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THE PRO-p IWAHORI HECKE ALGEBRA OF A REDUCTIVE p-ADIC GROUP, V (PARABOLIC INDUCTION)

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I dedicate this work to the memory of Robert Steinberg, having in mind both a nice encounter in Los Angeles and the representations named after him, which play such a fundamental role in the representation theory of reductive p-adic groups.

We give basic properties of the parabolic induction and coinduction functors associated to R-algebras modelled on the pro-p Iwahori Hecke R-algebras $\mathcal{H}_R(G)$ and $\mathcal{H}_R(M)$ of a reductive p-adic group G and of a Levi subgroup M when R is a commutative ring. We show that the parabolic induction and coinduction functors are faithful, have left and right adjoints that we determine, respect finitely generated R-modules, and that the induction is a twisted coinduction.

1.	Introduction	499
2.	Levi algebra	503
3.	Induction and coinduction	517
4.	Parabolic induction and coinduction from $\mathcal{H}_M$ to $\mathcal{H}$	520
Acknowledgements		528
References		528

1. Introduction

We give basic properties of the parabolic induction and coinduction functors associated to R-algebras modelled on the pro-p Iwahori Hecke R-algebras $\mathcal{H}_R(G)$ and $\mathcal{H}_R(M)$ of a reductive p-adic group G and of a Levi subgroup M when R is a commutative ring. We show that the parabolic induction and coinduction functors are faithful, have left and right adjoints that we determine, respect finitely generated R-modules, and that the induction is a twisted coinduction.

When R is an algebraically closed field of characteristic p, Abe [2014, §4] proved that the induction is a twisted coinduction when he classified the simple $\mathcal{H}_R(G)$-modules in terms of supersingular simple $\mathcal{H}_R(M)$-modules. In two forthcoming articles [Ollivier and Vignéras $\geq$ 2015; Abe et al. $\geq$ 2015], we will use this paper

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to compute the images of an irreducible admissible R-representation of G by the basic functors: invariants by a pro-p-Iwahori subgroup, left or right adjoint of the parabolic induction.

Let R be a commutative ring and let $\mathcal{H}$ be a pro-p Iwahori Hecke R-algebra, associated to a pro-p Iwahori Weyl group $W(1)$ and parameter maps $\mathfrak{S} \xrightarrow{\mathfrak{q}} R$, $\mathfrak{S}(1) \xrightarrow{\mathfrak{c}} R[Z_k]$ [Vignéras 2013a, §4.3; 2015b].

For the reader unfamiliar with these definitions, we recall them briefly. The pro-p Iwahori Weyl group $W(1)$ is an extension of an Iwahori–Weyl group W by a finite commutative group Z_k, and $X(1)$ denotes the inverse image in $W(1)$ of a subset X of W. The Iwahori–Weyl group contains a normal affine Weyl subgroup W^{aff}; $\mathfrak{S}$ is the set of all affine reflections of W^{aff}, and $\mathfrak{q}$ is a W-equivariant map $\mathfrak{S} \to R$, with W acting by conjugation on $\mathfrak{S}$ and trivially on R; $\mathfrak{c}$ is a $(W(1) \times Z_k)$-equivariant map $\mathfrak{S}(1) \to R[Z_k]$, with $W(1)$ acting by conjugation and Z_k by multiplication on both sides.

The Iwahori–Weyl group is a semidirect product $W = \Lambda \rtimes W_0$, where Λ is the (commutative finitely generated) subgroup of translations and W_0 is the finite Weyl subgroup of W^{aff}.

Let S^{aff} be a set of generators of W^{aff} such that $(W^{\text{aff}}, S^{\text{aff}})$ is an affine Coxeter system and $(W_0, S := S^{\text{aff}} \cap W_0)$ is a finite Coxeter system. The Iwahori–Weyl group is also a semidirect product $W = W^{\text{aff}} \rtimes \Omega$, where Ω denotes the normalizer of S^{aff} in W. Let ℓ denote the length of $(W^{\text{aff}}, S^{\text{aff}})$ extended to W and then inflated to $W(1)$ such that $\Omega \subset W$ and $\Omega(1) \subset W(1)$ are the subsets of length-0 elements.

Let $\tilde{w} \in W(1)$ denote a fixed but arbitrary lift of $w \in W$.

The subset $\mathfrak{S} \subset W^{\text{aff}}$ of all affine reflections is the union of the W^{aff}-conjugates of S^{aff} and the map $\mathfrak{q}$ is determined by its values on S^{aff}; the map $\mathfrak{c}$ is determined by its values on any set $\tilde{S}^{\text{aff}} \subset S^{\text{aff}}(1)$ of lifts of S^{aff} in $W(1)$.

Definition 1.1. The R-algebra $\mathcal{H}$ associated to $(W(1), \mathfrak{q}, \mathfrak{c})$ and S^{aff} is the free R-module of basis $(T_{\tilde{w}})_{\tilde{w} \in W(1)}$ and relations generated by the braid and quadratic relations

$$T_{\tilde{w}} T_{\tilde{w}'} = T_{\tilde{w}\tilde{w}'}, \quad T_{\tilde{s}}^2 = \mathfrak{q}(s)(\tilde{s})^2 + \mathfrak{c}(\tilde{s}) T_{\tilde{s}}$$

for all $\tilde{w}, \tilde{w}' \in W(1)$ with $\ell(w) + \ell(w') = \ell(ww')$ and all $\tilde{s} \in S^{\text{aff}}(1)$.

By the braid relations, the map $R[\Omega(1)] \to \mathcal{H}$ sending $\tilde{u} \in \Omega(1)$ to $T_{\tilde{u}}$ identifies $R[\Omega(1)]$ with a subring of $\mathcal{H}$ containing $R[Z_k]$. This identification is used in the quadratic relations. The isomorphism class of $\mathcal{H}$ is independent of the choice of S^{aff}.

Let S_M be a subset of S. We recall the definitions of the pro-p Iwahori Weyl group $W_M(1)$, the parameter maps $\mathfrak{S}_M \xrightarrow{\mathfrak{q}_M} R$, $\mathfrak{S}_M(1) \xrightarrow{\mathfrak{c}_M} R[Z_k]$ and S_M^{aff} given in [Vignéras 2015b].

The set S_M generates a finite Weyl subgroup $W_{M,0}$ of W_0, $W_M := \Lambda \rtimes W_{M,0}$ is a subgroup of W, $W_M(1)$ is the inverse image of W_M in $W(1)$, $\mathfrak{S}_M(1) =$

$\mathfrak{S}(1) \cap W_M(1)$, $\mathfrak{q}_M$ is the restriction of $\mathfrak{q}$ to $\mathfrak{S}_M$, and $\mathfrak{c}_M$ is the restriction of $\mathfrak{c}$ to $\mathfrak{S}_M(1)$. The subgroup $W_M^{\mathrm{aff}} := W^{\mathrm{aff}} \cap W_M \subset W_M$ is an affine Weyl group and S_M^{aff} denotes the set of generators of W_M^{aff} containing S_M such that $(W_M^{\mathrm{aff}}, S_M^{\mathrm{aff}})$ is an affine Coxeter system.

Definition 1.2. For $S_M \subset S$, the R-algebra $\mathcal{H}_M$ associated to $(W_M(1), \mathfrak{q}_M, \mathfrak{c}_M)$ and S_M^{aff} is called a Levi algebra of $\mathcal{H}$.

Let $(T^M_{\tilde{w}})_{\tilde{w} \in W_M(1)}$ denote the basis of $\mathcal{H}_M$ associated to $(W_M(1), \mathfrak{q}_M, \mathfrak{c}_M)$ and S_M^{aff} and ℓ_M the length of $W_M(1)$ associated to S_M^{aff}.

Remark 1.3. When $S_M = S$, we have $\mathcal{H}_M = \mathcal{H}$, and when $S_M = \varnothing$, we have $\mathcal{H}_M = R[\Lambda(1)]$.

In general when $S_M \neq S$, S_M^{aff} is not $W_M \cap S^{\mathrm{aff}}$, and $\mathcal{H}_M$ is not a subalgebra of $\mathcal{H}$; it embeds in $\mathcal{H}$ only when the parameters $\mathfrak{q}(s) \in R$ for $s \in S^{\mathrm{aff}}$ are invertible.

As in the theory of Hecke algebras associated to types, one introduces the subalgebra $\mathcal{H}_M^+ \subset \mathcal{H}_M$ of basis $(T^M_{\tilde{w}})_{\tilde{w} \in W_{M^+}(1)}$ associated to the positive monoid

$$W_{M^+} := \{w \in W_M \mid w(\Sigma^+ - \Sigma_M^+) \subset \Sigma^{\mathrm{aff},+}\},$$

where $\Sigma_M \subset \Sigma$ are the reduced root systems defining $W_M^{\mathrm{aff}} \subset W^{\mathrm{aff}}$, the upper index indicates the positive roots with respect to S^{aff}, S_M^{aff}, and Σ^{aff} is the set of affine roots of Σ. One chooses an element $\tilde{\mu}_M$ central in $W_M(1)$, in particular of length $\ell_M(\tilde{\mu}_M) = 0$, lifting a strictly positive element μ_M in $\Lambda_{M^+} := \Lambda \cap W_{M^+}$. The element $T^M_{\tilde{\mu}_M}$ of $\mathcal{H}_M$ is invertible of inverse $T^M_{\tilde{\mu}_M^{-1}}$, but in general $T_{\tilde{\mu}_M}$ is not invertible in $\mathcal{H}$.

Theorem 1.4. (i) *The R-submodule $\mathcal{H}_{M^+}$ of basis $(T^M_{\tilde{w}})_{\tilde{w} \in W_{M^+}(1)}$ is a subring of $\mathcal{H}_M$, called the positive subalgebra of $\mathcal{H}_M$.*

(ii) *The R-algebra $\mathcal{H}_M = \mathcal{H}_{M^+}[(T^M_{\tilde{\mu}_M})^{-1}]$ is a localization of $\mathcal{H}_{M^+}$ at $T^M_{\tilde{\mu}_M}$.*

(iii) *The injective linear map $\mathcal{H}_M \xrightarrow{\theta} \mathcal{H}$ sending $T^M_{\tilde{w}}$ to $T_{\tilde{w}}$ for $\tilde{w} \in W_M(1)$ restricted to $\mathcal{H}_{M^+}$ is a ring homomorphism.*

(iv) *As a $\theta(\mathcal{H}_{M^+})$-module, $\mathcal{H}$ is the almost localization of a left free $\theta(\mathcal{H}_{M^+})$-module $\mathcal{V}_{M^+}$ at $T_{\tilde{\mu}_M}$.*

The theorem was known in special cases. Part (iv) means that $\mathcal{H}$ is the union over $r \in \mathbb{N}$ of

$$_r\mathcal{V}_{M^+} := \{x \in \mathcal{H} \mid T^r_{\tilde{\mu}_M} x \in \mathcal{V}_{M^+}\}, \quad \mathcal{V}_{M^+} = \oplus_{d \in {}^M W_0} \theta(\mathcal{H}_{M^+}) T_{\tilde{d}}.$$

Here ${}^M W_0$ is the set of elements of minimal lengths in the cosets $W_{M,0} \backslash W_0$ and $\tilde{d} \in W(1)$ is an arbitrary lift of d. The theorem admits a variant for the subalgebra $\mathcal{H}_{M^-} \subset \mathcal{H}_M$ associated to the negative submonoid W_{M^-}, inverse of W_{M^+}, for the linear map $\mathcal{H}_M \xrightarrow{\theta^*} \mathcal{H}$ sending $(T^M_{\tilde{w}})^*$ to $T^*_{\tilde{w}}$ for $\tilde{w} \in W_M(1)$ [Vignéras 2013a, Proposition 4.14], and with *left* replaced by *right* in (iv): $\mathcal{H}_M = \mathcal{H}_{M^-}[T^M_{\tilde{\mu}_M}]$, θ^* restricted to $\mathcal{H}_{M^-}$ is a ring homomorphism, and the right $\theta^*(\mathcal{H}_{M^-})$-module $\mathcal{H}$ is the almost localisation at $T^*_{\tilde{\mu}_M^{-1}}$ of a right free $\theta^*(\mathcal{H}_{M^-})$-module $\mathcal{V}^*_{M^-}$ of rank $|W_{M,0}|^{-1}|W_0|$, meaning that $\mathcal{H}$ is the union over $r \in \mathbb{N}$ of

$$ {}_r\mathcal{V}^*_{M^-} := \{x \in \mathcal{H} \mid x(T^*_{\tilde{\mu}_M^{-1}})^r \in \mathcal{V}^*_{M^-}\}, \quad \mathcal{V}^*_{M^-} := \sum_{d \in W_0^M} T^*_{\tilde{d}}\, \theta^*(\mathcal{H}_{M^-}). $$

Here W_0^M is the inverse of ${}^M W_0$.

For a ring A, let Mod_A denote the category of right A-modules and ${}_A\mathrm{Mod}$ the category of left A-modules. Given two rings $A \subset B$, the induction $- \otimes_A B$ and the coinduction $\mathrm{Hom}_A(B, -)$ from Mod_A to Mod_B are the left and the right adjoint of the restriction Res^B_A. The ring B is considered as a left A-module for the induction, and as a right A-module for the coinduction.

Property (iv) and its variant describe $\mathcal{H}$ as a left $\theta(\mathcal{H}_{M^+})$-module and as a right $\theta^*(\mathcal{H}_{M^-})$-module. The linear maps θ and θ^* identify the subalgebras $\mathcal{H}_{M^+}, \mathcal{H}_{M^-}$ of $\mathcal{H}_M$ with the subalgebras $\theta(\mathcal{H}_{M^+}), \theta^*(\mathcal{H}_{M^-})$ of $\mathcal{H}$.

Definition 1.5. The parabolic induction and coinduction from $\mathrm{Mod}_{\mathcal{H}_M}$ to $\mathrm{Mod}_{\mathcal{H}}$ are the functors $I^{\mathcal{H}}_{\mathcal{H}_M} = - \otimes_{\mathcal{H}_{M^+},\theta} \mathcal{H}$ and $\mathbb{I}^{\mathcal{H}}_{\mathcal{H}_M} = \mathrm{Hom}_{\mathcal{H}_{M^-},\theta^*}(\mathcal{H}, -)$.

We show the following:

Theorem 1.6. *The parabolic induction $I^{\mathcal{H}}_{\mathcal{H}_M}$ is faithful, transitive, respects finitely generated R-modules, and admits a right adjoint* $\mathrm{Hom}_{\mathcal{H}_{M^+}}(\mathcal{H}_M, -)$.

If R is a field, the right adjoint functor respects finite dimension.

The transitivity of the parabolic induction means that for $S_M \subset S_{M'} \subset S$,

$$ I^{\mathcal{H}}_{\mathcal{H}_M} = I^{\mathcal{H}}_{\mathcal{H}_{M'}} \circ I^{\mathcal{H}_{M'}}_{\mathcal{H}_M} : \mathrm{Mod}_{\mathcal{H}_M} \to \mathrm{Mod}_{\mathcal{H}_{M'}} \to \mathrm{Mod}_{\mathcal{H}}\,. $$

Let w_0 denote the longest element of W_0, $S_{w_0(M)}$ the subset $w_0 S_M w_0$ of S, and $w_0^M := w_0 w_{M,0}$, where $w_{M,0}$ is the longest element of $W_{M,0}$. A lift $\tilde{w}_0^M \in W_0(1)$ of w_0^M defines an R-algebra isomorphism

$$ \mathcal{H}_M \to \mathcal{H}_{w_0(M)}, \qquad T^M_{\tilde{w}} \mapsto T^{w_0(M)}_{\tilde{w}_0^M \tilde{w} (\tilde{w}_0^M)^{-1}} \quad \text{for } \tilde{w} \in W_M(1), \tag{1} $$

inducing an equivalence of categories

$$ \mathrm{Mod}_{\mathcal{H}_M} \xrightarrow{\tilde{\mathfrak{w}}_0^M} \mathrm{Mod}_{\mathcal{H}_{w_0(M)}} $$

of inverse $\tilde{\mathfrak{w}}_0^{w_0(M)}$ defined by the lift $(\tilde{w}_0^M)^{-1} \in W_0(1)$ of $w_0^{w_0(M)} = (w_0^M)^{-1}$.

Definition 1.7. The w_0-twisted parabolic induction and coinduction from $\mathrm{Mod}_{\mathcal{H}_M}$ to $\mathrm{Mod}_{\mathcal{H}}$ are the functors $I^{\mathcal{H}}_{\mathcal{H}_{w_0(M)}} \circ \tilde{\mathfrak{w}}_0^M$ and $\mathbb{I}^{\mathcal{H}}_{\mathcal{H}_{w_0(M)}} \circ \tilde{\mathfrak{w}}_0^M$.

Up to modulo equivalence, these functors do not depend on the choice of the lift of w_0^M used for their construction.

Theorem 1.8. *The parabolic induction* (*resp. coinduction*) *is equivalent to the* w_0*-twisted parabolic coinduction* (*resp. induction*):

$$\mathbb{I}_{\mathcal{H}_M}^{\mathcal{H}} \simeq I_{\mathcal{H}_{w_0(M)}}^{\mathcal{H}} \circ \tilde{\mathfrak{w}}_0^M, \quad I_{\mathcal{H}_M}^{\mathcal{H}} \simeq \mathbb{I}_{\mathcal{H}_{w_0(M)}}^{\mathcal{H}} \circ \tilde{\mathfrak{w}}_0^M.$$

Using that the coinduction admits a left adjoint and that the induction is a twisted coinduction, one proves the following:

Theorem 1.9. *The parabolic induction* $I_{\mathcal{H}_M}^{\mathcal{H}}$ *admits a left adjoint equivalent to*

$$\tilde{\mathfrak{w}}_0^{w_0(M)} \circ (- \otimes_{\mathcal{H}_{w_0(M)^-},\theta^*} \mathcal{H}_{w_0(M)}) : \mathrm{Mod}_{\mathcal{H}} \to \mathrm{Mod}_{\mathcal{H}_{w_0(M)}} \to \mathrm{Mod}_{\mathcal{H}_M} .$$

When R is a field, *the left adjoint functor respects finite dimension.*

The coinduction satisfies the same properties as the induction:

Corollary 1.10. *The coinduction* $\mathbb{I}_{\mathcal{H}_M}^{\mathcal{H}}$ *is faithful*, *transitive*, *respects finitely generated R-modules*, *and admits a left and a right adjoint. When R is a field*, *the left and right adjoint functors respect finite dimension.*

Note that the induction and the coinduction are exact functors, as they admit a left and a right adjoint.

We prove Theorem 1.4 in Section 2, and Theorems 1.6, 1.8 and 1.9 in Section 4.

Remark 1.11. One cannot replace $(\mathcal{H}, \mathcal{H}_M, \mathcal{H}_M^+)$ by $(\mathcal{H}, \mathcal{H}_M, \mathcal{H}_M^-)$ to define the induction $I_{\mathcal{H}_M}^{\mathcal{H}}$.

When no nonzero element of the ring R is infinitely p-divisible, is the parabolic induction functor

$$\mathrm{Mod}_{\mathcal{H}_M} \xrightarrow{I_{\mathcal{H}_M}^{\mathcal{H}}} \mathrm{Mod}_{\mathcal{H}}$$

fully faithful? The answer is yes for the parabolic induction functor

$$\mathrm{Mod}_R^\infty(M) \xrightarrow{\mathrm{Ind}_P^G} \mathrm{Mod}_R^\infty(G)$$

when M is a Levi subgroup of a parabolic subgroup P of a reductive p-adic group G and $\mathrm{Mod}_R^\infty(G)$ the category of smooth R-representations of G [Vignéras 2014, Theorem 5.3].

2. Levi algebra

We prove Theorem 1.4 and its variant on the subalgebra $\mathfrak{H}_M^\epsilon \subset \mathfrak{H}_M$, its image in $\mathcal{H}$, on $\mathfrak{H}_M$ as a localisation of $\mathfrak{H}_M^\epsilon$ and on $\mathcal{H}$ as an almost left localisation of $\theta(\mathfrak{H}_M^+)$, and almost left localisation of $\theta^*(\mathfrak{H}_M^-)$.

2A. ***Monoid*** W_{M^ϵ}**.** Let $S_M \subset S$ and $\epsilon \in \{+,-\}$. To S^{aff} is associated a submonoid $W_{M^\epsilon} \subset W_M$ defined as follows.

Let Σ denote the reduced root system of affine Weyl group W^{aff}, V the real vector space of dual generated by Σ, $\Sigma^{\text{aff}} = \Sigma + \mathbb{Z}$ the set of affine roots of Σ and $\mathfrak{H} = \{\text{Ker}_V(\gamma) \mid \gamma \in \Sigma^{\text{aff}}\}$ the set of kernels of the affine roots in V. We fix a W_0-invariant scalar product on V. The affine Weyl group W^{aff} identifies with the group generated by the orthogonal reflections with respect to the affine hyperplanes of $\mathfrak{H}$.

Let $\mathfrak{A}$ denote the alcove of vertex 0 of $(V, \mathfrak{H})$ such that S^{aff} is the set of orthogonal reflections with respect to the walls of $\mathfrak{A}$ and S is the subset associated to the walls containing 0. An affine root which is positive on $\mathfrak{A}$ is called positive. Let $\Sigma^{\text{aff},+}$ denote the set of positive affine roots, $\Sigma^+ := \Sigma \cap \Sigma^+_{\text{aff}}$, $\Sigma^{\text{aff},-} := -\Sigma^{\text{aff},-}$, and $\Sigma^- := -\Sigma^+$.

Let Δ_M denote the set of positive roots $\alpha \in \Sigma^+$ such that $\text{Ker}\,\alpha$ is a wall of $\mathfrak{A}$ and the orthogonal reflection s_α of V with respect to $\text{Ker}\,\alpha$ belongs to S_M, $\Sigma_M \subset \Sigma$ the reduced root system generated by Δ_M, and $\Sigma^\epsilon_M := \Sigma_M \cap \Sigma^\epsilon_{\text{aff}}$.

Definition 2.1. The positive monoid $W_{M^+} \subset W_M$ is

$$\{w \in W_M \mid w(\Sigma^+ - \Sigma^+_M) \subset \Sigma^{\text{aff},+}\}.$$

The negative monoid $W_{M^-} := \{w \in W_M \mid w^{-1} \in W_{M^+}\}$ is the inverse monoid.

It is well known that the finite Weyl group $W_{M,0}$ is the W_0-stabilizer of $\Sigma^\epsilon - \Sigma^\epsilon_M$. This implies

$$W_{M^\epsilon} = \Lambda_{M^\epsilon} \rtimes W_{M,0}, \qquad \text{where } \Lambda_{M^\epsilon} := \Lambda \cap W_{M^\epsilon}.$$

Let $\Lambda \xrightarrow{\nu} V$ denote the homomorphism such that $\lambda \in \Lambda$ acts on V by translation by $\nu(\lambda)$.

Lemma 2.2. $\Lambda_{M^\epsilon} = \{\lambda \in \Lambda \mid -(\gamma \circ \nu)(\lambda) \geq 0$ *for all* $\gamma \in \Sigma^\epsilon - \Sigma^\epsilon_M\}$.

Proof. Let $\lambda \in \Lambda$. By definition, $\lambda \in \Lambda_{M^+}$ if and only if $\lambda(\gamma)$ is positive for all $\gamma \in \Sigma^+ - \Sigma^+_M$. We have $\lambda(\gamma) = \gamma - \nu(\lambda)$. The minimum of the values of γ on $\mathfrak{A}$ is 0 [Vignéras 2013a, (35)]. So $\gamma(v - \nu(\lambda)) \geq 0$ for $\gamma \in \Sigma^+ - \Sigma^+_M$ and $v \in \mathfrak{A}$ is equivalent to $-(\gamma \circ \nu)(\lambda) \geq 0$ for all $\gamma \in \Sigma^+ - \Sigma^+_M$. □

When $S_M \subset S_{M'} \subset S$, we have the inclusion $\Sigma^\epsilon_M \subset \Sigma^\epsilon_{M'}$, the inverse inclusion $\Sigma^\epsilon - \Sigma^\epsilon_M \subset \Sigma^\epsilon - \Sigma^\epsilon_{M'}$, and the inclusions $W_M \subset W_{M'}$ and $W_{M^\epsilon} \subset W^\epsilon_{M'}$.

Remark 2.3. Set $\mathcal{D}^\epsilon := \{v \in V \mid \gamma(v) \geq 0 \text{ for } \gamma \in \Sigma^\epsilon\}$ and $\Lambda^\epsilon := (-\nu)^{-1}(\mathcal{D}^\epsilon)$. The antidominant Weyl chamber of V is $\mathcal{D}^-$ and the dominant Weyl chamber is $\mathcal{D}^+$. Careful: [Vignéras 2015a, §1.2(v)] uses a different notation: $\Lambda^\epsilon = (\nu)^{-1}(\mathcal{D}^\epsilon)$.

The Bruhat order $\leq$ of the affine Coxeter system $(W^{\text{aff}}, S^{\text{aff}})$ extends to W: for $w_1, w_2 \in W^{\text{aff}}$, $u_1, u_2 \in \Omega$, we have $w_1u_1 \leq w_2u_2$ if $u_1 = u_2$ and $w_1 \leq w_2$ [Vignéras 2006, Appendice]. We write $w < w'$ if $w \leq w'$ and $w \neq w'$ for $w, w' \in W$. Careful:

the Bruhat order $\leq_M$ on W_M associated to $(W_M^{\mathrm{aff}}, S_M^{\mathrm{aff}})$ is not the restriction of $\leq$ when S_M^{aff} is not contained in S^{aff} [Vignéras 2015b].

Remark 2.4. The basic properties of $(W^{\mathrm{aff}}, S^{\mathrm{aff}})$ extend to W:

(i) If $x \leq y$ for $x, y \in W$ and $s \in S^{\mathrm{aff}}$,

$$sx \leq (y \text{ or } sy), \quad xs \leq (y \text{ or } ys), \quad (x \text{ or } sx) \leq sy, \quad (x \text{ or } xs) \leq ys$$

[Vignéras 2015a, Lemma 3.1, Remark 3.2].

(ii) $W = \bigsqcup_{\lambda\in\Lambda^\epsilon} W_0\lambda W_0$ [Henniart and Vignéras 2015, §6.3, Lemma].

(iii) For $\lambda \in \Lambda^+$, $W_0\lambda W_0$ admits a unique element of maximal length $w_\lambda = w_0\lambda$, where w_0 is the unique element of maximal length in W_0, and $\ell(w_\lambda) = \ell(w_0) + \ell(\lambda)$ [Vignéras 2015a, Lemma 3.5].

(iv) For $\lambda \in \Lambda^+$, $\{w \in W \mid w \leq w_\lambda\} \supset \bigsqcup_{\mu\in\Lambda^+, \mu\leq\lambda} W_0\mu W_0$ [Vignéras 2015a, Lemma 3.5].

Remark 2.5. The set $\{w \in W \mid w \leq w_\lambda\}$ is a union of (W_0, W_0)-classes only if $\lambda, \mu \in \Lambda^+$, $\mu \leq w_0\lambda$ implies $\mu \leq \lambda$. I see no reason for this to be true.

Lemma 2.6. *The monoid W_{M^ϵ} is a lower subset of W_M for the Bruhat order $\leq_M$: for $w \in W_{M^\epsilon}$, any element $v \in W_M$ such that $v \leq_M w$ belongs to W_{M^ϵ}.*

Proof. See [Abe 2014, Lemma 4.1]. □

An element $w \in W$ admits a reduced decomposition in (W, S^{aff}), $w = s_1 \cdots s_r u$ with $s_i \in S^{\mathrm{aff}}$, $u \in \Omega$. As in [Vignéras 2013a], we set for $w, w' \in W$,

$$q_w := \mathfrak{q}(s_1)\cdots\mathfrak{q}(s_r), \quad q_{w,w'} := (q_w q_{w'} q_{ww'}^{-1})^{1/2}. \tag{2}$$

This is independent of the choice of the reduced decomposition. For $w, w' \in W_M$ and $s_i \in S_M^{\mathrm{aff}}$, $u \in \Omega_M$, let $q_{M,w},\ q_{M,w,w'}$ denote the similar elements. They may be different from $q_w,\ q_{w,w'}$.

Lemma 2.7. *We have $S_M^{\mathrm{aff}} \cap W_{M^\epsilon} \subset S^{\mathrm{aff}}$ and $q_{w,w'} = q_{M,w,w'}$ if $w, w' \in W_{M^\epsilon}$.*

In particular, $\ell_M(w)+\ell_M(w')-\ell_M(ww')=\ell(w)+\ell(w')-\ell(ww')$ if $w, w'\in W_{M^\epsilon}$.

Proof. See [Abe 2014, Lemma 4.4, proof of Lemma 4.5]. □

An element $\lambda \in \Lambda_{M^\epsilon}$ such that all the inequalities in Lemma 2.2 are strict is called strictly positive if $\epsilon = +$, and strictly negative if $\epsilon = +$. We choose

a central element $\tilde{\mu}_M$ of $W_M(1)$ lifting a strictly positive element μ_M of Λ.

We set $\tilde{\mu}_{M^+} := \tilde{\mu}_M$ and $\tilde{\mu}_{M^-} := \tilde{\mu}_M^{-1}$. The center of the pro-$p$ Iwahori Weyl group $W_M(1)$ is the set of elements in the center of $\Lambda(1)$ fixed by the finite Weyl group $W_{M,0}$ [Vignéras 2014]. Hence $\tilde{\mu}_{M^\epsilon}$ is an element of the center of $\Lambda(1)$ fixed by $W_{M,0}$ and $-\gamma \circ \nu(\mu_{M^\epsilon}) > 0$ for all $\gamma \in \Sigma^\epsilon - \Sigma^\epsilon_M$. We have $\gamma \circ \nu(\mu_{M^\epsilon}) = 0$ for $\gamma \in \Sigma_M$. The length of μ_{M^ϵ} is 0 in W_M, and is positive in W when $S_M \neq S$.

Let $\mathcal{H}_{M^\epsilon}$ denote the R-submodule of the Iwahori–Hecke R-algebra $\mathcal{H}_M$ of M of basis $(T^M_{\tilde{w}})_{\tilde{w} \in W_{M^\epsilon}(1)}$, and $\mathcal{H}_M \xrightarrow{\theta} \mathcal{H}$ the linear map sending $T^M_{\tilde{w}}$ to $T_{\tilde{w}}$ for $\tilde{w} \in W_M(1)$.

The proofs of the properties (i), (ii), (iii) of Theorem 1.4 and its variant are as follows:

(i) $\mathcal{H}_{M^\epsilon}$ is a subring of $\mathcal{H}_M$, because $T^M_{\tilde{w}} T^M_{\tilde{w}'}$ is a linear combination of elements $T_{\tilde{v}}$ such that $v \leq_M ww'$ [Vignéras 2013a].

(iii) We have $\theta(T^M_{\tilde{w}_1} T^M_{\tilde{w}_2}) = T_{\tilde{w}_1} T_{\tilde{w}_2}$ and $\theta^*((T^M_{\tilde{w}_1})^* (T^M_{\tilde{w}_2})^*) = T^*_{\tilde{w}_1} T^*_{\tilde{w}_2}$ for $w_1, w_2 \in W_{M^\epsilon}$. This follows from the braid relations if $\ell_M(w_1) + \ell_M(w_2) = \ell_M(w_1 w_2)$ because $\ell(w_1) + \ell(w_2) = \ell(w_1 w_2)$ (Lemma 2.7). If $w_2 = s \in S^{\mathrm{aff}}_M$ with $\ell_M(w_1) - 1 = \ell_M(w_1 s)$, this follows from the quadratic relations

$$T_{\tilde{w}_1} T_{\tilde{s}} = T_{\tilde{w}_1 \tilde{s}^{-1}} \big(\mathfrak{q}(s)(\tilde{s})^2 + T_{\tilde{s}} \mathfrak{c}(\tilde{s})\big) = \mathfrak{q}(s) T_{\tilde{w}_1 \tilde{s}} + T_{\tilde{w}_1} \mathfrak{c}(\tilde{s}),$$
$$T^*_{\tilde{w}_1} T^*_{\tilde{s}} = \mathfrak{q}(s) T^*_{\tilde{w}_1 \tilde{s}} - T^*_{\tilde{w}_1} \mathfrak{c}(\tilde{s}),$$

$s \in S^{\mathrm{aff}}$, $\ell(w_1) - 1 = \ell(w_1 s)$ (Lemma 2.7) and $\mathfrak{q}(s) = \mathfrak{q}_M(s)$, $\mathfrak{c}(\tilde{s}) = \mathfrak{c}_M(\tilde{s})$ [Vignéras 2015b]. In general the formula is proved by induction on $\ell_M(w_2)$ [Abe 2014, §4.1]. The proof of [Abe 2014, Lemma 4.5] applies.

(ii) $\mathcal{H}_M = \mathcal{H}_{M^\epsilon}[(T^M_{\tilde{\mu}_{M^\epsilon}})^{-1}]$, because for $w \in W_M$, there exists $r \in \mathbb{N}$ such that $\mu^{\epsilon r}_M w \in W_{M^\epsilon}$.

Remark 2.8. If the parameters $\mathfrak{q}(s)$ are invertible in R, then $\mathcal{H}_{M^+} \xrightarrow{\theta} \mathcal{H}$ extends uniquely to an algebra homomorphism $\mathcal{H}_M \hookrightarrow \mathcal{H}$, sending $T^M_{\tilde{\mu}^{-\epsilon r}_M \tilde{w}}$ to $T^{-r}_{\tilde{\mu}_{M^\epsilon}} T_{\tilde{w}}$ for $\tilde{w} \in W_{M^+}(1)$, $r \in \mathbb{N}$.

Remark 2.9. The trivial character $\chi_1 : \mathcal{H} \to R$ of $\mathcal{H}$ is defined by

$$\chi_1(T_{\tilde{w}}) = q_w \quad (\tilde{w} \in W(1)).$$

When $\mathcal{H}$ is the Hecke algebra of the pro-p-Iwahori subgroup of a reductive p-adic group G, we know that $\mathcal{H}$ acts on the trivial representation of G by χ_1. Note that the restriction of the trivial character of $\mathcal{H}_M$ to $\theta(\mathcal{H}_{M^+})$ is not equal to $\chi_1 \circ \theta$ when $\ell_M(\mu_M) = 0$, $\ell(\mu_M) \neq 0$.

2B. *An anti-involution ζ*. The R-linear bijective map

(3) $$\mathcal{H} \xrightarrow{\zeta} \mathcal{H} \quad \text{such that} \quad \zeta(T_{\tilde{w}}) = T_{\tilde{w}^{-1}} \quad \text{for } \tilde{w} \in W(1)$$

is an anti-involution when $\zeta(h_1 h_2) = \zeta(h_2)\zeta(h_1)$ for $h_1, h_2 \in \mathcal{H}$ because $\zeta \circ \zeta = \mathrm{id}$. For $S_M \subset S$, let $\mathcal{H} \xrightarrow{\zeta_M} \mathcal{H}_M$ denote the linear map such that $\zeta(T^M_{\tilde{w}}) = T^M_{\tilde{w}^{-1}}$ for $\tilde{w} \in W_M(1)$.

Lemma 2.10. 1. *The following properties are equivalent*:

(i) ζ *is an anti-involution.*

(ii) $\zeta(\mathfrak{c}(\tilde{s})) = c_{(\tilde{s})^{-1}}$ *for* $\tilde{s} \in S^{\text{aff}}(1)$.

(iii) $\zeta \circ \mathfrak{c} = \mathfrak{c} \circ (-)^{-1}$, *where* $\mathfrak{S}(1) \xrightarrow{\mathfrak{c}} R[Z_k]$ *is the parameter map.*

2. *If* ζ *is an anti-involution then* ζ_M *is an anti-involution.*

Proof. Let $\tilde{w} = \tilde{s}_1 \cdots \tilde{s}_{\ell(w)}\tilde{u}$ be a reduced decomposition, $\tilde{s}_i \in S^{\text{aff}}(1)$, $\tilde{u} \in W(1)$, $\ell(\tilde{u}) = 0$ and let $\tilde{s} \in S^{\text{aff}}(1)$. Then,

$$\zeta(T_{\tilde{w}}) = T_{(\tilde{w})^{-1}} = T_{(\tilde{u})^{-1}} T_{\tilde{s}^{-1}_{\ell(w)}} \cdots T_{\tilde{s}^{-1}_1} = \zeta(T_{\tilde{u}})\zeta(T_{\tilde{s}_{\ell(w)}}) \cdots \zeta(T_{\tilde{s}_1}),$$

$$(\zeta(T_{\tilde{s}}))^2 = T^2_{\tilde{s}^{-1}} = \mathfrak{q}(s)\tilde{s}^{-2} + \mathfrak{c}(\tilde{s}^{-1})T_{\tilde{s}^{-1}}.$$

The map ζ is an antiautomorphism if and only if $\zeta(\mathfrak{c}(\tilde{s})) = \mathfrak{c}(\tilde{s}^{-1})$ for $\tilde{s} \in S^{\text{aff}}(1)$. This is equivalent to $\zeta \circ \mathfrak{c} = \mathfrak{c} \circ (-)^{-1}$ because $\mathfrak{S}(1)$ is the union of the $W(1)$-conjugates of $S^{\text{aff}}(1)$, $\mathfrak{c}$ is $W(1)$-equivariant and ζ commutes with the conjugation by $W(1)$.

If $\mathfrak{c}$ satisfies (iii), its restriction $\mathfrak{c}_M$ to $\mathfrak{S}_M(1)$ satisfies (iii). □

Lemma 2.11. *When* $\mathcal{H} = \mathcal{H}(G)$ *is the pro-p Iwahori Hecke R-algebra of a reductive p-adic group G, we have that* ζ *is an anti-involution.*

Proof. Let $s \in \mathfrak{S}$, $\tilde{s}$ be an admissible lift and $t \in Z_k$. Then $\mathfrak{c}(\tilde{s})$ is invariant by ζ [Vignéras 2013a, Proposition 4.4]. If $u \in U^*_\gamma$ for $\gamma = \alpha + r \in \Phi^{\text{aff}}_{\text{red}}$, then $u^{-1} \in U^*_\gamma$ and $m_\alpha(u)^{-1} = m_\alpha(u^{-1})$. Hence the set of admissible lifts of s is stable by the inverse map. As the group Z_k is commutative, we have

$$(\zeta \circ c)(t\tilde{s}) = \zeta(tc(s)) = t^{-1}c(s) = c(s)t^{-1} = c(t\tilde{s})^{-1}. \qquad \square$$

From now on, we suppose that ζ *is an anti-involution.* We recall the involutive automorphism [Vignéras 2013a, Proposition 4.24]

$$\mathcal{H} \xrightarrow{\iota} \mathcal{H} \quad \text{such that} \quad \iota(T_{\tilde{w}}) = (-1)^{\ell(w)}T^*_{\tilde{w}} \quad \text{for } \tilde{w} \in W(1),$$

and [Vignéras 2013a, Proposition 4.13 2)]:

$$(4) \quad T^*_{\tilde{s}} := T_{\tilde{s}} - \mathfrak{c}(\tilde{s}) \quad \text{for } \tilde{s} \in S^{\text{aff}}(1), \qquad T^*_{\tilde{w}} := T^*_{\tilde{s}_1} \cdots T^*_{\tilde{s}_r} T_{\tilde{u}} \quad \text{for } \tilde{w} \in W(1)$$

of reduced decomposition $\tilde{w} = \tilde{s}_1 \cdots \tilde{s}_{\ell(w)}\tilde{u}$.

Remark 2.12. We have $\zeta(T^*_{\tilde{w}}) = T^*_{(\tilde{w})^{-1}}$ for $\tilde{w} \in W(1)$, ζ and ι commute, $\zeta_M(\mathcal{H}_{M^\epsilon}) = \mathcal{H}^{-\epsilon}_M$ and $\theta \circ \zeta_M = \zeta \circ \theta$, $\theta^* \circ \zeta_M = \zeta \circ \theta^*$.

2C. *ϵ-alcove walk basis.* We define a basis of $\mathcal{H}$ associated to $\epsilon \in \{+, -\}$ and an orientation o of $(V, \mathfrak{H})$, which we call an ϵ-alcove walk basis associated to o.

For $s \in S^{\text{aff}}$, let α_s denote the positive affine root such that s is the orthogonal reflection with respect to $\operatorname{Ker} \alpha_s$. For an orientation o of $(V, \mathfrak{H})$, let $\mathcal{D}_o$ denote the corresponding (open) Weyl chamber in $(V, \mathfrak{H})$, $\mathfrak{A}_o$ the (open) alcove of vertex 0 contained in $\mathcal{D}_o$, and $o.w$ the orientation of Weyl chamber $w^{-1}(\mathfrak{D}_o)$ for $w \in W$. We recall [Vignéras 2013a]:

Definition 2.13. The following properties determine uniquely elements $E_o(\tilde{w}) \in \mathcal{H}$ for any orientation o of $(V, \mathfrak{H})$ and $\tilde{w} \in W(1)$. For $\tilde{w} \in W(1)$, $\tilde{s} \in S^{\text{aff}}(1)$, $\tilde{u} \in \Omega(1)$,

$$E_o(\tilde{s}) = \begin{cases} T_{\tilde{s}} & \text{if } \alpha_s \text{ is negative on } \mathfrak{A}_o, \\ T^*_{\tilde{s}} = T_{\tilde{s}} - \mathfrak{c}(\tilde{s}) & \text{if } \alpha_s \text{ is positive on } \mathfrak{A}_o, \end{cases} \tag{5}$$

$$E_o(\tilde{u}) = T_{\tilde{u}}, \tag{6}$$

$$E_o(\tilde{s}) E_{o.s}(\tilde{w}) = q_{s,w} E_o(\tilde{s}\tilde{w}). \tag{7}$$

They imply, for $w' \in W$, $\lambda \in \Lambda$,

$$E_o(\tilde{w}') E_{o.w'}(\tilde{w}) = q_{w',w} E_o(\tilde{w}'\tilde{w}), \quad E_o(\tilde{\lambda}) E_o(\tilde{w}) = q_{\lambda,w} E_o(\tilde{\lambda}\tilde{w}). \tag{8}$$

We recall that λ acts on V by translation by $\nu(\lambda)$. The Weyl chamber $\mathcal{D}_o$ of the orientation o is characterized by

$$E_o(\tilde{\lambda}) = T_{\tilde{\lambda}} \text{ when } \nu(\lambda) \text{ belongs to the closure of } \mathcal{D}_o. \tag{9}$$

The alcove walk basis of $\mathcal{H}$ associated to o is $(E_o(\tilde{w}))_{\tilde{w} \in W(1)}$ [Vignéras 2013a]. The Bernstein basis $(E(\tilde{w}))_{\tilde{w} \in W(1)}$ is the alcove walk basis associated to the antidominant orientation (of Weyl chamber $\mathcal{D}^-$). By Remark 2.3 and [Vignéras 2013a],

$$E(\tilde{w}) = T_{\tilde{w}} \quad \text{for } w \in \Lambda^+ \cup W_0, \qquad E(\tilde{w}) = T^*_{\tilde{w}} \quad \text{for } w \in \Lambda^-.$$

Definition 2.14. The ϵ-alcove walk basis $(E^{\epsilon}_o(\tilde{w}))_{\tilde{w} \in W(1)}$ of $\mathcal{H}$ associated to o is

$$E^{\epsilon}_o(\tilde{w}) := \begin{cases} E_o(\tilde{w}) & \text{if } \epsilon = +, \\ \zeta(E_o(\tilde{w}^{-1})) & \text{if } \epsilon = -. \end{cases} \tag{10}$$

Lemma 2.15. *The elements $E^-_o(\tilde{w})$ for any orientation o of $(V, \mathcal{H})$ and $\tilde{w} \in W(1)$ are determined by the following properties. For $\tilde{w} \in W(1)$, $\tilde{s} \in S^{\text{aff}}(1)$, $\tilde{u} \in \Omega(1)$,*

$$E^-_o(\tilde{s}) = E_o(\tilde{s}), \quad E^-_o(\tilde{u}) = E_o(\tilde{u}), \tag{11}$$

$$E^-_{o.s}(\tilde{w}) E^-_o(\tilde{s}) = q_{w,s} E^-_o(\tilde{w}\tilde{s}). \tag{12}$$

They imply, for $w' \in W$, $\lambda \in \Lambda$,

$$E^-_{o.w'^{-1}}(\tilde{w}) E^-_o(\tilde{w}') = q_{w,w'} E^-_o(\tilde{w}\tilde{w}'), \quad E^-_o(\tilde{w}) E^-_o(\tilde{\lambda}) = q_{w,\lambda} E^-_o(\tilde{w}\tilde{\lambda}). \tag{13}$$

Proof.

$$\begin{aligned} E^-_o(\tilde{s}) &= \zeta(E_o((\tilde{s})^{-1})) = E_o(\tilde{s}), \\ E^-_o(\tilde{w}\tilde{u}) &= \zeta(E_o((\tilde{w}\tilde{u})^{-1})) = \zeta(E_o((\tilde{u})^{-1}(\tilde{w})^{-1})) = \zeta(T_{(\tilde{u})^{-1}} E_o((\tilde{w})^{-1})) \\ &= \zeta(E_o((\tilde{w})^{-1})) T_{\tilde{u}} = E^-_o(\tilde{w}) T_{\tilde{u}}, \end{aligned}$$

$$E^-_{o.s}(\tilde{w})E^-_o(\tilde{s}) = \zeta(E_{o.s}((\tilde{w})^{-1}))\zeta(E_o((\tilde{s})^{-1})) = \zeta(E_o((\tilde{s})^{-1})E_{o.s}((\tilde{w})^{-1}))$$
$$= q_{s,w^{-1}}\zeta(E_o((\tilde{s})^{-1}(\tilde{w})^{-1})) = q_{w,s}\zeta(E_o((\tilde{w}\tilde{s})^{-1})) = q_{w,s}E^-_o(\tilde{w}\tilde{s}).$$

We used that $q_w = q_{w^{-1}}$ implies

$$q_{w_1^{-1},w_2^{-1}} = (q_{w_1^{-1}}q_{w_2^{-1}}q^{-1}_{w_1^{-1}w_2^{-1}})^{1/2} = (q_{w_1}q_{w_2}q^{-1}_{w_2w_1})^{1/2} = q_{w_2,w_1}$$

for $w_1, w_2 \in W$. □

The ϵ-alcove walk bases satisfy the triangular decomposition

$$E^\epsilon_o(\tilde{w}) - T_{\tilde{w}} \in \sum_{\tilde{w}'\in W(1),\, \tilde{w}'<\tilde{w}} RT_{\tilde{w}'}. \tag{14}$$

Remark 2.16. The basis $E_-(\tilde{w})$ introduced in [Abe 2014] is the $-$ alcove walk basis associated to the dominant Weyl chamber, satisfying $E_-(\tilde{w}) = T^*_{\tilde{w}}$ if $w \in W_0$ and $E_-(\tilde{\lambda}) = T_{\tilde{\lambda}}$ if $\lambda \in \Lambda^-$.

Let V_M be the real vector space of dual generated by Σ_M with a $W_{M,0}$-invariant scalar product and the corresponding set $\mathfrak{H}_M$ of affine hyperplanes.

Lemma 2.17. *If $\epsilon, \epsilon' \in \{+,-\}$ and o_M is any orientation of $(V_M, \mathfrak{H}_M)$, then $(E^{\epsilon'}_{o_M}(\tilde{w}))_{\tilde{w}\in W_{M^\epsilon}(1)}$ is a basis of $\mathcal{H}_{M^\epsilon}$.*

When $\mathfrak{q}(s) = 0$, see [Abe 2014, Lemma 4.2].

Proof. A basis of $\mathcal{H}_{M^\epsilon}$ is $(T^M_{\tilde{w}})_{\tilde{w}\in W_{M^\epsilon}(1)}$. As $w <_M w'$ and $w' \in W_{M^\epsilon}$ implies $w \in W_{M^\epsilon}$ (Lemma 2.6), the triangular decomposition (14) implies that $(E^{\epsilon'}_{o_M}(\tilde{w}))_{\tilde{w}\in W_{M^\epsilon}(1)}$ is a basis of $\mathcal{H}_{M^\epsilon}$. □

Lemma 2.18. *The ϵ-Bernstein basis satisfies $E^\epsilon(\tilde{w}) = T_{\tilde{w}}$ if $w \in \Lambda^\epsilon \cup W_0$.*

Proof. The inverse of $\Lambda^+ \cup W_0$ is $\Lambda^- \cup W_0$; hence

$$E^-(\tilde{w}) = \zeta(E((\tilde{w})^{-1})) = \zeta(T_{(\tilde{w})^{-1}}) = T_{\tilde{w}}. \qquad \square$$

The ϵ-Bernstein elements on $W_{M^\epsilon}(1)$ are compatible with θ and θ^*:

Proposition 2.19 [Ollivier 2010, Proposition 4.7; 2014, Lemma 3.8; Abe 2014, Lemma 4.5].

$$\theta(E^\epsilon_M(\tilde{w})) = \theta^*(E^\epsilon_M(\tilde{w})) = E^\epsilon(\tilde{w}) \quad \textit{for } \tilde{w} \in W_{M^\epsilon}(1).$$

Proof. It suffices to prove the proposition when the $\mathfrak{q}(s)$ are invertible. Let $\tilde{w} \in W(1)$. We write $\tilde{w} = \tilde{\lambda}\tilde{u} = \tilde{\lambda}_1(\tilde{\lambda}_2)^{-1}\tilde{u}$ with $u \in W_0$, and λ_1, λ_2 in Λ^ϵ. We have

$$E(\tilde{\lambda}_1)E((\tilde{\lambda}_2)^{-1}) = q_{\lambda_1,\lambda_2^{-1}}E(\tilde{\lambda}), \quad E(\tilde{\lambda}_2)E((\tilde{\lambda}_2)^{-1}) = q_{\lambda_2,\lambda_2^{-1}} = q_{\lambda_2},$$
$$E(\tilde{\lambda}_1)E((\tilde{\lambda}_2)^{-1})E(\tilde{u}) = q_{\lambda_1,\lambda_2^{-1}}q_{\lambda,u}E(\tilde{w}).$$

We suppose the $\mathfrak{q}(s)$ are invertible. Then,

$$E(\tilde{w}) = q_{\lambda_2}(q_{\lambda_1,\lambda_2^{-1}}q_{\lambda,u})^{-1}E(\tilde{\lambda}_1)E(\tilde{\lambda}_2)^{-1}E(\tilde{u}), \tag{15}$$
$$= q_{\lambda_2}(q_{\lambda_1,\lambda_2^{-1}}q_{\lambda,u})^{-1}\begin{cases} T_{\tilde{\lambda}_1}T_{\tilde{\lambda}_2}^{-1}T_{\tilde{u}} & \text{if } \epsilon = +, \\ T^*_{\tilde{\lambda}_1}(T^*_{\tilde{\lambda}_2})^{-1}T_{\tilde{u}} & \text{if } \epsilon = -. \end{cases}$$

We suppose now $w \in W_{M^\epsilon}$, that is, $\lambda \in \Lambda_{M^\epsilon}$, $u \in W_{M,0}$. Note $\Lambda^\epsilon \subset \Lambda_{M^\epsilon}$ and $q_{M,\lambda,u} = q_{\lambda,u}$ (Lemma 2.7). If $\epsilon = +$, we have

$$E_M(\tilde{w}) = q_{M,\lambda_2}(q_{M,\lambda_1,\lambda_2^{-1}}q_{\lambda,u})^{-1}T^M_{\tilde{\lambda}_1}(T^M_{\tilde{\lambda}_2})^{-1}T^M_{\tilde{u}},$$

and

$$\begin{aligned}\theta(E_M(\tilde{w})) &= q_{M,\lambda_2}(q_{M,\lambda_1,\lambda_2^{-1}}q_{\lambda,u})^{-1}T_{\tilde{\lambda}_1}T_{\tilde{\lambda}_2}^{-1}T_{\tilde{u}} \\ &= q_{M,\lambda_2}(q_{M,\lambda_1,\lambda_2^{-1}}q_{\lambda,u})^{-1}q_{\lambda_2}^{-1}q_{\lambda_1,\lambda_2^{-1}}q_{\lambda,u}E(\tilde{w}) \\ &= q_{M,\lambda_2}(q_{M,\lambda_1,\lambda_2^{-1}}q_{\lambda_2})^{-1}q_{\lambda_1,\lambda_2^{-1}}E(\tilde{w}).\end{aligned}$$

The triangular decomposition of $E_M(\tilde{w})$ and $E(\tilde{w})$ implies

$$q_{M,\lambda_2}(q_{M,\lambda_1,\lambda_2^{-1}}q_{\lambda_2})^{-1}q_{\lambda_1,\lambda_2^{-1}} = 1$$

and $\theta(E_M(\tilde{w})) = E(\tilde{w})$ for $w \in W_{M^+}$. If $\epsilon = -$, the same argument applied to θ^* gives $\theta^*(E_M(\tilde{w})) = E(\tilde{w})$ for $w \in W_{M^-}$.

By Remark 2.12, $\zeta \circ \theta = \theta \circ \zeta_M$, $\zeta \circ \theta^* = \theta \circ \zeta_M^*$, $W_{M^{-\epsilon}}$ is the inverse of W_{M^ϵ} and $E^-(\tilde{w}) = \zeta(E((\tilde{w})^{-1}))$. Hence for $w \in W_{M^-}$,

$$E^-(\tilde{w}) = (\zeta \circ \theta)(E_M((\tilde{w})^{-1})) = (\theta \circ \zeta_M)(E_M((\tilde{w})^{-1})) = \theta(E_M^-(\tilde{w})).$$

Similarly, for $w \in W_{M^+}$, we have $E^-(\tilde{w}) = \theta^*(E_M^-(\tilde{w}))$. □

2D. *w_0-twist.* Let $S_M \subset S$, w_0 denote the longest element of W_0 and $S_{w_0(M)} = w_0 S_M w_0 \subset w_0 S w_0 = S$. The longest element $w_{M,0}$ of $W_{M,0}$ satisfies $w_{M,0}(\Sigma_M^\epsilon) = \Sigma_M^{-\epsilon}$, and $w_{M,0}(\Sigma^\epsilon - \Sigma_M^\epsilon) = \Sigma^\epsilon - \Sigma_M^\epsilon$. The longest element $w_{w_0(M),0}$ of $W_{w_0(M),0}$ is $w_0 w_{M,0} w_0$.

Let $w_0^M := w_0 w_{M,0}$. Its inverse ${}^M w_0 := w_{M,0} w_0$ is $w_0^{w_0(M)}$ and $w_0^M(\Sigma_M^\epsilon) = \Sigma_{w_0(M)}^\epsilon$. This implies that $w_0^M(\Sigma_M^{\mathrm{aff},\epsilon}) = \Sigma_{w_0(M)}^{\mathrm{aff},\epsilon}$. Indeed the image by w_0^M of the simple roots of Σ_M is the set of simple roots of $\Sigma_{w_0(M)}$, and this remains true for the simple affine roots which are not roots. Note that the irreducible components $\Sigma_{M,i}$ of Σ_M have a unique highest root $a_{M,i}$, and that the $-a_{M,i}+1$ are the simple affine roots of Σ which are not roots. We have $w_0^M(-a_{M,i}+1) = w_0 w_{M,0}(-a_{M,i}+1) = w_0(a_{M,i})+1$. The irreducible components of $\Sigma_{w_0(M)}$ are the $w_0(\Sigma_{M,i})$ and $-w_0(a_{M,i})$ is the highest root of $w_0(\Sigma_{M,i})$.

We deduce

$$w_0^M S_M^{\mathrm{aff}}(w_0^M)^{-1} = S_{w_0(M)}^{\mathrm{aff}},$$
$$w_0^M W_{M,0}^{\mathrm{aff}}(w_0^M)^{-1} = W_{w_0(M,)0}^{\mathrm{aff}}, \quad w_0^M W_{M,0}(w_0^M)^{-1} = W_{w_0(M,)0}.$$

We have $\Lambda = w_0^M \Lambda (w_0^M)^{-1}$ and $w_0^M \Lambda_M^{\epsilon}(w_0^M)^{-1} = \Lambda_{w_0(M)}^{-\epsilon}$. Recalling $W_M = \Lambda \rtimes W_{M,0}$, $W_{M^\epsilon} = \Lambda_{M^\epsilon} \rtimes W_{M,0}$ and the group Ω_M of elements which stabilize $\mathfrak{A}_M$, we deduce

$$w_0^M W_M (w_0^M)^{-1} = W_{w_0(M)},$$
$$w_0^M \Omega_M (w_0^M)^{-1} = \Omega_{w_0(M)}, \quad w_0^M W_{M^\epsilon}(w_0^M)^{-1} = W_{w_0(M)}^{-\epsilon}. \tag{16}$$

Let ν_M denote the action of W_M on V_M and $\mathfrak{A}_M$ the dominant alcove of $(V_M, \mathfrak{H}_M)$. The linear isomorphism

$$V_M \xrightarrow{w_0^M} V_{w_0(M)}, \qquad \langle \alpha, x\rangle = \langle w_0^M(\alpha), w_0^M(x)\rangle \quad \text{for } \alpha \in \Sigma_M,$$

satisfies

$$w_0^M \circ \nu_M(w) = \nu_{w_0(M)}(w_0^M w (w_0^M)^{-1}) \circ w_0^M \quad \text{for } w \in W_M.$$

It induces a bijection $\mathfrak{H}_M \to \mathfrak{H}_{w_0(M)}$ sending $\mathfrak{A}_M$ to $\mathfrak{A}_{w_0(M)}$, a bijection $\mathfrak{D}_M \mapsto w_0^M(\mathfrak{D}_M)$ between the Weyl chambers, and a bijection $o_M \mapsto w_0^M(o_M)$ between the orientations such that $\mathfrak{D}_{w_0^M(o_M)} = w_0^M(\mathfrak{D}_{o_M})$.

Proposition 2.20. *Let $\tilde{w}_0^M \in W_0(1)$ be a lift of w_0^M. The R-linear map*

$$\mathcal{H}_M \xrightarrow{j} \mathcal{H}_{w_0(M)}, \qquad T_{\tilde{w}}^M \mapsto T^{w_0(M)}_{\tilde{w}_0^M \tilde{w} (\tilde{w}_0^M)^{-1}} \quad \textit{for } \tilde{w} \in W_M(1),$$

is an R-algebra isomorphism sending $\mathcal{H}_{M^\epsilon}$ onto $\mathcal{H}_{w_0(M)^{-\epsilon}}$ and respecting the ϵ'-alcove walk basis

$$j(E_{o_M}^{\epsilon'}(\tilde{w})) = E^{\epsilon'}_{w_0^M(o_M)}(\tilde{w}_0^M \tilde{w} (\tilde{w}_0^M)^{-1}) \quad \textit{for } \tilde{w} \in W_M(1)$$

for any orientation o_M of $(V_M, \mathfrak{H}_M)$ and $\epsilon, \epsilon' \in \{+,-\}$.

Proof. The proof is formal using the properties given above the proposition and the characterization of the elements in the ϵ'-alcove walks bases given by (5), (6), (7) if $\epsilon' = +$ and (11), (12) if $\epsilon' = -$. □

We study now the transitivity of the w_0-twist. Let $S_M \subset S_{M'} \subset S$. We have the subset $w_{M',0} S_M w_{M',0} = S_{w_{M',0}(M)}$ of S and we associate to the conjugation by a lift $\tilde{w}_{M',0}$ of $w_{M',0}$ in $W(1)$ an isomorphism $\mathcal{H}_M \xrightarrow{j'} \mathcal{H}_{w_{M',0}(M)}$ similar to $\mathcal{H}_M \xrightarrow{j} \mathcal{H}_{w_0(M)}$ in Proposition 2.20. We will show that j factorizes by j'.

We have $w_0^M = w_0^{M'} w_{M'}^M$, where $w_{M'}^M := w_{M',0} w_{M,0}$ (equal to w_0^M if $S = S_{M'}$),

$$W_{w_{M',0}(M)} = w_{M'}^M W_M (w_{M'}^M)^{-1},$$
$$W_{w_0(M)} = w_0^{M'} W_{w_{M',0}(M)} (w_0^{M'})^{-1} = w_0^M W_M (w_0^M)^{-1}.$$

For $S_{M_1} \subset S_{M'}$, let $W_{M_1^{\epsilon,M'}} \subset W_{M_1}$ denote the submonoid associated to $S_{M'}^{\mathrm{aff}}$ as in Definition 2.1 and replace the pair $(\Sigma^+ - \Sigma^+_{M_1}, \Sigma^{\mathrm{aff},+})$ by $(\Sigma^+_{M'} - \Sigma^+_{M_1}, \Sigma^{\mathrm{aff},+}_{M'})$. We note that

$$W_{w_{M',0}(M)^{-\epsilon,M'}} = w_{M'}^M W_{M^\epsilon} (w_{M'}^M)^{-1},$$
$$W_{w_0(M)^{-\epsilon}} = w_0^{M'} W_{w_{M',0}(M)^{-\epsilon,M'}} (w_0^{M'})^{-1} = w_0^M W_{M^\epsilon} (w_0^M)^{-1}.$$

Let $\tilde{w}_0^M, \tilde{w}_0^{M'}, \tilde{w}_{M'}^M$ be in $W_0(1)$ lifting $w_0^M, w_0^{M'}, w_{M'}^M$ and satisfying $\tilde{w}_0^M = \tilde{w}_0^{M'} \tilde{w}_{M'}^M$. The algebra isomorphisms

$$\mathcal{H}_M \xrightarrow{j'} \mathcal{H}_{w_{M',0}(M)}, \quad \mathcal{H}_{M'} \xrightarrow{j''} \mathcal{H}_{w_0(M')}, \quad \mathcal{H}_M \xrightarrow{j} \mathcal{H}_{w_0(M)}$$

defined by $\tilde{w}_{M'}^M$, $\tilde{w}_0^{M'}$, $\tilde{w}_0^M$ respectively, as in Proposition 2.20, send the ϵ-subalgebra to the $-\epsilon$-subalgebra and are compatible with the ϵ'-Bernstein bases. We cannot compose j' with the map j'' defined by $\tilde{w}_0^{M'}$, but we can compose j' with the bijective R-linear map defined by the conjugation by $\tilde{w}_0^{M'}$ in $W(1)$

$$\mathcal{H}_{w_{M',0}(M)} \xrightarrow{k''} \mathcal{H}_{w_0(M)}, \qquad T_{\tilde{w}}^{w_{M',0}(M)} \mapsto T^{w_0(M)}_{\tilde{w}_0^{M'} \tilde{w} (\tilde{w}_0^{M'})^{-1}} \quad \text{for } \tilde{w} \in W_{w_{M',0}(M)}(1).$$

Proposition 2.21. *We have $j = k'' \circ j'$ and k'' is an R-algebra isomorphism respecting the ϵ-subalgebras and the ϵ-Bernstein bases*: $k''(\mathcal{H}_{w_{M',0}(M)^\epsilon}) = \mathcal{H}_{w_0(M)^\epsilon}$ *and* $k''(E^\epsilon_{w_{M',0}(M)}(\tilde{w})) = E^\epsilon_{w_0(M)}(\tilde{w}_0^{M'} \tilde{w} (\tilde{w}_0^{M'})^{-1})$ *for* $\epsilon \in \{+, -\}$, $w \in W_{w_{M',0}(M)}$.

Proof. The relations between the groups W_* and W_{*^ϵ} imply obviously that $j = k'' \circ j'$ and that k'' respects the ϵ-subalgebras.

Now, k'' is an algebra isomorphism respecting the ϵ'-Bernstein bases because j, j' are algebra isomorphisms respecting the ϵ'-Bernstein bases and $k'' = j \circ (j')^{-1}$. □

2E. *Distinguished representatives of W_0 modulo $W_{M,0}$.* The classical set ${}^M W_0$ of representatives on $W_{M,0} \backslash W_0$ is equal to ${}_M D_1 = {}_M D_2$, where

(17) $\qquad {}_M D_1 := \{d \in W_0 \mid d^{-1}(\Sigma^+_M) \in \Sigma^+\},$

(18) $\qquad {}_M D_2 := \{d \in W_0 \mid \ell(wd) = \ell(w) + \ell(d) \text{ for all } w \in W_{M,0}\}$

[Carter 1985, §2.3.3]. The properties of ${}^M W_0$ used in this article that we are going to prove are probably well known. Note that the classical set of representatives of $W_0 \backslash W$ is studied in [Vignéras 2015a], that $+$ can be replaced by $\epsilon \in \{+, -\}$ in the definition of ${}_M D_1$, that ${}^M w_0 = w_{M,0} w_0 \in {}^M W_0$ and that ${}^M W_0 \cap S = S - S_M$.

Taking inverses, we get the classical set W_0^M of representatives on $W_0/W_{M,0}$ equal to $D_{M,1} = D_{M,2}$, where

$$D_{M,1} := \{d \in W_0 \mid d(\Sigma_M^+) \subset \Sigma^+\}, \tag{19}$$

$$D_{M,2} := \{d \in W_0 \mid \ell(dw) = \ell(d) + \ell(w) \text{ for all } w \in W_{M,0}\}. \tag{20}$$

The length of an element of W is equal to the length of its inverse, and [Vignéras 2013a, Corollary 5.10] gives that for $\lambda \in \Lambda$, $w \in W_0$,

$$\ell(\lambda w) = \sum_{\beta \in \Sigma^+ \cap w(\Sigma^+)} |\beta \circ \nu(\lambda)| + \sum_{\beta \in \Phi_w} |-\beta \circ \nu(\lambda) + 1|, \tag{21}$$

where $\Phi_w := \Sigma^+ \cap w(\Sigma^-)$. If $w = s_1 \cdots s_{\ell(w)}$ is a reduced decomposition in (W_0, S), $\Phi_w = \{\alpha_{s_1}\} \cup s_1(\Phi_{s_1 w})$ and $\ell(w)$ is the order of Φ_w. If $w \in W_{M,0}$, we have $\Phi_w \subset \Sigma_M^+$. Let $\ell_\beta(\lambda w)$ denote the contribution of $\beta \in \Sigma^+$ to the right side of (21).

We show now that $W_{M,0}$ can be replaced by W_{M^+} in (18) and by W_{M^-} in (20) (taking the inverses). It is also a variant of the equivalence $\ell(\lambda w) < \ell(\lambda) + \ell(w) \Leftrightarrow \beta \circ \nu(\lambda) > 0$ for some $\beta \in \Phi_w$ for λ, w as in (21).

Lemma 2.22.

(i)
$$\ell(wd) = \ell(w) + \ell(d) \quad \textit{for } w \in W_{M^+} \textit{ and } d \in {}^M W_0,$$
$$\ell(dw) = \ell(d) + \ell(w) \quad \textit{for } w \in W_{M^-} \textit{ and } d \in W_0^M.$$

(ii) *If* $\lambda \in \Lambda$, $w \in W_{M,0}$, $d \in {}^M W_0$, *then* $\ell(\lambda w d) < \ell(\lambda w) + \ell(d)$ *is equivalent to*

$$w(\beta) \circ \nu(\lambda) > 0 \quad \textit{and} \quad d^{-1}(\beta) \in \Sigma^- \quad \textit{for some } \beta \in \Sigma^+ - \Sigma_M^+.$$

Proof. [Ollivier 2010, Lemma 2.3; Abe 2014, Lemma 4.8]. Let $\lambda \in \Lambda$, $w \in W_{M,0}$, $d \in {}^M W_0$ and $\beta \in \Sigma^+$.

Suppose $\beta \in \Sigma_M^+$. Then $\ell_\beta(d) = 0$, $\Phi_d = \varnothing$ because $d^{-1}(\Sigma_M^\epsilon) \subset \Sigma^\epsilon$ by (17), and $\ell_\beta(\lambda w d) = \ell_\beta(\lambda w)$ because $w^{-1}(\beta) \in \Sigma^\epsilon \Leftrightarrow w^{-1}(\beta) \in \Sigma_M^\epsilon \Rightarrow d^{-1} w^{-1}(\beta) \in \Sigma^\epsilon$ by (17).

Suppose $\beta \in \Sigma^+ - \Sigma_M^+$. Then $w^{-1}(\beta) \in \Sigma^+ - \Sigma_M^+$ and $\ell_\beta(\lambda w) = |\beta \circ \nu(\lambda)|$.

The number $\ell(d)$ of $\beta \in \Sigma^+ - \Sigma_M^+$ such that $d^{-1}(\beta) \in \Sigma^-$ is equal to the number of $\beta \in \Sigma^+ - \Sigma_M^+$ such that $(wd)^{-1}(\beta) \in \Sigma^-$.

When $\lambda \in \Lambda_{M^+}$ and $(wd)^{-1}(\beta) \in \Sigma^-$, we have $\beta \circ \nu(\lambda) \leq 0$ and $\ell_\beta(\lambda w d) = |\beta \circ \nu(\lambda)| + 1$. Therefore $\ell(\lambda w d) = \ell(\lambda w) + \ell(d)$, which gives (i).

When $\lambda \notin \Lambda - \Lambda_{M^+}$, $\ell(\lambda w d) < \ell(\lambda w) + \ell(d)$ if and only if there exists $\beta \in \Sigma^+ - \Sigma_M^+$ such that $\beta \circ \nu(\lambda) > 0$ and $d^{-1} w^{-1}(\beta) \in \Sigma^-$. This gives (ii) because $\beta \mapsto w^{-1}(\beta)$ is a permutation map of $\Sigma^+ - \Sigma_M^+$. □

Lemma 2.23. (i) *For* $\lambda \in \Lambda$, $w \in W_0$, *we have* $q_\lambda = q_{w\lambda w^{-1}}$, $q_w = q_{w_0 w w_0}$, *and*

$$\ell(w_0) = \ell(w) + \ell(w^{-1} w_0) = \ell(w_0 w^{-1}) + \ell(w).$$

(ii) *For* $w \in W_{M,0}$, *we have* $q_w = q_{w_0^M w (w_0^M)^{-1}}$.

Proof. (i) See [Vignéras 2013a, Proposition 5.13]. The length on W_0 is invariant by inverse and by conjugation by w_0 because $w_0 S w_0 = S$ and by [Bourbaki 1968, VI, §1, Corollaire 3].

(ii) We have $q_w = q_{w_{M,0} w w_{M,0}^{-1}} = q_{w_0^M w (w_0^M)^{-1}}$ for $w \in W_{M,0}$. □

Lemma 2.24. $$W_0^M = W_0^{w_0(M)} w_0^M = w_0 W_0^M w_{M,0}.$$

Proof. By (19),

$$d \in W_0^M \Longleftrightarrow d(\Sigma_M^+) \subset \Sigma^+ \Longleftrightarrow d(w_0^M)^{-1}(\Sigma_{w_0(M)}^+) \subset \Sigma^+ \Longleftrightarrow d(w_0^M)^{-1} \in W_0^{w_0(M)}.$$

This proves the equality $W_0^M = W_0^{w_0(M)} w_0^M$. The equality $W_0^M = w_0 W_0^M w_{M,0}$, follows from

$$\begin{aligned} d(w_0^M)^{-1}(\Sigma_{w_0(M)}^+) \subset \Sigma^+ &\Longleftrightarrow w_0 d w_{M,0} w_0(\Sigma_{w_0(M)}^+) \subset \Sigma^- \\ &\Longleftrightarrow w_0 d w_{M,0}(\Sigma_M^-) \subset \Sigma^- \Longleftrightarrow w_0 d w_{M,0} \in W_0^M. \end{aligned}$$ □

Remark 2.25. $W_M = \Lambda \rtimes W_{M,0}$ but $q_{\lambda w} = q_{w_0^M \lambda w (w_0^M)^{-1}}$ could be false for $\lambda \in \Lambda$, $w \in W_{M,0}$ such that $\ell(\lambda w) < \ell(\lambda) + \ell(w)$.

Lemma 2.26. *We have* $\ell(w_0^M) = \ell(w_0^M d^{-1}) + \ell(d)$ *for any* $d \in W_0^M$.

Proof. For $d \in W_0^M$, we have $\ell(d w_{M,0}) = \ell(d) + \ell(w_{M,0})$ by (20) and $w = w_0^M d^{-1}$ satisfies $w_0 = w d w_{M,0}$ and $\ell(w_0) = \ell(w) + \ell(d w_{M,0})$. We have $w_0^M = w_0 w_{M,0} = wd$ and $\ell(w_0^M) = \ell(w_0) - \ell(w_{M,0}) = \ell(w) + \ell(d)$. □

The Bruhat order $x \leq x'$ in W_0 is defined by the following equivalent two conditions:

(i) There exists a reduced decomposition of x' such that by omitting some terms one obtains a reduced decomposition of x.

(ii) For any reduced decomposition of x', by omitting some terms one obtains a reduced decomposition of x.

A reduced decomposition of $w \in W_0$ followed by a reduced decomposition of $w' \in W_0$ is a reduced decomposition of ww' if and only $\ell(ww') = \ell(w) + \ell(w')$. A reduced decomposition of $d \in W_0^M$ cannot end by a nontrivial element $w \in W_{M,0}$.

Lemma 2.27. *For* $w, w' \in W_{M,0}$, $d, d' \in W_0^M$, *we have* $dw \leq d'w'$ *if and only if there exists a factorisation* $w = w_1 w_2$ *such that* $\ell(w) = \ell(w_1) + \ell(w_2)$, $dw_1 \leq d'$ *and* $w_2 \leq w'$.

Proof. We prove the direction "only if" (the direction "if" is obvious). If $dw \leq d'w'$, a reduced decomposition of dw is obtained by omitting some terms of the product of a reduced decomposition of d' and of a reduced decomposition of w'. We have $dw = d_1 w_2$ with $d_1 \leq d'$, $w_2 \leq w'$ and $\ell(d_1 w_2) = \ell(d_1) + \ell(w_2)$. We have $d_1 =$

dw_1, $w_1 := ww_2^{-1}$. As $w, w_2 \in w_{M,0}$ and $d \in W_0^M$, we have $\ell(dw_1) = \ell(d) + \ell(w_1)$ and $\ell(dw) = \ell(d) + \ell(w)$. Hence $\ell(w_1) + \ell(w_2) = \ell(w)$. □

Lemma 2.28. *Let $d' \in {}^{w_0(M)}W_0$, $d \in W_0^M$.*

(i) *If there exists $u \in W_{M,0}$, $u' \in W_0^M$ such that $v = w_0^M u \leq w = du'$, then $d = w_0^M$.*

(ii) *We have $d'd \in w_0^M W_{M,0}$ if and only if $d'd = w_0^M$.*

Proof. (i) As $\ell(w) = \ell(d) + \ell(u')$, we have $u = u_1 u_2$ with $w_0^M u_1 \leq d$, $u_2 \leq u'$ and $u_1, u_2 \in W_{M,0}$ (Lemma 2.27). We have

$$\ell(w_0^M u_1) = \ell(w_0^M) + \ell(u_1) = \ell(w_0^M d^{-1}) + \ell(d) + \ell(u_1)$$

(Lemma 2.26). Hence $d = w_0^M$, $u_1 = 1$.

(ii) If there exists $u \in W_{M,0}$ such that $d = d'^{-1} w_0^M u$, we have $d = d'^{-1} w_0^M$ because $d'^{-1} w_0^M \in W_0^M$ (Lemma 2.24). □

2F. *$\mathcal{H}$ as a left $\theta(\mathcal{H}_{M^+})$-module and as a right $\theta^*(\mathcal{H}_{M^-})$-module.* We prove Theorem 1.4(iv) on the structure of the left $\theta(\mathcal{H}_{M^+})$-module $\mathcal{H}$ and its variant for the right $\theta^*(\mathcal{H}_{M^-})$-module $\mathcal{H}$. We suppose $S_M \neq S$.

Recalling the properties (i), (ii), (iii) of Theorem 1.4, $\mathcal{H}_M = \mathcal{H}_{M^+}[(T^M_{\tilde{\mu}_M})^{-1}]$ is the localisation of the subalgebra $\mathcal{H}_{M^+}$ at the central element $T^M_{\tilde{\mu}_M}$. The algebra $\mathcal{H}_{M^+}$ embeds in $\mathcal{H}$ by θ. Recalling (17), (18) we choose a lift $\tilde{d} \in W(1)$ for any element d in the classical set of representatives ${}^M W_0$ of $W_{M,0} \backslash W_0$. We define

$$\mathcal{V}_{M^+} = \sum_{d \in {}^M W_0} \theta(\mathcal{H}_{M^+}) T_{\tilde{d}}. \tag{22}$$

Proposition 2.29. (i) *$\mathcal{V}_{M^+}$ is a free left $\theta(\mathcal{H}_{M^+})$-module of basis $(T_{\tilde{d}})_{d \in {}^M W_0}$.*

(ii) *For any $h \in \mathcal{H}$, there exists $r \in \mathbb{N}$ such that $T^r_{\tilde{\mu}_M} h \in \mathcal{V}_{M^+}$.*

(iii) *If $\mathfrak{q} = 0$, $T_{\tilde{\mu}_M}$ is a left and right zero divisor in $\mathcal{H}$.*

For $\mathrm{GL}(n, F)$, (ii) is proved in [Ollivier 2010, Proposition 4.7] for $(\mathfrak{q}(s)) = (0)$. When the $\mathfrak{q}(s)$ are invertible, $T_{\tilde{w}}$ is invertible in $\mathcal{H}$ for $\tilde{w} \in W(1)$.

Proof. (i) As ${}^M W_0$ is a set of representatives of $W_{M^+} \backslash W$, a set of representatives of $W_{M^+}(1) \backslash W(1)$ is the set $\{\tilde{d} \mid d \in {}^M W_0\}$ of lifts of ${}^M W_0$ in $W(1)$. The canonical bases of $\mathcal{H}_{M^+}$ and of $\mathcal{H}$ are respectively $(T_{\tilde{w}})_{(\tilde{w}) \in W_{M^+}(1)}$ and $(T_{\tilde{w}\tilde{d}})_{(\tilde{w},d) \in W_{M^+}(1) \times {}^M W_0}$, and $T_{\tilde{w}\tilde{d}} = T_{\tilde{w}} T_{\tilde{d}}$ by the additivity of lengths (Lemma 2.22).

(ii) We can suppose that h runs over in a basis of $\mathcal{H}$. We cannot take the Iwahori–Matsumoto basis $(T_{\tilde{w}})_{\tilde{w} \in W(1)}$ and we explain why. For $\tilde{w} = \tilde{w}_M \tilde{d}$ with $\tilde{w}_M \in W_{M^+}(1)$, $d \in {}^M W_0$, we choose $r \in \mathbb{N}$ such that $\tilde{\mu}^r_M \tilde{w}_M \in W_{M^+}(1)$. By the length additivity (Lemma 2.22) $T_{\tilde{\mu}^r_M \tilde{w}} = T_{\tilde{\mu}^r_M \tilde{w}_M} T_{\tilde{d}}$ lies in $\theta(\mathcal{H}_{M^+}) T_{\tilde{d}}$, but we cannot deduce that $T_{\tilde{\mu}^r_M} T_{\tilde{w}}$ lies in $\theta(\mathcal{H}_{M^+}) T_{\tilde{d}}$.

We take the Bernstein basis satisfying Lemma 2.18 and we suppose that $\mathfrak{q}(s)=\boldsymbol{q}_s$ is indeterminate (but not invertible) with the same arguments as in [Ollivier 2010, Proposition 4.8]. Then $E(\tilde{d})=T_{\tilde{d}}$ for $d\in {}^MW_0$. If we prove that $E(\tilde{\mu}_M^r\tilde{w})$ lies in $\theta(\mathcal{H}_{M^+})T_{\tilde{d}}$ then $E(\tilde{\mu}_M)^rE_o(\tilde{w})=\boldsymbol{q}_{\mu_M^r,w}E(\tilde{\mu}_M^r\tilde{w})$ lies also in $\theta(\mathcal{H}_{M^+})T_{\tilde{d}}$. This implies $T^r_{\tilde{\mu}_M}E_o(\tilde{w})\in\theta(\mathcal{H}_{M^+})T_{\tilde{d}}$.

Now we prove $E(\tilde{\mu}_M^r\tilde{w})\in\theta(\mathcal{H}_{M^+})T_{\tilde{d}}$. We write $\tilde{w}_M=\tilde{\lambda}\tilde{w}_{M,0}$, $\tilde{\lambda}\in\Lambda(1)$, $\tilde{w}_{M,0}\in W_{M,0}(1)$. Recalling $E(*)=T_*$ for $*\in W_0(1)$ and the additivity of the length (Lemma 2.22),

$$\boldsymbol{q}_{\mu_M^r\lambda,w_{M,0}d}E(\tilde{\mu}_M^r\tilde{w})=E(\tilde{\mu}_M^r\tilde{\lambda})E(\tilde{w}_{M,0}\tilde{d})=E(\tilde{\mu}_M^r\tilde{\lambda})T_{\tilde{w}_{M,0}\tilde{d}}=E(\tilde{\mu}_M^r\tilde{\lambda})T_{\tilde{w}_{M,0}}T_{\tilde{d}}$$
$$=\boldsymbol{q}_{\mu_M^r\lambda,w_{M,0}}E(\tilde{\mu}_M^r\tilde{w}_M)T_{\tilde{d}}.$$

The monoid W_{M^ϵ} is a lower subset of $(W_M,\leq_M)$ (Lemma 2.6). The triangular decomposition (14) implies $E_M(\tilde{\mu}_M^r\tilde{w}_M)\in\mathcal{H}_{M^+}$. By Proposition 2.19, $E(\tilde{\mu}_M^r\tilde{w}_M)\in\theta(\mathcal{H}_{M^+})$ and by the additivity of the length (Lemma 2.22),

$$\boldsymbol{q}_{w_{M,0}d}=\boldsymbol{q}_{w_{M,0}}\boldsymbol{q}_d,\quad \boldsymbol{q}_{\mu_M^r\lambda w_{M,0}d}=\boldsymbol{q}_{\mu_M^r\lambda w_{M,0}}\boldsymbol{q}_d,$$

implying

$$\boldsymbol{q}_{\mu_M^r\lambda}\boldsymbol{q}_{w_{M,0}d}\boldsymbol{q}^{-1}_{\mu_M^r\lambda w_{M,0}d}=\boldsymbol{q}_{\mu_M^r\lambda}\boldsymbol{q}_{w_{M,0}}\boldsymbol{q}^{-1}_{\mu_M^r\lambda w_{M,0}};$$

hence $\boldsymbol{q}_{\mu_M^r\lambda,w_{M,0}d}=\boldsymbol{q}_{\mu_M^r\lambda,w_{M,0}}$.

(iii) We have $\ell(\mu_M)\neq 0$ and equivalently, $\nu(\mu_M)\neq 0$ in V. We choose $w\in W_0$ with $w(\nu(\mu_M))\neq\nu(\mu_M)$. Then $\nu(w\mu_Mw^{-1})=w(\nu(\mu_M))$ and $\nu(\mu_M)$ belong to different Weyl chambers. The alcove walk basis $(E_o(\tilde{w}))_{\tilde{w}\in W(1)}$ of $\mathcal{H}$ associated to an orientation o of V of Weyl chamber containing $\nu(\mu_M)$ satisfies

$$E_o(\tilde{\mu}_M)=T_{\tilde{\mu}_M},$$
$$E_o(\tilde{\mu}_M)E_o(\tilde{w}\tilde{\mu}_M\tilde{w}^{-1})=E_o(\tilde{w}\tilde{\mu}_M\tilde{w}^{-1})E_o(\tilde{\mu}_M)=0. \tag{23}$$
□

The properties of the left $\theta(\mathcal{H}_{M^+})$-module $\mathcal{H}$ transfer to properties of the right $\theta^*(\mathcal{H}_{M^-})$-module $\mathcal{H}$, with the involutive antiautomorphism $\zeta\circ\iota$ of $\mathcal{H}$ (Remark 2.12) exchanging $T_{\tilde{w}}$ and $(-1)^{\ell(w)}T^*_{(\tilde{w})^{-1}}$ for $\tilde{w}\in W(1)$, $\theta(\mathcal{H}_{M^+})$ and $\theta^*(\mathcal{H}_{M^-})$, $\mathcal{V}_{M^+}$ and

$$\mathcal{V}^*_{M^-}:=\sum_{d\in W_0^M}T^*_{\tilde{d}}\theta^*(\mathcal{H}_{M^-}), \tag{24}$$

where $W_0^M=\{d'^{-1}\mid d'\in {}^MW_0\}$ is the set of classical representatives of $W_0/W_{M,0}$ (19), and $\tilde{d}=(\tilde{d}')^{-1}$ if $d=d'^{-1}$.

Corollary 2.30. (i) *$\mathcal{V}^*_{M^-}$ is a free right $\theta^*(\mathcal{H}_{M^-})$-module of basis $(T^*_{\tilde{d}})_{d\in W_0^M}$.*

(ii) *For any $h\in\mathcal{H}$, there exists $r\in\mathbb{N}$ such that $h(T^*_{(\tilde{\mu}_M)^{-1}})^r\in\mathcal{V}^*_{M^-}$.*

(iii) *If $\mathfrak{q}=0$, $T^*_{\tilde{\mu}_M^{-1}}$ is a left and right zero divisor in $\mathcal{H}$.*

3. Induction and coinduction

3A. *Almost localisation of a free module.* In this chapter, all rings have unit elements.

Definition 3.1. Let A be a ring and $a \in A$ a central nonzero divisor. We say that a left A-module B is an almost a-localisation of a left A-module $B_D \subset B$ of basis D when:

(i) D is a finite subset of B, and the map $\oplus_{d \in D} A \to B$, $(x_d) \to \sum x_d d$, is injective,

(ii) for any $b \in B$, there exists $r \in \mathbb{N}$ such that $a^r b$ lies in $B_D := \sum_{d \in D} Ad$.

Example 3.2. Our basic example is $(A, a, B, D) = (\mathcal{H}_{M^+}, T_{\mu_M}, \mathcal{H}, (T_{\tilde{d}})_{d \in {}^M W_0})$ (Proposition 2.29).

As a is central and not a zero divisor in A, the a-localisation of A is ${}_a A = A_a = \cup_{n \in \mathbb{N}} A a^{-n}$. The left multiplication by a in A is an injective A-linear endomorphism $A \to A$, $x \mapsto ax$, and the left multiplication by a in B is an A-linear endomorphism $a_B : x \mapsto ax$ of B which may be not injective; hence B may be not a flat A-module. The ring B is the union for $r \in \mathbb{N}$ of the A-submodules

$$ {}_r B_D := \{b \in B \mid a^r b \in B_D\}, $$

and looks like a localisation of B_D at a.

Definition 3.3. Let A be a ring and $a \in A$ a central nonzero divisor. We say that a right A-module B is an almost a-localisation of a right A-module ${}_D B$ of basis D if:

(i) D is a finite subset of B, and the map $\oplus_{d \in D} A \to B$, $(x_d) \to \sum d\, x_d$, is injective,

(ii) for any $b \in B$, there exists $r \in \mathbb{N}$ such that $ba^r \in {}_D B := \sum_{d \in D} dA$.

The ring B is the union for $r \in \mathbb{N}$ of the A-submodules

$$ {}_D B_r = \{b \in B \mid ba^r \in {}_D B\}. $$

Example 3.4. Our basic example is $(A, a, B, D) = (\mathcal{H}_{M^-}, T_{\mu_M^{-1}}, \mathcal{H}, (T_{\tilde{d}})_{d \in W_0^M})$ (Corollary 2.30).

We note that $(A_a, B) = (\mathcal{H}_M, \mathcal{H})$ in Example 3.2 and in Example 3.4.

3B. *Induction and coinduction.*

3B1. For a ring A, let Mod_A denote the category of right A-modules, and ${}_A\mathrm{Mod}$ the category of left A-modules. The A-duality $X \mapsto X^* := \mathrm{Hom}_A(X, A)$ exchanges left and right A-modules.

A functor from Mod_A to a category admits a left adjoint if and only if it is left exact and commutes with small direct products (small projective limits); it admits a right adjoint if and only if it is right exact and commutes with small direct sums (small injective limits) [Vignéras 2013b, Proposition 2.10].

For two rings $A \subset B$, we define two functors

$$\text{the induction } I_A^B := - \otimes_A B,$$
$$\text{the coinduction } \mathbb{I}_A^B := \text{Hom}_A(B, -) : \text{Mod}_A \to \text{Mod}_B,$$

where B is seen as an (A, B)-module for the induction, and as a (B, A)-module for the coinduction. For $\mathcal{M} \in \text{Mod}_A$, we have $(m \otimes x)b = m \otimes xb$, $(fb)(x) = f(bx)$ if $x, b \in B$ and $m \in \mathcal{M}$, $f \in \text{Hom}_A(B, \mathcal{M})$.

The restriction $\text{Res}_A^B : \text{Mod}_B \to \text{Mod}_A$ is equal to $\text{Hom}_B(B, -) = - \otimes_B B$, where B is seen first as an (A, B)-module and then as a (B, A)-module. The induction and the coinduction are the left and right adjoints of the restriction [Benson 1998, §2.8.2].

For two rings A and B and an (A, B)-module $\mathcal{J}$, the functor

$$- \otimes_A \mathcal{J} : \text{Mod}_A \to \text{Mod}_B \text{ is left adjoint to } \text{Hom}_B(\mathcal{J}, -) : \text{Mod}_B \to \text{Mod}_A.$$

Let $\mathcal{M} \in \text{Mod}_A$, $\mathcal{N} \in \text{Mod}_B$. The adjunction is given by the functorial isomorphism

$$\text{Hom}_B(\mathcal{M} \otimes_A \mathcal{J}, \mathcal{N}) \xrightarrow{\alpha} \text{Hom}_A(\mathcal{M}, \text{Hom}_B(\mathcal{J}, \mathcal{N})), \quad f(m \otimes x) = \alpha(f)(m)(x),$$

for $f \in \text{Hom}_B(\mathcal{M} \otimes_A \mathcal{J}, \mathcal{N})$, $m \in \mathcal{M}$, $x \in \mathcal{J}$ [Benson 1998, Lemma 2.8.2].

For three rings $A \subset B$, $A \subset C$, the isomorphism α applied to $\mathcal{M} = C$, $\mathcal{J} = B$ gives an isomorphism

$$\text{Hom}_B(C \otimes_A B, -) \simeq \text{Hom}_A(C, -) : \text{Mod}_B \to \text{Mod}_C .$$

3B2. Let $A \subset B$ be two rings and $a \in A$ a central nonzero divisor. Let $A_a = A[a^{-1}]$ denote the localisation of A at a. There is a natural inclusion $A \subset A_a$. The restriction $\text{Mod}_{A_a} \to \text{Mod}_A$ identifies Mod_{A_a} with the A-modules where the action of a is invertible. For $\mathcal{M}$, $\mathcal{M}'$ in Mod_{A_a}, we have

$$(25) \qquad \text{Hom}_{A_a}(\mathcal{M}, \mathcal{M}') = \text{Hom}_A(\mathcal{M}, \mathcal{M}'), \quad \mathcal{M} \otimes_{A_a} \mathcal{M}' = \mathcal{M} \otimes_A \mathcal{M}'.$$

For $f \in \text{Hom}_A(\mathcal{M}, \mathcal{M}')$, $m \in \mathcal{M}$, $m' \in \mathcal{M}'$, we have $f(aa^{-1}m) = af(a^{-1}m) \Rightarrow a^{-1}f(m) = f(a^{-1}m)$, and $m \otimes a^{-1}m' = ma^{-1}a \otimes a^{-1}m' = ma^{-1} \otimes m'$ in $\mathcal{M} \otimes_A \mathcal{M}'$. We view Mod_{A_a} as a full subcategory of Mod_A.

The restriction followed by the induction, respectively the coinduction, $\text{Mod}_A \to \text{Mod}_B$ defines an induction, respectively coinduction,

$$I_{A_a}^B = I_A^B \circ \text{Res}_A^{A_a} = - \otimes_A B, \quad \mathbb{I}_{A_a}^B = \mathbb{I}_A^B \circ \text{Res}_A^{A_a} = \text{Hom}_A(B, -) : \text{Mod}_{A_a} \to \text{Mod}_B,$$

even when A_a is not contained in B. The induction $I_{A_a}^B$ admits a right adjoint

$$\mathbb{I}_A^{A_a} \circ \text{Res}_A^B = \text{Hom}_A(A_a, -) : \text{Mod}_B \to \text{Mod}_{A_a}$$

because the restriction $\mathrm{Res}_A^{A_a}$ and the induction I_A^B admit a right adjoint: the coinduction $\mathbb{I}_A^{A_a}$ and the restriction Res_A^B. The coinduction $\mathbb{I}_{A_a}^B$ admits a left adjoint

$$I_A^{A_a} \circ \mathrm{Res}_A^B = - \otimes_A A_a : \mathrm{Mod}_B \to \mathrm{Mod}_{A_a}$$

because the restriction $\mathrm{Res}_A^{A_a}$ and the induction I_A^B admit a left adjoint: the induction $I_A^{A_a}$ and the corestriction Res_A^B.

When a is invertible in B, we have $A_a \subset B$ and they coincide with the induction and coinduction from A_a to B.

The induction and the coinduction of A_a seen as a right A_a-module, are the (A_a, B)-modules

$$I_{A_a}^B(A_a) = A_a \otimes_A B, \quad \mathbb{I}_{A_a}^B(A_a) = \mathrm{Hom}_A(B, A_a). \tag{26}$$

Lemma 3.5. *Let $\mathcal{M} \in \mathrm{Mod}_{A_a}$. Then $I_{A_a}^B(\mathcal{M}) = \mathcal{M} \otimes_{A_a} I_{A_a}^B(A_a)$ in Mod_B.*

Proof. $\mathcal{M} \otimes_A B = (\mathcal{M} \otimes_{A_a} A_a) \otimes_A B = \mathcal{M} \otimes_{A_a} (A_a \otimes_A B)$. ◻

3B3. Let (A, a, B, D) satisfy Definition 3.1. Let $\mathcal{M} \in \mathrm{Mod}_{A_a}$. As R-modules,

$$I_{A_a}^B(\mathcal{M}) = \mathcal{M} \otimes_A B_D \tag{27}$$

because the action of a on $\mathcal{M}$ is invertible; hence $\mathcal{M} \otimes_A {}_rB_D = \mathcal{M} \otimes_A B_D$ for $r \in \mathbb{N}$. In particular, we have the following:

Lemma 3.6. *The left A_a-module $I_{A_a}^B(A_a)$ is free of basis $(1 \otimes d)_{d \in D}$.*

Remark 3.7. The A-dual $(B_D)^*$ of the left A-module B_D is the right A-module $\oplus_{d \in D} d^* A$ of basis the dual basis $D^* = \{d^* \mid d \in D\}$ of D. Let $\mathcal{M} \in \mathrm{Mod}_{A_a}$. We have canonical isomorphisms of R-modules

$$\oplus_{d \in D} \mathcal{M} \xrightarrow{\simeq} \mathcal{M} \otimes_A B_D \xrightarrow{\simeq} \mathrm{Hom}_A((B_D)^*, \mathcal{M}),$$
$$(x_d) \mapsto \sum_{d \in D} x_d \otimes d \mapsto (d^* \mapsto x_d)_{d \in D}.$$

The tensor product over A by a free A-module is exact and faithful; hence the induction is exact and faithful.

Let $R \subset A$ be a subring central in B. The ring R is automatically commutative and a central subring of the localisation A_a of A. The modules over A_a or B are naturally R-modules.

Let $\mathcal{M} \in \mathrm{Mod}_{A_a}$ be a finitely generated R-module. The R-module $\mathcal{M} \otimes_{A_a} I_{A_a}^B(A_a)$ is finitely generated.

Let $\mathcal{N} \in \mathrm{Mod}_B$ be a finitely generated R-module. The R-module $\mathrm{Hom}_A(A_a, \mathcal{N})$ is finitely generated if R is a field by the Fitting lemma applied to the action of a on $\mathcal{N}$. There exists a positive integer n such that $\mathcal{N}$ is a direct sum $\mathcal{N} = \mathcal{N}_a \oplus \mathcal{N}'_a$, where a^n acts on $\mathcal{N}_a$ as an automorphism and a^n is 0 on $\mathcal{N}'_a$. Then, $\mathrm{Hom}_A(A_a, \mathcal{N}) \simeq \mathcal{N}_a$ is finite-dimensional.

We obtain the following:

Proposition 3.8. *Let (A, a, B, D) satisfy Definition 3.1. The induction functor*

$$I^B_{A_a} = - \otimes_A B : \mathrm{Mod}_{A_a} \to \mathrm{Mod}_B$$

is exact, faithful and admits a right adjoint $R^B_{A_a} := \mathrm{Hom}_A(A_a, -)$.

Let $R \subset A$ be a subring central in B. Then $I^B_{A_a}$ respects finitely generated R-modules. If R is a field, $R^B_{A_a}$ respects finite dimension over R.

3B4. Let (A, a, B, D) satisfy Definition 3.3.

For $\mathcal{M} \in \mathrm{Mod}_A$, the set $\mathcal{M}_d$ of $f \in \mathrm{Hom}_A({}_D B, \mathcal{M})$ vanishing on $D - \{d\}$ is isomorphic to $\mathcal{M}$ by the value at d. The A-dual $({}_D B)^*$ of ${}_D B$ is a free left A-module of basis D^*. We have

$$\mathrm{Hom}_A({}_D B, \mathcal{M}) = \oplus_{d \in D} \mathcal{M}_d \simeq \oplus_{d^* \in D^*} \mathcal{M} \otimes d^* = \mathcal{M} \otimes_A ({}_D B)^*. \tag{28}$$

The A-modules $\mathcal{M}_d$ and $\mathcal{M} \otimes d^*$ are isomorphic by $f \mapsto f(d) \otimes d^*$.

For $\mathcal{M} \in \mathrm{Mod}_{A_a}$, we have linear isomorphisms

$$\mathbb{I}^B_{A_a}(\mathcal{M}) = \mathrm{Hom}_A(B, \mathcal{M}) \simeq \mathrm{Hom}_A({}_D B, \mathcal{M}), \quad \mathcal{M} \otimes_A ({}_D B)^* = \mathcal{M} \otimes_A A_a \otimes_A ({}_D B)^*.$$

For $d \in D$, let $f_d \in \mathrm{Hom}_A(B, A_a)$ equal to 1 on d and 0 on $D - \{d\}$. We deduce from these arguments:

Lemma 3.9. *Let (A, a, B, D) satisfy Definition 3.3. The left A_a-module $\mathbb{I}^B_{A_a}(A_a)$ is free of basis $(f_d)_{d \in D}$ and $\mathbb{I}^B_{A_a}(\mathcal{M}) \simeq \mathcal{M} \otimes_{A_a} \mathbb{I}^B_A(A_a)$.*

Let $R \subset A$ be a subring central in B. Let $\mathcal{M} \in \mathrm{Mod}_{A_a}$ be a finitely generated R-module. The R-module $\mathcal{M} \otimes_{A_a} \mathbb{I}^B_{A_a}(A_a)$ is finitely generated. If R is a field, and the dimension of $\mathcal{N} \in \mathrm{Mod}_B$ is finite over R, then $\mathcal{N} \otimes_A A_a = \mathcal{N}_a \otimes_A A_a \simeq \mathcal{N}_a$ has finite dimension over R by the Fitting lemma, as in the proof of Proposition 3.8. We obtain the following:

Proposition 3.10. *Let (A, a, B, D) satisfy Definition 3.3. The coinduction*

$$\mathbb{I}^B_{A_a} = \mathrm{Hom}_A(B, -) : \mathrm{Mod}_{A_a} \to \mathrm{Mod}_B$$

is exact, faithful, and admits a left adjoint $L^B_{A_a} = - \otimes_A A_a$.

Let $R \subset A$ be a subring central in B. Then $\mathbb{I}^B_{A_a}$ respects finitely generated R-modules. If R is a field, $L^B_{A_a}$ respects finite dimension over R.

4. Parabolic induction and coinduction from $\mathcal{H}_M$ to $\mathcal{H}$

We prove Theorems 1.6, 1.8 and 1.9 giving the properties of the parabolic induction from $\mathcal{H}_M$ to $\mathcal{H}$.

4A. ***Basic properties of the parabolic induction and coinduction.*** Example 3.2 satisfies Definition 3.1 and Example 3.4 satisfies Definition 3.3. In these two examples, $(A_a, B) = (\mathcal{H}_M, \mathcal{H})$. The first one,

$$(A, a, D) = \big(\theta(\mathcal{H}_{M^+}), T_{\tilde{\mu}_M}, (T_{\tilde{d}})_{d\in {}^M W_0}\big),$$

where we identify $\mathcal{H}_{M^+}$ with $\theta(\mathcal{H}_{M^+})$, defines the parabolic induction $I_{\mathcal{H}_M}^{\mathcal{H}} = - \otimes_{\mathcal{H}_{M^+},\theta} \mathcal{H} : \mathrm{Mod}_{\mathcal{H}_M} \to \mathrm{Mod}_{\mathcal{H}}$. The second one,

$$(A, a, D) = \big(\theta^*(\mathcal{H}_{M^-}), T^*_{(\tilde{\mu}_M)^{-1}}, (T^*_{\tilde{d}})_{d\in W_0^M}\big),$$

where we identify $\mathcal{H}_{M^-}$ with $\theta^*(\mathcal{H}_{M^-})$, defines the parabolic coinduction $\mathbb{I}_{\mathcal{H}_M}^{\mathcal{H}} = \mathrm{Hom}_{\mathcal{H}_{M^-},\theta^*}(\mathcal{H}, -) : \mathrm{Mod}_{\mathcal{H}_M} \to \mathrm{Mod}_{\mathcal{H}}$. Propositions 3.8 and 3.10 imply:

Proposition 4.1. *The parabolic induction $I_{\mathcal{H}_M}^{\mathcal{H}}$ and the coinduction $\mathbb{I}_{\mathcal{H}_M}^{\mathcal{H}}$ are exact, faithful and respect finitely generated R-modules. The parabolic induction admits a right adjoint*

$$R_{\mathcal{H}_M}^{\mathcal{H}} = \mathrm{Hom}_{\mathcal{H}_{M^+},\theta}(\mathcal{H}_M, -) : \mathrm{Mod}_{\mathcal{H}} \to \mathrm{Mod}_{\mathcal{H}_M}.$$

The parabolic coinduction admits a left adjoint

$$\mathbb{L}_{\mathcal{H}_M}^{\mathcal{H}} := - \otimes_{\mathcal{H}_{M^-},\theta^*} \mathcal{H}_M : \mathrm{Mod}_{\mathcal{H}} \to \mathrm{Mod}_{\mathcal{H}_M}.$$

If R is a field, the adjoint functors $R_{\mathcal{H}_M}^{\mathcal{H}}$ and $\mathbb{L}_{\mathcal{H}_M}^{\mathcal{H}}$ respect finite dimension over R.

4B. ***Transitivity.*** Let $S_M \subset S_{M'} \subset S$. Let $W_{M^\epsilon,M'} = \Lambda_{M^\epsilon,M'} \rtimes W_{M,0}$ denote the submonoid of W_M associated to $S_{M'}^{\mathrm{aff}}$ as in Definition 2.1 (see before Proposition 2.21), and

$$\Lambda_{M^\epsilon,M'} = \Lambda \cap W_{M^\epsilon,M'} = \{\lambda \in \Lambda \mid -(\gamma \circ \nu)(\lambda) \geq 0 \text{ for all } \gamma \in \Sigma_{M'}^\epsilon - \Sigma_M^\epsilon\}.$$

By the properties (i), (ii), (iii) of Theorem 1.4, the R-submodule $\mathcal{H}_{M^\epsilon,M'}$ of $\mathcal{H}_M$ of basis $(T_{\tilde{w}}^M)_{\tilde{w}\in W_{M^\epsilon,M'}(1)}$, is a subring of $\mathcal{H}_M$, the restriction to $\mathcal{H}_{M^\epsilon,M'}$ of the injective linear map

$$\mathcal{H}_M \xrightarrow{\theta'} \mathcal{H}_{M'}, \qquad T_{\tilde{w}}^M \mapsto T_{\tilde{w}}^{M'} \quad \text{for } \tilde{w} \in W_M(1),$$

respects the product, and $\mathcal{H}_M = \mathcal{H}_{M^\epsilon,M'}[(T^M_{\tilde{\mu}_{M^\epsilon}})^{-1}]$. Obviously, the map $\mathcal{H}_M \xrightarrow{\theta} \mathcal{H}$ satisfies $\theta = \theta_{M'} \circ \theta'$ for the linear map

$$\mathcal{H}_{M'} \xrightarrow{\theta_{M'}} \mathcal{H}, \qquad T_{\tilde{w}}^{M'} \mapsto T_{\tilde{w}}, \quad \text{for } \tilde{w} \in W_{M'}(1).$$

Lemma 4.2. *We have*:

(i) $W_M \subset W_{M'}$, $W_{M^\epsilon} = W_{M^\epsilon,M'} \cap W_{M'^\epsilon}$, $\theta'(\mathcal{H}_{M^\epsilon}) = \theta'(\mathcal{H}_{M^\epsilon,M'}) \cap \mathcal{H}_{M'^\epsilon}$,

(ii) $\tilde{\mu}_{M^\epsilon}\tilde{\mu}_{M'^\epsilon}$ *is central in* $W_M(1)$, *satisfies* $-(\gamma \circ \nu)(\mu_{M^\epsilon}\mu_{M'^\epsilon}) > 0$ *for all* $\gamma \in \Sigma^\epsilon - \Sigma_M^\epsilon$, *and the additivity of the lengths* $\ell(\mu_{M^\epsilon}\mu_{M'^\epsilon}) = \ell(\mu_{M^\epsilon}) + \ell(\mu_{M'^\epsilon})$,

(iii) ${}^M W_0 = {}^M W_{M',0}\, {}^{M'} W_0$.

Proof. (i) We have $W_{M,0} \subset W_{M',0}$ and $\Lambda_{M^\epsilon} = \Lambda'_{M^\epsilon} \cap \Lambda_{M'^\epsilon}$. Therefore

$$W_M = \Lambda \rtimes W_{M,0} \subset \Lambda \rtimes W_{M',0} = W_{M'},$$

and

$$\begin{aligned} W_{M^\epsilon,M'} \cap W^\epsilon_{M'} &= (\Lambda'_{M^\epsilon} \rtimes W_{M,0}) \cap (\Lambda'_{M'^\epsilon} \rtimes W_{M',0}) \\ &= (\Lambda'_{M^\epsilon} \cap \Lambda_{M'^\epsilon}) \rtimes W_{M,0} \\ &= \Lambda_{M^\epsilon} \rtimes W_{M,0} = W_{M^\epsilon}. \end{aligned}$$

(ii) Now $\tilde{\mu}_{M'^\epsilon}$ is central in $W_{M'}(1)$, which contains $W_M(1)$, and $\tilde{\mu}_{M^\epsilon}$ is central in $W_M(1)$; hence $\tilde{\mu}_{M^\epsilon}\tilde{\mu}_{M'^\epsilon}$ is central in $W_M(1)$. We have

$$\begin{aligned} -(\gamma \circ \nu)(\mu_{M'^\epsilon}) > 0 &\quad \text{for all } \gamma \in \Sigma^\epsilon - \Sigma^\epsilon_{M'}, \\ -(\gamma \circ \nu)(\mu_{M'^\epsilon}) = 0 &\quad \text{for all } \gamma \in \Sigma_{M'}, \\ -(\gamma \circ \nu)(\mu_{M^\epsilon}\ ) > 0 &\quad \text{for all } \gamma \in \Sigma^\epsilon - \Sigma^\epsilon_{M}, \\ -(\gamma \circ \nu)(\mu_{M^\epsilon}\ ) = 0 &\quad \text{for all } \gamma \in \Sigma_{M}. \end{aligned}$$

Hence $-(\gamma \circ \nu)(\mu'_{M^\epsilon}\mu_{M'^\epsilon}) > 0$ for all $\gamma \in \Sigma^\epsilon - \Sigma^\epsilon_M$ and

$$\ell(\mu_{M^\epsilon}\mu_{M'^\epsilon}) = \ell(\mu_{M^\epsilon}) + \ell(\mu_{M'^\epsilon}).$$

(iii) Let $u \in {}^M W_{M',0}$, $v \in {}^{M'}W_0$ and let $w \in W_{M,0}$. We have

$$\ell(wuv) = \ell(wu) + \ell(v) = \ell(w) + \ell(u) + \ell(v) = \ell(w) + \ell(uv);$$

hence $uv \in {}^M W_0$. The injective map $(u, v) \mapsto uv : {}^M W_{M',0} \times {}^{M'}W_0 \to {}^M W_0$ is bijective because

$$|{}^M W_0| = |W_{M,0}\backslash W_0| = |W_{M,0}\backslash W_{M',0}|\,|W_{M',0}\backslash W_0| = |{}^M W_{M',0}|\,|{}^{M'}W_0|,$$

where $|X|$ denotes the number of elements of a finite set X. □

Proposition 4.3. *The induction is transitive*:

$$I^{\mathcal{H}}_{\mathcal{H}_M} = I^{\mathcal{H}}_{\mathcal{H}_{M'}} \circ I^{\mathcal{H}_{M'}}_{\mathcal{H}_M} : \mathrm{Mod}_{\mathcal{H}_M} \to \mathrm{Mod}_{\mathcal{H}_{M'}} \to \mathrm{Mod}_{\mathcal{H}}.$$

The coinduction is also transitive. This is proved at the end of this paper.

Proof. By Lemma 3.5, the proposition is equivalent to

$$\mathcal{H}_M \otimes_{\mathcal{H}_{M^+}} \mathcal{H} \simeq \mathcal{H}_M \otimes_{\mathcal{H}_{M^+,M'}} \mathcal{H}_{M'} \otimes_{\mathcal{H}_{M'^+}} \mathcal{H}$$

in $\mathrm{Mod}_{\mathcal{H}}$. As $\mathcal{H}_{M'} = \mathcal{H}_{M'^+}[(T^{M'}_{\tilde{\mu}_{M'^+}})^{-1}]$ is the localisation of the ring $\mathcal{H}_{M'^+}$ at the central element $T^{M'}_{\tilde{\mu}_{M'^+}} \in \mathcal{H}_{M'^+}$, the right $\mathcal{H}$-module $\mathcal{H}_{M'} \otimes_{\mathcal{H}_{M'^+}} \mathcal{H}$ is the inductive limit of $(T^{M'}_{\tilde{\mu}_{M'^+}})^{-r} \otimes \mathcal{H}$ for $r \in \mathbb{N}$ with the transition maps

$$(T^{M'}_{\tilde{\mu}_{M'^+}})^{-r} \otimes x \mapsto (T^{M'}_{\tilde{\mu}_{M'^+}})^{-r-1} \otimes T_{\tilde{\mu}_{M'^+}} x \quad \text{for } x \in \mathcal{H}.$$

As $\mathcal{H}_M = \mathcal{H}_{M^+,M'}[(T^M_{\tilde{\mu}_{M^+}})^{-1}]$ is the localisation of the ring $\mathcal{H}_{M^+,M'}$ at the central element $T^M_{\tilde{\mu}_{M^+}} \in \mathcal{H}_{M^+,M'}$, the right $\mathcal{H}$-module $\mathcal{H}_M \otimes_{\mathcal{H}_{M^+,M'}} \mathcal{H}_{M'} \otimes_{\mathcal{H}_{M'^+}} \mathcal{H}$ is the inductive limit of $(T^M_{\tilde{\mu}_{M^+}})^{-s} \otimes \mathcal{H}_{M'} \otimes_{\mathcal{H}_{M'^+}} \mathcal{H}$ for $s \in \mathbb{N}$ with the transition maps

$$(T^M_{\tilde{\mu}_{M^+}})^{-s} \otimes y \mapsto (T^M_{\tilde{\mu}_{M^+}})^{-s-1} \otimes T^{M'}_{\tilde{\mu}_{M^+}} y \quad \text{for } y \in \mathcal{H}_{M'} \otimes_{\mathcal{H}_{M'^+}} \mathcal{H}.$$

Using that $T^{M'}_{\tilde{\mu}_{M'^+}}$ is central in $\mathcal{H}_{M'}$ and $T^{M'}_{\tilde{\mu}_{M^+}} \in \mathcal{H}_{M'^+}$, we have, for $y = (T^{M'}_{\tilde{\mu}_{M'^+}})^{-r} \otimes x$,

$$T^{M'}_{\tilde{\mu}_{M^+}} y = T^{M'}_{\tilde{\mu}_{M^+}} (T^{M'}_{\tilde{\mu}_{M'^+}})^{-r} \otimes x = (T^{M'}_{\tilde{\mu}_{M'^+}})^{-r} T^{M'}_{\tilde{\mu}_{M^+}} \otimes x = (T^{M'}_{\tilde{\mu}_{M'^+}})^{-r} \otimes T_{\tilde{\mu}_{M^+}} x.$$

Altogether, the right $\mathcal{H}$-module $\mathcal{H}_M \otimes_{\mathcal{H}_{M^+,M'}} \mathcal{H}_{M'} \otimes_{\mathcal{H}_{M'^+}} \mathcal{H}$ is the inductive limit of $(T^M_{\tilde{\mu}_{M^+}})^{-s} \otimes (T^{M'}_{\tilde{\mu}_{M'^+}})^{-r} \otimes \mathcal{H}$ for $r, s \in \mathbb{N}$ with the transition maps

$$(T^M_{\tilde{\mu}_{M^+}})^{-s} \otimes (T^{M'}_{\tilde{\mu}_{M'^+}})^{-r} \otimes x \mapsto (T^M_{\tilde{\mu}_{M^+}})^{-s-1} \otimes (T^{M'}_{\tilde{\mu}_{M'^+}})^{-r} \otimes T_{\tilde{\mu}_{M^+}} x,$$
$$(T^M_{\tilde{\mu}_{M^+}})^{-s} \otimes (T^{M'}_{\tilde{\mu}_{M'^+}})^{-r} \otimes x \mapsto (T^M_{\tilde{\mu}_{M^+}})^{-s} \otimes (T^{M'}_{\tilde{\mu}_{M'^+}})^{-r-1} \otimes T_{\tilde{\mu}_{M'^+}} x.$$

The right $\mathcal{H}$-module $\mathcal{H}_M \otimes_{\mathcal{H}_{M^+,M'}} \mathcal{H}_{M'} \otimes_{\mathcal{H}_{M'^+}} \mathcal{H}$ is also the inductive limit of the modules $(T^M_{\tilde{\mu}_{M^+}})^{-r} \otimes (T^{M'}_{\tilde{\mu}_{M'^+}})^{-r} \otimes \mathcal{H}$ for $r \in \mathbb{N}$ with the transition maps

$$(T^M_{\tilde{\mu}_{M^+}})^{-r} \otimes (T^{M'}_{\tilde{\mu}_{M'^+}})^{-r} \otimes x \mapsto (T^M_{\tilde{\mu}_{M^+}})^{-r-1} \otimes (T^{M'}_{\tilde{\mu}_{M'^+}})^{-r-1} \otimes T_{\tilde{\mu}_{M^+}} T_{\tilde{\mu}_{M'^+}} x.$$

By Lemma 4.2(ii), $T_{\tilde{\mu}_{M^+}} T_{\tilde{\mu}_{M'^+}} = T_{\tilde{\mu}_{M^+}\tilde{\mu}_{M'^+}}$. Hence, in $\mathrm{Mod}_{\mathcal{H}}$ we have

$$\mathcal{H}_M \otimes_{\mathcal{H}_{M^+,M'}} \mathcal{H}_{M'} \otimes_{\mathcal{H}_{M'^+}} \mathcal{H} \simeq \varinjlim_{x \mapsto T_{\tilde{\mu}_{M^+}\tilde{\mu}_{M'^+}} x} \mathcal{H}.$$

On the other hand, $\mathcal{H}_M = \mathcal{H}_{M^+}[(T^M_{\tilde{\mu}_{M^+}} \tilde{\mu}_{M'^+})^{-1}]$ is the localisation of $\mathcal{H}_{M^+}$ at $T^M_{\tilde{\mu}_{M^+}\tilde{\mu}_{M'^+}}$ (Lemma 4.2); hence $\mathcal{H}_M \otimes_{\mathcal{H}_{M^+}} \mathcal{H}$ is the inductive limit of $(T^M_{\tilde{\mu}_{M^+}\tilde{\mu}_{M'^+}})^{-r} \otimes \mathcal{H}$ for $r \in \mathbb{N}$ with the transition maps

$$(T^M_{\tilde{\mu}_{M^+}\tilde{\mu}_{M'^+}})^{-r} \otimes x \mapsto (T^M_{\tilde{\mu}_{M^+}\tilde{\mu}_{M'^+}})^{-r-1} \otimes T_{\tilde{\mu}_{M^+}\tilde{\mu}_{M'^+}} x.$$

We deduce that

$$\mathcal{H}_M \otimes_{\mathcal{H}_{M^+}} \mathcal{H} \simeq \varinjlim_{x \mapsto T_{\tilde{\mu}_{M^+}\tilde{\mu}_{M'^+}} x} \mathcal{H}$$

is isomorphic to $\mathcal{H}_M \otimes_{\mathcal{H}_{M^+,M'}} \mathcal{H}_{M'} \otimes_{\mathcal{H}_{M'^+}} \mathcal{H}$ in $\mathrm{Mod}_{\mathcal{H}}$. □

4C. *w_0-twisted induction is equal to coinduction.* We prove Theorem 1.8. When $\mathcal{H} = \mathcal{H}_R(G)$ is the pro-p Iwahori Hecke algebra of a reductive p-adic group G over an algebraically closed field R of characteristic p, Theorem 1.8 is proved by Abe [2014, Proposition 4.14]. We will extend his arguments to the general algebra $\mathcal{H}$.

Let $\tilde{w}^M_0 \in W_0(1)$ lifting w^M_0. The algebra isomorphism $\mathcal{H}_M \simeq \mathcal{H}_{w_0(M)}$ defined by $\tilde{w}^M_0$ (Proposition 2.20) induces an equivalence of categories

$$\mathrm{Mod}_{\mathcal{H}_M} \xrightarrow{\tilde{\mathfrak{w}}^M_0} \mathrm{Mod}_{\mathcal{H}_{w_0(M)}} \tag{29}$$

called a w_0-twist. Let $\mathcal{M}$ be a right $\mathcal{H}_M$-module. The underlying R-module of $\tilde{\mathfrak{w}}_0^M(\mathcal{M})$ and of $\mathcal{M}$ is the same; the right action of $T^M_{\tilde{w}}$ on $\mathcal{M}$ is equal to the right action of $T^{w_0(M)}_{\tilde{w}_0^M \tilde{w}(\tilde{w}_0^M)^{-1}}$ on $\tilde{\mathfrak{w}}_0^M(\mathcal{M})$ for $\tilde{w} \in W_M(1)$. The inverse of $\tilde{\mathfrak{w}}_0^M$ is the algebra isomorphism induced by $(\tilde{w}_0^M)^{-1}$ lifting

$$^M w_0 := (w_0^M)^{-1} = w_{M,0} w_0 = w_0 w_0 w_{M,0} w_0 = w_0^{w_0(M)}.$$

Remark 4.4. The lifts of w_0^M are $t\tilde{w}_0^M = \tilde{w}_0^M t'$ with $t, t' \in Z_k$, the elements $T^M_{t'} \in \mathcal{H}_M$, $T^{w_0(M)}_t \in \mathcal{H}_{w_0(M)}$ are invertible, and the conjugation by T_t in $\mathcal{H}_M$, by $T^{w_0(M)}_t$ in $\mathcal{H}_{w_0(M)}$ induce equivalences of categories

$$\mathrm{Mod}_{\mathcal{H}_M} \xrightarrow{\mathfrak{t}'} \mathrm{Mod}_{\mathcal{H}_M}, \quad \mathrm{Mod}_{\mathcal{H}_{w_0(M)}} \xrightarrow{\mathfrak{t}} \mathrm{Mod}_{\mathcal{H}_{w_0(M)}}$$

such that $\mathfrak{t}\tilde{\mathfrak{w}}_0^M = \mathfrak{t} \circ \tilde{\mathfrak{w}}_0^M = \tilde{\mathfrak{w}}_0^M \circ \mathfrak{t}' = \tilde{\mathfrak{w}}_0^M \mathfrak{t}'$.

Remark 4.5. The trivial characters of $\mathcal{H}_M$ and $\mathcal{H}_{w_0(M)}$ correspond by $\tilde{\mathfrak{w}}_0^M$.

We will prove that, for all $S_M \subset S$, the coinduction

$$\mathrm{Mod}_{\mathcal{H}_M} \xrightarrow{\mathbb{I}^{\mathcal{H}}_{\mathcal{H}_M}} \mathrm{Mod}_{\mathcal{H}}$$

is equivalent to the w_0-twist induction

$$\mathrm{Mod}_{\mathcal{H}_M} \xrightarrow{\tilde{\mathfrak{w}}_0^M} \mathrm{Mod}_{\mathcal{H}_{w_0(M)}} \xrightarrow{I^{\mathcal{H}}_{\mathcal{H}_{w_0(M)}}} \mathrm{Mod}_{\mathcal{H}}\,.$$

This proves Theorem 1.8 because

$$\mathbb{I}^{\mathcal{H}}_{\mathcal{H}_M} \simeq I^{\mathcal{H}}_{\mathcal{H}_{w_0(M)}} \circ \tilde{\mathfrak{w}}_0^M \iff I^{\mathcal{H}}_{\mathcal{H}_M} \simeq \mathbb{I}^{\mathcal{H}}_{\mathcal{H}_{w_0(M)}} \circ \tilde{\mathfrak{w}}_0^M. \tag{30}$$

Indeed, if the left-hand side is true for all $S_M \subset S$, permuting M and $w_0(M)$ we have $\mathbb{I}_{\mathcal{H}_{w_0(M)}} \simeq I^{\mathcal{H}}_{\mathcal{H}_M} \circ \tilde{\mathfrak{w}}_0^{w_0(M)}$, and composing with $(\tilde{\mathfrak{w}}_0^{w_0(M)})^{-1}$, we get

$$I^{\mathcal{H}}_{\mathcal{H}_M} \simeq \mathbb{I}^{\mathcal{H}}_{\mathcal{H}_{w_0(M)}} \circ (\tilde{\mathfrak{w}}_0^{w_0(M)})^{-1} \simeq \mathbb{I}^{\mathcal{H}}_{\mathcal{H}_{w_0(M)}} \circ \tilde{\mathfrak{w}}_0^M$$

as $w_0^{w_0(M)} = (w_0^M)^{-1}$. The arguments can be reversed to get the equivalence.

Let $\mathcal{M} \in \mathrm{Mod}_{\mathcal{H}_M}$. We will construct an explicit functorial isomorphism in $\mathrm{Mod}_{\mathcal{H}}$:

$$(I^{\mathcal{H}}_{\mathcal{H}_{w_0(M)}} \circ \tilde{\mathfrak{w}}_0^M)(\mathcal{M}) \xrightarrow{\mathfrak{b}} \mathbb{I}^{\mathcal{H}}_{\mathcal{H}_M}(\mathcal{M}). \tag{31}$$

From Lemmas 3.5, 3.6, 3.9 and Examples 3.2, 3.4, we get

(i) $I^{\mathcal{H}}_{\mathcal{H}_{w_0(M)}}(\mathcal{H}_{w_0(M)}) = \mathcal{H}_{w_0(M)} \otimes_{\mathcal{H}_{w_0(M)^+},\theta} \mathcal{H}$ is a left free $\mathcal{H}_{w_0(M)}$-module of basis $1 \otimes T_{\tilde{d}'}$ for $d' \in {}^{w_0(M)}W_0$, and

$$(I^{\mathcal{H}}_{\mathcal{H}_{w_0(M)}} \circ \tilde{\mathfrak{w}}_0^M)(\mathcal{M}) = \tilde{\mathfrak{w}}_0^M(\mathcal{M}) \otimes_{\mathcal{H}_{w_0(M)}} I^{\mathcal{H}}_{\mathcal{H}_{w_0(M)}}(\mathcal{H}_{w_0(M)}).$$

(ii) $\mathbb{I}^{\mathcal{H}}_{\mathcal{H}_M}(\mathcal{H}_M) = \mathrm{Hom}_{\mathcal{H}_{M^-},\theta^*}(\mathcal{H}, \mathcal{H}_M)$, where $\mathcal{H}$ is seen as a right $\theta^*(\mathcal{H}_{M^-})$-module, is a left free $\mathcal{H}_M$-module of basis $(f^*_{\tilde{d}})_{d \in W_0^M}$, where $f^*_{\tilde{d}}(T^*_{\tilde{d}}) = 1$ and $f^*_{\tilde{d}}(T^*_{\tilde{x}}) = 0$ for $x \in W_0^M - \{d\}$, and

$$\mathbb{I}^{\mathcal{H}}_{\mathcal{H}_M}(\mathcal{M}) = \mathcal{M} \otimes_{\mathcal{H}_M} \mathbb{I}^{\mathcal{H}}_{\mathcal{H}_M}(\mathcal{H}_M).$$

It is an exercise to prove that the left $\mathcal{H}_M$-module $\mathbb{I}^{\mathcal{H}}_{\mathcal{H}_M}(\mathcal{H}_M)$ admits also the basis $(f_{\tilde{d}})_{d\in W_0^M}$, where $f_{\tilde{d}}(T_{\tilde{d}})=1$ and $f_{\tilde{d}}(T_{\tilde{x}})=0$ for $x\in W_0^M-\{d\}$. We will prove that the linear map

$$(32)\qquad m\otimes T_{\tilde{d}'}\mapsto m\otimes f_{\tilde{w}_0^M}T_{\tilde{d}'}:\oplus_{d'\in {}^{w_0(M)}W_0}\tilde{\mathfrak{w}}_0^M(\mathcal{M})\otimes T_{\tilde{d}'}\xrightarrow{\mathfrak{b}}\oplus_{d\in W_0^M}\mathcal{M}\otimes f_{\tilde{d}}$$

is a functorial isomorphism in $\mathrm{Mod}_{\mathcal{H}}$. The bijectivity follows from the bijectivity of the map $d'\mapsto d'^{-1}w_0^M : {}^{w_0(M)}W_0\to W_0^M$ (Lemma 2.24) and the following:

Lemma 4.6. *The map $f_{\tilde{w}_0^M}T_{\tilde{d}'}-f_{(d'^{-1}w_0^M)^{\sim}}$ lies in $\oplus_{x\in W_0^M,x<d'^{-1}w_0^M}\mathcal{M}\otimes f_{\tilde{x}}$.*

Proof. For $d\in W_0^M$, we have

$$(f_{\tilde{w}_0^M}T_{\tilde{d}'})(T_{\tilde{d}})=f_{\tilde{w}_0^M}(T_{\tilde{d}'}T_{\tilde{d}})=f_{\tilde{w}_0^M}(T_{\tilde{d}'\tilde{d}})+x,$$

where $x\in\sum Rf_{\tilde{w}_0^M}(T_{\tilde{w}})$ and the sum is over the $\tilde{w}\in W_0(1)$ with $w<d'd$ and $w\in w_0^M W_{M,0}$. If $d'd\notin w_0^M W_{M,0}$, there is no $w\in w_0^M W_{M,0}$ with $w<d'd$ (Lemma 2.26). We have $d'd\in w_0^M W_{M,0}$ if and only if $d=d'^{-1}w_0^M$ (part (ii) of Lemma 2.28). □

The restriction

$$\mathrm{Res}^{\mathcal{H}}_{\mathcal{H}_{w_0(M)^+},\theta}:\mathrm{Mod}_{\mathcal{H}}\to\mathrm{Mod}_{\mathcal{H}_{w_0(M)^+}}$$

is left adjoint to $-\otimes_{\mathcal{H}_{w_0(M)^+},\theta}\mathcal{H}$, and the $\mathcal{H}_{w_0(M)^+}$-equivariance of the linear map

$$(33)\qquad m\mapsto m\otimes f_{\tilde{w}_0^M}:\tilde{\mathfrak{w}}_0^M(\mathcal{M})\to\mathbb{I}^{\mathcal{H}}_{\mathcal{H}_M}(\mathcal{M})$$

implies the $\mathcal{H}$-equivariance of (31), i.e., of (32). Let $\mathcal{H}_M\xrightarrow{j}\mathcal{H}_{w_0(M)}$ denote the isomorphism induced by $\tilde{w}_0^M$ (Proposition 2.20), and θ_M the linear map $\mathcal{H}_M\xrightarrow{\theta}\mathcal{H}$. The $\mathcal{H}_{w_0(M)^+}$-invariance of the map $m\mapsto m\otimes f_{\tilde{w}_0^M}$ is equivalent to

$$(34)\qquad f_{\tilde{w}_0^M}\theta_{w_0(M)}(h)=j^{-1}(h)f_{\tilde{w}_0^M}\quad\text{for } h\in\mathcal{H}_{w_0(M)^+}.$$

We can suppose that h lies in the Bernstein basis of $\mathcal{H}_{w_0(M)^+}$. Let $\tilde{w}\in W_{w_0(M)^+}(1)$ and $h=E_{w_0(M)}(\tilde{w})$. As $\theta_{w_0(M)}(E_{w_0(M)}(\tilde{w}))=E(\tilde{w})$, and $j^{-1}(E_{w_0(M)}(\tilde{w}))$ is equal to $E_M((\tilde{w}_0^M)^{-1}\tilde{w}\tilde{w}_0^M)$, (34) is equivalent to the following:

Proposition 4.7. *For $w\in W_{w_0(M)^+}$, we have $f_{\tilde{w}_0^M}E(\tilde{w})=E_M((\tilde{w}_0^M)^{-1}\tilde{w}\tilde{w}_0^M)f_{\tilde{w}_0^M}$.*

Proof. By the usual reduction arguments, we suppose that the $\mathfrak{q}(s)$ are invertible in R. Using $W_{w_0(M)^+}=\Lambda_{w_0(M)^+}\rtimes W_{w_0(M),0}$, the product formula (8) and Lemma 2.23, we reduce to $w\in\Lambda_{w_0(M)^+}\cup W_{w_0(M),0}$. By induction on the length in $W_{w_0(M),0}$ with respect to $S_{w_0(M)}$, we reduce to $w\in\Lambda_{w_0(M)^+}\cup S_{w_0(M)}$.

Let $d\in W_0^M$. We have $(f_{\tilde{w}_0^M}E(\tilde{w}))(T_{\tilde{d}})=f_{\tilde{w}_0^M}(E(\tilde{w})T_{\tilde{d}})$ in $\mathcal{H}_M$. We must prove

$$(35)\qquad f_{\tilde{w}_0^M}(E(\tilde{w})T_{\tilde{d}})=\begin{cases}0 & \text{if } d\neq w_0^M,\\ E_M((\tilde{w}_0^M)^{-1}\tilde{w}\tilde{w}_0^M) & \text{if } \tilde{d}=\tilde{w}_0^M\end{cases}$$

for $w\in\Lambda_{w_0(M)^+}\cup S_{w_0(M)}$.

(i) Suppose $w=\lambda\in\Lambda_{w_0(M)^+}$. Let $\mathcal{A}$ denote the subalgebra of $\mathcal{H}$ of basis $(E(\tilde{x}))_{\tilde{x}\in\Lambda(1)}$ [Vignéras 2013a, Corollary 2.8]. By the Bernstein relations [Vignéras 2013a, Theorem 2.9], we have

$$E(\tilde{\lambda})T_{\tilde{d}}=T_{\tilde{d}}E((\tilde{d})^{-1}\tilde{\lambda}\tilde{d})+\sum T_{\tilde{w}}a_{\tilde{w}},$$

where $a_{\tilde{w}}\in\mathcal{A}$ and the sum is over $\tilde{w}\in W_0(1)$, $w<d$. If $d\neq w_0^M$, the image by $f_{\tilde{w}_0^M}$ of the right-hand side vanishes because $w\in w_0^M W_{M,0}$, $w\le d$ implies $w=d=w_0^M$; hence $f_{\tilde{w}_0^M}(E(\tilde{\lambda})T_{\tilde{d}})=0$ as we want. For $\tilde{d}=\tilde{w}_0^M$, using $(w_0^M)^{-1}\lambda\tilde{w}_0^M\in W_{w_0(M)^-}$, we have

$$\begin{aligned}f_{\tilde{w}_0^M}(E(\tilde{\lambda})T_{\tilde{w}_0^M})&=f_{\tilde{w}_0^M}(T_{\tilde{w}_0^M}E((\tilde{w}_0^M)^{-1}\tilde{\lambda}\tilde{w}_0^M)\\&=\theta^*(E((\tilde{w}_0^M)^{-1}\tilde{\lambda}\tilde{w}_0^M))\\&=E_M((\tilde{w}_0^M)^{-1}\tilde{\lambda}\tilde{w}_0^M).\end{aligned}$$

(ii) Suppose $w=s\in S_{w_0(M)}$. We have $w_0sw_0\in S_M$, $w_0sw_0w_{M,0}<w_{M,0}$ and

$$sw_0^M=sw_0w_{M,0}=w_0w_0sw_0w_{M,0}>w_0w_{M,0}=w_0^M.$$

Assume $sd<d$. We deduce $d\neq w_0^M$. Assume $\tilde{d}=\tilde{s}(\tilde{sd})$. Then

$$E(\tilde{s})T_{\tilde{d}}=T_{\tilde{s}}T_{\tilde{d}}=T_{\tilde{s}}^2T_{(\tilde{sd})}=(\mathfrak{q}(s)(\tilde{s})^2+\mathfrak{c}(\tilde{s})T_{\tilde{s}})T_{(\tilde{sd})}=\mathfrak{q}(s)(\tilde{s})^2T_{(\tilde{sd})}+\mathfrak{c}(\tilde{s})T_{\tilde{d}}.$$

We deduce that $f_{\tilde{w}_0^M}(E(\tilde{s})T_{\tilde{d}})=0$.

Assume $sd>d$. We write $\tilde{s}\,\tilde{d}=\tilde{d}_1\tilde{u}$ with $d_1\in W_0^M$, $u\in W_{M,0}$. Then $T_{\tilde{s}}T_{\tilde{d}}=T_{\tilde{s}\tilde{d}}=T_{\tilde{d}_1\tilde{u}}$. Therefore $f_{\tilde{w}_0^M}(E(\tilde{s})T_{\tilde{d}})=f_{\tilde{w}_0^M}(T_{\tilde{d}_1\tilde{u}})=0$ if $d_1\neq w_0^M$. We suppose now $d_1=w_0^M$. We have $d\le w_0^M\le sd$; hence $w_0^M=d$ or $w_0^M=sd$. In the latter case, a reduced decomposition of w_0^M starts by s. But this is incompatible with $s\in S_{w_0(M)}$ because $w_0^M={}^{w_0(M)}w_0$. We deduce that $d=w_0^M$. For $\tilde{d}=\tilde{w}_0^M$, we have

$$\begin{aligned}f_{\tilde{w}_0^M}(E(\tilde{s})T_{\tilde{w}_0^M})&=f_{\tilde{w}_0^M}(T_{\tilde{s}\,\tilde{w}_0^M})=f_{\tilde{w}_0^M}(T_{\tilde{w}_0^M}T_{(w_0^M)^{-1}\tilde{s}\tilde{w}_0^M})\\&=f_{\tilde{w}_0^M}(T_{\tilde{w}_0^M}E_{(w_0^M)^{-1}\tilde{s}\tilde{w}_0^M})=\theta^*(E_{(w_0^M)^{-1}\tilde{s}\tilde{w}_0^M}))\\&=E_M((\tilde{w}_0^M)^{-1}\tilde{s}\tilde{w}_0^M).\end{aligned}$$

This ends the proof of Proposition 4.7, and hence of Theorem 1.8. □

Corollary 4.8. *The right $\mathcal{H}$-modules $\mathcal{H}_M\otimes_{\mathcal{H}_{M^+},\theta}\mathcal{H}$ and $\operatorname{Hom}_{\mathcal{H}_{w_0(M)^-},\theta^*}(\mathcal{H},\mathcal{H}_{w_0(M)})$ are isomorphic.*

4D. ***Transitivity of the coinduction.*** Let $S_M\subset S_{M'}\subset S$. By Proposition 2.21, the algebra isomorphisms

$$\mathcal{H}_M\xrightarrow{j}\mathcal{H}_{w_0(M)},\quad \mathcal{H}_M\xrightarrow{j'}\mathcal{H}_{w_{M',0}(M)}\xrightarrow{k''}\mathcal{H}_{w_0(M)}$$

corresponding to $\tilde{w}_0^M$, $\tilde{w}_{M'}^M$, $\tilde{w}_0^{M'}$, $\tilde{w}_0^M=\tilde{w}_0^{M'}\tilde{w}_{M'}^M$, satisfy $j=k''\circ j'$. The associated equivalences of categories, denoted by

$$(36)\qquad \mathcal{M}_{\mathcal{H}_M}\xrightarrow{\tilde{\mathfrak{w}}_0^M}\mathcal{M}_{\mathcal{H}_{w_0(M)}},\qquad \mathcal{M}_{\mathcal{H}_M}\xrightarrow{\tilde{\mathfrak{w}}_{M'}^M}\mathcal{M}_{\mathcal{H}_{w_{M',0}(M)}}\xrightarrow{\tilde{\mathfrak{w}}_{0,k}^{M'}}\mathcal{M}_{\mathcal{H}_{w_0(M)}},$$

satisfy $\tilde{\mathfrak{w}}_0^M=\tilde{\mathfrak{w}}_{0,k}^{M'}\circ\tilde{\mathfrak{w}}_{M'}^M$. We refer to this as the transitivity of the w_0-twisting.

Lemma 4.9. *The functors* $\tilde{\mathfrak{w}}_0^{M'}\circ I_{\mathcal{H}_{w_{M',0}(M)}}^{\mathcal{H}_{M'}}$ *and* $I_{\mathcal{H}_{w_0(M)}}^{\mathcal{H}_{w_0(M')}}\circ\tilde{\mathfrak{w}}_{0,k}^{M'}$ *from* $\mathrm{Mod}_{\mathcal{H}_{w_{M',0}(M)}}$ *to* $\mathrm{Mod}_{\mathcal{H}_{w_0(M')}}$ *are isomorphic.*

The proof gives an explicit isomorphism.

Proof. Let $\mathcal{M}\in\mathrm{Mod}_{\mathcal{H}_{w_{M',0}(M)}}$. The R-module $\mathcal{M}\otimes_{\mathcal{H}_{w_{M',0}(M)^+},\theta}\mathcal{H}_{M'}$ with the right action of $\mathcal{H}_{w_0(M')}$ defined by

$$(x\otimes T_{\tilde{u}}^{M'})T^{w_0(M')}_{\tilde{w}_o^{M'}\tilde{v}(\tilde{w}_o^{M'})^{-1}}=x\otimes T_{\tilde{u}}^{M'}T_{\tilde{v}}^{M'}$$

for $x\in\mathcal{M}$, $u,v\in W_{M'}$, is $\tilde{\mathfrak{w}}_0^{M'}\circ I_{\mathcal{H}_{w_{M',0}(M)}}^{\mathcal{H}_{M'}}(\mathcal{M})$.

As $k''(\mathcal{H}_{w_{M',0}(M)^+})=\mathcal{H}_{w_0(M)^+}$ (Proposition 2.21), the R-linear map

$$\mathcal{M}\otimes_R\mathcal{H}_{M'}\to\tilde{\mathfrak{w}}_{0,k}^{M'}(\mathcal{M})\otimes_{\mathcal{H}_{w_0(M)^+},\theta}\mathcal{H}_{w_0(M')}$$

defined by $x\otimes T_{\tilde{u}}^{M'}\to x\otimes T^{w_0(M')}_{\tilde{w}_0^{M'}\tilde{u}(\tilde{w}_0^{M'})^{-1}}$ is the composite of the quotient map

$$\mathcal{M}\otimes_R\mathcal{H}_{M'}\to\tilde{\mathfrak{w}}_0^{M'}\circ\mathcal{M}\otimes_{\mathcal{H}_{w_{M',0}(M)^+}}\mathcal{H}_{M'},$$

and of the bijective linear map

$$\tilde{\mathfrak{w}}_0^{M'}\circ I_{\mathcal{H}_{w_{M',0}(M)}}^{\mathcal{H}_{M'}}(\mathcal{M})\to\tilde{\mathfrak{w}}_{0,k}^{M'}(\mathcal{M})\otimes_{\mathcal{H}_{w_0(M)^+},\theta}\mathcal{H}_{w_0(M')}.$$

The above map is clearly $\mathcal{H}_{w_0(M')}$-equivariant. ☐

Proposition 4.10. *The coinduction is transitive.*

Proof. By the transitivity of the w_0-twisting (36), Lemma 4.9, and the transitivity of the induction (Proposition 4.3), we have

$$\begin{aligned}\mathbb{I}_{\mathcal{H}_{M'}}^{\mathcal{H}}\circ\mathbb{I}_{\mathcal{H}_M}^{\mathcal{H}_{M'}}&=I_{\mathcal{H}_{w_0(M')}}^{\mathcal{H}}\circ\tilde{\mathfrak{w}}_0^{M'}\circ I_{\mathcal{H}_{w_0(M)}}^{\mathcal{H}_{w_0(M')M'}}\circ\tilde{\mathfrak{w}}_{M'}^M\\&=I_{\mathcal{H}_{w_0(M')}}^{\mathcal{H}}\circ I_{\mathcal{H}_{w_0(M)}}^{\mathcal{H}_{w_0(M')}}\circ\tilde{\mathfrak{w}}_{0,k}^{M'}\circ\tilde{\mathfrak{w}}_{M'}^M\\&=I_{\mathcal{H}_{w_0(M')}}^{\mathcal{H}}\circ I_{\mathcal{H}_{w_0(M)}}^{\mathcal{H}_{w_0(M')}}\circ\tilde{\mathfrak{w}}_0^M\\&=I_{\mathcal{H}_{w_0(M)}}^{\mathcal{H}}\circ\tilde{\mathfrak{w}}_0^M=\mathbb{I}_{\mathcal{H}_M}^{\mathcal{H}}.\end{aligned}$$ ☐

Proof of Theorem 1.9. The induction $I_{\mathcal{H}_M}^{\mathcal{H}}$ is equivalent to $\mathbb{I}_{\mathcal{H}_{w_0(M)}}^{\mathcal{H}}\circ\tilde{\mathfrak{w}}_0^M$. The coinduction $\mathbb{I}_{\mathcal{H}_M}^{\mathcal{H}}$ is the composite of the restriction $\mathrm{Mod}_{\mathcal{H}_M}\to\mathrm{Mod}_{\mathcal{H}_{M^-}}$ and of $\mathrm{Hom}_{\mathcal{H}_{M^-},\theta^*}(\mathcal{H},-):\mathrm{Mod}_{\mathcal{H}_{M^-}}\to\mathrm{Mod}_{\mathcal{H}}$. These functors admit left adjoints, the restriction $\mathrm{Mod}_{\mathcal{H}} \to \mathrm{Mod}_{\mathcal{H}_{M^-}}$ for $\mathrm{Hom}_{\mathcal{H}_{M^-},\theta^*}(\mathcal{H}, -)$, and the induction $-\otimes_{\mathcal{H}_{M^-}} \mathcal{H}_M : \mathrm{Mod}_{\mathcal{H}_{M^-}} \to \mathrm{Mod}_{\mathcal{H}_M}$ for the restriction $\mathrm{Mod}_{\mathcal{H}_M} \to \mathrm{Mod}_{\mathcal{H}_{M^-}}$; hence $-\otimes_{\mathcal{H}_{M^-},\theta^*} \mathcal{H}_M : \mathrm{Mod}_{\mathcal{H}} \to \mathrm{Mod}_{\mathcal{H}_M}$ for $\mathbb{I}^{\mathcal{H}}_{\mathcal{H}_M}$, and

$$(\tilde{\mathfrak{w}}_0^M)^{-1} \circ (-\otimes_{\mathcal{H}_{w_0(M)^-},\theta^*} \mathcal{H}_{w_0(M)}) \simeq \tilde{\mathfrak{w}}_0^{w_0(M)} \circ (-\otimes_{\mathcal{H}_{w_0(M)^-},\theta^*} \mathcal{H}_{w_0(M)})$$

for $\mathbb{I}^{\mathcal{H}}_{\mathcal{H}_{w_0(M)}} \circ \tilde{\mathfrak{w}}_0^M$. □

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Volume 279 No. 1-2 December 2015

In memoriam: Robert Steinberg

Robert Steinberg (1922–2014): In memoriam — 1
V. S. VARADARAJAN

Cellularity of certain quantum endomorphism algebras — 11
HENNING H. ANDERSEN, GUSTAV I. LEHRER and RUIBIN ZHANG

Lower bounds for essential dimensions in characteristic 2 via orthogonal representations — 37
ANTONIO BABIC and VLADIMIR CHERNOUSOV

Cocharacter-closure and spherical buildings — 65
MICHAEL BATE, SEBASTIAN HERPEL, BENJAMIN MARTIN and GERHARD RÖHRLE

Embedding functor for classical groups and Brauer–Manin obstruction — 87
EVA BAYER-FLUCKIGER, TING-YU LEE and RAMAN PARIMALA

On maximal tori of algebraic groups of type G_2 — 101
CONSTANTIN BELI, PHILIPPE GILLE and TING-YU LEE

On extensions of algebraic groups with finite quotient — 135
MICHEL BRION

Essential dimension and error-correcting codes — 155
SHANE CERNELE and ZINOVY REICHSTEIN

Notes on the structure constants of Hecke algebras of induced representations of finite Chevalley groups — 181
CHARLES W. CURTIS

Complements on disconnected reductive groups — 203
FRANÇOIS DIGNE and JEAN MICHEL

Extending Hecke endomorphism algebras — 229
JIE DU, BRIAN J. PARSHALL and LEONARD L. SCOTT

Products of partial normal subgroups — 255
ELLEN HENKE

Lusztig induction and ℓ-blocks of finite reductive groups — 269
RADHA KESSAR and GUNTER MALLE

Free resolutions of some Schubert singularities — 299
MANOJ KUMMINI, VENKATRAMANI LAKSHMIBAI, PRAMATHANATH SASTRY and C. S. SESHADRI

Free resolutions of some Schubert singularities in the Lagrangian Grassmannian — 329
VENKATRAMANI LAKSHMIBAI and REUVEN HODGES

Distinguished unipotent elements and multiplicity-free subgroups of simple algebraic groups — 357
MARTIN W. LIEBECK, GARY M. SEITZ and DONNA M. TESTERMAN

Action of longest element on a Hecke algebra cell module — 383
GEORGE LUSZTIG

Generic stabilisers for actions of reductive groups — 397
BENJAMIN MARTIN

On the equations defining affine algebraic groups — 423
VLADIMIR L. POPOV

Smooth representations and Hecke modules in characteristic p — 447
PETER SCHNEIDER

On CRDAHA and finite general linear and unitary groups — 465
BHAMA SRINIVASAN

Weil representations of finite general linear groups and finite special linear groups — 481
PHAM HUU TIEP

The pro-p Iwahori Hecke algebra of a reductive p-adic group, V (parabolic induction) — 499
MARIE-FRANCE VIGNÉRAS

Acknowledgement — 531

0030-8730(2015)279:1;1-1