ON THE EQUATIONS DEFINING
AFFINE ALGEBRAIC GROUPS

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In memory of Robert Steinberg

For the coordinate algebras of connected affine algebraic groups, we explore the problem of finding a presentation by generators and relations canonically determined by the group structure.

1. Introduction

Connected algebraic groups constitute a remarkable class of irreducible quasiprojective algebraic varieties. It contains the subclasses of abelian varieties and affine algebraic groups. These subclasses are basic: by Chevalley's theorem, every connected algebraic group G has a unique connected normal affine algebraic subgroup L such that G/L is an abelian variety, whence the variety G is an L-torsor over the abelian variety G/L. The varieties from these subclasses can be embedded in many ways as closed subvarieties in, respectively, projective and affine spaces. A natural question then arises as to whether there are distinguished embeddings and equations of their images, which are canonically determined by the group structure. For abelian varieties, this is the existence problem for canonically defined bases in linear systems and that of presenting homogeneous coordinate rings of ample invertible sheafs by generators and relations. These problems were explored and solved by D. Mumford [1966]. For affine algebraic groups, it is the existence problem of the canonically defined presentations of the coordinate algebras of such groups by generators and relations. We explore this problem in the present paper.

We fix as the base field an algebraically closed field k of arbitrary characteristic. In this paper, as in [Borel 1991], "variety" means "algebraic variety" in the sense of Serre [1955, Subsection 34]; every variety is taken over k.

Let G be a connected affine algebraic group and let $R_u(G)$ be its unipotent radical. In view of [Grothendieck 1958, Propositions 1, 2] and [Rosenlicht 1956, Theorem 10], the underlying variety of G is isomorphic to the product of that of

$G/R_u(G)$ and $R_u(G)$, and the latter is isomorphic to an affine space. Therefore, the problem under consideration is reduced to the case of reductive groups. Given this, henceforth G stands for a connected *reductive* algebraic group.

The simplest case of SL_2 is the guiding example. Take the polynomial k-algebra $k[x_1, x_2, x_3, x_4]$ in four variables x_i. The usual presentation of $k[SL_2]$ is given by the surjective homomorphism

$$(1) \qquad \mu : k[x_1, x_2, x_3, x_4] \to k[SL_2], \quad \mu(x_i)\left(\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}\right) = a_i,$$

whose kernel is the ideal $(x_1 x_4 - x_2 x_3 - 1)$. After rewriting, this presentation can be interpreted in terms of the group structure of SL_2 as follows.

We have $k[x_1, x_2, x_3, x_4] = k[x_1, x_3] \otimes_k k[x_2, x_4]$ and the restriction of μ to the subalgebra $k[x_1, x_3]$ (respectively, $k[x_2, x_4]$) is an isomorphism with the subalgebra $\mathcal{S}^+$ (respectively, $\mathcal{S}^-$) of $k[SL_2]$ consisting of all regular functions invariant with respect to the subgroup U^+ (respectively, U^-) of all unipotent upper (respectively, lower) triangular matrices acting by right translations. Hence (1) yields the following presentation of $k[SL_2]$ by generators and relations:

$$(2) \qquad \begin{aligned} k[SL_2] &\cong (\mathcal{S}^+ \otimes_k \mathcal{S}^-)/\mathcal{I}, \\ \mathcal{S}^+ &= k[\mu(x_1), \mu(x_3)] \cong k[x_1, x_3], \\ \mathcal{S}^- &= k[\mu(x_2), \mu(x_4)] \cong k[x_2, x_4], \\ \mathcal{I} &= \big(\mu(x_1) \otimes \mu(x_4) - \mu(x_2) \otimes \mu(x_3) - 1\big). \end{aligned}$$

The subgroups U^+, U^- are opposite maximal unipotent subgroups of SL_2. The subalgebras $\mathcal{S}^+$, $\mathcal{S}^-$ are stable with respect to SL_2 acting by left translations, and $f := \mu(x_1) \otimes \mu(x_4) - \mu(x_2) \otimes \mu(x_3) - 1$ is the unique element of $(\mathcal{S}^+ \otimes_k \mathcal{S}^-)^{SL_2}$ determined by the conditions $f(e, e) = 1$, $k[f] = (\mathcal{S}^+ \otimes_k \mathcal{S}^-)^{SL_2}$.

We show that there is an analogue of (2) for every connected reductive algebraic group G. Namely, we endow $k[G]$ with the G-module structure determined by left translations and fix in G a pair of opposite Borel subgroups B^+ and B^-. Let $U^\pm$ be the unipotent radical of $B^\pm$. Consider the G-stable subalgebras

$$(3) \qquad \begin{aligned} \mathcal{S}^+ &:= \{f \in k[G] \mid f(gu) = f(g) \text{ for all } g \in G, u \in U^+\}, \\ \mathcal{S}^- &:= \{f \in k[G] \mid f(gu) = f(g) \text{ for all } g \in G, u \in U^-\} \end{aligned}$$

of $k[G]$ and the natural multiplication homomorphism of k-algebras

$$(4) \qquad \mu : \mathcal{S}^+ \otimes_k \mathcal{S}^- \to k[G], \quad f_1 \otimes f_2 \mapsto f_1 f_2.$$

For $k = \mathbb{C}$, the following were put forward in [Flath and Towber 1992]:

Conjectures (D. E. Flath and J. Towber [1992]).

(S) *The homomorphism μ is surjective.*

(K) *The ideal $\ker \mu$ in $\mathcal{S}^+ \otimes_k \mathcal{S}^-$ is generated by $(\ker \mu)^G$.*

If these conjectures are true, then the problem under consideration is reduced to the following:

(a) Find the canonically defined generators of the k-algebra $(\ker \mu)^G$.

(b) Find the canonically defined presentations of $\mathcal{S}^{\pm}$ by generators and relations.

In [Flath and Towber 1992], Conjectures (S) and (K) were proved for $k = \mathbb{C}$ and $G = \mathrm{SL}_n, \mathrm{GL}_n, \mathrm{SO}_n, \mathrm{Sp}_n$ by means of lengthy direct computations of some Laplace decompositions, minors, and algebraic identities between them. In Theorems 3 and 9 below, we prove Conjectures (S) and (K) in full generality, with no restrictions on k and G.

In Theorems 11 and 20 below, we describe $\ker \mu$ as a vector space over k. In Theorem 21, we solve the above part (a) of the problem, finding the canonically defined generators of the k-algebra $(\ker \mu)^G$. We call them SL_2-*type relations* of the sought-for canonical presentation of $k[G]$ because for $G = \mathrm{SL}_2$, the element $\mu(x_1) \otimes \mu(x_4) - \mu(x_2) \otimes \mu(x_3) - 1$ is just such a generator of $\mathcal{I}$ (see (2)). All of them are inhomogeneous of degree 2. If G is semisimple, they are indexed by the elements of the Hilbert basis $\mathcal{H}$ of the monoid of dominant weights of G. Note that the cardinality $|\mathcal{H}|$ of $\mathcal{H}$ is at least $\mathrm{rank}\, G$ with equality for simply connected G, but in the general case it may be much bigger. For instance, if $G = \mathrm{PGL}_r$, then $|\mathcal{H}| \geq p(r) + \varphi(r) - 1$, where p and φ are, respectively, the classical partition function and the Euler function (see [Popov 2011, Example 3.15]). Note that the problem of determining a full set of generators of the ideal $\ker \mu$ was formulated in [Flath 1994, Section 4] and, for $k = \mathbb{C}$, $G = \mathrm{SL}_n, \mathrm{GL}_n, \mathrm{SO}_n, \mathrm{Sp}_n$, solved in [Flath and Towber 1992] by lengthy direct computations.

For a semisimple group G whose monoid of dominant weights is freely generated (i.e., with $|\mathcal{H}| = \mathrm{rank}\, G$), a solution to the above part (b) of the problem in characteristic 0 was obtained (but not published) by B. Kostant; his proof appeared in [Lancaster and Towber 1979, Theorem 1.1]. In arbitrary characteristic, such a solution is given by Theorems 1, 2, 22 below, which are heavily based on the main results of [Ramanan and Ramanathan 1985] and [Kempf and Ramanathan 1987]. All relations in this case are homogeneous of degree 2. We call them *Plücker-type relations* of the sought-for canonical presentation of $k[G]$ because the k-algebra $\mathcal{S}^{\pm}$ for $G = \mathrm{SL}_n$ is the coordinate algebra of the affine multicone over the flag variety, and if $\mathrm{char}\, k = 0$, these relations are generated by the classical Plücker-type relations, obtained by Hodge [1942; 1943], that determine this multicone (see Section 6). The set of these relations is a union of finite-dimensional vector spaces canonically determined by the group structure of G; these spaces are indexed by the elements of $\mathcal{H} \times \mathcal{H}$ and different spaces have zero intersection (see Theorem 22). Thus in this case, we obtain a canonical presentation of $k[G]$, in which all relations are quadratic and divided into two families: homogeneous relations of Plücker type and

inhomogeneous relations of SL_2-type. As a parallel, we recall that any abelian variety is canonically presented as an intersection of quadrics in a projective space given by the Riemann equations; see [Kempf 1989] and [Lange and Birkenhake 1992].

For an arbitrary reductive group G, let $\tau \colon \widehat{G} \to G$ be the universal covering. Then $\widehat{G} = Z \times C$, where Z is a torus, C is a simply connected semisimple group, $G = \widetilde{G}/\ker \tau$, and $\ker \tau$ is a finite central subgroup. The algebra $\mathcal{S}^{\pm}$ for $\widehat{G}$ is then the tensor product of $k[Z]$ and the algebra $\mathcal{S}^{\pm}$ for C. Since the presentation of $k[Z]$ is clear, and that of $\mathcal{S}^{\pm}$ for C are given by Theorems 1, 2, and 22, the above part (b) of the problem is reduced to finding a presentation for the invariant algebra of the finite abelian group $\ker \tau$.

As an illustration, in Section 6 we consider the example of $G = SL_n$, $\operatorname{char} k = 0$, and describe explicitly how the ingredients of our construction and the canonical presentation of $k[G]$ look in this case.

The preprints [Popov 1995; 2000] of these results in characteristic 0 have been disseminated long ago. The validity of the results in arbitrary characteristic was announced in [Popov 2000]. The author is happy to finally present the complete proofs in the volume dedicated to the memory of Robert Steinberg who made a great contribution to the theory of algebraic groups.

Notation and conventions. Below we use freely the standard notation and conventions of [Borel 1991; Jantzen 1987; Popov and Vinberg 1994; Shafarevich 2013]. In particular, the algebra of functions regular on a variety X is denoted by $k[X]$, the field of rational functions on an irreducible X is denoted by $k(X)$, and the local ring of X at a point x is denoted by $\mathcal{O}_{x,X}$. For a morphism $\varphi \colon X \to Y$ of varieties, $\varphi^* \colon k[Y] \to k[X]$ denotes its comorphism.

All topological terms refer to the Zariski topology; the closure of Z in X is denoted by $\overline{Z}$ (each time it is clear from the context what is X).

The fixed point set of an action of a group P on a set S is denoted by S^P. Every action $\alpha \colon H \times X \to X$ of an algebraic group H on a variety X is always assumed to be regular (the latter means that α is a morphism). For every $h \in H$, $x \in X$, we write $g \cdot x$ in place of $\alpha(g, x)$. The H-orbit and the H-stabilizer of x are denoted respectively by $H \cdot x$ and H_x. Every homomorphism of algebraic groups is assumed to be algebraic.

The additively written group of characters (i.e., homomorphisms to the multiplicative group of k) of an algebraic group H is denoted by $\mathrm{X}(H)$. The value of a character $\lambda \in \mathrm{X}(H)$ at an element $h \in H$ is denoted by h^λ. Given a kH-module M, its weight space with weight $\lambda \in X(H)$ is denoted by M_λ.

We fix in G the maximal torus

$$T := B^+ \cap B^-$$

and identify $\mathrm{X}(B^{\pm})$ with $\mathrm{X}(T)$ by means of the restriction isomorphisms $\mathrm{X}(B^{\pm}) \to \mathrm{X}(T)$, $\lambda \mapsto \lambda|_T$.

By $X(T)_+$ we denote the monoid of dominant weights of T determined by B^+. Below the highest weight of every simple G-module is assumed to be the highest weight with respect to T and B^+.

We denote by w_0 the longest element of the Weyl group of T and fix in the normalizer of T a representative $\dot{w}_0$ of w_0. We then have $\dot{w}_0 B^\pm \dot{w}_0^{-1} = B^\mp$ and $\dot{w}_0 U^\pm \dot{w}_0^{-1} = U^\mp$. For every $\lambda \in X(T)_+$, we put $\lambda^* := -w_0(\lambda) \in X(T)_+$.

The set of all nonnegative rational numbers is denoted by $\mathbb{Q}_{\geqslant 0}$ and we put $\mathbb{N} := \mathbb{Z} \cap \mathbb{Q}_{\geqslant 0}$.

If $m \in \mathbb{Z}$, $m > 0$, we put $[m] := \{a \in \mathbb{Z} \mid 1 \leqslant a \leqslant m\}$.

For $d \in \mathbb{N}$, we denote by $[m]_d$ the set of all increasing sequences of d elements of $[m]$ (if $d \notin [m]$, then $[m]_d = \varnothing$).

2. Proof of Conjecture (S)

For every $\lambda \in X(T)$, the spaces

(5)
$$\mathcal{S}^+(\lambda) := \{f \in \mathcal{S}^+ \mid f(gt) = t^\lambda f(g) \text{ for all } g \in G, t \in T\},$$
$$\mathcal{S}^-(\lambda) := \{f \in \mathcal{S}^- \mid f(gt) = t^{w_0(\lambda)} f(g) \text{ for all } g \in G, t \in T\}$$

are the finite-dimensional (see, e.g., [Jantzen 1987, I.5.12.c)]) G-submodules of the G-modules $\mathcal{S}^+$ and $\mathcal{S}^-$ respectively. Since $\mathcal{S}^-(\lambda)$ is the right translation of $\mathcal{S}^+(\lambda)$ by $\dot{w}_0$, these G-submodules are isomorphic. In the notation of [Jantzen 1987, II.2.2], we have

(6)
$$\mathcal{S}^-(\lambda) = H^0(\lambda^*),$$

so by (6) and [Jantzen 1987, II.2.6, 2.2, 2.3], the following properties hold:

(7)
$$\left.\begin{array}{l} \text{(i) } \mathcal{S}^\pm(\lambda) \neq 0 \Longleftrightarrow \lambda \in X(T)_+. \\ \text{(ii) } \mathrm{soc}_G\, \mathcal{S}^\pm(\lambda) \text{ is a simple } G\text{-module with the highest weight } \lambda^*. \end{array}\right\}$$

If $\mathrm{char}\, k = 0$, then the G-module $\mathcal{S}^+(\lambda)$ is semisimple and hence $\mathcal{S}^+(\lambda) = \mathrm{soc}_G\, \mathcal{S}^+(\lambda)$ by (7)(ii). If $\mathrm{char}\, k > 0$, then, in general, this equality does not hold. From (3), (5), and (7)(i) we infer that

(8)
$$\mathcal{S}^+ = \bigoplus_{\lambda \in X(T)_+} \mathcal{S}^+(\lambda), \quad \mathcal{S}^+(\lambda)\mathcal{S}^+(\mu) \subseteq \mathcal{S}^+(\lambda + \mu),$$
$$\mathcal{S}^- = \bigoplus_{\lambda \in X(T)_+} \mathcal{S}^-(\lambda), \quad \mathcal{S}^-(\lambda)\mathcal{S}^-(\mu) \subseteq \mathcal{S}^-(\lambda + \mu);$$

i.e., the decompositions (8) are the $X(T)_+$-gradings of the algebras $\mathcal{S}^+$ and $\mathcal{S}^-$. They are obtained from each other by the right translation by $\dot{w}_0$.

Theorem 1. *The linear span of $\mathcal{S}^\pm(\lambda)\mathcal{S}^\pm(\mu)$ over k is $\mathcal{S}^\pm(\lambda + \mu)$.*

Proof. This statement is the main result of [Ramanan and Ramanathan 1985]. Note that the difficulty lies in the case of positive characteristic: since $\mathcal{S}^{\pm}$ is an integral domain, if char $k = 0$, then the claim immediately follows from (7)(i) and the inclusions in (8) because then $\mathcal{S}^{\pm}(\lambda + \mu)$ is a simple G-module. $\square$

Theorem 2.   (i) *If $\mathcal{G}$ is a generating set of the semigroup $\mathrm{X}(T)_+$, then the k-algebra $\mathcal{S}^{\pm}$ is generated by the subspace $\bigoplus_{\lambda \in \mathcal{G}} \mathcal{S}^{\pm}(\lambda)$.*

(ii) *The k-algebras S^+ and S^- are finitely generated.*

Proof. Part (i) follows from (8) and Theorem 1. Being the intersection of the lattice $\mathrm{X}(T)$ with a convex cone in $\mathrm{X}(T) \otimes_{\mathbb{Z}} \mathbb{Q}$ generated by finitely many vectors, the semigroup $\mathrm{X}(T)_+$ is finitely generated. This, (i), and the inequality $\dim_k \mathcal{S}^{\pm}(\lambda) < \infty$ imply (ii). $\square$

Now we are ready to turn to the proof of Conjecture (S).

Theorem 3. *The homomorphism μ is surjective.*

Our proof of Theorem 3 is based on two general results. The first is the following well-known surjectivity criterion:

Lemma 4. *The following properties of a morphism $\varphi \colon X \to Y$ of affine algebraic varieties are equivalent*:

 (a) *φ is a closed embedding.*

 (b) *$\varphi^* \colon k[Y] \to k[X]$ is surjective.*

Proof. See, e.g., [Steinberg 1974, Section 1.5]. $\square$

The second is the closedness criterion for orbits of connected solvable affine algebraic groups that generalizes Rosenlicht's classical theorem [1961, Theorem 2] on the closedness of orbits of unipotent groups.

Theorem 5. *Let a connected solvable affine algebraic group S act on an affine algebraic variety Z. Let x be a point of Z. Consider the orbit morphism $\tau : S \to Z$, $s \mapsto s \cdot z$. Then the following properties are equivalent*:

 (a) *The orbit $S \cdot z$ is closed in Z.*

 (b) *The semigroup $\{\lambda \in \mathrm{X}(S) \mid$ the function $S \to k$, $s \mapsto s^{\lambda}$, lies in $\tau^*(k[Z])\}$ is a group.*

Proof. This is proved in [Popov 1989, Theorem 4] $\square$

Remark 6. Since $\mathrm{X}(S)$ in Theorem 5 is a finitely generated free abelian group, it can be naturally regarded as a lattice in $\mathrm{X}(S) \otimes_{\mathbb{Z}} \mathbb{Q}$. Hence the following general criterion is applicable for verifying condition (b).

Let M be a nonempty subset of a finite-dimensional vector space V over $\mathbb{Q}$. Let $\mathbb{Q}_{\geqslant 0} M$, conv M, and $\mathbb{Q}M$ be, respectively, the convex cone generated by M, the

convex hull of M, and the linear span of M in V. Then the following properties are equivalent (see [Popov 1989, p. 386]):

(i) 0 is an interior point of conv M.

(ii) $\mathbb{Q}_{\geqslant 0} M = \mathbb{Q} M$.

If M is a subsemigroup of V, then (i) and (ii) are equivalent to

(iii) M is a group.

Proof of Theorem 3. 1. We consider the action of G on its underlying algebraic variety by left translations. By Theorem 2, there is an irreducible affine algebraic variety X endowed with an action of G and a G-equivariant dominant morphism

(9) $\alpha: G \to X$ such that α^* is an isomorphism $k[X] \xrightarrow{\cong} \mathcal{S}^+$.

Let $x := \alpha(e)$. Since α is G-equivariant, we have

(10) $\alpha(g) = g \cdot x$ for every $g \in G$,

and since α is dominant, the orbit $G \cdot x$ is open and dense in X. Consider the canonical projection $\pi: G \to G/U^+$. It is the geometric quotient for the action of U^+ on G by right translations. Therefore, (3) yields the isomorphism

(11) $\pi^*: k[G/U^+] \xrightarrow{\cong} \mathcal{S}^+$,

and, since α is constant on the fibers π, there exists a G-equivariant morphism $\iota: G/U^+ \to X$ such that

(12) $\alpha = \iota \circ \pi$.

From (12) we infer that the image of ι is $G \cdot x$. Since the group U^+ is unipotent, the algebraic variety G/U^+ is quasiaffine (see [Rosenlicht 1961, Theorem 3]). Therefore, $k(G/U^+)$ is the field of fractions of $k[G/U^+]$. On the other hand, $k(X)$ is the field of fractions of $k[X]$ inasmuch as X is affine. Using that (12) and isomorphisms (9), (11) yield the isomorphism $\iota^*: k[X] \xrightarrow{\cong} k[G/U^+]$, we conclude that ι is a birational isomorphism. Therefore, for a point z in general position in $G \cdot x$, the fiber $\iota^{-1}(z)$ is a single point. Being G-equivariant, ι is then injective. Finally, since G is smooth, $k[G]$ is integrally closed; therefore, $\mathcal{S}^{\pm}$ is integrally closed as well in view of (3) (see, e.g., [Popov and Vinberg 1994, Theorem 3.16]). Thus X is normal, and hence by Zariski's Main Theorem, $\iota: G/U^+ \to G \cdot x$ is an isomorphism. Using that π is separable (see, e.g., [Borel 1991, II.6.5]), from this we infer that the following properties hold:

(i$_1$) $G_x = U^+$.

(ii$_1$) $G \to G \cdot x$, $g \mapsto \alpha(g) = g \cdot x$, is a separable morphism.

2. Let $y := \dot{w}_0 \cdot x$. Consider the G-equivariant morphism

$$(13) \qquad\qquad \beta \colon G \to X, \quad g \mapsto g \cdot y.$$

From (10), (13), (i_1), and (ii_1), we infer that the following properties hold:

(i_2) $G_y = U^-$.

(ii_2) $G \to G \cdot y$, $g \mapsto \beta(g) = g \cdot y$, is a separable morphism.

(iii_2) β^* is an isomorphism $k[X] \xrightarrow{\cong} \mathcal{S}^-$.

3. Now consider the G-equivariant morphism

$$(14) \qquad \gamma := \alpha \times \beta \colon G \to X \times X, \quad g \mapsto g \cdot z, \text{ where } z := (x, y).$$

From (14), (i_1), and (i_2), we obtain

$$(15) \qquad\qquad G_z = G_x \cap G_y = U^+ \cap U^- = \{e\},$$

and hence γ is injective. We claim that γ is a closed embedding, i.e.,

(a) $G \to G \cdot z$, $g \mapsto g \cdot z$, is an isomorphism;

(b) $G \cdot z$ is closed in $X \times X$.

If this claim is proved, then the proof of Theorem 3 is completed as follows. Consider the isomorphism

$$(16) \qquad\qquad k[X] \otimes_k k[X] \to k[X \times X], \quad f \otimes h \mapsto fh.$$

Then (4), (9), (iii_2), (14), (16) imply that μ is the composition of the homomorphisms

$$(17) \qquad \mathcal{S}^+ \otimes_k \mathcal{S}^- \xrightarrow[\cong]{(\alpha^*)^{-1} \otimes (\beta^*)^{-1}} k[X] \otimes_k k[X] \xrightarrow[\cong]{(16)} k[X \times X] \xrightarrow{\gamma^*} k[G].$$

Hence the surjectivity of μ is equivalent to the surjectivity of γ^*. By Lemma 4, the latter is equivalent to the property that γ is a closed embedding, i.e., that properties (a) and (b) hold.

Thus the proof of Theorem 3 is reduced to proving properties (a) and (b).

4. First, we shall prove property (a). Since γ is injective, this is reduced to proving the separability of γ. In turn, in view of (14), the latter is reduced to proving that $\ker d_e \gamma$ is contained in Lie G_z, i.e., that $\ker d_e \gamma = \{0\}$ because of (15) (see [Borel 1991, II.6.7]). Using [loc. cit.], from (10), (13), (i_1), (ii_1), (i_2), (ii_2) we infer that $\ker d_e \alpha \subseteq \text{Lie } U^+$, $\ker d_e \beta \subseteq \text{Lie } U^-$. In view of (14), we then have $\ker d_e \gamma = \ker d_e \alpha \cap \ker d_e \beta \subseteq \text{Lie } U^+ \cap \text{Lie } U^- = \{0\}$. This proves property (a).

5. Now we shall prove property (b). Actually, we shall prove the stronger property that the orbit $B^+ \cdot z$ is closed in $X \times X$: since the algebraic variety G/B^+ is complete, this stronger property implies property (b) (see [Steinberg 1974, Section 2.13, Lemma 2]). Using that B^+ is connected solvable, to this end we shall apply Theorem 5.

Namely, consider the morphism $\tau \colon B^+ \to X \times X$, $b \mapsto b \cdot z$ and the following subsemigroup M in $X(B^+)$:

$$M := \{\lambda \in X(B^+) \mid \text{ the function } B^+ \to k, b \mapsto b^\lambda \text{ lies in } \tau^*(k[X \times X])\}.$$

We identify $X(B^+)$ with the lattice in $L := X(B^+) \otimes_{\mathbb{Z}} \mathbb{Q}$. In view of Theorem 5 and Remark 6, the orbit $B^+ \cdot z$ is closed if and only if

$$\tag{18} \mathbb{Q}_{\geqslant 0} M = \mathbb{Q} M.$$

Given this, the problem is reduced to proving that property (18) holds. This is done below.

6. Since $\tau = \gamma|_{B^+}$, the algebra $\tau^*(k[X \times X])$ is the image of the homomorphism $\gamma^*(k[X \times X]) \to k[B^+]$, $f \mapsto f|_{B^+}$. From (17) we see that $\gamma^*(k[X \times X])$ contains $\mathcal{S}^+$ and $\mathcal{S}^-$. Hence the restrictions of $\mathcal{S}^+$ and $\mathcal{S}^-$ to B^+ lie in $\tau^*(k[X \times X])$. We shall exhibit some characters of B^+ lying in these restrictions.

First consider the restriction of $\mathcal{S}^+(\lambda)$ to B^+ for $\lambda \in X(T)_+$. Note that $\mathcal{S}^+(\lambda)$ contains a function f such that $f(e) \neq 0$. Indeed, in view of (7)(i) and Borel's fixed point theorem, $\mathcal{S}^+(\lambda)$ contains a B^--stable line ℓ. The group B^- acts on ℓ by means of a character $\nu \in X(B^-)$. Take a nonzero function $f \in \ell$. For every $b \in B^-, u \in U^+$, we then have $f(b^{-1}u) = b^\nu f(u) \overset{(3)}{=} b^\nu f(e)$, whence $f(e) \neq 0$ because $B^- U^+$ is dense in G. This proves the existence of f. Multiplying f by $1/f(e)$, we may assume that $f(e) = 1$. Then for every $b \in B^+$, we deduce from (3), (5) that $f(b) = b^\lambda f(e) = b^\lambda$, i.e., $f|_{B^+}$ is the character $B^+ \to k, b \mapsto b^\lambda$. This proves the inclusion

$$\tag{19} X(B^+)_+ \subseteq M.$$

Now consider the restriction of $\mathcal{S}^-(\lambda)$ to B^+ for $\lambda \in X(T)_+$. In view of (7)(ii), there is a B^+-stable line ℓ in $\mathcal{S}^-(\lambda)$, on which B^+ acts by the character $\lambda^* \in X(B^+)$. Take a nonzero function $f \in \ell$. We may assume that $f(e) = 1$: this is proved as above with $\nu = \lambda$, replacing B^- by B^+, and U^+ by U^-. For every $b \in B^+$, we then have $f(b^{-1}) = b^{\lambda^*}$, i.e., $f|_{B^+}$ is the character $B^+ \to k, b \mapsto b^{-\lambda^*} = b^{w_0(\lambda)}$. This proves the inclusion

$$\tag{20} -X(B^+)_+ \subseteq M.$$

Since $\mathbb{Q}_{\geqslant 0}(X(B^+)_+) - \mathbb{Q}_{\geqslant 0}(X(B^+)_+) = L$, the inclusions (19), (20) imply the equality $\mathbb{Q}_{\geqslant 0} M = L$, whence a fortiori the equality (18) holds. This completes the proof of Theorem 3. $\qquad \square$

3. Proof of Conjecture (K)

We now intend to describe the ideal $\ker \mu$ in $\mathcal{S}^+ \otimes_k \mathcal{S}^-$. This is done in Sections 3 and 4 in several steps: first in Theorem 9 we prove that $\ker \mu$ is generated by

$(\ker \mu)^G$, then in Theorem 11 we describe $\ker \mu$ as a vector space, and finally in Theorem 21 we find a standard finite generating set of $\ker \mu$.

The first step is based on the following general statement:

Theorem 7. *Let Z be an affine algebraic variety endowed with an action of a reductive algebraic group H. Let $a \in Z$ be a point such that the orbit morphism*

$$\varphi \colon H \to Z, \quad h \mapsto h \cdot a,$$

is a closed embedding. Then the ideal $\ker \varphi^*$ in $k[Z]$ is generated by $(\ker \varphi^*)^H$.

For the proof of Theorem 7, we need the following:

Lemma 8. *Let $\psi \colon Y \to Z$ be a morphism of irreducible affine algebraic varieties and let $z \in \psi(Y)$ be a smooth point of Z. Assume that for each point $y \in \psi^{-1}(z)$, the following hold*:

(i) *y is a smooth point of Y.*

(ii) *The differential $d_y \psi$ is surjective.*

Then the ideal $\{f \in k[Y] \mid f|_{\psi^{-1}(z)} = 0\}$ of $k[Y]$ is generated by $\psi^*(\mathfrak{m})$, where $\mathfrak{m} := \{h \in k[Z] \mid h(z) = 0\}$.

Proof. Given a nonzero function $f \in k[Y]$, below we denote by Y_f the principal open subset $\{y \in Y \mid f(y) \neq 0\}$ of Y; it is affine and $k[Y_f] = k[Y]_f$.

1. Let $s_1, \ldots, s_d$ be a system of generators of the ideal $\mathfrak{m}$ of $k[Z]$. Put $t_i := \psi^*(s_i)$. Then we have

$$(21) \qquad \{y \in Y \mid t_1(y) = \cdots = t_d(y) = 0\} = \psi^{-1}(z).$$

We claim that, for every point $a \in Y$, there is a function $h_a \in k[Y]$ such that the principal open subset $U = Y_{h_a}$ is a neighborhood of a and

$$I_U := \{f \in k[U] \mid f|_{\psi^{-1}(z) \cap U} = 0\}$$

is the ideal of $k[U]$ generated by $t_1|_U, \ldots, t_d|_U$.

Proving this, we consider two cases.

First, consider the case where $a \notin \psi^{-1}(z)$. Then any principal open neighborhood of a not intersecting $\psi^{-1}(z)$ may be taken as U because in this case $I_U = k[U]$ and, in view of (21) and Hilbert's Nullstellensatz, $k[U] = k[U]t_1|_U + \cdots + k[U]t_d|_U$.

Second, consider the case where $a \in \psi^{-1}(z)$. Let $n = \dim Y$, $m = \dim Z$. Since a and z are the smooth points, the assumption (ii) yields the equality

$$(22) \qquad\qquad\qquad \dim \ker d_a \psi = n - m.$$

The functions $s_1, \ldots, s_d$ generate the maximal ideal of $\mathcal{O}_{z,Z}$. Therefore, renumbering them if necessary, we may (and shall) assume that $s_1, \ldots, s_m$ is a system of local parameters of Z at z, i.e., $\bigcap_{i=1}^m \ker d_z s_i = \{0\}$. Since $d_a t_i = d_a \psi \circ d_z s_i$,

we then infer from (ii) that $\bigcap_{i=1}^{m} \ker d_a t_i = \ker d_a \psi$. In view of (22), the latter equality implies the existence of functions $f_1, \ldots, f_{n-m} \in \mathcal{O}_{a,Y}$ such that $t_1, \ldots, t_m, f_1, \ldots, f_{n-m}$ is a system of local parameters of Y at a. Let

$$(23) \qquad F := \{ y \in Y \mid t_1(y) = \cdots = t_m(y) = 0 \}.$$

By [Shafarevich 2013, Chapter II, Section 3.2, Theorem 2.13], there is a principal open neighborhood U of a such that $F \cap U$ is an irreducible smooth $(n-m)$-dimensional closed subvariety of U whose ideal in $k[U]$ is generated by $t_1|_U, \ldots, t_m|_U$. On the other hand, (21) and (23) yield $\psi^{-1}(z) \subseteq F$ and, by the fiber dimension theorem, every irreducible component of $\psi^{-1}(z)$ has dimension $\geqslant n - m$. Hence $U \cap F = \psi^{-1}(z) \cap U$. This and (21) prove the claim.

2. Using this claim, the proof of Lemma 8 is completed as follows. Since $Y = \bigcup_{a \in Y} Y_{h_a}$ and Y is quasicompact, there exists a finite set of points $a_1, \ldots, a_r \in Y$ such that

$$(24) \qquad Y = \bigcup_{i=1}^{r} Y_{h_i}, \quad \text{where } h_i := h_{a_i}.$$

Now, let $f \in k[Y]$ be a function such that $f|_{\psi^{-1}(z)} = 0$. Then, in view of the definition of h_a, for every $i = 1, \ldots, r$, we have

$$(25) \qquad f h_i^{b_i} = c_{i,1} t_1 + \cdots + c_{i,d} t_d \quad \text{for some } c_{i,j} \in k[Y] \text{ and } b_i \in \mathbb{N}.$$

From (24) and Hilbert's Nullstellensatz, we infer that there are functions $q_1, \ldots, q_r \in k[Y]$ such that

$$(26) \qquad 1 = q_1 h_1^{b_1} + \cdots + q_s h_s^{b_r}.$$

From (25) and (26), we then deduce that

$$f = \left(\sum_{i=1}^{r} q_i c_{i,1} \right) t_1 + \cdots + \left(\sum_{i=1}^{r} q_i c_{i,d} \right) t_d \in k[Y] t_1 + \cdots + k[Y] t_d. \qquad \square$$

Proof of Theorem 7. There is a closed equivariant embedding of Z in an affine space on which H operates linearly (see [Rosenlicht 1961, Lemma 2] and [Popov and Vinberg 1994, Theorem 1.5]). Hence we may (and shall) assume that Z is an irreducible smooth affine algebraic variety.

Since G is reductive, $k[Z]^G$ is a finitely generated k-algebra (see, e.g., [Mumford and Fogarty 1982, Theorem A.1.0] and the references therein). Denote by $Z /\!/ H$ the affine algebraic variety $\operatorname{Specm}(k[Z]^G)$ and by $\pi \colon Z \to Z /\!/ H$ the morphism corresponding to the inclusion homomorphism $k[Z]^G \hookrightarrow k[Z]$.

The condition on the point a implies that its H-stabilizer is trivial,

$$(27) \qquad H_a = \{ e \}.$$

Hence $H \cdot a$ is a closed H-orbit of maximal dimension. Taking into account that in every fiber of π there is a unique closed orbit lying in the closure of every orbit contained in this fiber (see [Mumford and Fogarty 1982, Corollaries 1.2, A.1.0]), from this we deduce the equality

$$(28) \qquad\qquad \pi^{-1}(\pi(a)) = H \cdot a.$$

Since the group $\{e\}$ is linearly reductive, from (27) and the separability of φ, we infer by [Bardsley and Richardson 1985, Proposition 7.6] that there is a smooth affine subvariety S of the H-variety Z, which is an étale slice at $a \in S$. In view of (27), this means the following:

(i) The morphisms

$$\pi|_S \colon S \to Z/\!\!/H \qquad \text{and} \qquad \psi \colon H \times S \to Z, \quad (h, s) \mapsto h \cdot s$$

are étale.

(ii) The diagram

$$
\begin{array}{ccc}
H \times S & \xrightarrow{\ \psi\ } & Z \\
{\scriptstyle \mathrm{pr}_2} \downarrow & & \downarrow {\scriptstyle \pi} \\
S & \xrightarrow{\ \pi|_S\ } & Z/\!\!/H
\end{array}
$$

is a Cartesian square; i.e., it is commutative and the morphism

$$H \times S \to S \times_{Z/\!\!/H} Z$$

determined by ψ and pr_2 is an isomorphism.

From (i) and (ii), we deduce that $\pi(a)$ is a smooth point of $Z/\!\!/H$ and the differentials $d_{(e,a)}\psi$, $d_a(\pi|_S)$ are isomorphisms. Since $d_{(e,a)}\mathrm{pr}_2$ is clearly surjective, (ii) then implies that $d_a\pi$ is surjective, too.

Now, in view of (28) and transitivity of the action of H on $H \cdot a$, we conclude that $d_z\pi$ is surjective for every point $z \in \pi^{-1}(\pi(a))$. In view of Lemma 8, this implies the claim of Theorem 7. $\qquad\square$

Theorem 9. *The ideal* $\ker\mu$ *in* $\mathcal{S}^+ \otimes_k \mathcal{S}^-$ *is generated by* $(\ker\mu)^G$.

Proof. In the proof of Theorem 3, we have shown that

— the homomorphism μ is the composition of the homomorphisms (17);

— the morphism γ is a closed embedding.

In view of these facts, Theorem 9 is equivalent to the claim that the ideal $\ker\gamma^*$ in $k[X \times X]$ is generated by $(\ker\gamma^*)^G$. This claim follows from Theorem 7. $\qquad\square$

4. Structure of $(\ker \mu)^G$

We shall use the following lemma for describing $(\ker \mu)^G$ as a vector space.

Lemma 10.

$$(29) \quad \dim(\mathcal{S}^+(\lambda) \otimes_k \mathcal{S}^-(\nu))^G = \begin{cases} 1 & \text{if } \nu = \lambda^*, \\ 0 & \text{if } \nu \neq \lambda^* \end{cases} \quad \text{for every } \lambda, \nu \in \mathrm{X}(T)_+,$$

$$(30) \qquad (\mathcal{S}^+ \otimes_k \mathcal{S}^-)^G = \bigoplus_{\lambda \in \mathrm{X}(T)_+} (\mathcal{S}^+(\lambda) \otimes_k \mathcal{S}^-(\lambda^*))^G.$$

Proof. In view of (8), the equality (30) follows from (29). To prove (29), we note that

$$(\mathcal{S}^+(\lambda) \otimes_k \mathcal{S}^-(\nu))^G \cong \mathrm{Hom}_G(\mathcal{S}^+(\lambda)^*, \mathcal{S}^-(\nu))$$

and, in view of (6), the G-module $\mathcal{S}^+(\lambda)^*$ is the universal highest weight module of weight λ (the Weyl module); in particular, for each G-module M, there is an isomorphism

$$(31) \qquad\qquad \mathrm{Hom}_G(\mathcal{S}^+(\lambda)^*, M) \xrightarrow{\;\cong\;} (M^{U^+})_\lambda,$$

where the right-hand side of (31) is the weight space of T (see [Jantzen 1987, II.2.13, Lemma]). Since $\mathcal{S}^-(\nu)^{U^+}$ is a line on which B^+ acts by means of ν^* (see [Jantzen 1987, II.2.2, Proposition]), this proves (29). □

We identify $k[G] \otimes_k k[G]$ with $k[G \times G]$ by the isomorphism

$$(32) \qquad k[G] \otimes_k k[G] \to k[G \times G], \quad f_1 \otimes f_2 \mapsto ((a, b) \mapsto f_1(a) f_2(b)).$$

Thus $\mathcal{S}^+ \otimes_k \mathcal{S}^-$ is regarded as a subalgebra of $k[G \times G]$, and (4), (32) yield the equality

$$(33) \qquad f(a, a) = \mu(f)(a) \text{ for every } f \in \mathcal{S}^+ \otimes_k \mathcal{S}^- \text{ and } a \in G.$$

Theorem 11. (i) *If $f \in (\mathcal{S}^+ \otimes_k \mathcal{S}^-)^G$, then $f - f(e, e) \in (\ker \mu)^G$.*

(ii) *Every $h \in (\ker \mu)^G$ can be uniquely written in the form*

$$(34) \qquad h = \sum (h_\lambda - h_\lambda(e, e)), \quad h_\lambda \in (\mathcal{S}^+(\lambda) \otimes_k \mathcal{S}^-(\lambda^*))^G,$$

where the sum is taken over a finite set of nonzero elements $\lambda \in \mathrm{X}(T)_+$.

Proof. (i) Since μ is G-equivariant, its restriction to $(\mathcal{S}^+ \otimes_k \mathcal{S}^-)^G$ is a homomorphism to $k[G]^G = k$. Hence $\mu(f)$ is a constant. In view of (33), this implies (i).

(ii) If (34) holds, then the decomposition (30) implies that h_λ is the natural projection of h to $(\mathcal{S}^+(\lambda) \otimes_k \mathcal{S}^-(\lambda^*))^G$ determined by this decomposition, whence the uniqueness of (34). To prove the existence, let h_λ be the aforementioned projection of h to $(\mathcal{S}^+(\lambda) \otimes_k \mathcal{S}^-(\lambda^*))^G$. Then $h = \sum_{\lambda \in F} h_\lambda$ for a finite set $F \subset \mathrm{X}(T)_+$.

Hence $0 = \mu(h) = \sum_{\lambda \in F} \mu(h_\lambda)$. As above, $\mu(h_\lambda) = h_\lambda(e, e)$; this implies equality (34), where the sum is taken over all $\lambda \in F$. Since h_0 is a constant, we may assume that F does not contain 0. This proves (ii).   □

In the next lemma, for brevity, we put (cf. [Jantzen 1987])

$$
(35) \qquad
\begin{gathered}
V(\lambda) := \mathcal{S}^-(\lambda)^* \cong \mathcal{S}^+(\lambda)^*, \quad L(\lambda) := V(\lambda)/\operatorname{rad}_G V(\lambda), \\
\pi_\lambda \colon V(\lambda) \to L(\lambda) \text{ is the canonical projection.}
\end{gathered}
$$

The G-module $V(\lambda)$ (hence $L(\lambda)$ as well) is generated by a B^+-stable line of weight λ (see [Jantzen 1987, II.2.13, Lemma]), whence $V(\lambda)$ is also generated by a B^--stable line of weight $-\lambda^*$.

Also, for the G-modules P and Q, we denote by $\mathcal{B}(P \times Q)$ the G-module of all bilinear maps $P \times Q \to k$; we then have the isomorphism of G-modules

$$
(36) \qquad P^* \otimes_k Q^* \overset{\cong}{\longrightarrow} \mathcal{B}(P \times Q), \quad f \otimes h \mapsto fh.
$$

Lemma 12. *For all elements $\lambda, \nu \in X(T)_+$, the following hold*:

(a) $\dim \mathcal{B}(V(\lambda) \times V(\nu))^G = \begin{cases} 1 & \text{if } \nu = \lambda^*, \\ 0 & \text{if } \nu \neq \lambda^*. \end{cases}$

(b) $\dim \mathcal{B}(L(\lambda) \times L(\nu))^G = \begin{cases} 1 & \text{if } \nu = \lambda^*, \\ 0 & \text{if } \nu \neq \lambda^*. \end{cases}$

(c) *Every nonzero element $\theta \in \mathcal{B}(L(\lambda) \times L(\lambda^*))^G$ is a nondegenerate pairing* $L(\lambda) \times L(\lambda^*) \to k$.

(d) *If $l^+ \in L(\lambda)$, $l^- \in L(\lambda^*)$ are the nonzero semi-invariants of, respectively, B^+ and B^-, then $\theta(l^+, l^-) \neq 0$ for θ from* (c). *For every nonzero element $\epsilon \in k$, there exists a unique θ such that $\theta(l^+, l^-) = \epsilon$.*

(e) *Every element $\vartheta \in \mathcal{B}(V(\lambda) \times V(\lambda^*))^G$ vanishes on $\ker \pi_\lambda \times \ker \pi_{\lambda^*}$. If $\vartheta \neq 0$, then ϑ is a nondegenerate pairing $V(\lambda) \times V(\lambda^*) \to k$.*

(f) *Let $v^+ \in V(\lambda)$ and $v^- \in V(\lambda^*)$ be, respectively, the nonzero B^+- and B^--semi-invariants of weights λ and $-\lambda$ that generate the G-modules $V(\lambda)$ and $V(\lambda^*)$. Then $\vartheta(v^+, v^-) \neq 0$ for every nonzero element $\vartheta \in \mathcal{B}(V(\lambda) \times V(\lambda^*))^G$.*

Proof. Part (a) follows from (29), (36), (35). Part (b) is proved similarly, using that $L(\lambda)$ is a simple G-module with highest weight λ (see [Jantzen 1987, II.2.4]). The simplicity of $L(\lambda)$ implies (c) because the left and right kernels of θ are G-stable.

Proving (d), take a basis $\{p_1, \dots, p_s\}$ of $L(\lambda)$ such that $p_1 = l^+$ and every p_i is a weight vector of T. Let $\{p_1^*, \dots, p_s^*\}$ be the basis of $L(\lambda^*)$ dual to $\{p_1, \dots, p_s\}$ with respect to θ. Let $L(\lambda)'$ be the linear span over k of all p_i with $i > 1$. Then $L(\lambda)'$ is B^--stable, and, for every element $u \in U^-$, we have $u \cdot p_1 = p_1 + p'$,

where $p' \in L(\lambda)'$ (see, e.g., [Steinberg 1974, Section 3.3, Proposition 2 and p. 84]). Then, for all elements $\alpha_1, \ldots, \alpha_s \in k$, we have

$$(u \cdot p_1^*)\left(\sum_{i=1}^{s} \alpha_i \, p_i\right) = p_1^*\left(\sum_{i=1}^{s} \alpha_i \,(u^{-1} \cdot p_i)\right)$$

$$= p_1^*(\alpha_1 \, p_1 + \text{an element of } L(\lambda)')$$

$$= \alpha_1 = p_1^*\left(\sum_{i=1}^{s} \alpha_i \, p_i\right),$$

whence $u \cdot p_1^* = p_1^*$. Therefore, $l^- = \lambda p_1^*$ for a nonzero $\lambda \in k$, and hence $\theta(l^+, l^-) = \lambda \neq 0$. This and (b) prove (d).

It follows from (35), (a), and (b) that the embedding

$$\mathcal{B}(L(\lambda) \times L(\lambda^*))^G \to \mathcal{B}(V(\lambda) \times V(\lambda^*))^G, \quad \theta \mapsto \theta \circ (\pi_\lambda \times \pi_{\lambda*}),$$

is an isomorphism. Part (e) follows from this and (c).

Part (f) follows from (d) and (e), because $\pi_\lambda(v^+)$ and $\pi_{\lambda*}(v^-)$ are, in view of (35), the nonzero semi-invariants of, respectively, B^+ and B^-. $\square$

Lemma 13. *Let an algebraic group H act on an algebraic variety Z and let V be a finite-dimensional submodule of the H-module $k[Z]$. Then the morphism*

(37) $$\varphi : Z \to V^*, \quad \varphi(a)(f) = f(a) \text{ for every } a \in Z, \, f \in V,$$

has the following properties:

  (i) *φ is H-equivariant.*

 (ii) *The restriction of φ^* to $(V^*)^*$ is an isomorphism $(V^*)^* \to V$.*

(iii) *φ^* exercises an isomorphism between $k[\overline{\varphi(Z)}]$ and the subalgebra of $k[Z]$ generated by V.*

Proof. Part (i) is proved by direct verification.

Every function $f \in V$ determines an element $l_f \in (V^*)^*$ by the formula $l_f(s) = s(f)$, $s \in V^*$. It is immediate that $V \to (V^*)^*$, $f \mapsto l_f$ is a vector space isomorphism and that (37) implies $\varphi^*(l_f) = f$. This proves (ii).

Let $\iota : (V^*)^* \to k[\overline{\varphi(Z)}]$ be the restriction homomorphism. The k-algebra $k[\overline{\varphi(Z)}]$ is generated by $\iota((V^*)^*)$. Part (iii) now follows from the fact that φ^* exercises an embedding of $k[\overline{\varphi(Z)}]$ in $k[Z]$ and, in view of (ii), the image of $\iota((V^*)^*)$ under this embedding is V. $\square$

Corollary 14. *In the notation of Lemma 13, let $V \neq \{0\}$ and let the orbit $H \cdot a$ be dense in Z. Then $\varphi(a) \neq 0$.*

We call the morphism (37) the *covariant determined by the submodule V*.

Lemma 15. *Let λ be an element of $X(T)_+$ and let*

$$\varphi^+\colon G \to \mathcal{S}^+(\lambda)^*, \quad \varphi^-\colon G \to \mathcal{S}^-(\lambda^*)^*$$

be the covariants determined by the submodules $\mathcal{S}^+(\lambda)$ and $\mathcal{S}^-(\lambda^*)$ of the G-module $k[G]$. Then $v^+ := \varphi^+(e)$ and $v^- := \varphi^-(e)$ are, respectively, the nonzero B^+- and B^--semi-invariants of weights λ and $-\lambda$.

Proof. First, we have $v^+ \neq 0$, $v^- \neq 0$ by Corollary 14. Next, for every $f \in \mathcal{S}^+(\lambda)$, $b \in B^+$, we have

$$(b \cdot v^+)(f) = \varphi^+(e)(b^{-1} \cdot f) \overset{(37)}{=\!=} (b^{-1} \cdot f)(e)$$

$$= f(b) \overset{(5)}{=\!=} b^\lambda f(e) \overset{(37)}{=\!=} (b^\lambda v^+)(f),$$

whence $b \cdot v^+ = b^\lambda v^+$; i.e., v^+ is a nonzero B^+-semi-invariant of weight λ, as claimed. For v^- the proof is similar. $\square$

Theorem 16. *The restriction of μ to $(\mathcal{S}^+(\lambda) \otimes_k \mathcal{S}^-(\lambda))^G$ for every $\lambda \in X(T)_+$ is an isomorphism $(\mathcal{S}^+(\lambda) \otimes_k \mathcal{S}^-(\lambda^*))^G \overset{\cong}{\longrightarrow} k[G]^G = k$.*

Proof. In view of (33) and Lemma 10, the proof is reduced to showing that there is a function $f \in (\mathcal{S}^+(\lambda) \otimes_k \mathcal{S}^-(\lambda^*))^G$ such that $f(e, e) \neq 0$.

Consider the covariants φ^+ and φ^- from Lemma 15 and the G-equivariant morphism

$$\varphi := \varphi^+ \times \varphi^-\colon G \times G \to \mathcal{S}^+(\lambda)^* \times \mathcal{S}^-(\lambda^*)^*.$$

Lemma 12(a) and (35) imply that $\mathcal{B}(\mathcal{S}^+(\lambda)^* \times \mathcal{S}^-(\lambda^*)^*)^G$ contains a nonzero element ϑ. By Lemma 13, the function $f := \vartheta \circ \varphi\colon G \times G \to k$ is contained in $(\mathcal{S}^+(\lambda) \otimes_k \mathcal{S}^-(\lambda^*))^G$. For this f, using Lemmas 15 and 12(f), we obtain

$$(38) \qquad f(e, e) = \vartheta(\varphi(e, e)) = \vartheta(\varphi^+(e), \varphi^-(e)) \neq 0.$$

This completes the proof. $\square$

Corollary 17. *For every element $\lambda \in X(T)_+$, there exists a unique element*

$$(39) \qquad s_\lambda \in (\mathcal{S}^+(\lambda) \otimes_k \mathcal{S}^-(\lambda^*))^G \subseteq k[G \times G] \quad \text{such that } s_\lambda(e, e) = 1.$$

If $\{f_1, \ldots, f_d\}$ and $\{h_1, \ldots, h_d\}$ are the bases of $\mathcal{S}^+(\lambda)$ and $\mathcal{S}^-(\lambda^*)$ dual with respect to a nondegenerate G-invariant pairing $\mathcal{S}^+(\lambda) \times \mathcal{S}^-(\lambda^*) \to k$ (the latter exists by (36) and Lemma 12), then $\varepsilon := \sum_{i=1}^d f_i(e)h_i(e) \neq 0$ and

$$s_\lambda = \varepsilon^{-1}\left(\sum_{i=1}^d f_i \otimes h_i\right).$$

Proof. First, note that if P, Q are the finite-dimensional kG-modules, $\theta \in \mathcal{B}(P, Q)^G$ is a nondegenerate pairing $P \times Q \to k$, and $\{p_1, \ldots, p_m\}$ and $\{q_1, \ldots, q_m\}$ are the bases of P and Q dual with respect to θ, then $\sum_{i=1}^m p_i \otimes q_i$ is a nonzero element

of $(P \otimes_k Q)^G$ (not depending on the choice of these bases). Indeed, θ determines the isomorphism of G-modules

(40)
$$\phi \colon P \otimes_k Q \to \mathrm{Hom}(P, P),$$
$$(\phi(p \otimes q))(p') = \theta(p', q)p, \quad \text{where } p,\, p' \in P,\, q \in Q.$$

From (40), we then obtain

$$\left(\phi\left(\sum_{i=1}^m p_i \otimes q_i\right)\right)(p_j) = \sum_{i=1}^m \theta(p_j, q_i)p_i = \sum_{i=1}^m \delta_{ij}\, p_i = p_j;$$

therefore, $\phi\left(\sum_{i=1}^m p_i \otimes q_i\right) = \mathrm{id}_P$, whence the claim.

For $P = \mathcal{S}^+(\lambda)$, $Q = \mathcal{S}^-(\lambda^*)$, it yields that $\sum_{i=1}^d f_i \otimes h_i$ is a nonzero element of $(\mathcal{S}^+(\lambda) \otimes_k \mathcal{S}^-(\lambda^*))^G$. Theorem 16 and (33) then complete the proof. $\square$

Remark 18. For char $k = 0$, there is another characterization of s_λ. Namely, let $\mathcal{U}$ be the universal enveloping algebra of Lie G. Every $\mathcal{S}^\pm(\lambda)$ is endowed with the natural $\mathcal{U}$-module structure. Let $\{x_1, \dots, x_n\}$ and $\{x_1^*, \dots, x_n^*\}$ be the bases of Lie G dual with respect to the Killing form Φ. Identify Lie T with its dual space by means of Φ. Let σ be the sum of all positive roots. For every $\lambda \in X(T)_+$, put

(41)
$$c_\lambda := \Phi(\lambda + \sigma, \lambda) + \Phi(\lambda^* + \sigma, \lambda^*)$$

and consider on the space $\mathcal{S}^+(\lambda) \otimes_k \mathcal{S}^-(\lambda^*)$ the linear operator

(42)
$$\Delta := \sum_{i=1}^n (x_i \otimes x_i^* + x_i^* \otimes x_i).$$

Proposition 19. *The following properties of an element* $t \in \mathcal{S}^+(\lambda) \otimes_k \mathcal{S}^-(\lambda^*)$ *are equivalent*:

  (i) $t = s_\lambda$.

  (ii) $\Delta(t) = -c_\lambda t$ *and* $t(e, e) = 1$.

Proof. By [Bourbaki 1975, Chapitre VIII, §6.4, Corollaire], the Casimir element $\Omega := \sum_{i=1}^n x_i x_i^* \in \mathcal{U}$ acts on any simple $\mathcal{U}$-module with the highest weight γ as scalar multiplication by $\Phi(\gamma + \sigma, \gamma)$. Since $\Phi(\gamma + \sigma, \gamma) > 0$ if $\gamma \neq 0$, the kernel of Ω in any finite-dimensional $\mathcal{U}$-module V coincides with V^G. We apply this to $V = \mathcal{S}^+(\lambda) \otimes_k \mathcal{S}^-(\lambda^*)$. For any elements $f \in \mathcal{S}^+(\lambda)$, $h \in \mathcal{S}^-(\lambda^*)$, we deduce from (41), (42) the following:

$$\Omega(f \otimes h) = \sum_{i=1}^n \left(x_i x_i^*(f) \otimes h + x_i^*(f) \otimes x_i(h) + x_i(f) \otimes x_i^*(h) + f \otimes x_i x_i^*(h)\right)$$
$$= \Omega(f) \otimes h + f \otimes \Omega(h) + \Delta(f \otimes h) = c_\lambda(f \otimes h) + \Delta(f \otimes h).$$

Now Corollary 17 and the aforesaid about $\ker \Omega$ complete the proof. $\square$

Theorem 20. *Let $\lambda_1, \ldots, \lambda_m$ be a system of generators of the monoid $X(T)_+$. Then $(\ker \mu)^G$ is the linear span over k of all monomials of the form*

$$(s_{\lambda_1} - 1)^{d_1} \cdots (s_{\lambda_m} - 1)^{d_m}, \quad \text{where } d_i \in \mathbb{N}, d_1 + \cdots + d_m > 0,$$

where s_{λ_i} is defined in Corollary 17.

Proof. By Theorem 11(i), the linear span L referred to in Theorem 20 is contained in $(\ker \mu)^G$. In view of Theorem 11, to prove the converse inclusion $(\ker \mu)^G \subseteq L$, we have to show that, for every function

$$(43) \qquad\qquad f \in (\mathcal{S}^+(\lambda) \otimes_k \mathcal{S}^-(\lambda^*))^G,$$

we have $f - f(e, e) \in L$. Since $\lambda_1, \ldots, \lambda_m$ is a system of generators of $X(T)_+$, there are integers $d_1, \ldots, d_m \in \mathbb{N}$ such that $\lambda = \sum_{i=1}^m d_i \lambda_i$. From (39) and (8) we then infer that $h := \prod_{i=1}^m s_{\lambda_i}^{d_i} \in (\mathcal{S}^+(\lambda) \otimes_k \mathcal{S}^-(\lambda^*))^G$ and $h(e, e) = 1$. This, (43), and (29) imply that $f = f(e, e)h$. Therefore,

$$(44) \qquad f - f(e, e) = f(e, e)(h - 1) = f(e, e)\left(\prod_{i=1}^m \left((s_{\lambda_i} - 1) + 1 \right)^{d_i} - 1 \right).$$

The right-hand side of (44) clearly lies in L. This completes the proof. $\square$

Theorem 21. *Let $\lambda_1, \ldots, \lambda_m$ be a system of generators of the monoid $X(T)_+$. Then the ideal $\ker \mu$ in $\mathcal{S}^+ \otimes_k \mathcal{S}^-$ is generated by $s_{\lambda_1} - 1, \ldots, s_{\lambda_m} - 1$, where s_{λ_i} is defined in Corollary 17.*

Proof. This follows from Theorems 9 and 20. $\square$

5. Presentation of $\mathcal{S}^\pm$

If the group G is semisimple, then the semigroup $X(T)_+$ has no units other than 0. Hence the set $\mathcal{H}$ of all indecomposable elements of $X(T)_+$ is finite,

$$(45) \qquad\qquad \mathcal{H} = \{\lambda_1, \ldots, \lambda_d\}$$

generates $X(T)_+$, and every generating set of $X(T)_+$ contains $\mathcal{H}$ (see, e.g., [Lorenz 2005, Lemma 3.4.3]). Note that $\mathcal{H}$, called the *Hilbert basis of* $X(T)_+$, in general is not a free generating system of $X(T)_+$ (i.e., it is not true that every element $\alpha \in X(T)_+$ may be uniquely expressed in the form $\alpha = \sum_{i=1}^d c_i \lambda_i$, $c_i \in \mathbb{N}$). Namely, it is free if and only if $G = G_1 \times \cdots \times G_s$, where every G_i is either a simply connected simple algebraic group or isomorphic to SO_{n_i} for an odd n_i (see [Steinberg 1975, §3], [Richardson 1979, Proposition 4.1], [Richardson 1982, Proposition 13.3] and [Popov 2011, Remark 3.16]). In particular, if G is simply connected, then $\mathcal{H}$ coincides with the set of all fundamental weights and generates $X(T)_+$ freely. Note that $\lambda_i^* \in \mathcal{H}$ for every i.

To understand the presentation of $\mathcal{S}^{\pm}$, denote respectively by $\operatorname{Sym} \mathcal{S}^{\pm}(\lambda_i)$ and $\operatorname{Sym}^m \mathcal{S}^{\pm}(\lambda_i)$ the symmetric algebra and the m-th symmetric power of $\mathcal{S}^{\pm}(\lambda_i)$. The naturally $\mathbb{N}^d$-graded free commutative k-algebra

$$(46) \qquad \mathcal{F}^{\pm} := \operatorname{Sym} \mathcal{S}^{\pm}(\lambda_1) \otimes_k \cdots \otimes_k \operatorname{Sym} \mathcal{S}^{\pm}(\lambda_d)$$

may be viewed as the algebra of regular functions $k[L^{\pm}]$ on the vector space

$$L^{\pm} := \mathcal{S}^{\pm}(\lambda_1)^* \oplus \cdots \oplus \mathcal{S}^{\pm}(\lambda_d)^*.$$

Let e_i be the i-th unit vector of $\mathbb{N}^d$ and let $\mathcal{F}_{p,q}^{\pm}$ be the homogeneous component of $\mathcal{F}^{\pm}$ of degree $e_p + e_q$. We have the natural isomorphisms of G-modules

$$(47) \qquad \varphi_{p,q}^{\pm} \colon \mathcal{F}_{p,q}^{\pm} \xrightarrow{\cong} \mathcal{S}_{p,q}^{\pm} := \begin{cases} \mathcal{S}^{\pm}(\lambda_p) \otimes_k \mathcal{S}^{\pm}(\lambda_q) & \text{if } p \neq q, \\ \operatorname{Sym}^2 \mathcal{S}^{\pm}(\lambda_p) & \text{if } p = q. \end{cases}$$

By Theorems 1 and 2, the natural multiplication homomorphisms

$$(48) \qquad \phi^{\pm} \colon \mathcal{F}^{\pm} \to \mathcal{S}^{\pm} \quad \text{and} \quad \psi_{p,q}^{\pm} \colon \mathcal{S}_{p,q}^{\pm} \to \mathcal{S}^{\pm}(\lambda_p + \lambda_q)$$

are surjective. Since $\mathcal{F}^{\pm}$ is a polynomial algebra, the surjectivity of $\phi^{\pm}$ reduces finding a presentation of $\mathcal{S}^{\pm}$ by generators and relations to describing $\ker \phi^{\pm}$. If $d = \dim T$, the following explicit description of $\ker \phi^{\pm}$ is available:

Theorem 22. *Let G be a connected semisimple group such that the Hilbert basis* (45) *freely generates the semigroup* $\mathrm{X}(T)_+$. *Then*

(i) *the ideal* $\ker \phi^{\pm}$ *of the* $\mathbb{N}^d$-*graded k-algebra* $\mathcal{F}^{\pm}$ *is homogeneous*;

(ii) *this ideal is generated by the union of all its homogeneous components of the total degree* 2;

(iii) *the set of these homogeneous components coincides with the set of all subspaces* $(\varphi_{p,q}^{\pm})^{-1}(\ker \psi_{p,q}^{\pm})$, $1 \leqslant p \leqslant q \leqslant d$.

Proof. This is the main result of [Kempf and Ramanathan 1987]. $\qquad \square$

Remark 23. In characteristic 0, for the first time the proof of Theorem 22 was obtained (but not published) by B. Kostant; his proof appeared in [Lancaster and Towber 1979, Theorem 1.1]. In this case, (47) and the surjectivity of $\psi_{p,q}^{\pm}$ yield that $\psi_{p,q}^{\pm}$ is the projection of $\mathcal{S}_{p,q}^{\pm}$ to the Cartan component of $\mathcal{S}_{p,q}^{\pm}$, and $\ker \psi_{p,q}^{\pm}$ is the unique G-stable direct complement to this component. The subspace $\ker \psi_{p,q}^{\pm}$ admits the following description using the notation of Remark 18 [loc. cit.]. Let $\{x_1, \ldots, x_n\}$ and $\{x_1^*, \ldots, x_n^*\}$ be the dual bases of $\operatorname{Lie} G$ with respect to Φ. Then $\ker \psi_{p,q}^+$ is the image of the linear transformation $\left(\sum_{s=1}^n (x_s \otimes x_s^* + x_s^* \otimes x_s)\right) - 2\Phi(\lambda_p^*, \lambda_q^*)\mathrm{id}$ of the vector space $\mathcal{S}_{p,q}^{\pm}$.

Summing up, if G is a connected semisimple group such that the Hilbert basis (45) freely generates the semigroup $\mathrm{X}(T)_+$, then the sought-for canonical presentation of $k[G]$ is given by the surjective homomorphism

$$(49) \qquad \phi := \phi^+ \otimes \phi^- : \mathcal{F} := \mathcal{F}^+ \otimes_k \mathcal{F}^- \to k[G]$$

of the polynomial k-algebra $\mathcal{F}$ and the following generating system $\mathcal{R}$ of the ideal $\ker\phi$. Identify $\mathcal{F}^+$ and $\mathcal{F}^-$ with subalgebras of $\mathcal{F}$ in the natural way. Then $\mathcal{R} = \mathcal{R}_1 \sqcup \mathcal{R}_2$, where

$$(50) \qquad \mathcal{R}_1 = \bigcup_{p,q} \left((\varphi_{p,q}^+)^{-1}(\ker\psi_{p,q}^+) \cup (\varphi_{p,q}^-)^{-1}(\ker\psi_{p,q}^-) \right)$$

(see the definition of $\varphi_{p,q}^{\pm}$, $\psi_{p,q}^{\pm}$ in (47), (48)) and

$$(51) \qquad \mathcal{R}_2 = \{s_{\lambda_1} - 1, \ldots, s_{\lambda_d} - 1\}$$

(see the definition of s_{λ_i} in Corollary 17). The elements of $\mathcal{R}_1$ (respectively, $\mathcal{R}_2$) are the Plücker-type (respectively, the SL_2-type) relations of the presentation.

The canonical presentation of $k[G]$ is redundant. To reduce the size of $\mathcal{R}_1$, we may replace every space $\ker\psi_{p,q}^{\pm}$ in (50) by a basis of this space. Finding such a basis falls within the framework of Standard Monomial Theory.

6. An example

As an illustration, here we explicitly describe the canonical presentation of $k[G]$ for $G = \mathrm{SL}_n$, $n \geqslant 2$, and $\operatorname{char} k = 0$.

Let T be the maximal torus of diagonal matrices in G, and let B^+ (respectively, B^-) be the Borel subgroup of lower (respectively, upper) triangular matrices in G. Then

$$\mathcal{H} = \{\varpi_1, \ldots, \varpi_{n-1}\}, \quad \text{where}$$

$$\varpi_d : T \to k, \quad \operatorname{diag}(a_1, \ldots, a_n) \mapsto a_{n-d+1} \cdots a_n.$$

Every pair $i_1, i_2 \in [n]$ determines the function

$$(52) \qquad x_{i_1,i_2} : G \to k, \quad \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \cdots\cdots\cdots \\ a_{n,1} & \cdots & a_{n,n} \end{pmatrix} \mapsto a_{i_1,i_2}.$$

The k-algebra generated by all functions (52) is $k[G]$.

For every $d \in [n-1]$ and every sequence $i_1, \ldots, i_d$ of d elements of $[n]$, put

$$f_{i_1,\ldots,i_d}^- := \det \begin{pmatrix} x_{i_1,1} & \cdots & x_{i_1,d} \\ \cdots\cdots\cdots\cdots \\ x_{i_d,1} & \cdots & x_{i_d,d} \end{pmatrix}, \quad f_{i_1,\ldots,i_d}^+ := \det \begin{pmatrix} x_{i_1,n-d+1} & \cdots & x_{i_1,n} \\ \cdots\cdots\cdots\cdots\cdots \\ x_{i_d,n-d+1} & \cdots & x_{i_d,n} \end{pmatrix}.$$

For every fixed d, all functions $f^-_{i_1,\ldots,i_d}$ (respectively, $f^+_{i_1,\ldots,i_d}$) such that $i_1 < \cdots < i_d$ are linearly independent over k and their linear span over k is the simple G-module $\mathcal{S}^-(\varpi_d)$ (respectively, $\mathcal{S}^+(\varpi_d)$); see, e.g., [Flath and Towber 1992, Proposition 3.2]. Therefore, denoting by $x^\pm_{i_1,\ldots,i_d}$ the element $f^\pm_{i_1,\ldots,i_d}$ of the k-algebra $\mathcal{F}^\pm$ defined by (46), we identify $\mathcal{F}^\pm$ with the polynomial k-algebra in variables $x^\pm_{i_1,\ldots,i_d}$, where d runs over $[n-1]$ and $i_1,\ldots,i_d$ runs over $[n]_d$. Correspondingly, the k-algebra $\mathcal{F}$ is identified with the polynomial k-algebra in the variables $x^-_{i_1,\ldots,i_d}$ and $x^+_{i_1,\ldots,i_d}$, the homomorphism (49) takes the form

$$\phi \colon \mathcal{F} \to k[G], \qquad x^+_{i_1,\ldots,i_d} \mapsto f^+_{i_1,\ldots,i_d}, \quad x^-_{i_1,\ldots,i_d} \mapsto f^-_{i_1,\ldots,i_d},$$

and $\phi^\pm = \phi|_{\mathcal{F}^\pm}$. Below the sets (50) and (51) are explicitly specified using this notation.

First, we will specify the Plücker-type relations. It is convenient to introduce the following elements of $\mathcal{F}^\pm$. Let $i_1,\ldots,i_d$ be a sequence of $d \in [n-1]$ elements of $[n]$, and let $j_1,\ldots,j_d$ be the nondecreasing sequence obtained from $i_1,\ldots,i_d$ by permutation. Then we put

$$x^\pm_{i_1,\ldots,i_d} = \begin{cases} \operatorname{sgn}(i_1,\ldots,i_d)x^\pm_{j_1,\ldots,j_d} & \text{if } i_p \neq i_q \text{ for all } p \neq q, \\ 0 & \text{otherwise.} \end{cases}$$

The k-algebra $\mathcal{S}^\pm$ is the coordinate algebra of the affine multicone over the flag variety; see [Towber 1979]. By the well-known classical Hodge's result [1942; 1943] (see also [Towber 1979, p. 434, Corollary 1]), the ideal $\ker \phi^\pm$ is generated by all elements of the form

$$(53) \qquad \sum_{l=1}^{q+1} (-1)^l x^\pm_{i_1,\ldots,i_{p-1},j_l}\, x^\pm_{j_1,\ldots,\hat{j}_l,\ldots,j_{q+1}},$$

where p and q run over $[n-1]$, $p \leqslant q$, and $i_1,\ldots,i_{p-1}$ and $j_1,\ldots,j_{q+1}$ run over $[n]_{p-1}$ and $[n]_{q+1}$ respectively. Since every element (53) is homogeneous of degree 2, this result together with Theorem 22 imply that, for every fixed $p,q \in [n-1]$, the set $(\varphi^\pm_{p,q})^{-1}(\ker \psi^\pm_{p,q})$ in (50) is the linear span of all elements (53), where $i_1,\ldots,i_{p-1}$ and $j_1,\ldots,j_{q+1}$ run over $[n]_{p-1}$ and $[n]_{q+1}$ respectively. This describes the Plücker-type relations (50).

Secondly, we will describe s_{ϖ_d}. If $\boldsymbol{i} \in [n]_{n-d}$ is a sequence $i_1,\ldots,i_{n-d}$, we put $x^\pm_{\boldsymbol{i}} := x^\pm_{i_1,\ldots,i_{n-d}}$ and denote by $\boldsymbol{i}^* \in [n]_d$ the unique sequence $j_1,\ldots,j_d$ whose intersection with $i_1,\ldots,i_{n-d}$ is empty. Let $\operatorname{sgn}(\boldsymbol{i},\boldsymbol{i}^*)$ be the sign of the permutation $(i_1,\ldots,i_{n-d},j_1,\ldots,j_d)$. Then by [Flath and Towber 1992, Theorem 3.1(b)],

$$s_{\varpi_d} = \sum_{\boldsymbol{i} \in [n]_{n-d}} \operatorname{sgn}(\boldsymbol{i},\boldsymbol{i}^*)x^-_{\boldsymbol{i}} x^+_{\boldsymbol{i}^*}.$$

This describes the SL_2-type relations (51).

A similar description of the presentation of $k[G]$ may be given for the classical groups G of several other types: for them, the Plücker-type (respectively, the SL_2-type) relations are obtained using [Lancaster and Towber 1979; 1985] (respectively, [Flath and Towber 1992]).

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