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Dedicated to the memory of Robert Steinberg.

For any ring $\widetilde{A}$ such that $\mathbb{Z}[q^{\pm 1/2}] \subseteq \widetilde{A} \subseteq \mathbb{Q}(q^{1/2})$, let $\Delta_{\widetilde{A}}(d)$ be an $\widetilde{A}$-form of the Weyl module of highest weight $d \in \mathbb{N}$ of the quantised enveloping algebra $U_{\widetilde{A}}$ of $\mathfrak{sl}_2$. For suitable $\widetilde{A}$, we exhibit for all positive integers r an explicit cellular structure for $\mathrm{End}_{U_{\widetilde{A}}}(\Delta_{\widetilde{A}}(d)^{\otimes r})$. This algebra and its cellular structure are described in terms of certain Temperley–Lieb-like diagrams. We also prove general results that relate endomorphism algebras of specialisations to specialisations of the endomorphism algebras. When ζ is a root of unity of order bigger than d we consider the U_{ζ}-module structure of the specialisation $\Delta_{\zeta}(d)^{\otimes r}$ at $q \mapsto \zeta$ of $\Delta_{\widetilde{A}}(d)^{\otimes r}$. As an application of these results, we prove that knowledge of the dimensions of the simple modules of the specialised cellular algebra above is equivalent to knowledge of the weight multiplicities of the tilting modules for $U_{\zeta}(\mathfrak{sl}_2)$. As an example, in the final section we independently recover the weight multiplicities of indecomposable tilting modules for $U_{\zeta}(\mathfrak{sl}_2)$ from the decomposition numbers of the endomorphism algebras, which are known through cellular theory.

1. Introduction

1A. *Notation.* Let A be the ring $\mathbb{Z}[q^{\pm 1/2}]$ where q is an indeterminate, and let U_A be the Lusztig A-form [1988; 1990; 1993] of the quantised enveloping algebra $U_q(\mathfrak{sl}_2)$ [Drinfeld 1987; Jimbo 1986; Chari and Pressley 1994], which has basis consisting of products of "divided powers" of the generators of $\mathfrak{sl}_2$ and binomials in the Cartan generators. Let $\Delta_A(d)$ be the Weyl module for U_A with highest weight $d \in \mathbb{N}$. This has dimension $d + 1$ and quantum dimension equal to the quantum number $[d + 1]$, where for any integer n,

$$[n] = [n]_q := \frac{q^n - q^{-n}}{q - q^{-1}}.$$

For any commutative A-algebra $\tilde{A}$, we write $U_{\tilde{A}} := \tilde{A} \otimes_A U_A$, and similarly for $\Delta_{\tilde{A}}(d)$, etc. For any positive integer r, let $E_r(d, \tilde{A}) := \operatorname{End}_{U_{\tilde{A}}}(\Delta_{\tilde{A}}(d)^{\otimes r})$.

Let $s_1, \ldots, s_{N-1}$ be the standard Coxeter generators of Sym_N. For $w \in \operatorname{Sym}_N$, write $\ell(w)$ for its length as a word in the generators s_i, and define the left set $L(w) := \{i \mid \ell(s_i w) < \ell(w)\}$; the right set $R(w)$ is defined similarly.

1B. ***The main result.*** Let $K = \mathbb{Q}(q^{1/2})$ be the field of fractions of A. Writing B_r for the r-string braid group (r a positive integer), it is known that there is an action of B_r on $\Delta_A(d)^{\otimes r}$, in which the standard generators of the braid group act on successive tensor factors via the R-matrix $\check{R}$. This is evident over K, and from [Lehrer and Zhang 2006; 2010] and [Andersen et al. 2008] or [Andersen 2012] (using [Kirillov and Reshetikhin 1990]) in the above integral form. This action respects the $U_{\tilde{A}}$-action on the tensor space, and so there is a homomorphism

$$\text{(1-1)} \qquad \eta : \tilde{A} B_r \longrightarrow \operatorname{End}_{U_{\tilde{A}}}(\Delta_{\tilde{A}}(d)^{\otimes r}) = E_r(d, \tilde{A}).$$

We define A using $q^{1/2}$ instead of q because then, with the usual definitions of U_q, the R-matrix is defined over A with respect to a basis of weight vectors.

In [Lehrer and Zhang 2006] it was shown that when $\tilde{A} = K$, η is surjective. This provides a means of studying the relevant endomorphism algebras. When $d = 2$ this surjectivity was proved in [Andersen 2012] for most $\tilde{A}$. We haven't been able to establish this result for $d > 2$. However, inspired in part by the methods used in [loc. cit.] we show in this paper that the endomorphism algebras have a nice cellular structure, even though the R-matrix generators satisfy a polynomial equation of degree $d + 1$.

We shall work with the Temperley–Lieb algebra $\operatorname{TL}_N(\tilde{A})$, which has generators $f_i, i = 1, \ldots, N - 1$ and relations

$$\begin{cases} f_i f_j f_i = f_i & \text{if } |i - j| = 1, \\ f_i f_j = f_j f_i & \text{if } |i - j| > 1, \\ f_i^2 = (q + q^{-1}) f_i. \end{cases}$$

This has an $\tilde{A}$-basis consisting of planar diagrams, as explained in [Graham and Lehrer 1996, §1] (see also [2003; 2004]); these are in one-to-one correspondence with the set of fully commutative elements of Sym_N; see [Fan and Green 1997].

Theorem 1.1. *Let $d \geq 1$ be an integer. For any $\tilde{A}$ such that $[d]!$ is invertible in $\tilde{A}$, the algebra $E_r(d, \tilde{A})$ is isomorphic to a cellular subalgebra of $\operatorname{TL}_{rd}(\tilde{A})$. In particular, it has an $\tilde{A}$-basis labelled by planar diagrams $D \in \operatorname{TL}_{rd}(\tilde{A})$ such that $L(D), R(D) \subseteq \{d, 2d, \ldots, (r-1)d\}$, where the left and right sets $L(D)$ and $R(D)$ are as in Definition 3.2 below.*

We remark that the cellular subalgebra in Theorem 1.1 has an identity different from that of $\operatorname{TL}_{rd}(\tilde{A})$, and is therefore not a unital subalgebra.

Note that the planar diagrams are labelled by the set Sym^c_{rd} of fully commutative elements in Sym_{rd}; the requirement in the theorem is equivalent to taking those $w \in \mathrm{Sym}^c_{rd}$ such that $L(w), R(w) \subseteq \{d, 2d, \ldots, (r-1)d\}$ (see [Fan and Green 1997]).

We shall give further details of the cellular structure below, both in terms of diagrams, and in terms of pairs of standard tableaux.

2. The case $d = 1$

2A. *The Temperley–Lieb action.* It is known (see, for example, [Lehrer and Zhang 2010, §3.4]) that in this case, the R-matrix acts on $\Delta_K(1)^{\otimes 2}$ with eigenvalues $q^{1/2}$ and $-q^{3/2}$. If we adjust the map η of (1-1) by sending the generators to $T_i := q^{1/2}R_i$, where R_i is the relevant R-matrix, then η factors through the algebra $H_r(A) := AB_r/\langle(T_i + q^{-1})(T_i - q)\rangle$, which is well known to be the Hecke algebra, and has A-basis $\{T_w \mid w \in \mathrm{Sym}_r\}$. We therefore have, after tensoring with $\tilde{A}$,

$$(2\text{-}1) \qquad \mu : H_r(\tilde{A}) \longrightarrow \mathrm{End}_{U_{\tilde{A}}}(\Delta_{\tilde{A}}(1)^{\otimes r}) = E_r(1, \tilde{A}).$$

Moreover it is a special case of the main result of [Du et al. 1998] (see also [Andersen et al. 2008]) that μ is surjective for any choice of $\tilde{A}$, even when $\tilde{A}$ is taken to be A. Further, the arguments in [Lehrer and Zhang 2010, Theorem 3.5], generalised to the integral case, show that the kernel of μ is the ideal generated by the element $a_3 := \sum_{w \in \mathrm{Sym}_3}(-q)^{-\ell(w)}T_w$; hence, for any $\tilde{A}$, we have an isomorphism

$$(2\text{-}2) \qquad \eta : H_r(\tilde{A})/\langle a_3 \rangle \cong \mathrm{TL}_r(\tilde{A}) \overset{\sim}{\longrightarrow} \mathrm{End}_{U_{\tilde{A}}}(\Delta_{\tilde{A}}(1)^{\otimes r}) = E_r(1, \tilde{A}),$$

where $\mathrm{TL}_r(\tilde{A}) := H_r(\tilde{A})/\langle a_3 \rangle$ is the r-string Temperley–Lieb algebra. The generator f_i acts as $q - T_i$ on $\Delta_{\tilde{A}}(1)^{\otimes r}$. It is easily shown that $f_i^2 = (q + q^{-1})f_i$, and that the other Temperley–Lieb relations are satisfied.

2B. *Projection to $\Delta_{\tilde{A}}(d)$.* Now it is elementary that

$$(2\text{-}3) \qquad \Delta_K(1)^{\otimes d} \cong \Delta_K(d) \oplus \Delta',$$

where Δ' is the direct sum of simple modules $\Delta_K(i)$ with $i < d$. We therefore have a canonical projection $p_d : \Delta_K(1)^{\otimes d} \longrightarrow \Delta_K(d)$, which may be considered an element of $E_d(1, K) = \mathrm{End}_{U_K}(\Delta_K(1)^{\otimes d})$.

Lemma 2.1. *The projection p_d is the image under μ (see (2-1)) of the element $e_d := P_d(q)^{-1} \sum_{w \in \mathrm{Sym}_d} q^{\ell(w)}T_w \in H_d(\tilde{A})$, where $P_d(q) = q^{d(d-1)/2}[d]!$.*

Proof. We begin by showing that for $i = 1, \ldots, d-1$,

$$(2\text{-}4) \qquad T_i p_d = p_d T_i = q p_d$$

as endomorphisms of $\Delta_K(1)^{\otimes d}$.

By symmetry, it suffices to prove (2-4) for $i = 1$. Now

$$\begin{aligned}
\Delta_K(1)^{\otimes d} &= \Delta_K(1) \otimes \Delta_K(1) \otimes \Delta_K(1)^{\otimes(d-2)} \\
&\cong (\Delta_K(0) \oplus \Delta_K(2)) \otimes \Delta_K(1)^{\otimes(d-2)} \\
&\cong (\Delta_K(0) \otimes \Delta_K(1)^{\otimes(d-2)}) \oplus (\Delta_K(2) \otimes \Delta_K(1)^{\otimes(d-2)})
\end{aligned}$$

But p_d acts as zero on the first summand (since the highest occurring weight is $d - 2$) and T_1 acts as q on the second summand. This proves the relation (2-4). Now since $f_i = \mu(q - T_i)$, this shows that p_d is the "Jones idempotent" of $\mathrm{TL}_d(K)$, defined by the relations $f_i p_d = p_d f_i = 0$ for all i.

It follows that if p'_d is the unique idempotent in $H_d(K)$ corresponding to the algebra homomorphism $T_w \mapsto q^{\ell(w)}$, then $p_d = \mu(p'_d)$. But this idempotent is precisely the element e_d in the statement. □

The next statement is immediate.

Corollary 2.2. *Let $\tilde{A} = A[[d]!^{-1}]$. Then*

$$(2\text{-}5) \qquad\qquad \Delta_{\tilde{A}}(1)^{\otimes rd} \cong \Delta_{\tilde{A}}(d)^{\otimes r} \oplus \Gamma,$$

where Γ is a $U_{\tilde{A}}$-submodule, and the corresponding projection $p \in \mathrm{End}_{rd}(1, \tilde{A})$ such that $p(\Delta_{\tilde{A}}(1)^{\otimes rd}) = \Delta_{\tilde{A}}(d)^{\otimes r}$ is given by $p = p_d^{\otimes r}$, where we now consider p_d as an element of $E_d(1, \tilde{A}) \subset E_d(1, K)$.

3. Endomorphisms of $\Delta_{\tilde{A}}(d)^{\otimes r}$

3A. *Identification of $E_r(d, \tilde{A})$.* Throughout this section we take $\tilde{A}$ to be $\tilde{A} = A[[d]!^{-1}]$. Recall that $E_r(d, \tilde{A}) = \mathrm{End}_{U_{\tilde{A}}}(\Delta_{\tilde{A}}(d)^{\otimes r})$. We are now in a position to identify $E_r(d, \tilde{A})$ on the nose, as a subalgebra of $\mathrm{TL}_{rd}(\tilde{A}) \cong \mathrm{End}_{U_{\tilde{A}}}(\Delta_{\tilde{A}}(1)^{\otimes rd})$. This will lead to the identification of the cellular structure on $E_r(d, \tilde{A})$.

Proposition 3.1. *There is an isomorphism $E_r(d, \tilde{A}) \xrightarrow{\sim} p\,\mathrm{TL}_{rd}(\tilde{A})p$, where p is the idempotent $p = p_d^{\otimes r}$ of $\mathrm{TL}_{rd}(\tilde{A})$ described above.*

Proof. For any endomorphism $\alpha \in E_r(d, \tilde{A})$ we obtain an endomorphism $\tilde{\alpha}$ of $\Delta_{\tilde{A}}(1)^{\otimes rd}$ by extending α by zero, using the decomposition (2-5), that is, by defining $\tilde{\alpha}$ to be zero on Γ. The map $\alpha \mapsto \tilde{\alpha}$ is an inclusion $E_r(d, \tilde{A}) \hookrightarrow E_{rd}(1, \tilde{A})$, and its image is clearly the space of endomorphisms $\beta \in E_{rd}(1, \tilde{A})$ such that $\ker(\beta) \supseteq \Gamma$ and $\mathrm{Im}(\beta) \subset \Delta_{\tilde{A}}(d)^{\otimes r}$ (as in the decomposition (2-5)). This image is $p\,\mathrm{TL}_{rd}(\tilde{A})p$. □

3B. *Temperley–Lieb diagrams.* The key step in proving cellularity is the identification of a certain $\tilde{A}$-basis of $p\,\mathrm{TL}_{rd}(\tilde{A})p$. This will be done in terms of certain diagrams. The Temperley–Lieb algebra $\mathrm{TL}_{rd}(\tilde{A})$ has $\tilde{A}$-basis consisting of planar diagrams from rd to rd, in the language of [Graham and Lehrer 1998]. These
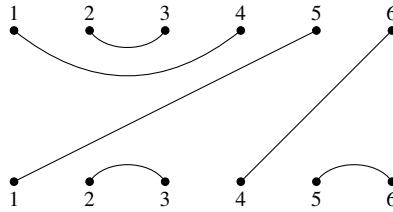
Figure 1. A planar diagram from 6 to 6.
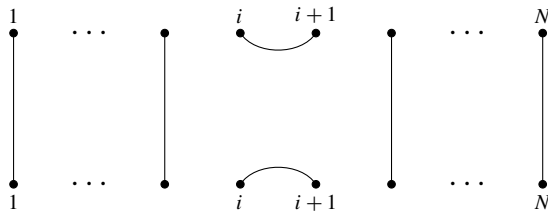
Figure 2. The generator f_i as a planar diagram from N to N.

diagrams are in bijection with the set Sym_{rd}^c of fully commutative elements [Fan and Green 1997] of Sym_{rd}, which in turn is in bijection with those elements of Sym_{rd} which correspond, under the Robinson–Schensted correspondence, to pairs of standard tableaux with two rows.

We shall describe now how to obtain a pair $(S(D), R(D))$ of standard tableaux directly from a planar diagram D. We use the planar diagram from 6 to 6 in Figure 1 to illustrate the description.

Each planar diagram from N to N consists of a set of N nonintersecting arcs. These may be through-arcs, joining an upper node to a lower node, or upper (top to top) or lower (bottom to bottom). The latter two are referred to as horizontal arcs. The diagrams are multiplied in the usual way, by concatenation, with each closed circle being replaced by $[2] = q + q^{-1}$. The generator f_i corresponds to the diagram in Figure 2. Note that if there are t through-arcs, then there are equally many top arcs and bottom arcs, and if this number is k, then $t + 2k = N$.

Now to each such planar diagram D, we associate an ordered pair $(S(D), T(D))$ of standard tableaux with two rows, as follows. Let $i_1, \ldots, i_k$ be the right nodes of the upper arcs written in ascending order. Then $S(D)$ has second row $i_1, \ldots, i_k$, and first row the complement of $\{i_1, \ldots, i_k\}$, written in ascending order. Note that the first row has $t + k \geq k$ elements. The tableau $T(D)$ is defined similarly, using the sequence $j_1, \ldots, j_k$ of right ends of the lower arcs. Note that both $S(D)$ and $T(D)$ correspond to the partition $(t + k, k)$, and hence the diagram corresponds via

the Robinson–Schensted correspondence to an element $w(D) \in \mathrm{Sym}_N$, which is fully commutative (see [Fan and Green 1997, Definition 3.3.1]).

Say that a horizontal arc is *small* if its vertices are $i, i+1$ for some i.

Definition 3.2. The *left set* $L(D)$ of a planar diagram D is the set of left vertices of the small upper arcs of D. Similarly, the *right set* $R(D)$ is the set of left vertices of the small lower arcs of D.

It is well known, and proved in a straightforward way using the Robinson–Schensted correspondence, that in the notation from Section 1A we have $L(D) = L(w(D))$, and similarly $R(D) = R(w(D))$.

For the diagram D in Figure 1, $L(D) = \{2\}$, while $R(D) = \{2, 5\}$. The tableaux $S(D)$ and $T(D)$ are given by

$$S(D) = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 5 & 6 \\ \hline 3 & 4 \\ \cline{1-2} \end{array} \quad \text{and} \quad T(D) = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 4 & 5 \\ \hline 3 & 6 \\ \cline{1-2} \end{array}.$$

Note that if $\mathscr{D}(S) := \{i \mid i+1 \text{ is in a lower row than } i\}$ is the descent set of a standard tableau S, then $L(D) = \mathscr{D}(S(D))$ and $R(D) = \mathscr{D}(T(D))$.

4. Proof of the main theorem

In this section we prove Theorem 1.1, and give some of its consequences. We keep the convention $\tilde{A} = A[([d]!)^{-1}]$ from Section 3.

4A. *A key lemma.* We begin by proving the following key result.

Lemma 4.1. *The $\tilde{A}$-algebra $p\,\mathrm{TL}_{dr}(\tilde{A})p$ has $\tilde{A}$-basis given by the set of elements pDp, where D is a diagram in $\mathrm{TL}_{dr}(\tilde{A})$ such that*

$$L(D) \cup R(D) \subseteq \{d, 2d, \dots, (r-1)d\}.$$

Proof. The $\tilde{A}$-algebra $E_r(d, \tilde{A}) \cong p\,\mathrm{TL}_{rd}(\tilde{A})p$ is evidently spanned by the elements pDp, where D ranges over planar diagrams from rd to rd. But for $i = 1, \dots, d-1$, we have seen that $p_d f_i = f_i p_d = 0$. It follows that $pDp = 0$ unless $L(D)$ and $R(D)$ are both contained in $\{d, 2d, \dots, (r-1)d\}$. Let $\mathscr{B}(d, r)$ be the set of planar diagrams satisfying these conditions. By the above remarks, it will suffice to show that

(4-1) $\{pDp \mid D \in \mathscr{B}(d, r)\}$ is linearly independent.

To prove (4-1) it suffices to work over the field K; in particular we are reduced to showing that

(4-2) $$|\mathscr{B}(d, r)| = \dim_K\left(\mathrm{End}_{\mathrm{U}_K}(\Delta_K(d)^{\otimes r})\right).$$

We shall prove (4-2) essentially by showing that both sides of (4-2) satisfy the same recurrence. Let us begin with the left side.

Observe that if a diagram $D \in \mathcal{B}(d, r)$ has t through-arcs, it may be thought of as a pair of diagrams D_1, D_2, where the D_i are monic diagrams from t to rd. Recall that a diagram from t to N ($t \leq N$) is monic if it has t through-arcs. One thinks of D_1 as the top half of D, and D_2 as the $*$ of the bottom half of D, where $*$ is the cellular involution on the Temperley–Lieb category that reflects diagrams in a horizontal line. It follows that if we write $|\mathcal{B}(d, r)| = b(d, r)$ and $|\mathcal{B}(d, r; t)| = b(d, r; t)$, where $\mathcal{B}(d, r; t)$ is the set of monic planar diagrams $D : t \to rd$ such that $L(D) \subseteq \{d, 2d, \dots, (r-1)d\}$, then

$$(4\text{-}3) \qquad b(d, r) = \sum_{0 \leq t \leq dr} b(d, r; t)^2.$$

Now consider the right side of (4-2). Define the positive integers $m(d, r; t)$ by

$$(4\text{-}4) \qquad \Delta_K(d)^{\otimes r} \cong \bigoplus_{t=0}^{dr} m(d, r; t)\Delta_K(t).$$

Thus the $m(d, r; t)$ are multiplicities, and $m(d, r; t) = 0$ unless $t \equiv rd$ (mod 2). Moreover, we obviously have, if $m(d, r) := \dim_K\big(\mathrm{End}_{\mathrm{U}_K}(\Delta_K(d)^{\otimes r})\big)$,

$$(4\text{-}5) \qquad m(d, r) = \sum_{0 \leq t \leq dr} m(d, r; t)^2.$$

It is clear that in view of (4-3) and (4-5), the lemma will follow if we prove that for all d, r and t,

$$(4\text{-}6) \qquad m(d, r; t) = b(d, r; t).$$

We shall prove (4-6) by induction on r. If $r = 1$, then

$$(4\text{-}7) \qquad m(d, 1; t) = b(d, 1; t) = \begin{cases} 0 & \text{if } t \neq d, \\ 1 & \text{if } t = d. \end{cases}$$

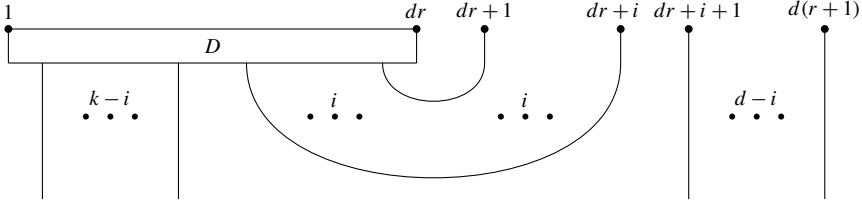Now by the Clebsch–Gordan formula, we have, for any integer n,

$$\Delta_K(d) \otimes \Delta_K(n) \cong \Delta_K(d+n) \oplus \Delta_K(d+n-2) \oplus \cdots \oplus \Delta_K(|d-n|).$$

It follows that

$$(4\text{-}8) \qquad m(d, r+1; t) = \sum_{s=t-d}^{t+d} m(d, r; s),$$

where $m(d, r; s) = 0$ if $s < 0$ or if $s > dr$.

We shall complete the proof of the lemma by showing that the numbers $b(d, r; t)$ satisfy a recurrence analogous to (4-8). For this observe that any diagram D in

Figure 3. From diagram D to diagram D'.

$\mathcal{B}(d, r; k)$ gives rise to a unique diagram in $\mathcal{B}(d, r+1; k+d-2i)$, for $0 \leq i \leq \min\{d, k\}$, as depicted in Figure 3, and each diagram $D' \in \mathcal{B}(d, r+1; t)$ arises in this way from a unique diagram in $\mathcal{B}(d, r; k)$ for a uniquely determined k. In fact, $k = t - d + 2i$ where i is the number of arcs in D' whose right vertices belong to $\{dr+1, \cdots, d(r+1)\}$. It follows that

$$(4\text{-}9) \qquad b(d, r+1; t) = \sum_{s=t-d}^{t+d} b(d, r; s),$$

where $b(d, r; s) = 0$ if $s < 0$ or if $s > dr$.

Comparing (4-8) with (4-9), and taking into account (4-7), it follows that $m(d, r; k) = b(d, r; k)$ for all d, r and k. This completes the proof of (4-6) above, and hence of the lemma. $\qquad\square$

4B. *Cellular structure.*

Proof of Theorem 1.1. We have seen that $E_r(d, \tilde{A}) \cong p\,\mathrm{TL}_{rd}(\tilde{A})p$, and that the latter algebra has the basis $\mathcal{B}(d, r)$, as stated in the theorem. It remains only to show that $p\,\mathrm{TL}_{rd}(\tilde{A})p$ has a cellular structure. Following [Graham and Lehrer 1996, Definition 1.1] we need to produce a cell datum $(\Lambda, M, C,^*)$ for $p\,\mathrm{TL}_{rd}(\tilde{A})p$.

Take Λ to be the poset $\{t \in \mathbb{Z} \mid 0 \leq t \leq dr$ and $dr - t \in 2\mathbb{Z}\}$, ordered as integers. For $t \in \Lambda$, let $M(t) := \mathcal{B}(d, r; t)$, the set of monic planar diagrams $D : t \to dr$ such that $L(D) \subseteq \{d, 2d, \ldots, (r-1)d\}$ (see Section 3B and the proof of Lemma 4.1). Then the map $C : \amalg_{t \in \Lambda} M(t) \times M(t) \longrightarrow p\,\mathrm{TL}_{rd}(\tilde{A})p$ is defined by $C(D_1, D_2) = pD_1 \circ D_2^* p$, where $\circ$ indicates concatenation of diagrams. We shall henceforth simply use juxtaposition to indicate composition in the Temperley–Lieb category. Since each diagram $D \in \mathcal{B}(r, d)$ is expressible uniquely as $D = D_1 D_2^*$ for some $t \in \Lambda$ and $D_1, D_2 \in M(t)$, it follows from Lemma 4.1 that C is a bijection from $\amalg_{t \in \Lambda} M(t) \times M(t)$ to a basis of $p\,\mathrm{TL}_{rd}(\tilde{A})p$. Finally, the anti-involution * is the restriction to $p\,\mathrm{TL}_{rd}(\tilde{A})p$ of the anti-involution on $\mathrm{TL}_{dr}(\tilde{A})$, namely, reflection in a horizontal line. Since $p^* = p$, we have $C(D_1, D_2)^* = (pD_1 D_2^* p)^* = pD_2 D_1^* p = C(D_2, D_1)$.

If $S, T \in M(t)$, we shall write $C(S, T) = C^t_{S,T}$, and for this proof only, write

$$\mathcal{A} = p\,\mathrm{TL}_{rd}(\tilde{A})\,p \quad \text{and} \quad \mathcal{A}(<i) = \sum_{\substack{j<i, \\ S,T \in M(j)}} \tilde{A}C^j_{S,T}.$$

It remains only to prove the axiom (C3) of [Graham and Lehrer 1996, Definition 1.1]. For this, let $S_1, S_2 \in M(s)$ and $T_1, T_2 \in M(t)$. Then

$$(4\text{-}10) \qquad\qquad C^s_{S_1,S_2} C^t_{T_1,T_2} = pS_1(S^*_2 pT_1)T^*_2 p,$$

so that if $s < t$, the left side is in $\mathcal{A}(<t)$, and there is nothing to prove. Hence we take $s \geq t$.

Now $S^*_2 pT_1$ is a morphism from t to s, and hence is an $\tilde{A}$-linear combination of planar diagrams D from t to s. Thus the left side of (4-10) is an $\tilde{A}$-linear combination of elements of the form $pS_1DT^*_2 p$. If D is not monic, then $pS_1DT^*_2 p \in \mathcal{A}(<t)$; if D is monic, then clearly $pS_1DT^*_2 p = pS'T^*_2 p$ for some monic $S' : t \to dr$.

It follows from (4-10) that modulo $\mathcal{A}(<t)$, $C^s_{S_1,S_2}C^t_{T_1,T_2} = \sum_{S \in \mathcal{B}(d,r;t)} a(S)C^t_{S,T_2}$, and $a(S)$ is independent of T_2. This proves the axiom (C3), and hence the cellularity of $\mathcal{A}$. The proof of Theorem 1.1 is now complete. $\qquad\qquad\square$

5. Endomorphism algebras and specialisation

We shall prove in this section results showing how the multiplicities of the indecomposable summands of the specialisations of $\Delta_A(d)^{\otimes r}$ corresponding to homomorphisms $A \to k$ where k is a field, relate to the dimensions of the simple modules for the corresponding endomorphism rings. It turns out that this is a consequence of a result on tilting modules which is valid for general quantum groups. Therefore in Sections 5A and 5B we deal with this general situation. Then in Section 5C we deduce the explicit consequences in our $\mathfrak{sl}_2$ case where we take advantage of our cellularity result from Section 4 on the endomorphism rings.

5A. *Integral endomorphism algebras and specialisation.* We now provide some rather general base change results for Hom-spaces between certain representations of quantum groups. So in this section we shall work with a general quantum group U_q over K with integral form U_A. We denote by k an arbitrary field (in this section k may even be any commutative noetherian A-algebra) made into an A-algebra by specializing q to $\zeta \in k \setminus \{0\}$ and set $\mathrm{U}_\zeta = \mathrm{U}_A \otimes_A k$. When M is a U_A-module we write M_q and M_ζ for the corresponding U_q- and U_ζ-modules, respectively.

For each dominant weight λ we write $\Delta_q(\lambda)$, $\Delta_A(\lambda)$ and $\Delta_\zeta(\lambda)$ for the Weyl modules for U_q, U_A and U_ζ respectively. Similarly, we have the dual Weyl modules $\nabla_q(\lambda)$, $\nabla_A(\lambda)$ and $\nabla_\zeta(\lambda)$ respectively. Then it is well known that, writing w_0 for

the longest element of the Weyl group,

$$\nabla_\zeta(\lambda) = \Delta_\zeta(-w_0\lambda)^*,$$

and similarly for $\nabla_A(\lambda)$ and $\nabla_q(\lambda)$.

We shall make repeated use of the following result. For any two weights $\lambda, \mu \in X$, we have

$$(5\text{-}1) \qquad \mathrm{Ext}^i_{U_A}(\Delta_A(\lambda), \nabla_A(\mu)) = \begin{cases} A & \text{if } \lambda = \mu \text{ and } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

This is proved exactly as in the corresponding classical case (see, for example, [Jantzen 2003, Proposition II.B.4]) by invoking the quantised Kempf vanishing theorem proved in general in [Ryom-Hansen 2003].

Lemma 5.1. *Let M, N be U_A-modules that are finitely generated as A-modules. If M has a filtration by Weyl modules $\Delta_A(\lambda)$ and N has a filtration by dual Weyl modules $\nabla_A(\mu)$, then $\mathrm{Hom}_{U_A}(M, N)$ is a free A-module of rank equal to $\dim_{\mathbb{Q}(q)} \mathrm{Hom}_{U_q}(M_q, N_q)$. Further, we have*

$$\mathrm{Hom}_{U_\zeta}(M_\zeta, N_\zeta) \simeq \mathrm{Hom}_{U_A}(M_A, N_A) \otimes_A k.$$

Proof. We have a spectral sequence with E_2-terms

$$E_2^{-p,q} = \mathrm{Tor}^A_p(\mathrm{Ext}^q_{U_A}(M, N), k)$$

converging to $\mathrm{Ext}^{q-p}_{U_\zeta}(M_\zeta, N_\zeta)$. By (5-1) we have $E_2^{-p,q} = 0$ if either $q > 0$ or $q = 0 < p$. Hence the spectral sequence collapses and we can read off the result. $\square$

Corollary 5.2. *Let V be a U_A-module which satisfies the assumption*

$$(5\text{-}2) \qquad\qquad V^* \otimes_A V \text{ has a } \nabla_A\text{-filtration.}$$

Then $\mathrm{End}_{U_\zeta}(V_\zeta^{\otimes r}) \simeq \mathrm{End}_{U_A}(V^{\otimes r}) \otimes_A k$.

Proof. We have $\mathrm{End}_{U_A}(V^{\otimes r}) \simeq \mathrm{Hom}_{U_A}(\Delta_A(0), (V^* \otimes V)^{\otimes r})$ because $\Delta_A(0)$ is the trivial U_A-module A. By the assumption (5-2), we may apply Lemma 5.1 to obtain the statement. $\square$

As usual we denote by ρ half the sum of the positive roots. Recall the concept of strongly multiplicity-free modules from [Lehrer and Zhang 2006]. A U_q-module V_q is *strongly multiplicity-free* if the weights of U_q occurring in V_q form a chain in the usual ordering on weights.

There are significant cases where the above result applies:

Proposition 5.3. *Suppose $V = \Delta_A(\lambda)$ for some dominant weight λ. Assume that V_q is strongly multiplicity-free, and that $-w_0\lambda + \mu + \rho$ is dominant for each weight μ of V. Then $V^* \otimes V$ has a ∇_A-filtration.*

Proof. Recall that U_A has a triangular decomposition $U_A = U_A^+ U_A^0 U_A^-$, and each weight μ defines a 1-dimensional representation of the subalgebra $U_A^0 U_A^-$, which we also denote by μ.

We have $V^* = \nabla_A(\lambda')$ where $\lambda' = -w_0\lambda$. Moreover ∇_A is realised as the induction functor $\mathrm{Ind}_{U_A^0 U_A^-}^{U_A}$. Hence by a standard property of induction,

$$V^* \otimes V = \mathrm{Ind}_{U_A^0 U_A^-}^{U_A}(\lambda') \otimes V = \mathrm{Ind}_{U_A^0 U_A^-}^{U_A}(\lambda' \otimes V),$$

where in this formula the last occurrence of V is its restriction to $U_A^0 U_A^-$. Now the hypothesis that V_q is strongly multiplicity-free implies that the weights of V are linearly ordered. But the weights of $\lambda' \otimes V$ are $\{\lambda' + \mu\}$, where μ runs over the weights of V. This set is therefore a linearly ordered chain, and accordingly, $\lambda' \otimes V$ has a $U_A^0 U_A^-$-module filtration

$$0 = F_0 \subset F_1 \subset \cdots \subset F_d = \lambda' \otimes V,$$

where $d = \dim V_q$, with the quotients F_i/F_{i-1} running over the $U_A^0 U_A^-$-modules $\lambda' + \mu$. Our hypothesis, together with (the quantised) Kempf's vanishing theorem imply that the higher (degree > 0) cohomology of the corresponding line bundles vanishes, and hence that induction is exact on this filtration. We therefore have a corresponding filtration of U_A-modules

$$0 \subset \nabla_A(F_1) \subset \cdots \subset \nabla_A(F_d) = \nabla_A(\lambda' \otimes V) = V^* \otimes V. \qquad \square$$

Corollary 5.4. *The conclusion of Proposition 5.3 holds in the following cases.*

(1) *V is a Weyl module with minuscule highest weight. This includes the natural modules in types A, C and D (but not type B).*

(2) *V is any Weyl module for $U_A(\mathfrak{sl}_2)$.*

(3) *V is the Weyl module in type G_2 with highest weight $2\alpha_1 + \alpha_2$, where α_1 and α_2 denote the two simple roots, with α_2 long.*

Proof. When V is minuscule, it is well known that for any weight μ of V we have $(\mu, \alpha^\vee) = \pm 1$ or 0, and hence (1) is clear. The case of $\mathfrak{sl}_2$ is evident, while in the case of type G_2, the weights of the Weyl module in question are the short roots, together with 0. This easily gives (3). $\qquad \square$

5B. *Multiplicities of tilting modules and dimensions of irreducibles.* In this section we shall prove some rather general results which will allow us to relate multiplicities of indecomposable tilting summands in tensor powers of certain representations of quantum groups to the dimensions of simple modules for the corresponding endomorphism algebras.

We note that the results of this section are similar in spirit to those of [Brundan and Kleshchev 1999, §3], which in turn have their genesis in some aspects of [Mathieu and Papadopoulos 1999, §3].

Theorem 5.5. *Let k be a field, U a k-algebra, and M a finite-dimensional (over k) U-module. Let $E = \mathrm{End}_U(M)$, and assume that for each indecomposable direct summand M' of M, we have $E'/\mathrm{Rad}\, E' \simeq k$ where $E' = \mathrm{End}_U(M')$. Then*

$$\frac{E}{\mathrm{Rad}\, E} \simeq \bigoplus_i M_{d_i}(k),$$

where $M_d(k)$ is the algebra of $n \times n$ matrices over k, i runs over the isomorphism classes of indecomposable U-modules (of course only a finite number occur), and the d_i are the multiplicities of the indecomposable summands of M.

Proof. Let $M = M_1 \oplus M_2 \oplus \cdots \oplus M_n$ be a decomposition of M into indecomposables. Then any endomorphism $\phi \in E$ may be written $\phi = (\phi_{ij})_{1 \le i,j \le n}$, where ϕ_{ij} is in $\mathrm{Hom}_U(M_j, M_i)$.

Now by Fitting's lemma, any endomorphism of M_i is either an automorphism or is nilpotent. With the notation $E_i := \mathrm{End}_U(M_i)$, it follows that for each i, the set $R_i := \{\psi \in E_i \mid \psi \text{ is not an automorphism}\}$ is a nilpotent ideal of E_i. In particular there is an integer N_i such that $R_i^{N_i} = 0$.

Next, suppose that we have a sequence $i = i_1, i_2, \ldots, i_{p+1} = i$, and $\phi_j := \phi_{i_j,i_{j+1}} \in \mathrm{Hom}_U(M_{i_{j+1}}, M_{i_j})$ for $j = 1, 2, \ldots, p$. Consider $\psi_1 := \phi_1 \ldots \phi_{p-1}\phi_p$ in $\mathrm{Hom}_U(M_i, M_i)$. We shall show that:

(5-3) ψ_1 *is an automorphism* $\implies$
                the M_{i_j} are all isomorphic, and ϕ_j is an isomorphism for each j.

To see (5-3), let $\psi_j = \phi_j \ldots \phi_p \phi_1 \ldots \phi_{j-1} \in \mathrm{Hom}(M_{i_j}, M_{i_j})$. If ψ_j is an automorphism for each j, then for each j, ϕ_{j-1} is injective and ϕ_j is surjective, whence each ϕ_j is an automorphism, and we are done. If not, then there is some j such that ψ_j is nilpotent. It follows that $\psi_1^N = 0$ for large N, which is a contradiction. This proves (5-3).

Now let J be the subspace of E consisting of the endomorphisms ϕ such that ϕ_{ij} is not invertible for each pair i, j. If

$$J_{ij} := \{\phi_{ij} \in \mathrm{Hom}_U(M_j, M_i) \mid \phi_{ij} \text{ is not invertible}\},$$

then again by Fitting's lemma, J_{ij} is an (E_i, E_j) bimodule, and using the observation (5-3) above, it is clear that J is an ideal of E. We shall show that J is nilpotent.

Let $\phi^{(1)}, \ldots, \phi^{(\ell)}$ be a sequence of elements of J. Then

$$(\phi^{(1)} \ldots \phi^{(\ell)})_{ij} = \sum_{k_1,k_2,\ldots,k_{\ell-1}} \phi^{(1)}_{ik_1} \phi^{(2)}_{k_1 k_2} \cdots \phi^{(\ell)}_{k_{\ell-1} j},$$

where the sum is over all sequences $k_1, k_2, \ldots, k_{\ell-1}$ with $1 \leq k_i \leq n$ for all i.

Now we have seen that for any j, if $R_j = \operatorname{Rad} E_j$, then there is an integer N_j such that $R_j^{N_j} = 0$. If we take $\ell \geq N_1 + N_2 + \cdots + N_n + 2$, then there some index a that occurs among the k_i at least $N_a + 1$ times. Then each summand in the expression for $(\phi^{(1)} \ldots \phi^{(\ell)})_{ij}$ contains a product of N_a noninvertible elements of E_a for some a, and hence is 0. Thus $J^{N_1 + \cdots + N_n + 2} = 0$.

Finally, it is clear that since $E_i / R_i \simeq k$ for each i, $E / J \simeq \bigoplus_{i=1}^n M_{d_i}(k)$.   □

The proof above actually yields the following corollary of the Artin–Wedderburn theorem.

Corollary 5.6. *Let M be as in Theorem 5.5 but drop the assumption on the endomorphism rings of direct summands of M. Then there are division rings D_i over k such that*

$$\frac{E}{\operatorname{Rad} E} \simeq \bigoplus_i M_{d_i}(D_i).$$

Proof. In this case Fitting's lemma yields that E_i / R_i is a division algebra D_i over k, and the argument above proves the assertion.   □

The application to our situation arises through the following property of finite-dimensional tilting modules for quantum groups. Let k be a field considered as an A-algebra via $q \mapsto \zeta \in k \setminus \{0\}$ and let U_ζ be as in Section 5A.

Proposition 5.7. *Let M be a finite-dimensional indecomposable tilting module for U_ζ and set $E = \operatorname{End}_{U_\zeta}(M)$. Then $E / \operatorname{Rad} E \simeq k$.*

Proof. By the Ringel–Donkin classification [Donkin 1993] (see [Andersen 1992] for the adaption to the quantum case) of indecomposable tilting modules we get that M has a unique highest weight $\lambda \in X^+$ and that the weight space M_λ is 1-dimensional. Therefore any $\varphi \in \operatorname{End}_{U_\zeta}(M)$ is given by a scalar $a \in k$ on M_λ. But then $\varphi - a \operatorname{id}_M$ is not an automorphism; i.e., $\varphi - a \operatorname{id}_M \in \operatorname{Rad} E$.   □

We denote the indecomposable tilting module for U_ζ with highest weight λ by $\mathcal{T}_\zeta(\lambda)$ and for an arbitrary tilting module $\mathcal{T}$ for U_ζ we write $(\mathcal{T} : \mathcal{T}_\zeta(\lambda))$ for the multiplicity with which $\mathcal{T}_\zeta(\lambda)$ occurs as a summand of $\mathcal{T}$. Then Theorem 5.5 together with Proposition 5.7 give the following result.

Corollary 5.8. *For any tilting module $\mathcal{T}$ for U_ζ and any $\lambda \in X^+$ we have*

$$(\mathcal{T} : \mathcal{T}_\zeta(\lambda)) = \dim_k L_\zeta(\lambda),$$

where $L_\zeta(\lambda)$ is the simple module for $E = \operatorname{End}_{U_\zeta}(\mathcal{T})$ corresponding to λ.

5C. *Multiplicities for* $\mathbf{U}_\zeta(\mathfrak{sl}_2)$. We now apply the above general results to $\mathfrak{sl}_2$. With k and ζ as above, the indecomposable tilting modules in this case are $\mathscr{T}_\zeta(m)$ with $m \in \mathbb{N}$. If ζ is not a root of unity in k then the category of finite-dimensional U_ζ-modules is semisimple and behaves exactly like the corresponding category for the generic quantum group U_q.

From now on we assume that ζ is a root of unity; for the specialisation U_ζ, etc., we assume that the homomorphism $A \to k$ is given by $q \mapsto \zeta$ (so $q^{1/2} \mapsto \sqrt{\zeta}$) and we set $\ell = \mathrm{ord}(\zeta^2)$. If d is a positive integer with $d < \ell$ we have $\Delta_\zeta(d) = \mathscr{T}_\zeta(d)$ and all the tensor powers $\mathscr{T}_r = \Delta_\zeta(d)^{\otimes r}$ are also tilting modules. We set $E_\zeta(d, r) = \mathrm{End}_{U_\zeta}(\mathscr{T}_r)$. By Lemma 5.1 we have

$$E_\zeta(d, r) = E_r(d, \tilde{A}) \otimes_{\tilde{A}} k,$$

where as before $\tilde{A} = A[([d]!)^{-1}]$. Note that our assumption $\ell > d$ ensures that the specialization $\phi_\zeta : A \to k$ factors through $\tilde{A}$ making k into an $\tilde{A}$-algebra.

Our cellularity results from Section 3 imply that

(5-4) $$E_\zeta(d, r) \cong p_\zeta \mathrm{TL}_{dr}(k) p_\zeta,$$

where p_ζ is the specialisation at $q = \zeta$ of the idempotent $p \in \mathrm{TL}_{dr}(\tilde{A})$. Note that in $\mathrm{TL}_{dr}(k) = \mathrm{TL}_{dr,\zeta}(k)$ the generators f_i satisfy $f_i^2 = (\zeta + \zeta^{-1}) f_i$.

The simple modules for the cellular algebra $p_\zeta \mathrm{TL}_{dr}(k) p_\zeta$ are parametrised by the poset $\Lambda = \{m \in \mathbb{Z} \mid 0 \leq m \leq dr \text{ and } dr - m \in 2\mathbb{Z}\}$; see Section 4B. We denote the simple module associated with $m \in \Lambda$ by $L_\zeta(m)$.

Theorem 5.9. *In the above notation, in particular assuming $\ell = \mathrm{ord}(\zeta^2) > d$, we have for $m \in \Lambda$,*

$$(\mathscr{T}_r : \mathscr{T}_\zeta(m)) = \dim_k L_\zeta(m).$$

This multiplicity is the rank of the matrix whose rows and columns are labelled by $\mathscr{B}(d, r; m)$ (see Section 4A) and whose (D_1, D_2)-entry is the coefficient of the identity map $m \to m$ (in the Temperley–Lieb category) in the expansion of $D_2^* p_\zeta D_1$ as a linear combination of diagrams from m to m.

Proof. The equality in the theorem is an immediate consequence of Corollary 5.8. To see the second statement note that $L_\zeta(m)$ is realised as follows: Let $W_\zeta(m)$ be the cell module corresponding to m. This has k-basis C_S, $S \in \mathscr{B}(d, r; m)$, the monic diagrams D from m to dr such that $L(D) \subseteq \{d, 2d, \dots, (r-1)d\}$. We may think of C_S as $p_\zeta S$, and then the $E_\zeta(d, r)$-action is by left composition: for $x \in E_\zeta(d, r)$, $xC_S = \sum_{T \in \mathscr{B}(d,r;m)} a(T, S)C_T$, where

$$x p_\zeta S = \sum_{T \in \mathscr{B}(d,r;m)} a(T, D) p_\zeta T + \text{lower terms},$$

where "lower" means "having fewer through-arcs".

There is an invariant form $(-,-)$ on $W_\zeta(m)$, defined by

(5-5) $C^m_{S,T}{}^2 \in (C_S, C_T) C^m_{S,T} + E_\zeta(d, r)(< m)$ for S, T in $\mathcal{B}(d, r; m)$.

The radical $\text{Rad}_\zeta(m)$ of this form is a submodule of $W_\zeta(m)$, and

$$L_\zeta(m) = W_\zeta(m)/\text{Rad}_\zeta(M).$$

It is therefore evident that $\dim L_\zeta(m)$ is equal to the rank of the Gram matrix $M_{m,\zeta}$, whose rows and columns are indexed by $\mathcal{B}(d, r; m)$, and whose (S, T)-entry is (C_S, C_T).

Finally, since $C^m_{S,T}{}^2 = p_\zeta S(T^* p_\zeta S) T^* p_\zeta$, and noting that $T^* p_\zeta S$ is a linear combination of diagrams from m to m, it follows from (5-5) that (C_S, C_T) is the coefficient of $\text{id}: m \to m$. □

Since $\dim W_\zeta(dr) = 1$ and the coefficient of $\text{id}: d \to d$ in $p_d(\zeta)$ is 1, it is immediate from the theorem that the multiplicity of $\mathcal{T}_\zeta(dr)$ is 1. We finish this section with a less trivial example.

Example 5.10. Take $k = dr - 2$ and recall that $d < \ell$. We shall compute the multiplicity of $\mathcal{T}_\zeta(k)$ in $\Delta_\zeta(d)^{\otimes r}$ for any d, r. Here $\mathcal{B}(d, r; dr-2) = \{S_1, S_2, \ldots, S_{r-1}\}$, where S_i is as shown in the figure:
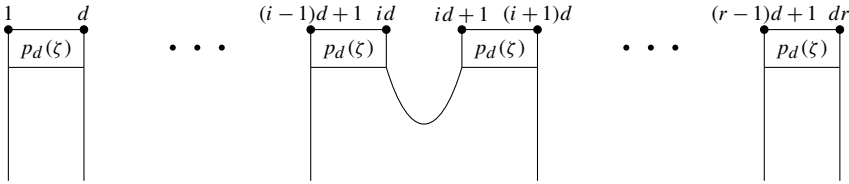
Now by repeated use of the diagrammatic recursion

it is straightforward to compute the Gram matrix $M_{dr-2,\zeta}$ of the invariant form (see the proof above). One shows that

$$(S_i, S_j) = \begin{cases} 0 & \text{if } j \neq i \text{ or } i \pm 1, \\ \dfrac{[2]_{\zeta^d}}{[d]_\zeta} & \text{if } j = i, \\ (-1)^{d+1}[d]_\zeta^{-1} & \text{if } j = i \pm 1. \end{cases}$$

Hence the Gram matrix of the invariant form is the $(r-1) \times (r-1)$ matrix

$$M_{dr-2,\zeta} = \frac{1}{[d]_\zeta} \begin{pmatrix} \delta & (-1)^{d+1} & 0 & \cdots & \cdots & 0 \\ (-1)^{d+1} & \delta & (-1)^{d+1} & 0 & \cdots & \vdots \\ 0 & (-1)^{d+1} & \delta & (-1)^{d+1} & 0 & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \\ & & & \ddots & (-1)^{d+1} \\ 0 & \cdots & \cdots & 0 & (-1)^{d+1} & \delta \end{pmatrix},$$

where $\delta = \zeta^d + \zeta^{-d} = [2]_{\zeta^d}$.

Now it is easily shown by induction that any $n \times n$ matrix of the form

$$A = \begin{pmatrix} a_1 & b_1 & 0 & \cdots & \cdots & 0 \\ 1 & a_2 & b_2 & 0 & \cdots & \vdots \\ 0 & 1 & a_3 & b_3 & 0 & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \\ & & & & \ddots & b_{n-1} \\ 0 & \cdots & \cdots & 0 & 1 & a_n \end{pmatrix}$$

with entries in a principal ideal domain may be transformed by row and column operations into

$$A' = \begin{pmatrix} 1 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & \vdots \\ 0 & 0 & 1 & 0 & 0 & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \\ & & & \ddots & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 0 & D \end{pmatrix},$$

where $D = \det(A)$. It follows that the rank of the Gram matrix $M_{dr-2,\zeta}$ is $r-1$ if $\det M_{dr-2,\zeta} \neq 0$, while if $\det M_{dr-2,\zeta} = 0$, the rank is $r-2$.

Now the determinant of $[d]_\zeta M_{dr-2,\zeta}$ is easily computed (cf. [Graham and Lehrer 1996, Equation 6.18.2]), and using this, we see that

$$\det M_{dr-2,\zeta} = (-1)^{(d+1)(r+1)} ([d]_\zeta)^{-(r-1)} [r]_{(-1)^{d+1}\zeta^d}.$$

It therefore follows that the multiplicity of $\mathcal{T}_\zeta(dr-2)$ in $\Delta_\zeta(d)^{\otimes r}$ is

$$\begin{cases} r-1 & \text{if } [r]_{(-1)^{d+1}\zeta^d} \neq 0, \\ r-2 & \text{otherwise.} \end{cases}$$

Finally, observe that

$$[r]_{(-1)^{d+1}\zeta^d} = 0 \quad \Longleftrightarrow \quad \zeta^{2dr} = 1.$$

Hence if we write (using the convention that for any root of unity ξ, we denote by $|\xi|$ or by $\mathrm{ord}(\xi)$ the multiplicative order of ξ)

$$(5\text{-}6) \qquad \ell = \begin{cases} |\zeta| & \text{if } |\zeta| \text{ is odd,} \\ \frac{1}{2}|\zeta| & \text{if } |\zeta| \text{ is even,} \end{cases}$$

then $\ell = |\zeta^2|$, whence the multiplicity of $\mathscr{T}_\zeta(dr-2)$ in $\Delta_\zeta(d)^{\otimes r}$ is given by

$$(5\text{-}7) \qquad (\mathscr{T}_r : \mathscr{T}_\zeta(dr-2)) = \begin{cases} r-1 & \text{if } \ell \nmid dr, \\ r-2 & \text{if } \ell \mid dr. \end{cases}$$

This shows also by standard cellular theory that the cell module $W_\zeta(dr-2)$ of $E_\zeta(d,r)$) is simple if $\ell \nmid dr$, while if $\ell \mid dr$, then $W_\zeta(dr-2)$ has composition factors $L_\zeta(d,r;dr-2)$ and $L_\zeta(d,r;dr)$ (the latter being the trivial module), each with multiplicity one.

6. Complex roots of unity

In this section we take $k = \mathbb{C}$ and fix a root of unity $\zeta \in \mathbb{C}$. As before we set $\ell = \mathrm{ord}(\zeta^2)$. In this case the structure of the tilting modules $\mathscr{T}_\zeta(m)$ is well understood, and hence, when $\ell > d$, provides an alternative approach to the computation of the multiplicities $\mu_\zeta(d,r;m) := (\Delta_\zeta(d)^{\otimes r} : \mathscr{T}_\zeta(m))$, and thus of the dimensions of the simple modules for the cellular algebra $E_\zeta(d,r)$ (see Theorem 5.9). In this section we demonstrate how this is done. We then show how these results on tilting modules may alternatively be deduced from results on the decomposition numbers of the algebras $E_\zeta(d,r)$, which are also proved in this section.

6A. *Structure of tilting modules.*

Proposition 6.1. *The indecomposable tilting module $\mathscr{T}_\zeta(m)$ for $U_\zeta = U_\zeta(\mathfrak{sl}_2)$ with highest weight m has the following description.*

(1) *If either $m < \ell$ or $m \equiv -1 \pmod{\ell}$ then $\mathscr{T}_\zeta(m) \simeq \Delta_\zeta(m)$ is irreducible.*

(2) *Write $m = a\ell + b$, where $a \geq 1$ and $0 \leq b < \ell - 1$. Then $\mathscr{T}_\zeta(m)$ is the unique nontrivial extension*

$$0 \longrightarrow \Delta_\zeta(m) \longrightarrow \mathscr{T}_\zeta(m) \longrightarrow \Delta_\zeta(m-2b-2) \longrightarrow 0.$$

Proof. This result is certainly well known and follows from the results of [Soergel 1998]. As we haven't been able to find a reference where this is explicitly stated we sketch the easy proof.

Denote by $\mathcal{L}_\zeta(m)$ the simple U_ζ-module with highest weight $m \in \mathbb{N}$ (not to be confused with the simple $E_\zeta(d, r)$-module $L_\zeta(m)$). It follows from the strong linkage principle [Andersen 2003] (or by direct calculations) that $\mathcal{L}_\zeta(m) = \Delta_\zeta(m)$ if and only if m satisfies the conditions in (1); in particular, (1) holds.

So assume $m = a\ell + b$ with a and b as in (2). The module $\Delta(a\ell - 1) \otimes_\mathbb{C} \Delta_\zeta(b+1)$ has a Weyl filtration with factors $\Delta_\zeta(m)$, $\Delta_\zeta(m-2)$, $\cdots$, $\Delta_\zeta(m - 2(b+1))$. Note that the first and the last factors belong to the same linkage class and that none of the other factors are in this class. Hence by the linkage principle [loc. cit.] there is a summand $\mathcal{T}$ of $\Delta_\zeta(a\ell - 1) \otimes_\mathbb{C} \Delta_\zeta(b+1)$ which has these two Weyl factors, i.e., fits into an exact sequence

$$0 \longrightarrow \Delta_\zeta(m) \longrightarrow \mathcal{T} \longrightarrow \Delta_\zeta(m - 2b - 2) \longrightarrow 0.$$

By case (1) we see that $\Delta_\zeta(a\ell - 1) \otimes_\mathbb{C} \Delta_\zeta(b+1)$ is tilting. Hence so is our summand $\mathcal{T}$. The proof of case (2) will therefore be complete if we prove that $\mathcal{T}$ is indecomposable. This in turn would follow if there were no nontrivial homomorphisms $\mathcal{T}$ of $\Delta_\zeta(a\ell - 1) \otimes_\mathbb{C} \Delta_\zeta(b+1) \longrightarrow \mathcal{L}_\zeta(m)$, for if the last sequence splits, there would be such a homomorphism. To check the last statement, we need the quantised Steinberg tensor product theorem [Andersen and Wen 1992, Theorem 1.10] for simple modules, $\mathcal{L}_\zeta(m) \simeq \mathcal{L}_\zeta(a\ell) \otimes \mathcal{L}_\zeta(b)$ (again in the case at hand this can alternatively be checked by direct calculations).

Using this together with the self-duality of simple modules and the result in (1) we get

$$\mathrm{Hom}_{U_\zeta}\big(\Delta_\zeta(a\ell - 1) \otimes_\mathbb{C} \Delta_\zeta(b+1), \mathcal{L}_\zeta(m)\big)$$

$$\simeq \mathrm{Hom}_{U_\zeta}\big(\mathcal{L}_\zeta(a\ell - 1) \otimes_\mathbb{C} \mathcal{L}_\zeta(b+1), \mathcal{L}_\zeta(m)\big)$$

$$\simeq \mathrm{Hom}_{U_\zeta}\big(\mathcal{L}_\zeta((a-1)\ell) \otimes_\mathbb{C} \mathcal{L}_\zeta(\ell - 1) \otimes_\mathbb{C} \mathcal{L}_\zeta(b+1), \mathcal{L}_\zeta(a\ell) \otimes_\mathbb{C} \mathcal{L}_\zeta(b)\big)$$

$$\simeq \mathrm{Hom}_{U_\zeta}\big(\mathcal{L}_\zeta((a-1)\ell) \otimes_\mathbb{C} \mathcal{L}_\zeta(b+1) \otimes_\mathbb{C} \mathcal{L}_\zeta(b), \mathcal{L}_\zeta(a\ell) \otimes_\mathbb{C} \mathcal{L}_\zeta(\ell - 1)\big)$$

$$\simeq \mathrm{Hom}_{U_\zeta}\big(\mathcal{L}_\zeta((a-1)\ell) \otimes_\mathbb{C} \mathcal{L}_\zeta(b+1) \otimes_\mathbb{C} \mathcal{L}_\zeta(b), \mathcal{L}_\zeta((a+1)\ell - 1)\big).$$

The last Hom-space is 0 because, by our condition on b, the weight $(a+1)\ell - 1$ is strictly larger than all weights of $\mathcal{L}_\zeta((a-1)\ell) \otimes_\mathbb{C} \mathcal{L}_\zeta(b+1) \otimes_\mathbb{C} \mathcal{L}_\zeta(b)$. $\qquad\square$

Since the weights of $\Delta_\zeta(m)$ are $m, m-2, \cdots, -m$, each occurring with multiplicity one, we deduce the following result.

Corollary 6.2. *We have*

$$\dim \mathcal{T}_\zeta(m)_t = \begin{cases} 1 & \text{if } t = m - 2i, \ 0 \le i \le m \text{ in case (1)}, \\ 2 & \text{if } t = m - 2j, \ b+1 \le j \le m - (b+1) \text{ in case (2)}, \\ 1 & \text{if } t = m - 2j, \text{ with } 0 \le j \le b \text{ or } m \ge j \ge m - b \text{ in case (2)}, \\ 0 & \text{otherwise.} \end{cases}$$

6B. *Multiplicities and dimensions.* Now the equation

$$(6\text{-}1) \qquad \Delta_\zeta(d)^{\otimes r} \cong \bigoplus_{m=0}^{dr} \mu_\zeta(d, r; m) \mathscr{T}_\zeta(m).$$

may be used to relate the multiplicities to the dimensions of the weight spaces. For this purpose, we make the following definitions.

Definition 6.3. (1) Let $w(d, r; m) := \dim(\Delta_\zeta(d)^{\otimes r})_m$. This is independent of ζ.

(2) Let $a_m = a_m(d, r) := \left|\left\{(i_1, \ldots, i_r) \mid 0 \le i_j \le d \text{ for all } j \text{ and } \sum_j i_j = m\right\}\right|$.

Note that $a_m = a_{dr-m}$ for all m.

Lemma 6.4. (1) *For* $0 \le m \le dr$, $m \equiv dr \pmod 2$, $w(d, r; m) = a_{(m+dr)/2}$.

(2) *We have*

$$w(d, r; m) = \mu_\zeta(d, r; m) + \sum_{j=1}^{(dr-m)/2} \dim \mathscr{T}_\zeta(m + 2j)_m \mu_\zeta(d, r; m + 2j).$$

The first statement follows easily from the fact that $\Delta_\zeta(d)^{\otimes r}$ has q-character $[d+1]^r$, while the second arises from (6-1) by taking the dimension of the m-weight spaces on both sides, taking into account that $\mathscr{T}_\zeta(t)$ has only weights m that satisfy $m = t - 2i$, $i \ge 0$, and $rd \ge m \ge -rd$.

Lemma 6.4(2) may be used to determine the multiplicities $\mu_\zeta(d, r; m)$ recursively. We shall do this for the case considered in Example 5.10.

Example 6.5. Let us compute $\mu_\zeta(d, r, dr - 2)$. By Lemma 6.4(2),

$$w(d, r; dr - 2) = \mu_\zeta(d, r; dr - 2) + \dim \mathscr{T}_\zeta(dr)_{dr-2}.$$

Moreover, it follows from Corollary 6.2 that

$$\dim \mathscr{T}_\zeta(dr)_{dr-2} = \begin{cases} 2 & \text{if } b = 0, \\ 1 & \text{if } b \ne 0. \end{cases}$$

Noting that by Lemma 6.4(1) we have $w(d, r, dr - 2) = a_{dr-1} = a_1 = r$, we get

$$\mu_\zeta(d, r; dr - 2) = \begin{cases} r - 1 & \text{if } \ell \nmid dr, \\ r - 2 & \text{if } \ell \mid dr, \end{cases}$$

in accord with (5-7).

Example 6.6. In Example 6.5 we considered multiplicities $\mu_\zeta(d, r; t)$, where t was large, namely $t = dr - 2$. We now consider the case where t is small.

Assume $t < \ell$. Then we may apply [Andersen and Paradowski 1995, Formula 3.20 (1)]. Using the notation from Section 4A this formula reads in our case

$$\mu_\zeta(d, r; t) = \sum_{j \ge 0} m(d, r; t + 2j\ell) - \sum_{i > 0} m(d, r; 2i\ell - t - 2).$$

Recall that the multiplicities $m(d, r; t)$ are given by the recursion relation (4-8); i.e., they may be calculated by induction on r.

In fact this formula is valid in general: maintaining the notation of Example 6.6 (except that the integer t below may now be arbitrary) we have the following result.

Proposition 6.7. *Let $t \in \mathbb{N}$.*

(1) *If $t \equiv -1 \pmod{\ell}$ then $\mu_\zeta(d, r; t) = m(d, r; t)$.*

(2) *If $t \not\equiv -1 \pmod{\ell}$ then, writing $t = a\ell + b$ with $0 \leq b \leq \ell - 2$, we have*

$$\mu_\zeta(d, r; t) = \sum_{j \geq 0} m(d, r; t + 2j\ell) - \sum_{i \geq 1} m(d, r; t - 2b - 2 + 2i\ell)$$

$$= \sum_{j \geq 0} m(d, r; t + 2j\ell) - \sum_{i \geq a+1} m(d, r; 2i\ell - t - 2).$$

Proof. This follows easily from the description of the indecomposable tilting modules $\mathcal{T}_\zeta(m)$ in Proposition 6.1 by taking characters in the relation $\Delta_\zeta(d)^{\otimes r} \cong \bigoplus_m \mu(d, r; m)\mathcal{T}_\zeta(m)$. Let $\mathscr{C}_1$ be the set of positive integers occurring in case (1) of Proposition 6.1, and similarly let $\mathscr{C}_2$ be those occurring in case (2).

If we denote by c_t the q-character of $\Delta_q(t)$, then Proposition 6.1 shows that if $t \in \mathscr{C}_1$, then $\mathrm{char}(\mathcal{T}_\zeta(t)) = c_t$, while if $t \in \mathscr{C}_2$, then $\mathrm{char}(\mathcal{T}_\zeta(t)) = c_t + c_{t-2b-2}$. Now substitute these values and compare coefficients of c_t in the equation

$$\sum_{t \in \mathbb{N}} m(d, r; t)c_t = \sum_{t \in \mathscr{C}_1} \mu_\zeta(d, r; t)\,\mathrm{char}(\mathcal{T}_\zeta(t)) + \sum_{t \in \mathscr{C}_2} \mu_\zeta(d, r; t)\,\mathrm{char}(\mathcal{T}_\zeta(t)).$$

One obtains $\mu_\zeta(d, r; t) = m(d, r; t)$ if $t \equiv -1 \pmod{\ell}$, while if $t = a\ell + b$ with $a \geq 0$ and $0 \leq b \leq \ell - 2$, we have

(6-2) $$m(d, r; t) = \mu_\zeta(d, r; t) + \mu_\zeta(d, r; (a+2)\ell - b - 2).$$

Now for any integer $t = a\ell + b \geq 0$ such that $t \not\equiv -1 \pmod{\ell}$, write $g(t) = (a+2)\ell - b - 2$; then $g(t) \not\equiv -1 \pmod{\ell}$, and the relation above reads $m(d, r; t) = \mu_\zeta(d, r; t) + \mu_\zeta(d, r; g(t))$. It follows that

$$\mu_\zeta(d, r; t) = \sum_{i \geq 0} m(d, r; g^{2i}(t)) - \sum_{j \geq 0} m(d, r; g^{2j+1}(t)).$$

The statements (1) and (2) are now immediate. $\square$

As these multiplicities are also dimensions of simple modules for our cellular algebra from Section 4, we may rewrite these formulae as follows (again using notation from Section 4A).

Corollary 6.8. *Let $t \in \mathbb{N}$.*

(1) *If $t \equiv -1 \pmod{\ell}$ then $\dim_{\mathbb{C}} L_\zeta(t) = b(d, r; t)$.*

(2) *If* $t \not\equiv -1 \pmod{\ell}$ *then, writing* $t = a\ell + b$ *with* $0 \le b \le \ell - 2$, *we have*

$$\dim_{\mathbb{C}} L_{\zeta}(t) = \sum_{j \ge 0} b(d, r; t + 2j\ell) - \sum_{i \ge a+1} b(d, r; 2i\ell - t - 2).$$

Note that the numbers $b(d, r; t)$ are dimensions of the cell modules of the cellular algebra $p\,\mathrm{TL}_{dr}(\tilde{A})p$ that do not change under specialisation.

6C. *Decomposition numbers.* In this section we shall determine the decomposition numbers of the cellular algebra $E_{\zeta}(d, r)$, and show how the weight multiplicities of the tilting modules are determined by these, giving an alternative proof of Corollary 6.2. The algebra has cell modules $W_{\zeta}(t)$ as implied in Section 4B and $\dim(W_{\zeta}(t)) = b(d, r; t)$. If $L_{\zeta}(t)$ is the corresponding simple module, we write $d_{st} = [W_{\zeta}(t) : L_{\zeta}(s)]$ for the multiplicity of $L_{\zeta}(s)$ in $W_{\zeta}(t)$. It is known by the theory of cellular algebras that the matrix (d_{st}) is lower unitriangular.

We have $\dim(L_{\zeta}(t)) = \mu_{\zeta}(d, r; t)$, and therefore we clearly have

(6-3) $$b(d, r; t) = \sum_{s \ge t} d_{st} \mu_{\zeta}(d, r; s).$$

Theorem 6.9. *Maintain the notation above. Suppose* $\ell \in \mathbb{N}$ *is such that* $\ell = \mathrm{ord}(\zeta^2)$ *and* $\ell > d$, *and write* $\mathbb{N} = \mathcal{N}_1 \sqcup \mathcal{N}_2$, *where* $\mathcal{N}_1 = \{t \in \mathbb{N} \mid t \equiv -1 \pmod{\ell}\}$ *and* $\mathcal{N}_2 = \mathbb{N} \setminus \mathcal{N}_1$. *Let* $g : \mathcal{N}_2 \longrightarrow \mathcal{N}_2$ *be the function defined in the proof of Proposition 6.7, viz. if* $t = a\ell + b$ *with* $0 \le b \le \ell - 2$, *then* $g(t) = (a + 1)\ell + \ell - b - 2$. *Observe that* $g(t) = t + 2(\ell - b - 1) \ge t + 2$, *and that* $g(t) \equiv t \pmod 2$.

(1) *For each* $t \in \mathcal{N}_2$ *such that* $0 \le t < g(t) \le dr$ *and* $t \equiv dr \pmod 2$, *there is a nonzero homomorphism* $\theta_t : W_{\zeta}(g(t)) \longrightarrow W_{\zeta}(t)$ *which is uniquely determined up to scalar multiplication.*

(2) *The* θ_t *are the only nontrivial homomorphisms between the cell modules of* $E_{\zeta}(d, r)$.

(3) *Let* $t \in \mathbb{N}$ *be such that* $0 \le t \le dr$ *and* $t \equiv dr \pmod 2$. *If* $t \in \mathcal{N}_2$ *and* $g(t) \le dr$, *then* $W_{\zeta}(t)$ *has composition factors* $L_{\zeta}(t)$ *and* $L_{\zeta}(g(t))$, *each with multiplicity* 1. *All other cell modules are simple.*

(4) *The decomposition numbers of* $E_{\zeta}(d, r)$ *are all equal to* 0 *or* 1.

Note that (3) and (4) are formal consequences of (1) and (2).

Proof. We begin by observing that the statement is true when $d = 1$. In this case $E_{\zeta}(1, r) = \mathrm{TL}_{r,\zeta}(\mathbb{C})$, the structure of whose cell modules (as well as all homomorphisms between them) is treated in [Graham and Lehrer 1998]. In particular, Theorem 5.3 of that reference asserts that (in our notation above) if $s \ne t$, then $L_{\zeta}(s)$ is a composition factor of $W_{\zeta}(t)$ if and only if s satisfies both (i) $t + 2\ell > s > t$

and (ii) $s + t + 2 \equiv 0 \pmod{2\ell}$. It is an easy exercise to show that (i) and (ii) are equivalent to (iii) $t \not\equiv -1 \pmod{\ell}$ and (iv) $s = g(t)$. This yields all the statements of the theorem for this case.

Next recall that $E_\zeta(d, r) \cong p_d(\zeta) \mathrm{TL}_{dr,\zeta}(\mathbb{C}) p_d(\zeta)$, where $p_d(\zeta)$ is the specialisation at ζ of the idempotent p_d. Thus we may define the exact functor $\mathscr{F}_d : \mathrm{Mod}(\mathrm{TL}_{dr,\zeta}(\mathbb{C})) \longrightarrow \mathrm{Mod}(E_\zeta(d, r))$ by $M \mapsto p_d(\zeta)M$, where Mod indicates the category of left modules for the relevant algebra. Now it is evident from the description in Section 4B of the cell module $W(t)$ and its basis $\mathscr{B}(d, r; t)$ that $\mathscr{F}_d(W_{\mathrm{TL}_{dr,\zeta}(\mathbb{C})}(t)) = W_{E_\zeta(d,r)}(t)$ for all t with $0 \leq t \leq dr$ and $t + dr \in 2\mathbb{Z}$.

Moreover by exactness, for any simple $\mathrm{TL}_{dr,\zeta}(\mathbb{C})$-module L, $\mathscr{F}_d(L)$ is either a simple $E_\zeta(d, r)$-module or zero. Thus it follows (also from the explicit diagrammatic description) that $\mathscr{F}_d(L_{\mathrm{TL}_{dr,\zeta}(\mathbb{C})}(t)) = L_{E_\zeta(d,r)}(t)$ whenever the latter is nonzero. Given the description in Section 4B of the cellular structure, and the fact that $\mathrm{TL}_{dr,\zeta}(\mathbb{C})$ is quasihereditary when $\zeta \neq \zeta_4 = \exp(\pi i / 2)$, $\mathscr{F}_d$ does not kill any nontrivial simple $\mathrm{TL}_{dr,\zeta}(\mathbb{C})$-module (this may be checked directly when $\zeta = \zeta_4$). The quasiheredity of $\mathrm{TL}_{dr,\zeta}(\mathbb{C})$ when $\zeta \neq \zeta_4$ is well known, but may be seen as follows.

Since $\zeta + \zeta^{-1} \neq 0$, if $t \in \mathbb{N}$, $0 \leq t \leq dr$, $t \equiv dr \pmod 2$, then for any monic diagram $u : t \to dr$, we have $u^* u = (\zeta + \zeta^{-1})^{(dr-t)/2} \mathrm{id}_t \neq 0$; hence, if u is thought of as an element of $W_\zeta(t)$, then $(u, u) \neq 0$. Thus, for any such t, $L_\zeta(t) \neq 0$. Although it is not needed for the proof of the theorem, the fact that if $L_{\mathrm{TL}_{dr,\zeta}(\mathbb{C})}(t) \neq 0$ then $\mathscr{F}_d(L_{\mathrm{TL}_{dr,\zeta}(\mathbb{C})}(t)) \neq 0$ is verified in the same way, but requires a computation, using the recurrence (5-6) in Example 5.10 above, to show that for a nonzero element $u = p_d D \in W_\zeta(t)$, where $D : t \to dr$ is a monic diagram, we have $(u, u) \neq 0$. That such elements exist is easily verified.

By the case $d = 1$ of Theorem 6.9 or, more precisely, [Graham and Lehrer 1998, Theorem 5.3] applied to $\mathrm{TL}_{dr,\zeta}(\mathbb{C})$, if $t \in \mathcal{N}_2$, $0 \leq t < g(t) \leq dr$ and $t \equiv dr \pmod 2$, then $W_{\mathrm{TL}_{dr,\zeta}(\mathbb{C})}(t)$ has composition factors $L_{\mathrm{TL}_{dr,\zeta}(\mathbb{C})}(t)$ and $L_{\mathrm{TL}_{dr,\zeta}(\mathbb{C})}(g(t))$. All other cell modules for $\mathrm{TL}_{dr,\zeta}(\mathbb{C})$ are simple. It follows from the previous paragraph that similarly, if $t \in \mathcal{N}_2$, $0 \leq t < g(t) \leq dr$ and $t \equiv dr \pmod 2$, then $W_{E_\zeta(d,r)}(t)$ has composition factors $L_{E_\zeta(d,r)}(t)$ and $L_{E_\zeta(d,r)}(g(t))$, and that other cell modules for $E_\zeta(d, r)$ are simple. All statements in the theorem are now easy consequences of standard cellular theory. $\qquad\square$

Remark 6.10. (1) From Theorem 6.9 it follows that (6-3) implies (6-2) and the other statements in Proposition 6.7. Thus the multiplicities $\mu_\zeta(d, r; t)$ are determined by Theorem 6.9.

(2) Since the dimensions $w(d, r; t)$ are known (Lemma 6.4(1)), it follows from Lemma 6.4(2) that the dimensions of the weight spaces $\mathscr{T}_\zeta(dr)_m$ are determined by Theorem 6.9.

(3) There are some analogies between this work and the modular theory developed by Erdmann [1995]. In the case $n = 2$, Erdmann dealt only with the 2-dimensional representation of $\mathfrak{gl}_2$. Nonetheless, there appear to be some similarities between her formulae and the Gram determinants of the cell modules in our situation.

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