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We study a new operation named pushforward on diffeological vector pseudo-bundles, which is left adjoint to the pullback. We show how to pushforward projective diffeological vector pseudo-bundles to get projective diffeological vector spaces, producing many concrete new examples, together with applications to smooth splittings of some projective diffeological vector spaces related to geometry. This brings new objects to diffeology from classical vector bundle theory.

1. Introduction

Diffeological spaces are elegant generalisations of smooth manifolds, including many infinite-dimensional spaces, like mapping spaces and diffeomorphism groups, and singular spaces, e.g., smooth manifolds with boundary or corners, orbifolds and irrational tori.

On diffeological spaces, one can still do some differential geometry and topology, such as differential forms and tangent bundles. These tangent bundles are, in general, no longer locally trivial. Instead, they are diffeological vector pseudo-bundles. We studied these objects and operations on them in [Christensen and Wu 2022], on which the current paper is based.

On the other hand, the theory of diffeological vector spaces and their homological algebra is intimately related to analysis and geometry; see [Wu 2015; Christensen and Wu 2016; 2021]. The projective objects there deserve special attention. However, in general, neither is it easy to test whether a given diffeological vector space is projective or not, nor is it straightforward to construct many concrete projective objects.

In this paper, we propose a way to use diffeological vector pseudo-bundles to study diffeological vector spaces. We generalise some results of projective objects for diffeological vector spaces to such bundles. In particular, we show that every classical vector bundle is such a projective object. We introduce a left adjoint called pushforward to the pullback on diffeological vector pseudo-bundles, we show that

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the free diffeological vector space generated by a diffeological space has a canonical bundle-theoretical explanation, and we show that pushforward preserves projectives. In this way, we construct many concrete projective diffeological vector spaces from classical vector bundle theory, together with applications of classical vector bundle theory to smooth splittings of some projective diffeological vector spaces.

Here is the structure of the paper. In Section 2, we briefly review some necessary background. In Section 3, we introduce pushforward on diffeological vector pseudo-bundles. Section 4 contains three parts, including necessary and sufficient conditions of smooth splittings of short exact sequences of diffeological vector pseudo-bundles, examples and properties of the projective objects, and preservation of projectives by pushforward. In particular, we get many new examples of projective diffeological vector spaces from classical vector bundles. In Section 5, we apply the established theory to smooth splittings of projective diffeological vector spaces. Readers interested in concrete examples are suggested to take a look at the last part of this section first.

2. Background

In this section, we give a very brief review, together with many related references.

Definition 2.1. A *diffeological space* is a set X together with a collection of maps $U \to X$ (called *plots*) from open subsets U of Euclidean spaces, such that:

(1) Every constant map is a plot.

(2) The composite $V \to U \to X$ is a plot if the first map is smooth between open subsets of Euclidean spaces and the second one is a plot.

(3) The map $U \to X$ is a plot if there is an open cover of U such that each restriction is a plot.

A *smooth map* $X \to Y$ between diffeological spaces is a map which sends plots of X to plots of Y. Diffeological spaces with smooth maps form a category denoted by Diff.

The idea of a diffeological space was introduced in [Souriau 1980], and [Iglesias-Zemmour 2013] is currently the standard reference for the subject. Also see [Christensen et al. 2014, Section 2] for a concise summary for the basics of diffeological spaces.

The category Diff has excellent properties. It contains the category of smooth manifolds as a full subcategory, and it is complete, cocomplete and cartesian closed. In particular, we have subspaces, quotient spaces and mapping spaces for diffeological spaces. Like charts for manifolds, we have various generating sets of plots for a diffeological space. Every diffeological space has a canonical topology called the D-topology; see [Iglesias-Zemmour 1985; Christensen et al. 2014]. Every diffeological space has a tangent bundle; see [Hector 1995; Christensen and Wu 2016; 2017]. Diffeological vector spaces are the vector space objects in Diff. Every vector space can be equipped with a smallest diffeology called the fine diffeology, making it a diffeological vector space; see [Iglesias-Zemmour 2007]. There are many other kinds of diffeological vector spaces in practice. Hierarchies of diffeological vector spaces were studied in [Christensen and Wu 2019], and homological algebra of diffeological vector spaces was developed in [Wu 2015]. The following two types of diffeological vector spaces will be needed:

Definition 2.2. A diffeological vector space V is called *projective* if for any linear subduction[1] $f : V_1 \to V_2$ and any smooth linear map $g : V \to V_2$, there exists a smooth linear map $h : V \to V_1$ such that $g = f \circ h$.

Proposition 2.3 [Wu 2015, Proposition 3.5]. *Given any diffeological space X, there exist a diffeological vector space V and a smooth map $i : X \to V$ satisfying the following universal property*: *for any diffeological vector space W and any smooth map $f : X \to W$, there exists a unique smooth linear map $g : V \to W$ such that $f = g \circ i$.*

The diffeological vector space V in the above proposition is unique up to isomorphism. We call it the *free diffeological vector space generated by X*, and we write $i_X : X \to F(X)$ for $i : X \to V$. As a model, $F(X) = \oplus_{x \in X}\mathbb{R}$ as a vector space, a plot $U \to F(X)$ locally factors via a smooth map through some $\mathbb{R} \times U_1 \times \cdots \times \mathbb{R} \times U_k \to F(X)$ with $(r_1, u_1, \ldots, r_k, u_k) \mapsto \sum_i r_i[p_i(u_i)]$ for some $k \in \mathbb{Z}^{>0}$ and plots $p_i : U_i \to X$, and $i_X(x) = [x]$, the element 1 in the copy of $\mathbb{R}$ corresponding to $x \in X$.

We recall the following concepts from [Christensen and Wu 2022]:

Definition 2.4. A *diffeological vector pseudo-bundle* over a diffeological space B is a smooth map $\pi : E \to B$ between diffeological spaces such that the following conditions hold:

(1) For each $b \in B$, $\pi^{-1}(b) =: E_b$ is a vector space.

(2) The fibrewise addition $E \times_B E \to E$ and the fibrewise scalar multiplication $\mathbb{R} \times E \to E$ are smooth.

(3) The zero section $\sigma : B \to E$ is smooth.

Definition 2.5. Given a diffeological space B, a *bundle map over B* is a commutative triangle

$$\begin{tikzcd} E_1 \arrow[rr, "f"] \arrow[dr, "\pi_1"'] & & E_2 \arrow[dl, "\pi_2"] \\ & B & \end{tikzcd}$$

[1]A *subduction* is a smooth map that is isomorphic to a quotient map in Diff.

where π_1, π_2 are diffeological vector pseudo-bundles over B, f is smooth and for each $b \in B$, the restriction $f|_{E_{1,b}} : E_{1,b} \to E_{2,b}$ is linear.

Such f is called a *bundle subduction* (respectively, *bundle induction*) *over* B if it is both a bundle map over B and a subduction (respectively, an induction[2]).

For a fixed diffeological space B, all diffeological vector pseudo-bundles over B and bundle maps over B form a category, denoted by DVPB_B. An isomorphism in DVPB_B is called a *bundle isomorphism over* B. A bundle map over B is a bundle isomorphism if and only if it is both a bundle induction and a bundle subduction over B.

Definition 2.6. A commutative square

$$\begin{tikzcd} E \arrow[r, "g"] \arrow[d, "\pi"'] & E' \arrow[d, "\pi'"] \\ B \arrow[r, "f"'] & B' \end{tikzcd}$$

in Diff, with π and π' being diffeological vector pseudo-bundles, is called a *bundle map*, if for each $b \in B$, the map $g|_{E_b} : E_b \to E'_{f(b)}$ is linear.

A bundle map (g, f), as above, is called a *bundle subduction* if both g and f are subductions.

All diffeological vector pseudo-bundles and bundle maps form a category denoted by DVPB.

Note that diffeological vector pseudo-bundles are neither diffeological fibre bundles in [Iglesias-Zemmour 1985; 2013], nor diffeological fibrations in [Christensen and Wu 2014]. They were introduced to encode tangent bundles of diffeological spaces [Christensen and Wu 2016]. Many operations on DVPB_B and DVPB were studied in [Christensen and Wu 2022], such as direct product, direct sum, free diffeological vector pseudo-bundle induced by a smooth map, tensor product, and exterior product. We will use the following construction later:

Proposition 2.7 [Christensen and Wu 2022, Proposition 3.3]. *Let $\pi : E \to B$ be a smooth map between diffeological spaces such that each fibre is a vector space. Then there is a smallest diffeology on E which contains the given diffeology and which makes π into a diffeological vector pseudo-bundle over B.*

We call the original $\pi : E \to B$ a *diffeological vector prebundle* , and the procedure in this proposition is called *dvsification*. More precisely, every plot in the new diffeology of the total space is locally of the following form: Given a plot $q : U \to B$, some $k \in \mathbb{N}$, plots $q_1, \dots, q_k : U \to E$ such that $\pi \circ q_i = q$ for all i, and plots $r_1, \dots, r_k : U \to \mathbb{R}$, the linear combination $U \to E$ with $u \mapsto \sum_i r_i(u) q_i(u) \in E_{q(u)}$

[2]An *induction* is a smooth map that is isomorphic to an inclusion of a subspace in Diff.

is a plot in the new diffeology. Note that when $k = 0$, this is $\sigma \circ q$ for the zero section $\sigma : B \to E$.

3. Pushforward

Recall from [Christensen and Wu 2022, Section 3.1] that one can pullback diffeological vector pseudo-bundles via smooth maps, i.e., a smooth map $f : B \to B'$ induces a functor $f^* : \mathrm{DVPB}_{B'} \to \mathrm{DVPB}_B$ by pullback. Now we define a related operation as follows:

Given a smooth map $f : B \to B'$ and a diffeological vector pseudo-bundle $\pi : E \to B$, we define

$$E' = \coprod_{b' \in B'} \Big(\bigoplus_{b \in f^{-1}(b')} E_b \Big). \tag{1}$$

Note that when $f^{-1}(b') = \varnothing$, the term in the above parentheses is $\mathbb{R}^0$. There are canonical maps $\pi_f : E' \to B'$ sending the fibre above b' to b', and $\alpha_f : E \to E'$ with $E_b \hookrightarrow \bigoplus_{\tilde{b} \in f^{-1}(f(b))} E_{\tilde{b}}$. We then have a natural commutative square

$$\begin{tikzcd} E \arrow[r, "\alpha_f"] \arrow[d, "\pi"'] & E' \arrow[d, "\pi_f"] \\ B \arrow[r, "f"'] & B' \end{tikzcd}$$

Hence, we can equip E' with the dvsification of the diffeology generated by the upper horizontal map α_f of the above square, making the right vertical map π_f a diffeological vector pseudo-bundle over B', and hence the above square becomes a bundle map from π to π_f. (As a *warning*, each fibre of E' may not be the direct sum of those of E as diffeological vector spaces; see Proposition 3.5. Also notice that the notation α_f will be used later in the paper.) More precisely, we have the following explicit description of a generating set of plots on E':

Lemma 3.1. *A plot on E' is locally of one of the following forms*:

(1) *$U \to E'$ defined by a finite sum $\sum_i \alpha_f \circ p_i$, where $p_i : U \to E$ are plots on E such that all $f \circ \pi \circ p_i$'s match*;

(2) *the composite of a plot of B' followed by the zero section $B' \to E'$.*

Proof. This is straightforward from the description of dvsification as recalled in the paragraph right after Proposition 2.7. □

It is straightforward to check that we get a functor $f_* : \mathrm{DVPB}_B \to \mathrm{DVPB}_{B'}$, called the *pushforward of f*, and we write E' above as $f_*(E)$. Moreover, from the above lemma, we have:

(1) $f'_* \circ f_* = (f' \circ f)_*$ for any smooth maps $f : B \to B'$ and $f' : B' \to B''$;

(2) $(1_B)_* =$ the identity on DVPB_B.

Example 3.2. Pushforward has been used implicitly in [Christensen and Wu 2022, Section 5]. For example, E_1 and E_2 in [Christensen and Wu 2022, Proposition 5.1] are the pushforward of the tangent bundle $\mathbb{R}^2 \cong T\mathbb{R} \to \mathbb{R}$ along the inclusions $\mathbb{R} \to X_g$ to the x-axis and the y-axis, respectively.

Here is the key result for pushforward:

Theorem 3.3. *Given a smooth map $f : B \to B'$, we have an adjoint pair of functors*

$$f_* : \mathrm{DVPB}_B \rightleftharpoons \mathrm{DVPB}_{B'} : f^*.$$

Proof. We show that there is a natural bijection

$$\mathrm{DVPB}_B(E, f^*(E')) \cong \mathrm{DVPB}_{B'}(f_*(E), E').$$

Given a bundle map $E \to f^*(E')$ over B, we have $E_b \to E'_{f(b)}$ for each $b \in B$, which induce $\bigoplus_{b\in f^{-1}(b')} E_b \to E'_{b'}$, and hence a map $f_*(E) \to E'$. This is clearly a bundle map over B'. Conversely, given a bundle map $f_*(E) \to E'$ over B', we have a map $\bigoplus_{b\in f^{-1}(b')} E_b \to E'_{b'}$ for each $b' \in \mathrm{Im}(f)$. It then induces a map $E_b \to E'_{f(b)}$, which together give a map $E \to f^*(E')$. It is straightforward to check that this is a bundle map over B. These procedures are inverses to each other, and therefore we proved the desired result. □

We have the following bundle-theoretical explanation of a free diffeological vector space:

Proposition 3.4. *For any diffeological space B, the total space of the pushforward of the trivial bundle $B \times \mathbb{R} \to B$ along the map $B \to \mathbb{R}^0$ is the free diffeological vector space $F(B)$.*

Proof. This follows directly from the diffeology of the total space of the pushforward (see Lemma 3.1) and the diffeology on free diffeological vector space, as recalled in the paragraph right after Proposition 2.3. □

From [Christensen and Wu 2022, Section 3], we know that the usual operations on diffeological vector pseudo-bundles have the obvious diffeology on each fibre indicated by the operation. But pushforward is an exception, although it is expected to be so.

Proposition 3.5. *Let $f : B \to B'$ be a smooth map, and let $E \to B$ be a diffeological vector pseudo-bundle. Then the diffeology on the fibre at b' of the pushforward $f_*(E)$ has the direct sum diffeology of the diffeological vector spaces E_b with $f(b) = b'$ if and only if $f^{-1}(b')$ as a subspace of B has the discrete diffeology.*

Proof. This follows directly from Lemma 3.1. □

Here is the universal property for pushforward:

Proposition 3.6. *Given a bundle map*

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "\pi"'] & E' \arrow[d, "\pi'"] \\ B \arrow[r, "g"'] & B' \end{tikzcd}$$

there exists a unique bundle map $\beta : g_*(E) \to E'$ over B' such that $f = \beta \circ \alpha_g$.

Proof. This is clear by the construction of pushforward, or from the adjoint (Theorem 3.3). □

Pushforward can send nonisomorphic bundles to isomorphic ones:

Example 3.7. Write B for the cross with the gluing diffeology, and write B' for the cross with the subset diffeology of $\mathbb{R}^2$. Then $B \to B'$ defined as the identity underlying set map is smooth, but its inverse is not; see [Christensen and Wu 2016, Example 3.19]. We show below that the induced map $F(B) \to F(B')$ between the free diffeological vector spaces, which is the identity for the underlying vector spaces, is indeed an isomorphism of diffeological vector spaces. This means that the pushforward of the two trivial bundles $B \times \mathbb{R} \to B$ and $B' \times \mathbb{R} \to B'$ along the maps $B \to \mathbb{R}^0$ and $B' \to \mathbb{R}^0$ are isomorphic, but clearly the two bundles are not.

By definition of a free diffeological vector space, every plot $p : U \to F(B')$ can be locally written as a finite sum $p(u) = \sum_i r_i(u)(p_{1i}(u), p_{2i}(u))$ for smooth maps r_i, p_{1i}, p_{2i} with codomain $\mathbb{R}$ satisfying $p_{1i}(u)p_{2i}(u) = 0$ for all u. It is enough to show that p can be viewed as a plot of $F(B)$. This is the case since $(p_{1i}(u), p_{2i}(u))$ can be written as $(p_{1i}(u), 0) + (0, p_{2i}(u)) - (0, 0)$, with each term viewed as a plot of B.

As a consequence of the above example, the canonical map $i_X : X \to F(X)$ from a diffeological space to the free diffeological vector space generated by it is *not* necessarily an induction.

On the other hand, we have:

Proposition 3.8. *The canonical map $i_X : X \to F(X)$ is an induction if and only if there exists a family of diffeological vector spaces $\{V_i\}_{i \in I}$ such that the diffeology on X is determined by the union of all $C^\infty(X, V_i)$, in the sense that $U \to X$ is a plot if and only if the composite $U \to X \to V_i$ is smooth for every smooth map $X \to V_i$.*

In particular, for every Frölicher space X (i.e., the diffeology on X is determined by $C^\infty(X, \mathbb{R})$), the canonical map $X \to F(X)$ is an induction. This applies to B' in Example 3.7.

Proof. This follows immediately from the universal property of the free diffeological vector space generated by a diffeological space. □

4. Projective diffeological vector pseudo-bundles

4A. ***Enough projectives.*** In this subsection, we will work in the category DVPB_B for a fixed diffeological space B. So we will omit the phrase "over B" in many places, as long as no confusion shall occur. Note that when we take $B = \mathbb{R}^0$, we recover the corresponding results for the category of diffeological vector spaces.

We first study smooth splittings of diffeological vector pseudo-bundles, which will be used later in the paper.

Definition 4.1. A diagram of morphisms

$$E_1 \xrightarrow{f} E_2 \xrightarrow{g} E_3$$

in DVPB_B is called a *short exact sequence* if f is a bundle induction, g is a bundle subduction and

$$E_{1,b} \xrightarrow{f_b} E_{2,b} \xrightarrow{g_b} E_{3,b}$$

is exact (i.e., $\ker(g_b) = \mathrm{Im}(f_b)$) for every $b \in B$.

As a direct consequence of the above definition, we have:

Corollary 4.2. *Given a short exact sequence*

$$E_1 \longrightarrow E_2 \longrightarrow E_3$$

of diffeological vector pseudo-bundles over B, we have a bundle isomorphism $E_2/E_1 \cong E_3$ over B.

The splitting of a short exact sequence goes as usual:

Theorem 4.3. *Assume that*

$$E_1 \xrightarrow{f} E_2 \xrightarrow{g} E_3$$

is a short exact sequence of diffeological vector pseudo-bundles over B. Then the following are equivalent:

(1) *There exists a bundle map $g' : E_3 \to E_2$ over B such that $g \circ g' = 1_{E_3}$.*

(2) *There exists a bundle map $f' : E_2 \to E_1$ over B such that $f' \circ f = 1_{E_1}$.*

(3) *There exists a bundle isomorphism $E_2 \to E_1 \oplus E_3$ over B making the following diagram commutative*:

$$\begin{tikzcd}
E_1 \arrow[r, "f"] \arrow[d, "="'] & E_2 \arrow[r, "g"] \arrow[d] & E_3 \arrow[d, "="] \\
E_1 \arrow[r, "i_1"'] & E_1 \oplus E_3 \arrow[r, "p_2"'] & E_3
\end{tikzcd}$$

If any one of the conditions holds in the theorem, we say that the short exact sequence *splits smoothly*, and that E_1 (respectively, E_3) is a *smooth direct summand* of E_2. Although every short exact sequence of vector spaces splits, it is not the case in DVPB_B, even when $B = \mathbb{R}^0$; see [Wu 2015, Example 4.3] or [Christensen and Wu 2019, Example 4.1].

Proof. We show below that (1) $\Longleftrightarrow$ (3), and (2) $\Longleftrightarrow$ (3) can be proved similarly.

(1) $\Longrightarrow$ (3): Since we have bundle maps $f : E_1 \to E_2$ and $g' : E_3 \to E_2$, we define the map $E_1 \oplus E_3 \to E_2$ by $(x_1, x_3) \mapsto f(x_1) + g'(x_3)$ for any $x_1 \in E_{1,b}$, $x_3 \in E_{3,b}$ and $b \in B$. This is clearly a bundle map over B. Its inverse is given by $x \mapsto (f^{-1}(x - g' \circ g(x)), g(x))$. It is straightforward to check that this is well defined, and it is smooth since f is an induction.

(3) $\Longrightarrow$ (1): The map g' is defined by the composite $E_3 \xrightarrow{i_2} E_1 \oplus E_3 \xrightarrow{\cong} E_2$. The rest are straightforward to check. $\square$

Now we can define projective diffeological vector pseudo-bundles and show that there are *enough* such objects.

Definition 4.4. A diffeological vector pseudo-bundle $E \to B$ is called *projective* if for any bundle subduction $f : E_1 \to E_2$ over B and any bundle map $g : E \to E_2$ over B, there exists a bundle map $h : E \to E_1$ over B making the triangle commutative:

$$\begin{tikzcd} & E \arrow[dl, dotted, "h"'] \arrow[d, "g"] \\ E_1 \arrow[r, "f"'] & E_2 \end{tikzcd}$$

Formally, we have the following basic properties:

Proposition 4.5. (1) *Let $\{E_i \to B\}$ be a family of diffeological vector pseudo-bundles. Then the direct sum $\bigoplus_i E_i \to B$ is projective if and only if each $E_i \to B$ is.*

(2) *The projectiveness of diffeological vector pseudo-bundles is inherited by taking retracts.*

(3) *Any bundle subduction to a projective diffeological vector pseudo-bundle splits smoothly.*

Recall from [Christensen and Wu 2022, Section 3.2.5] that given a smooth map $f : X \to B$, we get a diffeological vector pseudo-bundle $\pi : F_B(X) \to B$. More precisely, it is constructed as follows: For each $b \in B$, write X_b for $f^{-1}(b)$ with the subset diffeology of X. As a set $F_B(X) = \coprod_{b \in B} F(X_b)$, the disjoint union of the free diffeological vector spaces generated by these X_b and $\pi : F_B(X) \to B$ is the canonical projection. So we have the commutative triangle

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "f"'] & F_B(X) \arrow[d, "\pi"] \\ & B \end{tikzcd}$$

where the horizontal map is given by $x \in X_b \mapsto [x] \in F(X_b)$. The dvsification of i makes $\pi : F_B(X) \to B$ into a diffeological vector pseudo-bundle[3]. Then we have the following universal property for $\pi : F_B(X) \to B$: Given any diffeological vector pseudo-bundle $E \to B$ and any smooth map $h : X \to E$ over B, there is a unique bundle map $g : F_B(X) \to E$ over B such that $h = g \circ i$.

Lemma 4.6. *Let $f : X \to B$ be a smooth map. The corresponding diffeological vector pseudo-bundle $\pi : F_B(X) \to B$ is projective if and only if for any bundle subduction $\alpha : E_1 \to E_2$ over B and any smooth map $\beta : X \to E_2$ over B, there exists a smooth map $\gamma : X \to E_1$ over B such that $\beta = \alpha \circ \gamma$.*

Proof. As usual, this follows from the universal property of $\pi : F_B(X) \to B$. □

Proposition 4.7. *Every plot $U \to B$ induces a projective diffeological vector pseudo-bundle $F_B(U) \to B$.*

Proof. Given any bundle subduction $f : E_1 \to E_2$ over B and any smooth map $g : U \to E_2$ over B, we have smooth local liftings h_i of g to E_1. Let $\{\lambda_i\}$ be a smooth partition of unity subordinate to the corresponding open cover $\{U_i\}$ of U. Then $\sum_i \lambda_i \cdot h_i : U \to E_1$ is a global smooth lifting of g over B, where each $\lambda_i \cdot h_i : U \to E_1$ is defined as

$$(\lambda_i \cdot h_i)(u) = \begin{cases} \lambda_i(u) h_i(u), & \text{if } u \in U_i, \\ \sigma_1 \circ \pi_2 \circ g(u), & \text{else}, \end{cases}$$

with $\sigma_1 : B \to E_1$ denoting the zero section and $\pi_2 : E_2 \to B$ denoting the given diffeological vector pseudo-bundle. The result then follows from Lemma 4.6. □

As a direct consequence of the above proof, we have:

Corollary 4.8. *For every bundle subduction, a plot of the total space of the codomain **globally** lifts to a plot of the total space of the domain.*

Theorem 4.9. *For every diffeological space B, the category* DVPB$_B$ *has enough projectives, i.e., given any diffeological vector pseudo-bundle $E \to B$, there exists a projective diffeological vector pseudo-bundle $E' \to B$ together with a bundle subduction $E' \to E$ over B.*

[3]More precisely, the map i transfers the diffeology of X to the set $F_B(X)$, which makes $\pi : F_B(X) \to B$ into a diffeological vector prebundle because of the above commutative triangle. The dvsification is then applied to this prebundle.

Proof. We take $E' \to B$ to be the direct sum in DVPB_B of all $F_B(U) \to B$ indexed over all plots $U \to E$. By Proposition 4.7, each $F_B(U) \to B$ is projective, and hence by Proposition 4.5 (1), $E' \to B$ is projective. By the universal property of $F_B(U) \to B$, we get a bundle map $F_B(U) \to E$ over B, and hence a bundle map $E' \to E$ over B. By construction, this map is a subduction. □

In summary, for a fixed diffeological space B, the pair of projective diffeological vector pseudo-bundles over B and the bundle subductions over B forms a projective class.

4B. ***Examples and properties of projectives.*** We first give some examples of projective diffeological vector pseudo-bundles related to classical vector bundle theory. To do so, we need:

Lemma 4.10. *For a smooth map $f : B \to B'$, the pullback f^* sends a bundle subduction over B' to a bundle subduction over B, and hence it preserves short exact sequences.*

Proof. Let $g : E'_1 \to E'_2$ be a bundle subduction over B'. Then $f^*(E'_1) \to f^*(E'_2)$ is given by sending (b, x) to $(b, g(x))$. Every plot $p : U \to f^*(E'_2)$ gives rise to smooth maps $p_1 : U \to B$ and $p_2 : U \to E'_2$ via composition with the two projections. Since g is a bundle subduction, p_2 locally lifts as a smooth map to E'_1, which together with p_1 induces a local lifting of p to $f^*(E'_1)$, showing the first claim.

Since f^* is a right adjoint by Theorem 3.3, it preserves bundle inductions, which together with the first claim proves the second one. □

Remark 4.11. The above lemma also follows from the fact that the pullback $f^* : \mathrm{DVPB}_{B'} \to \mathrm{DVPB}_B$ has a right adjoint $f_!$. Given a diffeological vector pseudo-bundle $\pi : E \to B$, the bundle $f_!(E) \to B'$ is constructed as

$$f_!(E) = \coprod_{b' \in B'} \Gamma(\pi|_{f^{-1}(b')}).$$

When $f^{-1}(b') = \varnothing$, $\Gamma(\pi|_{f^{-1}(b')})$ is $\mathbb{R}^0$. A map $p : U \to f_!(E)$ is a plot if:

(1) The composite $U \xrightarrow{p} f_!(E) \xrightarrow{\tilde{\pi}} B'$ is a plot of B', where $\tilde{\pi}$ sends $\Gamma(\pi|_{f^{-1}(b')})$ to b'.

(2) For any smooth map $g : V \to U$ and any plot $h : V \to B$ such that the following diagram commutes:

$$\begin{tikzcd} V \arrow[r, "g"] \arrow[d, "h"'] & U \arrow[r, "p"] & f_!(E) \arrow[d, "\tilde{\pi}"] \\ B \arrow[rr, "f"'] & & B' \end{tikzcd}$$

the map $V \to E$ defined by $v \mapsto (p(g(v)))(h(v))$ is a plot of E.

It is straightforward to check that $\tilde{\pi}$ is a smooth map between diffeological spaces such that each fibre is a vector space. After dvsification, we get the desired diffeology on the total space $f_!(E)$. One can check that $f_!$ is a functor which is right adjoint to the pullback f^*. Moreover, each fibre of $f_!(E) \to B'$ has the diffeology of the section space; see [Christensen and Wu 2022, Section 3.1]. (I would like to thank J. Daniel Christensen for the suggestion of the set-theoretical construction of $f_!(E)$ in this remark from a type theory point of view.)

Projectiveness is local in the following sense:

Proposition 4.12. *Let $\pi : E \to B$ be a diffeological vector pseudo-bundle. Assume that there exists a D-open cover $\{B_j\}$ of B such that $i_j^*(E) \to B_j$ is projective in* DVPB_{B_j} *for each j, where $i_j : B_j \to B$ denotes the inclusion, together with a smooth partition of unity $\{\lambda_j : B \to \mathbb{R}\}$ subordinate to this cover. Then π is projective in* DVPB_B.

Proof. For any bundle subduction $f : E_1 \to E_2$ over B and any bundle map $g : E \to E_2$ over B, we get a diagram over B_j for each j:

$$\begin{tikzcd} & i_j^*(E) \arrow[d, "i_j^*(g)"] \\ i_j^*(E_1) \arrow[r, "i_j^*(f)"'] & i_j^*(E_2) \end{tikzcd}$$

Lemma 4.10 shows that the horizontal arrow is a bundle subduction over B_j. By assumption, we have a smooth lifting $h_j : i_j^*(E) \to i_j^*(E_1)$ over B_j. Then $\sum_j \lambda_j \cdot h_j : E \to E_1$ is a bundle map over B, as we desired. □

We also have the following expected result:

Proposition 4.13. *Let V be a projective diffeological vector space, and let B be a smooth manifold. Then the trivial bundle $B \times V \to B$ is projective.*

Surprisingly, note that the result can fail if B is an arbitrary diffeological space; see Example 4.27.

Proof. We first reduce the above statement to a special case. By Proposition 4.12, it is enough to prove this for the case when B is an open subset of a Euclidean space. Recall that every projective diffeological vector space is a smooth direct summand of direct sums of $F(U)$ for open subsets U of Euclidean spaces [Wu 2015, Corollary 6.15]. By Proposition 4.5 (1) and (2), it is enough to show this for the case when $V = F(U)$ for an open subset U of a Euclidean space.

Now we prove the statement for the special case when $V = F(U)$ and both B and U are Euclidean open subsets. As diffeological vector pseudo-bundles over B, we have isomorphisms $F_B(B \times U) \cong B \times F(U)$ of total spaces. The result then follows directly from Proposition 4.7. □

Combining the above two propositions together with the fact that every fine diffeological vector space is projective, we get:

Corollary 4.14. *Vector bundles in classical differential geometry are projective.*

However, a projective diffeological vector pseudo-bundle does not need to be locally trivial, even when the base space is Euclidean:

Example 4.15. Let $f : \mathbb{R} \to \mathbb{R}$ be the square function $x \mapsto x^2$. By Proposition 4.7, $F_{\mathbb{R}}(\mathbb{R}) \to \mathbb{R}$ is projective. Clearly, the fibre is $\mathbb{R}^0$ for $b < 0$, $\mathbb{R}$ for $b = 0$ and $\mathbb{R}^2$ for $b > 0$. Therefore, a projective diffeological vector pseudo-bundle does not need to be locally trivial.

Now we discuss some properties of projective diffeological vector pseudo-bundles.

Proposition 4.16. *Every projective diffeological vector pseudo-bundle $E \to B$ is a smooth direct summand of direct sum in* DVPB_B *of $F_B(U) \to B$ induced by some plots $U \to B$.*

Proof. By the proof of Theorem 4.9, we get a bundle subduction $E' \to E$ over B, with E' a direct sum in DVPB_B of $F_B(U) \to B$ induced by the plots $U \to E$ (and hence some plots $U \to B$, where repetition is allowed). Since $E \to B$ is projective, the result then follows from Proposition 4.5 (3). □

We are going to use the following notation from [Christensen and Wu 2019]: Let V be a diffeological vector space, and let X be a diffeological space.

(1) We say that all smooth linear functionals $V \to \mathbb{R}$ separate points of V, if for any $v \in V \setminus \{0\}$, there exists a smooth linear map $f : V \to \mathbb{R}$ such that $f(v) \neq 0$. Write $\mathcal{SV}$ for the family of all such diffeological vector spaces.

(2) We say that all smooth functions $X \to \mathbb{R}$ separate points of X, if for any $x, x' \in X$ with $x \neq x'$, there exists a smooth function $f : X \to \mathbb{R}$ such that $f(x) \neq f(x')$. Write $\mathcal{SD}'$ for the family of all such diffeological spaces.

Corollary 4.17. *Let $E \to B$ be a projective diffeological vector pseudo-bundle. Then $E_b \in \mathcal{SV}$ for every $b \in B$, i.e., the smooth linear functionals on E_b separate points.*

Proof. By Proposition 4.16, we know that E is a smooth direct summand of direct sums in DVPB_B of $F_B(U) \to B$ induced by some plots $U \to B$. As $\mathcal{SV}$ is closed under taking both smooth direct summands and direct sums [Christensen and Wu 2019, Proposition 3.11], it is enough to show the claim for the special case $F_B(U) \to B$ which is induced by a plot $p : U \to B$. In this case, the fibre at $b \in B$ is the free diffeological vector space generated by $p^{-1}(b)$ [Christensen and Wu 2022, Section 3.2.5], which is a subset of a Euclidean space, and hence $p^{-1}(b) \in \mathcal{SD}'$, i.e., the smooth functions on $p^{-1}(b)$ separate points. The result then follows from [Christensen and Wu 2019, Proposition 3.13]. □

One would expect that each fibre of a projective diffeological vector pseudo-bundle is a projective diffeological vector space. This is equivalent to the statement that the free diffeological vector space generated by *any* subset with the subset diffeology of a Euclidean space is projective, by a similar argument as above. But I don't know whether this is true or not. Nevertheless, we have:

Proposition 4.18. *Let B be a diffeological space. Then every fibre of a projective diffeological vector pseudo-bundle $E \to B$ is a projective diffeological vector space if and only if for every plot $p : U \to B$ and every $b \in B$, the free diffeological vector space generated by $p^{-1}(b)$ is projective.*

Proof. ($\Rightarrow$): This follows directly from Proposition 4.7.

($\Leftarrow$): The proof follows from a similar argument as the one in the proof of the above corollary. □

Proposition 4.19. *Let B be a discrete diffeological space*, i.e., *every plot is locally constant. Then a diffeological vector pseudo-bundle over B is projective if and only if each fibre is a projective diffeological vector space.*

Proof. ($\Rightarrow$): This follows from the definition of a discrete diffeological space, together with Proposition 4.18 and [Wu 2015, Corollary 6.4].

($\Leftarrow$): This follows from the fact that every diffeological vector pseudo-bundle over a discrete diffeological space is a coproduct in DVPB of diffeological vector spaces over a point. □

Also, we have the following results:

Proposition 4.20. *Let $\pi : E \to B$ be a projective diffeological vector pseudo-bundle, and let $\pi_1 \to \pi_2 \to \pi_3$ be a short exact sequence in* DVPB_B, *with $\pi_i : E_i \to B$. Then* $\mathrm{Hom}_B(\pi, \pi_1) \to \mathrm{Hom}_B(\pi, \pi_2) \to \mathrm{Hom}_B(\pi, \pi_3)$ *is also a short exact sequence in* DVPB_B.

Proof. By Proposition 4.16, we know that π is a smooth direct summand of direct sums of $F_B(U) \to B$ indexed by some plots $U \to B$. It is straightforward to check that both direct summand and direct product preserve short exact sequences in DVPB_B. For the direct product case, one needs Corollary 4.8 for the subduction part. By the universal property of a free bundle induced by a smooth map (see [Christensen and Wu 2022, Section 3.2.5] or the paragraph above Lemma 4.6), one has a bundle isomorphism over B from $\mathrm{Hom}_B(F_B(U), E_i)$ to the set $\mathrm{Hom}_B(U, E_i)$ of all smooth maps $U \to E_i$ preserving B, equipped with the subset diffeology of $C^\infty(U, E_i)$. Again by Corollary 4.8, it is direct to check that the functor $\mathrm{Hom}_B(U, -)$ preserves short exact sequences in DVPB_B. The result then follows by the above observations together with the first isomorphism in [Christensen and Wu 2022, Proposition 3.13] □

Remark 4.21. The converse of Proposition 4.20 is false. This is due to the fact that $\mathrm{Hom}_B(\pi, -)$ always preserves short exact sequences in DVPB_B for the trivial bundle $\pi : B \times \mathbb{R} \to B$, as it is naturally isomorphic to the identity functor. But the trivial bundle may not be projective; see Example 4.27.

As a consequence of Proposition 4.20 and [Christensen and Wu 2022, Proposition 3.12], we have:

Corollary 4.22. *If $E_1 \to B$ and $E_2 \to B$ are projective diffeological vector pseudo-bundles, then so is their tensor product $E_1 \otimes E_2 \to B$.*

Since $\bigwedge^k E$ is a smooth direct summand of $E^{\otimes k}$ (as a result of [Pervova 2019, Lemma 2.11] and Theorem 4.3), by the above corollary and Proposition 4.5 (2), we have:

Corollary 4.23. *If $E \to B$ is a projective diffeological vector pseudo-bundle, then so is each exterior product $\bigwedge^k E \to B$ for $k \geq 1$.*

4C. *Base change.*

Theorem 4.24. *The pushforward $f_* : \mathrm{DVPB}_B \to \mathrm{DVPB}_{B'}$ sends projectives in the domain to the projectives in the codomain.*

Proof. By the adjunction of Theorem 3.3, the following lifting problems are equivalent:

$$\begin{tikzcd} & f_*(E) \arrow[d] \arrow[dl, dotted] \\ E'_1 \arrow[r] & E'_2 \end{tikzcd} \quad \Longleftrightarrow \quad \begin{tikzcd} & E \arrow[d] \arrow[dl, dotted] \\ f^*(E'_1) \arrow[r] & f^*(E'_2) \end{tikzcd}$$

where $E'_1 \to E'_2$ is a bundle subduction over B'. By Lemma 4.10 and Definition 4.4, we know that the lifting problem on the right has a solution, and hence so is the one on the left. □

This theorem has several applications. We first give another class of examples of projective diffeological vector pseudo-bundles from tangent bundles of diffeological spaces. To do so, we need the following result:

Note that projective diffeological vector pseudo-bundles are defined in DVPB_B, but they have a similar property in DVPB as follows:

Proposition 4.25. *Given a bundle subduction $f : E'_1 \to E'_2$ over B' and a bundle map*

$$\begin{tikzcd} E \arrow[r, "g"] \arrow[d, "\pi"'] & E'_2 \arrow[d] \\ B \arrow[r, "l"'] & B' \end{tikzcd}$$

with π projective, there exists a bundle map $h : E \to E'_1$ such that $g = f \circ h$.

Proof. By the universal property of pushforward (Proposition 3.6), we can write g as a bundle map $\tilde{g} : l_*(E) \to E'_2$ over B' followed by the bundle map $\alpha_l : E \to l_*(E)$. By Theorem 4.24, the assumption that π is projective over B implies that $\pi_l : l_*(E) \to B'$ is projective over B'. Therefore, we have a bundle map $\tilde{h} : l_*(E) \to E'_1$ over B' such that $\tilde{g} = f \circ \tilde{h}$. Then the composite $\tilde{h} \circ \alpha_l$ is the bundle map h we are looking for. □

Let B be an arbitrary diffeological space, and let $b \in B$. The local structure of B at b is encoded by the pointed plot category whose objects are the pointed plots $(U, 0) \to (B, b)$ for open subsets U of some Euclidean spaces containing the origin 0, and whose morphisms are the obvious commutative triangles preserving the base points. The (internal) tangent space $T_b(B)$ is defined to be the colimit of the functor from the pointed plot category to the category of vector spaces by sending $p : (U, 0) \to (B, b)$ to $T_0(U)$. As a set, the total space TB of the (internal) tangent bundle of B is the disjoint union of all these $T_b(B)$, and $TB \to B$ is the obvious projection. Every plot $p : U \to B$ gives rise to a natural commutative square

$$\begin{tikzcd} TU \arrow[r, "Tp"] \arrow[d] & TB \arrow[d] \\ U \arrow[r, "p"'] & B \end{tikzcd}$$

Hector [1995] defined a diffeology on the set TB as the smallest one containing all such Tp, and we denote this diffeological space as $T^H B$. In this way, $T^H B \to B$ is in general a diffeological vector prebundle, but not necessarily a diffeological vector pseudo-bundle. Its dvsification is denoted by $T^{\mathrm{dvs}} B \to B$.

Equivalently, [Christensen and Wu 2016, Theorem 4.17] claims that every tangent bundle $T^{\mathrm{dvs}} B \to B$ of a diffeological space B is a colimit in DVPB of the tangent bundles $TU \to U$ indexed by the plots $U \to B$. Each $TU \to U$ is projective by Corollary 4.14. It is possible that some tangent bundles are projective. (But this is not always the case; see Example 4.27.) We show this by an example:

Example 4.26. Write B for the cross with the gluing diffeology. We show below that the tangent bundle $T^{\mathrm{dvs}} B \to B$ is projective.

Note that B is the pushout of

$$\mathbb{R} \xleftarrow{\;0\;} \mathbb{R}^0 \xrightarrow{\;0\;} \mathbb{R}$$

in Diff. It is straightforward to check that the tangent bundle $T^{\mathrm{dvs}}B \to B$ is the colimit of

$$\begin{tikzcd} T\mathbb{R} \arrow[d] & T\mathbb{R}^0 \arrow[l, "T0"'] \arrow[r, "T0"] \arrow[d] & T\mathbb{R} \arrow[d] \\ \mathbb{R} & \mathbb{R}^0 \arrow[l, "0"] \arrow[r, "0"'] & \mathbb{R} \end{tikzcd}$$

in DVPB. Write $Tx : T\mathbb{R} \to T^{\mathrm{dvs}}B$ and $Ty : T\mathbb{R} \to T^{\mathrm{dvs}}B$ for the two structural maps. Given a bundle subduction $f : E_1 \to E_2$ over B and a bundle map $g : T^{\mathrm{dvs}}B \to E_2$, since $T\mathbb{R} \to \mathbb{R}$ is projective, by Proposition 4.25 we have bundle maps $hx, hy : T\mathbb{R} \to E_1$ such that $g \circ Tx = f \circ hx$ and $g \circ Ty = f \circ hy$. By the universal property of pushout, we get a desired bundle map $h : T^{\mathrm{dvs}}B \to E_1$ over B with the required property.

As another consequence of Theorem 4.24, we have the following example which gives counterexamples to several arguments:

Example 4.27. If the free diffeological vector space $F(B)$ is not projective, then the trivial bundle $B \times \mathbb{R} \to B$ is not projective. This happens when the D-topology on B is not Hausdorff [Christensen and Wu 2019, Corollary 3.17]. The proof of the statement follows from Proposition 3.4 and Theorem 4.24.

This example shows that not every trivial bundle is projective, even when the fibre is a projective (or fine) diffeological vector space. It also shows that the pullback functor does *not* preserve projectives, since the trivial bundle $B \times \mathbb{R} \to B$ is the pullback of $\mathbb{R} \to \mathbb{R}^0$ along the map $B \to \mathbb{R}^0$. Furthermore, it shows that not every tangent bundle is projective. For example, $TB \to B$ is not projective when B is an irrational torus, since in this case $TB = B \times \mathbb{R}$ [Christensen and Wu 2016, combining Examples 3.23 and 4.19 (3) with Theorem 4.15] and the D-topology on B is not Hausdorff.

Moreover, via Theorem 4.24 and Section 4B, we get many examples of projective diffeological vector spaces from classical differential geometry!

5. Applications to smooth splittings of projective diffeological vector spaces

By [Christensen and Wu 2019, Proposition 3.14 and Theorem 4.2], we know that every finite-dimensional linear subspace of a projective diffeological vector space is a smooth direct summand; or in other words, the only indecomposable projective diffeological vector space is $\mathbb{R}$. In this section, we use classical smooth bundle theory, and the theory established so far, to get some general criteria and interesting examples of smooth splittings of projective diffeological vector spaces.

To simplify notation, we write V_π (or V_E when the bundle is understood) for the diffeological vector space obtained from the pushforward of the diffeological vector pseudo-bundle $\pi : E \to B$ along the map $B \to \mathbb{R}^0$.

5A. ***General theory.*** Here is the general setup. Given a classical fibre (respectively, principal) bundle $E \to B$, we get a linear subduction $F(E) \to F(B)$ of diffeological vector spaces which splits smoothly since $F(B)$ is projective. We aim to give a bundle-theoretical explanation of its kernel. In fact, we will prove more general results as follows:

Given a bundle map

$$\begin{tikzcd} E_1 \arrow[r, "g"] \arrow[d, "\pi_1"'] & E_2 \arrow[d, "\pi_2"] \\ B_1 \arrow[r, "f"'] & B_2 \end{tikzcd}$$

from a diffeological vector pseudo-bundle π_1 to another π_2, by Proposition 3.6, we get a bundle map $h : f_*(E_1) \to E_2$ over B_2 so that $g = h \circ \alpha_f$, where $\alpha_f : E_1 \to f_*(E_1)$ is the structural map introduced at the beginning of Section 3. Write $\pi : E \to B_2$ for the kernel of h.

Here is the key result:

Theorem 5.1. *Let $(g, f) : \pi_1 \to \pi_2$ be a bundle map as above, with E_1 locally Euclidean, and B_2 Hausdorff and filtered*[4]*. Then we have a smooth linear map $g_* : V_{\pi_1} \to V_{\pi_2}$ between diffeological vector spaces, whose kernel is isomorphic to V_π, with $\pi : E \to B_2$ defined above.*

Proof. By Proposition 3.6, we get a smooth linear map $g_* : V_{\pi_1} \to V_{\pi_2}$. Write K for its kernel. It consists of elements of finite sum $\sum_i e_i$ in V_{π_1}, with $e_i \in E_1$, such that for each $b_2 \in B_2$, the subsum $\sum_{i:\pi_2 \circ g(e_i)=b_2} g(e_i) = 0$. So there is a canonical isomorphism $\alpha : V_\pi \to K$ as vector spaces, which is smooth by Lemma 3.1.

Now we use all the extra assumptions to show that the inverse map α^{-1} is smooth. Take a plot $p : U \to K$ and fix $u_0 \in U$. Since the composite $U \to K \hookrightarrow V_{\pi_1}$ is smooth, by Lemma 3.1, there exist finitely many plots $p_i : U \to E_1$, by shrinking U around u_0 if necessary, such that $p(u) = \sum_i p_i(u)$ which satisfies that for each $b_2 \in B_2$, the subsum $\sum_{i: f \circ \pi_1 \circ p_i(u)=b_2} g(p_i(u)) = 0$ for every $u \in U$. Fix $b_2^0 \in B_2$. Since B_2 is Hausdorff, we may assume that the image of the composites $f \circ \pi_1 \circ p_i$ do not intersect if their value at u_0 are distinct. Now take all the index i so that $f \circ \pi_1 \circ p_i(u_0) = b_2^0$, and denote this index subset by I_{u_0, b_2^0}. Since E_1 is locally

[4]A diffeological space X is *filtered*, if for every $x \in X$, the germ category of X at x is filtered, i.e., every finite diagram in the germ category has a cocone. Here, the germ category is like the pointed plot category, with morphisms changed to be smooth germs at the base points instead of genuine pointed maps. See [Christensen and Wu 2017; 2022] for more details.

Euclidean and B_2 is filtered, there exist a pointed plot $q : (V, 0) \to (B_2, b_2^0)$ and smooth pointed germs $h_i : (E_1, p_i(u_0)) \to (V, 0)$, so that $q \circ h_i = f \circ \pi_1$ and $h_i \circ p_i$ is independent of i, for all $i \in I_{u_0, b_2^0}$. This then implies that $f \circ \pi_1 \circ p_i = q \circ h_i \circ p_i$ are independent of i for all $i \in I_{u_0, b_2^0}$, and hence follows the smoothness of α^{-1}. □

Proposition 5.2. *If $(g, f) : \pi_1 \to \pi_2$ is a bundle subduction, then we get a linear subduction $g_* : V_{\pi_1} \to V_{\pi_2}$ of diffeological vector spaces.*

Proof. This follows directly from Proposition 3.6 and Lemma 3.1. □

As a consequence of the above results, we have:

Corollary 5.3. *Let $(g, f) : \pi_1 \to \pi_2$ be a bundle subduction so that E_1 is locally Euclidean, and B_2 is Hausdorff and filtered. Then we have a short exact sequence of diffeological vector spaces*

$$0 \to V_\pi \to V_{\pi_1} \to V_{\pi_2} \to 0.$$

Now we discuss a special case:

$$\begin{tikzcd} Y \times \mathbb{R} \arrow[r, "f \times 1_{\mathbb{R}}"] \arrow[d, "\mathrm{Pr}_1"'] & B \times \mathbb{R} \arrow[d, "\mathrm{Pr}_1"] \\ Y \arrow[r, "f"'] & B \end{tikzcd} \tag{2}$$

where f is an arbitrary smooth map.

Observe that:

Proposition 5.4. *The pushforward of* $\mathrm{Pr}_1 : Y \times \mathbb{R} \to Y$ *along* $f : Y \to B$ *is exactly the free bundle* $F_B(Y) \to B$.

Proof. This follows directly from the definition of the free bundle (see [Christensen and Wu 2022, Section 3.2.5] or the paragraph right after Proposition 4.5) and the definition of pushforward of a diffeological vector pseudo-bundle from Section 3. □

Note that the bundle map $F_B(Y) \to B \times \mathbb{R}$ over B is given by $\sum_i r_i [y_i] \mapsto \big(b, \sum_i r_i\big)$, where $f(y_i) = b$ for all i. We write $\bar{f}_* : \bar{F}_B(Y) \to B$ for its kernel.

Remark 5.5. (1) This proposition generalises Proposition 3.4 by taking $B = \mathbb{R}^0$.

(2) From above, we know that $F(Y)$ always has a smooth direct summand $\mathbb{R}$ (i.e., $F(Y) \cong \mathbb{R} \oplus \bar{F}(Y)$), since $\mathbb{R}$ is a projective diffeological vector space. This can be viewed as a property of the free diffeological vector space, and not every diffeological vector space is free over some diffeological space.

On the contrary, not every trivial line bundle $B \times \mathbb{R} \to B$ is projective when $B \neq \mathbb{R}^0$ (see Example 4.27), so the free bundle $F_B(Y) \to B$ may not have a smooth direct summand $B \times \mathbb{R} \to B$.

In the current special case, we have:

Corollary 5.6. *Let $f : Y \to B$ be a smooth map, with Y locally Euclidean and B Hausdorff and filtered.*

(1) *The kernel of $f_* : F(Y) \to F(B)$ is isomorphic to $V_{\bar{f}_*}$ with $\bar{f}_* : \overline{F}_B(Y) \to B$ as defined above.*

(2) *If f is a subduction, then we get a short exact sequence of diffeological vector spaces*

$$0 \to V_{\bar{f}_*} \to F(Y) \to F(B) \to 0.$$

(3) *The pushforward of the free bundle $F_B(Y) \to B$ along $B \to \mathbb{R}^0$ is isomorphic to the free diffeological vector space $F(Y)$.*

Remark 5.7. To make $f_* : F(Y) \to F(B)$ a linear subduction, it is not necessary to require $f : Y \to B$ to be a subduction; see Example 3.7.

Now we discuss a more special case, which occurs often in practice: In the diagram (2), we further assume that f is a principal G-bundle[5] for some diffeological group G. We give an alternative description of the bundle $V_{\bar{f}_*}$ as follows.

As a setup, assume that G acts smoothly on Y on the right. Note that G acts smoothly on $F(G)$ on the left by $G \times F(G) \to F(G)$, given by $g \cdot \sum_i r_i[g_i] = \sum_i r_i[gg_i]$, and it passes to a smooth left action of G on $\overline{F}(G)$, where $\overline{F}(G)$ is the linear subspace of $F(G)$ consisting of elements of finite sum $\sum_i r_i[g_i]$ with $\sum_i r_i = 0$. So we get a commutative square in Diff:

$$\begin{tikzcd} Y \times \overline{F}(G) \arrow[r] \arrow[d] & \tilde{E} \arrow[d, "\tilde{\pi}"] \\ Y \arrow[r, "f"'] & B \end{tikzcd} \tag{3}$$

where $\tilde{E}$ is the quotient of $Y \times \overline{F}(G)$ with $(y, v) \sim (y \cdot g, g^{-1} \cdot v)$ for $y \in Y$, $g \in G$ and $v \in \overline{F}(G)$, and $\tilde{\pi}[y, v] = f(y)$.

Lemma 5.8. *With the above notations, $\tilde{\pi}$ is a vector bundle over B with fibre $\overline{F}(G)$.*

Proof. Let $p : U \to B$ be a plot. Since $f : Y \to B$ is a principal G-bundle, we may shrink U so that we have a pullback diagram:

$$\begin{tikzcd} U \times G \arrow[r, "\phi"] \arrow[d] & Y \arrow[d, "f"] \\ U \arrow[r, "p"'] & B \end{tikzcd}$$

[5] Principal bundle here is in the sense of [Iglesias-Zemmour 1985], i.e., pullback along every plot of the base space is locally trivial. The same applies to all principal (respectively, vector, fibre) bundles and coverings discussed afterwards.

We are left to show that there is an isomorphism $\alpha : P \to U \times \overline{F}(G)$ as diffeological vector pseudo-bundles over U, where P is the pullback of

$$U \xrightarrow{p} B \xleftarrow{\tilde{\pi}} \tilde{E}.$$

We define $\alpha(u, [y, v]) = (u, \theta(u, y) \cdot v)$, where $y = \phi(u, e) \cdot \theta(u, y)$ since $f(y) = p(u) = f(\phi(u, e))$, and e is the identity element in the group G. It is clear that α is smooth and fibrewise isomorphic as vector spaces. And α^{-1} is given by $(u, v) \mapsto (u, [\phi(u, e), v])$, which is obviously smooth. $\square$

It is straightforward to check that the above square (3) is a bundle map.

Proposition 5.9. *Recall that the kernel of the bundle map $F_B(Y) \to B \times \mathbb{R}$ over B is denoted by $\bar{f}_* : \overline{F}_B(Y) \to B$. It is isomorphic to $\tilde{\pi} : \tilde{E} \to B$ as vector bundles over B.*

Proof. The isomorphism as vector bundles over B is given by $\tilde{E} \to \overline{F}_B(Y)$ with $\left[y, \sum_i r_i[g_i]\right] \mapsto \sum_i r_i[y \cdot g_i]$, and it is easy to check all the required conditions. $\square$

As a consequence of the above results, we have:

Corollary 5.10. *Let $f : Y \to B$ be a principal G-bundle with Y being locally Euclidean, and B being Hausdorff and filtered. Then we have a short exact sequence of diffeological vector spaces*

$$0 \to V_{\tilde{\pi}} \to F(Y) \to F(B) \to 0.$$

Note that when $f : Y \to B$ is a classical fibre (respectively, principal) bundle, the conditions (f being a subduction, Y being locally Euclidean, and B being Hausdorff and filtered) are satisfied.

Proposition 5.11. *Let $\pi : E \to Y$ be a vector bundle of fibre type a diffeological vector space V, and let $f : Y \to B$ be a fibre bundle of fibre type a diffeological space X.*

(1) *If X is finite discrete*[6], *then the pushforward $f_*(E) \to B$ is a vector bundle with fibre type $F(X) \otimes V$.*

(2) *Assume that both π and f are locally trivial, and there exists a D-open covering $\{B_i\}_i$ of B which trivialises f and simultaneously the D-open covering $\{f^{-1}(B_i)\}_i$ trivialises π. Then the pushforward $f_*(E) \to B$ is also a locally trivial vector bundle of fibre type $F(X) \otimes V$.*

(3) *If π is trivial, then $f_*(E) \to B$ is a vector bundle of fibre type $F(X) \otimes V$.*

[6]When the fibre of a fibre bundle $f : Y \to B$ is discrete, f is also called a covering.

Proof. (1): Let $p : U \to B$ be a plot. Since $f : Y \to B$ is a covering with fibre type X, we may shrink U to get a pullback diagram:

$$\begin{tikzcd} U \times X \arrow[r, "\phi"] \arrow[d] & Y \arrow[d, "f"] \\ U \arrow[r, "p"'] & B \end{tikzcd}$$

Since $\pi : E \to Y$ is a vector bundle of fibre type V, for each $x \in X$, we may further shrink U to get a pullback diagram:

$$\begin{tikzcd} U \times \{x\} \times V \arrow[r, "\psi_x"] \arrow[d] & E \arrow[d, "\pi"] \\ U \times \{x\} \arrow[r, "\phi|_{U \times \{x\}}"'] & Y \end{tikzcd}$$

As X is finite discrete, we gather these together and get a pullback diagram:

$$\begin{tikzcd} U \times X \times V \arrow[r, "\psi"] \arrow[d] & E \arrow[d, "\pi"] \\ U \times X \arrow[r, "\phi"'] & Y \end{tikzcd}$$

Write P for the pullback of $U \xrightarrow{p} B \leftarrow f_*(E)$. Then P consists of elements of the form $\left(u, \sum_i e_{y_i}\right)$, with $p(u) = f(y_i)$ for all i. Define $U \times (F(X) \otimes V) \to P$ by linear expansion of $(u, [x] \otimes v) \mapsto (u, \psi(u, x, v))$. It is straightforward to check that this map is smooth and an isomorphism of vector spaces, and its inverse is also smooth.

(2) and (3) can be proved in a similar way. $\square$

Corollary 5.12. *If $f : Y \to B$ is a (locally trivial) fibre bundle of fibre type a diffeological space X, then $F_B(Y) \to B$ is a (locally trivial) vector bundle of fibre type $F(X)$.*

Proof. This follows immediately from Propositions 5.4 and 5.11. $\square$

5B. *Examples.* Now we deal with the case of a principal bundle whose group G is discrete. In this case, $F(G)$ is a fine diffeological vector space whose dimension matches the cardinality of G, and $\bar{F}(G)$ is a codimension-one linear subspace of $F(G)$, and hence also a fine diffeological vector space.

Example 5.13. For the principal $\mathbb{Z}/2\mathbb{Z}$-bundle $S^n \to \mathbb{R}P^n$, $F(\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{R}^2$ and $\bar{F}(\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{R}$. And therefore, the bundle $\tilde{\pi}$ in the commutative square (3) in the previous subsection can be viewed as the quotient of $S^n \times \mathbb{R}$ with the equivalence relation given by $(z, x) \sim (-z, -x)$, which is the tautological line bundle γ_n^1 on $\mathbb{R}P^n$. So we have an isomorphism

$$F(S^n) \cong F(\mathbb{R}P^n) \oplus V_{\gamma_n^1}. \tag{4}$$

Taking $n = 1$, γ_1^1 is the Möbius band. Moreover, since $\mathbb{R}P^1$ is diffeomorphic to S^1, we get

$$F(S^1) \cong F(S^1) \oplus V_{\gamma_1^1} \cong \cdots \cong F(S^1) \oplus (V_{\gamma_1^1})^m \tag{5}$$

for any $m \in \mathbb{N}$.

By some results from [Milnor and Stasheff 1974], we have:

Example 5.14. (1) Since the tangent bundle $TS^n \to S^n$ direct sum the normal bundle (which is the trivial line bundle) of S^n in $\mathbb{R}^{n+1}$ is a trivial bundle over S^n of rank $n+1$, we get

$$F(S^n)^{n+1} \cong F(S^n) \oplus V_{TS^n}.$$

Moreover, by [Adams 1962], V_{TS^n} has a smooth direct summand $F(S^n)^{\rho(n+1)-1}$, where $\rho(n+1) = 2^c + 8d$ with $n+1 = 2^b(2a+1)$, $b = c+4d$ and $0 \leq c \leq 3$.

(2) Since the tangent bundle $T\mathbb{R}P^n \to \mathbb{R}P^n$ direct sum the trivial line bundle over $\mathbb{R}P^n$ is isomorphic to the direct sum of $n+1$ copies of the tautological line bundle $\gamma_n^1 \to \mathbb{R}P^n$, we get

$$(V_{\gamma_n^1})^{n+1} \cong F(\mathbb{R}P^n) \oplus V_{T\mathbb{R}P^n}.$$

(3) The total space of the tangent bundle $TS^n \to S^n$ can be viewed as a submanifold of $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$, with the first component for the base and the second one for the tangent part. If we identify (x, v) with $(-x, -v)$ in TS^n, we get the total space of the tangent bundle $T\mathbb{R}P^n \to \mathbb{R}P^n$; if we identify (x, v) with $(-x, v)$ in TS^n, we get another locally trivial vector bundle $\pi : E \to \mathbb{R}P^n$ of rank n. (In the case $n = 1$, π is exactly the Möbius band over $\mathbb{R}P^1$; notice the difference from Example 5.13, based on the different meaning of the coordinates!) Write $f : S^n \to \mathbb{R}P^n$ for the quotient map. Note that $E \to f_*(TS^n)$ given by $[x, v] \mapsto (x, v) + (-x, v)$ is a bundle map over $\mathbb{R}P^n$, using Proposition 5.11 (1), which is the kernel of the canonical bundle map $f_*(TS^n) \to T\mathbb{R}P^n$. Hence, we have an isomorphism

$$V_{TS^n} \cong V_{T\mathbb{R}P^n} \oplus V_\pi,$$

which also recovers the first isomorphism in (5) in Example 5.13.

Therefore, if we combine the three isomorphisms in this example, we get

$$F(\mathbb{R}P^n) \oplus F(S^n)^{n+1} \cong F(S^n) \oplus V_\pi \oplus (V_{\gamma_n^1})^{n+1}.$$

By taking $n = 1$, we obtain

$$F(S^1)^3 \cong F(S^1) \oplus (V_{\gamma_1^1})^3 \cong F(S^1).$$

Remark 5.15. (1) The isomorphism $F(S^1)^3 \cong F(S^1)$ implies that pushforward can take nonisomorphic bundles over the same base space into isomorphic diffeological vector spaces.

(2) The isomorphism $F(S^1)^3 \cong F(S^1)$ can also be derived directly by considering the covering map $S^1 \to S^1$ with $z \mapsto z^3$.

(3) I don't know if $F(S^1)^2$ is isomorphic to $F(S^1)$ as diffeological vector spaces. If it is not, then there seems to be some connection with Bott periodicity in the complex case.

(4) I wonder if the approach here can lead to an alternative proof of the maximal number of linearly independent vector fields on spheres.

Finally, we show by the following example that the extra condition of filteredness added to the results in the previous subsection is necessary:

Example 5.16. Let $\mathbb{Z}/2\mathbb{Z}$ act on $\mathbb{R}$ by $\pm 1 \cdot x = \pm x$, and write B for the quotient space. Then B is weakly filtered but not filtered [Christensen and Wu 2017, Example 4.7], and B with the D-topology is homeomorphic to the subspace $[0, \infty)$ of $\mathbb{R}$ (hence is Hausdorff). Write $f : \mathbb{R} \to B$ for the quotient map, and write K for the kernel of $F(\mathbb{R}) \to F(B)$. It consists of elements of the form of a finite sum $\sum_i r_i [x_i]$ with $r_i, x_i \in \mathbb{R}$ such that for every fixed $x \in X$, the subsum $\sum_{i : x_i = \pm x} r_i = 0$. So, $p : \mathbb{R} \to K$ defined by $t \mapsto [t] - [-t]$ is a plot of K. On the other hand, the map $f_* : F_B(\mathbb{R}) \to B$ has fibre $\mathbb{R}$ over $[0] \in B$ and fibre $\mathbb{R}^2$ over $[b] \in B$ for $b \neq 0$. Hence, $\bar{f}_* : \bar{F}_B(\mathbb{R}) \to B$ has fibre $\mathbb{R}^0$ over $[0] \in B$ and fibre $\mathbb{R}$ over $[b] \in B$ for $b \neq 0$. The canonical smooth linear bijection $\alpha : V_{\bar{f}_*} \to K$ is not an isomorphism of diffeological vector spaces since $\alpha^{-1} \circ p$ is not a plot of $V_{\bar{f}_*}$. If it were, then by iterated use of Lemma 3.1, there exist finitely many smooth germs $(p^1_{i,j}, p^2_{i,j}) : \mathbb{R} \to \mathbb{R}_{(\text{base})} \times \mathbb{R}_{(\text{fibre})}$ at $0 \in \mathbb{R}$ such that

$$p(t) = \alpha\Big(\sum_{i,j} \alpha_g\big(\alpha_f(p^1_{i,j}(t), p^2_{i,j}(t))\big)\Big),$$

where $g : B \to \mathbb{R}^0$, both α_f and α_g are structural maps from Section 3, the range of j depends on i, $f \circ p^1_{i,j}$ is independent of j for any fixed i, $p^2_{i,j}(t) = 0$ whenever $p^1_{i,j}(t) = 0$ (by the description of $V_{\bar{f}_*}$), which causes the contradiction as follows: By evaluating at $t = 0$, we know that

$$\sum_{i,j : p^1_{i,j}(0) = x} p^2_{i,j}(0) = 0$$

for any fixed $x \in \mathbb{R} \setminus \{0\}$. By continuity of the $p^2_{i,j}$, we know that

$$\sum_{i,j : p^1_{i,j}(t) = t} p^2_{i,j}(t) \neq 1$$

for $t \neq 0$ but sufficiently close to 0, which implies that $\alpha^{-1} \circ p$ cannot be a plot.

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