proofs of some of the fundamental theorems of the subject. His monumental set of lecture notes on Chevalley groups [1968] has been studied by hundreds of mathematicians (I myself lectured twice on them) and will appear as a publication of the AMS. In spite of his greatness he was a gentle and modest man, aware of his gifts certainly, but accepting them and trying to get the job done.

His work is widely available in his *Collected papers* [1997] and its scope is extraordinary. It is a very difficult task to present his work in one short essay and I will not even attempt it, nor do I have the competence for it. But I will describe some highlights so that most of the readers will get some idea of what he achieved in his lifetime. I thank Professor Alexander Merkurjev for enlightening me on the impact of Steinberg's work on algebraic K-theory and other parts of mathematics.

Keywords: Robert Steinberg, memorial issue.

He told me that he and his collaborators have used every major theorem of Steinberg in their work.

I was spiritually close to Steinberg as a mathematician. In his words I was also a semisimple mathematician, as he said when he first introduced me to his close friend and collaborator Tonny Springer, a Dutch semisimple mathematician. However I was more interested in the transcendental aspects of real semisimple Lie groups, such as infinite dimensional representations and harmonic analysis.

After his beloved wife Maria passed away, he gradually lost the desire and will to do things, and I became closer to him in those days by visiting him as frequently as I could. His passing away was traumatic to his nephew and nieces and to all of his friends and relatives.

He was born in Romania but his parents settled in Canada very soon afterwards. I am sure he was deeply influenced by the wide open spaces of Canada and thereby acquired his lifelong love for long hikes and camping trips. He and Maria spent a part of almost every summer by hiking and camping in the high sierras. Maria's strength of mind and decisiveness blended well with his gentle personality, and they became one soul.

He studied under Richard Brauer and got his doctorate degree in 1948. He came to UCLA in 1948 and never left it. In 1985 he was given the Leroy P. Steele Prize of the AMS for lifetime achievement. He was elected to the National Academy of Sciences in the same year. He wrote a letter to me on that occasion and said that this proves he still has friends. He was awarded the Jeffery–Williams Prize of the Canadian Mathematical Society in 1990. He was an avid fan of basketball and hockey, and the Bruins and Lakers were his favorite teams, and Jerry West his all-time favorite player. He was generally taciturn but always charming, and could open up to close friends.

To understand roughly the scope of his achievement, it is essential to know what simple and semisimple groups are. In 1894, Elie Cartan classified all simple Lie algebras over $\mathbb{C}$, and found that they fall into four infinite families (the *classical algebras*), and five isolated ones (the *exceptional algebras*). This is the same as the classification of simply connected complex Lie groups which are essentially simple. The semisimple groups are, up to a cover, products of simple groups. The classical groups (so christened by Weyl) are the group of matrices of determinant 1, the orthogonal (or spin) groups, and the symplectic groups. These groups have the remarkable property that they make sense over any field or even any commutative ring with unit. Over a finite field they become finite groups which are almost simple and these were studied intensively by Dickson in the late nineteenth century. It is a natural question to ask if the exceptional groups also make sense over finite fields. In the early 1950s, Chevalley had started to study algebraic groups over fields of characteristic 0 by using the exponential map and coming down to the Lie algebras. But this method was not very successful and certainly could not touch the case when the field had positive characteristic. But Borel changed the entire landscape by studying the algebraic groups directly using algebraic geometric methods, proving the existence of what are now called Borel subgroups, and their conjugacy, over any algebraically closed field. Chevalley then used Borel's work as a starting point and completed the classification of all semisimple affine algebraic groups by methods of algebraic geometry (up to a finite cover, semisimple groups are products of simple groups, and reductive groups are products of semisimple groups and tori). He found that the simple groups are classified in the same way as Cartan's. He then discovered the further remarkable fact that any semisimple group is naturally a *group scheme* over $\mathbb{Z}$, and hence it makes sense to look at its points over any field (this is an oversimplification). In particular it makes sense to speak of the simple groups over finite fields, and this process led Chevalley to discover new simple finite groups hitherto unknown. The groups he constructed over any field became known as the *Chevalley groups*.

In my opinion, the fact that the semisimple groups are really group schemes over $\mathbb{Z}$ accounts for their great importance, depth, and vitality. Over arbitrary fields it led Borel, Chevalley, Tits, Steinberg, Lusztig, Deligne, Curtis, and others to erect a beautiful theory of their structure and representations. Over the real and p-adic fields they become Lie groups on which one could do geometry and analysis, as Weyl, Gel'fand, Mautner, Harish-Chandra, Mostow, Bruhat, Kazhdan, and others did. Over the adeles their structure and representation theory led Langlands to formulate his program linking the harmonic analysis on the adelic groups to the most fundamental aspects of algebraic number theory, the so-called Langlands program, which has inspired and animated a huge number of mathematicians of his and later generations.

Chevalley's discovery that semisimple groups are group schemes over $\mathbb{Z}$ was the mathematical context when Steinberg started his research. In his words, he wanted to become a semisimple mathematician, and soon became one. His field was the entire theory of Chevalley groups and the associated finite groups, their structure and their linear representations. He had important things to say on all aspects of these groups. But the striking fact was that he used only elementary methods, including basic algebraic geometry, and seldom ventured into the cohomological aspects. I feel he resembled Harish-Chandra in this: he got to where he wanted to go with very simple ideas and methods.

In his *Collected papers*, he discussed all his papers, elaborating some fine points and putting his work within the framework of current knowledge, occasionally adding some personal reminiscences. About one paper he wrote that it was entirely worked out in the High Sierras when he was in his sleeping bag looking at the stars! About another paper he wrote that this was his only paper for which he got money from the Russians when they translated it, and mentions that the translation of his *Lectures on Chevalley groups* [1975] fetched him no money as it was before *glasnost*!!

New finite simple groups. In what follows I shall describe some highlights of his vast opus. His first major work was a couple of papers starting with the famous *Variations on a theme of Chevalley* [1959] in the PJM, where he constructed new families of finite simple groups not covered by the Chevalley groups and obtained for them structure properties similar to those of the Chevalley groups. Suzuki and Ree and others followed him with further families of new finite simple groups, all of them collectively known as the *twisted Chevalley groups*. The Chevalley groups and their twists were called finite simple groups of Lie type, and the great classification theorem of finite simple groups is just the statement that apart from the cyclic groups of prime order p, the alternating groups A_n ($n \geq 5$), and 26 *sporadic* groups, a finite simple group is of Lie type.

Generators and relations for Chevalley groups. In the famous paper *Générateurs, relations et revêtements de groupes algébriques* [1962], Steinberg considers Chevalley groups corresponding to a root system Σ and field K. They are generated by unipotent elements $g_r(t)$ with $r \in \Sigma$, $t \in K$. Among all the relations between the generators there are (obvious) ones (R) that can be written uniformly for all Σ, K. He then considers the abstract group $\widehat{G}$ generated by symbols $x_r(t)$ ($r \in \Sigma$, $t \in K$) subject to the relations (R) and the natural surjective homomorphism

$$\pi : \widehat{G} \longrightarrow G.$$

Thus, $\mathrm{Ker}(\pi)$ describes all the relations between the generators modulo the obvious relations. Steinberg proves the remarkable result that the covering π is central, i.e., $\mathrm{Ker}(\pi)$ is contained in the center of $\widehat{G}$, and that π is a universal central extension. J. Milnor has used Steinberg's construction in the case of a general linear group over an arbitrary ring S to define the group $K_2(S)$ that describes the relations between the elementary matrices over S modulo the obvious relations. The corresponding group $\widehat{G}$ is known as the *Steinberg group* of S. Thus, this paper of Steinberg made a great impact on the development of higher algebraic K-theory. The kernel of π was studied in a profound manner by Moore and Matsumoto over a p-adic field, and their work led to deep relationships with the norm residue symbol of number theory. Among other things the work of Moore and Matsumoto highlighted the importance of the two-fold covering of the symplectic group, the so-called metaplectic group, over the local fields and the adeles. The adelic metaplectic group was the platform which Weil used in his reformulation of Siegel's work on quadratic forms.

Regular elements of semisimple algebraic groups [Steinberg 1965]**.** This is one of his most admired and beautiful papers. Here he studies conjugacy classes of regular elements in a semisimple group G. For simplicity let us assume that the ground field is algebraically closed. An element $g \in G$ is called *regular* if the dimension of the orbit of g under the action of G by conjugacy has maximal dimension (which is $\dim(G) - \mathrm{rank}(G)$). In this paper he proves that the regular conjugacy classes have an affine space section in the algebraic geometric sense, for every simply connected semisimple group. For example, for $\mathrm{SL}(n)$ we get the space of companion matrices, a result that goes back at least to Gantmacher. Actually he does not restrict himself to the algebraically closed ground field and proves that if G is a simply connected quasisplit group over a field K (that is, it contains a Borel subgroup defined over K), then every conjugacy class defined over K contains an element defined over K. As a consequence of the main result, Steinberg proves that every principal homogeneous space of a quasisplit semisimple group admits reduction to a maximal torus. This result yields the solution of the famous Serre conjecture:

> *If K is a field of cohomological dimension 1, then all principal homogeneous spaces of a connected algebraic group over K are trivial.*

The result that the regular conjugacy classes have a section in the algebraic geometric sense led to an interesting interaction between us. I was looking at this question on a semisimple *Lie algebra* over $\mathbb{C}$. Kostant had constructed a beautiful affine cross section for the regular orbits of the adjoint representation (which reduces to the companion matrices for $\mathfrak{sl}(n)$), roughly at the same time as Steinberg's work. When I looked at the Lie algebra problem, it occurred to me that by making use of some ideas of Harish-Chandra I could obtain a proof of many of Kostant's results in a very simple way. I had this published in the American Journal of Mathematics and left a reprint in Bob's mail box. He then asked me to come to his office and explained the corresponding global result. I treasure the memory of that discussion between us which had no element of condescension in it, when I was a young researcher and he was at the peak of his powers.

The Steinberg representation. The complex representations of the finite Chevalley groups are difficult to construct, even though Green had quite early worked out the irreducible characters of $\mathrm{GL}(n)$. The final results were obtained by Deligne and Lusztig who realized the representations using certain étale cohomology spaces. But Steinberg found one of the most important and ubiquitous ones very early in his career. It is now called the *Steinberg representation*, and one can find a masterful essay on its various incarnations in his *Collected papers*. For a Chevalley group G over a finite field, if B is a Borel subgroup, and 1_B^G is the representation of G induced by the trivial representation of B, then St is the unique irreducible component of 1_B^G which does not occur in any 1_P^G where P is any parabolic subgroup containing B properly. Correspondingly, there is a formula for its character as an alternating sum of the characters of the 1_P^G. Remarkably, this character formula makes sense in a p-adic field and its properties play a fundamental role in the harmonic analysis on the p-adic semisimple groups, as developed by Harish-Chandra, Jacquet, and others. Borel and Serre proved, using the cohomology of the Bruhat–Tits buildings, that *St* is an irreducible square integrable (hence unitary) representation of the p-adic group.

The Steinberg representation also plays a basic role in the Langlands correspondence. For example, an elliptic curve over Q has split multiplicative reduction at a prime p if and only if the unitary automorphic representation associated to it by the Langlands correspondence has for its component at p the Steinberg representation. In general, under the correspondence, ignoring scalar twists by one dimensional representations, a Steinberg representation at p corresponds to a Galois representation for which the image of a decomposition group at p contains a regular unipotent element.[1]

For lack of time I cannot discuss some of the other major discoveries in his work. I mention the new and easier proofs of the isomorphism and isogeny theorems of algebraic semisimple groups, which say that an isomorphism (isogeny) between semisimple algebraic groups is always induced by an isomorphism (isogeny) of their corresponding root data and conversely. The other item is his new and simpler counterexample to Hilbert's 14th problem, which asks one to prove that the ring of invariant polynomials of a linear action of *any* algebraic group is finitely generated. For semisimple groups over the complex field this was proved for SL(n) by Hilbert, and for all semisimple groups over a field of characteristic 0 by Weyl, as a consequence of his famous result that all finite dimensional representations of a semisimple Lie algebra are direct sums of irreducible representations. In prime characteristic the Weyl reducibility fails to hold and one needs a weakening of it, called *geometric reductivity*, conjectured by Mumford and proved by Haboush. The finite generation of invariants then follows from geometric reductivity, as was shown by Nagata. So to find counterexamples to the finite generation of invariants, one has to leave the category of semisimple or even reductive groups. Nagata found a counterexample for a finite-dimensional action of a product of the additive groups. In the late 1990s, Steinberg found much simpler classes of examples in all characteristics, and made a thorough analysis of the problem, sharpening Nagata's construction and relating the examples to plane cubic curves and their geometry.

I know I have given only a brief discussion of a very minute part of Steinberg's work which is astonishing in its scope, depth, and beauty. His profound insights about semisimple groups, and the easy grace and charm of his personality, cannot ever be forgotten by people who came into contact with him. I have known very few like him.

[1] These remarks on the Steinberg representation and elliptic curves were pointed out to me by Professor Don Blasius.

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