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ON MAXIMAL TORI OF ALGEBRAIC GROUPS OF TYPE G_2

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To the memory of Robert Steinberg

Given an octonion algebra C over a field k, its automorphism group is an algebraic semisimple k-group of type G_2. We study the maximal tori of G in terms of the algebra C.

1. Introduction

For classical algebraic groups, and in particular for arithmetic fields, the investigation of maximal tori is an interesting topic in the theory of algebraic groups and arithmetic groups; see [Prasad and Rapinchuk 2009, § 9; 2010] and also [Garibaldi and Rapinchuk 2013]. It is also related to the Galois cohomology of quasisplit semisimple groups by Steinberg's section theorem; that connection is an important ingredient of this paper.

Let k be a field, let k_s be a separable closure and denote by $\Gamma_k = \mathrm{Gal}(k_s/k)$ the absolute Galois group of k. In this paper, we study maximal tori of groups of type G_2. We recall that a semisimple algebraic k-group G of type G_2 is the group of automorphisms of a unique octonion algebra C [Knus et al. 1998, 33.24]. We come now to the following invariant of maximal tori [Gille 2004; Raghunathan 2004]. Given a k-embedding of $i : T \to G$ of a rank -2 torus, we have a natural action of Γ_k on the root system $\Phi(G_{k_s}, i(T_{k_s}))$, and the yoga of twisted forms defines then a cohomology class $\mathbf{type}(T, i) \in H^1(k, W_0)$, which is called the type of the couple (T, i). Here $W_0 \cong \mathbb{Z}/2\mathbb{Z} \times S_3$ is the Weyl group of the Chevalley group of type G_2. By Galois descent [Knus et al. 1998, 29.9], a W_0-torsor is nothing but a couple (k', l), where k' (resp. l) is a quadratic (resp. cubic) étale k-algebra. The main problem is then the following: given an octonion algebra C and such a couple (k', l), under which additional conditions is there a k-embedding $i : T \to G = \mathrm{Aut}(C)$ of type $[(k', l)] \in H^1(k, W_0)$?

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We give a precise answer when the cubic extension l is not a field (Section 4.4). When l is a field, we use subgroups of type A_2 of G to relate with maximal tori of special unitary groups where we can apply results of Knus, Haile, Rost and Tignol [Haile et al. 1996]. This provides a criterion which is quite complicated (see Proposition 5.2.6).

The problem above can be formulated in terms of existence of k-points for a certain homogeneous space X under G associated to k', l; see [Lee 2014, §1] or Section 2.6. We recall here Totaro's general question [2004, Question 0.2].

For a smooth connected affine k-group G over the field k and a homogeneous G-variety Y such that Y has a zero-cycle of degree $d > 0$, does Y necessarily have a closed étale point of degree dividing d?

Starting with Springer's odd extension theorem for quadratic forms, there are several cases where the question has a positive answer, mainly for principal homogeneous spaces (i.e., torsors). We quote here the results by Totaro [2004, Theorem 5.1] and Garibaldi and Hoffmann [2006] for certain exceptional groups, Black [2011] for classical adjoint groups and Black and Parimala [2014] for semisimple simply connected classical groups of rank ≤ 2.

If the base field k is large enough (e.g., $\mathbb{Q}(t)$, $\mathbb{Q}((t))$), we can construct a homogeneous space X under G of the shape above having a quadratic point and a cubic point but no k-point (Theorem 4.5.3). This provides a new class of counterexamples to the question in the case $d = 1$ which are geometrically speaking simpler than those of Florence [2004] and Parimala [2005].

Finally, in case of a number field, we show that this kind of variety satisfies the Hasse principle. In this case, our results are effective; that is, we can describe the type of the maximal tori of a given group of type G_2, for example, for the "compact" G_2 over the rational numbers (see Examples 6.4).

Let us review the contents of the paper. In Section 2, we recall the notion of type and oriented type for a k-embedding $i : T \to G$ of a maximal k-torus in a reductive k-group G. We study then the image of the map $H^1(k, T) \to H^1(k, G)$ of Galois cohomology and relate it, in the quasisplit case, with Steinberg's theorem on Galois cohomology. Section 3 gathers basic facts on octonion algebras which are used in the core of the paper, namely Sections 4 and 5. The number field case is considered in the short Section 6. Finally, the Appendix deals with the Galois cohomology of k-tori and quasisplit reductive k-groups over Laurent series fields.

A. Fiori [2015] investigated independently maximal tori of algebraic groups of type G_2 and their rational conjugacy classes. Though his scope is different, certain tools are common with our paper, for example, the definition and the study of the subgroup of type A_2 attached to a maximal torus (Proposition 5.5 in [loc. cit.], §5.1 here).

2. Maximal tori of reductive groups and image of the cohomology

Let G be a reductive k-group. We are interested in maximal tori of G and also in the images of the map $H^1(k,T) \to H^1(k,G)$. We shall discuss refinements of the application of Steinberg's theorem on rational conjugacy classes to Galois cohomology.

2.1. *Twisted root data.*

2.1.1. *Definition.* In [Lee 2014, 1.3] and [Gille 2014, §6.1], in the spirit of [Demazure and Grothendieck 1970a; 1970b; 1970c], the notion of twisted root data is defined over an arbitrary base scheme S. We focus here on the case of the base field k and use the equivalence of categories between étale sheaves over $\mathrm{Spec}(k)$ and the category of Galois sets, namely sets equipped with a continuous action of the absolute Galois group Γ_k.

We recall from [Springer 1998, §7.4] that a root datum is a quadruple $\Psi = (M, R, M^\vee, R^\vee)$, where M is a lattice, $M^\vee$ its dual, $R \subset M$ a finite subset (the roots), $R^\vee$ a finite subset of $M^\vee$ (the coroots), and a bijection $\alpha \mapsto \alpha^\vee$ of R onto $R^\vee$ which satisfy the next axioms (RD1) and (RD2).

For each $\alpha \in R$, we define endomorphisms s_α of M and $s_\alpha^\vee$ of $M^\vee$ by

$$s_\alpha(m) = m - \langle m, \alpha^\vee\rangle\alpha, \quad s_\alpha^\vee(f) = f - \langle \alpha, f\rangle\alpha^\vee \quad (m \in M, f \in M^\vee).$$

(RD1) For each $\alpha \in R$, $\langle \alpha, \alpha^\vee\rangle = 2$;

(RD2) For each $\alpha \in R$, $s_\alpha(R) = R$ and $s_\alpha^\vee(R^\vee) = R^\vee$.

We denote by $W(\Psi)$ the subgroup of $\mathrm{Aut}(M)$ generated by the s_α; it is called the Weyl group of Ψ.

2.1.2. *Isomorphisms, orientation.* An isomorphism of root data

$$\Psi_1 = (M_1, R_1, M_1^\vee, R_1^\vee) \xrightarrow{\sim} \Psi_2 = (M_2, R_2, {M_2}^\vee, {R_2}^\vee)$$

is an isomorphism $f : M_1 \xrightarrow{\sim} M_2$ such that f induces a bijection $R_1 \xrightarrow{\sim} R_2$ and f induces a dual isomorphism $f^\vee : M_2^\vee \xrightarrow{\sim} M_1^\vee$ such that $f^\vee$ induces a bijection ${R_2}^\vee \xrightarrow{\sim} R_1^\vee$. Let $\mathrm{Isom}(\Psi_1, \Psi_2)$ be the scheme of isomorphisms between Ψ_1 and Ψ_2. We define the quotient $\mathrm{Isomext}(\Psi_1, \Psi_2)$ by $\mathrm{Isomext}(\Psi_1, \Psi_2) = W(\Psi_2)\backslash \mathrm{Isom}(\Psi_1, \Psi_2)$, which is isomorphic to $\mathrm{Isom}(\Psi_1, \Psi_2)/W(\Psi_1)$.

An orientation u between Ψ_1 and Ψ_2 is an element $u \in \mathrm{Isomext}(\Psi_1, \Psi_2)$. We can then define the set $\mathrm{Isomint}_u(\Psi_1, \Psi_2)$ of inner automorphisms with respect to the orientation u as the preimage of u by the projection $\mathrm{Isom}(\Psi_1, \Psi_2) \to \mathrm{Isomext}(\Psi_1, \Psi_2)$.

We denote by $\mathrm{Aut}(\Psi) = \mathrm{Isom}(\Psi, \Psi)$ the group of automorphisms of the root datum Ψ, and we have an exact sequence

$$1 \to W(\Psi) \to \mathrm{Aut}(\Psi) \to \mathrm{Autext}(\Psi) \to 1,$$

where $\mathrm{Autext}(\Psi) = \mathrm{Isomext}(\Psi, \Psi)$ stands for the quotient group of automorphisms of Ψ (called the group of exterior or outer automorphisms of Ψ). The choice of an ordering on the roots permits us to define a set of positive roots Ψ_+, its basis and the Dynkin index $\mathrm{Dyn}(\Psi)$ of Ψ. Furthermore, we have an isomorphism $\mathrm{Aut}(\Psi, \Psi_+) \xrightarrow{\sim} \mathrm{Autext}(\Psi)$ so that the above sequence is split.

2.1.3. *Twisted version.* A twisted root datum is a root datum equipped with a continuous action of Γ_k. To distinguish from the absolute case, we shall use the notation $\underline{\Psi}$. The Weyl group $W(\underline{\Psi})$ is then a finite group equipped with an action of Γ_k. If $\underline{\Psi}_1$, $\underline{\Psi}_2$ are two twisted root data, the sets $\mathrm{Isom}(\underline{\Psi}_1, \underline{\Psi}_2)$, $\mathrm{Isomext}(\underline{\Psi}_1, \underline{\Psi}_2)$ are Galois sets. An orientation between $\underline{\Psi}_1$, $\underline{\Psi}_2$ is an element $u \in \mathrm{Isomext}(\underline{\Psi}_1, \underline{\Psi}_2)(k)$, and the set $\mathrm{Isomint}_u(\underline{\Psi}_1, \underline{\Psi}_2)$ is then a Galois set.

2.2. ***Type of a maximal torus.*** We denote by G_0 the split form of G. We denote by T_0 a maximal k-split torus of G_0 and by $\Psi_0 = \Psi(G_0, T_0)$ the associated root datum. We denote by W_0 the Weyl group of Φ_0 and by $\mathrm{Aut}(\Psi_0)$ its automorphism group.

Let $i : T \to G$ be a k-embedding as a maximal torus. The root datum

$$\underline{\Psi}(G, i(T)) = \Psi(G(T)_{k_s}, i(T)_{k_s})$$

is equipped with an action of the absolute Galois group Γ_k, so it defines a twisted root datum. It is a k-form of the constant root datum $\underline{\Psi}_0$ and we define the type of (T, i) as the isomorphism class of

$$[\underline{\Psi}(G, i(T))] \in H^1(k, \mathrm{Aut}(\Psi_0)).$$

Recall that by Galois descent, those k_s/k-forms are classified by the Galois cohomology pointed set $H^1(k, \mathrm{Aut}(\Psi_0))$.

If two embeddings i, j have the same image, then $\mathbf{type}(T, i) = \mathbf{type}(T, j) \in H^1(k, \mathrm{Aut}(\Psi_0))$. If we compose $i : T \to G$ by an automorphism $f \in \mathrm{Aut}(G)(k)$, we have $\mathbf{type}(T, i) = \mathbf{type}(T, f \circ i) \in H^1(k, \mathrm{Aut}(\Psi_0))$.

Remark 2.2.1. If G is semisimple and has no outer isomorphism (as is the case for groups of type G_2), $W_0 = \mathrm{Aut}(\Psi_0)$ and the next considerations will not add anything.

We would like to have an invariant with value in the Galois cohomology of some Weyl group. The strategy is to "rigidify" by adding an extra data to $i : T \to G$, namely an orientation with respect to a quasisplit form of G.

Given a k-embedding $i : T \to G$, we denote by $\underline{\mathrm{Dyn}}(G, i(T))$ the Dynkin diagram k-scheme of $\underline{\Psi}(G, i(T))$; it is finite étale and then encoded in the Galois set $\mathbf{Dyn}(G_{k_s}, i(T)_{k_s})$. There is a canonical isomorphism: $\underline{\mathrm{Dyn}}(G) \cong \underline{\mathrm{Dyn}}(G, i(T))$ [Demazure and Grothendieck 1970c, XXIV, 3.3].

We denote by G' a quasisplit k-form of G. Let (T', B') be a Killing couple of G', and denote by $\underline{\Psi}' = \underline{\Psi}(G', T')$ the associated twisted root datum and by $W' = N_{G'}(T')/T'$ its Weyl group, which is a twisted constant finite k-group.

Suppose that G is semisimple simply connected or adjoint; in this case, the homomorphism $\mathrm{Autext}(G) \to \mathrm{Aut}_{\mathrm{Dyn}}(\underline{\mathrm{Dyn}}(G))$ is an isomorphism [ibid., XXIV, 3.6]. We fix then an isomorphism $v : \underline{\mathrm{Dyn}}(G') \xrightarrow{\sim} \underline{\mathrm{Dyn}}(G)$. Together with the canonical isomorphism $\underline{\mathrm{Dyn}}(G) \cong \underline{\mathrm{Dyn}}(G, i(T))$, it induces an isomorphism $\tilde{v} : \underline{\mathrm{Dyn}}(G') \xrightarrow{\sim} \underline{\mathrm{Dyn}}(G, i(T))$. For G semisimple simply connected or adjoint, the isomorphism $\tilde{v}$ defines equivalently an orientation

$$u \in \mathrm{Isomext}\big(\underline{\Psi}(G', T')(k), \underline{\Psi}(G, i(T))\big).$$

Then the Galois set $\mathrm{Isomint}_u\big(\underline{\Psi}(G', T'), \underline{\Psi}(G, i(T))\big)$ is a right W'-torsor and its class in $H^1(k, W')$ is called the oriented type of $i : T \to G$ with respect to the orientation v. It is denoted by $\mathbf{type}_v(T, i)$ and we bear in mind that it depends on the choice of G' and on v.

2.3. ***The quasisplit case.*** We deal here with the quasisplit k-group G' and with the exact sequence $1 \to T' \to N_{G'}(T') \xrightarrow{\pi} W' \to 1$. Here we have a canonical isomorphism $\mathrm{id} : \underline{\mathrm{Dyn}}(G') \cong \underline{\mathrm{Dyn}}(G')$ and then a natural way to define an orientation for a k-embedding $j : E \to G'$ of a maximal k-torus. Keeping the notations above, let us state the following result.

Theorem 2.3.1 (Kottwitz). (1) *The map*

$$\mathrm{Ker}\big(H^1(k, N_{G'}(T')) \to H^1(k, G')\big) \xrightarrow{\pi_*} H^1(k, W')$$

is onto.

(2) *For each $\gamma \in H^1(k, W')$, there exists a k-embedding $j : E \to G'$ of a maximal k-torus such that* $\mathbf{type}_{\mathrm{id}}((E, j)) = \gamma$.

In [Kottwitz 1982, Corollary 2.2], this result occurs only as a result on embeddings of maximal tori. It was rediscovered by Raghunathan [2004] and independently by the second author [Gille 2004]. The proof of (1) uses Steinberg's theorem on rational conjugacy classes, and we can explain quickly how one can derive (2) from (1). Given $\gamma \in H^1(k, W')$, assertion (1) provides a principal homogeneous space P under $N' = N_{G'}(T')$ together with a trivialization $\phi : G' \xrightarrow{\sim} P \wedge^{N'} G'$ such that $\pi_*[P] = \gamma$. Then ϕ induces a trivialization at the level of twisted k-groups $\phi_* : G' \xrightarrow{\sim} {}^P G'$. Now if we twist $i' : T' \to G'$ by P, we get a k-embedding

$${}^P i' : {}^P T' \to {}^P G' \overset{\phi_*}{\underset{\sim}{\leftarrow}} G',$$

and one checks that $\mathbf{type}_{\mathrm{id}}\big({}^P T', {}^P i'\big) = \gamma$.

2.4. ***Image of the cohomology of tori.*** We give now a slightly more precise form of Steinberg's theorem [1965, Theorem 11.1]; see also [Serre 1994, III.2.3].

Theorem 2.4.1. *Let $[z] \in H^1(k, G')$. Let $i : T \to {}_zG'$ be a maximal k-torus of the twisted k-group ${}_zG'$. Then there exists a k-embedding $j : T \to G'$ and $[a] \in H^1(k, T)$ such that $j_*[a] = [z]$ and such that $\mathbf{type}_{\mathrm{can}}(T, i) = \mathbf{type}_{\mathrm{id}}(T, j)$.*

In the result, the first orientation is the canonical one, namely arising from the canonical isomorphism $\underline{\mathrm{Dyn}}(G') \xrightarrow{\sim} \underline{\mathrm{Dyn}}({}_z(G'))$.

Proof. If the base field is finite, there is nothing to do since $H^1(k, G') = 1$ by Lang's theorem. We can then assume that k is infinite. We denote by $P(z)$ the G'-homogeneous space defined by z and by $\phi : G'_{k_s} \xrightarrow{\sim} P(z)_{k_s}$, a trivialization satisfying $z_\sigma = \phi^{-1} \circ \sigma(\phi)$ for each $\sigma \in \Gamma_k$. It induces a trivialization $\varphi : G'_{k_s} \xrightarrow{\sim} ({}_z(G'))_{k_s}$ satisfying $\mathrm{int}(z_\sigma) = \varphi^{-1} \circ \sigma(\varphi)$ for each $\sigma \in \Gamma_k$.

We denote by $(G')^{\mathrm{sc}}$ the simply connected cover of DG' and by $f : (G')^{\mathrm{sc}} \to G'$ the natural k-homomorphism. Let T^{sc} be $({}_zf)^{-1}(i(T))$. Let g^{sc} be a regular element in $T^{\mathrm{sc}}(k)$ and consider the $G'^{\mathrm{sc}}(k_s)$-conjugacy class $\mathcal{C}$ of $\varphi^{-1}(g^{\mathrm{sc}})$ in $(G')^{\mathrm{sc}}(k_s)$. This conjugacy class is rational in the sense that it is stabilized by Γ_k since $(\varphi^{-1}(g^{\mathrm{sc}})) = z_\sigma {}^\sigma(\varphi^{-1}(g^{\mathrm{sc}}))z_\sigma^{-1}$ for each $\sigma \in \Gamma_k$. According to Steinberg [1965, Corollary 10.1] (and [Borel and Springer 1968, 8.6] in the nonperfect case), $\mathcal{C} \cap (G')^{\mathrm{sc}}(k)$ is not empty, so there exist $g_1^{\mathrm{sc}} \in (G')^{\mathrm{sc}}(k)$ and $h^{\mathrm{sc}} \in (G')^{\mathrm{sc}}(k_s)$ such that $\varphi^{-1}(g^{\mathrm{sc}}) = (h^{\mathrm{sc}})^{-1} g_1^{\mathrm{sc}} h^{\mathrm{sc}}$. We put $g = {}_zf(g^{\mathrm{sc}})$, $g_1 = f(g_1^{\mathrm{sc}})$, $h = f(h^{\mathrm{sc}})$, $T_1 = Z_{G'}(g_1)$ and $i_1 : T_1 \to G'$.

Since $g \in ({}_z(G'))(k)$ and $g_1 \in G'(k)$, we have $h^{-1}g_1h = z_\sigma {}^\sigma(h^{-1}g_1h)z_\sigma^{-1} = z_\sigma h^{-\sigma} g_1 {}^\sigma h z_\sigma^{-1}$ for each $s \in \Gamma_k$, whence

$$g_1 = a_\sigma g_1 a_\sigma^{-1},$$

where $a_\sigma = hz_\sigma h^{-\sigma}$ is a 1-cocycle cohomologous to z with values in $T_1(k_s) = Z_{G'}(g_1)(k_s)$. It remains to show the equality on the oriented types. By the rigidity trick (see the proof of Proposition 3.2 in [Gille 2004]), up to replacing k by the function field of the T_1-torsor defined by a, we can assume that $[a] = 1 \in H^1(k, T_1)$. We write $a_\sigma = b^{-1} {}^\sigma b$ for some $b \in T_1(k_s)$, and we have that $z_\sigma = (bh)^{-1} {}^\sigma(bh)$ and $\varphi^{-1}(g) = (bh)^{-1} g_1 bh$.

Putting $h_2 = bh \in G'(k_s)$, we have $z_\sigma = h_2^{-1} {}^\sigma h_2$ and $\varphi^{-1}(g) = h_2^{-1} g_1 h_2$. We get k-isomorphisms $\phi_2 = \phi \circ L_{h_2^{-1}} : G' \to P(z)$ and $\varphi_2 = \varphi \circ \mathrm{int}(h_2^{-1}) : G' \xrightarrow{\sim} {}_z(G')$ such that the following diagram commutes

$$\begin{array}{ccc} T_1 & \xrightarrow{\ i_1\ } & G' \\ \wr\big\downarrow \varphi_2 & & \wr\big\downarrow \varphi_2 \\ T & \xrightarrow{\ i\ } & {}_z(G') \end{array}$$

Thus $\mathbf{type}_{\mathrm{can}}(T, i) = \mathbf{type}_{\mathrm{id}}(T_1, i_1) \in H^1(k, W')$. □

2.5. *Image of the cohomology of tori, II.* Recall the following well-known fact.

Lemma 2.5.1. *Let H be a reductive k-group and T be a k-torus of the same rank as H. Let $i, j : T \to H$ be k-embeddings of a maximal k-torus T. If $j = \mathrm{Int}(h) \circ i$ for some $h \in H(k_s)$, then we have $h^{-1}\,{}^{\sigma}h \in i(T)(k_s)$ for all σ in the absolute Galois group Γ_k.*

Proof. For any $\sigma \in \Gamma$ and any $t \in T(k_s)$, we have $j({}^{\sigma}t) = {}^{\sigma}h \cdot i({}^{\sigma}t) \cdot {}^{\sigma}h^{-1}$. Therefore, we have $j = \mathrm{Int}({}^{\sigma}h) \circ i = \mathrm{Int}(h) \circ i$, and $h^{-1}\,{}^{\sigma}h$ is a k_s-point of the centralizer $C_H(i(T)) = i(T)$. □

Lemma 2.5.2. *Let H be a reductive k-group and let T be a k-torus of the same rank as H. Let v be an orientation of H with respect to a quasisplit form H'. Let $i, j : T \to H$ be k-embeddings of a maximal k-torus T which are $H(k_s)$-conjugate. Then we have* $\mathrm{Im}(i_*) = \mathrm{Im}(j_*) \subseteq H^1(k, H)$ *and* $\mathbf{type}_v(T, i) = \mathbf{type}_v(T, j)$.

Proof. Let $j = \mathrm{Int}(h) \circ i$ for some $h \in H(k_s)$. By Lemma 2.5.1, we have $h^{-1}\,{}^{\sigma}h \in i(T)(k_s)$. Let $[\alpha] \in \mathrm{Im}(j_*)$ and α be a cocycle with values in $j(T(k_s))$ which represents $[\alpha]$. Define $\beta_\sigma = h^{-1}\alpha_\sigma{}^{\sigma}h$. Then β is cohomologous to α and $\beta_\sigma = (h^{-1}\alpha_\sigma h) \cdot (h^{-1}\,{}^{\sigma}h) \in i(T(k_s))$. Hence $[\alpha] = [\beta] \in \mathrm{Im}(i_*)$, which shows that $\mathrm{Im}(i_*) = \mathrm{Im}(j_*) \subseteq H^1(k, H)$.

Let $T_1 = i(T)$ and $T_2 = j(T)$. Let $\mathrm{Transpt}_G(T_1, T_2)$ be the strict transporter from T_1 to T_2 [Demazure and Grothendieck 1970a, $\mathrm{VI_B}$, Définition 6.1(ii)]. Note that $\mathrm{Transpt}_G(T_1, T_2)$ is a right $N_G(T_1)$-torsor. We have a canonical isomorphism

$$\mathrm{Transpt}_G(T_1, T_2) \wedge \mathrm{Isomint}_v(\underline{\Psi}', \underline{\Psi}(G, T_1)) \xrightarrow{\sim} \mathrm{Isomint}_v(\underline{\Psi}', \underline{\Psi}(G, T_2)).$$

Since $j = \mathrm{Int}(h) \circ i$, we have $h \in \mathrm{Transpt}_G(T_1, T_2)(k_s)$ and h defines a trivialization $\phi_h : N_G(T_1) \to \mathrm{Transpt}_G(T_1, T_2)$ which sends the neutral element to h. Let $W_1 = N_G(T_1)/T_1$. Since $\phi_h^{-1} \circ \sigma(\phi_h) = h^{-1}\,{}^{\sigma}h \in T_1(k_s)$, the image of the class of $\mathrm{Transpt}_G(T_1, T_2)$ in $H^1(k, W_1)$ is trivial. Hence $\mathrm{Isomint}_v(\underline{\Psi}', \underline{\Psi}(G, T_1)) \simeq \mathrm{Isomint}_v(\underline{\Psi}', \underline{\Psi}(G, T_2))$; i.e., $\mathbf{type}_v(T, i) = \mathbf{type}_v(T, j)$. □

Proposition 2.5.3. *Let T be a k-torus of the same rank as G. Let $i_1, i_2 : T \to G$ be k-embeddings of T in G. Let v be an orientation of G with respect to a quasisplit form G'. If* $\mathbf{type}_v(T, i_1) = \mathbf{type}_v(T, i_2) \in H^1(k, W')$, *then there is a k-embedding $j : T \to G$ such that $j(T) = i_1(T)$ and j, i_2 are $G(k_s)$-conjugate. In particular, the images of $i_{1,*}, i_{2,*}, j : H^1(k, T) \to H^1(k, G)$ coincide.*

Proof. Let $T_1 = i_1(T)$ and $T_2 = i_2(T)$ and again put $W_i = N_G(T_i)/T_i$ for $i = 1, 2$. Let η denote the class of the $N_G(T_1)$-torsor $\mathrm{Transpt}_G(T_1, T_2)$ in $H^1(k, N_G(T_1))$ and $\bar{\eta}$ be the image of η in $H^1(k, W_1)$. We have a canonical isomorphism

$$\mathrm{Transpt}_G(T_1, T_2) \wedge \mathrm{Isomint}_v(\underline{\Psi}', \underline{\Psi}(G, T_1)) \xrightarrow{\sim} \mathrm{Isomint}_v(\underline{\Psi}', \underline{\Psi}(G, T_2)).$$

Since $\mathbf{type}_v(T, i_1) = \mathbf{type}_v(T, i_2)$, we have

$$\mathrm{Isomint}_v(\underline{\Psi}', \underline{\Psi}(G, T_1)) \simeq \mathrm{Isomint}_v(\underline{\Psi}', \underline{\Psi}(G, T_2)).$$

Hence $\bar{\eta}$ is the trivial class in $H^1(k, W_1)$. Thus the $N_G(T_1)$-torsor $\mathrm{Transpt}_G(T_1, T_2)$ admits a reduction to T_1. More precisely, there exist a T_1-torsor E_1 and an isomorphism $E_1 \wedge^{T_1} N_G(T_1) \xrightarrow{\sim} \mathrm{Transpt}_G(T_1, T_2)$ of $N_G(T_1)$-torsors. We take a point $e_1 \in E_1(k_s)$ and consider its image g in $G(k_s)$ under the mapping

$$E_1 \wedge^{T_1} N_G(T_1) \xrightarrow{\sim} \mathrm{Transpt}_G(T_1, T_2) \hookrightarrow G.$$

Then $h = g^{-1}\,{}^{\sigma}g$ is a k_s-point of the centralizer $C_G(T_1) = T_1$ for all $\sigma \in \Gamma_k$. We define a k-embedding $j : T \to G$ as $j(t) = (\mathrm{Int}(g^{-1}) \circ i_2)(t)$. To see that j is indeed defined over k, we check as follows:

$$\begin{aligned} j({}^{\sigma}t) &= (\mathrm{Int}(g^{-1}) \circ i_2)({}^{\sigma}t) \\ &= \mathrm{Int}(g^{-1})({}^{\sigma}i_2(t)) \\ &= h \cdot {}^{\sigma}((\mathrm{Int}(g^{-1}) \circ i_2)(t)) \cdot h^{-1} \\ &= {}^{\sigma}(j(t)). \end{aligned}$$

By our construction, we have $j(T) = i_1(T)$ and i_2, j are conjugated. Let $f = (j|_{T_1})^{-1} \circ i_1$. Then f is an automorphism of T and $i_1 = j \circ f$. Hence the images of $i_{1,*}$ and j_* coincide. By Lemma 2.5.2, the images of j and $i_{2,*}$ coincide. □

This applies to the quasisplit case and enables us to slightly refine Theorem 2.4.1.

Corollary 2.5.4. *With the notations of Theorem 2.4.1, for each class $\gamma \in H^1(k, W')$, choose (by Theorem 2.3.1) a k-embedding $i(\gamma) : E(\gamma) \to G'$ of oriented type γ. Then the map*

$$\bigsqcup_{\gamma \in H^1(k, W')} H^1(k, E(\gamma)) \xrightarrow{\sqcup i(\gamma)_*} H^1(k, G')$$

is onto.

2.6. ***Varieties of embedding k-tori.*** Let T be a k-torus and $\underline{\Psi}$ be a twisted root datum of Ψ_0 attached to T; i.e., the character group of T is isomorphic to the character group encoded in Ψ. In this section, we will define a k-variety X such that the existence of a k-point of X is equivalent to the existence of a k-embedding of T into G with respect to $\underline{\Psi}$.

We start with a functor. The *embedding functor* $\mathcal{E}(G, \underline{\Psi})$ is defined as follows: for any k-algebra C, $\mathcal{E}(G, \underline{\Psi})(C)$ is the set of all $f : T_C \hookrightarrow G_C$ such that f is both a closed immersion and a group homomorphism which induces an isomorphism $f^{\Psi} : \underline{\Psi}_C \xrightarrow{\sim} \underline{\Psi}(G_C, f(T_C))$ such that $f^{\Psi}(\alpha) = \alpha \circ f^{-1}|_{f(T_{C'})}$ is in $\underline{\Psi}(G_{C'}, f(T_{C'}))$ for all C'-roots α for all C-algebra C'. In fact, the functor $\mathcal{E}(\underline{\Psi}, G)$

is representable by a k-scheme [Lee 2014, Theorem 1.1]. Define the Galois set $\mathrm{Isomext}(\underline{\Psi}, G)$ by $\mathrm{Isomext}(\underline{\Psi}, G) = \mathrm{Isomext}(\underline{\Psi}, \underline{\Psi}(G, E))$, where E stands for an arbitrary maximal k-torus of G. Given an orientation $v \in \mathrm{Isomext}(\underline{\Psi}, G)(k)$, we define the *oriented embedding functor* as follows: for any k-algebra C,

$$\mathcal{E}(G, \underline{\Psi}, v)(C) = \big\{ f : T_C \hookrightarrow G_C \mid f \in \mathcal{E}(G, \underline{\Psi})(C) \text{ and } \\ \text{the image of } f^{\Psi} \text{ in } \mathrm{Isomext}(\underline{\Psi}, G)(C) \text{ is } v \big\}.$$

We have the following result:

Theorem 2.6.1. *In the sense of the étale topology, $\mathcal{E}(G, \underline{\Psi}, v)$ is a left homogeneous space under the adjoint action of G and a torsor over the variety of the maximal tori of G under the right $W(\underline{\Psi})$-action. Moreover, $\mathcal{E}(G, \underline{\Psi}, v)$ is representable by an affine k-scheme.*

Proof. We refer to [Lee 2014, Theorem 1.6]. □

Remark 2.6.2. The definition of varieties of embeddings is quite abstract but is simplified a lot if there is a k-embedding $i : T \to G$ of oriented type isomorphic to $(\underline{\Psi}, v)$. Indeed in this case, the k-variety $\mathcal{E}(G, \underline{\Psi}, v)$ is G-isomorphic to the homogeneous space $G/i(T)$, and we observe that the map $G/i(T) \to G/N_G(i(T))$ is a $W_G(i(T))$-torsor over the variety of maximal tori of G.

Remark 2.6.3. We sketch another way to prove Theorem 2.4.1. With the notations of that result, let $z \in Z^1(k, G')$ and put $G = {}_zG'$. Let T be a maximal k-torus of G and consider the twisted root data $\underline{\Psi} = \underline{\Psi}(G, T)$ attached to T. Let v be the canonical element in $\mathrm{Isomext}(\underline{\Psi}, G)(k)$ and let $v' = c \circ v$, where $c \in \mathrm{Isomext}(G, G')(k)$ corresponds to the canonical orientation $\underline{\mathrm{Dyn}}(G) \cong \underline{\mathrm{Dyn}}(G')$. We denote by X (resp. X') the k-variety of oriented embeddings of T in G (resp. G') with respect to $\underline{\Psi}$ and v (resp. v'). Note that G' acts on X' and we have a natural isomorphism $X \xrightarrow{\sim} {}_zX'$. Theorem 2.3.1(2) shows that $X'(k) \neq \varnothing$ and the choice of a k-point x' of X' defines a G'-equivariant isomorphism $G'/T \xrightarrow{\sim} X'$. In the other hand, the embedding i defines a k-point $x \in X(k)$. Since $X \cong {}_zX'$, we have that ${}_z(G'/T)(k) \neq \varnothing$; hence the class $[z] \in H^1(k, G)$ admits a reduction to $i' : T \hookrightarrow G'$ such that $\mathbf{type}_{\mathrm{can}}(T, i) = \mathbf{type}_{\mathrm{id}}(T, i') \in H^1(k, W')$.

3. Generalities on octonion algebras

Let C be an octonion algebra. We denote by G the automorphism group of C; it is a semisimple k-group of type G_2. We denote by N_C the norm of C; it is a 3-fold Pfister form. In particular, N_C is hyperbolic (equivalently isotropic) if and only if G is split (equivalently isotropic).

3.1. ***Behavior under field extensions.*** If l/k is a field extension of odd degree, the Springer odd extension theorem [Elman et al. 2008, 18.5] implies that C is split if and only if C_l is split. More generally, we have the following criterion.

Lemma 3.1.1. *Let $(k_j)_{j=1,\dots,n}$ be a family of finite field extensions such that* g.c.d.$([k_j:k])$ *is odd. Then C is split if and only if C_{k_j} is split for $j=1,\dots,n$.* □

Proof. The left implication is obvious. Conversely, assume that C_{k_j} is split for $j = 1,\dots,n$. Then there exists an index j such that $[k_j : k]$ is odd, hence C splits. □

Remark 3.1.2. This is a special case of the following more general result by Garibaldi and Hoffmann [2006, Theorem 0.3] answering positively Totaro's question. Let $(k_j)_{j=1,\dots,n}$ be a family of finite field extensions and put $d=$ g.c.d.$([k_j:k])$. Let C, C' be Cayley k-algebras such that C_{k_j} and C'_{k_j} are isomorphic for $j=1,\dots,n$. Then there exists a separable finite field extension K/k of degree dividing d such that C_K is isomorphic to C'_K. This is the case of groups of type G_2 in that theorem which includes also the case of certain groups of type F_4 and E_6.

We recall also the behavior with respect to quadratic étale algebras.

Lemma 3.1.3. *Let k'/k be a quadratic étale algebra. Then the following are equivalent*:

(i) *$C\otimes_k k'$ splits.*

(ii) *There is an isometry $(k', n_{k'/k}) \to (C, N_C)$, where $n_{k'/k} : k' \to k$ stands for the norm map.*

(iii) *There exists an embedding of unital composition k-algebras $k' \to C$.*

Proof. If C is split, all three facts hold so that we can assume that C is not split.

(i) ⇒ (ii): Since C is not split, it follows that k' is a field. Since N_C is split over k', there exists a nontrivial and nondegenerate symmetric bilinear form B such that $B\otimes n_{k'/k}$ is a subform of N_C [Elman et al. 2008, 34.8]. Since N_C is multiplicative, there is an isometry $(k', n_{k'/k}) \to (C, N_C)$.

(ii) ⇒ (iii): Since the orthogonal group $O(N_C)(k)$ acts transitively on the sphere $\{x \in C \mid N_C(x) = 1\}$, we can assume that our isometry $(k', n_{k'/k}) \to (C, N_C)$ maps $1_{k'}$ to 1_C. It is then a map of unital composition k-algebras.

(iii) ⇒ (i): If $k' = k\times k$, then N_C is isotropic and C is split. Hence k' is a field and N_C is k'-isotropic so that $C_{k'}$ is split. □

3.2. ***The Cayley–Dickson process.*** We know that C, up to k-isomorphism, can be obtained by the Cayley–Dickson doubling process; that is, $C\cong C(Q,c)=Q\oplus Qa$, where Q is a k-quaternion algebra and $c\in k^\times$ [Springer and Veldkamp 2000, § 1.5].

We denote by $\sigma_Q = \mathrm{trd}_Q - \mathrm{id}_Q$ the canonical involution of Q and recall that the multiplicativity rule on C, the norm N_C, and the canonical involution σ_C are given by

$$(x+ya)(u+va) = (xu + c\sigma_Q(v)y) + (vx + y\sigma_Q(u))a \quad (x,y,u,v \in Q),$$
$$N_C(x+ya) = N(x) - cN(y),$$
$$\sigma_C(x+ya) = \sigma_Q(x) - ya.$$

Then N_C is isometric to the 3-Pfister form $n_Q \otimes \langle 1, -c\rangle$ and that form determines the octonion algebra [ibid., Corollary 1.7.3]. Also it provides an embedding j of the k-group $H(Q) = (\mathrm{SL}_1(Q) \times_k \mathrm{SL}_1(Q))/\mu_2$ in $\mathrm{Aut}(C(Q,c))$. This map is given by $(g_1, g_2).(q_1, q_2) = (g_1 q_1 g_1^{-1}, g_2 q_2 g_1^{-1})$. Another corollary of the determination of an octonion algebra by its norm is the following well-known fact.

Corollary 3.2.1. *Let C be a octonion k-algebra and let Q be a quaternion algebra. Then the following are equivalent*:

(i) *There exists $c \in k^\times$ such that $C \cong C(Q,c)$.*

(ii) *There exists an isometry $(Q, N_Q) \to (C, N_C)$.*

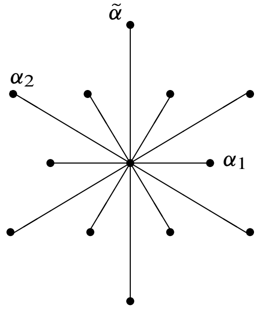
Proof. (i) $\Rightarrow$ (ii) is obvious. Assume that there exists an isometry $(Q, N_Q) \to (C, N_C)$. By the linkage property of Pfister forms [Elman et al. 2008, 24.1(1)], there exists a bilinear 1-Pfister form ϕ such that $N_C \cong N_Q \otimes \phi$. Since N_C represents 1, we can assume that ϕ represents 1 so that $\phi \cong \langle 1, -c\rangle$. Therefore C and $C(Q,c)$ have isometric norms and are isomorphic. □

Remark 3.2.2. In odd characteristic, Hooda provided an alternative proof, see [Hooda 2014, Theorem 4.3] and also a nice generalization [ibid., Proposition 4.2].

Lemma 3.2.3. *Let C be a nonsplit octonion k-algebra. If $D \subseteq C$ is a unital composition subalgebra and $u \in C \setminus D$ then $D \oplus Du$ is a unital composition subalgebra as well.*

Proof. Since C is nonsplit, the corresponding norm map N_C is anisotropic. Let b_C be the polar map of N_C. Since the map $x \mapsto b_C(u,x)$ is linear and the restriction of b_C on $D \times D$ is regular, there is $v \in D$ such that $b_C(v,x) = b_C(u,x)$ for all $x \in D$. Let $u' = u - v$. We have $b_C(u',x) = b_C(v,x) - b_C(u,x) = 0$ for all $x \in D$, so $u' \in D^\perp$. Since $v \in D$ and $u \notin D$, we have $u' \neq 0$, so $N_C(u') \neq 0$. By the doubling process [Springer and Veldkamp 2000, Proposition 1.5.1], we have that $D \oplus Du'$ is a unital composition subalgebra of C. But $u' = u - v$ and $v \in D$, so $D \oplus Du' = D \oplus Du$. □

3.3. ***On the dihedral group, I.*** In this case, $W_0 = \mathrm{Aut}(\Psi_0)$ and $W_0 = D_6 = \mathbb{Z}/6\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z} = C_2 \times S_3$ is the dihedral group of order 12. More precisely, $C_2 = \langle c\rangle$ stands for its center. The right way to see it is by its action on the root system $\Psi(G_0, T_0) \subset \widehat{T}_0 = \mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_2 = \mathbb{Z}^2$, as provided by the following picture:



where α_1, α_2 stand for a base of the root system G_2 and $\tilde{\alpha} = 3\alpha_1 + 2\alpha_2$.

Let $\{\epsilon_i\}_{i=1}^3$ be an orthonormal basis of $\mathbb{Q}^3$. We can view the root space of G_2 as the hyperplane in $\mathbb{Q}^3$ defined by $\{\sum_{i=1}^3 \xi_i \epsilon_i \mid \sum_{i=1}^3 \xi_i = 0\}$, and identify α_1, α_2 with $\epsilon_1 - \epsilon_2$ and $-2\epsilon_1 + \epsilon_2 + \epsilon_3$ respectively [Bourbaki 1981, planche IX]. For a root α, let s_α be the reflection orthogonal to α. Under the above identification, the element $c = s_{2\alpha_1+\alpha_2} s_{\alpha_2}$ acts on the roots by $-\mathrm{id}$ and $S_3 = \langle s_{\alpha_1}, s_{2\alpha_1+\alpha_2}\rangle$ acts by permuting the ϵ_i. Note that although $s_{2\alpha_1+\alpha_2} s_{\alpha_2}$ acts on the subspace $\{\sum_{i=1}^3 \xi_i \epsilon_i \mid \sum_{i=1}^3 \xi_i = 0\}$ by $-\mathrm{id}$, $s_{2\alpha_1+\alpha_2} s_{\alpha_2}$ does not act as $-\mathrm{id}$ on $\{\epsilon_i\}_{i=1}^3$.

Remarks 3.3.1. (a) In the G_2 root system, for any long root β and any short root α orthogonal to β, we have $s_\alpha \circ s_\beta = c$. Also observe that $\widehat{T}_0$ is a sublattice of index 2 of the lattice $\mathbb{Z}\frac{\alpha}{2} \oplus \mathbb{Z}\frac{\beta}{2}$. This is related to the fact that the morphism $\mathrm{SL}_2 \times \mathrm{SL}_2 \to G_0$ defined by the coroots $\alpha^\vee$ and $\beta^\vee$ has kernel equal to the diagonal subgroup μ_2.
(b) The roots α_1, $\tilde{\alpha}$ generate a closed symmetric subsystem of type $A_1 \times A_1$ of G_2. Any subroot system (not necessarily closed) of G_2 which is of type $A_1 \times A_1$ is a W_0-conjugate of the previous one.

3.4. *Subgroups of type $A_1 \times A_1$.*

Given an octonion k-algebra C, we relate Cayley–Dickson decomposition to subgroups of $G = \mathrm{Aut}(C)$.

Lemma 3.4.1. *Let H be a semisimple k-subgroup of G of type $A_1 \times A_1$. Then there exists a quaternion algebra Q, $c \in k^\times$, an isomorphism $C \cong C(Q, c)$ and an isomorphism $H \xrightarrow{\sim} H(Q)$ such that the following diagram commutes*:

$$\begin{array}{ccc} H & \hookrightarrow & G \\ \downarrow \wr & & \downarrow \wr \\ H(Q) & \overset{j}{\hookrightarrow} & \mathrm{Aut}(C(Q,c)) \end{array}$$

Proof. We start with a few observations on the split case $G = G_0 = \mathrm{Aut}(C_0)$, where we have the k-subgroup $H_0 = (\mathrm{SL}_2 \times \mathrm{SL}_2)/\mu_2$ acting on C_0. The root subsystem $\Phi(H_0, T_0)$ is $\mathbb{Z}\alpha_1 \oplus \mathbb{Z}\tilde{\alpha}$ so that the first (resp. the second) factor SL_2 of H_0 corresponds to a short (resp. long) root. We denote by $H_{0,<} \cong \mathrm{SL}_2$ (resp. $H_{0,>}$) the "short" subgroup (resp. the "long" one) of H_0. Taking the decomposition

$C_0 = M_2(k) \oplus M_2(k)_\sharp$, the point is that we have $M_2(k) = (C_0)^{H_{0,>}}$. In other words, we can recover the composition subalgebra $M_2(k)$ of C_0 from H_0.

We come now to our problem. We are given a k-subgroup H of $G = \mathrm{Aut}(C)$ of type $A_1 \times A_1$. Let T be a maximal k-torus of H. Then the root system $\Phi(H_{k_s}, T_{k_s})$ is a subsystem of $\Phi(G_{k_s}, T_{k_s}) \cong \Psi_0$ of type $A_1 \times A_1$; hence W_0-conjugated to the standard one (Remarks 3.3.1(b)). Since the Galois action preserves the length of a root, it follows that we can define by Galois descent the k-subgroups $H_<$ and $H_>$ of H. We define then $Q = (C)^{H_>}$. By Galois descent, it is a quaternion subalgebra of C which is normalized by H. It leads to a Cayley–Dickson decomposition $C = Q \oplus L$, where L is the orthogonal complement of Q in C. Then L is a right Q-module and we choose $a \in L$ such that $L = Qa$. The k-subgroup $H(Q)$ of $\mathrm{Aut}(C)$ is nothing but $\mathrm{Aut}(C, Q)$ [Springer and Veldkamp 2000, §2.1], so we have $H \subseteq H(Q)$. For dimension reasons, we conclude that $H = H(Q)$ as desired. □

4. Embedding a torus in a group of type G_2

We assume that G is a semisimple k-group of type G_2. As in Section 2, we denote its split form by G_0, and T_0, W_0, etc. are defined as before.

4.1. ***On the dihedral group, II.*** We continue to discuss the action of the dihedral group W_0 (of order 12) on the root system of type G_2 started in Section 3.3. Let $\oplus_{i=1}^3 \mathbb{Z}\epsilon_i$ be a W_0-lattice, where the S_3-component of W_0 acts by permuting the ϵ_i and the center acts by $-\mathrm{id}$. Note that G_0 is of type G_2, so G_0 is both adjoint and simply connected and the dual group of G_0 is isomorphic to G_0 itself. Hence we have the following exact sequence of W_0-lattices, where W_0 acts on $\mathbb{Z}$ through its center $\mathbb{Z}/2\mathbb{Z}$ by $-\mathrm{id}$:

$$0 \to \widehat{T}_0 \xrightarrow{f} \oplus_{i=1}^3 \mathbb{Z}\epsilon_i \xrightarrow{\deg} \mathbb{Z} \to 0,$$

where $f(\alpha_1) = \epsilon_1 - \epsilon_2$ and $f(\alpha_2) = -2\epsilon_1 + \epsilon_2 + \epsilon_3$. We also consider its dual sequence

$$0 \to \mathbb{Z} \to \oplus_{i=1}^3 \mathbb{Z}\epsilon_i^\vee \to \widehat{T}_0^\vee \simeq \widehat{T}_0 \to 0.$$

4.2. ***Subtori.*** Keep the notations in Section 3.3. Let us fix an isomorphism

$$\chi : \mathbb{Z}/2\mathbb{Z} \times S_3 \to \langle c \rangle \times \langle s_{\alpha_1}, s_{2\alpha_1+\alpha_2} \rangle = W_0,$$

where $\chi((-1, 1)) = c$, $\chi((1, (12))) = s_{\alpha_1}$ and $\chi((1, (23))) = s_{2\alpha_1+\alpha_2}$.

We identify $\mathbb{Z}/2\mathbb{Z} \times S_3$ with W_0 by χ in the rest of this paper. Under this identification, we have

$$H^1(k, W_0) = H^1(k, \mathbb{Z}/2\mathbb{Z}) \times H^1(k, S_3).$$

Hence a class of $H^1(k, W_0)$ is represented uniquely (up to k-isomorphism) by a couple (k', l), where k' is a quadratic étale algebra of k and l/k is a cubic étale algebra of k.

Given such a couple (k', l), we denote by $\underline{\Psi}_{(k',l)} = [(k', l)] \wedge^{W_0} \Psi_0$ the associated twisted root datum. Let $l' = l \otimes_k k'$ and define the k-torus

$$T^{(k',l)} = \mathrm{Ker}\big(R_{k'/k}(R^1_{l'/k'}(\mathbb{G}_{m,l'})) \xrightarrow{N_{k'/k}} R^1_{l/k}(\mathbb{G}_{m,l})\big).$$

In the following, we prove that the torus encoded in $\underline{\Psi}_{(k',l)}$ is indeed $T^{(k',l)}$. However, we should keep in mind that two nonisomorphic root data $\underline{\Psi}$ may encode the same torus (Remark 4.2.2).

Lemma 4.2.1. *Let T be a k-torus of rank 2 and let $i : T \to G$ be a k-embedding such that* **type**$(T, i) = [(k', l)]$. *Then*:

(1) *The k-torus T is k-isomorphic to $T^{(k',l)}$.*

(2) *If there exists a quadratic étale algebra l_2 such that $l = k \times l_2$, then there is a k-isomorphism*

$$T \cong \big(R^1_{k_1/k}(\mathbb{G}_m) \times_k R^1_{k_2/k}(\mathbb{G}_m)\big)/\mu_2,$$

where k_1, k_2 are quadratic étale algebras such that $k_2 = k'$ and $[k_1] = [k_2] + [l_2] \in H^1(k, \mathbb{Z}/2\mathbb{Z})$.

Proof. (1) We have $W_0 = \mathbb{Z}/2\mathbb{Z} \times S_3$ and from Section 4.1, we have a W_0-resolution

$$0 \to \mathbb{Z} \to \oplus_{i=1}^3 \mathbb{Z}\epsilon_i^\vee \to \widehat{T}_0 \to 0.$$

It follows that $\widehat{T}_0$ is isomorphic to the W_0-module $\oplus_{i=1}^3 \mathbb{Z}\epsilon_i^\vee / \langle (1, 1, 1) \rangle$.

Let N be the W_0-lattice $\oplus_{i=1}^3 \mathbb{Z}e_i / \langle (1, 1, 1) \rangle$, where S_3 acts by permuting the indices and $\mathbb{Z}/2\mathbb{Z}$ acts trivially. Note that as $\mathbb{Z}$-lattices, we can identify N with $\widehat{T}_0$. Let $M = N \oplus N$ and equip M with a W_0-action: S_3 acts on N diagonally and $\mathbb{Z}/2\mathbb{Z}$ acts on M by exchanging the two copies of N. Embed N diagonally into M and we get the exact sequence of W_0-modules

$$0 \to N \xrightarrow{f} M = N \oplus N \xrightarrow{g} \widehat{T}_0 \to 0,$$

where $f(x) = (x, x)$ and $g(x, y) = x - y$. After twisting the above exact sequence by the W_0-torsor attached to (k', l) and taking the corresponding tori, we have

$$1 \to T \to R_{k'/k}\big(R^1_{l'/k'}(\mathbb{G}_{m,l'})\big) \xrightarrow{n_{k'/k}} R^1_{l/k}(\mathbb{G}_{m,l}) \to 1.$$

Hence T is the k-torus $T^{(k',l)}$.

(2) If $l = k \times l_2$, then there is an injective homomorphism $\iota : \mathbb{Z}/2\mathbb{Z} \to S_3$ and a class $[z] \in \mathrm{im}(\iota_* : H^1(k, \mathbb{Z}/2\mathbb{Z}) \to H^1(k, S_3))$ such that l corresponds to $[z]$. Let α be a short root such that the corresponding reflection s_α is $\iota(-1)$, and let β be a

long root orthogonal to α. As we mentioned in Remarks 3.3.1(a), the center of W_0 is generated by $s_\alpha \circ s_\beta$. Therefore, the image of the map

$$\mathrm{Id} \times \iota : \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \hookrightarrow \mathbb{Z}/2\mathbb{Z} \times S_3 = W_0$$

is generated by $\{s_\alpha, s_\beta\}$. Let us call it $W^{(k',l_2)}$. Let $H_0 \simeq (\mathrm{SL}_2 \times_k \mathrm{SL}_2)/\mu_2$ be the subgroup of G_0 generated by T_0 and the root groups associated to $\pm\alpha$ and $\pm\beta$. Then H_0 is of type $A_1 \times A_1$ and the Weyl group of H_0 with respect to T_0 is exactly $W^{(k',l_2)}$. Hence there is $[x] \in \mathrm{im}(H^1(k, N_{H_0}(T_0)) \to H^1(k, G_0))$ such that $(G, i(T))$ is isomorphic to ${}_x(G_0, T_0)$. Moreover, the embedding i factorizes through $H = {}_x(H_0)$. Let the first (resp. second) copy of SL_2 of H_0 correspond to the root group $\pm\beta$ (resp. $\pm\alpha$). Let π be the projection from $N_{H_0}(T_0)$ to $N_{H_0}(T_0)/T_0 = W^{(k',l_2)}$. Since

$$([k'], [l_2]) \in H^1(k, \langle s_\beta \circ s_\alpha \rangle) \times H^1(k, \langle s_\alpha \rangle) = H^1(k, W^{(k',l_2)})$$

is equivalent to

$$([k'], [k'] + [l_2]) \in H^1(k, \langle s_\beta \rangle) \times H^1(k, \langle s_\alpha \rangle) = H^1(k, W^{(k',l_2)}),$$

we have

$$\pi_*([x]) = \big([k'] + [l_2], [k']\big) \in H^1(k, \langle s_\alpha \rangle) \times H^1(k, \langle s_\beta \rangle).$$

Therefore,

$$T \simeq {}_x(T_0) \cong \big(R^1_{k_1/k}(\mathbb{G}_m) \times_k R^1_{k_2/k}(\mathbb{G}_m)\big)/\mu_2,$$

where $[k_2] = k'$ and $[k_1] = [k_2] + [l_2]$. □

Remark 4.2.2. A natural question is whether the class of $[(k', l)]$ is determined by the isomorphism class of the torus $T^{(k',l)}$ as a k-torus. It is not the case; there are indeed examples of nonequivalent pairs (k', l) and $(k'_\sharp, l_\sharp)$ such that the k-tori $T^{(k',l)}$ and $T^{(k'_\sharp,l_\sharp)}$ are isomorphic whenever the field k admits a biquadratic field extension $k_1 \otimes_k k_2$. We put then $k_{1,\sharp} = k_2$ and $k_{2,\sharp} = k_1$. With the notations of the proof of Lemma 4.2.1(2), we consider the k-tori

$$T = \big(R^1_{k_1/k}(\mathbb{G}_m) \times_k R^1_{k_2/k}(\mathbb{G}_m)\big)/\mu_2,$$
$$T_\sharp = \big(R^1_{k_{1,\sharp}/k}(\mathbb{G}_m) \times_k R^1_{k_{2,\sharp}/k}(\mathbb{G}_m)\big)/\mu_2.$$

Then the k-tori T and $T_\sharp$ are obviously k-isomorphic. However, the root data $\underline{\Psi}_{(k',l)}$ and $\underline{\Psi}_{(k'_\sharp,l_\sharp)}$ are not isomorphic as $k_2 \not\cong k_{2,\sharp} = k_1$.

Since the pointed set $H^1(k, \mathrm{GL}_2(\mathbb{Z}))$ classifies two-dimensional k-tori, the map $H^1(k, W_0) \to H^1(k, \mathrm{GL}_2(\mathbb{Z}))$ is in this case not injective. It is due to the fact that the normalizer of $C_2 \times (1 \times \mathbb{Z}/2\mathbb{Z})$ in $\mathrm{GL}_2(\mathbb{Z})$ is larger than the normalizer in W_0.

We deal now with the Galois cohomology of those tori.

Lemma 4.2.3. (1) *We have an exact sequence*

$$0 \to \mathrm{Ker}(l^\times \to k^\times)/N_{l'/l}\big(\mathrm{Ker}\big((l')^\times \xrightarrow{n_{l'/k'}} (k')^\times\big)\big) \to H^1(k, T^{(k',l)}) \\ \to (k')^\times/N_{l'/k'}((l')^\times) \xrightarrow{n_{k'/k}} k^\times/N_{l/k}(l^\times) \to 0,$$

and the map $n_{k'/k}$ admits a section.

(2) *Assume that k' and l are fields. Then $H^1(k, (\widehat{T^{(k',l)}})^0) = 0$.*

Proof. We put $T = T^{(k',l)}$.

(1) The Hilbert theorem 90 produces an isomorphism

$$k^\times/N_{l/k}(l^\times) \xrightarrow{\sim} H^1(k, R^1_{l/k}(\mathbb{G}_{m,l})).$$

Combined with the Shapiro isomorphism, we get an isomorphism

$$(k')^\times/N_{l'/k'}(l'^\times) \xrightarrow{\sim} H^1\big(k', R^1_{l'/k'}(\mathbb{G}_{m,l'})\big) \xrightarrow{\sim} H^1\big(k, R_{k'/k}\big(R^1_{l'/k'}(\mathbb{G}_{m,l'})\big)\big).$$

Putting these two facts together, the long exact sequence of Galois cohomology is

$$\cdots \to \mathrm{Ker}\big((l')^\times \to (k')^\times\big) \xrightarrow{N_{l'/l}} \mathrm{Ker}(l^\times \to k^\times) \to H^1(k, T) \\ \to (k')^\times/N_{l'/k'}((l')^\times) \xrightarrow{n_{k'/k}} k^\times/N_{l/k}(l^\times) \to \cdots .$$

Since $k^\times/N_{l/k}(l^\times)$ is of 3-torsion, half of the "diagonal map" $k^\times/N_{l/k}(l^\times) \to (k')^\times/N_{l'/k'}((l')^\times)$ provides a section of $(k')^\times/N_{l'/k'}((l')^\times) \xrightarrow{n_{k'/k}} k^\times/N_{l/k}(l^\times)$.

(2) We have an exact sequence

$$0 \to \widehat{T}^0 \to \mathrm{Coind}_k^{k'}(I_{l'/k'}) \xrightarrow{n_{k'/k}} I_{l/k} \to 0$$

of Galois modules, where $I_{l/k} = \mathrm{Ker}\big(\mathrm{Coind}_k^l(\mathbb{Z}) \to \mathbb{Z}\big)$. It gives rise to the long exact sequence of groups

$$0 \to H^0(k, \widehat{T}^0) \to H^0(k, \mathrm{Coind}_k^{k'}(I_{l'/k'})) \to H^0(k, I_{l/k}) \to \cdots \\ \to H^1(k, \widehat{T}^0) \to H^1(k, \mathrm{Coind}_k^{k'}(I_{l'/k'})) \to H^1(k, I_{l/k}) \to \cdots .$$

We consider the exact sequence $0 \to I_{l/k} \to \mathrm{Coind}_k^l(\mathbb{Z}) \to \mathbb{Z} \to 0$ and the corresponding sequence

$$0 \to H^0(k, I_{l/k}) \to H^0(k, \mathrm{Coind}_k^l(\mathbb{Z})) \to H^0(k, \mathbb{Z}) \to \cdots \\ \to H^1(k, I_{l/k}) \to H^1(k, \mathrm{Coind}_k^l(\mathbb{Z})) \to H^1(k, \mathbb{Z}).$$

The group $\mathbb{Z} = H^0(k, \mathrm{Coind}_k^l(\mathbb{Z}))$ embeds in $\mathbb{Z}$ by multiplication by 3; also we have $H^1(k, \mathrm{Coind}_k^l(\mathbb{Z})) \xrightarrow{\sim} H^1(l, \mathbb{Z}) = 0$ by Shapiro's isomorphism. The above sequence induces an isomorphism $\mathbb{Z}/3\mathbb{Z} \xrightarrow{\sim} H^1(k, I_{l/k})$. On the other hand, we have $H^1(k, \mathrm{Ind}_k^{k'}(I_{l'/k'})) \xrightarrow{\sim} H^1(k', I_{l'/k'}) \xleftarrow{\sim} \mathbb{Z}/3\mathbb{Z}$. The norm map $n_{k'/k} : H^1(\mathrm{Coind}_k^{k'}(I_{l'/k'})) \to H^1(k, I_{l/k})$ is multiplication by 2 on $\mathbb{Z}/3\mathbb{Z}$. Hence it is injective. By using the starting exact sequence, we conclude that $H^1(k, \widehat{T}^0) = 0$ as desired. □

4.3. ***A necessary condition.*** There is a basic restriction on the types of maximal tori of G.

Proposition 4.3.1. (1) *Let T be a k-torus of rank two and let $i : T \to G$ be a k-embedding such that* **type**$(T, i) = [(k', l)]$. *Then $G \times_k k'$ is split.*

(2) *Assume that $l = k \times k \times k$. Then the following are equivalent*:

(i) *There exists a k-embedding $i : T \to G$ of a rank-2 torus T such that* **type**$(T, i) = [(k', k^3)]$.

(ii) *$G_{k'}$ splits.*

(iii) *There is an isometry $(k', n_{k'/k}) \hookrightarrow (C, N_C)$.*

Proof. (1) Since G is of type G_2, it is equivalent to show that $G \times_k k'$ is isotropic.

We may assume that $T = T^{(k',l)}$. We consider first the case when $l = k \times l_2$, where l_2 is a quadratic étale k-algebra. Then we have

$$T \times_k k' \xrightarrow{\sim} R^1_{l'/k'}(\mathbb{G}_{m,l'}) \xleftarrow{\sim} R_{l_2 \otimes k'/k'}(\mathbb{G}_{m,l_2 \otimes k'}).$$

Hence $T \times_k k'$ is isotropic.

It remains to consider the case when the cubic k-algebra l is a field. From the first case, we see that $G_{l'}$ is split. In other words, the k'-group $G_{k'}$ is split by the cubic field algebra $l = l \otimes_k k'$ of k'. Hence $C_{k'}$ is split, and hence C splits.

(2) (i) ⇒ (ii) follows from (1).

(ii) ⇒ (i): If G is split, (i) holds according to Theorem 2.3.1. We may assume that G is not split, and hence is anisotropic. In particular, k is infinite. Since $G_{k'}$ splits, k' is a field and we denote by $\sigma : k' \to k'$ the conjugacy automorphism. We use now a classical trick. Since $G(k')$ is Zariski dense in the Weil restriction $R_{k'/k}(G_{k'})$, there exists a Borel k-subgroup B of $R_{k'/k}(G_{k'})$ such that its conjugate $\sigma(B)$ is opposite to B. The k-group $T = B \cap \sigma(B) \cap G$ of G is then a rank-2 torus. If we write $B = R_{k'/k}(B')$, with B' a Borel k'-subgroup of $G_{k'}$, then $T_{k'}$ is a maximal torus of B'. We denote the natural embedding of the maximal torus T by $i : T \to G$. By seeing $i(T_{k'})$ as a maximal k'-torus of B', it follows that the action of σ on the root system $\Psi(G_{k'}, T')$ is by -1. Thus **type**$(T, i) = (k', k^3)$ as desired.

For the equivalence (ii) $\Longleftrightarrow$ (iii), see Lemma 3.1.3. □

Remark 4.3.2. Another proof of (2) is provided by the next Proposition 4.4.1; it is the case $k_1 = k_2$.

4.4. ***The biquadratic case.*** In the dihedral group $D_6 \subset \mathrm{GL}_2(\mathbb{Z})$, it is convenient to change coordinates by considering the diagonal subgroup $(\mathbb{Z}/2\mathbb{Z})^2 = \langle c_1, c_2\rangle$. The map $(\mathbb{Z}/2\mathbb{Z})^2 \xrightarrow{\sim} C_2 \times (1 \times \mathbb{Z}/2\mathbb{Z}) \subset C_2 \times S_3$ is given by $(c_1, c_2) \mapsto (c_1, c_1 c_2)$.

We are interested in the case when the class of (k', l) belongs to the image of $H^1(k, \mathbb{Z}/2\mathbb{Z}) \times H^1(k, \mathbb{Z}/2\mathbb{Z}) \to H^1(k, \mathbb{Z}/2\mathbb{Z}) \times H^1(k, S_3)$. In terms of étale algebras, it rephrases by saying that there are quadratic étale k-algebras k_1/k, k_2/k such that $k' = k_2$ and $l = k \times l_2$, where $[k_2] = [k_1] + [l_2]$. We call that case the biquadratic case. In that case, $T^{(k',l)}$ is k-isomorphic to

$$\big(R^1_{k_1/k}(\mathbb{G}_m) \times R^1_{k_2/k}(\mathbb{G}_m)\big)/\mu_2.$$

Proposition 4.4.1. *Let k_1, k_2 be quadratic étale k-algebras and denote by $\chi_1, \chi_2 \in H^1(k, \mathbb{Z}/2\mathbb{Z})$ their classes. We consider the couple $(k', l) = (k_2, k \times l_2)$, where $[l_2] = [k_1] + [k_2]$. We denote by $\underline{\Psi} = \underline{\Psi}_{(k',l)}$, defined in Section 4.2, and by $X = \mathcal{E}(G, \underline{\Psi})$ the K-variety of embeddings defined in Section 2.6.*

(a) *The following are equivalent*:

(1) $X(k) \neq \varnothing$*; that is, G admits a maximal k-torus of type* $[(k', l)]$.

(2) *$C \otimes_k k_j$ is split for $j = 1, 2$.*

(3) *C admits a quaternion subalgebra Q such that there exists $c \in k^\times$ satisfying*

$$[Q] = \chi_1 \cup (c) = \chi_2 \cup (c) \in {}_2\mathrm{Br}(k).$$

(b) *If the k-variety X has a zero-cycle of odd degree then it has a k-point.*

Proof. (a) If C is split, the statement is trivial since the three assertions hold. We can then assume that C is nonsplit. We choose scalars $a_1, a_2 \in k$ such that $k_j \cong k[t]/t^2 - a_j$ for $j = 1, 2$ if k is of odd characteristic and $k_j \cong k[t]/t^2 + t + a_j$ in the characteristic-two case.

(1) $\Rightarrow$ (2): We assume that $T = T^{k',l} \cong \big(R^1_{k_1/k}(\mathbb{G}_m) \times R^1_{k_2/k}(\mathbb{G}_m)\big)/\mu_2$ embeds in G. Then T_{k_j} is isotropic so that G_{k_j} is isotropic, and hence split for $j = 1, 2$. We conclude that C_{k_j} is split for $j = 1, 2$.

(2) $\Rightarrow$ (3): We shall construct a quaternion subalgebra Q of C which contains k_1 and k_2. Since C_{k_j} splits for $j = 1, 2$, we know that k_j embeds in C as a unital composition subalgebra (Lemma 3.1.3). If $k_1 = k_2$ then Q can be obtained from k_1 by the doubling process from [Springer and Veldkamp 2000, Proposition 1.2.3]. So we can assume that $k_1 \neq k_2$. Let $x \in k_2 \setminus k_1$. Then Lemma 3.2.3 shows that $Q = k_1 \oplus k_1 x$ is a unital composition subalgebra of C. It is of dimension 4, so it is a quaternion subalgebra which contains k_1 and k_2. The common slot lemma yields that there exists $c \in k^\times$ such that $[Q] = \chi_1 \cup (c) = \chi_2 \cup (c) \in \mathrm{Br}(k)$. In odd characteristic, a reference for the common slot lemma is [Lam 2005, Chapter III, Theorem 4.13]. A characteristic-free version is a consequence of a fact on Pfister forms pointed out by Garibaldi and Petersson [2011, Proposition 3.12]. The

1-Pfister quadratic forms $n_{k_1/k}$ and $n_{k_2/k}$ are subforms of the Pfister quadratic form N_Q, so there exists a bilinear quadratic Pfister form $h = \langle 1, c\rangle$ such that $N_Q \cong h \otimes n_{k_1/k} \cong N_Q = h \otimes n_{k_2/k}$. Thus $[Q] = \chi_1 \cup (c) = \chi_2 \cup (c) \in \mathrm{Br}(k)$ according to the characterization of quaternion algebras by their norm forms.

(3) $\Rightarrow$ (1): We have that $C \cong C(Q, c)$, so we get an embedding

$$(\mathrm{SL}_1(Q) \times \mathrm{SL}_1(Q))/\mu_2 \to \mathrm{Aut}(C(Q,c)) \xrightarrow{\sim} G.$$

By embedding k_1 in Q (resp. k_2 in Q), we get an embedding

$$R^1_{k_1/k}(\mathbb{G}_m) \times R^1_{k_2/k}(\mathbb{G}_m) \to \mathrm{SL}_1(Q) \times \mathrm{SL}_1(Q),$$

so that

$$i : (R^1_{k_1/k}(\mathbb{G}_m) \times R^1_{k_2/k}(\mathbb{G}_m))/\mu_2 \to (\mathrm{SL}_1(Q) \times \mathrm{SL}_1(Q))/\mu_2 \to G$$

is an embedding. By the computations of the proof of Lemma 4.2.1(2), it indeed has type $[(k', l)]$.

(b) Assume that X has a 0-cycle of odd degree; i.e., there are finite field extensions $K_1, \dots, K_r$ of k such that $X(K_i) \neq \varnothing$ for $i = 1, \dots, r$ and g.c.d.$([K_1 : K], \dots, [K_r : K])$ is odd. By (a), it follows that $C_{K_i \otimes_k k_1}$ and $C_{K_i \otimes_k k_2}$ are split for $i = 1, \dots, r$. Then there exists an index i such that $[K_i : k]$ is odd. If $k_1 = k \times k$, then C splits over K_i; it follows that C is split by Lemma 3.1.1, whence $X(k) \neq \varnothing$ by Theorem 2.3.1. We can then assume that k_1 is a field. Then $K_i \otimes_k k_1$ is a field extension of K_j so that $C_{K_j \otimes_k k_1}$ splits; since $[K_i \otimes_k k_1 : k_1]$ is odd, Lemma 3.1.1 shows then that C_{k_1} is split. Similarly C_{k_2} is split, and by (a), we conclude that $X(k) \neq \varnothing$. ☐

In the following, we consider a special case where k' and l have the same discriminant.

Corollary 4.4.2. *Let k'/k be a quadratic étale algebra and let l be a cubic étale k-algebra of discriminant k'. If C admits a maximal k-torus of type* $[(k', l)]$, *then C splits.*

Proof. First, assume that l is not a field, so that $l \cong k \times k'$. Then Proposition 4.4.1 yields that C is split by the quadratic étale k-algebra k_1 which satisfies $[k_1] = [k'] + [l_2] = 0$, whence C is split.

If l is a field, the octonion l-algebra C_l admits a maximal l-torus of type $[(k' \otimes_k l, l \otimes_k l)]$. Since $l \otimes_k l \xrightarrow{\sim} l \times (l \otimes_k k')$, the first case shows that C_l is split. We conclude that C is split by Lemma 3.1.1. ☐

Remark 4.4.3. Take $k = \mathbb{R}$ and let C be the "anisotropic" Cayley algebra (or we simply call it a Cayley algebra). We consider the case where $(k', l) = (\mathbb{C}, \mathbb{R} \times \mathbb{C})$. By Corollary 4.4.2, there is no $\mathbb{R}$-embedding of a maximal torus of type (k', l). However, $G_{k'}$ splits and this example shows that only the direct implication holds in Proposition 4.3.1(1). The only possible type is then $[(\mathbb{C}, \mathbb{R}^3)]$, which is realized according to Proposition 4.3.1(2).

We can now provide a description of such maximal tori.

Proposition 4.4.4. *Let k_1, k_2 be quadratic étale k-algebras. We consider the couple $(k', l) = (k_2, k \times l_2)$, where $[l_2] = [k_1] + [k_2]$, and we assume that C is split by k_1 and k_2. We put $T = (R^1_{k_1/k}(\mathbb{G}_m) \times R^1_{k_2/k}(\mathbb{G}_m))/\mu_2$ and consider a k-embedding $i : T \to G$ of type $[(k', l)]$. Then there exists a quaternion subalgebra Q of C containing k_1 and k_2 and a Cayley–Dickson decomposition $C \cong C(Q, c)$ such that $i : T \to G \cong \mathrm{Aut}(C(Q, c))$ factorizes by the k-subgroup $(\mathrm{SL}_1(Q) \times \mathrm{SL}_1(Q))/\mu_2$ of $\mathrm{Aut}(C(Q, c))$.*

Proof. Consider the case where $k_1 \otimes_k k_2$ is a field. We denote by $\Gamma = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ the Galois group of the biquadratic field extension $k_1 \otimes_k k_2$. This group acts on the root system $\Phi(G_{k_s}, i(T_{k_s}))$ through a W_0-conjugate of the standard subgroup $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ of W_0 generated by the central symmetry and the symmetry with the horizontal axis (see the figure in Section 3.3). It follows that Γ stabilizes a subroot system Φ_1 of type $A_1 \times A_1$ of $\Phi(G_{k_s}, T_{k_s})$. By Galois descent, the k_s-subgroup generated by the root subgroups of Φ_1 descends to a k-subgroup H of G which is semisimple of type $A_1 \times A_1$. Lemma 3.4.1 shows that there is a Cayley–Dickson decomposition $C = Q \oplus Q.a$ such that $H = H(Q)$. We have then a factorization of $i : T \to G$ by $H(Q) \xrightarrow{\sim} (\mathrm{SL}_1(Q) \times \mathrm{SL}_1(Q))/\mu_2$.

The other cases (k_1 or k_2 split, $k_1 = k_2$) are simpler, of the same flavor, and left to the reader. □

4.5. ***The cubic field case: a first example.*** Beyond the previous "equal discriminant case", the embedding problem for a given octonion algebra C and a couple (k', l) whenever l is a cubic field is much more complicated. The property to carry a maximal torus of "cubic type" encodes information on the relevant k-group, and we shall first investigate specific examples over Laurent series fields. The next fact is inspired by similar considerations on central simple algebras by Chernousov, Rapinchuk and Rapinchuk [Chernousov et al. 2013, §2].

Let us start with a more general setting. Let G_0 be a semisimple Chevalley group defined over $\mathbb{Z}$, equipped with a maximal split subtorus T_0. Denote by Ψ_0 the root datum attached to (G_0, T_0). Let G'/k be a quasisplit form of G_0 and denote by T' a maximal k-torus of G' which is the centralizer of a maximal k-split torus of G'. We denote by $W' = N_{G'}(T')/T'$ the Weyl group of T'.

Lemma 4.5.1. *Let $K = k((t))$. Let E be a W'-torsor defined over k and put $T = E \wedge^{W'} T'$. Assume that $H^1(k, \widehat{T}^0) = 0$, where $\widehat{T}^0$ is the Galois lattice of cocharacters of T. Let $z : \mathrm{Gal}(K_s/K) \to G'(K_s)$ be a Galois cocycle and put $G = {}_zG'/K$. Assume there is an embedding $i : T_K \to G$ satisfying* $\mathbf{type}_{\mathrm{can}}(i, T_K) = [E]_K \in$

$H^1(K, W')$. *Then* $[z]$ *is "unramified"; i.e.,* $[z] \in \mathrm{Im}(H^1(k, G') \to H^1(K, G'))$. *In particular, there exists a semisimple* k*-group* H *such that* $G \cong H \times_k K$.

Proof. By our form of Steinberg's theorem, Theorem 2.4.1, there is a k-embedding $i' : T_K \to G'_K$ such that the class $[z] \in H^1(K, G'_K)$ belongs to the image of $i'_* : H^1(K, T) \to H^1(K, G'_K)$, and furthermore $\mathbf{type}_{\mathrm{can}}(T_K, i) = \mathbf{type}_{\mathrm{id}}(T_K, i') = [E]_K \in H^1(K, W')$.

On the other hand, we know by Theorem 2.3.1 that there exists a k-embedding $j : T \to G'$ such that $\mathbf{type}_{\mathrm{id}}(T, j) = [E]$. By Proposition 2.5.3, the images of $(i')_*$ and $(j_K)_* : H^1(K, T) \to H^1(K, G')$ coincide. It follows that $[z] \in H^1(K, G')$ belongs to the image of $(j_K)_* : H^1(K, T) \to H^1(K, G')$. We appeal now to the localization sequence $0 \to H^1(k, T) \to H^1(K, T) \to H^1(k, \widehat{T}^0) \to 0$ provided by the Appendix (Lemma A.1). Using our vanishing hypothesis $H^1(k, \widehat{T}^0) = 0$ and the commutative diagram

$$\begin{array}{ccccc} H^1(k, T) & \longrightarrow & H^1(K, T) & \longrightarrow & 0 \\ \downarrow {\scriptstyle j_{*,k}} & & \downarrow {\scriptstyle j_{*,K}} & & \\ H^1(k, G') & \longrightarrow & H^1(K, G') & & \end{array}$$

we conclude that $[z]$ comes from $H^1(k, G')$. □

Since every semisimple K-group of type G_2 is an inner form of its split form, the following corollary follows readily.

Corollary 4.5.2. *Let* $K = k((t))$ *and let* G/K *be a semisimple* k*-group of type* G_2. *Consider a couple* (k', l) *such that* k'/k *is a quadratic étale algebra and* l/k *is a cubic field separable extension. Denote by* E/k *the* W_0*-torsor associated to* (k', l) *and put* $T/k = E \wedge^{W_0} T_0$. *If the* K*-torus* $T \times_k K$ *admits an embedding* i *in* G *such that* $\mathbf{type}_{\mathrm{can}}(T_K, i) = [(k', l)]$, *then there exists a semisimple* k*-group* H *of type* G_2 *such that* $G \cong H \times_k K$.

Proof. We can assume that $G = {}_z(G_0)/K$, where $z : \mathrm{Gal}(K_s/K) \to G(K_s)$ is a Galois cocycle. By Lemma 4.2.3(2), we have $H^1(k, \widehat{T}^0) = 0$. The corollary then follows from Lemma 4.5.1 applied to $G' = G_0/k$ and $T' = T_0$. □

Theorem 4.5.3. *Let* Q *be a quaternion division algebra over* k, k' *a quadratic étale subalgebra of* Q *and* l/k *a Galois cubic field extension. As before, let* $K = k((t))$, $K' = k'((t))$, $L = l((t))$. *Let* $C/K = C(Q_K, t)$ *be the octonion algebra built out from the Cayley–Dickson doubling process.*

Let $\underline{\Psi} = \underline{\Psi}_{(K',L)}$ *be as defined in Section 4.2, and let* $X = \mathcal{E}(G, \underline{\Psi})$ *be the* K*-variety of embeddings defined in Section 2.6. Then* $X(K) = \varnothing$, $X(K') \neq \varnothing$ *and* $X(L) \neq \varnothing$.

Proof. We have $N_C = N_{Q,K} \otimes \langle 1, t\rangle$. Since N_Q is an anisotropic k-form, the quadratic form N_C is anisotropic and cannot be defined over k according to Springer's decomposition theorem [Elman et al. 2008, §19]. It follows that the k-group $G = \mathrm{Aut}(C)$ cannot be defined over k; Lemma 4.5.1 shows there is no embedding of a k-torus with type $[(K', L)]$, and therefore $X(K) = \varnothing$.

Since K' splits C, $G \times_K K'$ is split so that we have $X(K') \neq \varnothing$ by Theorem 2.3.1. It remains to show that $X(L)$ is not empty. We have $[(K', L)] \otimes_K L \cong [K' \otimes_K L, L^3]$. Since K' splits C, $K' \otimes_K L$ splits C and Proposition 4.3.1(2) yields $X(L) \neq \varnothing$. □

Remarks 4.5.4. (a) The requirements on the field k are mild and are satisfied by any local or global field.

(b) Geometrically speaking, the variety X/K is a homogeneous space under a k-group of type G_2 whose geometric stabilizer is a maximal K-torus. As far as we know, it is the simplest example of homogeneous space under a semisimple simply connected group with a 0-cycle of degree one and no rational points; compare with [Florence 2004], where stabilizers are finite and noncommutative, and [Parimala 2005], where stabilizers are parabolic subgroups.

5. Étale cubic algebras and hermitian forms

Our goal is to further investigate the cubic case by using results of Haile, Knus, Rost and Tignol [Haile et al. 1996] on hermitian 3-forms.

Let C be an octonion algebra over k and put $G = \mathrm{Aut}(C)$. Let $i : T \to G$ be a k-embedding of a rank-2 torus, and we denote by $[(k', l)]$ its type.

We denote by $R_{>0}$ the subset of long roots of the root system $R = \Phi(G_{k_s}, i(T_{k_s}))$. Then $R_>$ is a root system of type A_2 and is Γ_k-stable, and hence defines a twisted datum. We consider the k_s-subgroup of G_{k_s} generated by T_{k_s} and the root groups attached to elements of $R_>$; it is semisimple simply connected of type A_2 and descends to a semisimple k-group $J(T, i)$ of G. Our goal is to study such embeddings (T, i) by means of the subgroup $J(T, i)$.

We shall see in the sequel that such a k-group $J(T, i)$ is a special unitary group for some hermitian 3-form for k'/k.

Remarks 5.0.5. (a) J.-P. Serre explained another way to construct the k-subgroup $J(T, i)$. Define the finite k-group of multiplicative type

$$\mu_{T,k_s} = \mathrm{Ker}\left(T_{k_s} \xrightarrow{\prod \alpha} \prod_{\alpha \in R_>} \mathbb{G}_{m,k_s} \right);$$

it descends to a k-subgroup μ_T of T. We claim that

$$J(T, i) = Z_G(\mu_T).$$

For checking that fact, it is harmless to assume that k is algebraically closed. For simplicity, we put $J = J(T, i)$; it is isomorphic to SL_3. Since $\Phi(J, i(T))) = R_>$, we have that $\mu_T = Z(J)$ [Demazure and Grothendieck 1970c, XIX, 1.10.3]; it follows that $\mu_T \cong \mu_3$ and that $J \subseteq Z_G(\mu_T)$. Since J is a semisimple subgroup of maximal rank of G, Borel and de Siebenthal's theorem provides a k-subgroup μ_n of T such that $J = Z_G(\mu_n)$ [Pépin Le Halleur 2012, Proposition 6.6]. Then $\mu_n \subseteq Z(J) \cong \mu_3$ so that $\mu_n = Z(J) = \mu_T$. Thus $J = Z_G(\mu_T)$.

(b) If k is of characteristic 3, we can associate to T another k-subgroup $J_<(T, i)$ of type A_2. Let $R_<$ be the subset of short roots of the root system $R = \Phi(G_{k_s}, i(T_{k_s}))$. It is a 3-closed symmetric subset [Pépin Le Halleur 2012, Lemma 2.4], so the k_s-subgroup of G_{k_s} generated by T_{k_s} and the root groups attached to elements of $R_<$ define a semisimple k_s-subgroup $J_<$ of G_{k_s} [ibid., Theorem 3.1]; furthermore, we have $\Phi(J, i(T_{k_s})) = R_<$. The k_s-group $J_<$ descends to a semisimple k-group $J_<(T, i)$. It is semisimple of type A_2 and adjoint since $R_<$ spans $\widehat{T}(k_s)$.

5.1. ***Rank-3 hermitian forms and octonions.*** Let k'/k be a quadratic étale algebra. From a construction of Jacobson [1958, §5] (see [Knus et al. 1994, §6] for the generalization to an arbitrary base field), we recall that we can attach to a rank-3 hermitian form (E, h) (for k'/k) with trivial (hermitian) discriminant an octonion k-algebra $C(k', E, h) = k' \oplus E$. Furthermore, the k-group $\mathrm{SU}(k', E, h)$ embeds in $\mathrm{Aut}(C(k', E, h))$ by $g.(x, e) = (x, g.e)$. We denote by $J(k', E, h)$ this k-subgroup and we observe that k' is the k-vector subspace of $C(k', E, h)$ of fixed points for the action of $J(k', E, h)$ on $C(k', E, h)$. Also $J(k', E, h)$ is the k-subgroup of $\mathrm{Aut}(C(k', E, h))$ acting trivially on k'.

In a converse way (see [Knus et al. 1998, Exercise 6(b), page 508]), if we are given an embedding of a unital composition k-algebra $k' \to C$, we denote by E the orthogonal subspace of k' for N_C. For any $x, y \in k'$ and $z \in E$, we have

$$0 = \langle xy, z\rangle_C = \langle y, \sigma_C(x)z\rangle_C$$

by using the identity [Springer and Veldkamp 2000, Lemma 1.3.2], so that the multiplication $C \times C \to C$ induces a bilinear k-map $k' \times E \to E$. Then E has a natural k'-structure and the restriction of N_C to E defines a hermitian form h (of trivial discriminant) such that $C = C(k', E, h)$.

Furthermore, if we have two subfields k'_1, k'_2 of C isomorphic to k', the "Skolem–Noether" property [Knus et al. 1998, 33.21] shows that there exists $g \in G(k)$ mapping k'_1 to k'_2. Hence the hermitian forms (E_1, h_1), (E_2, h_2) are isometric.

Remark 5.1.1. Of course, in such a situation, h can be diagonalized as $\langle -b, -c, bc\rangle$ and we have $n_{C(k',E,h)} = n_{k'/k} \otimes \langle\langle b, c\rangle\rangle$. If we take $\langle -1, -1, 1\rangle$, we get one form of the split octonion algebra C_0 and then a k-subgroup $J_0 = \mathrm{SL}_3$ of $\mathrm{Aut}(C_0)$.

Lemma 5.1.2. *In the above setting, we put $G = \mathrm{Aut}(C(k', E, h))$ and $J = J(k', E, h)$.*

(1) *There is a natural exact sequence of algebraic k-groups $1 \to J \to N_G(J) \to \mathbb{Z}/2\mathbb{Z} \to 1$.*

(2) *The map $N_G(J)(k) \to \mathbb{Z}/2\mathbb{Z}$ is onto and the induced action of $\mathbb{Z}/2\mathbb{Z}$ on k' is the Galois action.*

Proof. (1) We consider the commutative exact diagram of k-groups

$$\begin{array}{ccccccccc}
 & & & & & & 1 & & \\
 & & & & & & \downarrow & & \\
1 & \longrightarrow & Z(J) & \longrightarrow & J & \longrightarrow & J/Z(J) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & Z_G(J) & \longrightarrow & N_G(J) & \longrightarrow & \mathrm{Aut}(J) & \longrightarrow & 1 \\
 & & & & & & \downarrow & & \\
 & & & & & & \mathrm{Autext}(J) = \mathbb{Z}/2\mathbb{Z} & & \\
 & & & & & & \downarrow & & \\
 & & & & & & 1 & &
\end{array}$$

Let T be a maximal k-torus of J; it is still maximal in G. Then we have $Z_G(J) \subseteq Z_G(T) = T$, and hence $Z_G(J) \subseteq Z(J)$, so that $Z(J) = Z_G(J)$. The diagram provides then an exact sequence $1 \to J \to N_G(J) \to \mathbb{Z}/2\mathbb{Z}$. We postpone the surjectivity.

(2) Now by the "Skolem–Noether property" [Knus et al. 1998, 33.21], the Galois action $\sigma : k' \to k'$ extends to an element $g \in G(k)$. Given $u \in J(k)$, gug^{-1} is an element of $G(k)$ which acts trivially on k', so it belongs to $J(k)$. Since it holds for any field extension of k, we have that $g \in N_G(J)(k)$. We conclude that the map $N_G(J) \to \mathbb{Z}/2\mathbb{Z}$ is surjective and that the induced action of $\mathbb{Z}/2\mathbb{Z}$ on k' is the Galois action. □

Let C be an octonion algebra, put $G = \mathrm{Aut}(C)$ and let J be a semisimple k-subgroup of type A_2 of G. Then J is of maximal rank and we can appeal again to the Borel and de Siebenthal classification theorem [Pépin Le Halleur 2012, Theorem 3.1]. If the characteristic of k is not 3, then J is geometrically conjugated to the standard SL_3 in G_2 and is then simply connected. If the characteristic k is 3, then J may arise as in Remarks 5.0.5(b) from the short roots associated to a maximal k-torus of J; in that case, J is adjoint. We can make a similar statement to Lemma 3.4.1.

Lemma 5.1.3. *Let J be a semisimple simply connected k-subgroup of type A_2 of $G = \mathrm{Aut}(C)$ and we denote by k'/k the quadratic étale algebra attached to the quasisplit form of J. Then there exists a rank-3 hermitian form (E, h) for k'/k, an isomorphism $C \cong C(k', E, h)$, and an isomorphism $J \xrightarrow{\sim} J(k', E, h)$ such that the* *following diagram commutes*

$$\begin{array}{ccc} J & \hookrightarrow & G \\ \downarrow \wr & & \downarrow \wr \\ J(E,h) & \overset{j}{\hookrightarrow} & \mathrm{Aut}(C(k',E,h)). \end{array}$$

Proof. Given a k-maximal torus T of G, we consider the root system $\Psi(G_{k_s}, T_{k_s})$. There are exactly 6 long roots in $\Psi(G_{k_s}, T_{k_s})$ which form an A_2-subsystem of $\Psi(G_{k_s}, T_{k_s})$. Let H be the subgroup of G_{k_s} which is generated by T_{k_s} and the root groups of long roots. Since the Galois action preserves the length of a root, the group H is defined over k. Hence given a k-maximal torus T, there is exactly one subgroup H of G which is a twisted form of SL_3 and contains T. Since all maximal k-split tori are conjugated over k, the split group G_0 of type G_2 has one single conjugacy $G_0(k)$-class of k-subgroups isomorphic to SL_3. It follows that the couple (G, J) is isomorphic over k_s to the couple (G_0, J_0). In particular, by Galois descent, the subspace of fixed points of J on C is an étale subalgebra l of rank 2 which is a unital composition subalgebra of C. We define then the orthogonal subspace E of l in C. Then E has a natural structure of an l-vector space and carries a hermitian form h of trivial (hermitian) discriminant such that $C(l, E, h) = C$ (see [Knus et al. 1998, Exercise 6(b), page 508]). But J acts trivially on l, so that $J \subseteq J(l, E, h)$. For dimension reasons, we conclude that $J = J(l, E, h)$. Then l/k is the discriminant étale algebra of J, and hence $k' = l$. □

Remark 5.1.4. Note that in the above proof, we didn't put any assumption on the characteristic of k. However, in characteristic $\neq 2, 3$, Hooda [2014, Theorem 4.4] proved the above lemma in a quite different way.

5.2. ***Embedding maximal tori.*** From now on, we assume for simplicity that the characteristic exponent of k is not 2.

Lemma 5.2.1. *Let $G = \mathrm{Aut}(C)$ be a semisimple k-group of type G_2. Let k' (resp. l) be a quadratic (resp. cubic) étale algebra of k. Let $i : T \to G$ be a k-embedding of a maximal k-torus such that* $\mathbf{type}(T, i) = [(k', l)]$ *and denote by $J(T, i)$ the associated k-subgroup of G.*

(1) *The discriminant algebra of $J(T, i)$ is k'/k.*

(2) *By Lemma 5.1.3, we can write $C = C(k', E, h)$ and identify $J(T, i)$ with $J(k', E, h)$. Then there is a k'-embedding $f : k' \otimes_k l \to M_3(k')$ such that $f \circ (\sigma \otimes \mathrm{id}) = \tau_h \circ f$ on $k' \otimes_k l$, where τ_h is the involution on $M_3(k')$ induced by h.*

Proof. (1) We put $J = J(T, i)$. We consider the Galois action on the root system $\Psi(G_{k_s}, i(T)_{k_s})$ and its subroot system $\Psi(J_{k_s}, i(T)_{k_s}) = \Psi(G_{k_s}, i(T)_{k_s})_{>}$. It is given by a map $f : \Gamma_k \to \mathbb{Z}/2\mathbb{Z} \times S_3$ defining $[(k', l)]$. Since the Weyl group of $\Psi(J_{k_s}, i(T)_{k_s})$ is S_3, it follows that the $\star$-action of Γ_k on the Dynkin diagram A_2 is the projection $\Gamma_k \to \mathbb{Z}/2\mathbb{Z}$. Therefore the discriminant algebra of $J(T, i)$ is k'/k.

(2) We have then a k-embedding $i : T \to J = \mathrm{SU}(k', E, h)$. Its type (absolute with respect to J) is $[(k', l)] \in H^1(k, \mathbb{Z}/2\mathbb{Z} \times S_3)$. By [Lee 2014, Theorem 1.15(2)], there is a k'-embedding $k' \otimes_k l \to M_3(k')$ with respect to the conjugacy involution $\sigma \otimes \mathrm{id}$ on $k' \otimes_k l$ and the involution τ_h attached to h. □

Proposition 5.2.2. *Let $G = \mathrm{Aut}(C)$ be a semisimple k-group of type G_2. Let k' (resp. l) be a quadratic (resp. cubic) étale k-algebra. We denote by X the variety of k-embeddings of maximal tori in G attached to the twist of Ψ_0 by (k', l) (seen as a W_0-torsor). The following are equivalent*:

(i) *$X(k) \neq \varnothing$; that is, there exists an embedding $i : T \to G$ of a maximal k-torus of type $[(k', l)]$.*

(ii) *There exists a rank-3 hermitian form (E, h) for k'/k of trivial (hermitian) discriminant such that $C \cong C(k', E, h)$ and such that there exists a k'-embedding of $k' \otimes_k l \to \mathrm{End}_{k'}(E)$ with respect to the conjugacy involution on k' and the involution τ_h attached to h.*

(iii) *There exists a rank-3 hermitian form (E, h) for k'/k of trivial (hermitian) discriminant such that $C \cong C(k', E, h)$ and an element $\lambda \in l^\times$ such that $(l \otimes_k k', \mathrm{t}'_\lambda) \simeq (E, h)$, where $\mathrm{t}'_\lambda(x, y) = \mathrm{tr}_{l \otimes k'/k'}(\lambda x \sigma(y))$.*

Proof. The implication (i) ⇒ (ii) follows from Lemma 5.2.1(2). Conversely, we assume (ii). Then $G \cong \mathrm{Aut}(C(k', E, h))$ admits the k-subgroup $J(k', E, h) \xrightarrow{\sim} \mathrm{SU}(k', E, k)$. By [Lee 2014, Theorem 1.15(2)], there is a k-embedding $i : T \to \mathrm{SU}(k', E, k)$ of a maximal torus whose absolute type (with respect to J) is $[(k', l)]$. The k-embedding $i : T \to \mathrm{SU}(k', E, k) \to G$ also has absolute type $[(k', l)]$.

The equivalence (ii) ⟺ (iii) follows from the embedding criterion of $k' \otimes_k l \to \mathrm{End}_{k'}(E)$ given by [Bayer-Fluckiger et al. 2015, Proposition 1.3.1]. □

Let k', l be as in Proposition 5.2.2. Let $\delta \in k^\times/k^{\times 2}$ be the discriminant of l and $d \in k^\times/k^{\times 2}$ be the discriminant of k'. Let B be a central simple algebra over k' with an involution σ of the second kind. Let Trd be the reduced trace on B. Let $(B, \sigma)_+$ be the k-vector space of σ-symmetric elements of B. Let Q_σ be the quadratic form on $(B, \sigma)_+$ defined by

$$Q_\sigma(x, y) = \mathrm{Trd}(xy).$$

Let us recall some results in [Haile et al. 1996].

Lemma 5.2.3. *Assume that k is not of characteristic 2. Let B be a central simple K-algebra of odd degree $n = 2m - 1$ with involution σ of the second kind. There* *is a quadratic form q_σ of dimension $n(n-1)/2$ and trivial discriminant over k such that*

$$Q_\sigma \simeq \langle 1\rangle \perp \langle 2\rangle \cdot \langle\langle \alpha\rangle\rangle \otimes q_\sigma .$$

Proof. We refer to [ibid., Proposition 4]. □

Theorem 5.2.4. *Assume that k is not of characteristic 2 or 3. Let σ, τ be involutions of the second kind on a central simple algebra B of degree 3. Then σ and τ are isomorphic if and only if Q_σ and Q_τ are isometric.*

Proof. We refer to [ibid., Theorem 15]. □

Let (B, σ) be as in Lemma 5.2.3 with degree $B = 3$ and assume that 6 is invertible in k. Let b_0, $c_0 \in k^\times$ such that $q_\sigma \simeq \langle -b_0, -c_0, b_0c_0\rangle$. Define $\pi(B, \sigma)$ to be the Pfister form $\langle\langle d, b_0, c_0\rangle\rangle$. An involution σ of the second kind is called *distinguished* if $\pi(B, \sigma)$ is hyperbolic. Let (E, h) be a rank-3 hermitian form over k' with trivial (hermitian) discriminant. We can find b, $c \in k^\times$ such that $h \simeq \langle -b, -c, bc\rangle_{k'}$.

Now consider the special case where $(B, \sigma) = (\mathrm{End}_{k'}(E), \tau_h)$. Then we have $q_{\tau_h} = \langle -b, -c, bc\rangle$ and $\pi(\mathrm{End}_{k'}(E), \tau_h) = \langle\langle d, b, c\rangle\rangle$, which is the norm form of the octonion $C(k', E, h)$. It is then possible to recover with that method at least the two following facts.

Remarks 5.2.5. (a) Theorem 2.3.1 for G_2, i.e., all possible types of tori occur in the split case: Given a couple (k', l), we can write the split octonion algebra C as $C(k', E, h)$ for $E = (k')^3$ $h = \langle -1, -1, 1\rangle$. First we note that l can be embedded into $\mathrm{End}_{k'}(E)$ since $\mathrm{End}_{k'}(E)$ is split. As N_C is isotropic, we have that τ_h is distinguished. By [Haile et al. 1996, Corollary 18], every cubic étale algebra l can be embedded as a subalgebra in $\mathrm{End}_{k'}(E)$ with its image in $(\mathrm{End}_{k'}(E), \tau_h)_+$. By Proposition 5.2.2(2), there is an embedding $i : T \to G$ of type $[(k', l)] \in H^1(k, W_0)$.

(b) Corollary 4.4.2 for the "equal discriminant case", i.e., the discriminant algebra of l is k': In this case, there is an embedding $i : T \to G$ of type $[(k', l)]$ if and only if N_C is isotropic. For a proof in the present setting, we assume there is an embedding $i : T \to G$ of type $[(k', l)]$. According to Proposition 5.2.2(2), there exists a 3-hermitian form (E, h) of trivial determinant such that $C \cong C(k', E, h)$ and an embedding $l \otimes_k k' \to \mathrm{End}_{k'}(E)$ with respect to the conjugacy involution on k' and the involution τ_h attached to h. Then $(\mathrm{End}_{k'}(E), \tau_h)_+$ contains a cubic étale algebra isomorphic to l whose discriminant is d. By [ibid., Theorem 16(e)], we have $\pi(\mathrm{End}_{k'}(E), \tau_h) = N_C$ is isotropic. Thus C is split.

Proposition 5.2.6. *Assume that k is not of characteristic 2, 3. Let $G = \mathrm{Aut}(C)$ be a semisimple k-group of type G_2. Let k' (resp. l) be a quadratic (resp. cubic) étale k-algebra. Then there is a k-embedding $i : T \to G$ of type $[(k', l)] \in H^1(k, W_0)$ if and only if the following two conditions both hold*:

(i) *There is a rank-*3 k'/k*-hermitian form* (E, h) *of trivial* (*hermitian*) *discriminant such that* $C \simeq C(k', E, h)$.

(ii) *Let* $b, c \in k^\times$ *such that* $\langle -b, -c, bc\rangle_{k'}$ *is isometric to the form* h *in* (i). *Then there is* $\lambda \in l^\times$ *such that* $N_{l/k}(\lambda) \in k^{\times 2}$ *and the* k*-quadratic form* $\langle\langle d\rangle\rangle \otimes \langle\delta\rangle \cdot t_{l/k}(\langle\lambda\rangle)$ *is isometric to* $\langle\langle d\rangle\rangle \otimes \langle -b, -c, bc\rangle$, *where* $t_{l/k}$ *denotes the Scharlau transfer with respect to the trace map* $\mathrm{tr} : l \to k$.

Proof. Suppose that there is a k-embedding $i : T \to G$ of type $[(k', l)] \in H^1(k, W_0)$. By Proposition 5.2.2(2), there is a rank-3 (k'/k)-hermitian form (E, h) such that $C \simeq C(k', E, h)$, and there exists an embedding $\iota : k' \otimes_k l \to \mathrm{End}_{k'}(E)$ with respect to the conjugacy involution on k' and the involution τ_h attached to h. By [Haile et al. 1996, Corollary 12], we can find $\lambda \in l^\times$ such that $N_{l/k}(\lambda) \in k^{\times 2}$ and the q_{τ_h} in Lemma 5.2.3 is the k-quadratic form $\langle\delta\rangle \cdot t_{l/k}(\langle\lambda\rangle)$. Since

$$Q_{\tau_h} = 3\langle 1\rangle \perp \langle 2\rangle \cdot \langle\langle d\rangle\rangle \otimes \langle -b, -c, bc\rangle,$$

condition (ii) follows from the Witt cancellation.

Conversely, suppose that (i) and (ii) hold. By Proposition 5.2.2(2), it suffices to prove that there is a k-embedding of l into $(M_3(k'), \tau_h)_+$. Note that every cubic étale k-algebra l can be embedded into $M_3(k')$ as a k-algebra. By [ibid., Corollary 14], for every $\lambda \in l^\times$ such that $N_{l/k}(\lambda) \in k^{\times 2}$, there is an involution σ of the second kind on $M_3(k')$ leaving l elementwise invariant such that

$$Q_\sigma = \langle 1, 1, 1\rangle \perp \langle 2\rangle \cdot \langle\langle d\rangle\rangle \otimes \langle\delta\rangle \cdot t_{l/k}(\langle\lambda\rangle).$$

Condition (ii) implies that we can choose λ so that Q_σ and Q_{τ_h} are isometric. By Theorem 5.2.4, the involutions σ and τ_h are isomorphic, and hence there is a k-embedding of l into $(M_3(k'), \tau_h)_+$. □

6. Hasse principle

We assume that the base field k is a number field.

Proposition 6.1. *Let* (k', l) *be a couple where* k' *is a quadratic étale* k*-algebra and* l/k *is a cubic étale* k*-algebra. Let* G *be a semisimple* k*-group of type* G_2 *and let* X *be the* G*-homogeneous space of the embeddings of maximal tori with respect to the type* $[(k', l)]$. *Then* X *satisfies the Hasse principle.*

Proof. Since G_0 is simply connected, we have $H^1(k_v, G_0) = 1$ for each finite place v of k. The Hasse principle states that the map

$$H^1(k, G_0) \xrightarrow{\sim} \prod_{v \text{ real place}} H^1(k_v, G_0)$$

is bijective. If G is split, $X(k)$ is not empty (Theorem 2.3.1), so we may assume that G is not split. By [Lee 2014, Proposition 2.8], $X(k)$ is not empty if and only if the

Borovoi obstruction $\gamma \in Ш^2(k, T^{(k',l)})$ vanishes. There is a real place v such that G_{k_v} is not split and then is k_v-anisotropic. Since there is a k_v-embedding of $T^{(k',l)}$ in G_{k_v}, the torus $T^{(k',l)}$ is k_v-anisotropic. By a lemma due to Kneser [Sansuc 1981, lemme 1.9.3], we know that $Ш^2(k, T^{(k',l)}) = 0$, so that $\gamma = 0$. Thus $X(k) \neq \varnothing$. □

Remark 6.2. Under the hypothesis of Proposition 6.1, the existence of a k-point on X is controlled by the Borovoi obstruction. It follows from the restriction-corestriction principle in Galois cohomology that X has a k-point if and only if X has a 0-cycle of degree one. In other words, examples like those in Theorem 4.5.3 do not occur over number fields.

Corollary 6.3. *Let k be a number field and k' (resp. l) be quadratic (resp. cubic) étale algebra over k. Let $\delta \in k^\times/k^{\times 2}$ be the discriminant of l and $d \in k^\times/k^{\times 2}$ be the discriminant of k'. Let Σ be the set of (real) places where G is not split. Then $T^{(k',l)}$ can be embedded in G with respect to the type $[(k',l)]$ if and only if $d = -1 \in k_v^\times/k_v^{\times 2}$ and $\delta = 1 \in k_v^\times/k_v^{\times 2}$ for each $v \in \Sigma$.*

Proof. According to Proposition 6.1, $T^{(k',l)}$ can be embedded in G with respect to the type $[(k',l)]$ if and only if this holds everywhere locally or equivalently (by Theorem 2.3.1) if and only if this holds locally on Σ. The problem boils down to the real anisotropic case where the only type is $[(\mathbb{C}, \mathbb{R}^3)]$, according to Remark 4.4.3. □

Examples 6.4. Keep the notations in Corollary 6.3.

(a) Consider the special case where k is the field of rational numbers $\mathbb{Q}$. Suppose that G is anisotropic over $\mathbb{Q}$. Since there is only one real place of $\mathbb{Q}$, by Corollary 6.3, the torus $T^{(k',l)}$ can be embedded in G with respect to type $[(k',l)]$ if and only if k' is imaginary and the discriminant of l is positive.

(b) Let k be a number field. Suppose that G is anisotropic. Note that in this case, k is a real extension over $\mathbb{Q}$. Let k' be an imaginary field extension of k and let the discriminant of l be $[a] \in k^\times/k^{\times 2}$ for some positive $a \in \mathbb{Q}$. Then by Corollary 6.3, the torus $T^{(k',l)}$ can always be embedded in G with respect to type $[(k',l)]$.

Appendix: Galois cohomology of tori and semisimple groups over Laurent series fields

This appendix first provides a reference for a well-known fact on the Galois cohomology of tori in the vein of the short exact sequence computing the tame Brauer group of a Laurent series field. This fact is used in the proof of Lemma 4.5.1. Secondly we apply our version of Steinberg's theorem to Bruhat–Tits theory, answering a question of A. Merkurjev.

We recall that an affine algebraic k-group G is a k-torus if there exists a finite Galois extension k'/k such that $G \times_k k' \xrightarrow{\sim} (\mathbb{G}_{m,k'})^r$. If T is a k-torus, we consider its Galois lattice of characters $\widehat{T} = \mathrm{Hom}_{k_s-gp}(T_{k_s}, \mathbb{G}_{m,k_s})$ and its Galois lattice of cocharacters $\widehat{T}^0 = \mathrm{Hom}_{k_s-gp}(\mathbb{G}_{m,k_s}, T_{k_s})$.

Lemma A.1. *We put $K = k((t))$. Let T/k be an algebraic k-torus. Then we have a natural split exact sequence*

$$0 \to H^1(k,T) \to H^1(K,T) \xrightarrow{\partial} H^1(k,\widehat{T}^0) \to 0.$$

Proof. Let k' be a Galois extension which splits T. We put $\Gamma = \mathrm{Gal}(k'/k)$ and $K' = k'((t))$. We have the exact sequence [Serre 1994, I.2.6(b)]

$$0 \to H^1(\Gamma, T(k')) \to H^1(k,T) \to H^1(k',T).$$

Since $T_{k'}$ is split, Hilbert's theorem 90 shows that $H^1(k',T) = 0$, whence there is an isomorphism $H^1(\Gamma, T(k')) \xrightarrow{\sim} H^1(k,T)$. Similarly, we have $H^1(\Gamma, T(K')) \xrightarrow{\sim} H^1(K,T)$. We consider the ($\Gamma$-split) exact sequence

$$1 \to (k'[[t]])^\times \to (K')^\times \to \mathbb{Z} \to 0$$

induced by the valuation. Tensoring with $\widehat{T}^0$, we get a Γ-split exact sequence

$$1 \to T(k'[[t]]) \to T(K') \to \widehat{T}^0 \to 1.$$

It gives rise to a split exact sequence

$$0 \to H^1\big(\Gamma, T(k'[[t]])\big) \to H^1(\Gamma, T(K')) \to H^1(\Gamma, \widehat{T}^0) \to 0.$$

Now we use the filtration argument of [Gille and Szamuely 2006, 6.3.1] by putting

$$U^j = \{x \in k'[[t]]^\times \mid v_t(x-1) \geq j\}$$

for each $j \geq 0$. The $V^j = \widehat{T}^0 \otimes U^j$ filter $T(k'[[t]])$ and each $V^j/V^{j+1} \cong \widehat{T}^0 \otimes_k k'$ is a k'-vector space equipped with a semilinear action, and hence is Γ-acyclic.[1] According to the limit fact [Gille and Szamuely 2006, 6.3.2], we conclude that the specialization map $H^1\big(\Gamma, T(k'[[t]])\big) \to H^1(\Gamma, T(k'))$ is an isomorphism. We have then a split exact sequence

$$0 \to H^1(\Gamma, T(k')) \to H^1(\Gamma, T(K')) \to H^1(\Gamma, \widehat{T}^0) \to 0.$$

Since $H^1(k', \widehat{T}^0) = 0$, we have $H^1(\Gamma, \widehat{T}^0) \xrightarrow{\sim} H^1(k, \widehat{T}^0)$, whence the desired exact sequence. □

Now we relate Bruhat–Tits theory and our version of Steinberg's Theorem 2.4.1. Let G' be a quasisplit semisimple k-group equipped with a maximal k-split subtorus S'. We denote by W' the Weyl group of the maximal torus $T' = C_{G'}(T')$ of G'. Put $K = k((t))$ and denote by K_{nr} the maximal unramified closure of K.

[1] Speiser's lemma shows that $V^j/V^{j+1} = E_j \otimes_k k'$ for a k-vector space E_j on which Γ acts trivially.

Proposition A.2. *Let E be a G'_K-torsor. Then the following are equivalent*:

(i) $E(K_{nr}) \neq \varnothing$.

(ii) *There exists a k-torus embedding $i_0 : T_0 \to G'$ such that $[E]$ belongs to the image of $i_{0,*} : H^1(K, T_0) \to H^1(K, G)$.*

Proof. We denote by $G/K = {}^E G'$ the inner twist of G'_K by E.

(i) ⇒ (ii): Then G is split by the extension K_{nr}/K and the technical condition (DE) of Bruhat–Tits theory is satisfied [Bruhat and Tits 1984, Proposition 5.1.6]. It follows that G admits a maximal K-torus $j : T \to G$ which is split over K_{nr} [ibid., Corollary 5.1.2].

In particular, there exists a k-torus T_0 such that $T = T_{0,K}$. We consider now the oriented type $\gamma = \mathbf{type}_{\text{can}}(T, j) \in H^1(K, W')$ provided by the action of the absolute Galois group of K on the root system $\Phi(G_{K_s}, j(T)_{K_s})$. Since T and G are split by K_{nr}, it is given by the action of $\text{Gal}(K_{nr}/K) \cong \text{Gal}(k_s/k)$ on the root system $\Phi(G_{K_{nr}}, j(T)_{K_{nr}})$ and then defines a constant class $\gamma_0 \in H^1(k, W')$ such that $\gamma = (\gamma_0)_K$.

In the other hand, by the Kottwitz embedding (Theorem 2.3.1), there exists a k-embedding $i_0 : T_0 \to G'$ of oriented type γ_0. By Theorem 2.4.1, we conclude that $[E]$ belongs to the image of $i_{0,*} : H^1(K, T_0) \to H^1(K, G')$.

(ii) ⇒ (i): We assume there is a k-embedding $i_0 : T_0 \to G'$ such that $[E]$ belongs to the image of $i_{0,*} : H^1(K, T_0) \to H^1(K, G')$. Since $T_{0,K}$ is split by K_{nr}, the Hilbert theorem 90 shows that $H^1(K_{nr}, T_0) = 0$, whence $E(K_{nr}) \neq \varnothing$. □

Remarks A.3. (a) If k is perfect, we have that $\text{cd}(K_{nr}) = 1$ (by Lang, see [Gille and Szamuely 2006, Theorem 6.2.11]) so condition (i) is always satisfied according to Steinberg's theorem.

(b) If k is not perfect, there exist examples when condition (i) is not satisfied, even in the semisimple split simply connected case; see [Gille 2002, Proposition 3 and Theorem 1].

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Volume 279 No. 1-2 December 2015

In memoriam: Robert Steinberg

Robert Steinberg (1922–2014): In memoriam 1
V. S. VARADARAJAN

Cellularity of certain quantum endomorphism algebras 11
HENNING H. ANDERSEN, GUSTAV I. LEHRER and RUIBIN ZHANG

Lower bounds for essential dimensions in characteristic 2 via orthogonal representations 37
ANTONIO BABIC and VLADIMIR CHERNOUSOV

Cocharacter-closure and spherical buildings 65
MICHAEL BATE, SEBASTIAN HERPEL, BENJAMIN MARTIN and GERHARD RÖHRLE

Embedding functor for classical groups and Brauer–Manin obstruction 87
EVA BAYER-FLUCKIGER, TING-YU LEE and RAMAN PARIMALA

On maximal tori of algebraic groups of type G_2 101
CONSTANTIN BELI, PHILIPPE GILLE and TING-YU LEE

On extensions of algebraic groups with finite quotient 135
MICHEL BRION

Essential dimension and error-correcting codes 155
SHANE CERNELE and ZINOVY REICHSTEIN

Notes on the structure constants of Hecke algebras of induced representations of finite Chevalley groups 181
CHARLES W. CURTIS

Complements on disconnected reductive groups 203
FRANÇOIS DIGNE and JEAN MICHEL

Extending Hecke endomorphism algebras 229
JIE DU, BRIAN J. PARSHALL and LEONARD L. SCOTT

Products of partial normal subgroups 255
ELLEN HENKE

Lusztig induction and ℓ-blocks of finite reductive groups 269
RADHA KESSAR and GUNTER MALLE

Free resolutions of some Schubert singularities 299
MANOJ KUMMINI, VENKATRAMANI LAKSHMIBAI, PRAMATHANATH SASTRY and C. S. SESHADRI

Free resolutions of some Schubert singularities in the Lagrangian Grassmannian 329
VENKATRAMANI LAKSHMIBAI and REUVEN HODGES

Distinguished unipotent elements and multiplicity-free subgroups of simple algebraic groups 357
MARTIN W. LIEBECK, GARY M. SEITZ and DONNA M. TESTERMAN

Action of longest element on a Hecke algebra cell module 383
GEORGE LUSZTIG

Generic stabilisers for actions of reductive groups 397
BENJAMIN MARTIN

On the equations defining affine algebraic groups 423
VLADIMIR L. POPOV

Smooth representations and Hecke modules in characteristic p 447
PETER SCHNEIDER

On CRDAHA and finite general linear and unitary groups 465
BHAMA SRINIVASAN

Weil representations of finite general linear groups and finite special linear groups 481
PHAM HUU TIEP

The pro-p Iwahori Hecke algebra of a reductive p-adic group, V (parabolic induction) 499
MARIE-FRANCE VIGNÉRAS

Acknowledgement 531