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ELEMENTS OF HIGHER HOMOTOPY GROUPS UNDETECTABLE BY POLYHEDRAL APPROXIMATION

JOHN K. ACETI AND JEREMY BRAZAS

When nontrivial local structures are present in a topological space X, a common approach to characterizing the isomorphism type of the n-th homotopy group $\pi_n(X, x_0)$ is to consider the image of $\pi_n(X, x_0)$ in the n-th Čech homotopy group $\check{\pi}_n(X, x_0)$ under the canonical homomorphism $\Psi_n : \pi_n(X, x_0) \to \check{\pi}_n(X, x_0)$. The subgroup $\ker(\Psi_n)$ is the obstruction to this tactic as it consists of precisely those elements of $\pi_n(X, x_0)$, which cannot be detected by polyhedral approximations to X. In this paper, we use higher dimensional analogues of Spanier groups to characterize $\ker(\Psi_n)$. In particular, we prove that if X is paracompact, Hausdorff, and LC^{n-1}, then $\ker(\Psi_n)$ is equal to the n-th Spanier group of X. We also use the perspective of higher Spanier groups to generalize a theorem of Kozlowski–Segal, which gives conditions ensuring that Ψ_n is an isomorphism.

1. Introduction

When nontrivial local structures are present in a topological space X, a common approach to characterizing the isomorphism type of $\pi_n(X, x_0)$ is to consider the image of $\pi_n(X, x_0)$ in the n-th Čech (shape) homotopy group $\check{\pi}_n(X, x_0)$ under the canonical homomorphism $\Psi_n : \pi_n(X, x_0) \to \check{\pi}_n(X, x_0)$. The *$n$-th shape kernel* $\ker(\Psi_n)$ is the obstruction to this tactic as it consists of precisely those elements of $\pi_n(X, x_0)$, which cannot be detected by polyhedral approximations to X. This method has proved successful in many situations for both the fundamental group [Cannon and Conner 2006; Eda and Kawamura 1998; Fischer and Guilbault 2005; Fischer and Zastrow 2005] and higher homotopy groups [Brazas 2021; Eda and Kawamura 2000a; 2010; Eda et al. 2013; Kawamura 2003]. In this paper, we study the map Ψ_n and give a characterization the n-th shape kernel in terms of higher-dimensional analogues of Spanier groups.

The subgroups of fundamental groups, which are now commonly referred to as "Spanier groups," first appeared in E.H. Spanier's unique approach [1966] to

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covering space theory. If $\mathscr{U}$ is an open cover of a topological space X and $x_0 \in X$, then the *Spanier group with respect to* $\mathscr{U}$ is the subgroup $\pi_1^{Sp}(\mathscr{U}, x_0)$ of $\pi_1(X, x_0)$ generated by path-conjugates $[\alpha][\gamma][\alpha]^{-1}$ where α is a path starting at x_0 and γ is a loop based at $\alpha(1)$ with image being contained in some element of $\mathscr{U}$. These subgroups are particularly relevant to covering space theory since, when X is locally path-connected, a subgroup $H \leq \pi_1(X, x_0)$ corresponds to a covering map $p:(Y, y_0) \to (X, x_0)$ if and only if $\pi_1^{Sp}(\mathscr{U}, x_0) \leq H$ for some open cover $\mathscr{U}$ [Spanier 1966, 2.5.12]. The intersection $\pi_1^{Sp}(X, x_0) = \bigcap_{\mathscr{U}} \pi_1^{Sp}(\mathscr{U}, x_0)$ is called the *Spanier group of* (X, x_0) [Fischer et al. 2011]. The inclusion $\pi_1^{Sp}(X, x_0) \subseteq \ker(\Psi_1)$ always holds [Fischer and Zastrow 2007, Proposition 4.8]. It is proved in [Brazas and Fabel 2014, Theorem 6.1] that $\pi_1^{Sp}(X, x_0) = \ker(\Psi_1)$ whenever X is paracompact Hausdorff and locally path connected. The upshot of this equality is having a description of level-wise generators (for each open cover $\mathscr{U}$) whereas there may be no readily available generating set for the kernel of a homomorphism induced by a canonical map from X to the nerve $|N(\mathscr{U})|$. Indeed, 1-dimensional Spanier groups have proved useful in persistence theory [Virk 2020]. Since much of applied topology is based on a geometric refinement of polyhedral approximation from shape theory, there seems potential for higher dimensional analogues to be useful as well.

Higher dimensional analogues of Spanier groups recently appeared in [Bahredar et al. 2021] and are defined in a similar way: $\pi_n^{Sp}(\mathscr{U}, x_0)$ is the subgroup of $\pi_n(X, x_0)$ consisting of homotopy classes of path-conjugates $\alpha * f$ where α is a path starting at x_0 and $f : S^n \to X$ is based at $\alpha(1)$ with image being contained in some element of $\mathscr{U}$. Then $\pi_n^{Sp}(X, x_0)$ is the intersection of these subgroups. In this paper, we prove a higher-dimensional analogue of the 1-dimensional equality $\pi_1^{Sp}(X, x_0) = \ker(\Psi_1)$ from [Brazas and Fabel 2014].

A space X is LC^n if for every neighborhood U of a point $x \in X$, there is a neighborhood V of x in U such that every map $f : S^k \to V$, $0 \leq k \leq n$ is null-homotopic in U. When a space is LC^n, "small" maps on spheres of dimension $\leq n$ contract by null-homotopies of relatively the same size. Certainly, every locally n-connected space is LC^n. However, when $n \geq 1$, the converse is not true even for metrizable spaces. Our main result is the following.

Theorem 1.1. *Let $n \geq 1$ and $x_0 \in X$. If X is paracompact, Hausdorff, and LC^{n-1}, then $\pi_n^{Sp}(X, x_0) = \ker(\Psi_n)$.*

This result confirms that higher Spanier groups, like their 1-dimensional counterparts, often identify precisely those elements of $\pi_n(X, x_0)$ which can be detected by polyhedral approximations to X. More precisely, under the hypotheses of Theorem 1.1, $g \in \pi_n^{Sp}(X, x_0)$ if and only if $f_{\#}(g) = 0$ for every map $f : X \to K$ to a polyhedron K. A first countable path-connected space is LC^0 if and only if it is locally path connected. Hence, in dimension $n = 1$, Theorem 1.1 only expands [Brazas and Fabel 2014, Theorem 6.1] to some nonfirst countable spaces.

Regarding the proof of Theorem 1.1, the inclusion $\pi_n^{Sp}(X, x_0) \subseteq \ker(\Psi_n)$ was first proved for $n = 1$ in [Fischer and Zastrow 2007, Proposition 4.8] and for $n \geq 2$ in [Bahredar et al. 2021, Theorem 4.14]. We include this proof for the sake of completion (Corollary 3.11). The proof of the inclusion $\ker(\Psi_n) \subseteq \pi_n^{Sp}(X, x_0)$ appears in Section 5 and is more intricate, requiring a carefully chosen sequence of open cover refinements using the LC^{n-1} property. These refinements allow one to recursively extend maps on simplicial complexes skeleton-wise. These extension methods, established in Section 4, are similar to methods found in [Kozlowski and Segal 1977; 1978].

We also put these extension methods to work in Section 6 where we identify conditions that imply Ψ_n is an isomorphism. Kozlowski and Segal [1978], proved that if X is paracompact Hausdorff and LC^n, then Ψ_n is an isomorphism. Fischer and Zastrow [2007], generalized this result in dimension $n = 1$ by replacing "LC^1" with "locally path connected and semilocally simply connected." Similar, to the approach of Fischer and Zastrow, our use of Spanier groups shows that the existence of *small* null-homotopies of small maps $S^n \to X$ (specifically in dimension n) is not necessary to prove that Ψ_n is injective. We say a space X is *semilocally π_n-trivial* if for every $x \in X$ there exists an open neighborhood U of x such that every map $S^n \to U$ is null-homotopic in X. This definition is independent of lower dimensions but certainly $LC^n \Rightarrow (LC^{n-1}$ and semilocally π_n-trivial). Our second result proves Kozlowski–Segal's theorem under a weaker hypothesis and is stated as follows.

Theorem 1.2. *Let $n \geq 1$ and $x_0 \in X$. If X is paracompact, Hausdorff, LC^{n-1}, and semilocally π_n-trivial, then $\Psi_n : \pi_n(X, x_0) \to \check{\pi}_n(X, x_0)$ is an isomorphism.*

The hypotheses in Theorem 1.2 are the homotopical versions of the hypotheses used in [Mardešić 1959] to ensure that the canonical homomorphism $\varphi_* : H_n(X) \to \check{H}_n(X)$ is an isomorphism; see also [Eda and Kawamura 2000b] regarding the surjectivity of φ_*. Examples show that Ψ_n can fail to be an isomorphism if X is semilocally π_n-trivial but not LC^{n-1} (Example 7.4) or if X is LC^{n-1} but not semilocally π_n-trivial (Example 7.5).

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2. Preliminaries and notation

Throughout this paper, X is assumed to be a path-connected topological space with basepoint x_0. The unit interval is denoted I and S^n is the unit n-sphere with basepoint $d_0 = (1, 0, \ldots, 0)$. The n-th homotopy group of (X, x_0) is denoted

$\pi_n(X, x_0)$. If $f : (X, x_0) \to (Y, y_0)$ is a based map, then $f_\# : \pi_n(X, x_0) \to \pi_n(Y, y_0)$ is the induced homomorphism.

A *path* in a space X is a map $\alpha : I \to X$ from the unit interval. The *reverse* of α is the path given by $\alpha^-(t) = \alpha(1-t)$ and the concatenation of two paths α, β with $\alpha(1) = \beta(0)$ is denoted $\alpha \cdot \beta$. Similarly, if $f, g : S^n \to X$ are maps based at $x \in X$, then $f \cdot g$ denotes the usual n-loop concatenation and f^- denotes the reverse map. We may write $\prod_{i=1}^m f_i$ to denote an m-fold concatenation $f_1 \cdot f_2 \cdot \cdots \cdot f_m$.

2.1. ***Simplicial complexes.*** We make heavy use of standard notation and theory of abstract and geometric simplicial complexes, which can be found in texts such as [Mardešić and Segal 1982; Munkres 1984]. We briefly recall relevant notation.

For an abstract (geometric) simplicial complex K and integer $r \geq 0$, K_r denotes the r-skeleton of K. If K is abstract, $|K|$ denotes the geometric realization of K with the weak topology. If K is geometric, then $\mathrm{sd}^m K$ denotes the m-th barycentric subdivision of K and if v is a vertex of K, then $\mathrm{st}(v, K)$ denotes the open star of the vertex v. When $L \subseteq K$ is a subcomplex, $\mathrm{sd}^m L$ is a subcomplex of $\mathrm{sd}^m K$. If $\sigma = \{v_0, v_1, \dots, v_r\}$ is a r-simplex of K, then $[v_0, v_1, \dots, v_r]$ denotes the r-simplex of $|K|$ with the indicated orientation.

We frequently make use of the standard n-simplex Δ_n in $\mathbb{R}^n$ spanned by the origin $\boldsymbol{o}$ and standard unit vectors. Since the boundary $\partial\Delta_n = (\Delta_n)_{n-1}$ is homeomorphic to S^{n-1}, we fix a based homeomorphism $\partial\Delta_n \cong S^{n-1}$ that allows us to represent elements of $\pi_n(X, x_0)$ by maps $(\partial\Delta_{n+1}, \boldsymbol{o}) \to (X, x_0)$.

2.2. ***The Čech expansion and shape homotopy groups.*** We now recall the construction of the first shape homotopy group $\check{\pi}_1(X, x_0)$ via the Čech expansion. For more details; see [Mardešić and Segal 1982].

Let $\mathcal{O}(X)$ be the set of open covers of X directed by refinement; we write $\mathscr{V} \succeq \mathscr{U}$ when $\mathscr{V}$ refines $\mathscr{U}$. Similarly, let $\mathcal{O}(X, x_0)$ be the set of open covers with a distinguished element containing x_0, i.e., the set of pairs $(\mathscr{U}, U_0)$ where $\mathscr{U} \in \mathcal{O}(X)$, $U_0 \in \mathscr{U}$, and $x_0 \in U_0$. We say $(\mathscr{V}, V_0)$ refines $(\mathscr{U}, U_0)$ if $\mathscr{V} \succeq \mathscr{U}$ and $V_0 \subseteq U_0$.

The nerve of a cover $(\mathscr{U}, U_0) \in \mathcal{O}(X, x_0)$ is the abstract simplicial complex $N(\mathscr{U})$ whose vertex set is $N(\mathscr{U})_0 = \mathscr{U}$ and vertices $A_0, \dots, A_n \in \mathscr{U}$ span an n-simplex if $\bigcap_{i=0}^n A_i \neq \varnothing$. The vertex U_0 is taken to be the basepoint of the geometric realization $|N(\mathscr{U})|$. Whenever $(\mathscr{V}, V_0)$ refines $(\mathscr{U}, U_0)$, we can construct a simplicial map $p_{\mathscr{U}\mathscr{V}} : N(\mathscr{V}) \to N(\mathscr{U})$, called a *projection*, by sending a vertex $V \in N(\mathscr{V})$ to a vertex $U \in \mathscr{U}$ such that $V \subseteq U$. In particular, we make a convention that $p_{\mathscr{U}\mathscr{V}}(V_0) = U_0$. Any such assignment of vertices extends linearly to a simplicial map. Moreover, the induced map $|p_{\mathscr{U}\mathscr{V}}| : |N(\mathscr{V})| \to |N(\mathscr{U})|$ is unique up to based homotopy. Thus the homomorphism $p_{\mathscr{U}\mathscr{V}\#} : \pi_1(|N(\mathscr{V})|, V_0) \to \pi_1(|N(\mathscr{U})|, U_0)$ induced on fundamental groups is (up to coherent isomorphism) independent of the choice of simplicial map.

Recall that an open cover $\mathscr{U}$ of X is normal if it admits a partition of unity subordinated to $\mathscr{U}$. Let Λ be the subset of $\mathcal{O}(X, x_0)$ (also directed by refinement) consisting of pairs $(\mathscr{U}, U_0)$ where $\mathscr{U}$ is a normal open cover of X and such that there is a partition of unity $\{\phi_U\}_{U\in\mathscr{U}}$ subordinated to $\mathscr{U}$ with $\phi_{U_0}(x_0) = 1$. It is well-known that every open cover of a paracompact Hausdorff space X is normal. Moreover, if $(\mathscr{U}, U_0) \in \mathcal{O}(X, x_0)$, it is easy to refine $(\mathscr{U}, U_0)$ to a cover $(\mathscr{V}, V_0)$ such that V_0 is the only element of $\mathscr{V}$ containing x_0 and therefore $(\mathscr{V}, V_0) \in \Lambda$. Thus, for paracompact Hausdorff X, Λ is cofinal in $\mathcal{O}(X, x_0)$.

The *n-th shape homotopy group* is the inverse limit

$$\check{\pi}_n(X, x_0) = \varprojlim(\pi_n(|N(\mathscr{U})|, U_0), p_{\mathscr{U}\mathscr{V}\#}, \Lambda).$$

This group is also referred to as the n-th Čech homotopy group.

Given an open cover $(\mathscr{U}, U_0) \in \mathcal{O}(X, x_0)$, a map $p_{\mathscr{U}} : X \to |N(\mathscr{U})|$ is a (*based*) *canonical map* if $p_{\mathscr{U}}^{-1}(\mathrm{st}(U, N(\mathscr{U}))) \subseteq U$ for each $U \in \mathscr{U}$ and $p_{\mathscr{U}}(x_0) = U_0$. Such a canonical map is guaranteed to exist if $(\mathscr{U}, U_0) \in \Lambda$: find a locally finite partition of unity $\{\phi_U\}_{U\in\mathscr{U}}$ subordinated to $\mathscr{U}$ such that $\phi_{U_0}(x_0) = 1$. When $U \in \mathscr{U}$ and $x \in U$, determine $p_{\mathscr{U}}(x)$ by requiring its barycentric coordinate belonging to the vertex U of $|N(\mathscr{U})|$ to be $\phi_U(x)$. According to this construction, the requirement $\phi_{U_0}(x_0) = 1$ gives $p_{\mathscr{U}}(x_0) = U_0$.

A canonical map $p_{\mathscr{U}}$ is unique up to based homotopy and whenever $(\mathscr{V}, V_0)$ refines $(\mathscr{U}, U_0)$, the compositions $p_{\mathscr{U}\mathscr{V}} \circ p_{\mathscr{V}}$ and $p_{\mathscr{U}}$ are homotopic as based maps. Hence, for $n \geq 1$, the homomorphisms

$$p_{\mathscr{U}\#} : \pi_n(X, x_0) \to \pi_n(|N(\mathscr{U})|, U_0)$$

satisfy $p_{\mathscr{U}\mathscr{V}\#} \circ p_{\mathscr{V}\#} = p_{\mathscr{U}\#}$. These homomorphisms induce the following canonical homomorphism to the limit, which is natural in the continuous maps of based spaces:

$$\Psi_n : \pi_n(X, x_0) \to \check{\pi}_n(X, x_0) \quad \text{given by } \Psi_n([f]) = ([p_{\mathscr{U}} \circ f]).$$

The subgroup $\ker(\Psi_n)$, which we refer to as the *n-th shape kernel* is, in a rough sense, an algebraic measure of the n-dimensional homotopical information lost when approximating X by polyhedra. Since $(p_{\mathscr{U}})$ forms an HPol-expansion of X [Mardešić and Segal 1982, Appendix 1, Sectin 3.2, Theorem 8], we have $[f] \in \pi_n(X, x_0)\backslash \ker(\Psi_n)$ if and only if there exist a polyhedron K and a map $p : (X, x_0) \to (K, k_0)$ such that $p_\#([f]) \neq 0$ in $\pi_n(K, k_0)$. Of utmost importance is the situation when $\ker(\Psi_n) = 0$. In this case, $\pi_n(X, x_0)$ can be understood as a subgroup of $\check{\pi}_n(X, x_0)$, that is, the n-th shape group retains all the data in the n-th homotopy group of X. A space for which $\ker(\Psi_n) = 0$ is said to be π_n-*shape injective*.

3. Higher Spanier groups

To define higher Spanier groups as in [Bahredar et al. 2021], we briefly recall the action of the fundamental groupoid on the higher homotopy groups of a space. Fix a retraction $R : S^n \times I \to S^n \times \{0\} \cup \{d_0\} \times I$. Given a map $f : (S^n, d_0) \to (X, y_0)$ and a path $\alpha : I \to X$ with $\alpha(0) = x_0$ and $\alpha(1) = y_0$, define $F : S^n \times \{0\} \cup \{d_0\} \times I \to X$ so that $g(x, 0) = f(x)$ and $f(d_0, t) = \alpha(1-t)$. The *path-conjugate of f by α* is the map $\alpha * f : (S^n, d_0) \to (X, x_0)$ given by $\alpha * f(x) = F(R(x, 1))$.

Path-conjugation defines the basepoint-change isomorphism $\varphi_\alpha : \pi_n(X, y_0) \to \pi_n(X, x_0)$, $\varphi_\alpha([f]) = [\alpha * f]$. In particular, $[\alpha * f][\alpha * g] = [\alpha * (f \cdot g)]$. Additionally, if $[\alpha] = [\beta]$, which we write to mean that the paths α and β are homotopic relative to their endpoints, then $[\alpha * f] = [\beta * f]$. Note that when $n = 1$, $f : S^1 \to X$ is a loop and $\alpha * f \simeq \alpha \cdot f \cdot \alpha^-$.

Definition 3.1. Let $n \geq 1$ and $\alpha : (I, 0) \to (X, x_0)$ be a path and U be an open neighborhood of $\alpha(1)$ in X. Define

$$[\alpha] * \pi_n(U) = \{[\alpha * f] \in \pi_n(X, x_0) \mid f(S^n) \subseteq U,\ f(d_0) = \alpha(1)\}.$$

Since $[\alpha * f][\alpha * g] = [\alpha * (f \cdot g)]$, the set $[\alpha] * \pi_n(U)$ is a subgroup of $\pi_n(X, x_0)$.

Definition 3.2. Let $n \geq 1$, $\mathscr{U}$ be an open cover of X, and $x_0 \in X$. The *n-th Spanier group of (X, x_0) with respect to $\mathscr{U}$* is the subgroup $\pi_n^{Sp}(\mathscr{U}, x_0)$ of $\pi_n(X, x_0)$ generated by the subgroups $[\alpha] * \pi_n(U)$ for all pairs (α, U) with $\alpha(1) \in U$ and $U \in \mathscr{U}$. In short

$$\pi_n^{Sp}(\mathscr{U}, x_0) = \langle [\alpha] * \pi_n(U) \mid U \in \mathscr{U}, \alpha(1) \in U \rangle.$$

The *n-th Spanier group of (X, x_0)* is the intersection

$$\pi_n^{Sp}(X, x_0) = \bigcap_{\mathscr{U} \in O(X)} \pi_n^{Sp}(\mathscr{U}, x_0).$$

We may refer to subgroups of the form $\pi_n^{Sp}(\mathscr{U}, x_0)$ as *relative* Spanier groups and to $\pi_n^{Sp}(X, x_0)$ as the *absolute* Spanier group.

Remark 3.3. We note that our definition of n-th Spanier group is the "unbased" definition from [Bahredar et al. 2021]; see also [Fischer et al. 2011] for more on "based" Spanier groups, which is defined using covers of X by *pointed* open sets. The two notions agree for locally path connected spaces. When $n = 1$, Spanier groups (absolute and relative to a cover) are normal subgroups of $\pi_1(X, x_0)$. In the case $n = 1$, Spanier groups have been studied heavily due to their relationship to covering space theory [Spanier 1966].

Remark 3.4 (functorality). Let Top_* denote the category of based topological spaces and based continuous functions and Grp and Ab denote the usual categories of groups and abelian groups respectively. If $f : (X, x_0) \to (Y, y_0)$ is a map and $\mathscr{V}$ is an open cover of Y, then $\mathscr{U} = \{f^{-1}(V) \mid V \in \mathscr{V}\}$ is an open cover of X such that $f_{\#}(\pi_n(\mathscr{U}, x_0)) \subseteq \pi_n(\mathscr{V}, y_0)$. It follows that $f_{\#}(\pi_n^{Sp}(X, x_0)) \subseteq \pi_n^{Sp}(Y, y_0)$. Thus $(f_{\#})|_{\pi_n^{Sp}(X,x_0)} : \pi_n^{Sp}(X, x_0) \to \pi_n^{Sp}(Y, y_0)$ is well-defined showing that $\pi_1^{Sp} : \mathrm{Top}_* \to \mathrm{Grp}$ and $\pi_n^{Sp} : \mathrm{Top}_* \to \mathrm{Ab}$, $n \geq 2$, are functors [Bahredar et al. 2021, Theorem 4.2]. Moreover, if $g : (Y, y_0) \to (X, x_0)$ is a based homotopy inverse of f, then $(f_{\#})|_{\pi_n^{Sp}(X,x_0)}$ and $(g_{\#})|_{\pi_n^{Sp}(Y,y_0)}$ are inverse isomorphisms. Hence, these functors descend to functors $\mathrm{hTop}_* \to \mathrm{Grp}$ and $\mathrm{hTop}_* \to \mathrm{Ab}$ where hTop_* is the category of based spaces and basepoint-relative homotopy classes of based maps.

Remark 3.5 (basepoint invariance). Suppose $x_0, x_1 \in X$ and $\beta : I \to X$ is a path from x_1 to x_0, and $\varphi_\beta : \pi_n(X, x_0) \to \pi_n(X, x_1)$, $\varphi_\beta([g]) = [\beta * g]$ is the basepoint-change isomorphism. If $[\alpha * f]$ is a generator of $\pi_n^{Sp}(\mathscr{U}, x_0)$, then $\varphi_\beta([\alpha * f]) = [(\beta \cdot \alpha) * f]$ is a generator of $\pi_n^{Sp}(\mathscr{U}, x_1)$. It follows that $\varphi_\beta(\pi_n^{Sp}(\mathscr{U}, x_0)) = \pi_n^{Sp}(\mathscr{U}, x_1)$. Moreover, in the absolute case, we have $\varphi_\beta(\pi_n^{Sp}(X, x_0)) = \pi_n^{Sp}(X, x_1)$. In particular, changing the basepoint of X does not change the isomorphism type of the n-th Spanier group, particularly its triviality.

In terms of our choice of generators, a generic element of $\pi_n^{Sp}(\mathscr{U}, x_0)$ is a product $\prod_{i=1}^{m}[\alpha_i * f_i]$ where each map $f_i : S^n \to X$ has an image in some open set $U_i \in \mathscr{U}$ (see Figure 1). The next lemma identifies how such products might actually appear in practice and motivates the proof of our key technical lemma, Lemma 5.1. Recall that $(\mathrm{sd}^m\, \Delta_{n+1})_n$ is the union of the boundaries of the $(n+1)$-simplices in the m-th barycentric subdivision $\mathrm{sd}^m\, \Delta_{n+1}$.

Lemma 3.6. *For $m, n \in \mathbb{N}$, let $\mathscr{U}$ be an open cover of X and $f : ((\mathrm{sd}^m\, \Delta_{n+1})_n, \boldsymbol{o}) \to (X, x_0)$ be a map such that for every $(n+1)$-simplex σ of $\mathrm{sd}^m\, \Delta_{n+1}$, we have $f(\partial\sigma) \subseteq U$ for some $U \in \mathscr{U}$. Then $f_{\#}(\pi_n((\mathrm{sd}^m\, \Delta_{n+1})_n, \boldsymbol{o})) \subseteq \pi_n^{Sp}(\mathscr{U}, x_0)$.*

Proof. The case $n = 1$ is proved in [Brazas and Fabel 2014]. Suppose $n \geq 2$ and set $K = \mathrm{sd}^m\, \Delta_{n+1}$. The set $\mathscr{W} = \{f^{-1}(U) \mid U \in \mathscr{U}\}$ is an open cover of $K_n = (\mathrm{sd}^m\, \Delta_{n+1})_n$ such that $f_{\#}(\pi_n^{Sp}(\mathscr{W}, \boldsymbol{o})) \subseteq \pi_n^{Sp}(\mathscr{U}, x_0)$ and for every $(n+1)$-simplex σ in K, we have $\partial\sigma \subseteq f^{-1}(U)$ for some $U \in \mathscr{U}$. Thus it suffices to prove $\pi_n(K_n, \boldsymbol{o}) \subset \pi_n^{Sp}(\mathscr{W}, \boldsymbol{o})$. Let S be the set of $(n+1)$-simplices of K. Since $n \geq 2$, K_n is simply connected. Standard simplicial homology arguments give that the reduced singular homology groups of K_n are trivial in dimension $< n$ and $H_n(K_n)$ is a finitely generated free abelian group. A set of free generators for $H_n(K_n)$ can be chosen by fixing the homology class of a simplicial map $g_\sigma : \partial\Delta_{n+1} \to K_n$ that sends $\partial\Delta_{n+1}$ homeomorphically onto the boundary of an $(n+1)$-simplex $\sigma \in S$. Thus K_n is $(n-1)$-connected and the Hurewicz homomorphism $h : \pi_k(K_n, \boldsymbol{o}) \to H_k(K_n)$ is an isomorphism for all $1 \leq k \leq n$. In particular, let $p_\sigma : I \to K_n$ be any path from $\boldsymbol{o}$ to $g_\sigma(\boldsymbol{o})$. Then $\pi_n(K_n, \boldsymbol{o})$ is freely generated by the path-conjugates $[p_\sigma * g_\sigma]$, $\sigma \in S$. By assumption, for every $\sigma \in S$, $[p_\sigma * g_\sigma]$ is a generator of

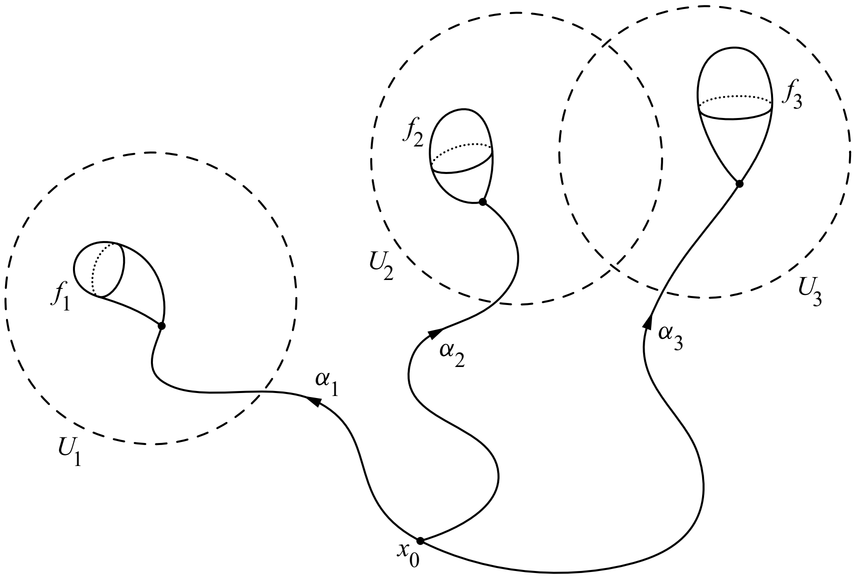

Figure 1. An element of $\pi_2^{Sp}(\mathscr{U}, x_0)$, which is a product of three path-conjugate generators $[\alpha_i * f_i]$.

$\pi_n^{Sp}(\mathscr{W}, \boldsymbol{o})$. Since $\pi_n^{Sp}(\mathscr{W}, \boldsymbol{o})$ contains all the generators of $\pi_n(K_n, \boldsymbol{o})$, the inclusion $\pi_n(K_n, \boldsymbol{o}) \subset \pi_n^{Sp}(\mathscr{W}, \boldsymbol{o})$ follows. □

To characterize the triviality of relative Spanier groups, we establish the following terminology.

Definition 3.7. Let $n \geq 0$ and $x \in X$. We say the space X is:

(1) *Semilocally π_n-trivial at x* if there exists an open neighborhood U of x in X such that every map $S^n \to U$ is null-homotopic in X.

(2) *Semilocally n-connected at x* if there exists an open neighborhood U of x in X such that every map $S^k \to X$, $0 \leq k \leq n$ is null-homotopic in X.

We say X is *semilocally π_n-trivial* (resp. semilocally n-connected) if it has this property at all of its points.

It is straightforward to see that X is semilocally n-connected at x if and only if X is semilocally π_k-trivial at x for all $0 \leq k \leq n$.

Remark 3.8. A space X is semilocally π_n-trivial if and only if X admits an open cover $\mathscr{U}$ such that $\pi_n^{Sp}(\mathscr{U}, x_0)$ is trivial [Bahredar et al. 2021, Theorem 3.7]. Moreover, X is semilocally n-connected if and only if X admits an open cover $\mathscr{U}$ such that $\pi_k^{Sp}(\mathscr{U}, x_0)$ is trivial for all $1 \leq k \leq n$. Note that local path connectivity is independent of the properties given in Definition 3.7.

Attempting a proof of Theorem 1.1, one should not expect the groups $\pi_n^{Sp}(\mathscr{U}, x_0)$ and $\ker(p_{\mathscr{U}\#})$ to agree "on the nose." Indeed, the following example shows that we should not expect the equality $\pi_n^{Sp}(\mathscr{U}, x_0) = \ker(p_{\mathscr{U}\#})$ to hold even in the "nicest" local circumstances.

Example 3.9. Let $X = S^2 \vee S^2$ and W be a contractible neighborhood of d_0 in S^2. Set $U_1 = S^2 \vee W$ and $U_2 = W \vee S^2$ and consider the open cover $\mathscr{U} = \{U_1, U_2\}$ of X. Then $\pi_3^{Sp}(\mathscr{U}, x_0) \cong \mathbb{Z}^2$ is freely generated by the homotopy classes of the two inclusions $i_1, i_2 : S^2 \to X$. However, $\pi_3(X) \cong \mathbb{Z}^3$ is freely generated by $[i_1]$, $[i_2]$, and the Whitehead product $[\![i_1, i_2]\!]$. However $|N(\mathscr{U})|$ is a 1-simplex and is therefore contractible. Thus $\ker(p_{\mathscr{U}\#})$ is equal to $\pi_3(X)$ and contains $[\![i_1, i_2]\!]$. Even though the spaces X, U_1, U_2 are locally contractible and the elements of $\mathscr{U}$ are 1-connected, $\pi_n^{Sp}(\mathscr{U}, x_0)$ is a proper subgroup of $\ker(p_{\mathscr{U}\#})$. One can view this failure as the result of two facts: (1) The sets U_i are not 2-connected and (2) the definition of Spanier group does not allow one to generate homotopy classes by taking Whitehead products of maps $S^2 \to U_i$ in the neighboring elements of $\mathscr{U}$.

First, we show the inclusion $\pi_n^{Sp}(X, x_0) \subseteq \ker(\Psi_n)$ holds in full generality. Recall that the intersections $\pi_n^{Sp}(X, x_0) = \bigcap_{\mathscr{U} \in O(X)} \pi_n^{Sp}(\mathscr{U}, x_0)$ and $\ker(\Psi_n) = \bigcap_{(\mathscr{U}, U_0) \in \Lambda} \ker(p_{\mathscr{U}\#})$ are formally indexed by different sets.

Lemma 3.10. *For every open cover $\mathscr{U}$ of X and canonical map $p_{\mathscr{U}} : X \to |N(\mathscr{U})|$, there exists a refinement $\mathscr{V} \succeq \mathscr{U}$ such that $\pi_n^{Sp}(\mathscr{V}, x_0) \subseteq \ker(p_{\mathscr{U}\#})$ in $\pi_n(X, x_0)$.*

Proof. Let $\mathscr{U} \in O(X)$. The stars $\mathrm{st}(U, |N(\mathscr{U})|)$, $U \in \mathscr{U}$ form an open cover of $|N(\mathscr{U})|$ by contractible sets and therefore $\mathscr{V} = \{p_{\mathscr{U}}^{-1}(\mathrm{st}(U, |N(\mathscr{U})|)) \mid U \in \mathscr{U}\}$ is an open cover of X. Since $p_{\mathscr{U}}$ is a canonical map, we have $p_{\mathscr{U}}^{-1}(\mathrm{st}(U, |N(\mathscr{U})|)) \subseteq U$ for all $U \in \mathscr{U}$. Thus $\mathscr{V}$ is a refinement of $\mathscr{U}$. A generator of $\pi_n^{Sp}(\mathscr{V}, x_0)$ is of the form $[\alpha * f]$ for a map $f : S^n \to p_{\mathscr{U}}^{-1}(\mathrm{st}(U, |N(\mathscr{U})|))$. However, $p_{\mathscr{U}} \circ f$ has image in the contractible open set $\mathrm{st}(U, |N(\mathscr{U})|)$ and is therefore null-homotopic. Thus $p_{\mathscr{U}\#}([\alpha * f]) = 0$. We conclude that $p_{\mathscr{U}\#}(\pi_n^{Sp}(\mathscr{V}, x_0)) = 0$. □

Corollary 3.11 [Bahredar et al. 2021, Theorem 4.14]. *Let $n \geq 1$. For any based space (X, x_0), we have $\pi_n^{Sp}(X, x_0) \subseteq \ker(\Psi_n)$.*

Proof. Suppose $[f] \in \pi_n^{Sp}(X, x_0)$. Given a normal, based open cover $(\mathscr{U}, U_0) \in \Lambda$ and any canonical map $p_{\mathscr{U}} : X \to |N(\mathscr{U})|$, Lemma 3.10 ensures we can find a refinement $\mathscr{V} \succeq \mathscr{U}$ such that $\pi_n^{Sp}(\mathscr{V}, x_0) \subseteq \ker(p_{\mathscr{U}\#})$. Thus $[f] \in \pi_n^{Sp}(\mathscr{V}, x_0) \subseteq \ker(p_{\mathscr{U}\#})$. Since $(\mathscr{U}, U_0)$ is arbitrary, we conclude that $[f] \in \ker(\Psi_n)$. □

Example 3.12 (higher earring spaces). An important space, which we will call upon repeatedly for examples, is the *n-dimensional earring space*

$$\mathbb{E}_n = \bigcup_{j \in \mathbb{N}} \{\boldsymbol{x} \in \mathbb{R}^{n+1} \mid \|\boldsymbol{x} - (1/j, 0, 0, \dots, 0)\| = 1/j\},$$

which is a shrinking wedge (one-point union) of n-spheres with basepoint the origin $\boldsymbol{o}$. It is known that $\mathbb{E}_n$ is $(n-1)$-connected, locally $(n-1)$-connected, and π_n-shape injective for all $n \geq 1$ [Eda and Kawamura 2000a; Morgan and Morrison 1986]. However, $\mathbb{E}_n$ is not semilocally π_n-trivial. Thus $\pi_n^{Sp}(\mathscr{U}, \boldsymbol{o}) \neq 0$ for any open cover $\mathscr{U}$ of $\mathbb{E}_n$ even though in the absolute case $\pi_n^{Sp}(\mathbb{E}_n, \boldsymbol{o})$ is trivial.

Example 3.13. Let $n \geq 3$ and notice that $\mathbb{E}_1 \vee \mathbb{E}_n$ is not semilocally π_1-trivial (since it has $\mathbb{E}_1$ as a retract) and therefore fails to be semilocally $(n-1)$-connected. However, it has recently been shown that $\pi_k(\mathbb{E}_1 \vee \mathbb{E}_n) = 0$ for $2 \leq k \leq n-1$ and that $\mathbb{E}_1 \vee \mathbb{E}_n$ is π_n-shape injective [Brazas 2021]. Thus $\mathbb{E}_1 \vee \mathbb{E}_n$ is semilocally π_k-trivial for all $k \leq n-1$ except $k=1$ and $\pi_n^{Sp}(\mathbb{E}_1 \vee \mathbb{E}_n, \boldsymbol{o}) = 0$. Thus the failure to be semilocally n-connected can occur at single dimension less than n.

4. Recursive extension lemmas

Toward a proof of the inclusion $\ker(\Psi_n) \subseteq \pi_n^{Sp}(X, x_0)$ for LC^{n-1} space X, we introduce some convenient notation and definitions. If $\mathscr{U}$ is an open cover and $A \subseteq X$, then $\mathrm{St}(A, \mathscr{U}) = \bigcup\{U \in \mathscr{U} \mid A \cap U \neq \varnothing\}$. Note that if $A \subseteq B$, then $\mathrm{St}(A, \mathscr{U}) \subseteq \mathrm{St}(B, \mathscr{U})$. Also if $\mathscr{V} \succeq \mathscr{U}$, then $\mathrm{St}(A, \mathscr{V}) \subseteq \mathrm{St}(A, \mathscr{U})$. We take the following terminology from [Willard 1970].

Definition 4.1. Let $\mathscr{U}, \mathscr{V} \in O(X)$:

(1) We say $\mathscr{V}$ is a *barycentric-star refinement* of $\mathscr{U}$ if for every $x \in X$, we have $\mathrm{St}(x, \mathscr{V}) \subseteq U$ for some $U \in \mathscr{U}$. We write $\mathscr{V} \succeq_* \mathscr{U}$.

(2) We say $\mathscr{V}$ is a *star refinement* of $\mathscr{U}$ if for every $V \in \mathscr{V}$, we have $\mathrm{St}(V, \mathscr{V}) \subseteq U$ for some $U \in \mathscr{U}$. We write $\mathscr{V} \succeq_{**} \mathscr{U}$.

Note that if $\mathscr{U} \preceq_* \mathscr{V} \preceq_* \mathscr{W}$, then $\mathscr{U} \preceq_{**} \mathscr{W}$.

Lemma 4.2 [Stone 1948]. *A T_1 space X is paracompact if and only if for every $\mathscr{U} \in O(X)$ there exists $\mathscr{V} \in O(X)$ such that $\mathscr{V} \succeq_* \mathscr{U}$.*

Definition 4.3. [Mardešić and Segal 1982, Chapter I, Section 3.2.5] Let $n \in \{0, 1, 2, 3, \dots, \infty\}$. A space X is LC^n at $x \in X$ if for every neighborhood U of x, there exists a neighborhood V of x such that $V \subseteq U$ and such that for all $0 \leq k \leq n$ ($k < \infty$ if $n = \infty$), every map $f : \partial\Delta_{k+1} \to V$ extends to a map $g : \Delta_{k+1} \to U$. We say X is LC^n if X is LC^n at all of its points.

We have the following evident implications for both the point-wise and global properties:

$$X \text{ is locally } n\text{-connected} \Rightarrow X \text{ is } LC^n \Rightarrow X \text{ is semilocally} n\text{-connected.}$$

For first countable spaces, the LC^n property is equivalent to the "n-tame" property in [Brazas 2021] defined in terms of shrinking sequences of maps.

Definition 4.4. Suppose $\mathscr{V} \succeq \mathscr{U}$ in $O(X)$:

(1) We say $\mathscr{V}$ is an *n-refinement of $\mathscr{U}$*, and write $\mathscr{V} \succeq^n \mathscr{U}$, if for all $0 \le k \le n$, $V \in \mathscr{V}$, and maps $f : \partial\Delta_{k+1} \to V$, there exists $U \in \mathscr{U}$ with $V \subseteq U$ and a continuous extension $g : \Delta_{k+1} \to U$ of f.

(2) We say $\mathscr{V}$ is an *n-barycentric-star refinement of $\mathscr{U}$*, and write $\mathscr{V} \succeq^n_* \mathscr{U}$, if for every $0 \le k \le n$, for every $x \in X$, and every map $f : \partial\Delta_{k+1} \to \mathrm{St}(x, \mathscr{V})$, there exists $U \in \mathscr{U}$ with $\mathrm{St}(x, \mathscr{V}) \subseteq U$ and a continuous extension $g : \Delta_{k+1} \to U$ of f.

Note that if $\mathscr{V} \succeq^n \mathscr{U}$ (resp. $\mathscr{V} \succeq^n_* \mathscr{U}$), then $\mathscr{V} \succeq^k \mathscr{U}$ (resp. $\mathscr{V} \succeq^k_* \mathscr{U}$) for all $0 \le k \le n$.

Lemma 4.5. *Suppose X is paracompact, Hausdorff, and LC^n. For every $\mathscr{U} \in O(X)$, there exists $\mathscr{V} \in O(X)$ such that $\mathscr{V} \succeq^n_* \mathscr{U}$.*

Proof. Let $\mathscr{U} \in O(X)$. Since X is LC^n, for every $U \in \mathscr{U}$ and $x \in U$, there exists an open neighborhood $W(U, x)$ such that $W(U, x) \subseteq U$ and such that for all $0 \le k \le n$, each map $f : \partial\Delta_{k+1} \to W(U, x)$ extends to a map $g : \Delta_{k+1} \to U$. Let $\mathscr{W} = \{W(U, x) \mid U \in \mathscr{U}, x \in U\}$ and note $\mathscr{W} \succeq^n \mathscr{U}$. Since X is paracompact Hausdorff, by Lemma 4.2, there exists $\mathscr{V} \in O(X)$ such that $\mathscr{V} \succeq_* \mathscr{W}$.

Fix $x' \in X$. Then $\mathrm{St}(x', \mathscr{V}) \subseteq W(U, x)$ for some $x \in U \in \mathscr{U}$. Then $\mathrm{St}(x', \mathscr{V}) \subseteq U$. Moreover, if $0 \le k \le n$ and $f : \partial\Delta_{k+1} \to \mathrm{St}(x', \mathscr{V})$ is a map, then since f has image in $W(U, x)$, there is an extension $g : \Delta_{k+1} \to U$. This verifies that $\mathscr{V} \succeq^n_* \mathscr{U}$. □

For the next two lemmas, we fix $n \in \mathbb{N}$, a geometric simplicial complex K with $\dim K = n + 1$, and a subcomplex $L \subseteq K$ with $\dim L \le n$. Let $M[k] = L \cup K_k$ denote the union of L and the k-skeleton of K. Since $L \subseteq K_n$, $M[n] = K_n$ is the union of the boundaries of the $(n+1)$-simplices of K. Later we will consider the cases where (1) $K = \mathrm{sd}^m \Delta_{n+1}$ and $L = \mathrm{sd}^m \partial\Delta_{n+1}$ and (2) $K = \mathrm{sd}^m \partial\Delta_{n+2}$ and $L = \{\boldsymbol{o}\}$.

Lemma 4.6 (recursive extensions). *Suppose $1 \le k \le n$, $\mathscr{U} \preceq_* \mathscr{V} \preceq^{k-1}_* \mathscr{W}$, $m \in \mathbb{N}$, and $f : M[k-1] \to X$ is a map such that for every $(n+1)$-simplex σ of K, we have $f(\sigma \cap M[k-1]) \subseteq W_\sigma$ for some $W_\sigma \in \mathscr{W}$. Then there exists a continuous extension $g : M[k] \to X$ of f such that for every $(n+1)$-simplex σ of K, we have $g(\sigma \cap M[k]) \subseteq U_\sigma$ for some $U_\sigma \in \mathscr{U}$.*

Proof. Supposing the hypothesis, we must extend f to the k-simplices of $M[k]$ that do not lie in L. Let τ be a k-simplex of $M[k]$ that does not lie in L and let S_τ be the set of $(n+1)$-simplices in K that contain τ. By assumption, S_τ is nonempty. We make some general observations first. Since f maps the $(k-1)$-skeleton of each $(n+1)$-simplex $\sigma \in S_\tau$ into W_σ and $\partial\tau$ lies in this $(k-1)$-skeleton, we have

$f(\partial\tau) \subseteq \bigcap_{\sigma\in S_\tau} W_\sigma$. Thus, for all τ, we have

$$f(\partial\tau) \subseteq \bigcap_{\sigma\in S_\tau} \mathrm{St}(W_\sigma, \mathscr{V}).$$

Fix a vertex v_τ of τ and let $x_\tau = f(v_\tau)$. Then $x_\tau \in W_\sigma \subseteq \mathrm{St}(x_\tau, \mathscr{W})$ whenever $\sigma \in S_\tau$. Since $\mathscr{W} \succeq_*^{k-1} \mathscr{V}$, we may find $V_\tau \in \mathscr{V}$ such that $\mathrm{St}(x_\tau, \mathscr{W}) \subseteq V_\tau$ and such that every map $\partial\Delta_k \to \mathrm{St}(x_\tau, \mathscr{W})$ extends to a map $\Delta_k \to V_\tau$. In particular, $f|_{\partial\tau} : \partial\tau \to W_\sigma$ extends to a map $\tau \to V_\tau$. We define $g : M[k] \to X$ so that it agrees with f on $M[k-1]$ and so that the restriction of g to τ is a choice of continuous extension $\tau \to V_\tau$ of $f|_{\partial\tau}$.

We now choose the sets U_σ. Fix an $(n+1)$-simplex σ of K. If the k-skeleton of σ lies entirely in L, we choose any $U_\sigma \in \mathscr{U}$ satisfying $W_\sigma \subseteq U_\sigma$. Suppose there exists at least one k-simplex in σ not in L. Then whenever τ is a k-simplex of σ not in L, we have $W_\sigma \subseteq \mathrm{St}(x_\tau, \mathscr{W}) \subseteq V_\tau$. Fix a point $y_\sigma \in W_\sigma$. The assumption that $\mathscr{V} \succeq_* \mathscr{U}$ implies that there exists $U_\sigma \in \mathscr{U}$ such that $\mathrm{St}(y_\sigma, \mathscr{V}) \subseteq U_\sigma$. In this case, we have $W_\sigma \subseteq V_\tau \subseteq U_\sigma$ whenever τ is a k-simplex of σ not in L.

Finally, we check that g satisfies the desired property. Again, fix an $(n+1)$-simplex σ of K. If τ is a k-simplex of σ not in L, our definition of g gives $g(\tau) \subseteq V_\tau \subseteq U_\sigma$. If τ' is a k-simplex in $\sigma \cap L$, then $g(\tau') = f(\tau') \subseteq W_\sigma \subseteq U_\sigma$. Overall, this shows that $g(\sigma \cap M[k]) \subseteq U_\sigma$ for each $(n+1)$-simplex σ of K. □

A direct, recursive application of the previous lemma is given in the following statement.

Lemma 4.7. *Suppose there is a sequence of open covers*

$$\mathscr{W}_n \preceq_* \mathscr{V}_n \preceq_*^{n-1} \mathscr{W}_{n-1} \preceq_* \cdots \preceq_*^2 \mathscr{W}_2 \preceq_* \mathscr{V}_2 \preceq_*^1 \mathscr{W}_1 \preceq_* \mathscr{V}_1 \preceq_*^0 \mathscr{W}_0$$

and a map $f_0 : M[0] \to X$ such that for every $(n+1)$-simplex σ of K, we have $f_0(\sigma \cap M[0]) \subseteq W$ for some $W \in \mathscr{W}_0$. Then there exists an extension $f_n : M[n] \to X$ of f_0 such that for every $(n+1)$-simplex σ of K, we have $f_n(\partial\sigma) \subseteq U$ for some $U \in \mathscr{W}_n$.

5. A proof of Theorem 1.1

We apply the extension results of the previous section in the case where $K = \mathrm{sd}^m \Delta_{n+1}$ for some $m \in \mathbb{N}$ and $L = \mathrm{sd}^m \partial\Delta_{n+1}$ so that $M[k] = L \cup K_k$ consists of the n-simplices of the boundary of Δ_{n+1} and the k-simplices of $\mathrm{sd}^m \Delta_{n+1}$ not in the boundary. Note that $M[n]$ is the union of the boundaries of the $(n+1)$-simplices of $\mathrm{sd}^m \Delta_{n+1}$.

Lemma 5.1. *Let $n \geq 1$. Suppose X is paracompact, Hausdorff, and LC^{n-1}. Then for every open cover $\mathscr{U}$ of X, there exists $(\mathscr{V}, V_0) \in \Lambda$ such that $\ker(p_{\mathscr{V}\#}) \subseteq \pi_n^{Sp}(\mathscr{U}, x_0)$.*

Proof. Suppose $\mathscr{U} \in O(X)$. Since X is paracompact, Hausdorff, and LC^{n-1}, we may apply Lemmas 4.2 and 4.5 to first find a sequence of refinements

$$\mathscr{U} = \mathscr{U}_n \preceq_* \mathscr{V}_n \preceq_*^{n-1} \mathscr{U}_{n-1} \preceq_* \cdots \preceq_*^2 \mathscr{U}_2 \preceq_* \mathscr{V}_2 \preceq_*^1 \mathscr{U}_1 \preceq_* \mathscr{V}_1 \preceq_*^0 \mathscr{U}_0$$

and then one last refinement $\mathscr{U}_0 \preceq_* \mathscr{V}_0 = \mathscr{V}$. Let $V_0 \in \mathscr{V}$ be any set containing x_0 and recall that since X is paracompact Hausdorff $(\mathscr{V}, V_0) \in \Lambda$. We will show that $\ker(p_{\mathscr{V}\#}) \subseteq \pi_n^{Sp}(\mathscr{U}, x_0)$. Note that $p_{\mathscr{V}}^{-1}(\mathrm{st}(V, N(\mathscr{V}))) \subseteq V$ by the definition of canonical map $p_{\mathscr{V}}$.

Suppose $[f] \in \ker(p_{\mathscr{V}\#})$ is represented by a map $f : (|\partial\Delta_{n+1}|, \boldsymbol{o}) \to (X, x_0)$. We will show that $[f] \in \pi_n^{Sp}(\mathscr{U}, x_0)$. Then $p_{\mathscr{V}} \circ f : |\partial\Delta_{n+1}| \to |N(\mathscr{V})|$ is null-homotopic and extends to a map $h : |\Delta_{n+1}| \to |N(\mathscr{V})|$. Set $Y_V = h^{-1}(\mathrm{st}(V, N(\mathscr{V})))$ so that $\mathscr{Y} = \{Y_V \mid V \in \mathscr{V}\}$ is an open cover of $|\Delta_{n+1}|$.

We find a particular simplicial approximation for h using the cover $\mathscr{Y}$ [Munkres 1984, Theorem 16.1]: let λ be a Lebesgue number for $\mathscr{Y}$ so that any subset of Δ_{n+1} of diameter less than λ lies in some element of $\mathscr{Y}$. Find $m \in \mathbb{N}$ such that each simplex in $\mathrm{sd}^m \Delta_{n+1}$ has diameter less than $\lambda/2$. Thus the star $\mathrm{st}(a, \mathrm{sd}^m \Delta_{n+1})$ of each vertex a in $\mathrm{sd}^m \Delta_{n+1}$ lies in a set $Y_{V_a} \in \mathscr{Y}$ for some $V_a \in \mathscr{V}$. The assignment $a \mapsto V_a$ on vertices extends to a simplicial approximation $h' : \mathrm{sd}^m \Delta_{n+1} \to N(\mathscr{V})$ of h, i.e., a simplicial map h' such that

$$h(\mathrm{st}(a, \mathrm{sd}^m \Delta_{n+1})) \subseteq \mathrm{st}(h'(a), N(\mathscr{V})) = \mathrm{st}(V_a, N(\mathscr{V}))$$

for each vertex a [Munkres 1984, Lemma 14.1].

Let $K = \mathrm{sd}^m \Delta_{n+1}$ and $L = \mathrm{sd}^m \partial\Delta_{n+1}$ so that $M[k] = L \cup K_k$. First, we extend $f : L \to X$ to a map $f_0 : M[0] \to X$. For each vertex a in K, pick a point $f_0(a) \in V_a$. In particular, if $a \in L$, take $f_0(a) = f(a)$. This choice is well defined since, for a boundary vertex $a \in L$, we have $p_{\mathscr{V}} \circ f(a) = h(a) \in \mathrm{st}(V_a, |N(\mathscr{V})|)$ and thus $f(a) \in p_{\mathscr{V}}^{-1}(\mathrm{st}(V_a, |N(\mathscr{V}|))) \subseteq V_a$.

Note that h' maps every simplex $\sigma = [a_0, a_1, \dots, a_k]$ of K to the simplex of $N(\mathscr{V})$ spanned by $\{h'(a_i) \mid 0 \le i \le k\} = \{V_{a_i} \mid 0 \le i \le k\}$. By definition of the nerve, we have $\bigcap\{V_{a_i} \mid 0 \le i \le k\} \ne \varnothing$. Pick a point $x_\sigma \in \bigcap\{V_{a_i} \mid 0 \le i \le k\}$.

By our initial choice of refinements, we have $\mathscr{U}_0 \preceq_* \mathscr{V}$. If $\sigma = [a_0, a_1, \dots, a_{n+1}]$ is an $(n+1)$-simplex of K, then $\mathrm{St}(x_\sigma, \mathscr{V}) \subseteq U_\sigma$ for some $U_\sigma \in \mathscr{U}$. In particular $\{f_0(a_i) \mid 0 \le i \le n+1\} \subseteq \bigcup\{V_{a_i} \mid 0 \le i \le n+1\} \subseteq U_\sigma$. Thus f_0 maps the 0-skeleton of σ into U_σ. If $1 \le k \le n$, τ is a k-face of $\sigma \cap L$ with $a_i \in \tau$, then $p_{\mathscr{V}} \circ f_0(\mathrm{int}(\tau)) = p_{\mathscr{V}} \circ f(\mathrm{int}(\tau)) = h(\mathrm{int}(\tau)) \subseteq h(\mathrm{st}(a_i, K)) \subseteq \mathrm{st}(V_{a_i}, |N(\mathscr{V})|)$. It follows that

$$f_0(\tau) \subseteq p_{\mathscr{V}}^{-1}(\mathrm{st}(V_{a_i}, |N(\mathscr{V})|)) \subseteq V_{a_i} \subseteq U_\sigma.$$

Thus for every n-simplex in $\sigma \cap L$, we have $f_0(\tau) \subseteq U_\sigma$. We conclude that for every $(n+1)$-simplex σ of K, we have $f_0(\sigma \cap M[0]) \subseteq U_\sigma$.

By our choice of sequence of refinements, we are precisely in the situation to apply Lemma 4.7. Doing so, we obtain an extension $f_n : M[n] \to X$ of f such that for every $(n+1)$-simplex σ of K, we have $f_n(\partial\sigma) \subseteq \boldsymbol{U}_\sigma$ for some $\boldsymbol{U}_\sigma \in \mathscr{U}_n = \mathscr{U}$. By Lemma 3.6, we have $[f] = [f_n|_{\partial\Delta_{n+1}}] \in \pi_n^{Sp}(\mathscr{U}, x_0)$. $\square$

Finally, both inclusions have been established and provide a proof of our main result.

Proof of Theorem 1.1. The inclusion $\pi_n^{Sp}(X, x_0) \subseteq \ker(\Psi_n)$ holds in general by Corollary 3.11. Under the given hypotheses, the inclusion $\ker(\Psi_n) \subseteq \pi_n^{Sp}(X, x_0)$ follows from Lemma 5.1. $\square$

When considering examples relevant to Theorem 1.1, it is helpful to compare π_n-shape injectivity with the following weaker property from [Ghane and Hamed 2009].

Definition 5.2. We say a space X is *n-homotopically Hausdorff at $x \in X$* if no nontrivial element of $\pi_n(X, x)$ has a representing map in every neighborhood of x. We say X is *n-homotopically Hausdorff* if it is n-homotopically Hausdorff at each of its points.

Clearly, π_n-shape injectivity $\Rightarrow$ n-homotopically Hausdorff. The next example, which highlights the effectiveness of Theorem 1.1, shows the converse is not true even for LC^{n-1} Peano continua.

Example 5.3. Fix $n \geq 2$ and let $\ell_j : S^n \to \mathbb{E}_n$ be the inclusion of the j-th sphere and define $f : \mathbb{E}_n \to \mathbb{E}_n$ to be the shift map given by $f \circ \ell_j = \ell_{j+1}$. Let $M_f = \mathbb{E}_n \times [0, 1]/\sim$, $(x, 0) \sim (f(x), 1)$ be the mapping torus of f. We identify $\mathbb{E}_n$ with the image of $\mathbb{E}_n \times \{0\}$ in M_f and take $\boldsymbol{o}$ to be the basepoint of M_f. Let $\alpha : I \to M_f$ be the loop where $\alpha(t)$ is the image of $(\boldsymbol{o}, t)$. Then M_f is locally contractible at all points other than those in the image of α. Also, every point $\alpha(t)$ has a neighborhood that deformation retracts onto a homeomorphic copy of $\mathbb{E}_n$. Thus, since $\mathbb{E}_n$ is LC^{n-1}, so is X. It follows from Theorem 1.1 that $\pi_n^{Sp}(M_f, \boldsymbol{o}) = \ker(\pi_n(M_f, \boldsymbol{o}) \to \check{\pi}_n(M_f, \boldsymbol{o}))$. In particular, the Spanier group of M_f contains all elements $[\alpha^k * g]$ where $g : S^n \to \mathbb{E}_n$ is a based map and $k \in \mathbb{Z}$. Using the universal covering map $E \to M_f$ that "unwinds" α and the relation $[g] = [\alpha * (f \circ g)]$ in $\pi_n(M_f, \boldsymbol{o})$, it is not hard to show that these are, in fact, the only elements of the n-th Spanier group. Hence,

$$\ker(\pi_n(M_f, \boldsymbol{o}) \to \check{\pi}_n(M_f, \boldsymbol{o})) = \{[\alpha^k * g] \mid [g] \in \pi_n(\mathbb{E}_n, \boldsymbol{o}), k \in \mathbb{Z}\},$$

which is an uncountable subgroup. Moreover, since M_f is shape equivalent to the aspherical space S^1, we have $\check{\pi}_n(M_f, \boldsymbol{o}) = 0$ and thus $\pi_n(M_f, \boldsymbol{o}) = \{[\alpha^k * g] \mid [g] \in \pi_n(\mathbb{E}_n, \boldsymbol{o}), k \in \mathbb{Z}\}$.

It follows from this description that, even though M_f is not π_n-shape injective, M_f is n-homotopically Hausdorff. Indeed, it suffices to check this at the points $\alpha(t)$, $t \in I$. We give the argument for $\alpha(0) = \boldsymbol{o}$, the other points are similar. If $0 \neq h \in \pi_n(M_f, \boldsymbol{o})$ has a representative in every neighborhood of $\boldsymbol{o}$ in M_f, then clearly $h \in \ker(\Psi_n)$. Hence, $h = [\alpha^k * g]$ for $[g] \in \pi_n(\mathbb{E}_n, \boldsymbol{o})$ and $k \in \mathbb{Z}$. Since M_f retracts onto the circle parametrized by α, the hypothesis on h can only hold if $k = 0$. However, there is a basis of neighborhoods of $\boldsymbol{o}$ in M_f that deformation retract onto an open neighborhood of $\boldsymbol{o}$ in $\mathbb{E}_n$. Thus $[g]$ has a representative in every neighborhood of $\boldsymbol{o}$ in $\pi_n(\mathbb{E}_n, \boldsymbol{o})$, giving $h = [g] \in \ker(\pi_n(\mathbb{E}_n, \boldsymbol{o}) \to \check{\pi}_n(\mathbb{E}_n, \boldsymbol{o})) = 0$.

It is an important feature of Example 5.3 that M_f is not simply connected and has multiple points at which it is not semilocally π_n-trivial. This motivates the following application of Theorem 1.1, which identifies a partial converse of the implication π_n-shape injective $\Rightarrow$ n-homotopically Hausdorff.

Corollary 5.4. *Let $n \geq 2$ and X be a simply connected, LC^{n-1}, compact Hausdorff space such that X fails to be semilocally π_n-trivial only at a single point $x \in X$. Then for every element $g \in \ker(\Psi_n) \subseteq \pi_n(X, x)$ and neighborhood V of x, g is represented by a map with image in V. In particular, if X is n-homotopically Hausdorff at x, then X is π_n-shape injective.*

Proof. Let $0 \neq g \in \ker(\Psi_n) \subseteq \pi_n(X, x)$. By Theorem 1.1, $g \in \pi_n^{Sp}(X, x)$. Since X is compact Hausdorff, we may replace $O(X)$ by the cofinal subdirected order $O_F(X)$ consisting of finite open covers $\mathscr{U}$ of X with the property that there is a unique $A_{\mathscr{U}} \in \mathscr{U}$ with $x \in A_{\mathscr{U}}$. For each $\mathscr{U} \in O_F(X)$, we can write $g = \prod_{i=1}^{m_{\mathscr{U}}} [\alpha_{\mathscr{U},i} * f_{\mathscr{U},i}]$ where $f_{\mathscr{U},i} : S^n \to U_{\mathscr{U},i}$ is a non-nullhomotopic map for some $U_{\mathscr{U},i} \in \mathscr{U}$ and $\alpha_{\mathscr{U},i}$ is a path from x to $f_{\mathscr{U},i}(d_0)$.

Let V be an open neighborhood of x. We check that g is represented by a map with image in V. Since X is LC^0 at x, there exists an open neighborhood V' of x such that any two points of V' may be connected by a path in V. Fix $\mathscr{U}_0 \in O_F(X)$ such that $A_{\mathscr{U}_0} \subseteq V'$. Then $A_{\mathscr{V}} \subseteq V'$ whenever $\mathscr{V} \in O_F(X)$ refines $\mathscr{U}_0$.

We claim that for sufficiently refined $\mathscr{V}$, all of the maps $f_{\mathscr{V},i}$ have image in V'. Suppose, to obtain a contradiction, there is a subset $T \subseteq \{\mathscr{V} \in O_F(X) \mid \mathscr{V} \succeq \mathscr{U}_0\}$, which is cofinal in $O_F(X)$ and such that for every $\mathscr{V} \in T$ there exists $i_{\mathscr{V}} \in \{1, 2, \dots, m_{\mathscr{V}}\}$ and $d_{\mathscr{V}} \in S^n$ such that $f_{\mathscr{V},i_{\mathscr{V}}}(d_{\mathscr{V}}) \in U_{\mathscr{V},i} \backslash V' \subseteq U_{\mathscr{V},i} \backslash A_{\mathscr{U}_0}$. Since X is compact, we may replace $\{f_{\mathscr{V},i_{\mathscr{V}}}(d_{\mathscr{V}})\}$ with a subnet $\{x_j\}_{j \in J}$ that converges to a point $y \in X$. Here, $x_j = f_{\mathscr{V}_j,i_{\mathscr{V}_j}}(d_{\mathscr{V}_j})$ for some directed set J and monotone, final function $J \to T$ given by $j \mapsto \mathscr{V}_j$. Let Y be an open neighborhood of y in X. Find $\mathscr{W} \in O_F(X)$ such that there exists $W_0 \in \mathscr{W}$ such that $y \in W_0$ and $\mathrm{St}(W_0, \mathscr{W}) \subseteq Y$. Since $\{x_j\}$ is subnet that converges to y, there exists $k \in J$ such that $\mathscr{V}_k \succeq \mathscr{W}$ and $x_k \in W_0$. We have $x_k \in \mathrm{Im}(f_{\mathscr{V}_k,i_{\mathscr{V}_k}}) \subseteq U_{\mathscr{V}_k,i_{\mathscr{V}_k}} \subseteq W$ for some $W \in \mathscr{W}$ and thus $\mathrm{Im}(f_{\mathscr{V}_k,i_{\mathscr{V}_k}}) \subseteq U_{\mathscr{V}_k,i} \subseteq \mathrm{St}(W_0, \mathscr{W}) \subseteq Y$. However, for every $\mathscr{V} \in O_F(X)$, $f_{\mathscr{V},i_{\mathscr{V}}}$ is not null-homotopic in X. Thus, since Y represents an arbitrary neighborhood of y, X is not semilocally π_n-trivial at y. By assumption, we must have $x = y$. Since $\{x_j\} \to x$, the same argument, but where Y is replaced by V', shows that there exists sufficiently refined $\mathscr{V}$ for which $\mathrm{Im}(f_{\mathscr{V},i_{\mathscr{V}}}) \subseteq V'$; a contradiction. Since the claim is proved, there exists $\mathscr{U}_1 \succeq \mathscr{U}_0$ in $O_F(X)$ such that whenever $\mathscr{V} \succeq \mathscr{U}_1$, we have $\mathrm{Im}(f_{\mathscr{V},i}) \subseteq V'$ for all $i \in \{1, 2, \dots, m_{\mathscr{V}}\}$.

Fix $\mathscr{V} \succeq \mathscr{U}_1$ in $O_F(X)$. For all $i \in \{1, 2, \dots, m_{\mathscr{V}}\}$, we may find a path $\beta_{\mathscr{V},i} : I \to V$ from x to $f_{\mathscr{V},i}(d_0)$. Since X is simply connected, we have $[\alpha_{\mathscr{V},i} * f_{\mathscr{U},i}] = [\beta_{\mathscr{V},i} * f_{\mathscr{U},i}]$ for all i. Thus g is represented by $\prod_{i=1}^{m_{\mathscr{V}}} \beta_{\mathscr{V},i} * f_{\mathscr{V},i}$, which has image in V. □

Remark 5.5 (topologies on homotopy groups). Given a group G and a collection of subgroups $\{N_j \mid j \in J\}$ of G such that for all $j, j' \in J$, there exists $k \in J$ such that $N_k \subseteq N_j \cap N_{j'}$, we can generate a topology on G by taking the set $\{gN_j \mid j \in J, g \in G\}$ of left cosets as a basis. We can apply this to both the collection of Spanier subgroups $\pi_n^{Sp}(\mathscr{U}, x_0)$ and the collection of kernels $\ker(p_{\mathscr{U}\#})$ to define two natural topologies on $\pi_n(X, x_0)$:

(1) The *Spanier topology* on $\pi_n(X, x_0)$ is generated by the left cosets of Spanier groups $\pi_n(\mathscr{U}, x_0)$ for $\mathscr{U} \in O(X)$.

(2) The *shape topology* on $\pi_n(X, x_0)$ is generated by left cosets of the kernels $\ker(p_{\mathscr{U}\#})$ where $(\mathscr{U}, U_0) \in \Lambda$. Equivalently, the shape topology is the initial topology with respect to the map Ψ_n where the groups $\pi_n(|N(\mathscr{U})|, U_0)$ are given the discrete topology and $\check{\pi}_n(X, x_0)$ is given the inverse limit topology.

Lemma 3.10 ensures the Spanier topology is always finer than the shape topology. Lemma 5.1 then implies that, whenever X is paracompact, Hausdorff, and LC^{n-1}, the two topologies agree. Moreover, $\pi_n(X, x_0)$ is Hausdorff in the shape topology if and only if X is π_n-shape injective.

6. When is Ψ_n an isomorphism?

It is a result of Kozlowski and Segal [1978] that if X is paracompact Hausdorff and LC^n, then $\Psi_n : \pi_n(X, x) \to \check{\pi}_n(X, x)$ is an isomorphism. This result was first proved for compact metric spaces in [Kuperberg 1975]. The assumption that X is LC^n assumes that small maps $S^n \to X$ may be contracted by small null-homotopies. However, if $\mathbb{E}_n$ is the n-dimensional earring space, then the cone $C\mathbb{E}_n$ is LC^{n-1} but not LC^n. However, $C\mathbb{E}_n$ is contractible and so Ψ_n is an isomorphism of trivial groups. Certainly, many other examples in this range exist. Our Spanier group-based approach allows us to generalize Kozlowski–Segal's theorem in a way that includes this example by removing the need for "small" homotopies in dimension n. In this section, when $\mathscr{U}$ is an open cover of a space X and a distinguished element

$U_0 \in \mathscr{U}$ containing the basepoint x_0 has been established or is clear from context, we will often write $\mathscr{U}$ to represent the pair $(\mathscr{U}, U_0) \in \Lambda$.

Lemma 6.1. *Let $n \geq 1$. Suppose that X is paracompact, Hausdorff, and LC^{n-1}. If $([f_{\mathscr{U}}])_{\mathscr{U}\in\Lambda} \in \check{\pi}_1(X, x_0)$, then for every $\mathscr{U} \in \Lambda$, there exists $[g] \in \pi_n(X, x)$ such that $(p_{\mathscr{U}})_{\#}([g]) = [f_{\mathscr{U}}]$.*

Proof. With $(\mathscr{U}, U_0) \in \Lambda$ and $p_{\mathscr{U}}$ fixed, consider a representing map

$$f_{\mathscr{U}} : (|\partial\Delta_{n+1}|, \boldsymbol{o}) \to (|N(\mathscr{U})|, U_0).$$

Let $\mathscr{U}' = \{p_{\mathscr{U}}^{-1}(\mathrm{st}(U, |N(\mathscr{U})|)) \mid U \in \mathscr{U}\}$. Since $p_{\mathscr{U}}^{-1}(\mathrm{st}(U, |N(\mathscr{U})|)) \subseteq U$ for all $U \in \mathscr{U}$, we have $\mathscr{U} \preceq \mathscr{U}'$. Applying Lemmas 4.2 and 4.5 we can choose the following sequence of refinements of $\mathscr{U}'$. First, we choose a star refinement $\mathscr{U}' \preceq_{**} \mathscr{W}$ so that for every $W \in \mathscr{W}$, there exists $U' \in \mathscr{U}'$ such that $\mathrm{St}(W, \mathscr{W}) \subseteq U'$. In this case, we can choose the projection map $p_{\mathscr{U}'\mathscr{W}} : |N(\mathscr{W})| \to |N(\mathscr{U}')|$ so that if $p_{\mathscr{U}'\mathscr{W}}(W) = U'$ on vertices, then $\mathrm{St}(W, \mathscr{W}) \subseteq U'$ in X. This choice will be important near the end of the proof.

To construct g, we must take further refinements. First, we choose a sequence of a refinements

$$\mathscr{W} = \mathscr{W}_n \preceq_* \mathscr{V}_n \preceq_*^{n-1} \mathscr{W}_{n-1} \preceq_* \cdots \preceq_*^2 \mathscr{W}_2 \preceq_* \mathscr{V}_2 \preceq_*^1 \mathscr{W}_1 \preceq_* \mathscr{V}_1 \preceq_*^0 \mathscr{W}_0$$

followed by one last refinement $\mathscr{W}_0 \preceq_* \mathscr{V}_0 = \mathscr{V}$. Let $V_0 \in \mathscr{V}$ be any set containing x_0 and recall that since X is paracompact Hausdorff $(\mathscr{V}, V_0) \in \Lambda$. For some choice of canonical map $p_{\mathscr{V}}$, we have $p_{\mathscr{V}}^{-1}(\mathrm{st}(V, N(\mathscr{V}))) \subseteq V$ for all $V \in \mathscr{V}$.

Recall that we have assumed the existence of a map

$$f_{\mathscr{V}} : (\partial\Delta_{n+1}, \boldsymbol{o}) \to (|N(\mathscr{V})|, V_0)$$

such that $p_{\mathscr{U}\mathscr{V}\#}([f_{\mathscr{V}}]) = [f_{\mathscr{U}}]$. Set $Y_V = f_{\mathscr{V}}^{-1}(\mathrm{st}(V, N(\mathscr{V})))$ so that $\mathscr{Y} = \{Y_V \mid V \in \mathscr{V}\}$ is an open cover of $\partial\Delta_{n+1}$. As before, we find a simplicial approximation for $f_{\mathscr{V}}$. Find $m \in \mathbb{N}$ such that the star $\mathrm{st}(a, \mathrm{sd}^m \partial\Delta_{n+1})$ of each vertex a in $\mathrm{sd}^m \partial\Delta_{n+1}$ lies in a set $Y_{V_a} \in \mathscr{Y}$ for some $V_a \in \mathscr{V}$. Since $f_{\mathscr{V}}(\boldsymbol{o}) = V_0$, we may take $V_{\boldsymbol{o}} = V_0$. The assignment $a \mapsto V_a$ on vertices extends to a simplicial approximation $f' : \mathrm{sd}^m \partial\Delta_{n+1} \to |N(\mathscr{V})|$ of $f_{\mathscr{V}}$, i.e., a simplicial map f' such that

$$f_{\mathscr{V}}(\mathrm{st}(a, \mathrm{sd}^m \partial\Delta_{n+1})) \subseteq \mathrm{st}(f'(a), |N(\mathscr{V})|) = \mathrm{st}(V_a, |N(\mathscr{V})|)$$

for each vertex a.

We begin to define g with the constant map $\{\boldsymbol{o}\} \to X$ sending $\boldsymbol{o}$ to x_0. In preparation for applications of Lemma 4.6, set $K = \mathrm{sd}^m \partial\Delta_{n+1}$ and $L = \{\boldsymbol{o}\}$ so that $K[k] = K_k$. First, we define a map $g_0 : M[0] \to X$ by picking, for each vertex $a \in K_0$, a point $g_0(a) \in V_a$. In particular, set $g_0(\boldsymbol{o}) = x_0$. This choice is well defined since we have $p_{\mathscr{V}}(x_0) = V_0 \in \mathrm{st}(V_{\boldsymbol{o}}, N(\mathscr{V}))$ and thus $g_0(\boldsymbol{o}) = x_0 \in p_{\mathscr{V}}^{-1}(\mathrm{st}(V_{\boldsymbol{o}}, N(\mathscr{V}))) \subseteq V_{\boldsymbol{o}}$. Note that f' maps every simplex $\sigma = [a_0, a_1, \ldots, a_k]$ of K to the simplex of $|N(\mathscr{V})|$ spanned by $\{V_{a_i} \mid 0 \leq i \leq k\}$. By definition of the nerve, we have $\bigcap\{V_{a_i} \mid 0 \le i \le k\} \neq \varnothing$. Pick a point $x_\sigma \in \bigcap\{V_{a_i} \mid 0 \le i \le k\}$. By our initial choice of refinements, we have $\mathscr{U}_0 \preceq_* \mathscr{V}$. If $\sigma = [a_0, a_1, \dots, a_n]$ is a n-simplex of K, then $\mathrm{St}(x_\sigma, \mathscr{V}) \subseteq U_{0,\sigma}$ for some $U_{0,\sigma} \in \mathscr{U}_0$. In particular $\{g_0(a_i) \mid 0 \le i \le n+1\} \subseteq \bigcup\{V_{a_i} \mid 0 \le i \le n\} \subseteq U_{0,\sigma}$. Thus g_0 maps the 0-skeleton of σ into $U_{0,\sigma}$. If $\boldsymbol{o} \in \sigma$, then $g_0(\boldsymbol{o}) \in p_{\mathscr{V}}^{-1}(\mathrm{st}(V_{\boldsymbol{o}}, N(\mathscr{V}))) \subseteq V_{\boldsymbol{o}} \subseteq U_{0,\sigma}$. Hence, for every n-simplex σ of K, we have $g_0(\sigma \cap M[0]) \subseteq U_{0,\sigma}$.

We are now in the situation to recursively apply Lemma 4.6. This is similar to the application in the proof of Lemma 5.1 with the dimension $n+1$ shifted down by one so we omit the details. Recalling that $M[n] = \mathrm{sd}^m\, \partial\Delta_{n+1}$, we obtain an extension $g : K = M[n] \to X$ of g_0 such that for every n-simplex σ of K, we have $g(\sigma) \subseteq W_\sigma$ for some $W_\sigma \in \mathscr{W} = \mathscr{U}_n$.

With g being defined, we seek show that $f_{\mathscr{U}} \simeq p_{\mathscr{U}} \circ g$. Since $f' \simeq f_{\mathscr{V}}$ (by simplicial approximation), $p_{\mathscr{U}\mathscr{V}} \simeq p_{\mathscr{U}\mathscr{U}'} \circ p_{\mathscr{U}'\mathscr{W}} \circ p_{\mathscr{W}\mathscr{V}}$ (for any choice of projection maps), and $p_{\mathscr{U}\mathscr{V}} \circ f_{\mathscr{V}} \simeq f_{\mathscr{U}}$ (for any choice of projection $p_{\mathscr{U}\mathscr{V}}$), it suffices to show that $p_{\mathscr{U}\mathscr{U}'} \circ p_{\mathscr{U}'\mathscr{W}} \circ p_{\mathscr{W}\mathscr{V}} \circ f' \simeq p_{\mathscr{U}} \circ g$. We do this by proving that the simplicial map $F = p_{\mathscr{U}\mathscr{U}'} \circ p_{\mathscr{U}'\mathscr{W}} \circ p_{\mathscr{W}\mathscr{V}} \circ f' : K \to |N(\mathscr{U})|$ is a simplicial approximation for $p_{\mathscr{U}} \circ g$. Recall that this can be done by verifying the "star-condition" $p_{\mathscr{U}} \circ g(\mathrm{st}(a, K)) \subseteq \mathrm{st}(F(a), |N(\mathscr{U})|)$ for any vertex $a \in K$ [Munkres 1984, Chapter 2, Section 14]. Since $n \ge 1$, we have $\mathscr{W} \preceq_{**} \mathscr{V}$. Hence, just like our choice of $p_{\mathscr{U}'\mathscr{W}}$, we may choose $p_{\mathscr{W}\mathscr{V}}$ so that whenever $p_{\mathscr{W}\mathscr{V}}(V) = W$, then $\mathrm{St}(V, \mathscr{V}) \subseteq W$. Also, we choose $p_{\mathscr{U}\mathscr{U}'}$ to map $p_{\mathscr{U}}^{-1}(\mathrm{st}(U, |N(\mathscr{U})|)) \mapsto U$ on vertices.

Fix a vertex $a_0 \in K$. To check the star-condition, we'll check that $p_{\mathscr{U}} \circ g(\sigma) \subseteq \mathrm{st}(F(a_0), |N(\mathscr{U})|)$ for each n-simplex σ having a_0 as a vertex. Pick an n-simplex $\sigma = [a_0, a_1, \dots, a_n] \subseteq K$ having a_0 as a vertex. Recall that $f'(a_i) = V_{a_i}$ for each i. Set $p_{\mathscr{W}\mathscr{V}}(V_{a_i}) = W_i$ and $p_{\mathscr{U}'\mathscr{W}}(W_i) = p_{\mathscr{U}}^{-1}(\mathrm{st}(U_i, |N(\mathscr{U})|)) \in \mathscr{U}'$ for some $U_i \in \mathscr{U}$. Then $F(a_i) = U_i$ for all i. It now suffices to check that $p_{\mathscr{U}} \circ g(\sigma) \subseteq \mathrm{st}(U_0, |N(\mathscr{U})|)$. Recall that by our choice of $p_{\mathscr{U}'\mathscr{W}}$, we have $\mathrm{St}(W_0, \mathscr{W}) \subseteq p_{\mathscr{U}}^{-1}(\mathrm{st}(U_0, |N(\mathscr{U})|))$. Thus it is enough to check that $g(\sigma) \subseteq \mathrm{St}(W_0, \mathscr{W})$. By construction of g, we have $g(\sigma) \subseteq W_\sigma$ for some $W_\sigma \in \mathscr{W}$. Since $g(a_0) \in W_0 \cap W_\sigma$, we have $g(\sigma) \subseteq W_\sigma \subseteq \mathrm{St}(W_0, \mathscr{W})$, completing the proof. □

Finally, we prove our second result, Theorem 1.2.

Proof of Theorem 1.2. Since X is paracompact, Hausdorff, LC^{n-1}, we have $\pi_n^{Sp}(X, x_0) = \ker(\Psi_n)$ by Theorem 1.1. Since X is semilocally π_n-trivial, we have $\pi_n^{Sp}(\mathscr{U}, x_0) = 1$ for some $\mathscr{U} \in \Lambda$. It follows that Ψ_n is injective. Moreover, by Lemma 5.1, we may find $\mathscr{V} \in \Lambda$ with $\ker(p_{\mathscr{V}\#}) \subseteq \pi_n^{Sp}(\mathscr{U}, x_0)$. Thus $p_{\mathscr{V}\#} : \pi_n(X, x_0) \to \pi_n(|N(\mathscr{V})|, V_0)$ is injective. Let $([f_{\mathscr{U}}])_{\mathscr{U}\in\Lambda} \in \check{\pi}_n(X, x_0)$. By Lemma 6.1, for each $\mathscr{U} \in \Lambda$, there exists $[g_{\mathscr{U}}] \in \pi_n(X, x_0)$ such that $p_{\mathscr{U}}([g_{\mathscr{U}}]) = [f_{\mathscr{U}}]$. If $\mathscr{V} \preceq \mathscr{W}$, then we have

$$p_{\mathscr{V}\#}([g_{\mathscr{V}}]) = [f_{\mathscr{V}}] = p_{\mathscr{V}\mathscr{W}\#}([f_{\mathscr{W}}]) = p_{\mathscr{V}\mathscr{W}\#} \circ p_{\mathscr{W}\#}([g_{\mathscr{W}}]) = p_{\mathscr{V}\#}([g_{\mathscr{W}}]).$$

Since $p_{\mathscr{V}\#}$ is injective, it follows that $[g_{\mathscr{W}}]=[g_{\mathscr{V}}]$ whenever $\mathscr{V}\preceq\mathscr{W}$. Setting $[g]=[g_{\mathscr{V}}]$ gives $\Psi_n([g])=([f_{\mathscr{U}}])_{\mathscr{U}\in\Lambda}$. Hence, Ψ_n is surjective. □

7. Examples

Example 7.1. Fix $n\geq 2$. When X is a metrizable LC^{n-1} space, the cone CX and unreduced suspension SX are LC^{n-1} and semilocally π_n-trivial but need not be LC^n. This occurs in the case $X=\mathbb{E}_n$ or if $X=Y\vee\mathbb{E}_n$ where Y is a CW-complex. In such cases, $\Psi_n:\pi_n(SX)\to\check{\pi}_n(SX)$ is an isomorphism. One point unions of such cones and suspensions, e.g., $CX\vee CY$ or $CX\vee SY$ also meet the hypotheses of Theorem 1.2 (checking this is fairly technical [Brazas 2021]) but need not be LC^n.

Example 7.2. The converse of Theorem 1.2 does not hold. For $n\geq 2$, $\mathbb{E}_n$ is LC^{n-1} but is not semilocally π_n-trivial at the wedgepoint x_0. However, $\Psi_n:\pi_n(\mathbb{E}_n,x_0)\to\check{\pi}_n(\mathbb{E}_n,x_0)$ is an isomorphism where both groups are canonically isomorphic to $\mathbb{Z}^{\mathbb{N}}$ [Eda and Kawamura 2000a]. Additionally, for the infinite direct product $\prod_{\mathbb{N}}S^n$, $\Psi_k:\pi_k\big(\prod_{\mathbb{N}}S^n,x_0\big)\to\check{\pi}_k\big(\prod_{\mathbb{N}}S^n,x_0\big)$ is an isomorphism for all $k\geq 1$ even though $\prod_{\mathbb{N}}S^n$ is not LC^{k-1} when $k-1\geq n$.

Example 7.3. We can also modify the mapping torus M_f from Example 5.3 so that Ψ_n becomes an isomorphism (recall that $n\geq 2$ is fixed). Let $X=M_f\cup C\mathbb{E}_n$ be the mapping cone of the inclusion $\mathbb{E}_n\to M_f$. For the same reason M_f is LC^{n-1}, the space X is LC^{n-1}. Moreover, if U is a neighborhood of $\alpha(t)$ that deformation retracts onto a homeomorphic copy of $\mathbb{E}_n$, then any map $S^n\to U$ may be freely homotoped “around” the torus and into the cone. It follows that X is semilocally π_n-trivial. We conclude from Theorem 1.2 that $\Psi_n:\pi_n(X)\to\check{\pi}_n(X)$ is an isomorphism. Since sufficiently fine covers of X always give nerves homotopy equivalent to $S^1\vee S^{n+1}$, we have $\check{\pi}_n(X)=0$. Thus $\pi_n(X)=0$.

Example 7.4. Let $n\geq 2$ and $X=\mathbb{E}_1\vee S^n$ (see Figure 2). Note that because $\mathbb{E}_1$ is aspherical [Cannon et al. 2002; Curtis and Fort 1957], X is semilocally π_n-trivial. However, X is not LC^1 because it has $\mathbb{E}_1$ as a retract. It is shown in [Brazas 2021] that $\pi_n(X)\cong\bigoplus_{\pi_1(\mathbb{E}_1)}\pi_n(S^n)\cong\bigoplus_{\pi_1(\mathbb{E}_1)}\mathbb{Z}$ and that $\Psi_n:\pi_n(X)\to\check{\pi}_n(X)$ is injective. In particular, we may represent elements of $\pi_n(X)$ as finite-support sums $\sum_{\beta\in\pi_1(\mathbb{E}_1)}m_\beta$ where $m_\beta\in\mathbb{Z}$. We show that Ψ_n is not surjective.

Identify $\pi_1(X)$ with $\pi_1(\mathbb{E}_1)$ and recall from [Eda 1992] that we can represent the elements of $\pi_1(\mathbb{E}_1)$ as countably infinite reduced words indexed by a countable linear order (with a countable alphabet $\beta_1,\beta_2,\beta_3,\dots$). Here β_j is represented by a loop $S^1\to\mathbb{E}_1$ going once around the j-th circle. Let X_j be the union of S^n and the largest j circles of $\mathbb{E}_1$ so that $X=\varprojlim_j X_j$. Identify $\pi_1(X_j)$ with the free group F_j on generators $\beta_1,\beta_2,\dots\beta_j$ and note that $\pi_n(X_j)\cong\bigoplus_{F_j}\mathbb{Z}$. Thus we may view an element of $\pi_n(X_j)$ as a finite-support sums $\sum_{w\in F_j}m_w$ of integers indexed over reduced words in F_j. Let $d_{j+1,j}:F_{j+1}\to F_j$ be the homomorphism that deletes the

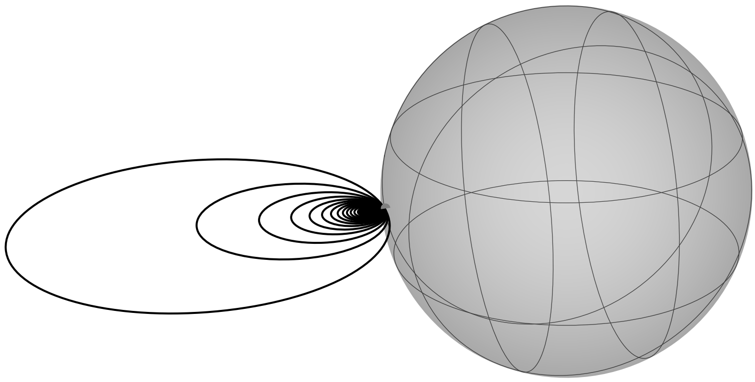

Figure 2. The one point union $\mathbb{E}_1 \vee S^2$.

letter β_{j+1}. Consider the inverse limit $\check{\pi}_1(X) = \varprojlim_j (F_j, d_{j+1,j})$. The map $X \to X_j$ that collapses all but the first j-circles of $\mathbb{E}_1$ induces a homomorphism $d_j : \pi_1(X) \to F_j$. There is a canonical homomorphism $\phi : \pi_1(X) \to \check{\pi}_1(X) = \varprojlim_j (F_j, d_{j+1,j})$ given by $\phi(\beta) = (d_1(\beta), d_2(\beta), \dots)$, which is known to be injective [Morgan and Morrison 1986] but not surjective. For example, if $x_k = \prod_{j=1}^{k} [\beta_1, \beta_j]$, then $(x_1, x_2, x_3, x_4, \dots)$ is an element of $\check{\pi}_1(X)$ not in the image of ϕ.

The bonding map $b_{j+1,j} : \pi_n(X_{j+1}) \to \pi_n(X_j)$ sends a sum $\sum_{w \in F_{j+1}} m_w$ to $\sum_{v \in F_j} p_v$ where $p_v = \sum_{d_{j+1,j}(w)=v} m_w$. Similarly, projection map $b_j : \pi_n(X) \to \pi_n(X_j)$ sends the sum $\sum_{\beta \in \pi_1(X)} n_\beta$ to $\sum_{v \in F_j} m_v$ where $m_v = \sum_{d_j(\beta)=v} m_\beta$. Let $y_j \in \pi_n(X)$ be the sum whose only nonzero coefficient is the x_j-coefficient, which is 1. Since $d_{j+1,j}(x_{j+1}) = x_j$, it's clear that $(y_1, y_2, y_3, \dots) \in \check{\pi}_n(X)$. Suppose $\Psi_n\big(\sum_\beta m_\beta\big) = (y_1, y_2, y_3, \dots)$. Writing $\sum_\beta m_\beta$ as a finite sum $\sum_{i=1}^{r} m_{\beta_i}$ for nonzero m_{β_i}, we must have $\sum_{d_j(\beta_i)=x_j} m_{\beta_i} = 1$ for all $j \in \mathbb{N}$. Since there are only finitely many β_i involved, there must exist at least one i for which $d_j(\beta_i) = x_j$ for infinitely many j. For such i, we have $\phi(\beta_i) = (x_1, x_2, x_3, \dots)$, which, as mentioned above, is impossible. Hence Ψ_n is not surjective.

The previous example shows why we cannot remove the LC^{n-1} hypothesis in Theorem 1.2. Since we weakened the hypothesis from [Kozlowski and Segal 1978] in dimension n and no hypothesis in dimension n is required for Theorem 1.1, one might suspect that we might be able to remove the dimension n hypothesis completely. The next example, which is a higher analogue of the harmonic archipelago [Bogley and Sieradski 1998; Conner et al. 2015; Karimov and Repovš 2012] shows why this is not possible.

Example 7.5. Let $n \geq 2$ and $\ell_j : S^n \to \mathbb{E}_n$ be the inclusion of the j-th n-sphere in $\mathbb{E}_n$. Let X be the space obtained by attaching $(n+1)$-cells to $\mathbb{E}_n$ using the attaching maps ℓ_j. Since $\mathbb{E}_n$ is LC^{n-1}, it follows that X is LC^{n-1}. However, X is not semilocally π_n-trivial at the wedgepoint $\boldsymbol{o}$ of $\mathbb{E}_n$. Indeed, the infinite concatenation maps $\prod_{j\geq k} \ell_j = \ell_k \cdot \ell_{k+1} \cdots$ are not null-homotopic (using a standard argument that works for the harmonic archipelago) but are all homotopic to each other. Thus, $\pi_n(X, \boldsymbol{o}) \neq 0$. However, for sufficiently fine open covers $\mathscr{U} \in O(X)$, $|N(\mathscr{U})|$ is homotopy equivalent to a wedge of $(n+1)$-spheres and thus $\check{\pi}_n(X, \boldsymbol{o}) = 0$. Therefore, despite X being LC^{n-1}, Ψ_n is not an isomorphism. In fact, $\pi_n(X, \boldsymbol{o}) = \pi_n^{Sp}(X, \boldsymbol{o}) = \ker(\Psi_n)$. The reader might also note that since $\mathbb{E}_{n-1}$ is $(n-1)$-connected and $\pi_n(\mathbb{E}_n, \boldsymbol{o}) \cong H_n(\mathbb{E}_n) \cong \mathbb{Z}^{\mathbb{N}}$, X will also be $(n-1)$-connected. A Meyer–Vietoris sequence argument similar to that in [Karimov and Repovš 2012] can then be used to show $\pi_n(X, \boldsymbol{o}) \cong H_n(X) \cong \mathbb{Z}^{\mathbb{N}} / \bigoplus_{\mathbb{N}} \mathbb{Z}$.

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REGULARITY FOR FREE MULTIPLICATIVE CONVOLUTION ON THE UNIT CIRCLE

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Suppose that μ_1 and μ_2 are Borel probability measures on the unit circle, both different from unit point masses, and let μ denote their free multiplicative convolution. We show that μ has no continuous singular part (relative to arclength measure) and that its density can only be locally unbounded at a finite number of points, entirely determined by the point masses of μ_1 and μ_2. Analogous results were proved earlier for the free additive convolution on $\mathbb{R}$ and for the free multiplicative convolution of Borel probability measures on the positive half-line.

1. Introduction

It has been known for some time that free convolutions have a strong regularizing effect. The earliest instances of this phenomenon were observed in [Voiculescu 1993; Bercovici and Voiculescu 1998; Biane 1997]. For the additive case (see [Voiculescu 1986; Bercovici and Voiculescu 1993; Voiculescu et al. 1992] for definitions), it was shown in [Belinschi 2008; 2014] that, given Borel probability measures μ_1, μ_2 on $\mathbb{R}$, neither of which is a point mass, the free convolution $\mu = \mu_1 \boxplus \mu_2$ has no singular continuous part relative to the Lebesgue measure, and its density is analytic wherever positive and finite. In addition, this density is locally bounded unless $\mu_1(\{\alpha_1\}) + \mu_2(\{\alpha_2\}) \geq 1$ for some $\alpha_1, \alpha_2 \in \mathbb{R}$. The atomic part of μ has finite support and was determined earlier [Bercovici and Voiculescu 1998]. Analogous results have been obtained in [Ji 2021] for the free multiplicative convolution of Borel probability measures on $[0, +\infty)$. Despite a strong similarity between these operations, the corresponding result for free multiplicative convolutions of Borel probability measures on the unit circle $\mathbb{T}$ in the complex plane is still missing. Recent results on Denjoy–Wolff points [Belinschi et al. 2022, Corollary 3.3] allow us to rectify this omission in Theorem 3.2.

The necessary background on subordination is given in Section 2, and the main result is proved in Section 3. An application in Section 4 yields a strengthening of

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the results of [Bercovici and Wang 2008] concerning indecomposable measures relative to free convolution.

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2. Analytic subordination for free multiplicative convolution

We begin by recalling the analytical apparatus for the calculation of free multiplicative convolutions on the unit circle $\mathbb{T} = \{z \in \mathbb{C} \colon |z| = 1\}$. An arbitrary Borel probability measure μ on $\mathbb{T}$ is uniquely determined by its *moments*

$$m_n(\mu) = \int_{\mathbb{T}} t^n \, d\mu(t), \quad n \in \mathbb{N},$$

and these moments are encoded in the *moment generating function*

$$\psi_\mu(z) = \int_{\mathbb{T}} \frac{tz}{1-tz} \, d\mu(t) = \sum_{n=1}^{\infty} m_n(\mu) z^n.$$

The formal series ψ_μ actually converges for z in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and

$$\psi_\mu(\mathbb{D}) \subset \left\{ z \in \mathbb{C} \colon \Re z > -\tfrac{1}{2} \right\}.$$

Observe that

$$2\Re\psi_\mu(z) + 1 = \int_{\mathbb{T}} \Re\left(\frac{\bar{\zeta} + z}{\bar{\zeta} - z}\right) d\mu(\zeta) = \int_{\mathbb{T}} \Re\left(\frac{\zeta + z}{\zeta - z}\right) d\mu(\bar{\zeta}), \quad z \in \mathbb{D}, \tag{2-1}$$

and the last term above is precisely a Poisson integral. It follows that μ can be recovered from ψ_μ by taking radial limits

$$2\pi \, d\mu(e^{-i\theta}) = \lim_{r \uparrow 1} \bigl(2\Re\psi_\mu(e^{i\theta}) + 1\bigr) \, d\theta.$$

(See, for instance, [Akhiezer 1965, Chapter 5], [Belinschi and Bercovici 2005, Section 3], and [Garnett 1981, Chapter 1] for details.) In particular, if μ^{s} denotes the singular part of the measure μ, (2-1) shows that

$$\lim_{r \uparrow 1} \Re\psi_\mu(r\bar{\zeta}) = +\infty \quad \text{for } \mu^{\mathrm{s}}\text{-almost all } \zeta \in \mathbb{T}. \tag{2-2}$$

We note for further use the following consequence of (2-1):

Lemma 2.1. *If ψ_μ is a bounded function on $\mathbb{D}$, then μ is absolutely continuous relative to arclength measure and its density is bounded.*

Consider now two Borel probability measures μ_1, μ_2 on $\mathbb{T} = \{z \in \mathbb{C} \colon |z| = 1\}$, and denote by $\mu = \mu_1 \boxtimes \mu_2$ their free multiplicative convolution. This was first defined in [Voiculescu 1987] using the multiplication of *-free unitary operators, and its calculation — in case the two measures have a nonzero first moment — relied on the analytic inverses of the functions ψ_{μ_1} and ψ_{μ_2} in the complex plane (see [Voiculescu et al. 1992] for the technical details). Subsequently, Biane [1998] discovered that ψ_μ is subordinate to ψ_{μ_j}, with $j = 1, 2$, in the sense of Littlewood. This result implies that — at least when μ_1 and μ_2 have nonzero first moments — one can describe the function ψ_μ as the unique solution of a system of implicit equations. This method for the calculation of ψ_μ does in fact extend to arbitrary μ_1 and μ_2, as seen in [Belinschi and Bercovici 2007]. We state the result below because it is instrumental in the proof of Theorem 3.2. We need the additional notation

$$\eta_\mu(z) = \frac{\psi_\mu(z)}{1+\psi_\mu(z)} \quad \text{and} \quad h_\mu(z) = \frac{\eta_\mu(z)}{z}.$$

It is easily seen that $\eta_\mu(\mathbb{D}) \subset \mathbb{D}$, $\eta_\mu(0) = 0$, $\eta'_\mu(0) = m_1(\mu)$, and h_μ extends to an analytic function from $\mathbb{D}$ to $\overline{\mathbb{D}}$. If the function h_μ takes values in $\mathbb{T}$, then it is constant and this happens precisely when μ is a point mass. The following statement combines [Belinschi and Bercovici 2007, Theorem 3.2] and [Belinschi et al. 2022, Corollary 3.3]:

Theorem 2.2. *Consider Borel probability measures μ_1, μ_2 on $\mathbb{T}$ and their free multiplicative convolution $\mu = \mu_1 \boxtimes \mu_2$. There exist unique continuous functions $\omega_1, \omega_2 : \mathbb{D} \cup \mathbb{T} \to \mathbb{D} \cup \mathbb{T}$ that are analytic on $\mathbb{D}$ and, in addition*:

(1) $\omega_1(0) = \omega_2(0) = 0$.

(2) $z\eta_\mu(z) = z\eta_{\mu_1}(\omega_1(z)) = z\eta_{\mu_2}(\omega_2(z)) = \omega_1(z)\omega_2(z)$, $\omega_1(z) = zh_2(\omega_2(z))$, *and* $\omega_2(z) = zh_1(\omega_1(z))$ *for every* $z \in \mathbb{D} \cup \mathbb{T}$. *In particular*, η_μ *extends continuously to* $\mathbb{T}$. *When either* $\omega_1(z)$ *or* $\omega_2(z)$ *belongs to* $\mathbb{T}$, *the values* $\eta_{\mu_j}(\omega_j(z))$ *are understood as radial limits*, *that is*,

$$\eta_{\mu_j}(\omega_j(z)) = \lim_{r \uparrow 1} \eta_{\mu_j}(r\omega_j(z)).$$

(3) *If* $m_1(\mu_1) = m_1(\mu_2) = 0$, *the functions* η_μ, ψ_μ, ω_1, *and* ω_2 *are identically zero.*

3. Boundedness and the lack of a singular continuous part

We are ready now to identify the singular behavior of a free multiplicative convolution on $\mathbb{T}$. Of course, part (1) was proved in [Belinschi 2003].

Lemma 3.1. *Suppose that μ_1 and μ_2 are Borel probability measures on $\mathbb{T}$, neither of which is a unit point mass, set $\mu = \mu_1 \boxtimes \mu_2$, and let $\alpha \in \mathbb{T}$.*

(1) *If* $\mu(\{\alpha\}) > 0$, *then there exist* $\alpha_1, \alpha_2 \in \mathbb{T}$ *such that* $\alpha_1\alpha_2 = \alpha$ *and*

$$\mu_1(\{\alpha_1\}) + \mu_2(\{\alpha_2\}) = 1 + \mu(\{\alpha\}).$$

(2) *If* ψ_μ *is unbounded near* $1/\alpha$, *then there exist* $\alpha_1, \alpha_2 \in \mathbb{T}$ *such that* $\alpha_1\alpha_2 = \alpha$ *and*

$$\mu_1(\{\alpha_1\}) + \mu_2(\{\alpha_2\}) \geq 1.$$

Proof. We only prove (2). As already mentioned, if $m_1(\mu_1) = m_1(\mu_2) = 0$, then μ is the Haar measure on $\mathbb{T}$, which has no singular part and a density identically equal to $1/2\pi$. Indeed, by Theorem 2.2 (3), ψ_μ is identically zero; in particular, bounded. For the remainder of the proof, we assume that at least one of $m_1(\mu_1), m_1(\mu_2)$ is nonzero, and thus the functions $\psi_\mu, \omega_1, \omega_2$ of Theorem 2.2 are not constant. Suppose now that $\beta = 1/\alpha$ is such that $\eta_\mu(\beta) = 1$ or, equivalently,

$$\psi_\mu(\beta) = \lim_{r\uparrow 1} \psi_\mu(r\beta) = \infty.$$

Setting $\alpha_1 = \omega_1(\beta)$ and $\alpha_2 = \omega_2(\beta)$,Theorem 2.2 (2) yields the equality $\alpha_1\alpha_2 = \beta$. Since $|\alpha_j| \le 1$, it follows that, in fact, $\alpha_j \in \mathbb{T}$ for $j = 1, 2$. The subordination in Theorem 2.2 (2) also yields

$$\lim_{z\to\beta} \eta_{\mu_j}(\omega_j(z)) = \eta_\mu(\beta) = 1, \quad j = 1, 2,$$

and then

$$\lim_{r\uparrow 1} \eta_{\mu_j}(r\alpha_j) = 1, \quad j = 1, 2,$$

by Lindelöf's Theorem (see [Collingwood and Lohwater 1966, Theorem 2.3]).

An application of the dominated convergence theorem shows that

$$\lim_{r\uparrow 1}(1-r)\psi_{\mu_j}(r\alpha_j) = \mu\Big(\Big\{\frac{1}{\alpha_j}\Big\}\Big) \in [0, 1), \quad j = 1, 2.$$

In terms of the functions η_{μ_j}, this amounts to

$$\lim_{r\uparrow 1} \frac{\eta_{\mu_j}(r\alpha_j) - 1}{r-1} = \frac{1}{\mu_j(\{1/\alpha_j\})}, \quad j = 1, 2,$$

where the right-hand side is understood as ∞ if $\mu_j(\{1/\alpha_j\}) = 0$. Using Julia–Carathéodory derivatives (see, for instance, [Garnett 1981, Chapter I, Exercise 7]) this relation can be rewritten as $\eta'_\mu(\omega_1(\alpha)) = 1/(\mu_j(\{1/\alpha_j\}))$. Properties of this derivative imply now that

$$\begin{aligned}
\frac{1}{\mu_1(\{1/\alpha_1\})} - 1 &= \liminf_{w\to\alpha_1} \frac{|\eta_{\mu_1}(w)| - 1}{|w|-1} - 1 \\
&= \liminf_{w\to\alpha_1} \frac{|\eta_{\mu_1}(w)| - |w|}{|w|-1} \\
&\le \liminf_{z\to\beta} \frac{|\eta_{\mu_1}(\omega_1(z))| - |\omega_1(z)|}{|\omega_1(z)|-1} \quad \text{(substituting } w = \omega_1(z)\text{)} \\
&= \liminf_{z\to\beta} \frac{|\omega_1(z)|}{|z|} \frac{|\omega_2(z)| - |z|}{|\omega_1(z)|-1} \quad \text{(using Theorem 2.2)} \\
&= \liminf_{z\to\beta} \frac{|\omega_2(z)| - |z|}{|\omega_1(z)|-1} \\
&\le \liminf_{z\to\beta} \frac{1 - |\omega_2(z)|}{1 - |\omega_1(z)|}.
\end{aligned}$$

Switching the roles of μ_1 and μ_2, we obtain

$$\begin{aligned}\frac{1}{\mu_2(\{1/\alpha_2\})}-1 &\le \liminf_{z\to\beta}\frac{1-|\omega_1(z)|}{1-|\omega_2(z)|}\\ &= \left[\limsup_{z\to\beta}\frac{1-|\omega_2(z)|}{1-|\omega_1(z)|}\right]^{-1}\\ &\le \left[\liminf_{z\to\beta}\frac{1-|\omega_2(z)|}{1-|\omega_1(z)|}\right]^{-1}\\ &\le \left[\frac{1}{\mu_1(\{1/\alpha_1\})}-1\right]^{-1}.\end{aligned}$$

A simple calculation shows now that the inequality

$$\left(\frac{1}{\mu_2(\{1/\alpha_2\})}-1\right)\left(\frac{1}{\mu_1(\{1/\alpha_1\})}-1\right)\le 1$$

is equivalent to $\mu_1(\{1/\alpha_1\})+\mu_2(\{1/\alpha_2\})\ge 1$, thus concluding the proof. □

We are now ready to state and prove the main result of this paper.

Theorem 3.2. *Consider the Borel probability measures μ_1, μ_2 on $\mathbb{T}$ and their free multiplicative convolution $\mu=\mu_1\boxtimes\mu_2$. Suppose that neither μ_1 nor μ_2 is a point mass. Then*:

(1) *The singular continuous part of μ relative to the arclength measure is zero.*

(2) *If we have*

$$\max\{\mu_1(\{\alpha_1\})+\mu_2(\{\alpha_2\}):\alpha_1,\alpha_2\in\mathbb{T}\}\le 1, \tag{3-1}$$

then μ is absolutely continuous relative to the arclength measure.

(3) *If* (3-1) *is strict*, *then the density of μ relative to the arclength measure is bounded.*

Remark 3.3. It is remarkable that, for all free convolutions (see [Belinschi 2014; Ji 2021]), only the atomic parts of μ_1, μ_2 have an impact on the local boundedness of the density of their convolution.

Proof. The set $\{(\alpha_1,\alpha_2)\in\mathbb{T}^2:\mu_1(\{\alpha_1\})+\mu_2(\{\alpha_2\})\ge 1\}$ is obviously finite. By Lemma 3.1 (2), the set $S=\{\alpha\in\mathbb{T}:\eta_\mu(\{1/\alpha\})=1\}$ is finite as well. Since (2-2) implies that the support of the singular summand of μ is contained in S, it follows that this summand is a finite sum of point masses. This proves (1). Suppose now that (3-1) holds. Then Lemma 3.1 (1) shows that μ is absolutely continuous. Finally, suppose that (3-1) is strict. Then Lemma 3.1 (2) implies that η_μ does not take the value 1 at any point on $\mathbb{T}$. Since η_μ is continuous on $\overline{\mathbb{D}}$, it must be bounded away from 1. Thus $\psi_\mu=\eta_\mu/(1-\eta_\mu)$ is a bounded function. Then (3) follows from Lemma 2.1. □

Remark 3.4. Suppose that $\mu_1(\{\alpha_1\}) + \mu_2(\{\alpha_2\}) = 1$ for some $\alpha_1, \alpha_2 \in \mathbb{T}$. It was shown in [Belinschi 2003] that, setting $\beta_j = 1/\alpha_j$ and $\beta = \beta_1\beta_2$, we have $\omega_j(\beta) = \beta_j$ for $j = 1, 2$, but, of course, $\mu(\{1/\beta\}) = 0$. (This can also be proved using the results of [Belinschi et al. 2022] and the "chain rule" for Julia–Carathéodory derivatives.) In all computable examples, the density of μ is unbounded near $1/\beta$. We suspect that this is true in full generality.

4. An application

The following statement extends the main result of [Bercovici and Wang 2008] for probability measures on the circle. Nearly identical proofs yield the corresponding extensions for free additive convolutions and for free multiplicative convolutions on the positive half-line. For these two convolutions, it is not necessary to assume that one of the convolved measures has more than two points in its support. The condition $\eta_\mu(\alpha) = 1$ in the statement amounts to the requirement that either γ is an atom of μ, or the density of μ is unbounded near γ (or both).

Theorem 4.1. *Consider Borel probability measures μ_1, μ_2 on $\mathbb{T}$, different from point masses, and set $\mu = \mu_1 \boxtimes \mu_2$. Suppose that $J \subset \mathbb{T}$ is an open arc such that each endpoint α of J satisfies $\eta_\mu(\alpha) = 1$. If either μ_1 or μ_2 has more than two points in its support, then $\mu(J) > 0$.*

Proof. Let α and β be the two endpoints of J, and let ω_j denote the subordination function of η_μ relative to η_{μ_j}. By Lemma 3.1, the points $\alpha_j = \omega_j(\alpha)$ and $\beta_j = \omega_j(\beta)$ satisfy

$$\mu_1(\{\alpha_1\}) + \mu_2(\{\alpha_2\}) \ge 1 \quad \text{and} \quad \mu_1(\{\beta_1\}) + \mu_2(\{\beta_2\}) \ge 1.$$

The hypothesis implies that either $\alpha_1 = \beta_1$ or $\alpha_2 = \beta_2$. Indeed, otherwise, it would follow that the support of μ_j is $\{\alpha_j, \beta_j\}$, for $j = 1, 2$. Switching, if necessary, the roles of μ_1 and μ_2, we may assume that $\alpha_1 = \beta_1$, so $\omega_1(\alpha) = \omega_1(\beta)$.

Suppose now that $\mu(J) = 0$. Then $|\eta_\mu(\zeta)| = 1$ for every $\zeta \in J$. The equation $\eta_\mu(\zeta) = \eta_{\mu_1}(\omega_1(\zeta))$ and the Schwarz lemma (which applies because $\eta_\mu(0) = 0$), imply that

$$|\eta_\mu(z)| \le |\omega_1(z)|$$

for every $z \in \mathbb{D}$. Letting z approach a point $\zeta \in J$, we see that $|\omega_1(\zeta)| = 1$. Now, ω_1 is not constant, and therefore $\omega_1(\zeta)$ moves counterclockwise as $\zeta \in J$ does so. By the Schwarz reflection principle, ω_1 is analytic and, thanks to the Julia–Carathéodory Theorem, it is locally injective on J. The equation $\omega_1(\alpha) = \omega_1(\beta)$ allows us to conclude that $\omega_1(J) \supseteq \mathbb{T} \setminus \{\omega_1(\alpha)\}$. Moreover, the fact that $|\eta_{\mu_1}(\omega_1(\zeta))| = 1$ for $\zeta \in J$ shows that the support of μ_1 is contained in $\mathbb{T} \setminus \omega_1(J) \subseteq \{\omega_1(\alpha)\}$, contrary to the hypothesis. This contradiction yields the desired conclusion that $\mu(J) \neq 0$. □

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INVARIANT THEORY FOR THE FREE LEFT-REGULAR BAND AND A Q-ANALOGUE

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To the memory of Georgia Benkart

We examine from an invariant theory viewpoint the monoid algebras for two monoids having large symmetry groups. The first monoid is the *free left-regular band* on *n* letters, defined on the set of all *injective* words, that is, the words with at most one occurrence of each letter. This monoid carries the action of the symmetric group. The second monoid is one of its *q*-analogues, considered by K. Brown, carrying an action of the finite general linear group. In both cases, we show that the invariant subalgebras are semisimple commutative algebras, and characterize them using *Stirling* and *q-Stirling numbers*.

We then use results from the theory of random walks and random-to-top shuffling to decompose the entire monoid algebra into irreducibles, simultaneously as a module over the invariant ring and as a group representation. Our irreducible decompositions are described in terms of *derangement symmetric functions*, introduced by Désarménien and Wachs.

1. Introduction

Motivated by results on mixing times for shuffling algorithms on permutations, Bidigare [1997] and Bidigare, Hanlon, and Rockmore [Bidigare et al. 1999] developed a complete spectral analysis for a class of random walks on chambers of a hyperplane arrangement. Their work relied heavily on the *Tits semigroup* structure on the cones of the arrangement. Later, Brown [2000] generalized their analysis to random walks coming from semigroups $\mathcal{F}$ which form a *left-regular band* (LRB), meaning that $x^2 = x$ for all x and $xyx = xy$ for all x, y in $\mathcal{F}$.

Here we study two examples of left-regular bands M, related to those discussed by Brown, having actions of large groups of monoid automorphisms G:

- the *free LRB on n letters* [Brown 2000, §1.3], denoted $\mathcal{F}_n$, with G the symmetric group $\mathfrak{S}_n$, and

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- a q-analogue $\mathcal{F}_n^{(q)}$ related to monoids in [Brown 2000], and G the general linear group $\mathrm{GL}_n := \mathrm{GL}_n(\mathbb{F}_q)$.

For both monoids $M = \mathcal{F}_n, \mathcal{F}_n^{(q)}$, we examine the *monoid algebra* $R := \boldsymbol{k}M$ with coefficients in a commutative ring $\boldsymbol{k}$, and answer the two *main questions of invariant theory* for G acting on R:

Question 1.1. What is the structure of the invariant subalgebra R^G?

Question 1.2. What is the structure of R, simultaneously as an R^G-module and a G-representation?

Section 2 answers Question 1.1 with our first main result, using the combinatorics of *Stirling* and *q-Stirling numbers*. We paraphrase it here; see Theorem 2.9 for a more precise statement.

Theorem 1.3. *Consider either monoid $M = \mathcal{F}_n, \mathcal{F}_n^{(q)}$ with symmetry groups $G = \mathfrak{S}_n, \mathrm{GL}_n$, and assume that $\boldsymbol{k}$ is a field in which $|G|$ is invertible.*

(1) *The invariant subalgebra R^G is a commutative subalgebra of R generated by a single element*; *call this element x for $M = \mathcal{F}_n$ and $x^{(q)}$ for $M = \mathcal{F}_n^{(q)}$.*

(2) *The elements $x, x^{(q)}$ have minimal polynomials*

$$f(X) = \begin{cases} X(X-1)(X-2)\cdots(X-n), & \text{if } M = \mathcal{F}_n, \\ X(X-[1]_q)(X-[2]_q)\cdots(X-[n]_q), & \text{if } M = \mathcal{F}_n^{(q)}, \end{cases}$$

where $[m]_q := 1 + q + \cdots + q^{m-1}$ is a standard q-analogue of the integer $m \geq 0$.

(3) *In particular, $R^G \cong \boldsymbol{k}[X]/(f(X))$, and R^G acts semisimply on R, with*
    - *x-eigenvalues $0, 1, 2, \ldots, n$ on $R = \boldsymbol{k}\mathcal{F}_n$,*
    - *$x^{(q)}$-eigenvalues $[0]_q, [1]_q, \ldots, [n]_q$ on $R = \boldsymbol{k}\mathcal{F}_n^{(q)}$.*

Since the above hypothesis that $|G|$ is invertible in $\boldsymbol{k}$ also implies that $\boldsymbol{k}G$ acts semisimply by Maschke's theorem, this leads to our next goal: a complete answer to Question 1.2 above, decomposing the monoid algebra R into simple modules for the simultaneous (commuting) actions of R^G and G. The fact that R^G is generated by a single, semisimple element x (respectively, $x^{(q)}$) reduces this problem to understanding each eigenspace of x (respectively, $x^{(q)}$) as a $\boldsymbol{k}G$-module.

To describe these $\boldsymbol{k}G$-modules, recall that irreducible representations $\{\chi^\lambda\}$ of $\mathfrak{S}_n$ are indexed by partitions λ of n and let $\mathcal{C}(\mathfrak{S}) := \bigoplus_{n=0}^{\infty} \mathcal{C}(\mathfrak{S}_n)$, where $\mathcal{C}(\mathfrak{S}_n)$ denotes the $\mathbb{Z}$-module of *virtual characters* of $\mathfrak{S}_n$. Then the classical *Frobenius characteristic map* ch is an algebra isomorphism between $\mathcal{C}(\mathfrak{S})$ and the ring of symmetric functions Λ. It has $\mathrm{ch}(\chi^\lambda) = s_\lambda$, the *Schur function*, and the trivial representation $\mathbf{1}_n$ has $\mathrm{ch}(\mathbf{1}_n) = h_n$, the *complete homogeneous* symmetric function.

There is a parallel and q-analogous story for a subset of irreducible representations $\{\chi_q^\lambda\}$ of GL_n called the *unipotent representations*, also indexed by partitions λ of n. These are the irreducible constituents of the GL_n-permutation action on the set $\mathrm{GL}_n/B = \mathcal{F}(V)$ of *complete flags of subspaces* in $V = (\mathbb{F}_q)^n$. Here, too, there is a *q-Frobenius characteristic map* ch_q that defines an algebra isomorphism between $\mathcal{C}(\mathrm{GL}) := \bigoplus_{n=0}^{\infty} \mathcal{C}(\mathrm{GL}_n)$ and Λ, where $\mathcal{C}(\mathrm{GL}_n)$ is the free $\mathbb{Z}$-submodule of the class functions on GL_n spanned by the unipotent characters $\{\chi_q^\lambda\}$. As one might hope, $\mathrm{ch}_q(\chi_q^\lambda) = s_\lambda$ and $\mathrm{ch}_q(\mathbf{1}_{\mathrm{GL}_n}) = h_n$, where $\mathbf{1}_{\mathrm{GL}_n}$ is the trivial representation of GL_n.

This allows us to phrase parallel answers to Question 1.2, in terms of an important family of symmetric functions introduced by Désarménien and Wachs [1988], which we will call the *Désarménien–Wachs derangement symmetric functions* $\{\mathfrak{d}_n\}_{n=0,1,2,\ldots}$, reviewed in Section 3C. Here $\mathfrak{d}_n$ is both the Frobenius image of an $\mathfrak{S}_n$-representation $\mathcal{D}_n$ that we call the *Derangement representation*, as well as the q-Frobenius image of a q-analogous GL_n-representation $\mathcal{D}_n^{(q)}$. As the name suggests, these representations have dimensions counted by the *derangement numbers* and *q-derangement numbers*, respectively[1]. They have irreducible decomposition

$$\mathcal{D}_n \cong \bigoplus_Q \chi^{\lambda(Q)} \quad \text{and} \quad \mathcal{D}_n^{(q)} \cong \bigoplus_Q \chi_q^{\lambda(Q)},$$

where Q runs through all standard Young tableaux of size n whose *first ascent* is even [Reiner and Webb 2004]. Derangement symmetric functions have connections to many well-studied objects in combinatorics such as the complex of injective words [Reiner and Webb 2004], random-to-top and random-to-random shuffling [Uyemura-Reyes 2002], higher Lie characters [Uyemura-Reyes 2002], and configuration spaces [Hersh and Reiner 2017]; see Section 3C. We add to this list by showing they form crucial building blocks for the invariant theory of $\boldsymbol{k}\mathcal{F}_n$ and $\boldsymbol{k}\mathcal{F}_n^{(q)}$.

Section 4 derives the following answer to Question 1.2, paraphrased here — see Theorem 4.11 for a more precise statement:

Theorem 1.4. *Let $\boldsymbol{k}$ be a field whose characteristic does not divide $|G|$. Then when $x, x^{(q)}$ act on $\boldsymbol{k}\mathcal{F}_n, \boldsymbol{k}\mathcal{F}_n^{(q)}$, for each $j = 0, 1, 2, \ldots, n$, the j-eigenspace for x and $[j]_q$-eigenspace for $x^{(q)}$ carry G-representations with the same Frobenius map images*

$$\mathrm{ch}\ker\big((x-j)|_{\boldsymbol{k}\mathcal{F}_n}\big) = \sum_{\ell=j}^{n} h_{n-\ell}\cdot h_j \cdot \mathfrak{d}_{\ell-j} = \mathrm{ch}_q \ker\big((x^{(q)} - [j]_q)|_{\boldsymbol{k}\mathcal{F}_n^{(q)}}\big).$$

Our proofs use techniques that go back to a discussion between Michelle Wachs and Reiner in the analysis of random-to-top shuffling, and have been employed

[1]There are two natural families of $\boldsymbol{k}\mathfrak{S}_n$-modules whose dimensions are the derangement numbers, discussed in [Hersh and Reiner 2017, Theorem 1.2]. The representation $\mathcal{D}_n$ here is the one with character $\widehat{\mathrm{Lie}}_n$ in the notation of [Hersh and Reiner 2017, Equation (1)].

more recently by Dieker and Saliola [2018] and Lafrenière [2020] in the analysis of random-to-random shuffling and a generalization. The method constructs eigenvectors of $x, x^{(q)}$ acting on $\mathcal{F}_n, \mathcal{F}_n^{(q)}$ from null vectors associated to the analogous operators for smaller values of n. Combining these ideas with various filtrations on $\boldsymbol{k}M$ allows us to describe the eigenspaces as parabolic inductions of derangement representations in a conceptual way, avoiding character computations.

The remainder of the paper proceeds as follows: Section 2 introduces the monoid algebras of interest, $R = \boldsymbol{k}\mathcal{F}_n, \boldsymbol{k}\mathcal{F}_n^{(q)}$, and proves Theorem 1.3, describing in parallel the invariant subalgebras R^G for $G = \mathfrak{S}_n, \mathrm{GL}_n$. Section 3 reviews the relation between symmetric functions, representations of $\mathfrak{S}_n$ and unipotent representations of GL_n. It also introduces the derangement symmetric functions $\mathfrak{d}_n$, and describes some of their many definitions and guises. Section 4 proves Theorem 1.4, simultaneously decomposing the monoid algebra R into simple modules for R^G and $\boldsymbol{k}G$, with arguments in parallel for $R = \boldsymbol{k}\mathcal{F}_n$ and $R = \boldsymbol{k}\mathcal{F}_n^{(q)}$.

2. Definitions, background, and the answer to Question 1.1

We introduce the monoids $M = \mathcal{F}_n, \mathcal{F}_n^{(q)}$, the symmetries $G = \mathfrak{S}_n, \mathrm{GL}_n$ of the monoid algebras $R = \boldsymbol{k}M$, and analyze the invariant rings R^G. Useful references are Brown [2000] and B. Steinberg [2016].

2A. *The monoids $\mathcal{F}_n$ and $\mathcal{F}_n^{(q)}$.*

Definition 2.1. The *free left-regular band* (or LRB) on n letters $\mathcal{F}_n$ (see [Brown 2000, §1.3] and [Steinberg 2016, §14.3.1]) consists, as a set, of all words $\boldsymbol{a} = (a_1, a_2, \dots, a_\ell)$ with letters a_i from $\{1, 2, \dots, n\}$ and no repeated letters, that is, $a_i \neq a_j$ for $1 \leq i < j \leq n$. Here the *length* $\ell(\boldsymbol{a}) := \ell$ lies anywhere in the range $0 \leq \ell \leq n$. The set $\mathcal{F}_n$ becomes a semigroup under the following operation: if $\boldsymbol{b} = (b_1, \dots, b_m)$ is another word in $\mathcal{F}_n$, then their product is

$$\boldsymbol{a} \cdot \boldsymbol{b} := (a_1, \dots, a_\ell, b_1, \dots, b_m)^\wedge,$$

where we have borrowed the notation from Brown [2000] that for a sequence $\boldsymbol{c} = (c_1, \dots, c_p)$, the subsequence $\boldsymbol{c}^\wedge = (c_1, \dots, c_p)^\wedge$ is obtained by removing any letter c_i that appears already in the prefix $(c_1, c_2, \dots, c_{i-1})$. One can check that the empty word ( ) is an identity element for this operation, and hence $\mathcal{F}_n$ is not only a semigroup, but a *monoid.*

Definition 2.2. The q-analogue of $\mathcal{F}_n$ that we will consider will be denoted $\mathcal{F}_n^{(q)}$. As a set, it consists of all partial flags of subspaces $\boldsymbol{A} = (A_1, A_2, \dots, A_\ell)$, where A_i is an i-dimensional $\mathbb{F}_q$-linear subspace of $(\mathbb{F}_q)^n$, and $A_1 \subset A_2 \subset \cdots \subset A_\ell$. Again the length $\ell(\boldsymbol{A}) := \ell$ lies in the range $0 \leq \ell \leq n$. The set $\mathcal{F}_n^{(q)}$ becomes a semigroup under the following operation: if $\boldsymbol{B} = (B_1, \dots, B_m)$ is another such flag in $\mathcal{F}_n^{(q)}$, then

$$\boldsymbol{A} \cdot \boldsymbol{B} := (A_1, \dots, A_\ell, A_\ell + B_1, A_\ell + B_2, \dots, A_\ell + B_m)^\wedge$$

using a similar notation as before: for a sequence $\boldsymbol{C} = (C_1, \dots, C_p)$ of nested subspaces $C_1 \subseteq C_2 \subseteq \cdots \subseteq C_p$, the subsequence $\boldsymbol{C}^\wedge$ is obtained by removing any subspace C_i that appears already in the prefix $(C_1, C_2, \dots, C_{i-1})$. As above, $\mathcal{F}_n^{(q)}$ is not only a semigroup, but a *monoid*, since the empty flag ( ) is an identity element.

Remark 2.3. *Warning*: Brown [2000, §1.4 and §5] introduced two other monoids $\mathcal{F}_{n,q}$ and $\overline{\mathcal{F}}_{n,q}$, closely related to $\mathcal{F}_n^{(q)}$. All three are different q-analogues of $\mathcal{F}_n$, related as follows:

Considered as a set, Brown's first q-analogue $\mathcal{F}_{n,q}$ consists of all sequences $\boldsymbol{v} = (v_1, v_2, \dots, v_\ell)$ of linearly independent vectors in $(\mathbb{F}_q)^n$. For another sequence $\boldsymbol{v}' = (v_1', v_2', \dots, v_m')$, one defines their product

$$\boldsymbol{v} \cdot \boldsymbol{v}' := (v_1, v_2, \dots, v_\ell, v_1', v_2' \dots, v_m')^\wedge,$$

where $(u_1, \dots, u_p)^\wedge$ is obtained by removing any u_i which is dependent upon the preceding vectors $(u_1, \dots, u_{i-1})$. One may regard the monoid $\mathcal{F}_n^{(q)}$ as a quotient monoid of $\mathcal{F}_{n,q}$ via the surjection

$$\mathcal{F}_{n,q} \twoheadrightarrow \mathcal{F}_n^{(q)}, \quad (v_1, v_2, \dots, v_\ell) \mapsto (A_1, A_2, \dots, A_\ell),$$

where $A_i := \mathbb{F}_q v_1 + \mathbb{F}_q v_2 + \cdots + \mathbb{F}_q v_i$.

Brown's second q-analogue $\overline{\mathcal{F}}_{n,q}$ turns out to be a further quotient of either $\mathcal{F}_{n,q}$ or $\mathcal{F}_n^{(q)}$, whose motivation he explains in [Brown 2000, §5.1 and §5.2]. It is q-analogous to a certain quotient monoid of $\mathcal{F}_n$ that he denotes $\overline{\mathcal{F}}_n$, which one could define as follows: the monoid quotient map $\mathcal{F}_n \twoheadrightarrow \overline{\mathcal{F}}_n$ identifies the longest words, those of length n, with their prefix word of length $n-1$,

$$\overline{(a_1, a_2, \dots, a_{n-1}, a_n)} = \overline{(a_1, a_2, \dots, a_{n-1})}.$$

One can then define Brown's second q-analogue $\overline{\mathcal{F}}_{n,q}$ as a quotient of $\mathcal{F}_n^{(q)}$, where the monoid quotient map $\mathcal{F}_n^{(q)} \twoheadrightarrow \overline{\mathcal{F}}_{n,q}$ identifies a complete flag of length n with the flag of length $n-1$ that omits the (improper) subspace $(\mathbb{F}_q)^n$ at the end:

$$\overline{(A_1, A_2, \cdots, A_{n-1}, (\mathbb{F}_q)^n)} = \overline{(A_1, A_2, \dots, A_{n-1})}.$$

2B. ***Symmetries of the monoid algebras.*** Let $\boldsymbol{k}$ be a commutative ring with 1. For any finite monoid M (such as $M = \mathcal{F}_n, \mathcal{F}_n^{(q)}$), the *monoid algebra* $R = \boldsymbol{k}M$ is the free $\boldsymbol{k}$-module with basis elements given by the elements $\boldsymbol{a}$ of M, and multiplication extended $\boldsymbol{k}$-linearly from the monoid operation on the basis elements

$$\Big(\sum_{\boldsymbol{a}} p_{\boldsymbol{a}}\ \boldsymbol{a}\Big)\Big(\sum_{\boldsymbol{b}} q_{\boldsymbol{b}}\ \boldsymbol{b}\Big) = \sum_{\boldsymbol{a},\boldsymbol{b}} p_{\boldsymbol{a}} q_{\boldsymbol{b}}\ \boldsymbol{a} \cdot \boldsymbol{b} = \sum_{\boldsymbol{c}} \Big(\sum_{\boldsymbol{a}\cdot\boldsymbol{b}=\boldsymbol{c}} p_{\boldsymbol{a}} q_{\boldsymbol{b}}\Big)\boldsymbol{c}.$$

Note that any group G of monoid automorphisms of M acts as ring automorphisms on $R = \boldsymbol{k}M$. In particular, the symmetric group $\mathfrak{S}_n$ permuting letters $\{1, 2, \dots, n\}$ acts on $\mathcal{F}_n$ via

$$w(a_1, \dots, a_\ell) = (w(a_1), \dots, w(a_\ell)).$$

Similarly, the finite general linear group $\mathrm{GL}_n := \mathrm{GL}_n(\mathbb{F}_q)$ acts on $\mathcal{F}_n^{(q)}$ by

$$g(A_1, \dots, A_\ell) = (g(A_1), \dots, g(A_\ell)).$$

Our first goal is to analyze the G-invariant subalgebras R^G in both cases.

2C. ***The invariant subalgebras R^G and Question 1.1.*** Since the groups G permute the monoid elements M, the monoid algebra $R = \boldsymbol{k}M$ becomes a permutation representation of G. Therefore, the invariant subalgebra R^G has as a $\boldsymbol{k}$-basis the orbit sums $\left\{\sum_{\boldsymbol{a} \in \mathcal{O}} \boldsymbol{a}\right\}$ as one runs through all G-orbits $\mathcal{O}$ on M. For both of the monoids $M = \mathcal{F}_n, \mathcal{F}_n^{(q)}$, one can easily identify the G-orbits, since the groups $G = \mathfrak{S}_n$ and GL_n act transitively on the subsets

$$\begin{aligned} \mathcal{F}_{n,\ell} &:= \{\boldsymbol{a} \in \mathcal{F}_n : \ell(\boldsymbol{a}) = \ell\}, \\ \mathcal{F}_{n,\ell}^{(q)} &:= \{\boldsymbol{A} \in \mathcal{F}_n^{(q)} : \ell(\boldsymbol{A}) = \ell\}. \end{aligned}$$

Thus the G-invariant subalgebras R^G have $\boldsymbol{k}$-bases $\{x_\ell\}_{\ell=0,1,\dots,n}$, and $\{x_\ell^{(q)}\}_{\ell=0,1,\dots,n}$, defined by

$$x_\ell := \sum_{\boldsymbol{a} \in \mathcal{F}_{n,\ell}} \boldsymbol{a} \quad \text{and} \quad x_\ell^{(q)} := \sum_{\boldsymbol{A} \in \mathcal{F}_{n,\ell}^{(q)}} \boldsymbol{A}. \tag{1}$$

Example 2.4. Let $q = 2$, $n = 3$, $\ell = 1$, and let e_1, e_2, e_3 be standard basis vectors for $V = (\mathbb{F}_2)^3$. Using the notation $\langle v_1, v_2, \dots, v_m\rangle$ for the $\mathbb{F}_q$-span of the vectors $\{v_1, v_2, \dots, v_m\}$ in V, one has

$$x_1^{(2)} = (\langle e_1\rangle)+(\langle e_2\rangle)+(\langle e_3\rangle)+(\langle e_1{+}e_2\rangle)+(\langle e_1{+}e_3\rangle)+(\langle e_2{+}e_3\rangle)+(\langle e_1{+}e_2{+}e_3\rangle).$$

It will be convenient to adopt the convention that $x_{n+1} := 0 =: x_{n+1}^{(q)}$.

Using the $\boldsymbol{k}$-bases in (1) for $(\boldsymbol{k}\mathcal{F}_n)^{\mathfrak{S}_n}$ and $(\boldsymbol{k}\mathcal{F}_n^{(q)})^{\mathrm{GL}_n}$, there is a simple rule for multiplication by the elements

$$\begin{aligned} x := x_1 &= \sum_{i=1}^{n} (i) = (1) + (2) + \cdots + (n), \\ x^{(q)} := x_1^{(q)} &= \sum_{\text{lines } L \subset (\mathbb{F}_q)^n} (L). \end{aligned}$$

To state the rule, recall a standard q-analogue of nonnegative integers

$$[n]_q := 1 + q + q^2 + \cdots + q^{n-1}.$$

Lemma 2.5. *Inside R^G for the monoid algebras $R = \boldsymbol{k}M$ with $M = \mathcal{F}_n, \mathcal{F}_n^{(q)}$, the elements x and $x^{(q)}$ act on the (ordered) $\boldsymbol{k}$-bases* (1) *as follows: for $\ell = 0, 1, \ldots, n$,*

$$x \cdot x_\ell = \ell x_\ell + x_{\ell+1},$$
$$x^{(q)} \cdot x_\ell^{(q)} = [\ell]_q \, x_\ell^{(q)} + q^\ell \, x_{\ell+1}^{(q)}.$$

In other words, x and $x^{(q)}$ act on R^G, in the ordered bases above, via the matrices:

$$x = \begin{bmatrix} 0 & & & & & \\ 1 & 1 & & & & \\ & 1 & 2 & & & \\ & & 1 & \ddots & & \\ & & & & n-1 & \\ & & & & 1 & n \end{bmatrix} \quad \text{and} \quad x^{(q)} = \begin{bmatrix} [0]_q & & & & & \\ q^0 & [1]_q & & & & \\ & q^1 & [2]_q & & & \\ & & q^2 & \ddots & & \\ & & & & [n-1]_q & \\ & & & & q^{n-1} & [n]_q \end{bmatrix}.$$

Proof. Note that the product $x \cdot x_\ell$ is G-invariant, and is a sum of terms $\boldsymbol{a}$ of length ℓ or $\ell+1$, so it must have the form $c \cdot x_\ell + d \cdot x_{\ell+1}$ for some constants c, d in $\boldsymbol{k}$. The constant $d = 1$, since any word $\boldsymbol{a} = (a_1, a_2, \ldots, a_{\ell+1})$ of length $\ell+1$ arises uniquely as $(a_1) \cdot (a_2, \ldots, a_{\ell+1})$. The constant $c = \ell$, since any word $(a_1, a_2, \ldots, a_\ell)$ of length ℓ arises in ℓ ways, from these products:

$$\begin{gathered}(a_1) \cdot (\underline{a_1}, a_2, a_3, a_4, \ldots, a_\ell),\\ (a_1) \cdot (a_2, \underline{a_1}, a_3, a_4, \ldots, a_\ell),\\ (a_1) \cdot (a_2, a_3, \underline{a_1}, a_4, \ldots, a_\ell),\\ \vdots\\ (a_1) \cdot (a_2, a_3, a_4, \ldots, a_\ell, \underline{a_1}).\end{gathered}$$

For the q-analogous formula, one argues similarly that

$$x^{(q)} \cdot x_\ell^{(q)} = c \cdot x_\ell^{(q)} + d \cdot x_{\ell+1}^{(q)}$$

for some constants c, d in $\boldsymbol{k}$. We first show that the constant $d = q^\ell$. Any flag $\boldsymbol{A} = (A_1, A_2, \ldots, A_{\ell+1})$ of length $\ell+1$ arises from products of the form $(A_1) \cdot (B_1, B_2 \ldots, B_\ell)$, where the flag $B_1 \subset B_2 \subset \cdots \subset B_\ell$ satisfies $A_1 + B_i = A_{i+1}$ for $i = 1, 2, \ldots, \ell$. If one picks $B_1, B_2, \ldots, B_\ell$ sequentially, then having chosen B_{i-1}, one must choose B_i so that B_i / B_{i-1} is any line inside the 2-dimensional quotient space A_{i+1}/B_{i-1} other than the line $(A_1 + B_{i-1})/B_{i-1}$. Since there are $q+1$ lines in A_{i+1}/B_{i-1}, this gives q choices for B_i, and q^ℓ sequential choices in total for $B_1, B_2, \ldots, B_\ell$.

We next argue that the constant $c = [\ell]_q$. Any flag $\boldsymbol{A} = (A_1, A_2, \ldots, A_\ell)$ of length ℓ arises from products of the form $(A_1) \cdot (B_1, B_2 \ldots, B_\ell)$ in which the flag $B_1 \subset B_2 \subset \cdots \subset B_\ell$ has $A_1 \subseteq B_\ell$ (else, $(A_1) \cdot (B_1, B_2 \ldots, B_\ell)$ has length $\ell+1$, not ℓ).

Letting i_0 be the smallest index for which $A_1 \subseteq B_{i_0}$, one finds that $1 \le i_0 \le \ell$. Having fixed i_0, the B_i for i in the range $i_0 \le i \le \ell$ are completely determined by $B_i = A_1 + B_i = A_i$. Meanwhile, for i in the range $1 \le i \le i_0 - 1$, as in the argument for the constant $d = q^\ell$ above, one can sequentially choose each of $B_1, B_2, \ldots, B_{i_0-1}$ in q ways so that they satisfy $A_1 + B_i = A_{i+1}$. This gives q^{i_0-1} choices, which when summed over $i_0 = 1, 2, \ldots, \ell$ gives $1 + q + q^2 + \cdots + q^{\ell-1} = [\ell]_q$ sequential choices in total. $\square$

Lemma 2.5 allows us to connect R^G to the *Stirling* and *q-Stirling numbers*, briefly reviewed here.

Definition 2.6. The classical *Stirling numbers of the second kind* $(S(n,k))_{k,n=0,1,\ldots}$ have two closely related families of q-analogues $S_q(n,k)$, $\tilde{S}_q(n,k)$, introduced by Carlitz [1933, §4] and studied by many others, e.g., Cai, Ehrenborg, and Readdy [Cai et al. 2018], Garsia and Remmel [1986], Gould [1961], de Médicis and Leroux [1993], Milne [1978; 1982], Sagan and Swanson [2022], Wachs and White [1991], among others. Using the notation[2] in [Milne 1978], all three are doubly indexed triangles defined for (n,k) with $n, k \ge 0$, having initial conditions that set them all equal to 1 when $(n,k) = (0,0)$, and vanishing whenever $n + k \ge 1$ but either $k = 0$ or $n = 0$. When both $n, k \ge 1$, they are then defined by the recursions

$$
\begin{aligned}
S(n,k) &= S(n-1,k-1) + k \cdot S(n-1,k),\\
\tilde{S}_q(n,k) &= \tilde{S}_q(n-1,k-1) + [k]_q \cdot \tilde{S}_q(n-1,k),\\
S_q(n,k) &= q^{k-1} \cdot S_q(n-1,k-1) + [k]_q \cdot S_q(n-1,k).
\end{aligned}
\tag{2}
$$

An easy induction using the recursion lets one check that, for all n and k, one has the relation

$$S_q(n,k) = q^{\binom{k}{2}} \tilde{S}_q(n,k),$$

and for $n \ge 1$, one has

$$S(n,1) = S_q(n,1) = \tilde{S}_q(n,1) = 1, \quad S(n,n) = \tilde{S}_q(n,n) = 1, \quad S_q(n,n) = q^{\binom{n}{2}}. \tag{3}$$

Remark 2.7. Alternatively, one can consider $S(n,k)$, $\tilde{S}_q(n,k)$, $S_q(n,k)$ as change-of-basis matrices in the polynomial rings $\boldsymbol{k}[t]$ with $\boldsymbol{k} = \mathbb{Z}, \mathbb{Z}[q], \mathbb{Z}[q,q^{-1}]$, respectively. Consider the obvious ordered $\boldsymbol{k}$-basis of $\boldsymbol{k}[t]$ given by the powers $(t^n)_{n=0}^{\infty} = (1, t, t^2, \ldots)$, versus these ($q$-)*falling factorial* $\boldsymbol{k}$-bases,

$$
\begin{aligned}
(t)_n &:= t(t-1)(t-2)\cdots(t-(n-1)), && \text{in } \mathbb{Z}[t],\\
(t)_{n,q} &:= t(t-[1]_q)(t-[2]_q)\cdots(t-[n-1]_q), && \text{in } \mathbb{Z}[q][t] \text{ or } \mathbb{Z}[q,q^{-1}][t].
\end{aligned}
$$

[2]Notational conflicts are unavoidable. E.g., our $S_q(n,k)$, $\tilde{S}_q(n,k)$ here equal $\bar{S}[n,k]$, $S[n,k]$, respectively, in [Sagan and Swanson 2022].

Then one has these change-of-basis formulas (see[3] Gould [1961, §3], Milne [1978, Equation (1.14)], and [Garsia and Remmel 1986, Equation (I.17)]):

$$
\begin{aligned}
t^n &= \sum_k S(n,k)\cdot(t)_k, && \text{in } \mathbb{Z}[t],\\
t^n &= \sum_k \tilde{S}_q(n,k)\cdot(t)_{k,q}, && \text{in } \mathbb{Z}[q][t],\\
t^n &= \sum_k S_q(n,k)q^{-\binom{k}{2}}\cdot(t)_{k,q}, && \text{in } \mathbb{Z}[q,q^{-1}][t].
\end{aligned}
\tag{4}
$$

We next show that $S(n,k)$, $S_q(n,k)$ also mediate a natural change-of-basis within R^G.

Corollary 2.8. *Let $\boldsymbol{k}$ be a commutative ring with* 1, *and let $R=\boldsymbol{k}M$ with $M=\mathcal{F}_n$ or $\mathcal{F}_n^{(q)}$. Then the (q-)Stirling numbers $S(m,k)$ and $S_q(m,k)$ are the expansion coefficients for the powers $\{x^m\}_{m=0,1,\dots,n}$ and $\{(x^{(q)})^m\}_{m=0,1,\dots,n}$ in the orbit-sum $\boldsymbol{k}$-bases $\{x_k\}_{k=0,1,\dots,n}$ and $\{x_k^{(q)}\}_{k=0,1,\dots,n}$ of R^G:*

$$x^m=\sum_k S(m,k)\,x_k \quad \textit{and} \quad (x^{(q)})^m=\sum_k S_q(m,k)\,x_k^{(q)}.$$

Thus unitriangularity of $\{S(m,k)\}$ shows $\{x^k\}_{k=0,1,\dots,n}$ always gives a $\boldsymbol{k}$-basis for R^G, while triangularity of $\{S_q(m,k)\}$ shows $\{(x^{(q)})^k\}_{k=0,1,\dots,n}$ is a $\boldsymbol{k}$-basis for R^G if and only if q lies in $\boldsymbol{k}^\times$.

Proof. Both expansions follow by induction on m. Here is the inductive step calculation in the q-Stirling case, applying induction, Lemma 2.5, and (2) for equalities $(*)$, $(**)$, and $(***)$, respectively:

$$
\begin{aligned}
(x^{(q)})^m = x^{(q)}\cdot(x^{(q)})^{m-1} &\overset{(*)}{=} x^{(q)}\cdot\sum_k S_q(m-1,k)\;x_k^{(q)}\\
&= \sum_k S_q(m-1,k)\;x^{(q)}\cdot x_k^{(q)}\\
&\overset{(**)}{=} \sum_k S_q(m-1,k)\big([k]_q x_k^{(q)}+q^k x_{k+1}^{(q)}\big)\\
&= \sum_k\big([k]_q S_q(m-1,k)+q^{k-1}S_q(m-1,k-1)\big)x_k^{(q)}\\
&\overset{(***)}{=} \sum_k S_q(m,k)x_k^{(q)}.
\end{aligned}
$$

The q-expansion is invertible only when q lies in $\boldsymbol{k}^\times$ due to triangularity and $S_q(m,m)=q^{\binom{m}{2}}$. □

This leads to our answer for Question 1.1.

[3]The formulas as discussed by Milne [1978, (1.14)] use the notation $[x]=(y-1)/(q-1)$, where $y=q^x$ is regarded as an indeterminate. To agree with notation and (4) here, one should substitute $t=[x]=(y-1)/(q-1)$, so that $y=1+t(q-1)$.

Theorem 2.9. *Let $\boldsymbol{k}$ be any commutative ring with* 1, *and let $R = \boldsymbol{k}M$ for either of the monoids $M = \mathfrak{F}_n, \mathfrak{F}_n^{(q)}$, with symmetry groups $G = \mathfrak{S}_n, \mathrm{GL}_n$. If $M = \mathfrak{F}_n^{(q)}$, assume further that q is in $\boldsymbol{k}^\times$.*

(i) *The unique $\boldsymbol{k}$-algebra map $\boldsymbol{k}[X] \xrightarrow{\gamma} R$ defined by*

$$X \mapsto \begin{cases} x, & \text{if } M = \mathfrak{F}_n, \\ x^{(q)}, & \text{if } M = \mathfrak{F}_n^{(q)}, \end{cases}$$

induces an algebra isomorphism $\boldsymbol{k}[X]/(f(X)) \cong R^G$, where

$$f(X) := \begin{cases} X(X-1)(X-2)\cdots(X-n), & \text{if } M = \mathfrak{F}_n, \\ X(X-[1]_q)(X-[2]_q)\cdots(X-[n]_q), & \text{if } M = \mathfrak{F}_n^{(q)}. \end{cases}$$

Hence, R^G is commutative and generated by x or $x^{(q)}$.

(ii) *If $\boldsymbol{k}$ is a field, where $|G|$ is invertible, then x or $x^{(q)}$ acts semisimply on any finite-dimensional R^G-module, with eigenvalues contained in the lists*

$$\begin{cases} 0, 1, 2, \ldots, n, & \text{if } M = \mathfrak{F}_n, \\ [0]_q, [1]_q, [2]_q, \ldots, [n]_q, & \text{if } M = \mathfrak{F}_n^{(q)}. \end{cases}$$

Proof. For (i), note that Lemma 2.5 shows that x or $x^{(q)}$ acts on R^G with characteristic polynomial $f(X)$. Consequently, the kernel of the algebra map $\boldsymbol{k}[X] \xrightarrow{\gamma} R^G$ contains $f(X)$, and γ descends to a map on the quotient $\boldsymbol{k}[X]/(f(X)) \xrightarrow{\gamma} R^G$. Moreover, since $f(X)$ is monic of degree $n+1$, the quotient $\boldsymbol{k}[X]/(f(X))$ has $\boldsymbol{k}$-basis $(1, X, X^2, \ldots, X^n)$, and Corollary 2.8 shows that γ maps this onto a $\boldsymbol{k}$-basis of powers $\{x^k\}_{k=0}^n$ or $\{(x^{(q)})^k\}_{k=0}^n$ for R^G. Hence, γ is an algebra isomorphism.

For (ii), assume that $\boldsymbol{k}$ is a field where the roots of the characteristic polynomial $f(X)$ of x or $x^{(q)}$ acting on R^G are all distinct. This means that $f(X)$ must also be the minimal polynomial for x, or $x^{(q)}$ acting on R^G, and that it acts semisimply in any finite dimensional R^G-module, with eigenvalues contained in that set of roots. Lastly, note the groups G have cardinalities

$$|G| = \begin{cases} |\mathfrak{S}_n| = n!, & \text{for } M = \mathfrak{F}_n, \\ |\mathrm{GL}_n| = q^{\binom{n}{2}}(q-1)^n[n]!_q & \\ \qquad = (q^n-1)(q^n-q)(q^n-q^2)\cdots(q^n-q^{n-1}), & \text{for } M = \mathfrak{F}_n^{(q)}, \end{cases}$$

where the q-factorial $[n]!_q$ is defined by

$$[n]!_q := [n]_q[n-1]_q \cdots [2]_q[1]_q. \tag{5}$$

One can then check that the invertibility of $n!$ in $\boldsymbol{k}$ and distinctness of $0, 1, 2, \ldots, n$ are both equivalent to $\boldsymbol{k}$ having characteristic zero or a prime $p > n$, while invertibility of $|\mathrm{GL}_n|$ in $\boldsymbol{k}$ and distinctness of $[0]_q, [1]_q, [2]_q, \ldots, [n]_q$ are both equivalent to $\boldsymbol{k}$ having characteristic zero or characteristic coprime to q and to $[m]_q$ for $m = 1, 2, \ldots, n$. $\square$

We close this section with some remarks on Brown's other q-analogues of $\mathcal{F}_n$.

Remark 2.10. The analysis in Lemma 2.5 can be lifted to an analogous (and even simpler) computation in Brown's first q-analogue $\mathcal{F}_{n,q}$. Denoting the orbit sum $\boldsymbol{k}$-basis in $\boldsymbol{k}\mathcal{F}_{n,q}$ by $y_0, y_1, \ldots, y_n$, multiplication by the element $y := y_1 = \sum_{v\in(\mathbb{F}_q)^n\setminus\{\mathbf{0}\}}(v)$ acts on that basis as follows:

$$y \cdot y_\ell = (q^\ell - 1)y_\ell + y_{\ell+1}. \tag{6}$$

Bearing in mind that the monoid surjection $\mathcal{F}_{n,q} \overset{\pi}{\twoheadrightarrow} \mathcal{F}_n^{(q)}$ described in Remark 2.3 has exactly

$$(q-1)(q^2-q)\cdots(q^\ell - q^{\ell-1}) = (q-1)^\ell q^{\binom{\ell}{2}}$$

preimages $(v_1, v_2, \ldots, v_\ell)$ for every flag $\boldsymbol{A} = (A_1, A_2, \ldots, A_\ell)$, one can check that (6) maps under the linearization $\boldsymbol{k}\mathcal{F}_{n,q} \overset{\pi}{\twoheadrightarrow} \boldsymbol{k}\mathcal{F}_n^{(q)}$ to a formula consistent with the second formula in Lemma 2.5.

Remark 2.11. It is also easy to check that Lemma 2.5 gives similar computations in the other monoids $\overline{\mathcal{F}}_n$ and $\overline{\mathcal{F}}_{n,q}$ considered by Brown, discussed in Remark 2.3. Specifically, in $\boldsymbol{k}\overline{\mathcal{F}}_n$, one has

$$\bar{x} \cdot \bar{x}_\ell = \begin{cases} \ell\bar{x}_\ell + \bar{x}_{\ell+1}, & \text{if } 0 \le \ell < n-1, \\ n\bar{x}_{n-1}, & \text{if } \ell = n-1, \end{cases}$$

and in $\boldsymbol{k}\overline{\mathcal{F}}_{n,q}$, one has

$$\bar{x}^{(q)} \cdot \bar{x}_\ell^{(q)} = \begin{cases} [\ell]_q\bar{x}_\ell^{(q)} + q^\ell\bar{x}_{\ell+1}^{(q)}, & \text{if } 0 \le \ell < n-1, \\ [n]_q\bar{x}_{n-1}^{(q)}, & \text{if } \ell = n-1. \end{cases}$$

The point is that when one $\boldsymbol{k}$-linearizes the monoid surjection $\mathcal{F} \to \overline{\mathcal{F}}_n$ it maps $x_\ell \longmapsto \bar{x}_\ell$ for $i \le n-2$, and maps $x_{n-1}, x_n \longmapsto \bar{x}_{n-1}$. An analogous statement holds for $\mathcal{F}^{(q)} \to \overline{\mathcal{F}}_{n,q}$. One can then check that applying these linearized surjections to Lemma 2.5 gives the above formulas.

3. Representation-theoretic preliminaries

Having answered Question 1.1 by describing the structure of R^G, the next few subsections collect and review some facts regarding representations of $G = \mathfrak{S}_n$ and $G = \mathrm{GL}_n$ that will help us answer Question 1.2 in Section 4 on the structure of R, simultaneously as an R^G-module and a G-representation.

3A. *Semisimplicity, filtrations, and eigenspaces.* In what follows, we will be examining various modules V over the monoid algebra $R = \boldsymbol{k}M$ for the two monoids $M = \mathcal{F}_n, \mathcal{F}_n^{(q)}$, carrying $\boldsymbol{k}G$-module structures for the automorphism groups $G = \mathfrak{S}_n, \mathrm{GL}_n$. In all cases, the G-actions on R and V will be *compatible* in the sense that

$$g(r \cdot v) = g(r) \cdot g(v) \qquad \text{for all } r \in R,\ v \in V,\ g \in G.$$

Note that in this setting, V carries commuting actions of R^G and of $\boldsymbol{k}G$, and we will wish to describe it simultaneously as a module over both.

Henceforth, assume that $\boldsymbol{k}$ is a field in which $|G|$ is invertible, and take V to be finite-dimensional over $\boldsymbol{k}$. This implies that V is semisimple both as an R^G-module due to Theorem 2.9 (ii), and as a $\boldsymbol{k}G$-module by Maschke's Theorem.

In order to answer Question 1.2, we will utilize two important features of our setting:

(1) Semisimplicity implies that given a filtration by R^G-submodules and $\boldsymbol{k}G$-submodules V_i

$$\{0\} = V_0 \subset V_1 \subset \cdots \subset V_r = V,$$

one actually has an R^G-module and $\boldsymbol{k}G$-module isomorphism

$$V \cong \bigoplus_i V_i / V_{i-1}.$$

This will play a crucial role in Section 4B (specifically, in our proof of Theorem 1.4), where we will define filtrations on $\boldsymbol{k}\mathcal{F}_n$ and $\boldsymbol{k}\mathcal{F}_n^{(q)}$ that significantly simplify the analysis.

(2) By Theorem 2.9 (ii), we have that R^G is generated by the single element x or $x^{(q)}$, which acts diagonalizably with certain eigenvalues λ all lying in $\boldsymbol{k}$. It follows that in order to understand the R^G and $\boldsymbol{k}G$-module structure of any module V, it suffices to decompose the eigenspaces $\ker((x-\lambda)|_V)$ as $\boldsymbol{k}G$-modules.

Hence, we will answer Question 1.2 by describing the j-eigenspaces of $\boldsymbol{k}\mathcal{F}_n$ as $\mathfrak{S}_n$-representations and the $[j]_q$-eigenspaces of $\boldsymbol{k}\mathcal{F}_n^{(q)}$ as GL_n- representations for $j = 0, 1, \ldots, n$.

3B. ***Symmetric functions, $\mathfrak{S}_n$-representations, and unipotent GL_n-representations.*** We review here the relation between the *ring of symmetric functions* Λ and representations of $\mathfrak{S}_n$; see Sagan [1991] and Stanley [1999] as references, and for undefined terminology. We then review the parallel story for R. Steinberg's *unipotent representations* of GL_n; see [Grinberg and Reiner 2014, §4.2, §4.6, and §4.7] as a reference.

The *ring of symmetric functions* Λ (of bounded degree, in infinitely many variables) may be viewed as a polynomial algebra $\mathbb{Z}[h_1, h_2, \ldots] = \mathbb{Z}[e_1, e_2, \ldots]$, where h_n and e_n are the *complete homogeneous* and *elementary* symmetric functions of degree n. One may view Λ as a graded $\mathbb{Z}$-algebra $\Lambda = \bigoplus_{n=0}^{\infty} \Lambda^n$, which we wish to relate to the direct sum

$$\mathcal{C}(\mathfrak{S}) := \bigoplus_{n=0}^{\infty} \mathcal{C}(\mathfrak{S}_n),$$

where $\mathcal{C}(\mathfrak{S}_n)$ denotes the $\mathbb{Z}$-module of *virtual characters* of $\mathfrak{S}_n$. That is, $\mathcal{C}(\mathfrak{S}_n)$ is the free $\mathbb{Z}$-module on the basis of irreducible characters $\{\chi^\lambda\}$ indexed by the partitions λ of n, or alternatively, the $\mathbb{Z}$-submodule of class functions on $\mathfrak{S}_n$ of the form $\chi - \chi'$ for genuine characters χ, χ'. One makes $\mathcal{C}(\mathfrak{S})$ into a graded algebra via the *induction product* defined by

$$\text{(7)}\quad \mathcal{C}(\mathfrak{S}_{n_1}) \times \mathcal{C}(\mathfrak{S}_{n_2}) \to \mathcal{C}(\mathfrak{S}_{n_1+n_2}), \quad (f_1, f_2) \mapsto f_1 * f_2 := (f_1 \otimes f_2) \uparrow_{\mathfrak{S}_{n_1} \times \mathfrak{S}_{n_2}}^{\mathfrak{S}_{n_1+n_2}},$$

where $(-) \uparrow_H^G$ is the usual *induction* of class functions on a subgroup H to class functions on G.

For later use, we note that since $[\mathfrak{S}_{n_1+n_2} : \mathfrak{S}_{n_1} \times \mathfrak{S}_{n_2}] = \binom{n_1+n_2}{n_1}$, whenever f_1, f_2 are genuine characters, one has the formula for the degree of $f_1 * f_2$:

$$\deg(f_1 * f_2) = \binom{n_1+n_2}{n_1} \deg(f_1) \deg(f_2). \tag{8}$$

One then has the *Frobenius characteristic isomorphism* of $\mathbb{Z}$-algebras $\mathcal{C}(\mathfrak{S}) \xrightarrow{\text{ch}} \Lambda$, mapping

$$\mathcal{C}(\mathfrak{S}) \xrightarrow{\text{ch}} \Lambda, \quad \mathbf{1}_{\mathfrak{S}_n} \mapsto h_n,\ \text{sgn}_{\mathfrak{S}_n} \mapsto e_n,\ \chi^\lambda \mapsto s_\lambda.$$

Here, s_λ is the *Schur function*. For a composition $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_\ell$, we use the standard shorthand

$$h_\alpha := h_{\alpha_1} h_{\alpha_2} \cdots h_{\alpha_\ell}.$$

For later use, we note that one can express the regular representation $\boldsymbol{k}\mathfrak{S}_n = \mathbf{1}_{\mathfrak{S}_1} * \mathbf{1}_{\mathfrak{S}_1} * \cdots * \mathbf{1}_{\mathfrak{S}_1}$, implying

$$\text{ch}\ \boldsymbol{k}\mathfrak{S}_n = h_1^n = h_{1^n}. \tag{9}$$

There is a parallel story for a certain subset of GL_n-representations. Specifically, there is a collection of irreducible GL_n-representations $\{\chi_q^\lambda\}$, indexed by partitions λ of n, which are the irreducible constituents occurring within the GL_n-permutation action on the set GL_n / B of *complete flags of subspaces* $\mathcal{F}(V)$ in $V = (\mathbb{F}_q)^n$. They were studied by R. Steinberg [1951], and are now called the *unipotent characters* of GL_n. Letting $\mathcal{C}(\mathrm{GL}_n)$ represent the free $\mathbb{Z}$-submodule of the class functions on GL_n with unipotent characters $\{\chi_q^\lambda\}$ as a basis, one can define the *parabolic* or *Harish–Chandra induction* product on the direct sum $\mathcal{C}(\mathrm{GL}) := \bigoplus_{n=0}^{\infty} \mathcal{C}(\mathrm{GL}_n)$ as follows:

$$\begin{aligned} &\mathcal{C}(\mathrm{GL}_{n_1}) \times \mathcal{C}(\mathrm{GL}_{n_2}) \to \mathcal{C}(\mathrm{GL}_{n_1+n_2}), \\ &\qquad (f_1, f_2) \mapsto f_1 * f_2 := \big((f_1 \otimes f_2) \Uparrow_{\mathrm{GL}_{n_1} \times \mathrm{GL}_{n_2}}^{P_{n_1,n_2}}\big) \uparrow_{P_{n_1,n_2}}^{\mathrm{GL}_{n_1+n_2}}. \end{aligned}$$

Here, P_{n_1,n_2} is the maximal parabolic subgroup of $\mathrm{GL}_{n_1+n_2}$ setwise stabilizing the $\mathbb{F}_q$-span of the first n_1 standard basis vectors, and $(-)\Uparrow_{\mathrm{GL}_{n_1}\times\mathrm{GL}_{n_2}}^{P_{n_1,n_2}}$ is the *inflation* operation that creates a $\mathrm{GL}\, n_1 \times \mathrm{GL}_{n_2}$-representation from a P_{n_1,n_2}-representation, by precomposing with the surjective homomorphism $P_{n_1,n_2} \twoheadrightarrow \mathrm{GL}_{n_1} \times \mathrm{GL}_{n_2}$ sending $\left[\begin{smallmatrix} A & B \\ 0 & C \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} A & 0 \\ 0 & C \end{smallmatrix}\right]$. For later use, we note that since the inflation operation does not change the degree of a representation, and since

$$[\mathrm{GL}_{n_1+n_2} : P_{n_1,n_2}] = \begin{bmatrix} n_1+n_2 \\ n_1 \end{bmatrix}_q = \frac{[n_1+n_2]!_q}{[n_1]!_q[n_2]!_q},$$

(with $[n]!_q$ as in (5)) when f_1, f_2 are genuine characters, one has this degree formula for $f_1 * f_2$:

$$\deg(f_1 * f_2) = \begin{bmatrix} n_1+n_2 \\ n_1 \end{bmatrix}_q \deg(f_1)\deg(f_2). \tag{10}$$

This parabolic induction operation turns out to make $\mathcal{C}(\mathrm{GL})$ into an associative, commutative $\mathbb{Z}$-algebra. One then has a q-analogue of the Frobenius isomorphism $\mathcal{C}(\mathrm{GL}) \xrightarrow{\mathrm{ch}_q} \Lambda$ sending[4]

$$\mathcal{C}(\mathrm{GL}) \xrightarrow{\mathrm{ch}_q} \Lambda, \quad \mathbf{1}_{\mathrm{GL}_n} \mapsto h_n,\ \chi_q^\lambda \mapsto s_\lambda.$$

Note that the permutation representation $\boldsymbol{k}[\mathrm{GL}_n /B]$ of GL_n on the complete flags can be expressed as $\mathbf{1}_{\mathrm{GL}_1} * \mathbf{1}_{\mathrm{GL}_1} * \cdots * \mathbf{1}_{\mathrm{GL}_1}$, and therefore one has this q-analogue of (9):

$$\mathrm{ch}_q\ \boldsymbol{k}[\mathrm{GL}_n /B] = h_1^n = h_{1^n}. \tag{11}$$

3C. ***(q-)derangement numbers and representations.*** A central role in this story is played by the classical *derangement numbers d_n* and the *q-derangement numbers $d_n(q)$* of Wachs [1989]:

$$\begin{aligned} d_n &:= n!\sum_{k=0}^{n}\frac{(-1)^k}{k!} = n!\left(\frac{1}{1!}-\frac{1}{2!}+\frac{1}{3!}-\frac{1}{4!}+\cdots+\frac{(-1)^n}{n!}\right), \\ d_n(q) &:= [n]!_q\sum_{k=0}^{n}\frac{(-1)^k}{[k]!_q}. \end{aligned} \tag{12}$$

There are two well-known combinatorial models for d_n counting permutations in $\mathfrak{S}_n$:

- *derangements*, which are the fixed-point free permutations, or

[4] One might wonder which GL_n-character maps under ch_q to the elementary symmetric function e_n; it is the *Steinberg representation*, in which GL_n acts on the top homology of the *Tits building*, which is the simplicial complex of flags of nonzero proper subspaces in $(\mathbb{F}_q)^n$.

- *desarrangements*, which are permutations $w = (w_1, w_2, \dots, w_n)$ whose first ascent position i with $w_i < w_{i+1}$ (using $w_{n+1} = n+1$ by convention) occurs for an even position i.

Wachs [1989], and later Désarménien and Wachs [1993], gave various interpretations for $d_n(q)$. In particular, $d_n(q)$ is still closely related to derangements and desarrangements. Letting D_n and E_n denote the derangements and desarrangements in S_n, and defining the *major index* statistic of a permutation $w = (w_1, \dots, w_n)$ as $\mathrm{maj}(\sigma) = \sum_{i:w_i > w_{i+1}} i$, one has

$$d_n(q) = \sum_{\sigma \in D_n} q^{\mathrm{maj}(\sigma)} = \sum_{\sigma \in E_n} q^{\mathrm{maj}(\sigma^{-1})}.$$

These d_n and $d_n(q)$ are the dimensions for a pair of representations of $\mathfrak{S}_n$ and GL_n, which we call the *derangement representation* $\mathcal{D}_n$ and its (unipotent) q-analogue $\mathcal{D}_n^{(q)}$. Both have the same symmetric function image $\mathfrak{d}_n$ under the Frobenius maps ch and ch_q, a symmetric function with many equivalent descriptions. For the reader's convenience, and for future use, we will compile these descriptions in Proposition 3.1, after first reviewing terminology.

Define for a permutation $w = (w_1, w_2, \dots, w_n)$ in $\mathfrak{S}_n$ its *descent set*

$$\mathrm{Des}(w) := \big\{ i \in \{1, 2, \dots, n-1\} : w_i > w_{i+1} \big\}.$$

For example, $w = (6, 3, 5, 2, 1, 4)$ has $\mathrm{Des}(w) = \{1, 3, 4\}$. Note that the definition of a *desarrangement* given above may be rephrased as a permutation w in $\mathfrak{S}_n$ for which the smallest element of $\{1, 2, \dots, n\} \setminus \mathrm{Des}(w)$ is even. Thus $w = (6, 3, 5, 2, 1, 4)$ is a desarrangement, since $\min(\{1, 2, 3, 4, 5, 6\} \setminus \{1, 3, 4\}) = 2$ is even.

Given a standard Young tableau Q with n cells written in English notation, its *descent set* is

$$\mathrm{Des}(w) := \{ i \in \{1, 2, \dots, n-1\} : i+1 \text{ appears south and weakly west of } i \text{ in } Q \}.$$

For example,

$$Q = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 6 \\ \hline 4 \\ \cline{1-1} 5 \\ \cline{1-1} \end{array}$$

has $\mathrm{Des}(Q) = \{1, 3, 4\}$. Define a *desarrangement tableau* to be a standard Young tableau Q with n cells for which the smallest element of $\{1, 2, \dots, n\} \setminus \mathrm{Des}(Q)$ is even. Thus, the example tableau Q given above is a desarrangement tableau.

Finally, for integers $n \geq 1$ and $D \subseteq [n]$, define *Gessel's fundamental quasisymmetric function*

$$L_{n,D} := \sum_{\substack{1 \leq i_1 \leq i_2 \leq \cdots \leq i_n \\ i_j < i_{j+1} \text{ if } j \in D}} x_{i_1} x_{i_2} \cdots x_{i_n},$$

which is a formal power series in $x_1, x_2, \dots$ and is homogeneous of degree n. For w in $\mathfrak{S}_n$, let $\lambda(w)$ denote its cycle type partition of n. For any partition λ of n, the *higher Lie character* of Thrall [1942] or the *Gessel–Reutenauer symmetric function* $\mathfrak{L}_\lambda$ (see [Gessel and Reutenauer 1993], [Grinberg and Reiner 2014, §6.6], and [Stanley 1999, Exercise 7.89]) can be defined as

$$\mathfrak{L}_\lambda := \sum_{\substack{w \in \mathfrak{S}_n: \\ \lambda(w)=\lambda}} L_{n,\mathrm{Des}(w)}.$$

Proposition 3.1. *With the convention that $\mathfrak{d}_0 := 1$, the following definitions of a sequence of symmetric functions $\{\mathfrak{d}_n\}_{n=0,1,2,\dots}$ are all equivalent*:

(A) $\mathfrak{d}_n = h_1 \mathfrak{d}_{n-1} + (-1)^n e_n$ *for* $n \geq 1$;

(B) $\mathfrak{d}_n = \sum_{k=0}^{n} (-1)^k e_k \cdot h_{1^{n-k}}$;

(C) $\mathfrak{d}_n = h_{1^n} - \sum_{j=0}^{n-1} \mathfrak{d}_j h_{n-j}$ $\big($*or equivalently*, $h_{1^n} = \sum_{j=0}^{n} \mathfrak{d}_j h_{n-j}\big)$ *for* $n \geq 1$;

(D) $\mathfrak{d}_n = \sum_Q s_{\lambda(Q)}$, *where Q runs through the desarrangement tableaux of size n*;

(E) $\mathfrak{d}_n = \sum_w L_{n,\mathrm{Des}(w)}$, *where w runs through all desarrangements in $\mathfrak{S}_n$*;

(F) $\mathfrak{d}_n = \sum_w L_{n,\mathrm{Des}(w)}$, *where w runs through all derangements in $\mathfrak{S}_n$*;

(G) $\mathfrak{d}_n = \sum_w \mathfrak{L}_{\lambda(w)}$, *where w runs through all derangements in $\mathfrak{S}_n$.*

We will mainly need definition (C) for $\mathfrak{d}_n$. However, we wish to point out that part (D) decomposes $\mathfrak{d}_n$ very explicitly into Schur functions, illustrated in Table 1 for $n = 0, 1, 2, 3, 4$.

n	desarrangement tableaux Q	$\mathfrak{d}_n$
0	$\varnothing$	1
1	none	0
2	$\begin{array}{c} 1 \\ 2 \end{array}$	$s_{(1,1)}$
3	$\begin{array}{cc} 1 & 3 \\ 2 & \end{array}$	$s_{(2,1)}$
4	$\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array}$ $\begin{array}{cc} 1 & 3 \\ 2 & \\ 4 & \end{array}$ $\begin{array}{cc} 1 & 3 \\ 2 & 4 \end{array}$ $\begin{array}{ccc} 1 & 3 & 4 \\ 2 & & \end{array}$	$s_{(1,1,1,1)} + s_{(2,1,1)} + s_{(2,2)} + s_{(3,1)}$

Table 1. Decomposition of $\mathfrak{d}_n$ into Schur functions for $n = 0, 1, 2, 3, 4$.

Sketch proof of Proposition 3.1.. We sketch some of the equivalences here. The equivalence of (A) and (B) is straightforward. Defining $\{\mathfrak{d}_n\}$ by (A), note they satisfy definition (C) by induction on n:

$$
\begin{aligned}
(13)\qquad \sum_{j=0}^{n} \mathfrak{d}_j h_{n-j} = \Big(\sum_{j=1}^{n} \mathfrak{d}_j h_{n-j}\Big) + h_n \;&=\; \Big(\sum_{j=1}^{n} (h_1 \mathfrak{d}_{j-1} + (-1)^j e_j)\cdot h_{n-j}\Big) + h_n \\
&=\; h_1 \sum_{j=1}^{n} \mathfrak{d}_{j-1} h_{n-j} \overset{(*)}{=} \sum_{j=0}^{n} (-1)^j e_j h_{n-j} \\
&\overset{(**)}{=} h_1 \cdot h_{1^{n-1}} + 0 = h_{1^n}.
\end{aligned}
$$

Here, equality $(*)$ used $\sum_{j=0}^{n}(-1)^j e_j h_{n-j} = 0$ for $n \geq 1$, and equality $(**)$ used induction. Consequently, (A) and (C) define the same sequence of polynomials $\{\mathfrak{d}_n\}$, and so (A), (B), and (C) coincide.

Defining $\{\mathfrak{d}_n\}$ by the explicit sum (D), let us check that they also satisfy the recursive definition (A) by induction on n. In the base case $n = 0$, both have $\mathfrak{d}_0 = 1$, since the unique (empty) tableau of size 0 is a desarrangement tableau. In the inductive step, using the Pieri formula shows that $h_1 \cdot \mathfrak{d}_{n-1}$ is the sum over all standard tableaux of size n obtained from a desarrangement tableau Q of size $n-1$ by adding n in any corner cell. This produces all desarrangement tableaux of size n, except the single column tableau Q_0 which:

- is produced for n odd, but is *not* a desarrangement tableaux, and
- is not produced for n even, but *is* a desarrangement tableau.

These exceptions are corrected by $(-1)^n e_n$ in the formula $\mathfrak{d}_n = h_1 \mathfrak{d}_{n_1} + (-1)^n e_n$ in (A). Consequently, (A) and (D) define the same sequence of polynomials $\{\mathfrak{d}_n\}$.

The equivalence of (D) and (E) uses two facts. First, applying the Robinson–Schensted bijection to w to obtain a pair of standard Young tableaux (P, Q), one has $\mathrm{Des}(w) = \mathrm{Des}(Q)$; see [Stanley 1999, Lemma 7.23.1]. Thus, w is a desarrangement if and only if Q is a desarrangement tableau[5]. Second, $s_\lambda = \sum_P L_{\mathrm{Des}(P)}$, where P runs over standard Young tableaux of shape λ, by [Stanley 1999, Theorem 7.19.7].

The equivalence of (E) and (F) was proven by Désarménien and Wachs [1988], where they showed that both families of symmetric functions defined in (E) and (F) satisfy the recursive definition (C). Their proof also used the equivalence of (F) and (G) that follows from the definition of $\mathfrak{L}_\lambda$. $\square$

Note that part (B) of Proposition 3.1 generalizes the formulas in (12), upon taking dimensions of the various representations and using (8) and (10). Similarly,

[5] Our earlier examples $w = (6, 3, 5, 2, 1, 4)$ and Q also exemplify this, as $w \mapsto (P, Q)$ with $Q = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 6 \\ \hline 4 & \\ \cline{1-1} 5 & \\ \cline{1-1}\end{array}$ and $P = \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 & \\ \cline{1-1} 6 & \\ \cline{1-1}\end{array}$.

part (C) corresponds to the formulas:

$$\dim_{\boldsymbol{k}} \boldsymbol{k}\mathfrak{S}_n = n! = \sum_{j=0}^{n} d_{n-j}\binom{n}{j},$$
$$\dim_{\boldsymbol{k}} \boldsymbol{k}[\mathrm{GL}_n/B] = [n]!_q = \sum_{j=0}^{n} d_{n-j}(q)\begin{bmatrix} n \\ j \end{bmatrix}_q, \tag{14}$$

after taking into account (9) and (11).

We conclude this section with some further historical remarks and context on the derangement representations $\mathcal{D}_n$ and symmetric functions $\mathfrak{d}_n$.

Remark 3.2. We are claiming no originality in Proposition 3.1. As mentioned in its proof, the equivalence of (C), (E), (F), and (G) is work of Désarménien and Wachs [1988]. In [Reiner and Webb 2004, Propositions 2.2, 2.1, and 2.3], it is noted that one can repackage their results to include part (D). It was also noted there that the tensor product $\mathrm{sgn} \otimes \mathcal{D}_n$ of $\mathcal{D}_n$ with the one-dimensional sign representation sgn of $\mathfrak{S}_n$, carries the same $\boldsymbol{k}\mathfrak{S}_n$-module as the homology of the *complex of injective words* on n letters. Therefore, after tensoring with the sign character of $\mathfrak{S}_n$ or applying the fundamental involution ω on symmetric functions, parts (A), (C), and (D) above correspond to [Reiner and Webb 2004, Propositions 2.2, 2.1, and 2.3].

Remark 3.3. It was noted in [Hersh and Reiner 2017] that $\mathcal{D}_n$ occurs naturally in the *representation stability and FI-module structure* (as in Church, Ellenberg, and Farb [Church et al. 2015]) on the cohomology of the configuration space of n labeled points in $\mathbb{R}^d$ for d odd. Specifically, $\mathcal{D}_n$ is the $\boldsymbol{k}\mathfrak{S}_n$-module on the subspace of *FI-module generators* for this cohomology, denoted $\widehat{\mathrm{Lie}}_n$ in [Hersh and Reiner 2017, Theorems 1.2 and 1.3].

Remark 3.4. As hinted at in Section 1, $\mathcal{D}_n$ also occurs as the $\boldsymbol{k}\mathfrak{S}_n$-module on the kernel of two shuffling operators on $\boldsymbol{k}\mathfrak{S}_n$, both studied by Uyemura-Reyes: *random-to-top* shuffles [2002, §1.1.7, §3.2.2, and §4.5.3] (also known as the *Tsetlin library*) and *random-to-random* shuffles [2002, Chapter 5]; see also [Steinberg 2016, Proposition 14.5] and Section 4A below. More generally, Uyemura-Reyes [2002, Theorem 4.1] described the $\boldsymbol{k}\mathfrak{S}_n$-module structure on the eigenspaces for *all* Bidigare–Hanlon–Rockmore shuffling operators that carry $\mathfrak{S}_n$-symmetry. Among these are random-to-top shuffles, whose eigenvalue multiplicities had previously been computed by Phatarfod [1991], ignoring the $\boldsymbol{k}\mathfrak{S}_n$-module structure. See also the discussion by Hanlon and Hersh [2004, §3] and by Saliola, Welker, and Reiner [Reiner et al. 2014, §VI.9].

Remark 3.5. In unpublished notes, Garsia [2012] (see also Tian [2016]), studied the *top-to-random* shuffling operator, which is adjoint or transpose to the random-to-top operator. There he sketched a proof that its minimal polynomial is $X(X-1)(X-2)\cdots(X-n)$. The element x acts as (rescaled) random-to-top on the chamber space of $\mathcal{F}_n$ (see 4A). In light of the fact that an operator and its transpose have the same minimal polynomial, Garsia's sketch is closely related to the part of our proof of Theorem 2.9 dealing with $M = \boldsymbol{k}\mathcal{F}_n$.

4. Answering Question 1.2

Our goal here is to answer Question 1.2, by describing the $\boldsymbol{k}G$-module decompositions on the eigenspaces of x, $x^{(q)}$ as they act on $\boldsymbol{k}M$ for $M = \mathcal{F}_n, \mathcal{F}_n^{(q)}$.

Recall the ($\boldsymbol{k}$-vector space) direct sum decompositions by length:

$$\boldsymbol{k}\mathcal{F}_n = \bigoplus_{\ell=0}^{n} \boldsymbol{k}\mathcal{F}_{n,\ell}, \quad \text{where } \mathcal{F}_{n,\ell} := \{\boldsymbol{a} \in \mathcal{F}_n : \ell(\boldsymbol{a}) = \ell\},$$
$$\boldsymbol{k}\mathcal{F}_n^{(q)} = \bigoplus_{\ell=0}^{n} \boldsymbol{k}\mathcal{F}_{n,\ell}^{(q)}, \quad \text{where } \mathcal{F}_{n,\ell}^{(q)} := \{\boldsymbol{A} \in \mathcal{F}_n^{(q)} : \ell(\boldsymbol{A}) = \ell\}.$$

Following Brown [2000], we call the monoid elements of $\mathcal{F}_{n,n}$ and $\mathcal{F}_{n,n}^{(q)}$ of maximum length *chambers*. Their $\boldsymbol{k}$-spans $\boldsymbol{k}\mathcal{F}_{n,n}$ and $\boldsymbol{k}\mathcal{F}_{n,n}^{(q)}$, which we call the *chamber spaces*, form submodules for the action of both the monoid algebras $\boldsymbol{k}M$ and the group algebras $\boldsymbol{k}G$. We first analyze the structure of these chamber spaces in Section 4A, and then use this to analyze the entire semigroup algebra $\boldsymbol{k}M$ in Section 4B.

4A. ***The chamber spaces.*** The chamber space $\boldsymbol{k}\mathcal{F}_{n,n}$ consists of all words of length n. Thus, as a $\boldsymbol{k}\mathfrak{S}_n$ module it is isomorphic to the left regular-representation $\boldsymbol{k}\mathfrak{S}_n$. Similarly, $\boldsymbol{k}\mathcal{F}_{n,n}^{(q)}$ has as a $\boldsymbol{k}$-basis the set $\mathcal{F}(V) = \{\boldsymbol{A} = (A_1, \dots, A_n)\}$ of all complete flags $A_1 \subset \cdots \subset A_{n-1} \subset A_n (= V)$, and is isomorphic to the coset representation of GL_n on $\boldsymbol{k}[\mathrm{GL}_n / B]$.

We start with an old observation: multiplication by x acts on $\mathcal{F}_{n,n}$ as a (rescaled) version of the random-to-top operator on $\boldsymbol{k}\mathfrak{S}_n$; see, for instance, B. Steinberg [2016, Proposition 14.5].

Example 4.1. If $n = 4$ and $w = (3, 1, 4, 2)$ in $\mathcal{F}_{4,4}$, then

$$\begin{aligned} x \cdot w &= ((1) + (2) + (3) + (4)) \cdot (3, 1, 4, 2) \\ &= (1, 3, 4, 2) + (2, 3, 1, 4) + (3, 1, 4, 2) + (4, 3, 1, 2), \end{aligned}$$

which $\left(\text{after scaling by } \frac{1}{4}\right)$ is the result of random-to-top shuffling on w as an element of $\boldsymbol{k}\mathfrak{S}_4$.

In this sense, the results in this section for the chamber space $\boldsymbol{k}\mathcal{F}_{n,n}$ are repackaging previously mentioned results on random-to-top shuffling and the $\mathfrak{S}_n$-action on its eigenspaces, due to Uyemura-Reyes [2002, Theorem 4.19], building on the computation of Phatarfod [1991] of the eigenvalue multiplicities. On the other hand, as far as we aware, our results for the q-analogue $\boldsymbol{k}\mathcal{F}_{n,n}^{(q)}$ in Theorem 4.2 are new.

We record here the action of $x^{(q)}$ on a complete flag $\boldsymbol{A}$ in $V=(\mathbb{F}_q)^n$, using Definition 2.2:

$$x^{(q)}\cdot \boldsymbol{A}=\sum_{\text{lines } L\in V}(L)\cdot \boldsymbol{A}=\sum_{\text{lines } L\in V}(L,L+A_1,L+A_2,\dots,L+A_{n-1},L+A_n)^{\wedge}.$$

For $j=0,1,\dots n$, we will write the j- and $[j]_q$-eigenspaces of the chamber spaces $\boldsymbol{k}\mathcal{F}_{n,n}$ and $\boldsymbol{k}\mathcal{F}_{n,n}^{(q)}$ as

$$\ker\bigl((x-j)|_{\boldsymbol{k}\mathcal{F}_{n,n}}\bigr)\quad\text{and}\quad \ker\bigl((x^{(q)}-[j]_q)|_{\boldsymbol{k}\mathcal{F}^{(q)}_{n,n}}\bigr).$$

In Theorem 4.2 below, we relate these j- and $[j]_q$-eigenspaces to $\mathcal{D}_{n-j}$ and $\mathcal{D}^{(q)}_{n-j}$. Our proof depends crucially on Proposition 4.5, Proposition 4.7, and Lemma 4.8 (all proved in Section 4A1) wherein we explicitly construct eigenvectors for the action of x and $x^{(q)}$ on the chamber spaces $\boldsymbol{k}\mathcal{F}_{n,n}$ and $\boldsymbol{k}\mathcal{F}_{n,n}^{(q)}$ from the null vectors of the same operators for smaller n.

Theorem 4.2. *When x and $x^{(q)}$ act on $\boldsymbol{k}\mathcal{F}_{n,n}$ and $\boldsymbol{k}\mathcal{F}_{n,n}^{(q)}$, for each $j=0,1,2,\dots,n$, their eigenspaces carry representations with the same Frobenius map images*

$$\operatorname{ch}\ker\bigl((x-j)|_{\boldsymbol{k}\mathcal{F}_{n,n}}\bigr)=h_j\cdot\mathfrak{d}_{n-j}=\operatorname{ch}_q\ker\bigl((x^{(q)}-[j]_q)|_{\boldsymbol{k}\mathcal{F}^{(q)}_{n,n}}\bigr).$$

In other words, one has $\boldsymbol{k}G$-module isomorphisms:

$$\ker\bigl((x-j)|_{\boldsymbol{k}\mathcal{F}_{n,n}}\bigr)\cong \mathbf{1}_{\mathfrak{S}_j}*\mathcal{D}_{n-j},$$
$$\ker\bigl((x^{(q)}-[j]_q)|_{\boldsymbol{k}\mathcal{F}^{(q)}_{n,n}}\bigr)\cong \mathbf{1}_{\mathrm{GL}_j}*\mathcal{D}^{(q)}_{n-j}.$$

Proof. Lemma 4.8 below exhibits G-equivariant injections

$$\begin{aligned}\mathbf{1}_{\mathfrak{S}_j}*\ker(x|_{\boldsymbol{k}\mathcal{F}_{n-j,n-j}})&\hookrightarrow\ker\bigl((x-j)|_{\boldsymbol{k}\mathcal{F}_{n,n}}\bigr),\\ \mathbf{1}_{\mathrm{GL}_j}*\ker\bigl(x^{(q)}|_{\boldsymbol{k}\mathcal{F}^{(q)}_{n-j,n-j}}\bigr)&\hookrightarrow\ker\bigl((x^{(q)}-[j]_q)|_{\boldsymbol{k}\mathcal{F}^{(q)}_{n,n}}\bigr).\end{aligned}\tag{15}$$

We now use facts proven by Phatarfod [1991] for $q=1$ and by Brown [2000, §5.2] for the q-analogue[6]:

$$\dim_{\boldsymbol{k}}\ker(x|_{\boldsymbol{k}\mathcal{F}_{n,n}})=d_n\quad\text{and}\quad\dim_{\boldsymbol{k}}\ker\bigl(x^{(q)}|_{\boldsymbol{k}\mathcal{F}^{(q)}_{n,n}}\bigr)=d_n(q).$$

Hence, the spaces on the left sides in (15) have dimensions $d_{n-j}\binom{n}{j}$ and $d_{n-j}(q)\left[{n\atop j}\right]_q$, respectively. However, since eigenspaces for distinct eigenvalues are always linearly independent, and since

$$\boldsymbol{k}\mathcal{F}_{n,n}\cong\boldsymbol{k}\mathfrak{S}_n\quad\text{and}\quad\boldsymbol{k}\mathcal{F}^{(q)}_{n,n}\cong\boldsymbol{k}[\mathrm{GL}_n/B]$$

[6]A minor discrepancy here is that Brown analyzes the action of $x^{(q)}$ not on the chamber space of $\boldsymbol{k}\mathcal{F}_n^{(q)}$ itself, but rather on the chamber space of the quotient $\boldsymbol{k}\overline{\mathcal{F}}_n^{(q)}$ discussed in Remark 2.3 above. However, just as Brown points out for $\mathcal{F}_n$ and $\overline{\mathcal{F}}_n$ in [Brown 2000, Remark, p. 888], the bijection $(A_1,A_2,\dots,A_{n-1},V)\mapsto(A_1,A_2,\dots,A_{n-1})$ between chambers of $\mathcal{F}_n^{(q)}$ and those of $\boldsymbol{k}\overline{\mathcal{F}}_n^{(q)}$ will commute with both the action of GL_n and with multiplication by $x^{(q)}$.

have dimensions $n!$ and $[n]!_q$, the equations in (14) imply that the injections in (15) must all be isomorphisms.

It also follows from the above analysis, or from Theorem 2.9 (ii), that

$$\boldsymbol{k}\mathcal{F}_{n,n} = \bigoplus_{j=0}^{n} \ker\bigl((x-j)|_{\boldsymbol{k}\mathcal{F}_{n,n}}\bigr) \quad \text{and} \quad \boldsymbol{k}\mathcal{F}_{n,n}^{(q)} = \bigoplus_{j=0}^{n} \ker\bigl((x^{(q)}-[j]_q)|_{\boldsymbol{k}\mathcal{F}_{n,n}^{(q)}}\bigr).$$

Then using (9) and (11) and comparing with Proposition 3.1 (C), the theorem follows. □

4A1. *Constructing eigenvectors from null vectors: proof of Lemma 4.8.* The goal of this subsection is to prove Lemma 4.8. It relies on parallel constructions[7] of eigenvectors for x and $x^{(q)}$ acting on the spaces $\boldsymbol{k}\mathcal{F}_{n,n}$ and $\boldsymbol{k}\mathcal{F}_{n,n}^{(q)}$ from null vectors for the same operators for smaller n.

Definition 4.3. Let $[n] := \{1, 2, \ldots, n\}$, and fix a j-element subset U of $\{1, 2, \ldots, n\}$. Let $\mathfrak{S}_{[n]\setminus U}$ denote the permutations $\boldsymbol{a} = (a_1, a_2, \ldots, a_{n-j})$ of the complementary subset $[n] \setminus U$, written in one-line notation. On the $\boldsymbol{k}$-vector space $\boldsymbol{k}[\mathfrak{S}_{[n]\setminus U}]$ having these permutations as a $\boldsymbol{k}$-basis, define two maps $\Psi_U, \Phi_U : \boldsymbol{k}[\mathfrak{S}_{[n]\setminus U}] \to \boldsymbol{k}[\mathfrak{S}_n]$ by extending these rules $\boldsymbol{k}$-linearly:

$$\Psi_U(\boldsymbol{a}) := \sum_{\boldsymbol{b} \in \mathfrak{S}_U} (b_1, b_2, \ldots, b_j, a_1, a_2, \ldots, a_{n-j}),$$

$$\Phi_U(\boldsymbol{a}) := \sum_{\boldsymbol{b} \in \mathfrak{S}_U} (a_1, b_1, b_2, \ldots, b_j, a_2, \ldots, a_{n-j}),$$

where the summation indices $\boldsymbol{b}$ run over all permutations $\boldsymbol{b} = (b_1, b_2, \ldots, b_j)$ in $\mathfrak{S}_U$.

Example 4.4. Let $n = 5$ and $U = \{4, 5\}$. Then

$$\Psi_U((1, 2, 3)) = (\mathbf{4}, \mathbf{5}, 1, 2, 3) + (\mathbf{5}, \mathbf{4}, 1, 2, 3),$$

$$\Phi_U((1, 2, 3)) = (1, \mathbf{4}, \mathbf{5}, 2, 3) + (1, \mathbf{5}, \mathbf{4}, 2, 3).$$

To state the next proposition, introduce for $U \subseteq [n]$ the *free left-regular band $\mathcal{F}_U$ on U*, having an obvious isomorphism $\mathcal{F}_U \cong \mathcal{F}_j$ if $j = |U|$. Also let $x_U := \sum_{i \in U} (i)$ inside $\boldsymbol{k}\mathcal{F}_U$.

Proposition 4.5. *Fix a j-element subset U of $[n]$ and a permutation $\boldsymbol{a}$ in $\mathfrak{S}_{[n]\setminus U}$. Then*

$$x \cdot \Psi_U(\boldsymbol{a}) = j \cdot \Psi_U(\boldsymbol{a}) + \Phi_U(x_{[n]\setminus U} \cdot \boldsymbol{a}).$$

Consequently, if v in $\boldsymbol{k}\mathcal{F}_{[n]\setminus U, n-j}$ has $x_{[n]\setminus U} \cdot v = 0$, then $\Psi_U(v)$ is a j-eigenvector for x on $\boldsymbol{k}\mathcal{F}_{n,n}$:

$$x \cdot \Psi_U(v) = j \cdot \Psi_U(v).$$

[7] Reiner is grateful to Michelle Wachs for explaining to him the $\boldsymbol{k}\mathcal{F}_n$ version of this construction (the operator Ψ_U) in 2002, in the context of random-to-top shuffling.

Proof. One can calculate that

$$x \cdot \Psi_U(\boldsymbol{a}) = \sum_{i=1}^{n} (i) \cdot \Psi_U(\boldsymbol{a}) = \sum_{i \in U} (i) \cdot \Psi_U(\boldsymbol{a}) + \sum_{i \in [n] \setminus U} (i) \cdot \Psi_U(\boldsymbol{a})$$
$$= j \cdot \Psi_U(\boldsymbol{a}) + \Phi_U(x_{[n]\setminus U} \cdot \boldsymbol{a}),$$

where we explain here the two substitutions in the last equality. The fact that the left sum equals $j \cdot \Psi_U(\boldsymbol{a})$ follows from the last equation $x \cdot x_j = j \cdot x_j$ in Lemma 2.5 applied to $\boldsymbol{k}\mathcal{F}_U \cong \boldsymbol{k}\mathcal{F}_j$. The fact that the right sum is $\Phi_U(x_{[n]\setminus U} \cdot \boldsymbol{a})$ follows via direct calculation from the definitions. □

We next introduce two q-analogous maps $\Psi_U^{(q)}$ and $\Phi_U^{(q)}$.

Definition 4.6. Fix U a j-dimensional $\mathbb{F}_q$-linear subspace of $V = (\mathbb{F}_q)^n$. Let $\mathcal{F}(V/U)$ denote the set of maximal flags in the quotient space V/U

$$\boldsymbol{A} = (A_1, A_2 \ldots, A_{n-j-1}, \underbrace{A_{n-j}}_{=V/U}).$$

On the space $\boldsymbol{k}[\mathcal{F}(V/U)]$ with these flags as a $\boldsymbol{k}$-basis, we define the maps $\Psi_U^{(q)}, \Phi_U^{(q)} : \boldsymbol{k}[\mathcal{F}(V/U)] \to \boldsymbol{k}[\mathcal{F}(V)]$ by extending the following rules $\boldsymbol{k}$-linearly:

$$\Psi_U^{(q)}(\boldsymbol{A}) := \sum_{\boldsymbol{B} \in \mathcal{F}(U)} (B_1, B_2, \ldots, B_{j-1}, U, A_1 + U, A_2 + U, \ldots, A_{n-j-1} + U, V),$$

$$\Phi_U^{(q)}(\boldsymbol{A}) := \sum_{\substack{\text{lines } L: \\ L \subset U + A_1, \\ L \not\subset U}} \sum_{\boldsymbol{B} \in \mathcal{F}(U)} (L, L + B_1, \ldots, L + B_{j-1}, \overbrace{L + U}^{=L+U+A_1}, L + U + A_2, \ldots, L + U + A_{n-j-1}, V),$$

where the summation indices $\boldsymbol{B}$ run over all complete flags $\boldsymbol{B} = (B_1, \ldots, B_{j-1}, U)$ in $\mathcal{F}(U)$.

To state the next proposition, introduce for any $\mathbb{F}_q$-vector space U of dimension j the monoid $\mathcal{F}_U^{(q)} \cong \mathcal{F}_j^{(q)}$ by identifying $U \cong \mathbb{F}_q^j$. Also introduce the element of the monoid algebra $\boldsymbol{k}\mathcal{F}_U^{(q)}$

$$x_U^{(q)} := \sum_{\text{lines } L \text{ in } U} (L).$$

Proposition 4.7. *For a j-dimensional subspace U of $V = (\mathbb{F}_q)^n$ and complete flag $\boldsymbol{A}$ in $\mathcal{F}(V/U)$,*

$$x^{(q)} \cdot \Psi_U^{(q)}(\boldsymbol{A}) = [j]_q \cdot \Psi_U^{(q)}(\boldsymbol{A}) + \Phi_U^{(q)}\big(x_{V/U}^{(q)} \cdot \boldsymbol{A}\big).$$

Hence if v in $\boldsymbol{k}\mathcal{F}_{V/U,n-j}^{(q)}$ has $x_{V/U}^{(q)} \cdot v = 0$, then $\Psi_U^{(q)}(v)$ is a $[j]_q$-eigenvector for $x^{(q)}$ on $\boldsymbol{k}\mathcal{F}_{n,n}^{(q)}$:

$$x^{(q)} \cdot \Psi_U^{(q)}(v) = [j]_q \cdot \Psi_U^{(q)}(v).$$

Proof. One can calculate that

$$x^{(q)} \cdot \Psi_U^{(q)}(\boldsymbol{A}) = \sum_{\substack{\text{lines } L \\ \text{in } V}} (L) \cdot \Psi_U^{(q)}(\boldsymbol{A}) = \sum_{\substack{\text{lines } L \\ \text{in } U}} (L) \cdot \Psi_U^{(q)}(\boldsymbol{A}) + \sum_{\substack{\text{lines } L \\ \text{not in } U}} (L) \cdot \Psi_U^{(q)}(\boldsymbol{A})$$
$$= [j]_q \cdot \Psi_U^{(q)}(\boldsymbol{A}) + \Phi_U^{(q)}(x_{V/U}^{(q)} \cdot \boldsymbol{A}),$$

where we explain here the two substitutions in the last equality. The fact that the left sum equals $[j]_q \cdot \Psi_U^{(q)}(\boldsymbol{A})$ follows from the last equation $x^{(q)} \cdot x_j^{(q)} = [j]_q \cdot x_j^{(q)}$ in Lemma 2.5 applied to $\boldsymbol{k}\mathcal{F}_U^{(q)} \cong \boldsymbol{k}\mathcal{F}_j^{(q)}$. To check the substitution made for the right sum, one calculates directly that

$$\begin{aligned}
&\Phi_U^{(q)}(x_{V/U}^{(q)} \cdot \boldsymbol{A}) \\
&\quad = \sum_{\substack{\text{lines } \bar{L} \\ \text{in } V/U}} \Phi_U^{(q)}((\bar{L}) \cdot \boldsymbol{A}) \\
&\quad = \sum_{\substack{\text{lines } \bar{L} \\ \text{in } V/U}} \Phi_U^{(q)}\big((\bar{L}, \bar{L} + A_1, \bar{L} + A_2, \ldots, \bar{L} + A_{n-j-1}, V/U)^\wedge\big) \\
&\quad = \sum_{\substack{\text{lines } \bar{L} \\ \text{in } V/U}} \sum_{\substack{\text{lines } L \text{ in } V \\ L \subset U + \bar{L} \\ L \not\subset U}} \sum_{\boldsymbol{B} \in \mathcal{F}(U)} (L, L + B_1, \ldots, L + B_{j-1}, \overbrace{L + B_j}^{=L+U}, \\
&\qquad\qquad\qquad\qquad L + U + A_1, \ldots, L + U + A_{n-j-1}, V)^\wedge \\
&\quad = \sum_{\substack{\text{lines } L \text{ in } V \\ L \not\subset U}} \sum_{\boldsymbol{B} \in \mathcal{F}(U)} (L) \cdot (B_1, \ldots, B_{j-1}, \overbrace{B_j}^{=U}, A_1 + U, A_2 + U, \ldots, A_{n-j-1} + U, V) \\
&\quad = \sum_{\substack{\text{lines } L \\ \text{not in } U}} (L) \cdot \Psi_U^{(q)}(\boldsymbol{A}). \qquad \square
\end{aligned}$$

We are at last ready to prove Lemma 4.8.

Lemma 4.8. *With our usual notation of $G = \mathfrak{S}_n, \mathrm{GL}_n$ acting on $\boldsymbol{k}M$ for $M = \mathcal{F}_n, \mathcal{F}_n^{(q)}$, one has G-equivariant injections for $j = 0, 1, 2, \ldots, n$:*

$$\mathbf{1}_{\mathfrak{S}_j} * \ker(x|_{\boldsymbol{k}\mathcal{F}_{n-j,n-j}}) \hookrightarrow \ker\big((x - j)|_{\boldsymbol{k}\mathcal{F}_{n,n}}\big),$$
$$\mathbf{1}_{\mathrm{GL}_j} * \ker(x^{(q)}|_{\boldsymbol{k}\mathcal{F}^{(q)}_{n-j,n-j}}) \hookrightarrow \ker\big((x^{(q)} - [j]_q)|_{\boldsymbol{k}\mathcal{F}^{(q)}_{n,n}}\big).$$

Proof. We give the proof for $\mathcal{F}_n^{(q)}$; the proof for $\mathcal{F}_n$ is analogous, but easier.

For each j-dimensional subspace U of $V = (\mathbb{F}_q)^n$, define a subspace $E(U)$ of $\boldsymbol{k}\mathcal{F}_{n,n}^{(q)}$ as the image under $\Psi_U^{(q)}$ of the nullspace for $x^{(q)} = x_{V/U}^{(q)}$ acting on $\boldsymbol{k}\mathcal{F}_{V/U,n-j}^{(q)} \cong \boldsymbol{k}\mathcal{F}_{n-j,n-j}^{(q)}$:

$$E(U) := \Psi_U^{(q)}\big(\ker x^{(q)}|_{\boldsymbol{k}\mathcal{F}^{(q)}_{V/U,n-j}}\big).$$

According to Proposition 4.7, each $E(U)$ is a subspace of the $[j]_q$-eigenspace $\ker\big((x^{(q)} - [j]_q)|_{\boldsymbol{k}\mathcal{F}^{(q)}_{n,n}}\big)$. Note that vectors in $E(U)$ are sums of complete flags $\boldsymbol{A} = (A_1, \ldots, A_n)$ that pass through $A_j = U$, and hence for $U \neq U'$, they are supported on basis elements of $\boldsymbol{k}\mathcal{F}_{n,n}^{(q)}$ indexed by disjoint sets of complete flags. Therefore, the subspace sum of all $E(U)$ is a direct sum $\bigoplus_U E(U)$ inside this $[j]_q$-eigenspace for x. It only remains to produce an isomorphism of GL_n-representations

$$\bigoplus_U E(U) \cong \mathbf{1}_{\mathrm{GL}_j} * \ker\big(x^{(q)}|_{\boldsymbol{k}\mathcal{F}^{(q)}_{n-j,n-j}}\big). \tag{16}$$

Recall that GL_n acts transitively on j-subspaces U. Fix the particular subspace U_0 spanned by the first j standard basis vectors in $V = (\mathbb{F}_q)^n$, whose GL_n-stabilizer is the maximal parabolic subgroup $P_{j,n-j}$. It follows (see, e.g., Webb [2016, Proposition 4.3.2]) that $\bigoplus_U E(U)$ carries the GL_n-representation induced from $P_{j,n-j}$ acting on $E(U_0)$. However, because elements in $E(U_0)$ are supported on flags $\boldsymbol{A}$ in $E(U_0)$ that all pass through $A_j = U_0$, this $P_{j,n-j}$-action is inflated through the surjection $P_{j,n-j} \twoheadrightarrow \mathrm{GL}_j \times \mathrm{GL}_{n-j}$. Furthermore, the definition of $\Psi^{(q)}_{U_0}(-)$ as a symmetrized sum over complete flags in U_0 shows that GL_j fixes elements of $E(U_0)$ pointwise, while elements of GL_{n-j} act as they do on $\ker\big(x^{(q)}|_{\boldsymbol{k}\mathcal{F}^{(q)}_{n-j,n-j}}\big)$. Comparing with (7) proves the desired isomorphism (16). □

4B. ***The entire semigroup algebra.*** Having described the eigenspaces of the chamber spaces $\boldsymbol{k}\mathcal{F}_{n,n}$ and $\boldsymbol{k}\mathcal{F}_{n,n}^{(q)}$ as G-representations, we now turn to the entire semigroup algebras $\boldsymbol{k}\mathcal{F}_n$ and $\boldsymbol{k}\mathcal{F}_n^{(q)}$.

Our strategy here will be to introduce filtrations on $\boldsymbol{k}\mathcal{F}_n$ and $\boldsymbol{k}\mathcal{F}_n^{(q)}$, and study the action of x and $x^{(q)}$ on the associated graded modules with respect to these filtrations. (Recall from the discussion in Section 3A that by semisimplicity, this is an equivalent way to understand the R^G and $\boldsymbol{k}G$-module structures on $\boldsymbol{k}\mathcal{F}_n$ and $\boldsymbol{k}\mathcal{F}_n^{(q)}$.)

Recall that for $\boldsymbol{a} \in \mathcal{F}_n$ and $\boldsymbol{A} \in \mathcal{F}_n^{(q)}$ the length of $\boldsymbol{a}$ is $\ell(\boldsymbol{a})$ and the length of $\boldsymbol{A}$ is $\ell(\boldsymbol{A})$.

Definition 4.9. Define

$$\begin{aligned}\boldsymbol{k}\mathcal{F}_{n,\geq \ell} &:= \mathrm{span}_{\boldsymbol{k}}\{\boldsymbol{a} \in \mathcal{F}_n : \ell(\boldsymbol{a}) \geq \ell\},\\ \boldsymbol{k}\mathcal{F}^{(q)}_{n,\geq \ell} &:= \mathrm{span}_{\boldsymbol{k}}\{\boldsymbol{A} \in \mathcal{F}^{(q)}_n : \ell(\boldsymbol{A}) \geq \ell\}.\end{aligned}$$

In other words, $\boldsymbol{k}\mathcal{F}_{n,\geq \ell}$ and $\boldsymbol{k}\mathcal{F}^{(q)}_{n,\geq \ell}$ are the $\boldsymbol{k}$-spans of the monoid elements of length at least ℓ.

We then introduce filtrations $\{\boldsymbol{k}\mathcal{F}_{n,\geq \ell}\}_{\ell=0,1,\ldots,n,n+1}$ and $\{\boldsymbol{k}\mathcal{F}^{(q)}_{n,\geq \ell}\}_{\ell=0,1,\ldots,n,n+1}$:

$$\begin{aligned}\{0\} = \boldsymbol{k}\mathcal{F}_{n,\geq n+1} \subset \boldsymbol{k}\mathcal{F}_{n,\geq n} \subset \cdots \subset \boldsymbol{k}\mathcal{F}_{n,\geq 1} \subset \boldsymbol{k}\mathcal{F}_{n,\geq 0} = \boldsymbol{k}\mathcal{F}_n,\\ \{0\} = \boldsymbol{k}\mathcal{F}^{(q)}_{n,\geq n+1} \subset \boldsymbol{k}\mathcal{F}^{(q)}_{n,\geq n} \subset \cdots \subset \boldsymbol{k}\mathcal{F}^{(q)}_{n,\geq 1} \subset \boldsymbol{k}\mathcal{F}^{(q)}_{n,\geq 0} = \boldsymbol{k}\mathcal{F}^{(q)}_n.\end{aligned} \tag{17}$$

Since $\ell(\boldsymbol{a} \cdot \boldsymbol{b}) \geq \ell(\boldsymbol{b})$, it is easily seen that each $\boldsymbol{k}\mathcal{F}_{n,\geq \ell}$ is a $\boldsymbol{k}\mathcal{F}_n$-submodule, and a $\boldsymbol{k}\mathfrak{S}_n$-submodule. Analogously, $\boldsymbol{k}\mathcal{F}^{(q)}_{n,\geq \ell}$ is a $\boldsymbol{k}\mathcal{F}^{(q)}_n$-submodule, and a $\boldsymbol{k}\,\mathrm{GL}_n$-submodule.

Recall that for $U \subset [n]$ of size j one has $\mathcal{F}_U \cong \mathcal{F}_j$ and $x_U = \sum_{i \in U} (i)$. Analogously, recall that for U a j-dimensional subspace of V, one has $\mathcal{F}_U^{(q)} \cong \mathcal{F}_j^{(q)}$ and

$$x_U^{(q)} = \sum_{\text{lines } L \in U} (L).$$

Both $\boldsymbol{k}\mathcal{F}_U$ and $\boldsymbol{k}\mathcal{F}_U^{(q)}$ have $\boldsymbol{k}$-vector space direct sum decompositions defined by length of words, so that one can identify $\boldsymbol{k}\mathcal{F}_{U,\ell} \cong \boldsymbol{k}\mathcal{F}_{j,\ell}$ and $\boldsymbol{k}\mathcal{F}_{U,\ell}^{(q)} \cong \boldsymbol{k}\mathcal{F}_{j,\ell}^{(q)}$ for $\ell = 0, 1, \ldots, j$.

As $\boldsymbol{k}$-vector spaces, one has a direct sum decomposition for the filtration factors

$$\begin{aligned}
\boldsymbol{k}\mathcal{F}_{n,\geq \ell}/\boldsymbol{k}\mathcal{F}_{n,\geq \ell+1} &= \bigoplus_{\substack{U \subseteq \{1,2,\ldots,n\} \\ |U| = \ell}} \overline{\boldsymbol{k}\mathcal{F}_{U,\ell}}, \\
\boldsymbol{k}\mathcal{F}_{n,\geq \ell}^{(q)}/\boldsymbol{k}\mathcal{F}_{n,\geq \ell+1}^{(q)} &= \bigoplus_{\substack{\mathbb{F}_q\text{-subspaces } U \subseteq (\mathbb{F}_q)^n \\ \dim(U) = \ell}} \overline{\boldsymbol{k}\mathcal{F}_{U,\ell}^{(q)}},
\end{aligned} \tag{18}$$

where $\overline{\boldsymbol{k}\mathcal{F}_{U,\ell}}$ and $\overline{\boldsymbol{k}\mathcal{F}_{U,\ell}^{(q)}}$ denote the image of the subspaces $\boldsymbol{k}\mathcal{F}_{U,\ell}$ and $\boldsymbol{k}\mathcal{F}_{U,\ell}^{(q)}$ within the quotient on the left. The next proposition is a simple but crucial observation about these summands in (18) that is used in the proof of Theorem 1.4.

Proposition 4.10. *Consider the summands on the right sides of* (18).

- *Each* $\overline{\boldsymbol{k}\mathcal{F}_{U,\ell}}$ *is a* $\boldsymbol{k}\mathcal{F}_n$*-submodule of* $\boldsymbol{k}\mathcal{F}_{n,\geq \ell}/\boldsymbol{k}\mathcal{F}_{n,\geq \ell+1}$, *annihilated by* (j) *for* $j \notin U$.
- *Each* $\overline{\boldsymbol{k}\mathcal{F}_{U,\ell}^{(q)}}$ *is a* $\boldsymbol{k}\mathcal{F}_n^{(q)}$*-submodule of* $\boldsymbol{k}\mathcal{F}_{n,\geq \ell}^{(q)}/\boldsymbol{k}\mathcal{F}_{n,\geq \ell+1}^{(q)}$, *annihilated by* (L) *for lines* $L \not\subset U$.

Consequently, one has

$$\begin{aligned}
x \cdot \bar{\boldsymbol{a}} &= x_U \cdot \bar{\boldsymbol{a}}, \quad \textit{for } \bar{\boldsymbol{a}} \textit{ in } \overline{\boldsymbol{k}\mathcal{F}_{U,\ell}}, \\
x^{(q)} \cdot \overline{\boldsymbol{A}} &= x_U^{(q)} \cdot \overline{\boldsymbol{A}}, \quad \textit{for } \overline{\boldsymbol{A}} \textit{ in } \overline{\boldsymbol{k}\mathcal{F}_{U,\ell}^{(q)}}.
\end{aligned}$$

Proof by example. Consider $n = 3$, with $\ell = 2$ and $U = \{1, 2\}$. Then working in the quotient $\overline{\boldsymbol{k}\mathcal{F}_{U,2}}$, because $3 \notin U$, the element (3) of $\boldsymbol{k}\mathcal{F}_3$ will annihilate the element $\overline{(1, 2)}$ of $\boldsymbol{k}\mathcal{F}_{3,\geq 2}/\boldsymbol{k}\mathcal{F}_{3,\geq 3}$. One has

$$(3) \cdot \overline{(1, 2)} = \overline{(3, 1, 2)} = 0 \quad \text{in } \boldsymbol{k}\mathcal{F}_{3,\geq 2}/\boldsymbol{k}\mathcal{F}_{3,\geq 3},$$

because $\ell(3, 1, 2) = 3 > 2 = \ell$. Thus, $x = (1) + (2) + (3)$ acts on $\overline{(1, 2)}$ as

$$\begin{aligned}
x \cdot \overline{(1, 2)} &= ((1) + (2) + (3)) \cdot \overline{(1, 2)} \\
&= \overline{(1, 2)} + \overline{(2, 1)} + \overline{(3, 1, 2)} = \overline{(1, 2)} + \overline{(2, 1)} = x_U \cdot \overline{(1, 2)}.
\end{aligned}$$

The proof for $\mathcal{F}_n^{(q)}$ is analogous: one has $\ell((L) \cdot \boldsymbol{A}) > \ell(\boldsymbol{A}) = \ell$ for lines $L \not\subset U$ and $\boldsymbol{A} \in \mathcal{F}_{U,\ell}^{(q)}$. $\square$

We now prove our main result of this section, encompassing Theorem 1.4 from Section 1.

Theorem 4.11. *Let $\boldsymbol{k}$ be a field in which $|G|$ is invertible. Then x and $x^{(q)}$ act diagonalizably on $\boldsymbol{k}\mathcal{F}_n$ and $\boldsymbol{k}\mathcal{F}_n^{(q)}$, and for each $j = 0, 1, 2, \ldots, n$, their eigenspaces carry representations with the same Frobenius map images*

$$\operatorname{ch}\ker\big((x-j)|_{\boldsymbol{k}\mathcal{F}_n}\big) = \sum_{\ell=j}^{n} h_{(n-\ell,j)} \cdot \mathfrak{d}_{\ell-j} = \operatorname{ch}_q \ker\big((x^{(q)}-[j]_q)|_{\boldsymbol{k}\mathcal{F}_n^{(q)}}\big).$$

In other words, one has $\boldsymbol{k}G$-module isomorphisms

$$\ker\big((x-j)|_{\boldsymbol{k}\mathcal{F}_n}\big) \cong \bigoplus_{\ell=j}^{n} \mathbf{1}_{\mathfrak{S}_{n-\ell}} * \mathbf{1}_{\mathfrak{S}_j} * \mathcal{D}_{\ell-j},$$
$$\ker\big((x^{(q)}-[j]_q)|_{\boldsymbol{k}\mathcal{F}_n^{(q)}}\big) \cong \bigoplus_{\ell=j}^{n} \mathbf{1}_{\mathrm{GL}_{n-\ell}} * \mathbf{1}_{\mathrm{GL}_j} * \mathcal{D}_{\ell-j}^{(q)}.$$

Proof. The filtrations in (17) show that

$$\begin{aligned}\ker\big((x-j)|_{\boldsymbol{k}\mathcal{F}_n}\big) &\cong \bigoplus_{\ell=0}^{n} \ker\big((x-j)|_{\boldsymbol{k}\mathcal{F}_{n,\geq\ell}/\boldsymbol{k}\mathcal{F}_{n,\geq\ell+1}}\big),\\ \ker\big((x^{(q)}-[j]_q)|_{\boldsymbol{k}\mathcal{F}_n^{(q)}}\big) &\cong \bigoplus_{\ell=0}^{n} \ker\big((x^{(q)}-[j]_q)|_{\boldsymbol{k}\mathcal{F}^{(q)}_{n,\geq\ell}/\boldsymbol{k}\mathcal{F}^{(q)}_{n,\geq\ell+1}}\big).\end{aligned} \tag{19}$$

It remains to analyze each summand on the right.

We have seen that (18) is also a direct sum decomposition as $\boldsymbol{k}M$-modules for $M = \boldsymbol{k}\mathcal{F}_n, \boldsymbol{k}\mathcal{F}^{(q)}$. For $G = \mathfrak{S}_n, \mathrm{GL}_n$, the action of $\boldsymbol{k}M$ and $\boldsymbol{k}G$ on both sides in (18) commute.

In the case of $M = \mathcal{F}_n$, this leads to the following equalities and isomorphisms of $\boldsymbol{k}\mathfrak{S}_n$-modules, explained below. Let $U_0 := \{1, 2, \ldots, \ell\}$. Then:

$$\begin{aligned}\ker\big((x-j)|_{\boldsymbol{k}\mathcal{F}_{n,\geq\ell}/\boldsymbol{k}\mathcal{F}_{n,\geq\ell+1}}\big) &\overset{\text{(i)}}{=} \bigoplus_{\substack{U\subseteq\{1,2,\ldots,n\}:\\ |U|=\ell}} \ker((x-j)|_{\overline{\boldsymbol{k}\mathcal{F}_{U,\ell}}})\\ &\overset{\text{(ii)}}{=} \bigoplus_{\substack{U\subseteq\{1,2,\ldots,n\}:\\ |U|=\ell}} \ker((x_U-j)|_{\overline{\boldsymbol{k}\mathcal{F}_{U,\ell}}})\\ &\overset{\text{(iii)}}{\cong} \mathbf{1}_{\mathfrak{S}_{n-\ell}} * \ker\big((x_{U_0}-j)|_{\boldsymbol{k}\mathcal{F}_{\ell,\ell}}\big)\\ &\overset{\text{(iv)}}{\cong} \begin{cases} 0, & \text{if } \ell < j\\ \mathbf{1}_{\mathfrak{S}_{n-\ell}} * \mathbf{1}_{\mathfrak{S}_j} * \mathcal{D}_{\ell-j}, & \text{if } \ell \geq j.\end{cases}\end{aligned}$$

- Equality (i) is the restriction of the $\boldsymbol{k}\mathfrak{S}_n$-module isomorphism (18) to j-eigenspaces for x.
- Equality (ii) arises because x acts the same as x_U on $\overline{\mathcal{F}_{U,\ell}}$, by Proposition 4.10.

- Isomorphism (iii) arises because the summands indexed by U, with $|U| = \ell$, are permuted transitively by $\mathfrak{S}_n$ with the typical summand for $U_0 = \{1, 2, \dots, \ell\}$ stabilized by the subgroup $\mathfrak{S}_{U_0} \cong \mathfrak{S}_\ell$. Thus, this is an induced $\boldsymbol{k}\mathfrak{S}_n$-module, e.g., by applying [Webb 2016, Proposition 4.3.2].
- Isomorphism (iv) comes from applying Theorem 4.2 to $\boldsymbol{k}\mathcal{F}_\ell$.

The argument for $M = \mathcal{F}_n^{(q)}$ is similar. In particular, setting U_0 to be the $\mathbb{F}_q$-span of the first ℓ standard basis vectors $e_1, e_2, \dots, e_\ell$ in $(\mathbb{F}_q)^n$, one has equalities and isomorphisms of $\boldsymbol{k}\,\mathrm{GL}_n$-modules:

$$\begin{aligned}
\ker\big((x^{(q)} - [j]_q)|_{\boldsymbol{k}\mathcal{F}^{(q)}_{n,\geq \ell}/\boldsymbol{k}\mathcal{F}^{(q)}_{n,\geq \ell+1}}\big) &\overset{\text{(i)}}{=} \bigoplus_{\substack{U \subseteq (\mathbb{F}_q)^n: \\ \dim(U)=\ell}} \ker\big((x^{(q)} - [j]_q)|_{\overline{\boldsymbol{k}\mathcal{F}^{(q)}_{U,\ell}}}\big) \\
&\overset{\text{(ii)}}{=} \bigoplus_{\substack{U \subseteq (\mathbb{F}_q)^n: \\ \dim(U)=\ell}} \ker\big((x^{(q)}_U - [j]_q)|_{\overline{\boldsymbol{k}\mathcal{F}^{(q)}_{U,\ell}}}\big) \\
&\overset{\text{(iii)}}{\cong} \mathbf{1}_{\mathrm{GL}_{n-\ell}} * \ker\big((x^{(q)}_{U_0} - [j]_q)|_{\boldsymbol{k}\mathcal{F}^{(q)}_{\ell,\ell}}\big) \\
&\overset{\text{(iv)}}{\cong} \begin{cases} 0, & \text{if } \ell < j, \\ \mathbf{1}_{\mathrm{GL}_{n-\ell}} * \mathbf{1}_{\mathrm{GL}_j} * \mathcal{D}^{(q)}_{\ell-j}, & \text{if } \ell \geq j, \end{cases}
\end{aligned}$$

where isomorphisms (i), (ii), and (iv) are justified exactly as in the proof of $q = 1$ above. For isomorphism (iii), note (as in the proof of Lemma 4.8) that GL_n acts transitively on ℓ-subspaces U, and that U_0 has GL_n-stabilizer subgroup $P_{\ell,n-\ell,}$, so that by [Webb 2016, Proposition 4.3.2],

$$\bigoplus_{\substack{U \subseteq (\mathbb{F}_q)^n: \\ \dim(U)=\ell}} \ker\big((x^{(q)}_U - [j]_q)|_{\overline{\boldsymbol{k}\mathcal{F}^{(q)}_{U,\ell}}}\big)$$

has the GL_n-representation induced from the $P_{\ell,n-\ell}$-action on $\ker\big((x^{(q)}_{U_0} - [j]_q)|_{\boldsymbol{k}\mathcal{F}^{(q)}_{\ell,\ell}}\big)$. Since every $\boldsymbol{A} \in \boldsymbol{k}\mathcal{F}^{(q)}_{U_0,\ell} \cong \boldsymbol{k}\mathcal{F}^{(q)}_{\ell,\ell}$ is a flag $(A_1, \dots, A_\ell)$, with $A_\ell = U_0$, it follows that this $P_{\ell,n-\ell}$-action is inflated through the surjection $P_{\ell,n-\ell} \twoheadrightarrow \mathrm{GL}_\ell \times \mathrm{GL}_{n-\ell}$, where the action of GL_ℓ is as $\ker\big((x^{(q)}_{U_0} - [j]_q)|_{\boldsymbol{k}\mathcal{F}^{(q)}_{\ell,\ell}}\big)$ and the action of $\mathrm{GL}_{n-\ell}$ is trivial. $\square$

Example 4.12. We illustrate Theorem 1.4 by computing the Frobenius map image for each j-eigenspace of x on $\boldsymbol{k}\mathcal{F}_n$, or equivalently the q-Frobenius map image for each $[j]_q$-eigenspace of $x^{(q)}$ on $\boldsymbol{k}\mathcal{F}_n^{(q)}$. For $n = 2, 3$, Tables 2 and 3 show these symmetric functions in their j-th row, decomposed into columns labeled by ℓ, indexing each filtration factor from (18) that contributes a term.

	$\ell=0$	$\ell=1$	$\ell=2$
$j=0$	$h_2\cdot\mathfrak{d}_0$ $=h_2\cdot s_{()}$ $=s_{(2)}$	$h_1\cdot\mathfrak{d}_1$ $=h_1\cdot 0$ $=0$	$h_0\cdot\mathfrak{d}_2$ $=h_0\cdot s_{(1,1)}$ $=s_{(1,1)}$
$j=1$		$h_{(1,1)}\cdot\mathfrak{d}_0$ $=h_{(1,1)}\cdot s_{()}$ $=s_{(1,1)}+s_{(2)}$	$h_1\cdot\mathfrak{d}_1$ $=h_1\cdot 0$ $=0$
$j=2$			$h_2\cdot\mathfrak{d}_0$ $=h_2\cdot s_{()}$ $=s_{(2)}$

Table 2. Frobenius map images for eigenspaces of x and $x^{(q)}$ on $\boldsymbol{k}\mathcal{F}_2$ and $\boldsymbol{k}\mathcal{F}_2^{(q)}$.

	$\ell=0$	$\ell=1$	$\ell=2$	$\ell=3$
$j=0$	$h_3\cdot\mathfrak{d}_0$ $=h_3\cdot s_{()}$ $=s_{(3)}$	$h_2\cdot\mathfrak{d}_1$ $=h_2\cdot 0$ $=0$	$h_1\cdot\mathfrak{d}_2$ $=h_1\cdot s_{(1,1)}$ $=s_{(2,1)}+s_{(1,1,1)}$	$h_0\cdot\mathfrak{d}_3$ $=h_0\cdot s_{(2,1)}$ $=s_{(2,1)}$
$j=1$		$h_{(2,1)}\cdot\mathfrak{d}_0$ $=h_{(2,1)}\cdot s_{()}$ $=s_{(3)}+s_{(2,1)}$	$h_{(1,1)}\cdot\mathfrak{d}_1$ $=h_{(1,1)}\cdot 0$ $=0$	$h_1\cdot\mathfrak{d}_2$ $=h_1\cdot s_{(1,1)}$ $=s_{(2,1)}+s_{(1,1,1)}$
$j=2$			$h_{(2,1)}\cdot\mathfrak{d}_0$ $=h_{(2,1)}\cdot s_{()}$ $=s_{(3)}+s_{(2,1)}$	$h_2\cdot\mathfrak{d}_1$ $=h_2\cdot 0$ $=0$
$j=3$				$h_3\cdot\mathfrak{d}_0$ $=h_3\cdot s_{()}$ $=s_{(3)}$

Table 3. Frobenius map images for eigenspaces of x and $x^{(q)}$ on $\boldsymbol{k}\mathcal{F}_3$ and $\boldsymbol{k}\mathcal{F}_3^{(q)}$.

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IRREDUNDANT BASES FOR FINITE GROUPS OF LIE TYPE

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We prove that the maximum length of an irredundant base for a primitive action of a finite simple group of Lie type is bounded above by a function which is a polynomial in the rank of the group. We give examples to show that this type of upper bound is best possible.

1. Introduction

1A. ***Main results.*** Let G be a group acting on a set Ω. Let ℓ be a nonnegative integer and let $\Lambda = [\omega_1, \dots, \omega_\ell]$ be a sequence of points $\omega_1, \dots, \omega_\ell$ drawn from Ω; we write $G_{(\Lambda)}$ or $G_{\omega_1,\omega_2,\dots,\omega_\ell}$ for the pointwise stabilizer. If $\ell = 0$, so Λ is empty, then we set $G_{(\Lambda)} = G$.

The sequence Λ is called a *base* if $G_{(\Lambda)} = \{1\}$; the sequence Λ is called *irredundant* if

$$G_{\omega_1,\dots,\omega_{k-1}} \gneq G_{\omega_1,\dots,\omega_{k-1},\omega_k}$$

for all $k = 1, \dots, \ell$. The size of the longest possible irredundant base is denoted $\mathrm{I}(G, \Omega)$.

The main result of this paper shows that for any primitive action of a simple group of Lie type, the size of an irredundant base is bounded by a polynomial function of the rank of the group.

Theorem 1. *If G is a simple group of Lie type of rank r acting primitively on a set Ω, then $\mathrm{I}(G, \Omega) \le Cr^8$, where C is an absolute constant. This holds with $C = 174$.*

The degree 8 of the polynomial bound is probably far from sharp but, as discussed in Section 1B, there are examples showing that this degree must be at least 2. Also there is no general complementary lower bound for $\mathrm{I}(G, \Omega)$ that grows with r, as shown by Example 4.5.

An upper bound on $\mathrm{I}(G, \Omega)$ implies an upper bound on a host of other statistics associated with the action of G on Ω. Consider, again, the sequence Λ, defined above. We call Λ a *minimal base* if it is a base and, furthermore, no proper

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subsequence of Λ is a base. We denote the minimum size of a minimal base $\mathrm{b}(G, \Omega)$, and the maximum size of a minimal base $\mathrm{B}(G, \Omega)$.

We say that Λ is *independent* if, for all $k = 1, \dots, \ell$, we have $G_{(\Lambda)} \neq G_{(\Lambda \setminus \omega_k)}$. We define the *height* of G to be the maximum size of an independent sequence, and we denote this quantity $\mathrm{H}(G, \Omega)$.

The last statistic of interest to us is the *relational complexity* of the action of G on Ω, denoted $\mathrm{RC}(G, \Omega)$. The definition of this is slightly involved and can be found in [8] where it is given the name *arity*.

It is easy to verify the inequalities [10]

$$\mathrm{b}(G, \Omega) \leq \mathrm{B}(G, \Omega) \leq \mathrm{H}(G, \Omega) \leq \mathrm{I}(G, \Omega). \tag{1-1}$$

Less obvious, but still rather elementary is the inequality [10]

$$\mathrm{RC}(G, \Omega) \leq \mathrm{H}(G, \Omega) + 1. \tag{1-2}$$

Theorem 1 and inequalities (1-1) and (1-2) immediately yield the following corollary.

Corollary 2. *If G is simple of Lie type of rank r acting primitively on a set Ω, then each of* $\mathrm{b}(G, \Omega)$, $\mathrm{B}(G, \Omega)$, $\mathrm{H}(G, \Omega)$ *and* $\mathrm{I}(G, \Omega)$ *is at most Cr^8 while* $\mathrm{RC}(G, \Omega)$ *is less than $Cr^8 + 1$, where C is as in Theorem 1.*

We can also deduce an upper bound for primitive actions of *almost* simple groups:

Corollary 3. *Let G be an almost simple group, with socle a simple group of Lie type of rank r over $\mathbb{F}_q$, where $q = p^f$ (p prime). If G acts primitively on a set Ω, then*

$$\mathrm{I}(G, \Omega) \leq 177r^8 + \pi(f),$$

where $\pi(f)$ is the number of primes, counted with multiplicity, dividing the integer f.

Example 5.1 shows that the term $\pi(f)$ in the upper bound cannot be avoided.

Our main tool for proving Theorem 1 is the following result on maximal subgroups of finite groups of Lie type. In the statement, we let $G(q) = (\overline{G}^F)'$ be a simple group of Lie type over $\mathbb{F}_q$, where $\overline{G}$ is the corresponding simple adjoint algebraic group over $\overline{\mathbb{F}_q}$ and F is a Frobenius endomorphism. Let p be the characteristic of $\mathbb{F}_q$. For a rational representation $\rho : \overline{G} \mapsto GL_n(\overline{\mathbb{F}_q})$, and a closed subgroup $\overline{H}$ of $\overline{G}$, we define $\deg_\rho(\overline{H})$ to be the degree of the image $\rho(\overline{H})$ as a subvariety of $\mathrm{GL}_n(\overline{\mathbb{F}_q})$. We give some basic definitions and results about degree in Section 2. Also denote by $\overline{H}^0$ the connected component of $\overline{H}$.

Theorem 4. *Let $G(q) = (\overline{G}^F)'$ be a finite simple group of Lie type as above, and let G be an almost simple group with socle $G(q)$. Let M be a maximal subgroup of G, and set $M_0 = M \cap G(q)$. Let $d = \dim \overline{G}$. Then one of the following holds:*

(1) $M_0 = \overline{M}^F \cap G(q)$, *where* $\overline{M}$ *is a closed* F*-stable subgroup of* $\overline{G}$ *of positive dimension*; *moreover*,

(a) $|\overline{M} : \overline{M}^0| \leq |W(\overline{G})|$, *the order of the Weyl group of* $\overline{G}$, *and*

(b) *excluding the cases where* $(\overline{G}, \overline{M}, p) = (C_r, D_r, 2)$ *or* $(C_3, G_2, 2)$, *if we let* $\mathrm{ad} : \overline{G} \mapsto \mathrm{GL}(L(\overline{G}))$ *be the adjoint representation*, *then*

$$\deg_{\mathrm{ad}}(\overline{M}) \leq |W(\overline{G})| \deg_{\mathrm{ad}}(\overline{G}) \leq |W(\overline{G})|\, 2^{d^2}.$$

(2) $M_0 = G(q_0)$, *a subgroup of the same type as* G (*possibly twisted*) *over a subfield* $\mathbb{F}_{q_0}$ *of* $\mathbb{F}_q$.

(3) $|M_0| \leq 2^{d^2}$.

1B. ***Context for, and possible improvements to, Theorem 1.*** We think of Theorem 1 as being a version of the Cameron–Kantor conjecture for irredundant bases. The Cameron–Kantor conjecture, which was stated in [6; 7] and proved in [20], asserts the existence of an absolute upper bound for $\mathrm{b}(G, \Omega)$ for the nonstandard actions of the almost simple groups. (A *standard action* of an almost simple group G with socle S is a transitive action where either $S = A_n$ and the action is on subsets or uniform partitions of $\{1, \dots, n\}$, or G is classical and the action is a subspace action.)

In Section 6 we explain exactly how Theorem 1 is connected to the Cameron–Kantor conjecture and we give a number of examples that clarify why Theorem 1 is, in a certain sense, the best possible "Cameron–Kantor-like statement" that can be made for irredundant bases. In particular, we give examples to show that:

(i) Even for nonstandard actions, the bound Cr^8 in Theorem 1 really needs to depend on r and is not absolute.

(ii) Theorem 1 only holds for primitive actions of *simple* groups of Lie type – it does not extend to actions of almost simple groups in general (although we do prove Corollary 3 for these).

(iii) Likewise, Theorem 1 does not extend to transitive actions of simple groups of Lie type in general.

Although (i) implies that the upper bound given in Theorem 1 is necessarily a function of r, it is undoubtedly true that the particular function of r we have given — $174r^8$ — can be improved. A construction of Freedman, Kelsey and Roney-Dougal (personal communication) implies that any polynomial upper bound must have degree at least 2; our guess is that an upper bound which is quadratic in r may hold in general.

A heuristic supporting this guess follows from the fact that $\mathrm{I}(G, \Omega) \leq \ell(G)$, where $\ell(G)$ is the maximum length of a subgroup chain in the simple group of Lie type G. Writing p for the field characteristic, U for a Sylow p-subgroup of G, and Φ^+ for the associated set of positive roots, we know that there exist constants c_1, c_2 such that

$$c_1 r^2 \log_p q \le |\Phi^+| \log_p q = \ell(U) < \ell(G) < \log_2 |G| \le c_2 r^2 \log_2 q.$$

More information about $\ell(G)$ can be found in [23].

Theorem 1 is the second recent success in trying to extend well-known results about bases to statements about irredundant bases; the first was achieved by Kelsey and Roney-Dougal [12] extending a result of Liebeck [14].

1C. ***Proofs and the structure of the paper.*** In Section 2 we present a number of definitions and results pertaining to the degree of an affine variety; these include, in particular, a statement of (one version of) Bézout's theorem on the degree of the intersection of a number of algebraic varieties.

In Section 3 we prove Theorem 4. The proof uses various results from the literature on the subgroup structure of algebraic groups [15; 18].

In Section 4 we prove Theorem 1; the proof makes use of both Theorem 4 and Bézout's theorem. Corollary 3 is deduced in Section 5.

The comparison of Theorem 1 with the Cameron-Kantor conjecture, and the relevant examples mentioned above, are given in Section 6.

2. Degree of an affine variety

Our proof of Theorem 1 is carried out by combining Theorem 4 with Bézout's theorem on the degree of the intersection of a number of algebraic varieties. We need a version of Bézout's theorem that holds for affine varieties and is due to Heintz [11].

In what follows we consider subsets of some affine space, $\mathbb{A}^n$, over an algebraically closed field k. A set X in $\mathbb{A}^n$ is called *locally closed* if $X = V \cap W$, where V is open and W is closed (in the Zariski topology). A set X is called *constructible* if it is a finite disjoint union of locally closed sets. Note that the intersection of a finite number of constructible sets is constructible. Note too that any variety in $\mathbb{A}^n$ is constructible. From here on X is a constructible set.

Definition 2.1 [11, Definition 1 and Remark 2]. If X is an irreducible variety of dimension r in $\mathbb{A}^n$, then the *degree* of X, written $\deg(X)$, is defined to be

$$\sup\{|E\cap X| : E \text{ is an } (n-r)\text{-dimensional affine subspace of } \mathbb{A}^n \text{ with } E \cap X \text{ finite}\}.$$

If X is a constructible set and $\mathcal{C}$ is the set of irreducible components of the closure of X, then we define

$$\deg(X) = \sum_{C \in \mathcal{C}} \deg(C). \tag{2-1}$$

Note that if X is an irreducible variety of dimension 0, then we have $\deg(X) = 1$. Thus, if X is any variety of dimension 0, irreducible or not, $\deg(X) = |X|$.

Now the main result that we need concerning degree is the following version of Bézout's theorem.

Proposition 2.2 [11, Theorem 1]. *Let X and Y be constructible sets in $\mathbb{A}^n$. Then*

$$\deg(X \cap Y) \leq \deg(X) \cdot \deg(Y).$$

This proposition obviously generalizes to the intersection of more than two varieties: If $X_1, X_2, \ldots, X_k$ are constructible sets in $\mathbb{A}^n$, then

$$\deg(X_1 \cap X_2 \cap \cdots \cap X_k) \leq \deg(X_1) \cdot \deg(X_2) \cdots \deg(X_k).$$

(We are implicitly using the fact that the intersection of two constructible sets is constructible.)

A useful corollary of Proposition 2.2 is the following fact connecting the degree of an affine variety to the degree of its defining polynomials. We make use of the fact, noted by Heintz [11, p. 247], that the degree of a hypersurface in $\mathbb{A}^n$ is equal to the degree of its defining polynomial.

Lemma 2.3. *Suppose that an affine variety X in $\mathbb{A}^n$ is defined by polynomials $f_1, \ldots, f_r$ of degree at most e. Then*

$$\deg(X) \leq e^r.$$

Proof. By definition $X = V(f_1, \ldots, f_r) = \bigcap_{i=1}^r V(f_i)$ where, for $i = 1, \ldots, r$, $V(f_i)$ is the hypersurface defined by the polynomial f_i. We noted that $\deg(V(f_i)) = \deg(f_i)$, and hence Proposition 2.2 implies that

$$\deg(X) \leq \deg(V(f_1)) \cdots \deg(V(f_r)) = \deg(f_1) \cdots \deg(f_r) \leq e^r. \qquad \square$$

As mentioned in the introduction, if $\overline{G}$ is an affine algebraic group over an algebraically closed field k, then for a rational representation $\rho : \overline{G} \mapsto GL_n(k)$, and a closed subgroup $\overline{H}$ of $\overline{G}$, we define $\deg_\rho(\overline{H})$ to be the degree of the image $\rho(\overline{H})$ as a subvariety of $\mathrm{GL}_n(k)$. From (2-1), we have

$$\deg_\rho(\overline{H}) = |\overline{H} : \overline{H}^0| \deg_\rho(\overline{H}^0) \geq \deg_\rho(\overline{H}^0). \tag{2-2}$$

3. Proof of Theorem 4

As in Theorem 4, let $G(q) = (\overline{G}^F)'$ be a simple group of Lie type over $\mathbb{F}_q$, where $\overline{G}$ is a simple algebraic group over $K = \overline{\mathbb{F}_q}$, and let G be an almost simple group with socle $G(q)$. Let M be a maximal subgroup of G, and set $M_0 = M \cap G(q)$. Let $d = \dim \overline{G}$ and let p be the characteristic of $\mathbb{F}_q$. Denote by $\mathrm{Lie}(p)$ the set of finite simple groups of Lie type over fields of characteristic p.

Suppose first that $G(q)$ is a classical group, so that $\overline{G}$ is the corresponding classical algebraic group. Let V be the natural module for $\overline{G}$, and let $n = \dim V$. We shall apply [15, Theorems 1′ and 2]. We postpone consideration of the cases where $G(q) = \mathrm{PSL}_n(q)$, $\mathrm{Sp}_4(2^e)$ or $P\Omega_8^+(q)$ and the group G contains an element in the coset of a graph automorphism (a triality graph automorphism in the last case). Assuming that these cases do not pertain, in [15], six classes $\mathcal{C}_i$ of closed subgroups of $\overline{G}$ are defined, and it is proved that one of the following holds:

(i) $M_0 = \overline{M}^F \cap G(q)$ for some F-stable member $\overline{M} \in \mathcal{C} := \bigcup_1^6 \mathcal{C}_i$.

(ii) $M_0 = G(q_0)$, a subgroup of the same type as $G(q)$ (possibly twisted) over a subfield $\mathbb{F}_{q_0}$ of $\mathbb{F}_q$.

(iii) M_0 is almost simple, and $F^*(M_0)$ is irreducible on V (and not of the same type as $G(q)$).

In case (ii), conclusion (2) of Theorem 4 holds.

Consider now case (i). The only finite members of $\mathcal{C}$ are:

- Subgroups of type $O_1(K) \operatorname{wr} S_n = 2^n.S_n$ in $O_n(K)$ with $p \neq 2$ (these lie in the class $\mathcal{C}_2$).
- Extraspecial-type subgroups $r^{2m}.\mathrm{Sp}_{2m}(r)$ (r prime, $n = r^m$) or $2^{2m}.O_{2m}^{\pm}(2)$ ($n = 2^m$) (these lie in the class $\mathcal{C}_5$).

A simple check shows that these subgroups have order less than 2^{d^2}, as required for conclusion (3) of Theorem 4.

All the other members of $\mathcal{C}$ are infinite, in which case

$$M_0 = \overline{M}^F \cap G(q), \tag{3-1}$$

where $\overline{M}$ is a maximal closed F-stable subgroup of $\overline{G}$ of positive dimension, as in (1) of Theorem 4.

Now consider case (iii) above. If $F^*(M) \notin \mathrm{Lie}(p)$, then an unpublished manuscript of Weisfeiler [24], subsequently improved and developed in [9], shows that $|M| < n^4(n+2)!$, which is less than 2^{d^2}, as in (3) of Theorem 4. And if $F^*(M) \in \mathrm{Lie}(p)$, then [22, Theorem 1] shows that (3-1) holds.

To complete the proof of Theorem 4 in the case where G is classical (apart from the postponed cases), it remains to prove the bounds for $|\overline{M} : \overline{M}^0|$, $\deg_{\mathrm{ad}}(\overline{M})$ and $\deg_{\mathrm{ad}}(\overline{G})$ for $\overline{M}$ in (1) of Theorem 4. The bound $|\overline{M} : \overline{M}^0| \leq |W(\overline{G})|$ follows by simply inspecting the structure of the members of $\mathcal{C}$; equality occurs when $\overline{M} = N_{\overline{G}}(T)$, where T is a maximal torus (these subgroups are in class $\mathcal{C}_2$ for $\mathrm{SL}(V)$ and $\mathrm{SO}(V)$).

To establish the degree bounds, we first prove:

Claim: Let $M_0 = \overline{M}^F \cap G(q)$ be as in (3-1). Then with two exceptions, $\overline{M}^0$ acts reducibly on some $\overline{G}$-composition factor of the adjoint module $L(\overline{G})$. The two exceptions are $(\overline{G}, \overline{M}, p) = (\mathrm{Sp}_n, \mathrm{SO}_n, 2)$ or $(\mathrm{Sp}_6, G_2, 2)$.

Proof of Claim. The composition factors of $L(\overline{G})$ are given in [16, Proposition 1.10]. Also $L(\overline{M}) \subseteq L(\overline{G})$. First consider $M_0 = \overline{M}^F \cap G(q)$ as in (i). Inspecting $\overline{M}^0$ for $\overline{M} \in \mathcal{C}$, we see that $L(\overline{M})$ maps to a proper subspace of some composition factor of $L(\overline{G})$, with the exception of $(\overline{G}, \overline{M}, p) = (\mathrm{Sp}_n, \mathrm{SO}_n, 2)$, proving the claim for M_0 as in (i). Finally, for M_0 as in (iii), the group $\overline{M}^0$ is simple, and [16, Theorem 4] shows that the only case where $L(\overline{M})$ does not map to a proper subspace of some composition factor of $L(\overline{G})$ is $(\overline{G}, \overline{M}, p) = (\mathrm{Sp}_6, G_2, 2)$. This completes the proof of the Claim.

We now use the Claim to deduce the required degree bounds. Let M, $\overline{M}$ be as in (3-1), and exclude the exceptions in the Claim, so that $\overline{M}^0$ acts reducibly on some composition factor of $L(\overline{G})$. If also $\overline{M}$ is reducible, then as it is maximal there is a subspace W of $L(\overline{G})$ such that

$$\overline{M} = \mathrm{stab}_{\overline{G}}(W).$$

This defines $\overline{M}$ by the polynomials defining $\overline{G}$ in the adjoint representation, together with some linear equations, and hence by Lemma 2.3, we have

$$\deg_{\mathrm{ad}}(\overline{M}) \leq \deg_{\mathrm{ad}}(\overline{G}).$$

On the other hand, if $\overline{M}$ acts irreducibly on every composition factor of $L(\overline{G})$, then by the Claim, there is a composition factor V such that $V \downarrow \overline{M}^0 = \bigoplus_1^t V_i$, where each V_i is irreducible for $\overline{M}^0$ and $t \geq 2$. Set

$$\overline{M}^1 = \bigcap_1^t \mathrm{stab}(V_i),$$

so that $\overline{M}^0 \leq \overline{M}^1 \lhd \overline{M}$. As above we see that $\deg_{\mathrm{ad}}(\overline{M}^1) \leq \deg_{\mathrm{ad}}(\overline{G})$, and so by the remarks after Lemma 2.3, we have $\deg_{\mathrm{ad}}(\overline{M}) \leq |\overline{M} : \overline{M}^1| \deg_{\mathrm{ad}}(\overline{G})$. We have seen that $|\overline{M} : \overline{M}^0| \leq |W(\overline{G})|$, so it follows that

$$\deg_{\mathrm{ad}}(\overline{M}) \leq |W(\overline{G})| \deg_{\mathrm{ad}}(\overline{G}),$$

as required for (1) of Theorem 4. Finally, in the adjoint representation, $\overline{G}$ is defined by d^2 quadratic polynomials expressing preservation of the Lie bracket on $L(\overline{G})$, so $\deg_{\mathrm{ad}}(\overline{G}) \leq 2^{d^2}$. Note that the exceptional cases $(\mathrm{Sp}_n, \mathrm{SO}_n, 2)$, $(\mathrm{Sp}_6, G_2, 2)$ in the Claim are also excepted in part (i)(b) of Theorem 4. Hence the proof of the theorem for G classical is now complete, apart from the postponed cases where $G(q) = \mathrm{PSL}_n(q)$, $\mathrm{Sp}_4(2^e)$ or $P\Omega_8^+(q)$ and G contains an element in the coset of a graph automorphism.

Now consider the excluded cases. Suppose first that $G(q) = \mathrm{PSL}_n(q)$. In this case, the collection $\mathcal{C}$ is extended in [15] to a collection $\mathcal{C}'$, and it is proved that conclusion (i), (ii) or (iii) above holds, with $\mathcal{C}'$ replacing $\mathcal{C}$. The only subgroups in $\mathcal{C}' \setminus \mathcal{C}$ are stabilizers of pairs $\{U, W\}$ of subspaces of V such that either $U \subseteq W$ or $V = U \oplus W$. The above proof shows that these subgroups satisfy (1) of Theorem 4. In the other cases, where $G(q) = \mathrm{Sp}_4(2^e)$ or $P\Omega_8^+(q)$, the maximal subgroups of G are listed in [1, Tables 8.14, 8.50]. Inspection of these lists shows that (1), (2) or (3) of Theorem 4 holds (using the same argument as above to bound the degree of $\overline{M}$). This completes the proof of Theorem 4 for $G(q)$ a classical group.

Suppose finally that $G(q)$ is an exceptional group of Lie type. The proof runs along similar lines. First we use [17, Theorem 8], which gives the possibilities for the maximal subgroup M. These are:

(i) $M_0 = \overline{M}^F \cap G(q)$, where $\overline{M}$ is a maximal closed F-stable subgroup of $\overline{G}$ of positive dimension.

(ii) $M_0 = G(q_0)$, a subgroup of the same type as G (possibly twisted) over a subfield $\mathbb{F}_{q_0}$ of $\mathbb{F}_q$.

(iii) M_0 is an "exotic local" subgroup:

$$3^3.\,\mathrm{SL}_3(3) < F_4, \quad 3^{3+3}.\,\mathrm{SL}_3(3) < E_6, \quad 5^3.\,\mathrm{SL}_3(5) < E_8 \quad \text{or} \quad 2^{5+10}.\,\mathrm{SL}_5(2) < E_8.$$

(iv) M_0 is the "Borovik subgroup" $(\mathrm{Alt}_5 \times \mathrm{Alt}_6).2^2 < E_8$.

(v) M_0 is almost simple with socle M_1, and one of the following holds:

(a) $M_1 \notin \mathrm{Lie}(p)$: the possibilities for M_0 are listed in [17, Theorem 4].

(b) $M_1 = M(q_1) \in \mathrm{Lie}(p)$, $\mathrm{rank}(M_1) \le \frac{1}{2}\,\mathrm{rank}(\overline{G})$ satisfying

- $q_1 \le 9$,
- $M_1 = A_2^{\pm}(16)$,
- M_1 has rank 1 and $q_1 \le (2, p-1)\cdot t(\overline{G})$, where $t(\overline{G}) = 12, 68, 124, 388, 1312$ according to $\overline{G} = G_2, F_4, E_6, E_7, E_8$, respectively.

In cases (iii), (iv) and (v) we check that $|M_0| < 2^{d^2}$, as in (3) of Theorem 4; and case (ii) is (2) of the theorem. Finally, in case (i), the list of possibilities for $\overline{M}$ is given in [17, Theorem 8]. We can check that $|\overline{M} : \overline{M}^0| \le |W(\overline{G})|$, and also that $\overline{M}^0$ acts reducibly on some $\overline{G}$-composition factor of $L(\overline{G})$ (see also [19] for this). Now we can argue exactly as in the classical case to obtain the required bounds on $\deg_{\mathrm{ad}}(\overline{G})$ for $\overline{M}$ for (1) of Theorem 4. This completes the proof of Theorem 4. □

4. Proof of Theorem 1

Let G be a simple group of Lie type of rank r over $\mathbb{F}_q$ with $G = (\overline{G}^F)'$, where $\overline{G}$ is the corresponding simple algebraic group over $\overline{\mathbb{F}_q}$ and F is a Frobenius endomorphism. Let $d = \dim \overline{G}$ and $p = \mathrm{char}(\mathbb{F}_q)$.

We write G_1 for a maximal subgroup of G. We consider the action of G on Ω, the set of cosets of G_1. We suppose that we have a stabilizer chain,

$$G > G_1 > G_2 > \cdots > G_k = \{1\} \tag{4-1}$$

where $G_i = G_{i-1} \cap G_1^{g_i}$ for some $g_i \in G$ $(i = 1, \dots, k)$.

Theorem 4 gives three possibilities for G_1.

4A. *Case 1 of Theorem 4.* In this case we have $G_1 = \overline{G_1}^F \cap G$ where $\overline{G_1}$ is a closed F-stable subgroup of $\overline{G}$ of positive dimension. We start by proving three lemmas where, in fact, the maximality assumption for G_1 is not necessary.

Set ρ to be a rational representation of $\overline{G}$ and let c be an upper bound for $\deg_\rho(\overline{G_1})$; note that, by (2-2), we also have $|\overline{G_1} : (\overline{G_1})^0| \le c$.

For each $i = 2, \dots, k$, we define $\overline{G_i} = \overline{G_{i-1}} \cap \overline{G_1}^{g_i}$ where g_i is the element of G mentioned above. Thus we have a chain of subgroups

$$\overline{G} > \overline{G_1} \ge \overline{G_2} \ge \cdots \ge \overline{G_k}. \tag{4-2}$$

Lemma 4.1. *The subgroups $G_1, \dots, G_k$ in* (4-1) *satisfy $G_i = \overline{G_i}^F \cap G$ for each $i = 1, \dots, k$.*

Proof. We proceed by induction on i. The result is true for $i = 1$. We assume the result is true for i and prove it for $i + 1$. Note that $G_{i+1} = G_i \cap G_1^{g_{i+1}}$ and $\overline{G_{i+1}} = \overline{G_i} \cap \overline{G_1}^{g_{i+1}}$.

Let $x \in \overline{G_{i+1}}^F \cap G$. This is equivalent to

$$\begin{aligned} x \in (\overline{G_i} \cap \overline{G_1}^{g_{i+1}})^F \cap G &\Leftrightarrow x \in (\overline{G_i}^F \cap (\overline{G_1}^{g_{i+1}})^F) \cap G \\ &\Leftrightarrow x \in (\overline{G_i}^F \cap G) \cap ((\overline{G_1}^{g_{i+1}})^F \cap G) \\ &\Leftrightarrow x \in (\overline{G_i}^F \cap G) \cap ((\overline{G_1}^F)^{g_{i+1}} \cap G) \\ &\Leftrightarrow x \in (\overline{G_i}^F \cap G) \cap (\overline{G_1}^F \cap G)^{g_{i+1}} \\ &\Leftrightarrow x \in G_i \cap G_1^{g_{i+1}} = G_{i+1}. \end{aligned}$$ □

The lemma implies, in particular, that all of the containments in (4-2) are proper. Let $d_1 = \dim(\overline{G_1})$. Then of course $d_1 < d = \dim \overline{G}$. Note that $\overline{G_1}$ is the largest group in the chain (4-2) of dimension d_1.

Now let $k_1, \dots, k_s$ be the points in the chain (4-2) where the dimension drops: that is, $k_1 = 1$, and for each $i \ge 2$, $\overline{G_{k_i}}$ is the largest group in the chain such that $\dim \overline{G_{k_i}} < \dim \overline{G_{k_i - 1}}$. Obviously $s \le d_1 + 1 \le d$.

Lemma 4.2. *We have* $\deg_\rho \overline{G_{k_i}} \le c^i$.

Proof. We proceed by induction on i. For $i = 1$, $\overline{G_{k_1}} = \overline{G_1}$ and this has degree at most c. We assume the result is true for i and prove it for $i + 1$. In particular, this means that $\overline{G_{k_i}}$ has degree at most c^i. Consider the chain

$$\overline{G_{k_i}} > \overline{G_{k_i+1}} > \overline{G_{k_i+2}} > \cdots > \overline{G_{k_{i+1}}}.$$

Notice that, all but the last listed group have the same dimension, and so have the same identity component; what is more the number of components decreases as we descend the chain from $\overline{G_{k_i}}$ to $\overline{G_{k_{i+1}-1}}$. Thus (2-2) implies that

$$\deg_\rho(\overline{G_{k_{i+1}-1}}) \leq \deg_\rho(\overline{G_{k_i}}) \leq c^i.$$

Now $\overline{G_{k_{i+1}}}$ is the intersection of $\overline{G_{k_{i+1}-1}}$ and a conjugate of $\overline{G_1}$. The former has degree at most c^i, and the latter has degree at most c. Hence Proposition 2.2 implies that $\deg_\rho(\overline{G_{k_{i+1}}}) \leq c^{i+1}$, as required. □

Lemma 4.3. *The length k of the stabilizer chain* (4-1) *satisfies*

$$k \leq d + \tfrac{1}{2}d(d+1)\log_2 c.$$

Proof. The previous lemma asserts that the degree of $\overline{G_{k_i}}$ is at most c^i and so we also know that $|\overline{G_{k_i}} : (\overline{G_{k_i}})^0| \leq c^i$. Now, for each $i = 1, \ldots, s$, we know that

$$\overline{G_{k_i}} > \overline{G_{k_i+1}} > \overline{G_{k_i+2}} > \cdots > \overline{G_{k_{i+1}-1}} \geq (\overline{G_{k_i}})^0,$$

where $\overline{G_{k_i}}^0$ is the identity component of all of the groups in this chain. Since $|\overline{G_{k_i}} : (\overline{G_{k_i}})^0| \leq c^i$, the length of the chain

$$\overline{G_{k_i}} > \overline{G_{k_i+1}} > \overline{G_{k_i+2}} > \cdots > \overline{G_{k_{i+1}-1}}$$

is at most $\log_2(c^i) = i\log_2 c$; in particular, for $i = 1, \ldots, s$, the length of the chain from $\overline{G_{k_i}}$ to $\overline{G_{k_{i+1}}}$ is at most $i\log_2 c + 1$. There are two further parts of the chain that we have not considered.

First, at the top of the chain, the containment $G > G_1 = G_{k_1}$ adds 1 to the total length. Second, at the bottom of the chain, $\overline{G_{k_s}}$ is of dimension 0 and degree at most c^s; in other words $\overline{G_{k_s}}$ has cardinality at most c^s and so there at most $\log_2(c^s)$ further containments at the end of the chain from $\overline{G_{k_s}}$ to $\{1\}$.

Our total chain length is, then, at most

$$1 + \sum_{i=1}^{s-1}(i\log_2 c + 1) + s\log_2 c = s + \tfrac{1}{2}s(s+1)\log_2 c.$$

Since $s \leq d$, the conclusion follows. □

We are ready to complete the proof of Theorem 1 in this case. We reinstate the maximality supposition on G_1. We consider the adjoint representation, ad, of $\overline{G}$ and we set

$$c = |W(\overline{G})| \cdot 2^{d^2}.$$

For the moment we exclude the exceptional cases $(\overline{G}, \overline{G_1}, p) = (C_n, D_n, 2)$ or $(C_3, G_2, 2)$ in Theorem 4(1)(b); then, by Theorem 4(1), c is an upper bound for $\deg_{\mathrm{ad}}(\overline{G_1})$ and also, by (2-2), for $|\overline{G_1} : (\overline{G_1})^0|$.

Recall that r is the rank of $\overline{G}$, and that $d = \dim \overline{G}$, so that $d \le 4r^2$. Also

$$c = |W(\overline{G})| \cdot 2^{d^2} \le 2^{r^2+d^2} \le 2^{r^2+16r^4}.$$

Hence Lemma 4.3 gives

$$k \le 4r^2 + \tfrac{1}{2}(4r^2)(4r^2+1)(r^2+16r^4).$$

The right-hand side is at most Cr^8 with $C = 174$, as required for Theorem 1.

It remains to deal with the excluded cases $(\overline{G}, \overline{G_1}, p) = (C_n, D_n, 2)$ or $(C_3, G_2, 2)$. In the former case [10, Lemma 6.11] implies that $\mathrm{I}(G, \Omega) \le 2r+1$ and the conclusion holds. In the latter case the action of $G = C_3(q)$ on $\Omega = (C_3(q) : G_2(q))$ is contained in $(D_4(q) : (D_4(q) : B_3(q))$, since there is a factorization $D_4(q) = AB$, where $A \cong B \cong B_3(q)$ and $A \cap B \cong G_2(q)$ (see [21, p. 105]). For this action of $X := D_4(q)$, we have $\mathrm{I}(X, \Omega) \le 15$ by [12, Theorem 3.1]. Hence $\mathrm{I}(G, \Omega) \le 15$.

This completes the proof of Theorem 1. □

4B. ***Case 2 of Theorem 4.*** In this case we have $G_1 = G(q_0)$, a subgroup of the same type as G (possibly twisted) over a subfield $\mathbb{F}_{q_0}$ of $\mathbb{F}_q$. Writing $G = (\overline{G}^F)'$ as before, there is a Frobenius endomorphism F_0 of $\overline{G}$ such that $G_1 = \overline{G}^{F_0} \cap G$, where $F_0^r = F$ for some integer $r \ge 2$.

Lemma 4.4. *For $x \in G$ we have*

$$G_1 \cap G_1^x = C_{G_1}(x^{-1}x^{F_0}) = (C_{\overline{G}}(x^{-1}x^{F_0}))^{F_0}.$$

Note that the group $C_{\overline{G}}(x^{-1}x^{F_0})$ may not be F_0-stable.

Proof. We have

$$\begin{aligned} g \in G_1 \cap G_1^x \Leftrightarrow g, g^{x^{-1}} \in G_1 &\Leftrightarrow g^{F_0} = g \ \text{ and } \ (xgx^{-1})^{F_0} = xgx^{-1} \\ &\Leftrightarrow g^{F_0} = g \ \text{ and } \ x^{F_0}gx^{-F_0} = xsx^{-1} \\ &\Leftrightarrow g \in C_{G_1}(x^{-1}x^{F_0}). \end{aligned}$$ □

Recall that we have a stabilizer chain $G > G_1 > G_2 > \cdots > G_k = 1$, where $G_i = G_{i-1} \cap G_1^{g_i}$ for each i, and $g_i \in G$. Define

$$\overline{G_1} = \overline{G}, \quad \overline{G_2} = C_{\overline{G}}(g_2^{-1}g_2^{F_0}),$$

and for $2 \le j \le k$,

$$\overline{G_j} = \bigcap_{i=2}^{j} C_{\overline{G}}(g_i^{-1}g_i^{F_0}).$$

Then by Lemma 4.4, we have $G_j = \overline{G_j}^{F_0}$ for $1 \le j \le k$, and so

$$\overline{G} = \overline{G_1} > \overline{G_2} > \cdots > \overline{G_k}.$$

Given $x \in \overline{G}$, we of course have $C_{\overline{G}}(x) = \{g \in \overline{G} : gx = xg\}$, so this centralizer consists of solutions of a system of linear equations in the entries of g, and hence $\deg_{\mathrm{ad}} C_{\overline{G}}(x) \le \deg_{\mathrm{ad}} \overline{G}$. Now we can bound the length k of the chain exactly as in Case 1, and the proof is complete. □

4C. ***Case 3 of Theorem 4.*** This case is a triviality: clearly if $|G_1| \le 2^{d^2}$, then a stabilizer chain has length at most d^2. This observation completes the proof of Theorem 1. □

Example 4.5. Here is an example that shows there is no general complementary *lower bound* to go with the upper bound given in Theorem 1. Let $G = \mathrm{SL}_r(2)$ acting on Ω, the set of cosets of H where H is the normalizer of a Singer cycle, with r an odd prime. Then $H \cong C_{2^r-1} \rtimes C_r$ and H is maximal in G for $r \ge 13$ (see [13, Table 3.5A]). Since distinct conjugates of the Singer cycle C_{2^r-1} intersect trivially, it follows that for this action we have $\mathrm{I}(G, \Omega) \le 3$. In particular, $\mathrm{I}(G, \Omega)$ does not necessarily grow as the rank increases, even when G is simple and the action is primitive.

Remark 4.6. It is possible to improve the polynomial bound of Theorem 1 in particular cases. For example, consider parabolic actions of $G = \mathrm{PSL}_n(q)$, i.e., transitive actions for which the stabilizer G_1 is a parabolic subgroup. Set $\overline{G} = \mathrm{SL}_n(\overline{\mathbb{F}_q})$ and let ρ be the usual n-dimensional rational representation. In this situation, parabolic subgroups $\overline{G}_1$ satisfy $\deg_\rho(\overline{G}_1) \le n$ and so Lemma 4.3 gives $\mathrm{I}(G, \Omega) \le n^4 \log_2 n$.

5. Almost simple groups: proof of Corollary 3

Let G be an almost simple group, with socle $S = G(q)$, a simple group of Lie type of rank r over $\mathbb{F}_q$, where $q = p^f$ (p prime). Let G act primitively on a set Ω, with point-stabilizer G_1, and let $M_1 = G_1 \cap S$. Note that $G = G_1 S$, and so $G_1/M_1 \cong G/S$, a subgroup of $\mathrm{Out}(S)$.

Now let $G > G_1 > G_2 > \cdots > G_k = \{1\}$ be a stabilizer chain, where $G_i = G_{\alpha_1 \cdots \alpha_i}$ for $1 \le i \le k$. Define $M_i = G_i \cap S$. We obtain two chains:

$$S > M_1 \ge M_2 \ge \cdots \ge M_k = \{1\},$$
$$G/S = G_1/M_1 \ge G_2/M_2 \ge \cdots \ge G_k/M_k = \{1\}.$$

Observe that, for $i = 1, \ldots, k-1$, if $M_i = M_{i+1}$, then $G_i/M_i > G_{i+1}/M_{i+1}$. By [13, Tables 5.1A, 5.1B], the order of $\mathrm{Out}(S)$ divides kf, for some integer $k \le 6r$, and hence a proper subgroup chain in G/S has length at most $\log_2(6r) + \pi(f)$. Now define

$$I = \{i : 1 \le i \le k-1 \text{ and } M_i > M_{i+1}\}$$

and write $I = \{i_1, \ldots, i_{\ell-1}\}$ where $i_j < i_{j+1}$ for $j = 1, \ldots, \ell-2$. Setting $i_\ell = k$ we have, firstly, that

$$\ell \ge k - \log_2(6r) - \pi(f) \tag{5-1}$$

and, secondly, that

$$S > M_{i_1} > M_{i_2} > \cdots > M_{i_\ell} = \{1\}. \tag{5-2}$$

Note that $i_1 = 1$, and (5-2) is the stabilizer chain $S > S_{\alpha_1} > S_{\alpha_1\alpha_{i_2}} > \cdots$ for the action of S on Ω.

Now Theorem 4 tells us that S_{α_1} satisfies (1), (2) or (3) of the conclusion of that theorem. Hence, arguing exactly as in the proof of Theorem 1 we obtain that $\ell \le 174r^8$. Combining this bound with (5-1) yields $k \le 174r^8 + \log_2(6r) + \pi(f)$, which is less than $177r^8 + \pi(f)$. This completes the proof of Corollary 3. □

Example 5.1. Here is an example that shows that the term $\pi(f)$ in the upper bound in Corollary 3 cannot be avoided.

Let $G = \mathrm{P\Gamma L}_2(q)$ with $q = p^f$, and consider the action of G on the set of 1-subspaces of $V = (\mathbb{F}_q)^2$. We claim that $\mathrm{I}(G, \Omega) = 3 + \pi(f)$. To see this, write the prime factorization of f as $f = r_1 r_2 \cdots r_\ell$ where $\ell = \pi(f)$, write $\{e_1, e_2\}$ for the natural basis of V over $\mathbb{F}_q$, and consider the stabilizer chain obtained by successively stabilizing the following 1-spaces (in order):

$$\langle e_1\rangle, \langle e_2\rangle, \langle e_1 + e_2\rangle, \langle e_1 + \zeta_1 e_2\rangle, \langle e_1 + \zeta_2 e_2\rangle, \ldots, \langle e_1 + \zeta_\ell e_2\rangle,$$

where, for $i = 1, 2, \ldots, \ell$, ζ_i is a primitive element of $\mathbb{F}_{p^{r_1 r_2 \cdots r_i}}$. This stabilizer chain establishes that $\mathrm{I}(G, \Omega) \ge 3 + \pi(f)$; on the other hand, the 3-transitivity of the action of G implies that the stabilizer of any 3 distinct points is isomorphic to C_f and this implies that $\mathrm{I}(G, \Omega) \le 3 + \pi(f)$.

It seems possible, however, that one could do better for $\mathrm{B}(G, \Omega)$ and/or $\mathrm{H}(G, \Omega)$. In the proof of Lemma 6.3 below we shall show that there exists a primitive action of $G = \mathrm{P\Gamma L}_2(q)$ for which $\mathrm{B}(G, \Omega) \ge \pi_d(f)$, where $\pi_d(f)$ is the number of *distinct* primes dividing the integer f.

Conjecture 5.2. *There exists a function $g : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that if G is an almost simple group of Lie type of rank r over a field of order p^f acting primitively on a set Ω, then*

$$\mathrm{B}(G, \Omega) \le \mathrm{H}(G, \Omega) < g(r) + \pi_d(f),$$

where $\pi_d(f)$ is the number of distinct primes dividing the integer f.

6. Theorem 1 and the Cameron–Kantor conjecture

The Cameron–Kantor conjecture (now a theorem due to Liebeck and Shalev [20]) asserts the following:

> *There exists a constant $c > 0$ such that if G is an almost simple primitive nonstandard permutation group on a set Ω, then $\mathrm{b}(G, \Omega) \leq c$.*

(A *standard action* of an almost simple group G with socle S is a transitive action where either $S = A_n$ and the action is on subsets or uniform partitions of $\{1, \dots, n\}$, or G is classical and the action is a subspace action; see [2] for more detail.) This statement is now known to be true with $c = 7$, by [2; 3; 4; 5].

Colva Roney-Dougal asked us whether a statement like the Cameron–Kantor conjecture might be true for any of the statistics $\mathrm{B}(G, \Omega)$, $\mathrm{H}(G, \Omega)$ or $\mathrm{I}(G, \Omega)$ and Theorem 1 was our answer to this question. One naturally wonders, though, whether it is possible to do better — to investigate this, given (1-1), the first question one should ask is whether a stronger statement can be proved for $\mathrm{B}(G, \Omega)$ (since any such statement for $\mathrm{H}(G, \Omega)$ or $\mathrm{I}(G, \Omega)$ is necessarily true for $\mathrm{B}(G, \Omega)$). To investigate this we need to clarify some things.

Primitivity and transitivity. Suppose that G is a transitive permutation group on Ω and identify Ω with $(G : H)$ where H is the stabilizer of a point. Now let $F \leq H$ and let $\Gamma = (G : F)$. Then it is true that $\mathrm{b}(G, \Gamma) \leq \mathrm{b}(G, \Omega)$ and hence, in particular, the Cameron–Kantor conjecture gives information about all transitive almost simple permutation groups G for which a point-stabilizer is a subgroup of a maximal subgroup that is a point stabilizer for a nonstandard primitive action.

Things are more complicated for us because it is not necessarily true that $\mathrm{B}(G, \Gamma) \leq \mathrm{B}(G, \Omega)$, that $\mathrm{H}(G, \Gamma) \leq \mathrm{H}(G, \Omega)$ or that $\mathrm{I}(G, \Gamma) \leq \mathrm{I}(G, \Omega)$; the examples below demonstrate this. Hence in investigating how to extend the statement of the Cameron–Kantor conjecture we need to distinguish between statements involving primitive groups and those involving transitive groups.

Rank-dependent constant versus absolute constant. Our investigations will focus on almost simple groups with socle a group of Lie type. Our first example will establish that it is not possible to give an absolute upper bound for $\mathrm{B}(G, \Omega)$, even for nonstandard actions. In light of this it is worth clarifying what the Cameron–Kantor conjecture implies with regard to a rank-dependent upper bound:

> *For every positive integer r there exists a constant $c > 0$ such that if G is an almost simple primitive permutation group on a set Ω, with socle a group of Lie type of rank at most r, then $\mathrm{b}(G, \Omega) \leq c$.*

The point we are making here is that, if we allow our upper bound to be rank-dependent, then we do not need to distinguish between standard and nonstandard actions — it is easy enough to establish that the standard actions also satisfy the given statement. (For the $\mathcal{C}_8$ standard actions of $\mathrm{Sp}_{2m}(q)$ this follows from [10, Lemma 6.11]; for the $\mathcal{C}_1$ standard actions of the classical groups this follows from [12, Theorem 3.1].)

Note, finally, that we have not considered the question of Cameron–Kantor-like statements for irredundant bases of primitive actions of the alternating groups.

6A. ***Simple, primitive, absolute upper bound.*** In this subsection we show that the following possible extension of the Cameron–Kantor conjecture is false:

> *There exists a constant* $c > 0$ *such that if* G *is a simple primitive nonstandard permutation group on a set* Ω*, then* $\mathrm{B}(G, \Omega) \leq c$.

The key point here is that an upper bound on $\mathrm{B}(G, \Omega)$ in this setting must depend on r.

Lemma 6.1. *For every* $n \geq 13,\ q \geq 5$, *there exists a nonstandard primitive action* $(\mathrm{PSL}_n(q), \Omega)$ *such that* $\mathrm{B}(\mathrm{PSL}_n(q), \Omega) \geq n - 1$.

Proof. We consider the action of $G = \mathrm{SL}_n(q)$ acting on the cosets of a $\mathcal{C}_2$-maximal subgroup that is the normalizer of a split torus. For $q \geq 5,\ n \geq 13$ this induces a primitive nonstandard action of $\mathrm{PSL}_n(q)$ (see [13, Table 3.5A]); furthermore this action of G is equivalent to the action of G on decompositions of $V = (\mathbb{F}_q)^n$ as a direct sum of n 1-dimensional subspaces.

Let $\{e_1, \dots, e_n\}$ be a basis of V over $\mathbb{F}_q$. For $i = 1, \dots, n-1$, we define a decomposition $\mathcal{D}_i$ of V as

$$\mathcal{D}_i = \langle e_1 \rangle \oplus \langle e_2 \rangle \oplus \cdots \oplus \langle e_{i-1} \rangle \oplus \langle e_i + e_{i+1} \rangle \oplus \langle e_{i+1} \rangle \oplus \langle e_{i+2} \rangle \oplus \cdots \oplus \langle e_n \rangle.$$

Suppose, first, that $g \in G$ fixes $\mathcal{D}_1, \dots, \mathcal{D}_{n-1}$. This implies that g fixes the space $\langle e_n \rangle$ (since it is the only 1-space appearing in all $n-1$ decompositions); similarly, for $j = 1, \dots, n-1,\ g$ fixes the space $\langle e_j \rangle$ (since it is the only 1-space appearing in all $n-1$ decompositions except for $\mathcal{D}_j$). Thus, for $j = 1, \dots, n$, there exists $\lambda_j \in \mathbb{F}_q$ such that $e_j^g = \lambda_j e_j$. But now, for $j = 1, \dots, n-1$, the space $\langle e_j + e_{j+1} \rangle$ occurs in decomposition $\mathcal{D}_j$ and no others, hence this 1-space too is fixed by g. This implies, finally, that, for $j = 1, \dots, n-1,\ \lambda_j = \lambda_{j+1}$ and so g acts as a scalar. In particular, the set $\{\mathcal{D}_1, \dots, \mathcal{D}_{n-1}\}$ is a base for the induced action of $\mathrm{PSL}_n(q)$.

On the other hand, for $j \in 1, \dots, n-1$, define

$$\Lambda_j = \{\mathcal{D}_1, \dots, \mathcal{D}_{j-1}, \mathcal{D}_{j+1}, \dots, \mathcal{D}_{n-1}\}$$

and set g_j to be an element of G that swaps $\langle e_j \rangle$ and $\langle e_n \rangle$ while fixing $\langle e_i \rangle$ for $i = 1, \dots, j-1, j+1, \dots, n-1$. It is straightforward to check that g fixes all of the decompositions in Λ_j. We conclude that Λ is a minimal base for this action of size $n-1$. $\square$

In light of this lemma our remaining investigations will focus on almost simple groups where the socle is a group of Lie type of bounded rank.

6B. ***Simple, transitive, rank-dependent upper bound.*** In this subsection we show that the following possible extension of the Cameron–Kantor conjecture is false:

> *For every positive integer r there exists a constant $c > 0$ such that if G is a simple transitive permutation group on a set Ω, with socle a group of Lie type of rank at most r, then $\mathrm{B}(G, \Omega) \leq c$.*

The next lemma does the job:

Lemma 6.2. *For every integer $c > 1$, there exists a transitive action $(\mathrm{SL}_2(2^c), \Omega)$, such that $\mathrm{B}(\mathrm{SL}_2(2^c), \Omega) \geq c$.*

Proof. Let $q = 2^c$, let $G = \mathrm{SL}_2(q)$, let U be a Sylow 2-subgroup of G, let H be an index 2 subgroup of U and let Ω be the set of right cosets of H in G. Since $H = 2^{c-1}$ it is clear that $\mathrm{B}(G, \Omega) \leq \mathrm{I}(G, \Omega) \leq c$. We claim that, in fact, $\mathrm{B}(G, \Omega) = c$.

To show this, let $B = N_G(U)$ and let Δ be the set of right cosets of H in B. Since $\mathrm{B}(B, \Delta) \leq \mathrm{B}(G, \Omega)$ it is sufficient to show that $\mathrm{B}(B, \Delta) \geq c$.

Consider U as a c-dimensional vector space over $\mathbb{F}_2$. The action of B on Δ is isomorphic to the action of B on the set of all *affine hyperplanes* — these are the usual linear hyperplanes as well as their translates. Since we are working over $\mathbb{F}_2$, each hyperplane has 2 cosets (itself and one other) thus $|\Delta| = 2(q-1)$.

Observe that if H_1 is a linear hyperplane, then the stabilizer of H_1 in B is H_1 itself (in particular, H_1 is a conjugate of H). Let $e_1, \dots, e_c$ be the usual vectors in the natural basis of U (so e_i has 0's in all places except the i-th where the entry is 1). For $i = 1, \dots, c$, define

$$H_i := \langle e_1, \dots, e_{i-1}, e_{i+1}, \dots, e_c \rangle.$$

Then $H_1, \dots, H_c$ are linear hyperplanes in U hence are elements of Δ and conjugates of H. For $i = j, \dots, c$, define $\Lambda_j = \{H_1, \dots, H_{j-1}, H_{j+1}, \dots, H_c\}$ and observe that $B_{(\Lambda_j)} = \langle e_j \rangle$. Thus $\Lambda = \{H_1, \dots, H_c\}$ is a minimal base of size c. $\square$

6C. ***Almost simple, primitive, rank-dependent upper bound.*** Here we show that the following possible extension of the Cameron–Kantor conjecture is false:

> *For every positive integer r there exists a constant $c > 0$ such that if G is an almost simple primitive permutation group on a set Ω, with socle a group of Lie type of rank at most r, then $\mathrm{B}(G, \Omega) \leq c$.*

The next lemma does the job:

Lemma 6.3. *For all $c > 0$, there exists a nonstandard primitive action* $(\mathrm{P\Gamma L}_2(q), \Omega)$, *for some q, such that* $\mathrm{B}(\mathrm{P\Gamma L}_2(q), \Omega) > c$.

Proof. Let $G = \Gamma\mathrm{L}_2(q)$ and consider the action on cosets of the normalizer of a split torus. For $q > 11$ this induces a primitive nonstandard action of $\mathrm{P\Gamma L}_2(q)$; furthermore, this action of G is equivalent to the action of G on decompositions of $V = (\mathbb{F}_q)^2$ as a direct sum of two 1-dimensional subspaces. Let $q = p^d$ and assume that $d = f_1 \cdots f_k$ where $k \geq 3$ and $f_1, \ldots, f_k$ are distinct primes.

Let $\{e_1, e_2\}$ be the natural basis for V over $\mathbb{F}_q$: $e_1 = (1\ 0)$ and $e_2 = (0\ 1)$. We define decompositions $\mathcal{D}_i$ for $i = 1, \ldots, k$ as

$$\mathcal{D}_i : \langle e_1 \rangle \oplus \langle e_1 + \zeta_i e_2 \rangle,$$

where ζ_i is a primitive element in $\mathbb{F}_{p^{f_i}}$. To see that $\mathcal{D}_1, \ldots, \mathcal{D}_k$ form an independent set we consider the action $F = \langle \sigma \rangle < G$ where σ is the field automorphism that acts on vectors by raising each entry to the p-th power.

For $j \in 1, \ldots, k$, define $\Lambda_j = \{\mathcal{D}_1, \ldots, \mathcal{D}_{j-1}, \mathcal{D}_{j+1}, \ldots, \mathcal{D}_k\}$. The pointwise-stabilizer of Λ_j in F is $\langle \sigma^{d/f_j} \rangle$ and so the pointwise-stabilizers of Λ_j are distinct for $j = 1, \ldots, k$; in particular we obtain that $\Lambda = \{\mathcal{D}_1, \ldots, \mathcal{D}_k\}$ is an independent set of size k.

We claim that, in fact, Λ is a minimal base. To see this, we must prove that the pointwise-stabilizer of Λ is trivial. Let $g \in G_{(\Lambda)}$ and write $g = \sigma^r x$ where r is some positive integer and $x \in \mathrm{GL}_2(q)$; without loss of generality we can assume that r divides d. It is clear that $\langle e_1 \rangle^g = \langle e_1 \rangle$, so there exists $\lambda_0 \in \mathbb{F}_q$ such that

$$\lambda_0 e_1 = e_1^g = e_1^{\sigma^r x} = e_1^x.$$

Similarly, for $i = 1, \ldots, k$, there exists $\lambda_i \in \mathbb{F}_q$ such that

$$\lambda_i(e_1 + \zeta_i e_2) = (e_1 + \zeta_i e_2)^g = e_1^g + (\zeta_i e_2)^g = e_1^x + \zeta_i^{\sigma^r} e_2^x = \lambda_0 e_1 + \zeta_i^{p^r} e_2^x.$$

Rearranging we obtain that

$$e_2^x = \lambda_i \zeta_i^{1-p^r} e_2 + \zeta_i^{-p^r} (\lambda_i - \lambda_0)\, e_1.$$

We conclude that, for distinct $i, j \in \{1, \ldots, k\}$ we have

$$\lambda_i \zeta_i^{1-p^r} = \lambda_j \zeta_j^{1-p^r} \quad \text{and} \quad \zeta_i^{-p^r} (\lambda_i - \lambda_0) = \zeta_j^{-p^r} (\lambda_j - \lambda_0).$$

The latter equation yields that

$$\lambda_i = \left(\frac{\zeta_i}{\zeta_j}\right)^{p^r} \lambda_j + \left(1 - \left(\frac{\zeta_i}{\zeta_j}\right)^{p^r}\right)\lambda_0,$$

while the former yields that

$$\lambda_i = \frac{\zeta_i^{p^r-1}}{\zeta_j^{p^r-1}}\lambda_j.$$

Combining these two identities and rearranging yields

$$\left(\frac{\zeta_j/\zeta_i - 1}{(\zeta_j/\zeta_i)^{p^r} - 1}\right)\lambda_j = \lambda_0.$$

If we fix j and choose $\ell, m \in \{1, \dots, k\}$ such that j, ℓ and m are all distinct, then we obtain that

$$\frac{\zeta_j/\zeta_\ell - 1}{(\zeta_j/\zeta_\ell)^{p^r} - 1} = \frac{\zeta_j/\zeta_m - 1}{(\zeta_j/\zeta_m)^{p^r} - 1}$$

and, rearranging, we have

$$\left(\frac{\zeta_j/\zeta_\ell - 1}{\zeta_j/\zeta_m - 1}\right)^{p^r-1} = 1.$$

We claim that the smallest field containing the quantity in parenthesis is either $\mathbb{F}_{p^{f_j f_\ell f_m}}$ or $\mathbb{F}_{p^{f_\ell f_m}}$. To see this, denote this quantity η and suppose that η is contained in $\mathbb{F}_{p^{f_j f_\ell}}$. Rearranging we obtain

$$\zeta_m = \frac{\zeta_j \zeta_\ell \eta}{\zeta_j - \zeta_\ell + \zeta_\ell \eta} \in \mathbb{F}_{p^{f_j f_\ell}},$$

a contradiction. A similar argument allows us to conclude that this quantity is not contained in $\mathbb{F}_{p^{f_j f_m}}$ and the claim follows.

We obtain that r is divisible by both f_ℓ and f_m. Repeating this argument we obtain that r is divisible by all primes $f_1, \dots, f_k$ and thus $g = x$. But this implies that $\lambda_i = \lambda_j = \lambda_0$ for all $i, j = 1, \dots, k$ and g is a scalar, as required.

We conclude that Λ is a minimal base for this action. Since $|\Lambda| = k$, we need only choose $k > c$ to obtain that $\mathrm{B}(G, \Omega) \geq k > c$ as required. □

6D. ***Simple, primitive, rank-dependent upper bound.*** In light of the examples given in the preceding sections, this is the only setting where a direct extension of Cameron–Kantor conjecture is possible. As mentioned above, if we allow our upper bound to be rank-dependent, then we can ignore the distinction between standard and nonstandard actions, and hence the statement we end up with has the form of Theorem 1.

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LOCAL EXTERIOR SQUARE AND ASAI L-FUNCTIONS FOR GL(n) IN ODD CHARACTERISTIC

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Let F be a nonarchimedean local field of odd characteristic $p>0$. We consider local exterior square L-functions $L(s,\pi,\wedge^2)$, Bump–Friedberg L-functions $L(s,\pi,\mathrm{BF})$, and Asai L-functions $L(s,\pi,\mathrm{As})$ of an irreducible admissible representation π of $\mathrm{GL}_m(F)$. In particular, we establish that those L-functions, via the theory of integral representations, are equal to their corresponding Artin L-functions $L\big(s,\wedge^2(\phi(\pi))\big)$, $L\big(s+\frac{1}{2},\phi(\pi)\big)L\big(2s,\wedge^2(\phi(\pi))\big)$, and $L\big(s,\mathrm{As}(\phi(\pi))\big)$ of the associated Langlands parameter $\phi(\pi)$ under the local Langlands correspondence. These are achieved by proving the identity for irreducible supercuspidal representations, exploiting the local-to-global argument due to Henniart and Lomelí.

1. Introduction

Let F be a nonarchimedean local field of positive characteristic $p\neq 2$. Let π be an irreducible admissible representation of $\mathrm{GL}_m(F)$, where m is a positive integer. The local Langlands correspondence provides a bijection between the set of equivalence classes of irreducible admissible (complex-valued) representations of $\mathrm{GL}_m(F)$ and the set of equivalence classes of m-dimensional Weil–Deligne representations of the Weil–Deligne group W'_F of F. Let r denote either the exterior square representation $\wedge^2:\mathrm{GL}_m(\mathbb{C})\to\mathrm{GL}_{m(m-1)/2}(\mathbb{C})$ or the Asai representation (the twisted tensor induction) $\mathrm{As}:\mathrm{GL}_m(\mathbb{C})\to\mathrm{GL}_{m^2}(\mathbb{C})$ (see [Anandavardhanan and Rajan 2005, §2.1] and [Shankman 2018, §1.2]) of the Langlands dual group $\mathrm{GL}_m(\mathbb{C})$ of $\mathrm{GL}_m(F)$. Let $L(s,(r\circ\phi)(\pi))$, where $s\in\mathbb{C}$, be the Artin local L-function defined by Deligne and Langlands [Tate 1979], where $\phi(\pi):W'_F\to\mathrm{GL}_m(\mathbb{C})$ is the Weil–Deligne representation corresponding to π under the local Langlands correspondence. In this paper, we address that L-factors, in the cases of exterior square, Bump–Friedberg, and Asai local L-functions, are compatible with the local Langlands correspondence, and establish a series of equalities of local L-functions:

- Jacquet–Shalika cases, Theorem 3.8:

$$L(s,\pi,\wedge^2)=L\big(s,\wedge^2(\phi(\pi))\big);$$

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- Bump–Friedberg cases, Theorem 4.6:

$$L(s,\pi,\mathrm{BF}) = L\big(s+\tfrac{1}{2},\phi(\pi)\big)L\big(2s,\wedge^2(\phi(\pi))\big);$$

- Flicker cases, Theorem 4.7:

$$L(s,\pi,\mathrm{As}) = L\big(s,\mathrm{As}(\phi(\pi))\big);$$

where L-factors on the left-hand sides are defined by the theory of integral representations in positive characteristic.

In the late 1980s, global zeta integrals for (partial) Asai L-functions for GL_m appeared in the work of Flicker [1988; 1993]. Around that time, Jacquet and Shalika [1990] and Bump and Friedberg [1990] independently constructed two different integral representations for an (incomplete) exterior square L-function associated to a cuspidal automorphic representation on GL_m over a global field. Recently there has been renewed interest in the local theory of Asai and exterior square L-functions via Rankin–Selberg methods. In characteristic 0, the identities were shown for Jacquet–Shalika integrals by the author [Jo 2020a], and by Matringe for Bump–Friedberg integrals [Matringe 2015] and Flicker integrals [Matringe 2009; 2011]. As a matter of fact, these results improve discrete series cases of Kewat and Raghunathan [2012] for Jacquet–Shalika integrals and of Anandavardhanan and Rajan [2005] for Flicker integrals. In the positive characteristic $p>0$, Artin L-factors $L\big(s,\wedge^2(\phi(\pi))\big)$ and $L\big(s,\mathrm{As}(\phi(\pi))\big)$ coincide with $\mathcal{L}(s,\pi,\wedge^2)$ and $\mathcal{L}(s,\pi,\mathrm{As})$, respectively, via the Langlands–Shahidi method by a sequence of work by Henniart and Lomelí [2011; 2013a; 2013b]. Similar problems have been worked out by Henniart [2010] in the characteristic zero cases.

The method to prove the matching was developed by Cogdell and Piatetski-Shapiro [2017] in the framework of local L-functions of pairs of irreducible generic representations (π_1,π_2). The computation of local Rankin–Selberg L-functions boils down to decomposing it as the product of what is called the exceptional L-functions (in the sense of [Cogdell and Piatetski-Shapiro 2017]) $L_{\mathrm{ex}}(s,\pi_1^{(k_1)}\times\pi_2^{(k_2)})$ for pairs of Bernstein–Zelevinsky's derivatives $(\pi_1^{(k_1)},\pi_2^{(k_2)})$. The "derivative" in the sense of Bernstein–Zelevinsky $\pi^{(k)}$ is given by representations of smaller groups $\mathrm{GL}_{m-k}(F)$. The advantage of adapting such derivatives enables us to proceed by induction on the rank $m-k$ of general linear groups $\mathrm{GL}_{m-k}(F)$.

Each pole of exceptional L-functions, which we often refer to as an *exceptional pole*, is astonishingly characterized by local distinctness or existence of certain models. The classification of irreducible generic distinguished representations has been widely explored in various works. Indeed, topics of the classification are brought to light for $(S_{2n}(F),\Theta)$-distinguished representations (Shalika models) in [Matringe 2017], for $H_m(F)$-distinguished representation (linear and Friedberg–Jacquet models) in [Matringe 2015], for $\mathrm{GL}_m(F)$-distinguished representations

(Flicker–Rallis models) in [Matringe 2011], and for $(\mathrm{GL}_m(F), \theta)$-distinguished representations (Bump–Ginzburg models) in [Kaplan 2017]. In particular, the main results are summarized as, so to speak, "the hereditary property of models" motivated from the classification of irreducible admissible generic representations of $\mathrm{GL}_m(F)$ (Whittaker models) due to Rodier [1973, p. 427].

When we combine these two ingredients, the factorization of local L-functions and the classification of distinguished representations, we obtain major applications: the inductive relation of local L-functions for irreducible generic representations $\pi = \mathrm{Ind}_{\mathrm{Q}}^{\mathrm{GL}_m}(\Delta_1 \otimes \Delta_2 \otimes \cdots \otimes \Delta_t)$ (Corollary 2.14):

$$L(s, \pi, \wedge^2) = \prod_{1\le k\le t} L(s, \Delta_k, \wedge^2) \prod_{1\le i<j\le t} L(s, \Delta_i \times \Delta_j), \tag{1-1}$$

and the weak multiplicativity of γ-factors for parabolically normalized induced (not necessarily irreducible) representations $\pi = \mathrm{Ind}_{\mathrm{Q}}^{\mathrm{GL}_m}(\Delta_1 \otimes \Delta_2 \otimes \cdots \otimes \Delta_t)$ (Theorem 2.12):

$$\gamma(s, \pi, \wedge^2, \psi) \sim \prod_{1\le k\le t} \gamma(s, \Delta_k, \wedge^2, \psi) \prod_{1\le i<j\le t} \gamma(s, \Delta_i \times \Delta_j, \psi), \tag{1-2}$$

where $\sim$ means the equality up to a unit in $\mathbb{C}[q^{\pm s}]$ and the Δ_i are discrete series representations. Building upon (1-1), we can incorporate the Langlands classification of irreducible admissible representations in terms of discrete series ones into the theory of local L-functions. In turn, (1-2) allows us to compute local L-functions further in accordance with the Bernstein–Zelevinsky classification of discrete series representations in terms of irreducible supercuspidal ones. As a consequence, we express all exterior square L-factors for irreducible admissible representations in terms of L-factors for irreducible supercuspidal ones in a purely local mean. This comes down to reducing our main questions to all irreducible supercuspidal representations, which eventually serve as building blocks.

We emphasize that the third identity in the Flicker cases is not new and can be found in [Anandavardhanan et al. 2021, Appendix A]. However, we discovered that our technique seems to carry out uniformly to other L-functions for GL_m. As an application of our approach, the main result of [Anandavardhanan et al. 2021] immediately implies the agreement of local Asai L-factors for irreducible supercuspidal representations, which is sufficient to extend it to all irreducible admissible representations, reflecting on the local Langlands correspondence. At this point, unlike [Anandavardhanan et al. 2021, Appendix A], additional globalizations are not required to generalize the equality unconditionally. In the course of following the direction taken in [Anandavardhanan et al. 2021, Appendix A], we encountered a few stumbling blocks. In contrast to characteristic 0 cases, we could not find a good way to adjust the globalization of discrete series representations in [Gan and Lomelí 2018, Proposition 8.2] to our circumstance. As seen in several other's work [Anandavardhanan and Rajan 2005; Kewat and Raghunathan 2012; Kable 2004; Yamana 2017], there might not be a guarantee that the different places v_1 and v_2 are coprime in order to conclude that $\log(q_{v_1})/\log(q_{v_2})$ is irrational, and this coprimality condition may prompt an issue in characteristic $p > 0$. We propose to resolve all these difficulties by globalizing irreducible supercuspidal representations in [Henniart and Lomelí 2011; 2013b, Theorem 3.1] and controlling all but one place in which we are interested.

In practice, we demonstrate the identity sequentially for irreducible supercuspidal representations and eventually for discrete series representations under the working hypothesis analogous to Kaplan's inquiry [2017, Remark 4.18] that $(\mathrm{GL}_m(F), \theta)$-distinguished discrete series representations in positive characteristic are self-dual. Thankfully, we remove the hypothesis by investigating irreducible generic subquotients of principal series representations. We expect to overcome Kaplan's issue beyond the principal series cases by reconciling the different definitions of local symmetric square L-functions possessing their own insights about representations. The poles of $L(s, \pi, \mathrm{Sym}^2)$ can be determined, by means of the Rankin–Selberg method, using the occurrence of $(\mathrm{GL}_m(F), \theta)$-distinguished representations [Yamana 2017], whereas the symmetric square L-functions through the Langlands–Shahidi method $\mathcal{L}(s, \pi, \mathrm{Sym}^2)$ can be related to the presence of the self-duality $\pi \simeq \tilde{\pi}$, using the Rankin–Selberg L-factor $L(s, \pi \times \pi)$ as a product of $\mathcal{L}(s, \pi, \wedge^2)$ and $\mathcal{L}(s, \pi, \mathrm{Sym}^2)$ [Henniart and Lomelí 2011; 2013b]. Taking it for granted that $L(s, \pi, \mathrm{Sym}^2)$ can be factored in terms of exceptional L-factors for derivatives (see [Jo 2021, Theorem 3.15]), our discourse sheds light on some impetus toward systematic development of symmetric square L-factors via integral representations [Yamana 2017] in number theoretic aspects and the classification of $(\mathrm{GL}_m(F), \theta)$-distinguished representations over local function fields [Kaplan 2017] in representation theoretic perspectives. We will return to these matters in the near future.

Finally, it is worth pointing out that the main result of this paper will be used to prove the claim in the preprint by Chen and Gan [2021, Theorem 1.1], that *the exterior square L-function can be equivalently defined by the Langlands–Shahidi method or the local zeta integrals of Jacquet and Shalika* [1990] in positive characteristic.

Let us overview the content of this paper. Section 2A begins with a summary of the theory of derivatives of Bernstein and Zelevinsky and the basic existence theorem of Jacquet–Shalika integrals. Section 2B is devoted to classifying all irreducible generic distinguished representations with respect to given closed algebraic subgroups, especially H_{2n} and S_{2n}, due to Matringe. By combining the factorization of Section 2A, the classification of Section 2B, and the method of deformations and specializations, we prove a weak version of multiplicativity of γ-factors and the inductive relation of L-factors. Using the globalization of irreducible supercuspidal representations presented in Section 3B, if necessary, we complete computing local exterior square L-functions at the end of Section 3B, local Bump–Friedberg L-functions in Section 4A, and local Asai L-functions in Section 4B.

2. Jacquet–Shalika zeta integrals

2A. ***Derivatives and exceptional poles.*** Let F be a nonarchimedean local field of characteristic $p \neq 0, 2$. We let $\mathcal{O}$ denote its ring of integers, $\mathfrak{p}$ its maximal ideal, and q the cardinality of its residual field. We will let ϖ denote a uniformizer, so $\mathfrak{p} = (\varpi)$. We normalize the absolute value by $|\varpi|^{-1} = |\mathcal{O}/\mathfrak{p}|$. The character of GL_m given by $g \mapsto |\det(g)|$ is denoted by ν.

For the group $\mathrm{GL}_m := \mathrm{GL}_m(F)$, we often confront the two cases: m is even and m is odd. For the former, we let $m = 2n$, and for the latter $m = 2n+1$. Let σ_m be the permutation matrix given by

$$\sigma_{2n} = \left(\begin{array}{cccc|cccc} 1 & 2 & \cdots & n & n+1 & n+2 & \cdots & 2n \\ 1 & 3 & \cdots & 2n-1 & 2 & 4 & \cdots & 2n \end{array}\right)$$

when $m = 2n$ is even, and by

$$\sigma_{2n+1} = \left(\begin{array}{cccc|ccccc} 1 & 2 & \cdots & n & n+1 & n+2 & \cdots & 2n & 2n+1 \\ 1 & 3 & \cdots & 2n-1 & 2 & 4 & \cdots & 2n & 2n+1 \end{array}\right)$$

when $m = 2n+1$ is odd. Let B_m be the Borel subgroup consisting of the upper triangular matrices with Levi subgroup A_m of diagonal matrices and unipotent radical N_m. We let Z_m denote the center consisting of scalar matrices. We define P_m to be the mirabolic subgroup given by

$$P_m = \left\{ \begin{pmatrix} g & {}^t u \\ & 1 \end{pmatrix} \;\middle|\; g \in \mathrm{GL}_{m-1},\ u \in F^{m-1} \right\}.$$

We denote by U_m the unipotent radical of P_m. As a group, P_m has a structure of a semidirect product $P_m = \mathrm{GL}_{m-1} \ltimes U_m$. We let $\mathcal{M}_m$ be the set of $m \times m$ matrices and $\mathcal{N}_m$ be the subspace of upper triangular matrices of $\mathcal{M}_m$. Let $\{e_i \mid 1 \leq i \leq m\}$ be the standard low basis of F^m.

We let ψ_F denote a nontrivial additive character of F. We let ψ denote the character of N_m defined by

$$\psi(n) = \psi_F\left(\sum_{i=1}^{n-1} n_{i,i+1}\right), \quad n = (n_{i,j}) \in N_m.$$

We denote by $\mathcal{A}_F(m)$ the set of equivalence classes of all admissible representations of GL_m on complex vector spaces. Furthermore, we say that a representation $\pi \in \mathcal{A}_F(m)$ is called *generic* if $\mathrm{Hom}_{N_m}(\pi, \psi) \neq \{0\}$. We say that a representation $\pi \in \mathcal{A}_F(m)$ is of *Whittaker type* if

$$\dim_{\mathbb{C}} \mathrm{Hom}_{N_m}(\pi, \psi) = 1.$$

For any character χ of $F^\times$, χ can be uniquely decomposed as $\chi = \chi_0 \nu^{s_0}$, where χ_0 is a unitary character and s_0 is a real number. We use the notation $s_0 = \mathrm{Re}(\chi)$ for the real part of the exponent of the character χ.

If $\pi \in \mathcal{A}_F(m)$ is irreducible and generic, it is known that π is of Whittaker type [Gelfand and Kajdan 1975]. By Frobenius reciprocity, there exists a unique embedding of π into $\mathrm{Ind}_{N_m}^{\mathrm{GL}_m}(\psi)$ up to scalar. The image $\mathcal{W}(\pi, \psi)$ of V_π is called the *Whittaker model* of π. For a nonzero functional $\lambda \in \mathrm{Hom}_{N_m}(\pi, \psi)$, we define the *Whittaker function* $W_v \in \mathcal{W}(\pi, \psi)$ associated to $v \in V_\pi$ by

$$W_v(g) = \lambda(\pi(g)v), \quad g \in \mathrm{GL}_m .$$

We set $W := W_v$. It follows from [Bernstein and Zelevinsky 1976, Lemma 4.5] and [Zelevinsky 1980, §9] that if $\Delta_1, \Delta_2, \dots, \Delta_t$ are irreducible *essentially square integrable*, which we call *discrete series representations*, then the representation of the form $\mathrm{Ind}_{\mathrm{Q}}^{\mathrm{GL}_m}(\Delta_1 \otimes \Delta_2 \otimes \cdots \otimes \Delta_t)$ is a representation of Whittaker type, where the induction is the normalized parabolic induction from the standard parabolic subgroup Q attached to the partition $(m_1, m_2, \dots, m_t)$ of m and $\Delta_i \in \mathcal{A}_F(m_i)$. Also, whenever the parabolic subgroup Q and ambient group GL_m are clear from the context, we simply write $\mathrm{Ind}(\Delta_1 \otimes \Delta_2 \otimes \cdots \otimes \Delta_t)$.

Let $\mathrm{Rep}(G)$ denote the category of smooth representations of an l-group G. There are four functors Ψ^-, Ψ^+, Φ^-, and Φ^+. The functor Ψ^- is a normalized Jacquet functor and Φ^- is a normalized ψ-twisted Jacquet functor from $\mathrm{Rep}(P_m)$ to $\mathrm{Rep}(\mathrm{GL}_{m-1})$ and $\mathrm{Rep}(P_{m-1})$, respectively. Given $\tau \in \mathrm{Rep}(P_m)$ on the space V_τ, $\Psi^-(\tau)$ is realized on the space $V_\tau / V_\tau(U_m, \mathbf{1})$ with the action

$$\Psi^-(\tau)(g)(v + V_\tau(U_m, \mathbf{1})) = |\det(g)|^{-1/2}\big(\tau(g)v + V_\tau(U_m, \mathbf{1})\big)$$

and the subspace $V_\tau(U_m, \mathbf{1}) = \langle \tau(u)v - v \mid v \in V_\tau,\ u \in U_m \rangle$. Likewise $\Phi^-(\tau)$ is realized on the space $V_\tau / V_\tau(U_m, \psi)$ with the action

$$\Phi^-(\tau)(p)(v + V_\tau(U_m, \psi)) = |\det(p)|^{-1/2}\big(\tau(p)v + V_\tau(U_m, \psi)\big)$$

and the subspace $V_\tau(U_m, \psi) = \langle \tau(u)v - \psi(u)v \mid v \in V_\tau, u \in U_m \rangle$. The functors Ψ^+ and Φ^+ are normalized and compactly supported inductions from $\mathrm{Rep}(\mathrm{GL}_{m-1})$ and $\mathrm{Rep}(P_{m-1})$, respectively, to $\mathrm{Rep}(P_m)$. Given $\sigma \in \mathrm{Rep}(\mathrm{GL}_{m-1})$,

$$\Psi^+(\sigma) = \mathrm{ind}_{\mathrm{GL}_{m-1}\, U_m}^{P_m}\big(|\det(g)|^{1/2}\sigma \otimes \mathbf{1}\big) = |\det(g)|^{1/2}\sigma \otimes \mathbf{1}$$

is realized on the space V_σ, where ind denotes a compactly supported induction. If $\sigma \in \mathrm{Rep}(P_{m-1})$, then $\Phi^+(\sigma) = \mathrm{ind}_{P_{m-1} U_m}^{P_m}\big(|\det(g)|^{1/2}\sigma \otimes \psi\big)$.

For $\tau \in \mathrm{Rep}(P_m)$, four functors are utilized to define what is called the *Bernstein–Zelevinsky k-th derivatives* $\tau^{(k)}$. Let $\tau^{(k)} \in \mathrm{Rep}(\mathrm{GL}_{m-k})$ be $\tau^{(k)} = \Psi^-(\Phi^-)^{k-1}(\tau)$

for $1 \le k \le m$. The smooth representation τ affords a natural filtration by P_m-modules

$$0 \subseteq \tau_m \subseteq \tau_{m-1} \subseteq \cdots \subseteq \tau_1 = \tau$$

such that $\tau_k/\tau_{k+1} = (\Phi^+)^{k-1}\Psi^+(\tau^{(k)})$ and $\tau_k = (\Phi^+)^{k-1}(\Phi^-)^{k-1}(\tau)$. Let $\pi = \mathrm{Ind}_{\mathrm{Q}}^{\mathrm{GL}_m}(\Delta_1 \otimes \cdots \otimes \Delta_t)$ be a parabolically induced representation, where Δ_i is an irreducible essentially square integrable representation of GL_{m_i} so that $m = m_1 + \cdots + m_t$. Then $\pi^{(k)}$ has a filtration whose successive quotients are isomorphic to $\mathrm{Ind}(\Delta_1^{(k_1)} \otimes \cdots \otimes \Delta_t^{(k_t)})$, with $k = k_1 + \cdots + k_t$ [Bernstein and Zelevinsky 1977, Theorem 4.4 and Lemma 4.5]. For every $0 \le k \le m-1$, let $(\omega_{\pi_{i_k}^{(k)}})_{i_k=1,2,\ldots,r_k}$ be the family of the central characters of nonzero successive quotient of the form $\pi_{i_k}^{(k)} = \mathrm{Ind}(\Delta_1^{(k_1)} \otimes \cdots \otimes \Delta_t^{(k_t)})$.

Let $\mathcal{S}(F^n)$ be the space of smooth locally constant compactly supported functions on F^n. For each Whittaker function $W \in \mathcal{W}(\pi, \psi)$ and Schwartz–Bruhat function $\Phi \in \mathcal{S}(F^n)$, we define the Jacquet–Shalika integrals:

$$\begin{aligned} &J(s, W, \Phi) \\ &\quad = \int_{N_n \backslash \mathrm{GL}_n} \int_{\mathcal{N}_n \backslash \mathcal{M}_n} W\left(\sigma_{2n} \begin{pmatrix} I_n & X \\ & I_n \end{pmatrix} \begin{pmatrix} g & \\ & g \end{pmatrix}\right) \psi^{-1}(\mathrm{Tr}(X)) \Phi(e_n g) \, |\det(g)|^s \, dX \, dg \end{aligned}$$

in the even case $m = 2n$ and

$$\begin{aligned} J(s, W, \Phi) = \int_{N_n \backslash \mathrm{GL}_n} \int_{\mathcal{N}_n \backslash \mathcal{M}_n} \int_{F^n} W\left(\sigma_{2n+1} \begin{pmatrix} I_n & X & \\ & I_n & \\ & & 1 \end{pmatrix} \begin{pmatrix} g & & \\ & g & \\ & & 1 \end{pmatrix} \begin{pmatrix} I_n & & \\ & I_n & \\ & y & 1 \end{pmatrix}\right) \\ \times \psi^{-1}(\mathrm{Tr}(X)) \Phi(y) |\det(g)|^{s-1} \, dy \, dX \, dg \end{aligned}$$

in the odd case $m = 2n+1$. Several nice consequences follow from inserting an asymptotic formula over the torus for W into the local zeta integral $J(s, W, \Phi)$ [Jo 2020b, Theorem 3.3 and Lemma 3.10].

Theorem 2.1. *Let $\pi = \mathrm{Ind}_{\mathrm{Q}}^{\mathrm{GL}_m}(\Delta_1 \otimes \Delta_2 \otimes \cdots \otimes \Delta_t)$ be a parabolically induced representation. Let $W \in \mathcal{W}(\pi, \psi)$ and $\Phi \in \mathcal{S}(F^n)$.*

- (i)-(1) *(Even case, $m = 2n$) If we have,* $\mathrm{Re}(s) > -\frac{1}{k}\omega_{\pi_{i_{2n-2k}}^{(2n-2k)}}$, *for all $1 \le k \le n$ and all $1 \le i_{2k} \le r_{2k}$, then each local integral $J(s, W, \Phi)$ converges absolutely.*
- (i)-(2) *(Odd case, $m = 2n+1$) If we have* $\mathrm{Re}(s) > -\frac{1}{k}\omega_{\pi_{i_{2n+1-2k}}^{(2n+1-2k)}}$, *for all $1 \le k \le n$ and all $1 \le i_{2k-1} \le r_{2k-1}$, then each local integral $J(s, W, \Phi)$ converges absolutely.*
- (ii) *Each $J(s, W, \Phi)$ is a rational function in $\mathbb{C}(q^{-s})$, hence $J(s, W, \Phi)$ as a function of s extends meromorphically to all $\mathbb{C}$.*
- (iii) *Each $J(s, W, \Phi)$ can be written with a common denominator determined by π. Hence the family has "bounded denominators".*

Let $\mathcal{J}(\pi)$ be the complex linear space of the local integrals $J(s, W, \Phi)$. The family of local integrals $\mathcal{J}(\pi)$ is a $\mathbb{C}[q^{\pm s}]$-fractional ideal of $\mathbb{C}(q^{-s})$ containing 1 [Jo 2020b, Theorems 3.6 and 3.9]. Since the ring $\mathbb{C}[q^s, q^{-s}]$ is a principal ideal domain, the fractional ideal $\mathcal{J}(\pi)$ has a generator. Since $1 \in \mathcal{J}(\pi)$, we can take a generator having numerator 1 and normalized (up to units) to be of the form $P(q^{-s})^{-1}$ with $P(X) \in \mathbb{C}[X]$ having $P(0) = 1$. The *local exterior square L-function*, or simply the *exterior square L-factor*,

$$L(s, \pi, \wedge^2) = \frac{1}{P(q^{-s})}$$

is defined to be the normalized generator of the fractional ideal $\mathcal{J}(\pi)$ spanned by the local zeta integrals $J(s, W, \Phi)$.

We define the Fourier transform on $\mathcal{S}(F^m)$ by

$$\hat{\Phi}(y) = \int_{F^n} \Phi(x)\psi(x\,{}^t y)\,dx.$$

We assume that the measure on F^m is the self-dual measure. Then the Fourier inversion takes the form $\hat{\hat{\Phi}}(x) = \Phi(-x)$. Let

$$w_m := \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}$$

denote the long Weyl element in GL_m. For $(\pi, V_\pi) \in \mathrm{Rep}(\mathrm{GL}_m)$, let π^ι denote the representation of GL_m on the same space V_π given by $\pi^\iota(g) = \pi({}^t g^{-1})$. If π is irreducible, it is known that $\pi^\iota \simeq \tilde{\pi}$, the contragredient representation of π. The parabolically induced representation $\pi^\iota = \mathrm{Ind}(\tilde{\Delta}_t \otimes \tilde{\Delta}_{t-1} \otimes \cdots \otimes \tilde{\Delta}_1)$ is, again, of Whittaker type. If $W \in \mathcal{W}(\pi, \psi)$, then $\widetilde{W}(g) := W(w_m\,{}^t g^{-1})$ belongs to $\mathcal{W}(\pi^\iota, \psi^{-1})$. We let τ_m be a matrix given by

$$\begin{pmatrix} & I_n \\ I_n & \end{pmatrix}, \text{ when } m = 2n, \qquad \begin{pmatrix} & I_n & \\ I_n & & \\ & & 1 \end{pmatrix}, \text{ when } m = 2n+1.$$

As a consequence of the uniqueness of bilinear forms on $\mathcal{W}(\pi, \psi) \times \mathcal{S}(F^n)$, we can define the local γ-factor, which gives rise to the local functional equation for our integrals $J(s, W, \Phi)$ [Cogdell and Matringe 2015; Matringe 2014] (see [Jo 2020a, Theorem 2.10, (2.1)]).

Theorem 2.2. *Let $\pi = \mathrm{Ind}_{\mathrm{Q}}^{\mathrm{GL}_m}(\Delta_1 \otimes \Delta_2 \otimes \cdots \otimes \Delta_t)$ be a parabolically induced representation of GL_m. Then there is a rational function $\gamma(s, \pi, \wedge^2, \psi) \in \mathbb{C}(q^{-s})$ such that for every W in $\mathcal{W}(\pi, \psi)$, and every Φ in $\mathcal{S}(F^n)$, we have*

$$J(1-s, \varrho(\tau_m)\widetilde{W}, \hat{\Phi}) = \gamma(s, \pi, \wedge^2, \psi) J(s, W, \Phi),$$

where ϱ denotes right translation.

An equally important local factor is the local ε-factor

$$\varepsilon(s,\pi,\wedge^2,\psi)=\gamma(s,\pi,\wedge^2,\psi)\frac{L(s,\pi,\wedge^2)}{L(1-s,\pi^\iota,\wedge^2)},$$

which is an invertible element $\varepsilon(s,\pi,\wedge^2,\psi)$ in $\mathbb{C}[q^{\pm s}]$. With the local ε-factor, the functional equation becomes

$$\frac{J(1-s,\varrho(\tau_m)\widetilde{W},\hat{\Phi})}{L(1-s,\pi^\iota,\wedge^2)}=\varepsilon(s,\pi,\wedge^2,\psi)\frac{J(s,W,\Phi)}{L(s,\pi,\wedge^2)}.$$

Let $\pi=\mathrm{Ind}_{\mathrm{Q}}^{\mathrm{GL}_{2n}}(\Delta_1\otimes\Delta_2\otimes\cdots\otimes\Delta_t)$ be a parabolically induced representation. Let $\mathcal{S}_0(F^n)$ be the subspace of $\Phi\in\mathcal{S}(F^n)$ for which $\Phi(0,0,\ldots,0)=0$. Suppose there exists a function in $\mathcal{J}(\pi)$ having a pole of order d_{s_0} at $s=s_0$. We investigate the rational function defined by an individual zeta integral $J(s,W,\Phi)$. Then the Laurent expansion about $s=s_0$ will take the form

$$J(s,W,\Phi)=\frac{B_{s_0}(W,\Phi)}{(q^s-q^{s_0})^{d_{s_0}}}+(\text{higher order terms}).$$

We define the Shalika subgroup S_{2n} of GL_{2n} by

$$S_{2n}=\left\{\begin{pmatrix}I_n & Z\\ & I_n\end{pmatrix}\begin{pmatrix}h & \\ & h\end{pmatrix}\;\middle|\; Z\in\mathcal{M}_n,\ h\in GL_n\right\}.$$

Let us denote an action of the Shalika subgroup S_{2n} on $\mathcal{S}(F^n)$ by

$$R\left(\begin{pmatrix}I_n & Z\\ & I_n\end{pmatrix}\begin{pmatrix}h & \\ & h\end{pmatrix}\right)\Phi(x)=\Phi(xh)$$

for $\Phi\in\mathcal{S}(F^n)$. The coefficient of the leading term, $B_{s_0}(W,\Phi)$, will define a nontrivial bilinear form on $\mathcal{W}(\pi,\psi)\times\mathcal{S}(F^n)$ enjoying the quasiinvariance

$$B_{s_0}\big(\varrho(g)W,R(g)\Phi\big)=|\det(h)|^{-s_0}\psi(\mathrm{Tr}(Z))B_{s_0}(W,\Phi)$$

for $g=\begin{pmatrix}I_n & Z\\ & I_n\end{pmatrix}\begin{pmatrix}h & \\ & h\end{pmatrix}\in S_{2n}$. The pole at $s=s_0$ of the family $\mathcal{J}(\pi)$ is called *exceptional* if the associated bilinear form $B_{s_0}(W,\Phi)$ vanishes identically on $\mathcal{W}(\pi,\psi)\times\mathcal{S}_0(F^n)$. If $s=s_0$ is an exceptional pole of $\mathcal{J}(\pi)$, then the bilinear form B_{s_0} factors to a nonzero bilinear form on $\mathcal{W}(\pi,\psi)\times\mathcal{S}(F^n)/\mathcal{S}_0(F^n)$. The quotient $\mathcal{S}(F^n)/\mathcal{S}_0(F^n)$ is isomorphic to $\mathbb{C}$ via the map $\Phi\mapsto\Phi(0)$. Let Θ be the character of S_{2n} given by

$$\Theta\left(\begin{pmatrix}I_n & Z\\ & I_n\end{pmatrix}\begin{pmatrix}h & \\ & h\end{pmatrix}\right)=\psi(\mathrm{Tr}(Z)).$$

We say that $\pi\in\mathcal{A}_F(2n)$ is (S_{2n},Θ)-*distinguished* if $\mathrm{Hom}_{S_{2n}}(\pi,\Theta)\neq\{0\}$. A nonzero linear functional Λ in $\mathrm{Hom}_{S_{2n}}(\pi,\Theta)$ (respectively, Λ_s in $\mathrm{Hom}_{S_{2n}}(\pi,\nu^{-s/2}\Theta)$) is called a *Shalika functional* (respectively, a *twisted Shalika functional*). If $s=s_0$ is an exceptional pole, then the form B_{s_0} can be written as $B_{s_0}(W, \Phi) = \Lambda_{s_0}(W)\Phi(0)$ with Λ_{s_0} the Shalika functional on $\mathcal{W}(\pi, \psi)$. Using the notation, we let

$$L_{\mathrm{ex}}(s, \pi, \wedge^2) = \prod_{s_0}(1 - q^{s_0}q^{-s})^{d_{s_0}},$$

where s_0 runs through all the exceptional poles of $\mathcal{J}(\pi)$ with d_{s_0} the maximal order of the pole at $s = s_0$. The factorization of local exterior square L-functions proposed by Cogdell and Piatetski-Shapiro asserts that it can be expressed in terms of the exceptional exterior square L-factors of the derivatives of π [Jo 2020b].

Theorem 2.3. *Let $\pi = \mathrm{Ind}_{\mathrm{Q}}^{\mathrm{GL}_m}(\Delta_1 \otimes \Delta_2 \otimes \cdots \otimes \Delta_t)$ be an irreducible generic representation of GL_m such that all the derivatives $\pi^{(k)}$ of π are completely reducible with irreducible generic constituents of the form $\pi_i^{(k)} = \mathrm{Ind}(\Delta_1^{(k_1)} \otimes \cdots \otimes \Delta_t^{(k_t)})$ with $k = k_1 + \cdots + k_t$. For each k, i is indexing the partition of k. Then*:

(i) $m = 2n$: $L(s, \pi, \wedge^2) = \mathrm{lcm}_{k,i}\left\{L_{\mathrm{ex}}(s, \pi_i^{(2k)}, \wedge^2)^{-1}\right\}$,

(ii) $m = 2n+1$: $L(s, \pi, \wedge^2) = \mathrm{lcm}_{k,i}\left\{L_{\mathrm{ex}}(s, \pi_i^{(2k+1)}, \wedge^2)^{-1}\right\}$,

where the least common multiple is with respect to divisibility in $\mathbb{C}[q^{\pm s}]$ and is taken over all k with $k = 0, 1, \ldots, n-1$ and for each k all constituents $\pi_i^{(2k)}$ (respectively, $\pi_i^{(2k+1)}$) of $\pi^{(2k)}$ (respectively, $\pi^{(2k+1)}$).

A similar definition for $L_{\mathrm{ex}}(s, \pi \times \sigma)$ and a factorization formula has been constructed by Cogdell and Piatetski-Shapiro in the context of local Rankin–Selberg L-functions for a pair of representations (π, σ) of GL_m [Cogdell and Piatetski-Shapiro 2017; Matringe 2015, §4.1].

2B. ***Classifications of distinguished representations.*** For $m = 2n$, we let M_{2n} denote the standard Levi subgroup of GL_{2n} associated with the partition (n, n) of $2n$. Let $w_{2n} = \sigma_{2n}$, and then we set $H_{2n} = w_{2n}M_{2n}w_{2n}^{-1}$. Let $w_{2n+1} = w_{2n+2}|_{\mathrm{GL}_{2n+1}}$ so that

$$w_{2n+1} = \left(\begin{array}{cccc|ccccc} 1 & 2 & \cdots & n+1 & n+2 & n+3 & \cdots & 2n & 2n+1 \\ 1 & 3 & \cdots & 2n+1 & 2 & 4 & \cdots & 2n-2 & 2n \end{array}\right).$$

In the odd case, $w_{2n+1} \neq \sigma_{2n+1}$, and we denote by M_{2n+1} the standard Levi subgroup attached to the partition $(n+1, n)$ of $2n+1$. We set $H_{2n+1} = w_{2n+1}M_{2n+1}w_{2n+1}^{-1}$. We observe that H_m is compatible in the sense that $H_m \cap \mathrm{GL}_{m-1} = H_{m-1}$. If α is a character of $F^\times$ and $\mathrm{diag}(g, g') \in M_m$, we denote by χ_α the character

$$\chi_\alpha : w_m \begin{pmatrix} g & \\ & g' \end{pmatrix} w_m^{-1} \mapsto \alpha\left(\frac{\det(g)}{\det(g')}\right)$$

of H_m. Let χ be a character of H_m. We say that $\pi \in \mathcal{A}_F(m)$ is (H_m, χ)-*distinguished* if $\mathrm{Hom}_{H_m}(\pi, \chi) \neq 0$. If χ is trivial, it is customary to say that π is H_m-*distinguished*. In order to classify all irreducible generic distinguished representations, we need to know that the induced representations of the form $\mathrm{Ind}_Q^{\mathrm{GL}_{2n}}(\Delta\otimes\tilde{\Delta})$ are distinguished. These types of properties over non-Archimedean local fields in characteristic zero were originally investigated by Cogdell and Piatetski-Shapiro [1994]. Afterwards the conjecture was settled by Matringe [2015; 2017]. Parts of the proof of [Matringe 2015, Proposition 3.8] contain inaccuracies, and subsequently it is clarified in [Matringe 2017, Proposition 5.3].

Proposition 2.4 (N. Matringe). *Let Δ be discrete series representations of* GL_n *and α a character of $F^\times$. Then irreducible generic representations of the form* $\mathrm{Ind}_Q^{\mathrm{GL}_{2n}}(\Delta\otimes\tilde{\Delta})$ *are both (H_{2n},χ_α)- and (S_{2n},Θ)-distinguished.*

Proof. We consider parabolically induced representations of the form

$$\Pi_s := \mathrm{Ind}_Q^{\mathrm{GL}_{2n}}(\Delta_0\nu^s\otimes\tilde{\Delta}_0\nu^{-s}),$$

with Δ_0 a unitary discrete series representations of GL_n and s a complex parameter. The proof in [Matringe 2015, Proposition 3.8] relies on Bernstein's analytic continuation principle for invariant linear forms. In order to apply it to positive characteristic, we need to explain that the space $\mathrm{Hom}_{S_{2n}}(\Pi_s,\Theta)$ is of dimension at most one for all s except the finite number for which Π_s is irreducible. However, if this is the case, $\mathrm{Hom}_{S_{2n}}(\Pi_s,\Theta)$ embeds as a subspace of $\mathrm{Hom}_{H_{2n}\cap P_{2n}}(\Pi_s,\mathbf{1}_{H_{2n}})$ via [Matringe 2014, Proposition 4.3] along with $\mathrm{Hom}_{S_{2n}}(\Pi_s,\Theta)\subseteq\mathrm{Hom}_{S_{2n}\cap P_{2n}}(\Pi_s,\Theta)$. Thanks to an auxiliary deformation parameter s, the proof of [Matringe 2015, Proposition 5.1-8] asserts that except for a finite number of s, the space $\mathrm{Hom}_{H_{2n}\cap P_{2n}}(\Pi_s,\mathbf{1}_{H_{2n}})$ is of dimension at most 1, as desired.

Alternatively, the quickest way is to use the equivalence between (H_{2n},χ_α)-distinctions and (S_{2n},Θ)-distinctions [Yang 2022, Corollary 3.6], which only depends on Gan's approach of theta correspondence [2019, Theorem 3.1]. This allows us to reduce to the case for $\alpha=0$, where the result is immediate from Blanc and Delorme [2008], as described in [Matringe 2014, §5]. We refer the interested reader to [Offen 2018, Proposition 3.2.15] for an expository construction of this open orbit contribution. □

We are now ready to introduce the classification of (H_{2n},χ_α)-distinguished representations that was established by Matringe [2015, Theorem 3.1]. The classification result holds in positive characteristic $p\neq 2$, though written in characteristic 0 only. Indeed, the proof relies crucially on Bernstein and Zelevinsky's version of Mackey's theorem [1977, Theorem 5.2], the explicit description of discrete series representations and their Jacquet modules [Zelevinsky 1980, Proposition 9.5], and the fact that a discrete series representation of GL_{2n+1} cannot be H_{2n+1}-distinguished [Matringe 2014, Theorem 3.1]. All the aforementioned properties are true in positive characteristic (see [Anandavardhanan et al. 2021, Appendix A] and [Gan 2019, §4]).

Theorem 2.5 (N. Matringe, $m = 2n$). *Let* $\pi = \mathrm{Ind}_{\mathrm{Q}}^{\mathrm{GL}_{2n}}(\Delta_1 \otimes \Delta_2 \otimes \cdots \otimes \Delta_t)$ *be an irreducible generic representation of* GL_{2n}. *Let* α *be a character of* $F^\times$ *with* $0 \le \mathrm{Re}(\alpha) \le \frac{1}{2}$. *Then* π *is* (H_{2n}, χ_α)*-distinguished if and only if there is a reordering of the* Δ_i *and an integer* r *between* 1 *and* $[t/2]$, *such that* $\Delta_{i+1} = \tilde{\Delta}_i$ *for* $i = 1, 3, \dots, 2r-1$, *and* Δ_i *is* (H_{2n_i}, χ_α)*-distinguished for* $i > 2r$.

For a discrete series representation Δ of GL_{2n}, Δ is H_{2n}-distinguished if and only if it is (S_{2n}, Θ)-distinguished. Matringe [2014, §5], using an analytic approach, and Gan [2019, Theorem 4.2], using the theta correspondence, individually settled this connection. Combining this with [Matringe 2017, Theorem 1.1 and Proposition 5.3], we classify the (S_{2n}, Θ)-distinguished generic representation of GL_{2n} in terms of (S_{2n_i}, Θ)-distinguished discrete series representations Δ_i [Matringe 2017, Corollary 1.1]. We refer the reader to [Matringe 2017] for further details of the proof.

Theorem 2.6 (N. Matringe). *Let* $\pi = \mathrm{Ind}_{\mathrm{Q}}^{\mathrm{GL}_{2n}}(\Delta_1 \otimes \Delta_2 \otimes \cdots \otimes \Delta_t)$ *be an irreducible generic representation of* GL_{2n}. *Then* π *is* (S_{2n}, Θ)*-distinguished if and only if there is a reordering of the* Δ_i *and an integer* r *between* 1 *and* $[t/2]$, *such that* $\Delta_{i+1} = \tilde{\Delta}_i$ *for* $i = 1, 3, \dots, 2r-1$, *and* Δ_i *is* (S_{2n_i}, Θ)*-distinguished for* $i > 2r$.

In the light of Theorem 2.5, Theorem 2.6, and [Gan 2019, Theorem 4.2], Matringe and Gan's equivalence is valid in more general setting of irreducible generic representations of GL_{2n}.

2C. ***Deformations and specializations.*** Let $\pi = \mathrm{Ind}_{\mathrm{Q}}^{\mathrm{GL}_m}(\Delta_1 \otimes \Delta_2 \otimes \cdots \otimes \Delta_t)$ be a parabolically induced representation of GL_m. Let $\mathcal{D}_\pi$ denote the complex manifold $\bigl(\mathbb{C}/(2\pi i/\log(q))\mathbb{Z}\bigr)^t$. The isomorphism $\mathcal{D}_\pi \to (\mathbb{C}^\times)^t$ is defined by

$$u := (u_1, u_2, \dots, u_t) \mapsto q^u := (q^{u_1}, q^{u_2}, \dots, q^{u_t}).$$

We use $q^{\pm u}$ as short for $(q^{\pm u_1}, q^{\pm u_2}, \dots, q^{\pm u_t})$. For $u \in \mathcal{D}_\pi$, we set

$$\pi_u = \mathrm{Ind}_{\mathrm{Q}}^{\mathrm{GL}_m}(\Delta_1 \nu^{u_1} \otimes \Delta_2 \nu^{u_2} \otimes \cdots \otimes \Delta_t \nu^{u_t}).$$

Let us set

$$\pi_u^{(k_1,k_2,\dots,k_t)} = \mathrm{Ind}_{\mathrm{Q}}^{\mathrm{GL}_m}(\Delta_1^{(k_1)} \nu^{u_1} \otimes \Delta_2^{(k_2)} \nu^{u_2} \otimes \cdots \otimes \Delta_t^{(k_t)} \nu^{u_t}).$$

Section 2C is indebted to Cogdell and Piatetski-Shapiro [2017], and we closely follow the path of the adaptation that was used in [Matringe 2009; 2015; Jo 2020a] to study the characteristic zero case. In particular, the deformation and specialization argument is widely available in the literature [Cogdell and Piatetski-Shapiro 2017; Matringe 2009; 2015; Jo 2020a]. Henceforth, we only remark on the nature of the difference but the reader should consult [Cogdell and Piatetski-Shapiro 2017; Matringe 2009] for complete details.

Definition 2.7. We say that $u = (u_1, u_2, \dots, u_t) \in \mathcal{D}_\pi$ is in *general position* if it satisfies the following properties:

(i) For every sequences of nonnegative integers $(k_1, k_2, \dots, k_t)$, a nonzero representation
$$\pi_u^{(k_1,k_2,\dots,k_t)} = \mathrm{Ind}\big(\Delta_1^{(k_1)}\nu^{u_1} \otimes \Delta_2^{(k_2)}\nu^{u_2} \otimes \cdots \otimes \Delta_t^{(k_t)}\nu^{u_t}\big)$$
is irreducible;

(ii) If $(a_1r_1, a_2r_2, \dots, a_tr_t)$ and $(b_1r_1, b_2r_2, \dots, b_tr_t)$ are two different sequences such that
$$\sum_{i=1}^{t} a_ir_i = \sum_{i=1}^{t} b_ir_i,$$
then two representations
$$\mathrm{Ind}\big(\Delta_1^{(a_1r_1)}\nu^{u_1} \otimes \Delta_2^{(a_2r_2)}\nu^{u_2} \otimes \cdots \otimes \Delta_t^{(a_tr_t)}\nu^{u_t}\big)$$
$$\mathrm{Ind}\big(\Delta_1^{(b_1r_1)}\nu^{u_1} \otimes \Delta_2^{(b_2r_2)}\nu^{u_2} \otimes \cdots \otimes \Delta_t^{(b_tr_t)}\nu^{u_t}\big)$$
possess distinct central characters;

(iii) If $(i, j, k, \ell) \in \{1, 2, \dots, t\}$, with $\{i, j\} \neq \{k, \ell\}$, then $L(s, \Delta_i\nu^{u_i} \times \Delta_j\nu^{u_j})$ and $L(s, \Delta_k\nu^{u_k} \times \Delta_\ell\nu^{u_\ell})$ have no common poles;

(iv) If $(i, j) \in \{1, 2, \dots, t\}$, with $i \neq j$, then $L(s, \Delta_i\nu^{u_i}, \wedge^2)$ and $L(s, \Delta_j\nu^{u_j}, \wedge^2)$ have no common poles;

(v) If $(i, j, k) \in \{1, 2, \dots, t\}$, with $i \neq j$, then $L(s, \Delta_i\nu^{u_i} \times \Delta_j\nu^{u_j})$ and $L(s, \Delta_k\nu^{u_k}, \wedge^2)$ have no common poles;

(vi) If $1 \le i \neq j \le t$ and $(\Delta_i^{(a_ir_i)})^\sim \simeq \Delta_j^{(a_jr_j)}\nu^e$ for some complex number e, then the dimension of the space
$$\mathrm{Hom}_{P_{2(n_i-a_ir_i)} \cap S_{2(n_i-a_ir_i)}}\big(\mathrm{Ind}\big(\Delta_i^{(a_ir_i)}\nu^{(u_i+u_j+e)/2} \otimes (\Delta_i^{(a_ir_i)}\nu^{(u_i+u_j+e)/2})^\sim\big), \Theta\big)$$
is at most 1.

We confirm that off a finite number of hyperplanes in u, the deformed representation π_u is in general position [Jo 2020a, Proposition 4.1]. The important point is that $u \in \mathcal{D}_\pi$ in general position depends only on the representation π, not $s \in \mathbb{C}$. The purpose of (ii) is that outside a finite number of hyperplanes, the central character of $\pi_u^{(a_1r_1,a_2r_2,\dots,a_tr_t)}$ are distinct and therefore there are only trivial extensions among these representation. As a result, off these hyperplanes, the derivatives $\pi_u^{(k)} = \oplus\pi_u^{(a_1r_1,a_2r_2,\dots,a_tr_t)}$ are completely reducible, where $k = \sum_{i=1}^{t} a_ir_i$ and each $\pi_u^{(a_1r_1,a_2r_2,\dots,a_tr_t)}$ are irreducible. Conditions (i) and (ii) ensure that Theorem 2.3 is applicable. The purpose of Condition (vi) is that the occurrence of the exceptional pole of $L(s, \pi, \wedge^2)$ at $s = 0$ can be determined by the existence of Shalika functional from [Jo 2020a, Lemma 3.2]. Throughout Section 2C, we assume the working hypothesis proposed by E. Kaplan [2017, Remark 4.18] for fields of odd characteristic.

Working Hypothesis. Let Δ be an (S_{2n}, Θ)-distinguished discrete series representation of GL_{2n}. Then Δ is self-dual. Namely, $\tilde{\Delta} \simeq \Delta$.

The following statement is a consequence of the working hypothesis along with Theorem 2.6:

Corollary 2.8. *Assume the working hypothesis. Let $\pi = \mathrm{Ind}_{\mathrm{Q}}^{\mathrm{GL}_{2n}}(\Delta_1 \otimes \Delta_2 \otimes \cdots \otimes \Delta_t)$ be an irreducible generic representation of GL_{2n}. If π is (S_{2n}, Θ)-distinguished, then π is self-dual. Namely, $\tilde{\pi} \simeq \pi$.*

The working hypothesis needs not be considered for the subclass of irreducible principal series representations induced from Borel subgroups due to Theorem 2.6, and we shall verify the presumption case-by-case in Section 2.

Proposition 2.9. *Let $\pi = \mathrm{Ind}_{B_{2n}}^{\mathrm{GL}_{2n}}(\chi_1 \otimes \chi_2 \otimes \cdots \otimes \chi_{2n})$ be a (S_{2n}, Θ)-distinguished irreducible principal series representation of GL_{2n}. Then π is self-dual. Namely, $\tilde{\pi} \simeq \pi$.*

Now we provide an interpretation of Theorem 2.6 in terms of local L-functions, which is analogous to [Matringe 2015, Proposition 4.13].

Proposition 2.10. *Let $\pi = \mathrm{Ind}_{\mathrm{Q}}^{\mathrm{GL}_{2n}}(\Delta_1 \otimes \Delta_2 \otimes \cdots \otimes \Delta_t)$ be an irreducible generic representation of GL_{2n}, where each Δ_i is a discrete series representation of GL_{n_i} with $2n = \sum_{i=1}^t n_i$ and $t \geq 2$. Suppose that $L_{\mathrm{ex}}(s, \pi, \wedge^2)$ has a pole at $s = s_0$. Then we are in one of the following cases*:

(i) *There are $(i, j) \in \{1, 2, \ldots, t\}$, with $i \neq j$, such that n_i and n_j are even, and $L_{\mathrm{ex}}(s, \Delta_i, \wedge^2)$ and $L_{\mathrm{ex}}(s, \Delta_j, \wedge^2)$ have $s = s_0$ as a common pole.*

(ii) *There are $(i, j, k, \ell) \in \{1, 2, \ldots, t\}$, with $\{i, j\} \neq \{k, \ell\}$, such that $L_{\mathrm{ex}}(s, \Delta_i \times \Delta_j)$ and $L_{\mathrm{ex}}(s, \Delta_k \times \Delta_\ell)$ have $s = s_0$ as a common pole.*

(iii) *There are $(i, j, k) \in \{1, 2, \ldots, t\}$, with $i \neq j$, such that n_k is even and $L_{\mathrm{ex}}(s, \Delta_i \times \Delta_j)$ and $L_{\mathrm{ex}}(s, \Delta_k, \wedge^2)$ have $s = s_0$ as a common pole.*

Proof. Suppose that $L_{\mathrm{ex}}(s, \pi, \wedge^2)$ has a pole at $s = s_0$. Since $L(s, \pi, \wedge^2) = L(s - s_0, \pi\nu^{s_0/2}, \wedge^2)$, the representation $\pi\nu^{s_0/2}$ admits a nontrivial Shalika functional. We know from Theorem 2.6 that $\pi\nu^{s_0/2}$ is isomorphic to

$$\mathrm{Ind}\Big(\big(\Delta_{i_1}\nu^{s_0/2} \otimes (\Delta_{i_1}\nu^{s_0/2})^{\sim}\big) \otimes \cdots \otimes \big(\Delta_{i_r}\nu^{s_0/2} \otimes (\Delta_{i_r}\nu^{s_0/2})^{\sim}\big) \otimes \Delta_{i_{r+1}}\nu^{s_0/2} \otimes \cdots \otimes \Delta_{i_t}\nu^{s_0/2}\Big),$$

with $0 \leq r \leq [t/2]$, where $\Delta_{i_j}\nu^{s_0/2}$ affords a Shalika functional and each n_{i_j} is even for all $j > r$. Putting it in a different way, $(\Delta_i\nu^{s_0/2})^{\sim} \simeq \Delta_j\nu^{s_0/2}$ with $i \neq j$, or $\Delta_k\nu^{s_0/2}$ owns a Shalika functional, where n_k is an even number.

According to [Matringe 2015, Proposition 4.6], $(\Delta_i \nu^{s_0/2})^{\sim} \simeq \Delta_j \nu^{s_0/2}$ or equivalently $\tilde{\Delta}_i \simeq \Delta_j \nu^{s_0}$ if and only if $L_{\mathrm{ex}}(s, \Delta_i \times \Delta_j)$ has a pole at $s = s_0$.

If $\Delta_k \nu^{s_0/2}$ has the Shalika functional, the space $\mathrm{Hom}_{S_{n_k}}(\Delta_k \nu^{s_0/2}, \Theta)$ is nontrivial and its central character $\omega_{\Delta_k \nu^{s_0/2}}$ is trivial. Since $\Delta_k \nu^{s_0/2}$ is the irreducible square integrable representation, we obtain from [Jo 2020a, Proposition 3.4] that $L_{\mathrm{ex}}(s, \Delta_k \nu^{s_0/2}, \wedge^2)$ has a pole at $s = 0$, or equivalently, $L_{\mathrm{ex}}(s, \Delta_k, \wedge^2)$ has a pole at $s = s_0$. Therefore $s = s_0$ is the common pole for either of three cases in Proposition 2.10. □

Let Δ be a discrete series representation. Such a representation Δ is the unique irreducible quotient of the form: $\mathrm{Ind}_{\mathrm{Q}}^{\mathrm{GL}_m}(\rho \otimes \rho\nu \otimes \cdots \otimes \rho\nu^{\ell-1})$, where the induction is a normalized parabolic induction from the standard parabolic subgroup Q attached to the partition $(r, r, \ldots, r)$ of $m = r\ell$ and $\rho \in \mathcal{A}_F(r)$ is irreducible and supercuspidal [Zelevinsky 1980]. We denote by $\Delta = [\rho, \rho\nu, \ldots, \rho\nu^{\ell-1}]$ such a quotient. Using Hartogs' theorem [Jo 2020a] is closer to the original spirit of the direction in [Cogdell and Piatetski-Shapiro 2017]. Nevertheless, we present an alternative approach employing Proposition 2.10 to keep uniformity with [Matringe 2009; 2015].

Proposition 2.11. *Assume the working hypothesis, and let us denote by* $\pi = \mathrm{Ind}_{\mathrm{Q}}^{\mathrm{GL}_m}(\Delta_1 \otimes \Delta_2 \otimes \cdots \otimes \Delta_t)$ *a parabolically induced representation of* GL_m. *Let* $u = (u_1, u_2, \ldots, u_t) \in \mathcal{D}_\pi$ *be in general position, and let*

$$\pi_u = \mathrm{Ind}_{\mathrm{Q}}^{\mathrm{GL}_m}(\Delta_1 \nu^{u_1} \otimes \Delta_2 \nu^{u_2} \otimes \cdots \otimes \Delta_t \nu^{u_t})$$

be the deformed representation. Then we have the following:

(i) $L(s, \pi_u, \wedge^2) = \prod_{1 \le k \le t} L(s + 2u_k, \Delta_k, \wedge^2) \prod_{1 \le i < j \le t} L(s + u_i + u_j, \Delta_i \times \Delta_j)$.

(ii) *There is a polynomial* $Q(X) \in \mathbb{C}[X]$ *such that*

$$L(s, \pi, \wedge^2) = Q(q^{-s}) \prod_{1 \le k \le t} L(s, \Delta_k, \wedge^2) \prod_{1 \le i < j \le t} L(s, \Delta_i \times \Delta_j).$$

Proof. Let us take Δ_i to be associated to the segment $[\rho_i, \rho_i\nu, \ldots, \rho_i\nu^{\ell_i - 1}]$, with ρ_i an irreducible supercuspidal representation of GL_{r_i}, $m_i = r_i \ell_i$, and $m = \sum_{i=1}^{t} r_i \ell_i$. Keeping Theorem 2.3 in mind, we set

$$L(s, \pi_u, \wedge^2)^{-1} = \mathrm{lcm}\left\{ L_{\mathrm{ex}}(s, \pi_u^{(a_1 r_1, a_2 r_2, \ldots, a_t r_t)}, \wedge^2)^{-1} \right\},$$

where $0 \le a_i \le \ell_i$, $m - \sum_{i=1}^{t} a_i r_i$ is an even number, and the least common multiple is taken in terms of divisibility in $\mathbb{C}[q^{\pm s}]$. Suppose that $L_{\mathrm{ex}}(s, \pi_u^{(a_1 r_1, a_2 r_2, \ldots, a_t r_t)}, \wedge^2)$ has a pole at $s = s_0$. If the number of indices i such that $r_i \ne \ell_i$ is more than 3, we deduce from Proposition 2.10 that:

(i) There are $(i, j) \in \{1, 2, \ldots, t\}$, with $i \ne j$, such that $m_i - a_i r_i$ and $m_j - a_j r_j$ are even, and $L\big(s, \Delta_i^{(a_i r_i)} \nu^{u_i}, \wedge^2\big)$ and $L\big(s, \Delta_j^{(a_j r_j)} \nu^{u_j}, \wedge^2\big)$ have $s = s_0$ as a common pole.

(ii) There are $(i, j, k, \ell) \in \{1, 2, \dots, t\}$, with $\{i, j\} \neq \{k, \ell\}$, such that the functions $L\big(s, \Delta_i^{(a_i r_i)} \nu^{u_i} \times \Delta_j^{(a_j r_j)} \nu^{u_j}\big)$ and $L\big(s, \Delta_k^{(a_k r_k)} \nu^{u_k} \times \Delta_\ell^{(a_\ell r_\ell)} \nu^{u_\ell}\big)$ have $s = s_0$ as a common pole.

(iii) There are $(i, j, k) \in \{1, 2, \dots, t\}$, with $i \neq j$, such that $m_k - a_k r_k$ is even, and $L\big(s, \Delta_i^{(a_i r_i)} \nu^{u_i} \times \Delta_j^{(a_j r_j)} \nu^{u_j}\big)$ and $L\big(s, \Delta_k^{(a_k r_k)} \nu^{u_k}, \wedge^2\big)$ have $s = s_0$ as a common pole.

However, Conditions (iii), (iv), and (v) of general positions ensure that the above scenario cannot happen as long as u is in general position, because exceptional poles are poles of original L-factors $L(s, \Delta_i \times \Delta_j)$ and $L(s, \Delta_k, \wedge^2)$. Owing to [Jo 2020a, Corollary 4.11], when there exists exactly one pair (i, j) of indices $i \neq j$ such that $r_i \neq \ell_i$ and $r_j \neq \ell_j$, we have

$$L_{\mathrm{ex}}\big(s, \mathrm{Ind}(\Delta_i^{(a_i r_i)} \nu^{u_i} \otimes \Delta_j^{(a_j r_j)} \nu^{u_j}), \wedge^2\big) = L_{\mathrm{ex}}\big(s, \Delta_i^{(a_i r_i)} \nu^{u_i} \times \Delta_j^{(a_j r_j)} \nu^{u_j}\big).$$

If i is the only index such that $r_i \neq \ell_i$, it is nothing but $L_{\mathrm{ex}}(s, \Delta_i^{(a_i r_i)} \nu^{u_i}, \wedge^2)$.

To summarize, $L_{\mathrm{ex}}(s, \pi_u^{(a_1 r_1, a_2 r_2, \dots, a_t r_t)}, \wedge^2)$ is equal to $L_{\mathrm{ex}}(s, \Delta_i^{(a_i r_i)} \nu^{u_i} \times \Delta_j^{(a_j r_j)} \nu^{u_j})$ for $i < j$ or $L_{\mathrm{ex}}(s, \Delta_i^{(a_i r_i)} \nu^{u_i}, \wedge^2)$. Following the rest of the proof in [Jo 2020a, Theorem 5.1], we arrive at

$$\begin{aligned} L(s, \pi_u, \wedge^2) &= \prod_{1 \le k \le t} L(s, \Delta_k \nu^{u_k}, \wedge^2) \prod_{1 \le i < j \le t} L(s, \Delta_i \nu^{u_i} \times \Delta_j \nu^{u_j}) \\ &= \prod_{1 \le k \le t} L(s + 2u_k, \Delta_k, \wedge^2) \prod_{1 \le i < j \le t} L(s + u_i + u_j, \Delta_i \times \Delta_j). \end{aligned}$$

Concerning the second part, let $\mathcal{W}_\pi^{(0)}$ be the Whittaker model associated to π_u [Cogdell and Piatetski-Shapiro 2017, §3.1]. For $W_u \in \mathcal{W}_\pi^{(0)}$, it follows from the standard Bernstein's principle of meromorphic continuation and rationality [Jo 2020a, Propositions 4.2 and 4.4] that $J(s, W_u, \Phi)$ defines a rational function in $\mathbb{C}(q^{-s}, q^{-u})$. We conclude (i), that the rational function

$$\frac{J(s, W_u, \Phi)}{\prod_{1 \le k \le t} L(s + 2u_k, \Delta_k, \wedge^2) \prod_{1 \le i < j \le t} L(s + u_i + u_j, \Delta_i \times \Delta_j)}$$

has no poles on the Zariski open set of u in general position. We can take one step further to assert that the ratio lies in $\mathbb{C}[q^{\pm s}, q^{\pm u}]$ by the proof of [Matringe 2015, Lemma 5.1] and [Jo 2020a, Proposition 5.3]. The statement is now an immediate consequence of specialization to $u = 0$. □

We denote by $P \sim Q$ that the ratio is a unit in $\mathbb{C}[q^{\pm s}]$ for two rational functions $P(q^{-s})$ and $Q(q^{-s})$ in $\mathbb{C}(q^{-s})$. As alluded in the Langlands–Shahidi method [Ganapathy and Lomelí 2015; Henniart and Lomelí 2011; 2013b; Lomelí 2016], the unit emerging in Theorem 2.12 (ii) will be presumably 1. This is so-called the *multiplicativity of γ-factors*. However, demonstrating the multiplicativity property requires manipulating integrals in a delicate manner. Nonetheless, it seems likely that the weaker one that is relevant to us is enough for the application therein.

Theorem 2.12. *Assume the working hypothesis. Let* $\pi = \mathrm{Ind}_{\mathrm{Q}}^{\mathrm{GL}_m}(\Delta_1 \otimes \Delta_2 \otimes \cdots \otimes \Delta_t)$ *be a parabolically induced representation of* GL_m. *Let* $u = (u_1, u_2, \ldots, u_t) \in \mathcal{D}_\pi$ *be in general position and* $\pi_u = \mathrm{Ind}_{\mathrm{Q}}^{\mathrm{GL}_m}(\Delta_1 \nu^{u_1} \otimes \Delta_2 \nu^{u_2} \otimes \cdots \otimes \Delta_t \nu^{u_t})$ *be the deformed representation. Then we have the following*:

(i) $\gamma(s, \pi_u, \wedge^2, \psi) \sim \prod_{1 \le k \le t} \gamma(s+2u_k, \Delta_k, \wedge^2, \psi) \prod_{1 \le i < j \le t} \gamma(s+u_i+u_j, \Delta_i \times \Delta_j, \psi)$,

(ii) $\gamma(s, \pi, \wedge^2, \psi) \sim \prod_{1 \le k \le t} \gamma(s, \Delta_k, \wedge^2, \psi) \prod_{1 \le i < j \le t} \gamma(s, \Delta_i \times \Delta_j, \psi)$.

Proof. The proof proceeds along the line of [Jo 2020a, Proposition 5.4] and [Matringe 2015, Proposition 5.5] by applying Theorem 2.2 and Proposition 2.11 to our framework, and this idea originated from Cogdell and Piatetski-Shapiro [2017, Proposition 4.3]. Statement (ii) can be shown by specializing to $u = 0$. □

To proceed further, we adopt the terminology from [Cogdell and Piatetski-Shapiro 2017; Matringe 2015]. We say that $\pi \in \mathcal{A}_F(m)$ is a representation of *Langlands type* if Ξ has the form $\mathrm{Ind}_{\mathrm{Q}}^{\mathrm{GL}_m}(\Delta_{\circ 1} \nu^{u_1} \otimes \Delta_{\circ 2} \nu^{u_2} \otimes \cdots \otimes \Delta_{\circ t} \nu^{u_t})$, where each $\Delta_{\circ i}$ is the irreducible square integrable representation of GL_{m_i}, $m_1 + m_2 + \cdots + m_t = m$, each u_i is real, and they are ordered so that $u_1 \ge u_2 \ge \cdots \ge u_t$. Let π be an irreducible admissible representation of GL_m. Regardless of being generic, π can be realized as the unique Langlands quotient of Langlands type $\Xi = \mathrm{Ind}_{\mathrm{Q}}^{\mathrm{GL}_m}(\Delta_{\circ 1} \nu^{u_1} \otimes \Delta_{\circ 2} \nu^{u_2} \otimes \cdots \otimes \Delta_{\circ t} \nu^{u_t})$ which is of Whittaker type. The exterior square L-factor is defined to be

$$L(s, \Xi, \wedge^2) = L(s, \pi, \wedge^2).$$

Theorem 2.13. *Assume the working hypothesis. Consider a representation of Langlands type of* GL_m, $\pi = \mathrm{Ind}_{\mathrm{Q}}^{\mathrm{GL}_m}(\Delta_{\circ 1} \nu^{u_1} \otimes \Delta_{\circ 2} \nu^{u_2} \otimes \cdots \otimes \Delta_{\circ t} \nu^{u_t})$. *Then we have*

$$L(s, \pi, \wedge^2) = \prod_{1 \le k \le t} L(s + 2u_k, \Delta_{\circ k}, \wedge^2) \prod_{1 \le i < j \le t} L(s + u_i + u_j, \Delta_{\circ i} \times \Delta_{\circ j}).$$

Proof. The proof is akin to those of [Cogdell and Piatetski-Shapiro 2017, Theorem 4.1], [Jo 2020a, Theorem 5.7], and [Matringe 2009, Theorem 4.26]. In order to be concise, we do not include the complete details. □

We pass to the case of irreducible generic representations.

Corollary 2.14. *Assume the working hypothesis. Let* $\pi = \mathrm{Ind}_{\mathrm{Q}}^{\mathrm{GL}_m}(\Delta_1 \otimes \Delta_2 \otimes \cdots \otimes \Delta_t)$ *be an irreducible generic representation of* GL_m. *Then we have*

$$L(s, \pi, \wedge^2) = \prod_{1 \le k \le t} L(s, \Delta_k, \wedge^2) \prod_{1 \le i < j \le t} L(s, \Delta_i \times \Delta_j).$$

Proof. Since π is irreducible, essentially square integrable representations Δ_i can be rearranged to be in Langlands order without changing π. □

We define the symmetric square L-factor to be the ratio of Rankin–Selberg L-factors for $\mathrm{GL}_m \times \mathrm{GL}_m$ by exterior square L-factors for GL_m:

$$L(s,\pi,\mathrm{Sym}^2)=\frac{L(s,\pi\times\pi)}{L(s,\pi,\wedge^2)}. \tag{2-1}$$

In comparison to [Matringe 2009; 2015], we pursue purely local means more to express a local exterior square L-function in terms of local L-functions for supercuspidal representations. Performing this step has the benefit of making the globalization result of Henniart and Lomelí [2011; 2013b] feasible, instead of globalizing discrete series representations [Kaplan 2017; Kewat and Raghunathan 2012; Matringe 2009] as a black box.

Theorem 2.15. *Assume that the working hypothesis holds for the subclass of all irreducible supercuspidal representations. Let* $\Delta_\circ=\big[\rho_\circ\nu^{-(\ell-1)/2},\dots,\rho_\circ\nu^{(\ell-1)/2}\big]$ *be an irreducible square integrable representation of* $\mathrm{GL}_{\ell r}$, *with* $\rho_\circ$ *an irreducible unitary supercuspidal representation of* GL_r.

(i) *Suppose that* ℓ *is even. Then we have*

$$L(s,\Delta_\circ,\wedge^2)=\prod_{i=1}^{\ell/2}L\big(s,\rho_\circ\nu^{(\ell+1)/2-i},\wedge^2\big)L\big(s,\rho_\circ\nu^{\ell/2-i},\mathrm{Sym}^2\big);$$
$$L(s,\Delta_\circ,\mathrm{Sym}^2)=\prod_{i=1}^{\ell/2}L\big(s,\rho_\circ\nu^{(\ell+1)/2-i},\mathrm{Sym}^2\big)L\big(s,\rho_\circ\nu^{\ell/2-i},\wedge^2\big).$$

(ii) *Suppose that* ℓ *is odd. Then we have*

$$L(s,\Delta_\circ,\wedge^2)=\prod_{i=1}^{(\ell+1)/2}L\big(s,\rho_\circ\nu^{(\ell+1)/2-i},\wedge^2\big)\prod_{i=1}^{(\ell-1)/2}L\big(s,\rho_\circ\nu^{\ell/2-i},\mathrm{Sym}^2\big);$$
$$L(s,\Delta_\circ,\mathrm{Sym}^2)=\prod_{i=1}^{(\ell+1)/2}L\big(s,\rho_\circ\nu^{(\ell+1)/2-i},\mathrm{Sym}^2\big)\prod_{i=1}^{(\ell-1)/2}L\big(s,\rho_\circ\nu^{\ell/2-i},\wedge^2\big);$$

Proof. Our proof is truly influenced by Shahidi [1992, Proposition 8.1]. By the uniqueness of the Whittaker functional, the Whittaker model for $\Delta_\circ$ agrees with that for $\xi=\mathrm{Ind}\big(\rho_\circ\nu^{-(\ell-1)/2}\otimes\cdots\otimes\rho_\circ\nu^{(\ell-1)/2}\big)$. Likewise the same feature holds for $\xi^\iota:=\mathrm{Ind}\big(\tilde\rho_\circ\nu^{-(\ell-1)/2}\otimes\cdots\otimes\tilde\rho_\circ\nu^{(\ell-1)/2}\big)$ and $\tilde\Delta_\circ$. This puts us in a position to manifest that

$$\gamma(s,\Delta_\circ,\wedge^2,\psi)=\gamma\big(s,\mathrm{Ind}\big(\rho_\circ\nu^{-(\ell-1)/2}\otimes\cdots\otimes\rho_\circ\nu^{(\ell-1)/2}\big),\wedge^2,\psi\big).$$

Let u be in general position and $\xi_u = \operatorname{Ind}\big(\rho_\circ \nu^{u_1-(\ell-1)/2} \otimes \cdots \otimes \rho_\circ \nu^{u_\ell+(\ell-1)/2}\big)$ its associated deformed representation. Upon noting the assumption that any (S_{2n}, Θ)-distinguished irreducible supercuspidal representation ρ is self-dual, we see that Proposition 2.10 to Theorem 2.12 can be completely carried over verbatim to the triple $(\Delta_\circ, \xi, \xi_u)$. The remainder of the proof is parallel to that of [Jo 2020a, Theorem 5.12] (cf. proof of Proposition 4.3), and we find

$$L(s, \Delta_\circ, \wedge^2) = \begin{cases} \displaystyle\prod_{i=1}^{\ell/2} L\big(s, \rho_\circ \nu^{(\ell+1)/2-i}, \wedge^2\big) L\big(s, \rho_\circ \nu^{\ell/2-i}, \mathrm{Sym}^2\big), & \ell \text{ even}, \\ \displaystyle\prod_{i=1}^{(\ell+1)/2} L\big(s, \rho_\circ \nu^{(\ell+1)/2-i}, \wedge^2\big) \prod_{i=1}^{(\ell-1)/2} L\big(s, \rho_\circ \nu^{\ell/2-i}, \mathrm{Sym}^2\big), & \ell \text{ odd}. \end{cases}$$

The expression of the local symmetric square L-function $L(s, \Delta_\circ, \mathrm{Sym}^2)$ is a direct consequence of the factorization $L(s, \Delta_\circ \times \Delta_\circ) = L(s, \Delta_\circ, \wedge^2) L(s, \Delta_\circ, \mathrm{Sym}^2)$, just as in (2-1). □

2D. ***The equality for principal series representations.*** We briefly review the Langlands–Shahidi method for the local exterior square L-function [Ganapathy and Lomelí 2015; Henniart and Lomelí 2011]. Let $\boldsymbol{G} = Sp_{2m}$ be a symplectic group over F in $2m$ variables. The group $\boldsymbol{M} \simeq \mathrm{GL}_m$ can be embedded as a Levi component of a maximal Siegel parabolic subgroup $\boldsymbol{P} = \boldsymbol{M}\boldsymbol{N}$ with unipotent radical $\boldsymbol{N}$. Let r be the adjoint representation of the L-group of $\boldsymbol{M}$ on ${}^L\mathfrak{n}$, the Lie algebra of the L-group of $\boldsymbol{N}$. We can check that $r = r_1 \oplus r_2$. The irreducible representation r_1 gives the standard γ-factor of GL_n and r_2 gives the *Langlands–Shahidi exterior square γ-factor*,

$$\gamma(s, \pi, r_2, \psi) = \gamma_{LS}(s, \pi, \wedge^2, \psi).$$

The γ-factor $\gamma_{LS}(s, \pi, \wedge^2, \psi)$ defined in [Henniart and Lomelí 2011] is a rational function in $\mathbb{C}(q^{-s})$. Let $P(X)$ be the unique polynomial in $\mathbb{C}[X]$ satisfying $P(0) = 1$ and such that $P(q^{-s})$ is the numerator of $\gamma_{LS}(s, \pi, \wedge^2, \psi)$. Whenever π is tempered, the *local Langlands–Shahidi exterior square L-function* is defined by

$$\mathcal{L}(s, \rho, \wedge^2) := P(q^{-s})^{-1}.$$

We observe that π tempered implies that $\mathcal{L}(s, \rho, \wedge^2)$ is holomorphic for $\mathrm{Re}(s) > 0$ [Henniart and Lomelí 2011, §4.6]. The *Langlands–Shahidi exterior square ε-factor* is defined to satisfy the relation

$$\varepsilon_{LS}(s, \pi, \wedge^2, \psi) = \gamma_{LS}(s, \pi, \wedge^2, \psi) \frac{\mathcal{L}(1-s, \tilde{\pi}, \wedge^2)}{\mathcal{L}(s, \pi, \wedge^2)}.$$

Besides, various types of L-factors $\mathcal{L}(s, \pi, \mathrm{Sym}^2)$ for $\boldsymbol{G} = SO_{2m+1}$, $\mathcal{L}(s, \pi \times \pi)$ for $\boldsymbol{G} = \mathrm{GL}_{2m}$, and $\mathcal{L}(s, \pi, \mathrm{As})$ for $\boldsymbol{G} = U_m$, can be extracted from [Henniart and Lomelí 2013b; Lomelí 2016].

Proposition 2.16. *Let Δ be a discrete series representation of the form*

$$[\chi, \chi\nu, \dots, \chi\nu^{\ell-1}], \tag{2-2}$$

where χ is a character of $F^{\times}$. Then we have

$$L(s, \Delta, \wedge^2) = \mathcal{L}(s, \Delta, \wedge^2).$$

As a consequence, if $\ell = 2n$ is even and Δ is (S_{2n}, Θ)-distinguished, then Δ is self-dual.

Proof. Just as observed in Proposition 2.9, the working hypothesis does not need to be checked for the character χ of $F^{\times}$, and Δ is automatically self-dual. As in the proof of Theorem 4.4, we can easily reduce it to the case where Δ is a unitary representation. We are then left with applying Theorem 2.15 to Δ, from which the equality shall follow by comparing it with the work of Shahidi [1992, Proposition 8.1]. □

Let us turn our attention to the subclass of irreducible generic subquotients of principal series representations. This class is not necessarily spherical.

Proposition 2.17. *Let π be an irreducible generic subquotient of a principal series representation of* GL_m. *Then we have*

$$L(s, \pi, \wedge^2) = \mathcal{L}(s, \pi, \wedge^2).$$

Proof. From [Bernstein and Zelevinsky 1977; Zelevinsky 1980], π is of the form $\mathrm{Ind}(\Delta_1 \otimes \Delta_2 \otimes \cdots \otimes \Delta_t)$, where each Δ_i is either a character χ_i of $F^{\times}$ or a discrete series representation given by the segment of the form (2-2). In considering Proposition 2.16, any (S_{2n_i}, Θ)-distinguished representations Δ_i satisfy the working hypothesis. The inductive relation formula, Corollary 2.14, is applicable, and it can be shown that

$$L(s, \pi, \wedge^2) = \prod_{1 \le k \le t} L(s, \Delta_k, \wedge^2) \prod_{1 \le i < j \le t} L(s, \Delta_i \times \Delta_j).$$

In the aspect of Proposition 2.16, we only need to compare it with [Ganapathy and Lomelí 2015, Theorem 3.1 (xi)]. □

The unramified character χ means that it is invariant under the maximal compact subgroup $\mathcal{O}^{\times}$ of $F^{\times}$. As before, the working hypothesis is no longer needed for the set of irreducible unramified representations. Hence, Corollary 2.14 in the preceding section Section 2C, has the following result:

Corollary 2.18. *Let $\pi = \mathrm{Ind}_{B_m}^{\mathrm{GL}_m}(\chi_1 \otimes \chi_2 \otimes \cdots \otimes \chi_m)$ be an irreducible full induced representation from the Borel subgroup of unramified character χ_i of $F^{\times}$. Then*

$$L(s, \pi, \wedge^2) = \prod_{1 \le i < j \le m} \frac{1}{1 - \chi_i(\varpi)\chi_j(\varpi)q^{-s}}.$$

3. Local to global argument

3A. ***Eulerian integral representations.*** We denote by $\mathbb{F}_q$ the residue field of F, and let $k = \mathbb{F}_q(t)$ be a (global) function field of the projective line $\mathbb{P}^1$ over $\mathbb{F}_q$. Let $\mathbb{A}$ denote its ring of adèles. Let (Π, V_Π) be a cuspidal automorphic representation of $\mathrm{GL}_m(\mathbb{A})$. We denote by $|\mathbb{P}^1|$ the set of closed points of $\mathbb{P}^1$. The set $|\mathbb{P}^1|$ is in bijection with the set of places of k. Hence we write by abuse of notation $|\mathbb{P}^1|$ for the set of places of k. Since Π is irreducible, we have restricted tensor product decomposition $\Pi = \bigotimes_v' \Pi_v$ with (Π_v, V_{Π_v}) irreducible admissible generic representations of $\mathrm{GL}_m(k_v)$ [Flath 1979], see [Cogdell 2003, §4]. Let its central character be ω_Π. We let $P_{n-1,1} = Z_n P_n$ be the standard parabolic subgroup associated to the partition $(n-1, 1)$ of n. Each $\Phi \in \mathcal{S}(\mathbb{A}^n)$ defines a smooth function on $\mathrm{GL}_n(\mathbb{A})$, left invariant by $P_n(\mathbb{A})$, by $g \mapsto \Phi(e_n g)$ for $g \in \mathrm{GL}_n(\mathbb{A})$. We consider the function

$$f(s, g; \Phi, \omega_\Pi) = |\det(g)|^s \int_{\mathbb{A}^\times} \omega_\Pi(z)\Phi(z e_n g)|z|^{ns}\, d^\times z,$$

with the absolute convergence of the integral [Jacquet and Shalika 1981, (4.1)]. We extend ω_π to a character of $P_{n-1,1}$ by $\omega_\Pi(p) = \omega_\Pi(a)$ for $p = \begin{pmatrix} h & u \\ & a \end{pmatrix} \in P_{n-1,1}$. We construct the Eisenstein series by

$$E(s, g; \Phi, \omega_\Pi) = \sum_{\gamma \in P_{n-1,1}(k)\backslash \mathrm{GL}_n(k)} F(s, \gamma g; \Phi, \omega_\Pi)$$

This series is convergent absolutely for $\mathrm{Re}(s) > 1$ [Jacquet and Shalika 1981, (4.1)]. The mirabolic (Godement–Jacquet) Eisenstein series $E(s, g; \Phi, \omega_\Pi)$ has a meromorphic continuation to all of $\mathbb{C}$ and satisfies the following functional equation [Jacquet and Shalika 1981, §4]:

$$E(s, g; \Phi, \omega_\Pi) = E(1-s, {}^\iota g; \hat{\Phi}, \omega_\Pi^{-1}), \tag{3-1}$$

where ${}^\iota g = {}^t g^{-1}$ and the Fourier transform on $\mathcal{S}(\mathbb{A}^n)$ is defined by

$$\hat{\Phi}(y) = \int_{\mathbb{A}^n} \Phi(x)\psi(x\, {}^t y)\, dx.$$

For $m = 2n$, $\Phi \in \mathcal{S}(\mathbb{A}^n)$, and $\varphi \in V_\Pi$, we let

$$I_\psi(s, \varphi, \Phi) = \int_{Z_n(k)\,\mathrm{GL}_n(k)\backslash \mathrm{GL}_n(\mathbb{A})} \int_{\mathcal{M}_n(k)\backslash \mathcal{M}_n(\mathbb{A})} \varphi\left(\begin{pmatrix} I_n & X \\ & I_n \end{pmatrix}\begin{pmatrix} g & \\ & g \end{pmatrix}\right) \times \psi^{-1}(\mathrm{Tr}(X)) E(s, g : \Phi, \omega_\Pi)\, dX\, dg.$$

For $m = 2n+1$, $\Phi \in \mathcal{S}(\mathbb{A}^n)$, and $\varphi \in V_\Pi$, we define a global integral as

$$\begin{aligned} &I_\psi(s,\varphi,\Phi) \\ &= \int_{\mathbb{A}^n}\int_{\mathrm{GL}_n(k)\backslash\mathrm{GL}_n(\mathbb{A})}\int_{\mathcal{M}_n(k)\backslash\mathcal{M}_n(\mathbb{A})}\int_{k^n\backslash\mathbb{A}^n}\varphi\left(\begin{pmatrix} I_n & X & Z \\ & I_n & \\ & & 1\end{pmatrix}\begin{pmatrix} g & & \\ & g & \\ & & 1\end{pmatrix}\begin{pmatrix} I_n & & \\ & I_n & \\ & y & 1\end{pmatrix}\right) \\ &\qquad\qquad\qquad\qquad \times \psi^{-1}(\mathrm{Tr}(X))\Phi(y)\,|\det(g)|^{s-1}\,dZ\,dX\,dg\,dy. \end{aligned}$$

The following theorem gives a meaning to these global integrals:

Theorem 3.1. *The integral $I_\psi(s,\varphi,\Phi)$ is convergent for* $\mathrm{Re}(s)$ *large enough, represents a meromorphic function on the entire plane, and satisfies the functional equation*

$$I_\psi(s,\varphi,\Phi) = I_{\psi^{-1}}(1-s,\varrho(\tau_m)\tilde{\varphi},\hat{\Phi}),$$

where ϱ denotes right translation and $\tilde{\varphi}(g) = \varphi({}^\iota g)$.

Proof. The analytic properties have been established for the even case $m = 2n$ in [Jacquet and Shalika 1990, §5] and the odd case $m = 2n+1$ in [Jacquet and Shalika 1990, §9]. The functional equation for $m = 2n$ follows immediately from that of the Eisenstein series $E(s, g : \Phi, \omega_\Pi)$ (3-1). See also [Kewat and Raghunathan 2012, Theorem 3.11]. We take this occasion to refine the elaboration for $m = 2n+1$ in [Cogdell and Matringe 2015, §3.5] thoroughly. If $\varphi \in V_\Pi$, then φ_1 and φ_2 are defined in [Jacquet and Shalika 1990, p. 219]:

$$\varphi_1(g) = \int_{\mathbb{A}^n}\varphi\left(g\begin{pmatrix} I_n & & \\ & I_n & \\ & y & 1\end{pmatrix}\right)\Phi(y)\,dy;\quad \varphi_2(g) = \int_{\mathbb{A}^n}\varphi\left(g\begin{pmatrix} I_n & & y \\ & I_n & \\ & & 1\end{pmatrix}\right)\hat{\Phi}(-{}^t y)\,dy,$$

where $\Phi \in \mathcal{S}(\mathbb{A}^n)$. We begin to deal with the equation on the bottom of page 219 in [Jacquet and Shalika 1990]:

$$\begin{aligned} &\int_{k^n\backslash\mathbb{A}^n}\int_{\mathcal{M}_n(k)\backslash\mathcal{M}_n(\mathbb{A})}\varphi_1\left(\begin{pmatrix} I_n & X & \\ & I_n & \\ & & 1\end{pmatrix}\begin{pmatrix} I_n & & Z \\ & I_n & \\ & & 1\end{pmatrix}\begin{pmatrix} g & & \\ & g & \\ & & 1\end{pmatrix}\right)\psi^{-1}(\mathrm{Tr}(X))\,dX\,dZ \\ &\qquad = \int_{k^n\backslash\mathbb{A}^n}\int_{\mathcal{M}_n(k)\backslash\mathcal{M}_n(\mathbb{A})}\varphi_2\left(\begin{pmatrix} I_n & X & \\ & I_n & \\ & & 1\end{pmatrix}\begin{pmatrix} I_n & & \\ & I_n & \\ & Z & 1\end{pmatrix}\begin{pmatrix} g & & \\ & g & \\ & & 1\end{pmatrix}\right) \\ &\qquad\qquad\qquad\qquad \times \psi^{-1}(\mathrm{Tr}(X))\,dX\,dZ\,|\det(g)|. \end{aligned}$$

(Here, φ in the corresponding formula in [Jacquet and Shalika 1990, p. 219] seems to be φ_2). As opposed to Jacquet and Shalika who conjugate them with the permutation matrix

$$\begin{pmatrix} & w_n & \\ w_n & & \\ & & 1\end{pmatrix},$$

we exploit τ_{2n+1}. This articulation is consistent with the shape of the local functional equation in [Cogdell and Matringe 2015, Theorem 3.1]. By applying $g \mapsto \tau_{2n+1}{}^{\iota}g\tau_{2n+1}^{-1}$, and then changing the variables $X \mapsto -X$ and $Z \mapsto -Z$, the above integral is written as

$$\int_{k^n\backslash\mathbb{A}^n}\int_{\mathcal{M}_n(k)\backslash\mathcal{M}_n(\mathbb{A})}\tilde{\varphi}_2\left(\begin{pmatrix} I_n & X & \\ & I_n & \\ & & 1\end{pmatrix}\begin{pmatrix} I_n & & Z \\ & I_n & \\ & & 1\end{pmatrix}\begin{pmatrix} {}^tg^{-1} & & \\ & {}^tg^{-1} & \\ & & 1\end{pmatrix}\tau_{2n+1}\right) \times \psi(\mathrm{Tr}(X))\,dX\,dZ\,|\det(g)|.$$

We insert the definitions of φ_1 and φ_2 and utilize the assignment $g \mapsto \tau_{2n+1}{}^{\iota}g\tau_{2n+1}^{-1}$ on the last matrix. After the change of variables $y \mapsto -y$, the identity becomes

$$\begin{aligned}\int_{\mathbb{A}^n}\int_{k^n\backslash\mathbb{A}^n}\int_{\mathcal{M}_n(k)\backslash\mathcal{M}_n(\mathbb{A})}\varphi\left(\begin{pmatrix} I_n & X & Z \\ & I_n & \\ & & 1\end{pmatrix}\begin{pmatrix} g & & \\ & g & \\ & & 1\end{pmatrix}\begin{pmatrix} I_n & & \\ & I_n & \\ & y & 1\end{pmatrix}\right)& \\ \times \psi^{-1}(\mathrm{Tr}(X))\Phi(y)\,dX\,dZ\,dy& \\ = \int_{\mathbb{A}^n}\int_{k^n\backslash\mathbb{A}^n}\int_{\mathcal{M}_n(k)\backslash\mathcal{M}_n(\mathbb{A})}\tilde{\varphi}\left(\begin{pmatrix} I_n & X & Z \\ & I_n & \\ & & 1\end{pmatrix}\begin{pmatrix} {}^tg^{-1} & & \\ & {}^tg^{-1} & \\ & & 1\end{pmatrix}\begin{pmatrix} I_n & & \\ & I_n & \\ & y & 1\end{pmatrix}\tau_{2n+1}\right)& \\ \times \psi(\mathrm{Tr}(X))\hat{\Phi}(y)\,dX\,dZ\,dy\,|\det(g)|&\end{aligned}$$

from which the desired global functional equation for integrals follows. □

Let

$$\begin{aligned} W_\varphi(g) &= \int_{N_m(k)\backslash N_m(\mathbb{A})}\varphi(ng)\psi^{-1}(n)\,dn, \\ \widetilde{W}_\varphi(g) &= \int_{N_m(k)\backslash N_m(\mathbb{A})}\tilde{\varphi}(w_m ng)\psi(n)\,dn \end{aligned}$$

be the associated Whittaker function of φ and $\tilde{\varphi}$, respectively. We have yet to check that our integrals are Eulerian.

Proposition 3.2 (Jacquet–Shalika). *For $\varphi \in V_\Pi$ and $\Phi \in \mathcal{S}(F^n)$, global Jacquet–Shalika integrals*

$$J_\psi(s, W_\varphi, \Phi) = \int_{N_n(\mathbb{A})\backslash \mathrm{GL}_n(\mathbb{A})}\int_{\mathcal{N}_n(\mathbb{A})\backslash\mathcal{M}_n(\mathbb{A})} W_\varphi\left(\begin{pmatrix} I_n & X \\ & I_n\end{pmatrix}\begin{pmatrix} g & \\ & g\end{pmatrix}\right) \times \psi^{-1}(\mathrm{Tr}(X))\Phi(e_n g)\,|\det(g)|^s\,dX\,dg$$

in the even case $m = 2n$ and

$$J_\psi(s, W_\varphi, \Phi) = \int_{N_n(\mathbb{A})\backslash \mathrm{GL}_n(\mathbb{A})} \int_{\mathcal{N}_n(\mathbb{A})\backslash \mathcal{M}_n(\mathbb{A})} \int_{\mathbb{A}^n} W_\varphi\left(\begin{pmatrix} I_n & X & \\ & I_n & \\ & & 1 \end{pmatrix}\begin{pmatrix} g & & \\ & g & \\ & & 1 \end{pmatrix}\begin{pmatrix} I_n & & \\ & I_n & \\ & y & 1 \end{pmatrix}\right) \times \psi^{-1}(\mathrm{Tr}(X))\Phi(y)|\det(g)|^{s-1}\, dy\, dX\, dg$$

in the odd case $m = 2n+1$ converge when $\mathrm{Re}(s)$ *is sufficiently large and, when this is the case, we have*

$$I_\psi(s, \varphi, \Phi) = J_\psi(s, W_\varphi, \Phi).$$

We suppose, in addition, that $W_\varphi(g) = \prod_{v\in|\mathbb{P}^1|} W_{\varphi_v}(g_v)$, $\psi(n) = \prod_{v\in|\mathbb{P}^1|} \psi(n_v)$, and $\Phi(g) = \prod_{v\in|\mathbb{P}^1|} \Phi_v(g_v)$. Then, when $\mathrm{Re}(s)$ *is sufficiently large,*

$$J_\psi(s, W_\varphi, \Phi) = \prod_{v\in|\mathbb{P}^1|} J_{\psi_v}(s, W_{\varphi_v}, \Phi_v).$$

Likewise, the right-hand side of the functional equation is also unfold and can be factored as

$$\begin{aligned} I_{\psi^{-1}}(1-s, \varrho(\tau_m)\tilde{\varphi}, \hat{\Phi}) &= J_{\psi^{-1}}(1-s, \varrho(\tau_m)\widetilde{W}_\varphi, \hat{\Phi}) \\ &= \prod_{v\in|\mathbb{P}^1|} J_{\psi^{-1}}(1-s, \varrho(\tau_m)\widetilde{W}_{\varphi_v}, \hat{\Phi}_v), \end{aligned}$$

with the convergence for $\mathrm{Re}(s) \ll 0$.

Proof. All these statements are drawn, with some minor changes of notation, from [Jacquet and Shalika 1990, Proposition 5 in §6] for $m = 2n$ and [Jacquet and Shalika 1990, §9.2] for $m = 2n+1$. □

Throughout, we will take $S \subset |\mathbb{P}^1|$ to be a finite set of places such that for all $v \notin S$, Π_v and ψ_v are all unramified and ψ_v normalized. The partial L-function is a product of local factors

$$L^S(s, \Pi, \wedge^2) = \prod_{v\notin S} L(s, \Pi_v, \wedge^2).$$

More precisely, this product converges for $\mathrm{Re}(s)$ large enough (see [Jacquet and Shalika 1990, §8–9]). The global L-function and ε-factors for Π are

$$L(s, \Pi, \wedge^2, S) = \prod_{v\in|\mathbb{P}^1|} L(s, \Pi_v, \wedge^2) = L^S(s, \Pi, \wedge^2)\prod_{v\in S} L(s, \Pi_v, \wedge^2)$$

and

$$\varepsilon(s, \Pi, \wedge^2, S) = \prod_{v\in|\mathbb{P}^1|} \varepsilon(s, \Pi_v, \wedge^2, \psi_v) = \prod_{v\in S}\varepsilon(s, \Pi_v, \wedge^2, \psi_v).$$

As for the ε-factor, we know that $\varepsilon(s, \Pi_v, \wedge^2, \psi_v) \equiv 1$ for $v \notin S$. The independence of $\varepsilon(s, \Pi, \wedge^2, S)$ from the choice of ψ can be seen as a consequence of the global functional equation below.

Theorem 3.3. *The global L-function $L(s, \Pi, \wedge^2, S)$ has a meromorphic continuation to the entire plane, and it satisfies the global functional equation*

$$L(s, \Pi, \wedge^2, S) = \varepsilon(s, \Pi, \wedge^2, S) L(1-s, \widetilde{\Pi}, \wedge^2, S),$$

where $\varepsilon(s, \Pi, \wedge^2, S)$ is entire and nonvanishing. This identity further implies that $\varepsilon(s, \Pi, \wedge^2, S)$ is independent of ψ as well.

Proof. From the unfolding in Proposition 3.2, and the local calculation of [Jacquet and Shalika 1990, §7.2 and §9.4] together with Corollary 2.18, we know that for $\mathrm{Re}(s)$ large and for appropriate choice of φ, we have

$$\begin{aligned} I_\psi(s, \varphi, \Phi) = J_\psi(s, W_\varphi, \Phi) &= \prod_{v \in |\mathbb{P}^1|} J_{\psi_v}(s, W_{\varphi_v}, \Phi_v) \\ &= \Big(\prod_{v \in S} J_{\psi_v}(s, W_{\varphi_v}, \Phi_v)\Big) L^S(s, \Pi, \wedge^2) \\ &= \Big(\prod_{v \in S} \frac{J_{\psi_v}(s, W_{\varphi_v}, \Phi_v)}{L(s, \Pi_v, \wedge^2)}\Big) L(s, \Pi, \wedge^2, S) \\ &= \Big(\prod_{v \in S} e_v(s, W_{\varphi_v}, \Phi_v)\Big) L(s, \Pi, \wedge^2, S), \end{aligned}$$

where $e_v(s, W_{\varphi_v}, \Phi_v) = J_{\psi_v}(s, W_{\varphi_v}, \Phi_v)/L(s, \Pi_v, \wedge^2)$. It follows from Theorem 2.1, $e_v(s, W_{\varphi_v}, \Phi_v)$ is entire. Therefore $L(s, \Pi, \wedge^2, S)$ has a meromorphic continuation, as the integral $I_\psi(s, \varphi, \Phi)$ is a meromorphic function on the entire plane from Theorem 3.1. While on the other side, we obtain

$$\begin{aligned} I_{\psi^{-1}}(1-s, \varrho(\tau_m)\tilde{\varphi}, \hat{\Phi}) &= J_{\psi^{-1}}(1-s, \varrho(\tau_m)\widetilde{W}_\varphi, \hat{\Phi}) \\ &= \Big(\prod_{v \in S} \tilde{e}_v(1-s, \varrho(\tau_m)\widetilde{W}_{\varphi_v}, \hat{\Phi}_v)\Big) L(1-s, \widetilde{\Pi}, \wedge^2, S), \end{aligned}$$

with $\tilde{e}_v(1-s, \varrho(\tau_m)\widetilde{W}_{\varphi_v}, \hat{\Phi}_v) = J_{\psi^{-1}}\big(1-s, \varrho(\tau_m)\widetilde{W}_{\varphi_v}, \hat{\Phi}_v\big)/L(1-s, \widetilde{\Pi}_v, \wedge^2)$. However we derive from the local functional equation, Theorem 2.2, that

$$\begin{aligned} \tilde{e}_v(1-s, \varrho(\tau_m)\widetilde{W}_{\varphi_v}, \hat{\Phi}_v) &= \frac{J_{\psi^{-1}}(1-s, \varrho(\tau_m)\widetilde{W}_{\varphi_v}, \hat{\Phi}_v)}{L(1-s, \widetilde{\Pi}_v, \wedge^2)} \\ &= \varepsilon(s, \Pi_v, \wedge^2, \psi_v) \frac{J_{\psi_v}(s, W_{\varphi_v}, \Phi_v)}{L(s, \Pi_v, \wedge^2)} \\ &= \varepsilon(s, \Pi_v, \wedge^2, \psi_v) e_v(s, W_{\varphi_v}, \Phi_v). \end{aligned}$$

Combining these all together, we get

$$\begin{aligned} L(s, \Pi, \wedge^2, S) &= \left(\prod_{v \in S} \varepsilon(s, \Pi_v, \wedge^2, \psi_v) \right) L(1-s, \widetilde{\Pi}, \wedge^2, S) \\ &= \varepsilon(s, \Pi, \wedge^2, S) L(1-s, \widetilde{\Pi}, \wedge^2, S), \end{aligned}$$

since for $v \notin S$ we know Π_v and ψ_v are unramified so that $\varepsilon(s, \Pi_v, \wedge^2, \psi_v) \equiv 1$. □

3B. ***The equality for discrete series representations.*** Let $k_0 = \mathbb{F}_q((t))$ be the completion of k at the point $0 \in |\mathbb{P}^1|$. We start with a local irreducible unitary supercuspidal representation $\rho_\circ$ and globalize it according to the result of Henniart and Lomelí [2011; 2013b, Theorem 3.1].

Theorem 3.4 (Henniart–Lomelí). *Let $\rho_\circ$ be an irreducible unitary supercuspidal representation of $\mathrm{GL}_m(F)$. We choose an isomorphism $\xi : F \xrightarrow{\sim} k_0$. Then there exists a cuspidal unitary automorphic representation $\Pi = \bigotimes_v' \Pi_v$ whose local components Π_v satisfy*:

- *$\rho_\circ$ corresponds to Π_0 via ξ*;
- *at the places $v \in |\mathbb{P}^1|$ away from 0, 1, and ∞, Π_v is irreducible and unramified*;
- *Π_1 is an irreducible generic subquotient of an unramified principal series representation*;
- *Π_∞ is an irreducible generic subquotient of a tamely ramified principal series representation.*

We have control at all places outside 0, which makes it possible to deduce the identity for irreducible supercuspidal representations.

Theorem 3.5 (supercuspidal cases). *Let ρ be an irreducible supercuspidal representation of GL_r. Then we have*

$$L(s, \rho, \wedge^2) = \mathcal{L}(s, \rho, \wedge^2).$$

As a consequence, if ρ is (S_{2n}, Θ)-distinguished, then ρ is self-dual.

Proof. Twisting by an unramified character does not affect the conclusion, so we can assume that $\rho = \rho_\circ$ is unitary. (See the proof of Theorem 4.4 for details, cf. [Lomelí 2016, §6.6]). We define the Langlands–Shahidi global L-function and ε-factors for Π by

$$\begin{aligned} \mathcal{L}(s, \Pi, \wedge^2, S) &= \prod_{v \in |\mathbb{P}^1|} \mathcal{L}(s, \Pi_v, \wedge^2), \\ \varepsilon_{LS}(s, \Pi, \wedge^2, \psi, S) &= \prod_{v \in |\mathbb{P}^1|} \varepsilon_{LS}(s, \Pi_v, \wedge^2, \psi_v). \end{aligned}$$

We choose a finite set $S = \{0, 1, \infty\}$ of places. Applying Theorem 3.4 to the irreducible unitary supercuspidal representation $\rho_\circ$, we obtain a cuspidal unitary automorphic representation Π. For our convenience, we rewrite the global functional equation in [Henniart and Lomelí 2011, §4.1 (vi)] as

$$\mathcal{L}(s, \Pi, \wedge^2, S) = \varepsilon_{LS}(s, \Pi, \wedge^2, S)\mathcal{L}(1-s, \widetilde{\Pi}, \wedge^2, S). \tag{3-2}$$

The function $\varepsilon_{LS}(s, \Pi, \wedge^2, S)$ is entire and nonvanishing. From the global functional equation given by Theorem 3.3 and (3-2), this means that the ratio of L-function satisfies

$$\frac{L(s, \Pi, \wedge^2, S)}{\mathcal{L}(s, \Pi, \wedge^2, S)} = \eta(s, \Pi, S)\frac{L(1-s, \widetilde{\Pi}, \wedge^2, S)}{\mathcal{L}(1-s, \widetilde{\Pi}, \wedge^2, S)},$$

where $\eta(s, \Pi, S) = \varepsilon(s, \Pi, \wedge^2, S)\varepsilon_{LS}(s, \Pi, \wedge^2, S)^{-1}$ is entire and nonvanishing. Applying the already established principal series representations in Corollary 2.18, along with Proposition 2.17, at the places $|\mathbb{P}^1| - \{0\}$ yields:

$$\prod_{v \notin \{0\}} L(s, \Pi_v, \wedge^2) = \prod_{v \notin \{0\}} \mathcal{L}(s, \Pi_v, \wedge^2),$$

$$\prod_{v \notin \{0\}} L(1-s, \widetilde{\Pi}_v, \wedge^2) = \prod_{v \notin \{0\}} \mathcal{L}(1-s, \widetilde{\Pi}_v, \wedge^2).$$

Therefore, at the remaining one place, we have

$$\frac{L(s, \rho_\circ, \wedge^2)}{\mathcal{L}(s, \rho_\circ, \wedge^2)} = \eta(s, \Pi, S)\frac{L(1-s, \tilde{\rho}_\circ, \wedge^2)}{\mathcal{L}(1-s, \tilde{\rho}_\circ, \wedge^2)}.$$

In view of [Ganapathy and Lomelí 2015] and [Kewat and Raghunathan 2012, Theorem 3.7], $L(s, \rho_\circ, \wedge^2)$ and $\mathcal{L}(s, \rho_\circ, \wedge^2)$ are regular and nonvanishing in the region $\mathrm{Re}(s) > 0$, whereas similar analytic properties for $L(1-s, \tilde{\rho}_\circ, \wedge^2)$ and $\mathcal{L}(1-s, \tilde{\rho}_\circ, \wedge^2)$ are valid in the half plane $\mathrm{Re}(s) < 1$. This forces that the ratio $L(s, \rho_\circ, \wedge^2)/\mathcal{L}(s, \rho_\circ, \wedge^2)$ is an entire and nonvanishing function, and hence it is a constant. Since these L-factors are normalized, these must be equal.

We now gain the full strength of flexibility to transport L-factors in the Langlands–Shahidi side to the Rankin–Selberg side. The L-factor $\mathcal{L}(s, \rho \times \rho)$ is decomposed as the product of $\mathcal{L}(s, \rho, \wedge^2)$ and $\mathcal{L}(s, \rho, \mathrm{Sym}^2)$ (see [Ganapathy and Lomelí 2015; Henniart and Lomelí 2011; Shahidi 1992, Corollary 8.2]). Then the pole of $\mathcal{L}(s, \rho, \wedge^2)$ at $s = 0$ detected by the existence of the Shalika functional [Jo 2020a, Theorem 3.6 (ii)] contributes the pole of $\mathcal{L}(s, \rho \times \rho)$. This is amount to saying that ρ is self-dual [Matringe 2015, Proposition 4.6]. □

Once we know the inductivity of ε-factors, we expect that $\eta(s, \Pi, S) \equiv 1$, independent of the choice of S. We now come to the case of discrete series representations.

Theorem 3.6 (discrete series cases). *Let Δ be a discrete series representation of* GL_m. *Then we have*

$$L(s, \Delta, \wedge^2) = \mathcal{L}(s, \Delta, \wedge^2).$$

As a consequence, if Δ is (S_{2n}, Θ)-distinguished, then Δ is self-dual.

Proof. As indicated in the proof of Theorem 4.4, after proper unramified twisting of Δ, we can easily reduce the equality to the case when Δ is a unitary representation of the form $\left[\rho_\circ \nu^{-(\ell-1)/2}, \ldots, \rho_\circ \nu^{(\ell-1)/2}\right]$ with $\rho_\circ$ a unitary irreducible supercuspidal representation of GL_r (cf. [Lomelí 2016, §6.6]). Taking advantage of Theorems 2.15 and 3.5, this finally matches with the expression in [Shahidi 1992, Proposition 8.1]. Concerning the second assertion, we literally reiterate the second part of the proof of Theorem 3.5 line-by-line, and therefore we omit thorough arguments entirely. □

The identity can be extended to the class of all irreducible admissible representations of GL_m.

Theorem 3.7. *Let π be an irreducible admissible representation of* GL_m. *Then*

$$L(s, \pi, \wedge^2) = \mathcal{L}(s, \pi, \wedge^2).$$

Proof. We realize π as the unique Langlands quotient of Langlands type $\Xi = \mathrm{Ind}_{\mathrm{Q}}^{\mathrm{GL}_m}(\Delta_{\circ 1}\nu^{u_1} \otimes \Delta_{\circ 2}\nu^{u_2} \otimes \cdots \otimes \Delta_{\circ t}\nu^{u_t})$, which is again of Whittaker type. Thanks to Theorem 3.6, the working hypothesis is not required to be checked for discrete series representations. Then Theorem 2.13 gives us that

$$L(s, \Xi, \wedge^2) = \prod_{1 \le k \le t} L(s + 2u_k, \Delta_{\circ k}, \wedge^2) \prod_{1 \le i < j \le t} L(s + u_i + u_j, \Delta_{\circ i} \times \Delta_{\circ j}),$$

which coincides with corresponding decompositions in Langlands–Shahidi theory [Ganapathy and Lomelí 2015, §3.1 (xi)]. □

By exploiting the main result of Henniart and Lomelí [2011], it can be summarized that the definition of local exterior square L-function via the theory of integral representations is compatible with the local Langlands correspondence. In what follows, we let W_F' denote the Weil–Deligne group of F, and let ϕ an m-dimensional (complex-valued) Frobenius semisimple representation of W_F'. We call this the *Weil–Deligne representation* of W_F'. Let $\wedge^2$ denote the exterior representation of $\mathrm{GL}_m(\mathbb{C})$. We then denote by $L(s, \wedge^2(\phi))$ the *Artin exterior square L-factor* attached to ϕ.

Theorem 3.8. *Let π be an irreducible admissible representation of* $\mathrm{GL}_m(F)$ *and $\phi(\pi)$ the Weil–Deligne representation of W_F' corresponding to π under the local Langlands correspondence. Then*

$$L(s, \pi, \wedge^2) = \mathcal{L}(s, \pi, \wedge^2) = L\big(s, \wedge^2(\phi(\pi))\big).$$

4. Bump–Friedberg and Flicker zeta integrals

4A. ***Bump–Friedberg L-factors.*** Define the embedding $J : \mathrm{GL}_n \times \mathrm{GL}_n \to \mathrm{GL}_m$ by

$$J(g, g')_{k,l} = \begin{cases} g_{i,j}, & \text{if } k = 2i-1,\ l = 2j-1, \\ g'_{i,j}, & \text{if } k = 2i,\ l = 2j, \\ 0, & \text{otherwise}, \end{cases}$$

for $m = 2n$ and $J : \mathrm{GL}_{n+1} \times \mathrm{GL}_n \to \mathrm{GL}_m$ by

$$J(g, g')_{k,l} = \begin{cases} g_{i,j}, & \text{if } k = 2i-1,\ l = 2j-1, \\ g'_{i,j}, & \text{if } k = 2i,\ l = 2j, \\ 0, & \text{otherwise}, \end{cases}$$

for $m = 2n+1$. As for the intention of holding onto coherent terminology with [Matringe 2015], interested readers may perceive that we interchange the role of g and g' in [Bump and Friedberg 1990]. Let $\pi = \mathrm{Ind}_{\mathrm{Q}}^{\mathrm{GL}_m}(\Delta_1 \otimes \cdots \otimes \Delta_t)$ be a parabolically induced representation. For each Whittaker function $W \in \mathcal{W}(\pi, \psi)$ and Schwartz–Bruhat function $\Phi \in \mathcal{S}(F^n)$, we define Bump–Friedberg integrals:

$$Z(s_1, s_2, W, \Phi) = \int_{N_n \backslash \mathrm{GL}_n} \int_{N_n \backslash \mathrm{GL}_n} W(J(g,g'))\Phi(e_m J(g,g'))|\det(g)|^{s_1 - 1/2} |\det(g')|^{1/2 + s_2 - s_1}\, dg\, dg'$$

when $m = 2n$ and

$$Z(s_1, s_2, W, \Phi) = \int_{N_n \backslash \mathrm{GL}_n} \int_{N_{n+1} \backslash \mathrm{GL}_{n+1}} W(J(g, g'))\Phi(e_m J(g, g'))| \det(g)|^{s_1} | \det(g')|^{s_2 - s_1}\, dg\, dg'$$

when $m = 2n+1$. If r is a real number, we denote by δ_r the character

$$\delta_r : J(g, g') \mapsto \left| \frac{\det(g)}{\det(g')} \right|^r.$$

We denote by χ_m and μ_m the characters of H_m:

$$\chi_m\Bigl(w_m \begin{pmatrix} g & \\ & g' \end{pmatrix} w_m^{-1}\Bigr) = \begin{cases} \mathbf{1}_{H_m}, & \text{for } m = 2n; \\ \left| \dfrac{\det(g)}{\det(g')} \right|, & \text{for } m = 2n+1; \end{cases}$$

$$\mu_m\Bigl(w_m \begin{pmatrix} g & \\ & g' \end{pmatrix} w_m^{-1}\Bigr) = \begin{cases} \left| \dfrac{\det(g)}{\det(g')} \right|, & \text{for } m = 2n; \\ \mathbf{1}_{H_m}, & \text{for } m = 2n+1. \end{cases}$$

We turn toward the case for $s_1 = s + \frac{1}{2}$ and $s_2 = 2s$. We unify Bump–Friedberg zeta integrals as one single integral of the form

$$Z(s, W, \Phi) = \int_{(N_m \cap H_m) \backslash H_m} W(h) \chi_m^{1/2}(h) \Phi(e_m h) | \det(h)|^s\, dh.$$

The twisted analogue of Bump–Friedberg zeta integrals attached to χ_α is defined by

$$Z(s, W, \Phi, \chi_\alpha) = \int_{(N_m \cap H_m)\backslash H_m} W(h)\chi_\alpha(h)\chi_m^{1/2}(h)\Phi(e_m h)|\det(h)|^s \, dh.$$

The integral $Z(s, W, \Phi, \chi_\alpha)$ converges absolutely for s of real part large enough. The $\mathbb{C}$-vector space generated by Bump–Friedberg zeta integrals

$$\big\langle Z(s, W, \Phi, \chi_\alpha) \mid W \in \mathcal{W}(\pi, \psi), \Phi \in \mathcal{S}(F^m)\big\rangle$$

is a $\mathbb{C}[q^{\pm s}]$-fractional ideal $\mathcal{I}(\pi, \chi_\alpha, \mathrm{BF})$ of $\mathbb{C}(q^{-s})$. The ideal $\mathcal{I}(\pi, \chi_\alpha, \mathrm{BF})$ is principal and has a unique generator of the form $P(q^{-s})^{-1}$, where $P(X)$ is a polynomial in $\mathbb{C}[X]$ with $P(0) = 1$. The *Bump–Friedberg L-factor* associated to π is defined by the unique normalized generator [Matringe 2015, Proposition 4.8]

$$L(s, \pi, \chi_\alpha, \mathrm{BF}) = \frac{1}{P(q^{-s})}.$$

If $\alpha = \mathbf{1}_{F^\times}$ is a trivial character, we write $L(s, \pi, \mathrm{BF})$ for $L(s, \pi, \chi_{\mathbf{1}_{F^\times}}, \mathrm{BF})$. The *Bump–Friedberg γ-factor*

$$\gamma(s, \pi, \mathrm{BF}, \psi) = \varepsilon(s, \pi, \mathrm{BF}, \psi)\frac{L(1/2 - s, \pi^\iota, \delta_{-1/2}, \mathrm{BF})}{L(s, \pi, \mathrm{BF})}$$

is a rational function in $\mathbb{C}(q^{-s})$ that depends on a choice of a nontrivial character ψ (see [Matringe 2015, Proposition 4.11]). While the proof of [Matringe 2014, Proposition 6.2] reflects the structure of Weil–Deligne representations, our aim is to show the factorization of $L(s, \pi, \chi_\alpha, \mathrm{BF})$ as a product of the standard L-factor $L(s + 1/2, \pi)$ and the exterior square L-factor $L(2s, \pi, \wedge^2)$ within the framework of the Rankin–Selberg method. Our approach here is more direct and concise.

Theorem 4.1 (supercuspidal cases). *Let ρ be an irreducible supercuspidal representation of* GL_r. *Then*

$$L(s, \rho, \mathrm{BF}) = L\big(s + \tfrac{1}{2}, \rho\big)L(2s, \rho, \wedge^2).$$

Proof. If $r = 1$, then ρ is a character of $F^\times$. The integral is just the Tate integral of the form $\int_{F^\times} \rho(z)\Phi(z)|z|^{s+1/2} \, d^\times z$, hence

$$L(s, \rho, \mathrm{BF}) = L\big(s + \tfrac{1}{2}, \rho\big) = L\big(s + \tfrac{1}{2}, \rho\big)L(2s, \rho, \wedge^2),$$

where the last equality comes from $L(2s, \rho, \wedge^2) = 1$ (see [Jo 2020a, Theorem 2.13]).

We deduce from Theorem 2.3 aligned with [Matringe 2015, Proposition 4.14] that all the poles of $L(s, \rho, \mathrm{BF})$ and $L(s, \rho, \wedge^2)$ are necessarily simple. Given $r = 2n+1$ with $n \geq 1$, the result of Matringe [2014, Theorem 3.1] (see Theorem 4.2) tells us that ρ cannot be H_{2n+1}-distinguished. According to [Jacquet 1979, §3.1] coupled with [Jo 2020a, Theorem 3.6 (ii)] and [Matringe 2015, Corollary 4.3], we have

$$L(s, \rho, \mathrm{BF}) = L(s, \rho) = L(s, \rho, \wedge^2) = 1.$$

Now we turn to the case when $r = 2n$. Analyzing poles of local L-functions is just a question of certain distinctions of representations. To be precise, [Jo 2020a, Theorem 3.6 (i)] together with [Matringe 2015, Corollary 4.3] and Section 2B lead us to the following equivalent statements:

(i) $L(2s, \rho, \wedge^2)$ has a pole at $s = s_0$;

(ii) $L(s, \rho, \mathrm{BF})$ has a pole at $s = s_0$;

(iii) $\rho\nu^{s_0}$ is (S_{2n}, Θ)-distinguished;

(iv) $\rho\nu^{s_0}$ is H_{2n}-distinguished.

The above characterization of poles of L-factors can be reinterpreted as

$$L(s, \rho, \mathrm{BF}) = L(2s, \rho, \wedge^2) = L\big(s + \tfrac{1}{2}, \rho\big)L(2s, \rho, \wedge^2),$$

where the last identity follows from $L\big(s + \frac{1}{2}, \rho\big) = 1$ (see [Jacquet 1979, §3.1]). □

Unlike the case of Jacquet and Shalika's zeta integrals Sections 2 and 3, it is necessary to additionally use the hereditary property of H_{2m+1}-distinguished representations due to Matringe [2015, Theorem 3.1].

Theorem 4.2 (N. Matringe, $m = 2n + 1$). *Let $\pi = \mathrm{Ind}_{\mathrm{Q}}^{\mathrm{GL}_{2n+1}}(\Delta_1 \otimes \Delta_2 \otimes \cdots \otimes \Delta_t)$ be an irreducible generic representation of GL_{2n+1}. Let α be a character of $F^\times$ with $0 \le \mathrm{Re}(\alpha) \le \frac{1}{2}$. Then π is $(H_{2n+1}, \chi_\alpha\delta_{-1/2})$-distinguished if and only if π is a parabolically induced representation of the form $\mathrm{Ind}_{P_{2n,1}}^{\mathrm{GL}_{2n+1}}(\pi' \otimes \alpha\nu^{-1/2})$, for π' an irreducible generic (H_{2n}, χ_α)-distinguished representation of GL_{2n} such that $\mathrm{Ind}_{P_{2n,1}}^{\mathrm{GL}_{2n+1}}(\pi' \otimes \alpha\nu^{-1/2})$ is still irreducible and generic.*

Throughout the rest of Section 4A, a variant of the systematic machinery developed in Section 2C (in particular, Proposition 2.10 to Theorem 2.12) should continue to work out in the context of Bump–Friedberg zeta integrals, and it is dealt with in [Matringe 2015, §4] in great detail and clarity. By doing so, Bump–Friedberg local L-functions are compatible with the classification of discrete series representation in terms of supercuspidal ones owing to Bernstein and Zelevinsky [1977] and [Zelevinsky 1980].

Proposition 4.3. *Let $\Delta_\circ = [\rho_\circ\nu^{-(\ell-1)/2}, \ldots, \rho_\circ\nu^{(\ell-1)/2}]$ be an irreducible square integrable representation of $\mathrm{GL}_{\ell r}$, with $\rho_\circ$ an irreducible unitary supercuspidal representation of GL_r.*

(i) *Suppose that ℓ is even. Then we have*

$$L(s, \Delta_\circ, \mathrm{BF}) = L(s + \ell/2, \rho_\circ)\prod_{i=1}^{\ell/2} L(2s, \rho_\circ\nu^{(\ell+1)/2-i}, \wedge^2)L(2s, \rho_\circ\nu^{\ell/2-i}, \mathrm{Sym}^2).$$

(ii) *Suppose that ℓ is odd. Then we have*

$$\begin{aligned}&L(s,\Delta_\circ,\mathrm{BF})\\&\quad=L\Big(s+\frac{\ell}{2},\rho_\circ\Big)\prod_{i=1}^{(\ell+1)/2}L(2s,\rho_\circ\nu^{(\ell+1)/2-i},\wedge^2)\prod_{i=1}^{(\ell-1)/2}L(2s,\rho_\circ\nu^{\ell/2-i},\mathrm{Sym}^2).\end{aligned}$$

Proof. By the uniqueness of the Whittaker functional, the Whittaker model for $\Delta_\circ$ coincides with that for $\mathrm{Ind}(\rho_\circ\nu^{-(\ell-1)/2}\otimes\cdots\otimes\rho_\circ\nu^{(\ell-1)/2})$. Likewise the same trait holds for dual objects provided by $\mathrm{Ind}(\tilde{\rho}_\circ\nu^{-(\ell-1)/2}\otimes\cdots\otimes\tilde{\rho}_\circ\nu^{(\ell-1)/2})$ and $\tilde{\Delta}_\circ$. According to [Matringe 2015, Proposition 5.5],

$$\begin{aligned}&\gamma(s,\Delta_\circ,\mathrm{BF},\psi)\\&\quad\sim\prod_{i=0}^{\ell-1}\gamma\Big(s+\frac{1-\ell}{2}+i,\rho_\circ,\mathrm{BF},\psi\Big)\prod_{0\le i<j\le\ell-1}\gamma(2s+1-\ell+i+j,\rho_\circ\times\rho_\circ,\psi).\end{aligned}$$

With the help of Theorem 4.1, the expression can be reformulated in terms of L-factors as

$$\begin{aligned}\gamma(s,\Delta_\circ,\mathrm{BF},\psi)\sim\prod_{i=0}^{\ell-1}\frac{L(-s-i+\ell/2,\tilde{\rho}_\circ)}{L(s+i+1-\ell/2,\rho_\circ)}\prod_{i=0}^{\ell-1}\frac{L(-2s+\ell-2i,\tilde{\rho}_\circ,\wedge^2)}{L(2s+1-\ell+2i,\rho_\circ,\wedge^2)}\\\times\prod_{0\le i<j\le\ell-1}\frac{L(-2s+\ell-i-j,\tilde{\rho}_\circ\times\tilde{\rho}_\circ)}{L(2s+1-\ell+i+j,\rho_\circ\times\rho_\circ)}.\end{aligned}$$

By virtue of [Jo 2020a, Lemma 5.11] combined with $L(-s,\rho_\circ)\sim L(s,\tilde{\rho}_\circ)$ (see [Jacquet et al. 1983, §8.2 (15)–(16)]), it may be written as

$$\begin{aligned}\gamma(s,\Delta_\circ,\mathrm{BF},\psi)\sim\prod_{i=0}^{\ell-1}\frac{L(s+i-\ell/2,\rho_\circ)}{L(s+i+1-\ell/2,\rho_\circ)}\prod_{i=0}^{\ell-1}\frac{L(2s-\ell+2i,\rho_\circ,\wedge^2)}{L(2s+1-\ell+2i,\rho_\circ,\wedge^2)}\\\times\prod_{0\le i<j\le\ell-1}\frac{L(2s-\ell+i+j,\rho_\circ\times\rho_\circ)}{L(2s+1-\ell+i+j,\rho_\circ\times\rho_\circ)}.\end{aligned}$$

We do the case where ℓ is even, the case where ℓ is odd is treated similarly. At this moment, we repeat the proof given in [Jo 2020a, Theorem 5.12] with adjusting s to $2s$. After canceling common factors, our quotient is simplified to

$$\begin{aligned}&\gamma(s,\Delta_\circ,\mathrm{BF},\psi)\\&\quad\sim\frac{L(s-\ell/2,\rho_\circ)}{L(s+\ell/2,\rho_\circ)}\prod_{i=0}^{(\ell/2)-1}\frac{L(2s-\ell+2i,\rho_\circ,\wedge^2)L(2s-\ell+2i+1,\rho_\circ,\mathrm{Sym}^2)}{L(2s+2i+1,\rho_\circ,\wedge^2)L(2s+2i,\rho_\circ,\mathrm{Sym}^2)}.\end{aligned}\tag{4-1}$$

Using [Matringe 2015, Corollary 4.1], $L\big(\frac{1}{2}-s,\tilde{\Delta}_\circ,\delta_{-1/2},\mathrm{BF}\big)^{-1}$ has zeros in the half plane $\mathrm{Re}(s)\ge\frac{1}{2}$, while $L(s,\Delta_\circ,\mathrm{BF})^{-1}$ has its zeros contained in the region $\mathrm{Re}(s)\le 0$. Since the half planes $\mathrm{Re}(s)\ge\frac{1}{2}$ and $\mathrm{Re}(s)\le 0$ are disjoint, they do not share factors in $\mathbb{C}[q^{\pm s}]$. As ρ is unitary, the poles of the product of L-factors in the numerator must lie on the line $\mathrm{Re}(s) = (\ell - i)/2$ for $i = 0, \dots, \ell-2, \ell-1$, while those in the denominator are located on the line $\mathrm{Re}(s) = -i/2$ for $i = 0, \dots, \ell-2, \ell-1, \ell$. Therefore, they do not have common factors at all. We establish the identity from the observation that the ratios (4-1) and $\gamma(s, \Delta_\circ, \mathrm{BF}, \psi) \sim L\big(\frac{1}{2}-s, \tilde{\Delta}_\circ, \delta_{-1/2}, \mathrm{BF}\big)/L(s, \Delta_\circ, \mathrm{BF})$ are all reduced and the indices i are rearranged. □

Theorem 4.4 is the key step to improve the factorization to the set of discrete series representations. If we can do this, then the application of the Langlands classification theorem allows us to extend it to all irreducible admissible representations.

Theorem 4.4 (discrete series cases). *Let Δ be an irreducible essentially square integrable representation of* GL_m. *Then*

$$L(s, \Delta, \mathrm{BF}) = L\big(s+\tfrac{1}{2}, \Delta\big)L(2s, \Delta, \wedge^2).$$

Proof. We choose an unramified quasicharacter ν^{s_1}, $s_1 \in \mathbb{C}$, so that $\Delta = \Delta_\circ \nu^{s_1}$, where $\Delta_\circ$ is an irreducible square integrable representation of GL_m. We can easily verify that $L(s, \Delta_\circ\nu^{s_1}, \mathrm{BF}) = L(s+s_1, \Delta_\circ, \mathrm{BF})$, $L\big(s+\frac{1}{2}, \Delta_\circ\nu^{s_1}\big) = L\big(s+s_1+\frac{1}{2}, \Delta_\circ\big)$, and $L(2s, \Delta_\circ\nu^{s_1}, \wedge^2) = L(2s+2s_1, \Delta_\circ, \wedge^2)$. Hence for the calculation, we may assume that $\Delta = \Delta_\circ$ is unitary. The representation Δ is the segment consisting of supercuspidal representations of the form $\Delta_\circ = [\rho_\circ\nu^{-(\ell-1)/2}, \dots, \rho_\circ\nu^{(\ell-1)/2}]$, where $\rho_\circ$ is an irreducible unitary supercuspidal representation of GL_r with $m = \ell r$. We replace σ by $\mathbf{1}_{F^\times}$ in [Cogdell and Piatetski-Shapiro 2017, Corollary in §2.6.2]. Then the formula becomes

$$L\big(s+\tfrac{1}{2}, \Delta_\circ\big) = L\big(s+\tfrac{1}{2}, \Delta_\circ \times \mathbf{1}_{F^\times}\big) = L\big(s+\tfrac{\ell}{2}, \rho_\circ\big).$$

This may also be seen from [Jacquet 1979, Proposition 3.1.3]. Now we are left with invoking Theorem 2.15. □

Finally, Theorem 4.5 renders the factorization result unconditional.

Theorem 4.5. *Let π be an irreducible admissible representation of* GL_m. *Then we have*

$$L(s, \pi, \mathrm{BF}) = L\big(s+\frac{1}{2}, \pi\big)L(2s, \pi, \wedge^2).$$

Proof. We realize π as the unique Langlands quotient of Langlands type $\Xi = \mathrm{Ind}_{\mathrm{Q}}^{\mathrm{GL}_m}(\Delta_{\circ 1}\nu^{u_1} \otimes \Delta_{\circ 2}\nu^{u_2} \otimes \cdots \otimes \Delta_{\circ t}\nu^{u_t})$, which is again of Whittaker type. The local Bump–Friedberg L-function is defined to be $L(s, \pi, \mathrm{BF}) = L(s, \Xi, \mathrm{BF})$. By [Matringe 2015, Theorem 5.2], we have the equality

$$L(s, \Xi, \mathrm{BF}) = \prod_{1 \le k \le t} L(s+u_k, \Delta_{\circ k}, \mathrm{BF}) \prod_{1 \le i < j \le t} L(2s+u_i+u_j, \Delta_{\circ i} \times \Delta_{\circ j}).$$

Applying Theorem 4.4, the product can be further decomposed as

$$L(s,\Xi,\mathrm{BF})=\prod_{1\le k\le t}L\big(s+u_k+\tfrac{1}{2},\Delta_{\circ k}\big)\prod_{1\le k\le t}L(2s+2u_k,\Delta_{\circ k},\wedge^2)\times\prod_{1\le i<j\le t}L(2s+u_i+u_j,\Delta_{\circ i}\times\Delta_{\circ j}).$$

Collecting the contributions for the first product $\prod L\big(s+u_k+\frac{1}{2},\Delta_{\circ k}\big)$ gives the standard L-factor $L\big(s+\frac{1}{2},\Xi\big)=L\big(s+\frac{1}{2},\pi\big)$ by [Jacquet 1979, Theorem 3.4], while gathering those for the rest of the product

$$\prod L(2s+2u_k,\Delta_{\circ k},\wedge^2)\prod L(2s+u_i+u_j,\Delta_{\circ i}\times\Delta_{\circ j})$$

yields the exterior square L-factor $L(2s,\Xi,\wedge^2)=L(2s,\pi,\wedge^2)$ by Theorem 2.13. □

We end this section with relating Bump–Friedberg L-factors to Galois theoretic counterparts. In conclusion, it is a consequence of the local Langlands correspondence that $L\big(s+\frac{1}{2},\pi\big)=L\big(s+\frac{1}{2},\phi(\pi)\big)$ combined with Theorem 3.8 and Theorem 4.5.

Theorem 4.6. *Let π be an irreducible admissible representation of $\mathrm{GL}_m(F)$ and $\phi(\pi)$ its associated Weil–Deligne representation of W'_F under the local Langlands correspondence. Then we have*

$$L(s,\pi,\mathrm{BF})=L\big(s+\tfrac{1}{2},\phi(\pi)\big)L\big(2s,\wedge^2(\phi(\pi))\big).$$

4B. ***Asai L-factors.*** Let E be a quadratic extension of F. We denote by $x\mapsto\bar{x}$ the nontrivial associated Galois action. We fix an element $z\in E^\times$ such that $\bar{z}=-z$ and a nontrivial character ψ_F of F. Let

$$\psi_E(x)=\psi_F\Big(\frac{x-\bar{x}}{z-\bar{z}}\Big),\qquad x\in E.$$

Then the additive character ψ_E of E is trivial on F and defines a character of $N_m(E)$, which by abuse of notation we again denote by ψ_E. We shall use the Fourier transform induced by the additive character ψ on the space of Schwartz–Bruhat space $\mathcal{S}(F^m)$. Let $\pi=\mathrm{Ind}_{\mathrm{Q}}^{\mathrm{GL}_m}(\Delta_1\otimes\cdots\otimes\Delta_t)\in\mathcal{A}_E(m)$ be a parabolically induced representation with an associated Whittaker model $\mathcal{W}(\pi,\psi_E)$. For each Whittaker function $W\in\mathcal{W}(\pi,\psi_E)$ and each Schwartz–Bruhat function $\Phi\in\mathcal{S}(F^m)$, we define the local Flicker integral [1988; 1993] by

$$\mathcal{Z}(s,W,\Phi)=\int_{N_m\backslash\mathrm{GL}_m}W(g)\Phi(e_mg)\,|\det(g)|^s\,dg,$$

which is absolutely convergent when the real part of s is sufficiently large. Each $\mathcal{Z}(s,W,\Phi)$ is a rational function of q^{-s}, and hence extends meromorphically to all of $\mathbb{C}$. These integrals $\mathcal{Z}(s,W,\Phi)$ span a fractional ideal $\mathcal{I}(\pi,\mathrm{As})$ of $\mathbb{C}[q^{\pm s}]$

generated by a normalized generator of the form $P(q^{-s})^{-1}$, where the polynomial $P(X) \in \mathbb{C}[X]$ satisfies $P(0) = 1$. The *local Asai L-function* attached to π is defined by such a unique normalized generator [Matringe 2009, Definition 3.1]

$$L(s, \pi, \mathrm{As}) = \frac{1}{P(q^{-s})}.$$

Let us define the *local Asai* ε-factor, as usual [Matringe 2015, §3] (see [Anandavardhanan et al. 2021, §8]), by

$$\varepsilon(s, \pi, \psi, \mathrm{As}) = \gamma(s, \pi, \psi, \mathrm{As}) \frac{L(s, \pi, \mathrm{As})}{L(1-s, \pi^{\iota}, \mathrm{As})}.$$

The Weil–Deligne group W'_E of E is of index two in the Weil–Deligne group W'_F of F, and the quotient W'_F/W'_E is naturally identified with $\mathrm{Gal}(E/F)$. We fix an element σ in W'_F which does not belong to W'_E once and for all. The image of σ in W'_F/W'_E is the nontrivial element of $\mathrm{Gal}(E/F)$, which by abuse of notation is also denoted by σ. Given an m-dimensional (complex) Frobenius semisimple representation ϕ of W'_E, the *Asai representation* $\mathrm{As}(\phi) : W'_F \to \mathrm{GL}(\mathbb{C}^m \otimes \mathbb{C}^m) \simeq \mathrm{GL}_{m^2}(\mathbb{C})$ given by (twisted) tensor induction of ϕ is defined as (see [Anandavardhanan and Rajan 2005, §2.1], [Krishnamurthy 2012, §2], and [Shankman 2018, §1.2]):

$$\mathrm{As}(\phi)(\tau)(v \otimes w) = \begin{cases} \phi(\tau)(v) \otimes \phi(\sigma\tau\sigma^{-1})(w), & \text{if } \tau \in W'_E, \\ \phi(\tau\sigma^{-1})(w) \otimes \phi(\sigma\tau)(v), & \text{if } \tau \notin W'_E. \end{cases}$$

We then denote by $L(s, \mathrm{As}(\phi))$ the *Artin L-factor* attached to the Asai representation.

The conjugation σ extends naturally to an automorphism of $\mathrm{GL}_m(E)$, which we also denote by σ. If $\pi \in \mathcal{A}_E(m)$, we denote by π^σ the representation $g \mapsto \pi(\sigma(g))$.

Theorem 4.7. *Let π be an irreducible admissible representation of $\mathrm{GL}_m(E)$ and $\phi(\pi)$ its associated Weil–Deligne representation of W'_E under the local Langlands correspondence. Then we have*

$$L(s, \pi, \mathrm{As}) = \mathcal{L}(s, \pi, \mathrm{As}) = L\big(s, \mathrm{As}(\phi(\pi))\big).$$

Proof. We first consider the case of irreducible unitary supercuspidal representations $\rho_\circ$ of GL_r. As a consequence of [Anandavardhanan et al. 2021] joined with [Anandavardhanan and Rajan 2005, Proposition 6] and [Henniart and Lomelí 2013a], we have

$$L(s, \rho_\circ, \mathrm{As}) = \mathcal{L}(s, \rho_\circ, \mathrm{As}).$$

Let Δ be a discrete series representation. In the spirit of twists by unramified characters for Langlands–Shahidi theoretic L-factors [Henniart and Lomelí 2013a, §3.1 (vi)] and Rankin–Selberg theoretic L-factors [Matringe 2009, Theorem 2.3], there is no harm to assume that $\Delta = \Delta_\circ$ is an irreducible square integrable representation of $\mathrm{GL}_{r\ell}$ associated to the segment $[\rho_\circ\nu^{-(\ell-1)/2}, \ldots, \rho_\circ\nu^{(\ell-1)/2}]$ with $\rho_\circ$ an irreducible unitary supercuspidal representation of GL_r. Let $\chi_{E/F}$ be an extension to $E^\times$ of the character $F^\times$ associated to E/F by the local class field theory. As explained in [Anandavardhanan et al. 2021, Appendix A], [Matringe 2009, Corollary 4.24] and [Matringe 2009, Theorem 4.26] driven from the Cogdell and Piatetski-Shapiro method similar to Section 2C depend on the complete classification of $\mathrm{GL}_m(F)$-distinguished representations [Matringe 2011]. Looking at the proof of this proposition, we need to check that the $\mathrm{GL}_m(F)$-distinguished representation, namely, $\mathrm{Hom}_{\mathrm{GL}_m(F)}(\pi, \mathbf{1}_{\mathrm{GL}_m(F)}) \neq \{0\}$, is still Galois self-dual, $\pi^\sigma \simeq \tilde{\pi}$, for any nonarchimedean local field of odd residual characteristic. It is presently written in this generality in the literature, see [Offen 2018, §3.2.12]. Counting on the weak multiplicativity of $\gamma(s, \pi, \mathrm{As}, \psi)$ [Matringe 2009, Corollary 4.24], we get the results below using arguments parallel to the one employed in the proof of Goldberg [1994, Theorem 5.6]:

$$L(s, \Delta_\circ, \mathrm{As}) = \prod_{i=1}^{\ell/2} L(s, \rho_\circ\nu^{(\ell+1)/2-i}, \mathrm{As}) L(s, \chi_{E/F} \otimes \rho_\circ\nu^{\ell/2-i}, \mathrm{As}),$$

when ℓ is even, and

$$L(s, \Delta_\circ, \mathrm{As}) = \prod_{i=1}^{(\ell+1)/2} L(s, \rho_\circ\nu^{(\ell+1)/2-i}, \mathrm{As}) \prod_{i=1}^{(\ell-1)/2} L(s, \chi_{E/F} \otimes \rho_\circ\nu^{\ell/2-i}, \mathrm{As}),$$

when ℓ is odd. The expression is similar to that in Theorem 2.15. This places us in a position to deduce

$$L(s, \Delta, \mathrm{As}) = \mathcal{L}(s, \Delta, \mathrm{As}) \tag{4-2}$$

for any discrete series representations Δ of $\mathrm{GL}_{r\ell}$.

In general, we realize π as the unique Langlands quotient of Langlands type $\Xi = \mathrm{Ind}_{\mathrm{Q}}^{\mathrm{GL}_m}(\Delta_{\circ 1}\nu^{u_1} \otimes \Delta_{\circ 2}\nu^{u_2} \otimes \cdots \otimes \Delta_{\circ t}\nu^{u_t})$. As such, by the inductive relation of $L(s, \pi, \mathrm{As})$ [Matringe 2009, Theorem 4.26], one has the equality

$$L(s, \pi, \mathrm{As}) = \prod_{1 \le k \le t} L(s + 2u_k, \Delta_{\circ k}, \wedge^2) \prod_{1 \le i < j \le t} L(s + u_i + u_j, \Delta_{\circ i} \times \Delta^\sigma_{\circ j}).$$

Consequently, the equality

$$L(s, \pi, \mathrm{As}) = \mathcal{L}(s, \pi, \mathrm{As})$$

follows from [Henniart and Lomelí 2013a, §4.2], along with (4-2) for all irreducible admissible representations π of $\mathrm{GL}_m(E)$. The remaining part is simply to quote the main theorem of Henniart and Lomelí [2013a, Theorem 4.3]. □

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ON WEAK CONVERGENCE OF QUASI-INFINITELY DIVISIBLE LAWS

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We study a new class of so-called quasi-infinitely divisible laws, which is a wide natural extension of the well-known class of infinitely divisible laws through the Lévy–Khinchin representations. We are interested in criteria of weak convergence within this class. Under rather natural assumptions, we state assertions, which connect a weak convergence of quasi-infinitely divisible distribution functions with one special type of convergence of their Lévy–Khinchin spectral functions. The latter convergence is not equivalent to the weak convergence. So we complement known results by Lindner, Pan, and Sato (2018) in this field.

1. Introduction

This paper is devoted to the questions concerning weak convergence within a new class of so-called quasi-infinitely divisible probability laws.

Let F be a distribution function of a probability law on the real line $\mathbb{R}$ with the characteristic function

$$f(t) := \int_{\mathbb{R}} e^{itx}\, dF(x), \quad t \in \mathbb{R}.$$

Recall that F (and the corresponding law) is called *infinitely divisible* if for every positive integer n there exists a distribution function $F_{1/n}$ such that $F = (F_{1/n})^{*n}$, where $*$ is the convolution, i.e., F is n-fold convolution power of $F_{1/n}$. It is well known that F is infinitely divisible if and only if the characteristic function f is represented by the *Lévy–Khinchin formula*:

$$f(t) = \exp\left\{it\gamma + \int_{\mathbb{R}}\left(e^{itx} - 1 - \frac{it}{\tau}\sin(\tau x)\right)\frac{1+x^2}{x^2}\, dG(x)\right\}, \quad t \in \mathbb{R}, \tag{1}$$

with some $\tau > 0$, *shift parameter* $\gamma \in \mathbb{R}$, and with a bounded nondecreasing *spectral function* $G : \mathbb{R} \to \mathbb{R}$ that is assumed to be right-continuous with condition $G(-\infty) = 0$ (all over the paper, $G(\pm\infty)$ denote the limits at $\pm\infty$ correspondingly).

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We use $u \mapsto \frac{1}{\tau}\sin(\tau u)$ as the *centering function* in the integral in (1) following Zolotarev [33; 34]. If (1) holds for some $\tau = \tau_0 > 0$, then it holds for any $\tau > 0$, where γ will depend on τ, but G will not. It is well known that the *spectral pair* (γ, G) is uniquely determined by f and hence by F. The Lévy–Khinchin formula plays a fundamental role in probability theory; it also has a lot of applications in related fields (see [3; 30; 31]).

It turns out that there exists a rather wide class of probability laws that are very similar to infinitely divisible laws. This class of so-called *quasi-infinitely divisible* laws was introduced by Lindner and Sato [22]. Following them, a distribution function F (and the corresponding law) is called *quasi-infinitely divisible* if its characteristic function f admits the representation (1) with some *shift parameter* $\gamma \in \mathbb{R}$, *spectral function* $G : \mathbb{R} \to \mathbb{R}$ of bounded variation on $\mathbb{R}$ (not necessarily monotone), and for some (any) $\tau > 0$. Here G is assumed to be right-continuous with condition $G(-\infty) = 0$ as before and so f (and F) uniquely determines the *spectral pair* (γ, G) (see [13, p. 80]). Observe that, due to the Jordan decomposition, we can represent

$$G(x) = G^{+}(x) - G^{-}(x), \quad x \in \mathbb{R},$$

with some bounded nondecreasing functions G^{+} and G^{-} on $\mathbb{R}$. Also we can always write $\gamma = \gamma^{+} - \gamma^{-}$ with some numbers γ^{+} and γ^{-} from $\mathbb{R}$. Then it is clear that

$$f(t) = f^{+}(t)/f^{-}(t), \quad t \in \mathbb{R},$$

where f^{+} and f^{-} are characteristic functions of some two infinitely divisible distribution functions F^{+} and F^{-} with the spectral pairs (γ^{+}, G^{+}) and (γ^{-}, G^{-}) correspondingly, and so $F * F^{-} = F^{+}$. Starting from this point of view, it is rather natural to call distribution function F (and the corresponding law) *rationally infinitely divisible*. So every infinitely divisible law is quasi-infinitely divisible, but the converse is not true. There are a lot of interesting examples of quasi-infinitely divisible laws, which are not infinitely divisible (see [13, pp. 82–83; 24, p. 165; 25, pp. 123–124]). Moreover, it seems that the class of quasi-infinitely divisible laws is essentially wider than the class of infinitely divisible ones. In particular, it is clearly seen within discrete probability laws (see [2; 18; 19; 23]).

Various forms of definition and the first detailed analysis of the class of quasi-infinitely divisible laws on $\mathbb{R}$ was performed in [23], the multivariate case is considered in the recent papers [6; 7; 21]. There are some results for discrete probability laws in this field (see [1; 2; 17; 18; 19]) and for mixed laws (see [4; 5]). It should be noted that quasi-infinitely divisible laws now have interesting applications in theory of stochastic processes (see [22; 28]), number theory (see [26]), physics (see [11; 12]), and insurance mathematics (see [32]).

We now focus on a weak convergence of quasi-infinitely divisible laws. Recall that, by definition, the sequence $(F_n)_{n\in\mathbb{N}}$ (where $\mathbb{N}$ is the set of positive integers) of distribution functions *weakly converges* to a distribution function F (we will write $F_n \xrightarrow{w} F,\ n \to \infty$) if

$$\int_{\mathbb{R}} h(x)\, dF_n(x) \to \int_{\mathbb{R}} h(x)\, dF(x), \quad n \to \infty \tag{2}$$

for any bounded continuous function $h : \mathbb{R} \to \mathbb{R}$. It is a well known fact that this is equivalent to the following convergence:

$$F_n(x) \to F(x), \quad n \to \infty \qquad \text{for any } x \in S, \tag{3}$$

where S is an arbitrary dense subset of $\mathbb{R}$ and, in particular, it can be chosen as the set of all continuity points of F. The latter convergence is usually called *weak* too (see [25]).

The weak convergence is also introduced for the class of real-valued functions of bounded variation on the real line (or for corresponding signed measures). Following Bogachev [10], it is analogously defined by (2), but instead of $F_n,\ n \in \mathbb{N}$, and F we write some functions of bounded variation $G_n,\ n \in \mathbb{N}$, and G correspondingly. We will save the notation $G_n \xrightarrow{w} G,\ n \to \infty$, in this case. It should be noted that weak convergence here is not equivalent to the analog of convergence (3) with functions of bounded variation (see [10, Section 1.4]).

There are rather general results by Lindner, Pan, and Sato in [23] concerning the weak convergence of quasi-infinitely divisible distribution functions. The authors state the conditions under which the weak convergence of distribution functions implies the weak convergence of the corresponding spectral functions together with the convergence of the shift parameters and vice versa. Namely, let $(F_n)_{n\in\mathbb{N}}$ be a sequence of quasi-infinitely divisible distribution functions and let (γ_n, G_n) be the spectral pair of F_n for every $n \in \mathbb{N}$. Let F be a quasi-infinitely divisible distribution function with the spectral pair (γ, G). Then the results from [23] are in fact the following: (1) If $\gamma_n \to \gamma$ and $G_n \xrightarrow{w} G,\ n \to \infty$, then $F_n \xrightarrow{w} F,\ n \to \infty$. (2) If we suppose $F_n \xrightarrow{w} F,\ n \to \infty$, then, under some assumptions on tightness and uniform boundedness for $(G_n)_{n\in\mathbb{N}}$, we have $\gamma_n \to \gamma$ and $G_n \xrightarrow{w} G,\ n \to \infty$. Here we omitted some details, the full formulation will be given in Section 3.

In this work we complement the results by Lindner, Pan, and Sato [23]. We connect the weak convergence of quasi-infinitely divisible distribution functions with one type of convergence of their spectral functions. The latter convergence is a special modification of the convergence (3) (see the next section for details), and we think that it is more natural and explicit than the weak convergence for the class of functions of bounded variation. A very similar convergence has appeared in [10, Theorem 1.4.7], but we are not aware of the existence of a definition for such a convergence. So we introduce the necessary definition in Section 2. We also show that the introduced convergence for functions of bounded variation follows from the pointwise convergence of their Fourier–Stieltjes transforms under some natural assumptions. This and other close propositions are key tools for our main results devoted to the weak convergence of quasi-infinitely divisible distribution functions. The main results are presented in Section 3. All proofs are provided in Section 4.

2. Preliminaries and tools

Let us consider the class of all functions $G : \mathbb{R} \to \mathbb{R}$ of bounded variations on $\mathbb{R}$. Since we will be interested in the functions G, which generate the Lebesgue–Stieltjes signed measures μ_G, we will focus only on right-continuous functions G. So for the measures we will have $\mu_G((a, b]) = G(b) - G(a)$ for all $a, b \in \mathbb{R}$ and $a \leqslant b$. Recall that all intervals $(a, b]$ constitute the generating semiring for μ_G. So if there are two right-continuous functions G_1 and G_2 of bounded variations such that $G_2(x) = G_1(x) + C,\ x \in \mathbb{R}$, where $C \in \mathbb{R}$ is a constant, then the corresponding measures are the same. Therefore we will consider only functions G that satisfy $G(-\infty) = 0$.

Let $\boldsymbol{V}$ denote the class of all functions $G : \mathbb{R} \to \mathbb{R}$ of bounded variation on $\mathbb{R}$, which are right-continuous at every point $x \in \mathbb{R}$ and satisfy $G(-\infty) = 0$. For every $G \in \boldsymbol{V}$ its total variation on $\mathbb{R}$ will be denoted by $\|G\|$ and the total variation on $(-\infty, x]$ by $|G|(x),\ x \in \mathbb{R}$. So we have

$$|G(x)| \leqslant |G|(x) \leqslant \|G\| \quad \text{for any } x \in \mathbb{R} \tag{4}$$

and $|G|(+\infty) = \|G\|$.

We now introduce a special type of convergence on the class $\boldsymbol{V}$. Suppose that a whole sequence $(G_n)_{n\in\mathbb{N}}$ and a function G are from $\boldsymbol{V}$. We say that $(G_n)_{n\in\mathbb{N}}$ *converges basically* to G, and write $G_n \rightrightarrows G,\ n \to \infty$, if each of its subsequences contains a further subsequence $(G_{n_k})_{k\in\mathbb{N}}$ such that

$$G_{n_k}(x_2) - G_{n_k}(x_1) \to G(x_2) - G(x_1), \quad k \to \infty$$

for any $x_1, x_2 \in \mathbb{R}$ except at most a countable set, which in general depends on the choice of the subsequences.

Let us show that the basic convergence is equivalent to the weak convergence for distribution functions. Let $(F_n)_{n\in\mathbb{N}}$ be a sequence of distribution functions and let F be a distribution function. Suppose that $F_n \xrightarrow{w} F,\ n \to \infty$. Then we have (3), where S is the set of all continuity points of F. Hence

$$F_n(x_2) - F_n(x_1) \to F(x_2) - F(x_1), \quad n \to \infty \qquad \text{for all } x_1, x_2 \in S.$$

Since $\mathbb{R}\backslash S$ is at most countable set, we conclude that $F_n \rightrightarrows F,\ n \to \infty$, by definition. We now suppose that $F_n \rightrightarrows F,\ n \to \infty$. Let $(F_{n_k})_{k\in\mathbb{N}}$ be an arbitrary subsequence of $(F_n)_{n\in\mathbb{N}}$ such that

$$F_{n_k}(x_2) - F_{n_k}(x_1) \to F(x_2) - F(x_1), \quad k \to \infty \tag{5}$$

for any $x_1, x_2 \in \mathbb{R}$ except at most a countable set D. Let us fix $\varepsilon > 0$ and choose $r_\varepsilon > 0$ such that $\pm r_\varepsilon \in \mathbb{R} \setminus D$ and $1 - F(r_\varepsilon) + F(-r_\varepsilon) < \varepsilon$. We define

$$T_k(r) := 1 - F_{n_k}(r) + F_{n_k}(-r), \quad k \in \mathbb{N},\ r > 0.$$

Due to (5), there exists $k_\varepsilon \in \mathbb{N}$ such that $T_k(r_\varepsilon) < \varepsilon$ for all $k \geqslant k_\varepsilon$. Taking r_ε greater to provide $T_k(r_\varepsilon) < \varepsilon$ for all $k < k_\varepsilon$, we obtain $\sup_{k\in\mathbb{N}} T_k(r_\varepsilon) < \varepsilon$ because, due to monotonicity of every F_{n_k}, the inequality $T_k(r_\varepsilon) < \varepsilon$ still holds for all $k \geqslant k_\varepsilon$. Thus $\sup_{k\in\mathbb{N}} T_k(r) \to 0,\ r \to \infty$, and in particular, $\sup_{k\in\mathbb{N}} F_{n_k}(-r) \to 0,\ r \to \infty$. Due to the latter, it is easy to check that (5) yields the convergence $F_{n_k}(x) \to F(x)$ for any $x \in \mathbb{R}$ except at most countable set D. Since $\mathbb{R} \setminus D$ is a dense subset of $\mathbb{R}$, we have $F_{n_k} \xrightarrow{w} F,\ k \to \infty$. Thus, according to definition of basic convergence, every subsequence of $(F_n)_{n\in\mathbb{N}}$ contains a further subsequence $(F_{n_k})_{k\in\mathbb{N}}$ that satisfies (5) and hence weakly converges to F. By the well-known fact in [8, p. 337], it means that the whole sequence $(F_n)_{n\in\mathbb{N}}$ weakly converges to F.

The proved assertion can be generalized for bounded nondecreasing functions $F \in \boldsymbol{V}$ and $F_n \in \boldsymbol{V},\ n \in \mathbb{N}$, but here the basic convergence $F_n \Rightarrow F,\ n \to \infty$, must be taken together with an additional condition that $F_n(+\infty) \to F(+\infty),\ n \to \infty$ (see [13, p. 39]). It should be noted that basic and weak convergences are not equivalent in a general case for functions from $\boldsymbol{V}$. Indeed, the weak convergence implies the basic one that will follow from Theorem 5 below, and also it is seen from Theorem 1.4.7 in [10]. However, the converse is not true. The latter assertion is concluded from the following simple examples. Below $\mathbb{1}_a$ with fixed $a \in \mathbb{R}$ denotes the following functions: $\mathbb{1}_a(x) = 1$ for $x \geqslant a$ and $\mathbb{1}_a(x) = 0$ for $x < a$.

Example 1. Let us define $G_n(x) := \mathbb{1}_n(x) - \mathbb{1}_{n+1}(x),\ x \in \mathbb{R},\ n \in \mathbb{N}$. It is easily seen that $G_n \in \boldsymbol{V},\ n \in \mathbb{N}$, and $G_n(x) \to 0,\ n \to \infty$ for all $x \in \mathbb{R}$. Setting $G(x) := 0,\ x \in \mathbb{R}$, we have the basic convergence $G_n \Rightarrow G,\ n \to \infty$. Here $G_n(+\infty) = G(+\infty) = 0$ and $\|G_n\| = 2,\ n \in \mathbb{N}$. However, for the continuous and bounded function $x \mapsto \cos(\pi x)$, $x \in \mathbb{R}$, we conclude that

$$\int_{\mathbb{R}} \cos(\pi x)\, dG_n(x) \nrightarrow \int_{\mathbb{R}} \cos(\pi x)\, dG(x) = 0, \quad n \to \infty.$$

Indeed,

$$\begin{aligned}\int_{\mathbb{R}} \cos(\pi x)\, dG_n(x) &= \cos(\pi n) - \cos(\pi(n+1)) \\ &= (-1)^n - (-1)^{n+1} = 2 \cdot (-1)^n \nrightarrow 0, \quad n \to \infty.\end{aligned}$$

Thus $(G_n)_{n\in\mathbb{N}}$ doesn't weakly converge to G. $\square$

Example 2. Let $G_n(x) := n\mathbb{1}_0(x) - n\mathbb{1}_{1/n^2}(x),\ x \in \mathbb{R},\ n \in \mathbb{N}$. So $G_n \in \boldsymbol{V},\ n \in \mathbb{N}$, and $G_n(x) \to 0,\ n \to \infty$ for all $x \neq 0$. We set $G(x) := 0,\ x \in \mathbb{R}$, and we obtain that $G_n \Rightarrow G,\ n \to \infty$. Observe that $\|G_n\| = 2n \to \infty,\ n \to \infty$. Hence $(G_n)_{n\in\mathbb{N}}$ cannot be weak convergent sequence, because, under the weak convergence, total variations must be uniformly bounded (see Proposition 1.4.4. in [10, p. 22]). Moreover, it even fails to hold that

$$\int_{\mathbb{R}} h(x)\, dG_n(x) \to \int_{\mathbb{R}} h(x)\, dG(x), \quad n \to \infty \tag{6}$$

for any continuous function h with compact support. Indeed, let $h(x) := \sqrt{x}$ for $x \in [0, 1],\ h(x) := \sqrt{2-x}$ for $x \in [1, 2]$, and $h(x) := 0$ for $x \notin [0, 2]$. Obviously, the function h satisfies the required properties. So we have

$$\int_{\mathbb{R}} h(x)\, dG_n(x) = h(0)\cdot n - h(1/n^2)\cdot n = 0 - \sqrt{1/n^2}\cdot n = -1 \quad \text{for every } n \in \mathbb{N},$$

but $\int_{\mathbb{R}} h(x)\, dG(x) = 0$. Thus (6) does not hold. However, it is interesting to note that there is a convergence of Fourier–Stieltjes transforms. Indeed, for any $t \in \mathbb{R}$

$$\int_{\mathbb{R}} e^{itx}\, dG_n(x) = (1 - e^{it/n^2})\cdot n = -\frac{it}{n}\,(1+o(1)) \to 0 = \int_{\mathbb{R}} e^{itx}\, dG(x), \quad n \to \infty. \ \square$$

The next example shows that the use of the subsequences in the definition of basic convergence is essential.

Example 3. For any $n \in \mathbb{N}$ we set $k_n \in \mathbb{N} \cup \{0\}$ satisfying $2^{k_n} \leqslant n < 2^{k_n+1}$. We define

$$G_n(x) := \mathbb{1}_{a_n}(x) - \mathbb{1}_{b_n}(x), \quad x \in \mathbb{R},$$

where

$$a_n := \frac{n - 2^{k_n}}{2^{k_n}} \quad \text{and} \quad b_n := \frac{n + 1 - 2^{k_n}}{2^{k_n}}, \qquad n \in \mathbb{N}.$$

It is seen that the interval $[a_n, b_n]$ is vanishing ($b_n - a_n = 2^{-k_n} \to 0$) and shifting over $[0, 1]$ as $n \to \infty$. Let $h : \mathbb{R} \to \mathbb{R}$ be a bounded continuous function. Due to the uniform continuity of h on $[0, 1]$, we have that

$$\int_{\mathbb{R}} h(x)\, dG_n(x) = h(a_n) - h(b_n) \to 0, \quad n \to \infty.$$

So $(G_n)_{n\in\mathbb{N}}$ weakly converges to $G(x) := 0$ for all $x \in \mathbb{R}$. Then, by the comments above, $(G_n)_{n\in\mathbb{N}}$ basically converges to G that can be also checked directly by definition. However, for any $x_0, x_1, x_2 \in [0, 1)$ there is no limit either for $G_n(x_0)$ or $G_n(x_2) - G_n(x_1)$ as $n \to \infty$, because $G_n(x)$ takes an infinite number of times each of the values 1 or 0, when $x \in [a_n, b_n)$ or not correspondingly. Note that, due to the weak convergence of $(G_n)_{n\in\mathbb{N}}$, there is a convergence of Fourier–Stieltjes transforms:

$$\int_{\mathbb{R}} e^{itx}\, dG_n(x) \to \int_{\mathbb{R}} e^{itx}\, dG(x) = 0 \quad \text{as } n \to \infty \qquad \text{for every } t \in \mathbb{R}. \qquad \square$$

The following example shows that it is important to use the differences of values of the functions at points x_1 and x_2 in the definition of the basic convergence in order to stay within $\boldsymbol{V}$.

Example 4. For every $n \in \mathbb{N}$ we define $G_n(x) := 1 + \frac{x}{n}$ for $x \in [-n, n]$, $G_n(x) = 0$ for $x < -n$, and $G_n(x) = 2$ for $x > n$. So G_n are nondecreasing continuous functions and $G_n(+\infty) = \|G_n\| = 2$, $n \in \mathbb{N}$. We see that $G_n(x) \to 1$ as $n \to \infty$ for any $x \in \mathbb{R}$. However, an identical 1 doesn't belong to $\boldsymbol{V}$ (it must be 0 at $-\infty$). At the same time for any real x_1 and x_2 we have $G_n(x_2) - G_n(x_1) \to 0$, $n \to \infty$, and we conclude that $(G_n)_{n\in\mathbb{N}}$ basically converges to the function $G(x) := 0$ for all $x \in \mathbb{R}$, which is from $\boldsymbol{V}$. Of course, $(G_n)_{n\in\mathbb{N}}$ doesn't weakly converge to G here, because

$$\int_{\mathbb{R}} dG_n(x) = 2 \nrightarrow \int_{\mathbb{R}} dG(x) = 0, \quad n \to \infty.$$

Note that for $t \neq 0$ we have

$$\int_{\mathbb{R}} e^{itx}\, dG_n(x) = \frac{1}{n}\int_{-n}^{n} e^{itx}\, dx = \frac{e^{itn} - e^{-itn}}{itn} \to 0, \quad n \to \infty$$

and

$$\int_{\mathbb{R}} e^{itx}\, dG_n(x)|_{t=0} = \int_{\mathbb{R}} dG_n(x) = G_n(+\infty) = 2, \quad n \in \mathbb{N}.$$

Thus the Fourier–Stieltjes transforms of G_n, $n \in \mathbb{N}$, pointwisely converge to the Fourier–Stieltjes transform of G (i.e., to identical 0) for almost all $t \in \mathbb{R}$. □

We now consider a general question about the relationship between the basic convergence of functions from $\boldsymbol{V}$ and the convergence of their Fourier–Stieltjes transforms. We do not pretend to study this question in full here; instead, we present only those assertions that will be used in the main results of the article.

Let $(G_n)_{n\in\mathbb{N}}$ be a sequence of functions from $\boldsymbol{V}$. Let us define the corresponding sequence of Fourier–Stieltjes integrals:

$$g_n(t) = \int_{\mathbb{R}} e^{itx}\, dG_n(x), \quad t \in \mathbb{R},\ n \in \mathbb{N}.$$

The results below in fact show that, under the rather weak and natural assumptions, the pointwise convergence of g_n implies the basic convergence of G_n as $n \to \infty$.

We will use the following assumption:

$$\overline{\lim_{n\to\infty}} \|G_n\| = B < \infty. \tag{7}$$

Theorem 5. *Let $(G_n)_{n\in\mathbb{N}}$ satisfy (7). Suppose that $g_n(t) \to g(t)$, $n \to \infty$, for almost all $t \in \mathbb{R}$ with some function $g : \mathbb{R} \to \mathbb{C}$. Then there exists a function $G \in \boldsymbol{V}$ such that $\|G\| \leqslant B$ and the equality*

$$g(t) = \int_{\mathbb{R}} e^{itx}\, dG(x) \tag{8}$$

holds for almost all $t \in \mathbb{R}$ including all continuity points of the function g. The function G is uniquely determined in the class $\boldsymbol{V}$, and $G_n \Rightarrow G$, $n \to \infty$. If also $g_n(0) \to g(0)$, $n \to \infty$, and g is continuous at $t = 0$, then additionally $G_n(+\infty) \to G(+\infty)$, $n \to \infty$.

We are not aware of any results with such assertion. There are some close remarks in [9] and [29]. It is seen that this theorem complements and partially generalizes the well-known Levy's continuity theorem, which was stated for sequences of probability distribution functions.

Suppose that the sequence $(G_n)_{n\in\mathbb{N}}$ weakly converges to a function G from $\boldsymbol{V}$ with Fourier–Stieltjes transform g. Then (7) is satisfied (see Proposition 1.4.4. in [10, p. 22]) and $g_n(t) \to g(t)$, $n \to \infty$ for every $t \in \mathbb{R}$. According to the theorem, we have the basic convergence $G_n \Rightarrow G$ and also $G_n(+\infty) \to G(+\infty)$, $n \to \infty$. Thus we showed that the weak convergence implies the basic convergence.

We next formulate the analog of Theorem 5 using the decompositions

$$G_n(x) = G_n^+(x) - G_n^-(x), \quad x \in \mathbb{R},\ n \in \mathbb{N}, \tag{9}$$

where G_n^+ and G_n^- are nondecreasing functions from $\boldsymbol{V}$. Note that (9) are not necessarily corresponding to the Hahn–Jordan decomposition (see (13) below). Here we assume

$$\varlimsup_{n\to\infty} G_n^-(+\infty) = M < \infty, \tag{10}$$

which can sometimes be more convenient for checking than (7).

Proposition 6. *Let $(G_n)_{n\in\mathbb{N}}$ satisfy (10) for some decompositions (9). Suppose that $g_n(0) \to c \in \mathbb{R}$, $n \to \infty$. Then (7) holds with some $B \leqslant c + 2M$.*

Thus if there is a convergence of g_n, $n \in \mathbb{N}$, at $t = 0$, then for some (9) assumptions (7) and (10) are equivalent. So we come to the following assertion.

Theorem 7. *Let $(G_n)_{n\in\mathbb{N}}$ satisfy (10) for some decompositions (9). Suppose that $g_n(t) \to g(t)$, $n \to \infty$, for almost all $t \in \mathbb{R}$ including $t = 0$ with some function $g : \mathbb{R} \to \mathbb{C}$. Then $g(0) \in \mathbb{R}$, the condition (7) is satisfied with $B \leqslant g(0) + 2M$, and the assertions of Theorem 5 hold. If also g is continuous at $t = 0$, then additionally $G_n(+\infty) \to G(+\infty)$, $n \to \infty$.*

3. Main results

Let $(F_n)_{n\in\mathbb{N}}$ be a sequence of quasi-infinitely divisible distribution function with corresponding sequence of characteristic function $(f_n)_{n\in\mathbb{N}}$. Let every f_n admit the representation

$$f_n(t) = \exp\left\{ it\gamma_n + \int_{\mathbb{R}} \left(e^{itx} - 1 - \frac{it}{\tau} \sin(\tau x) \right) \frac{1+x^2}{x^2}\, dG_n(x) \right\}, \quad t \in \mathbb{R},\ n \in \mathbb{N}, \tag{11}$$

where $\gamma_n \in \mathbb{R}$, $G_n \in \boldsymbol{V}$, $n \in \mathbb{N}$, and $\tau > 0$ is a fixed number. We are interested in criteria of the weak convergence of $(F_n)_{n\in\mathbb{N}}$ in terms of the spectral pairs (γ_n, G_n), $n \in \mathbb{N}$.

Assertions of the following Theorems 8 and 9 were obtained by Lindner, Pan, and Sato in [23] (where the results were presented in another form).

Theorem 8. *If $\gamma_n \to \gamma$ and $G_n \xrightarrow{w} G$, $n \to \infty$, with some $\gamma \in \mathbb{R}$ and $G \in \boldsymbol{V}$, then (γ, G) is the spectral pair for some quasi-infinitely divisible distribution function F, and $(F_n)_{n\in\mathbb{N}}$ weakly converges to F.*

We next use the decompositions

$$G_n(x) = G_n^+(x) - G_n^-(x), \quad x \in \mathbb{R},\ n \in \mathbb{N}, \tag{12}$$

where G_n^+ and G_n^- are nondecreasing functions from $\boldsymbol{V}$. There exists an important way of choosing G_n^+ and G_n^-. Let μ_{G_n} be the signed measure that is generated by G_n for every $n \in \mathbb{N}$, i.e., such that $\mu_{G_n}((a,b]) = G_n(b) - G_n(a)$ for all $a, b \in \mathbb{R}$, $a \leqslant b$, $n \in \mathbb{N}$. Every measure μ_{G_n} is uniquely represented by *the Hahn–Jordan decomposition* $\mu_{G_n} = \mu_{G_n}^+ - \mu_{G_n}^-$, where $\mu_{G_n}^+$ and $\mu_{G_n}^-$ are nonnegative finite measures concentrated on some disjoint sets (see [10, p. 3]). So we can choose

$$G_n^+(x) = \mu_{G_n}^+((-\infty, x]) \quad \text{and} \quad G_n^-(x) = \mu_{G_n}^-((-\infty, x]), \qquad x \in \mathbb{R},\ n \in \mathbb{N}. \tag{13}$$

In this case we will have (12) and additionally that $|G_n|(x) = G_n^+(x) + G_n^-(x)$, $x \in \mathbb{R}$, $n \in \mathbb{N}$.

Theorem 9. *Let F be a distribution function and $(F_n)_{n\in\mathbb{N}}$ weakly converge to F. Suppose that G_n^+ and G_n^- from* (12) *are defined according to the Hahn–Jordan decomposition by* (13) *for every $n \in \mathbb{N}$. Suppose that the sequence $(G_n^-)_{n\in\mathbb{N}}$ satisfies the assumptions*

$$\sup_{n\in\mathbb{N}} \|G_n^-\| < \infty \quad \textit{and} \quad \lim_{r\to\infty} \sup_{n\in\mathbb{N}} (1 - |G_n^-|(r) + |G_n^-|(-r)) = 0$$

(*uniform boundedness in variation and tightness*, *correspondingly*). *Then F is quasi-infinitely divisible with some spectral pair (γ, G). Moreover, $\gamma_n \to \gamma$ and $G_n \xrightarrow{w} G$, $n \to \infty$.*

Theorems 8 and 9 connect the weak convergence of quasi-infinitely divisible distribution functions with the weak convergence of their spectral functions. We are interested in analogs of these theorems but with the basic convergence of the spectral functions.

We will use the following assumption:

$$\overline{\lim_{n\to\infty}} \|G_n\| = B < \infty. \tag{14}$$

Theorem 10. *Suppose that* $(F_n)_{n\in\mathbb{N}}$ *satisfies* (14) *with some* $B \geqslant 0$. *Let* $(F_n)_{n\in\mathbb{N}}$ *weakly converge to a distribution function* F. *Then* F *is quasi-infinitely divisible with some spectral pair* (γ, G), *where* $\gamma \in \mathbb{R}$ *and* $G \in \boldsymbol{V}$ *with* $\|G\| \leqslant B$. *Moreover,* $\gamma_n \to \gamma$, $G_n \Rightarrow G$ *and* $G_n(+\infty) \to G(+\infty)$, $n \to \infty$.

The next theorem is an analog of this one, but with the assumption

$$\varlimsup_{n\to\infty} G_n^-(+\infty) = M < \infty \tag{15}$$

on decompositions (12) for G_n, $n \in \mathbb{N}$. If we choose G_n^+ and G_n^- according to the Hahn–Jordan decomposition by (13) for every $n \in \mathbb{N}$, then (15) is weaker than (14). Also observe that (15) is satisfied, when we deal with nondecreasing functions G_n, $n \in \mathbb{N}$. It should be noted, however, that it is not required in the theorems and corollaries below that G_n^+ and G_n^- in (12) must be chosen according to the Hahn–Jordan decomposition.

Theorem 11. *Suppose that* $(F_n)_{n\in\mathbb{N}}$ *satisfies* (15) *with some* $M \geqslant 0$ *and for some decompositions* (12). *Let* $(F_n)_{n\in\mathbb{N}}$ *weakly converge to a distribution function* F. *Then* (14) *holds for some* $B \geqslant 0$ *and all assertions of Theorem 10 are true. Also we have that* $B \leqslant G(+\infty) + 2M$.

Theorems 10 and 11 yield necessary conditions for the weak convergence within the class of quasi-infinitely divisible distribution functions under the assumption (14) or (15).

Corollary 12. *Suppose that* $(F_n)_{n\in\mathbb{N}}$ *satisfies* (14) *or* (15) *for some decompositions* (12). *Let* F *be a quasi-infinitely divisible distribution function* F *with spectral pair* (γ, G), *where* $\gamma \in \mathbb{R}$ *and* $G \in \boldsymbol{V}$. *If the sequence* $(F_n)_{n\in\mathbb{N}}$ *weakly converges to* F, *then* $\gamma_n \to \gamma$, $G_n \Rightarrow G$ *and* $G_n(+\infty) \to G(+\infty)$, $n \to \infty$.

Also Theorems 10 and 11 state sufficient conditions for membership of the class of quasi-infinitely divisible distribution functions.

Corollary 13. *A distribution function* F *is quasi-infinitely divisible if it is a weak limit of a sequence* $(F_n)_{n\in\mathbb{N}}$ *of quasi-infinitely divisible distribution functions* (*with characteristic functions* (11)), *which satisfies* (14) *or* (15) *for some decompositions* (12).

Note that this corollary is a stronger version of the same assertion in Theorem 9, because we don't assume the tightness for $(G_n^-)_{n\in\mathbb{N}}$ and we don't require the use of the Hahn–Jordan decomposition.

It is known (see [23, p. 17]) that a weak limit of quasi-infinitely divisible distribution functions is not necessarily quasi-infinitely divisible. Hence assumptions (14) or (15) cannot be simply omitted in Corollary 13. However, it seems that they can be done weaker (see [23, Example 4.4]).

We will use a notion of relative compactness for $(F_n)_{n\in\mathbb{N}}$ in the next theorem. Recall that $(F_n)_{n\in\mathbb{N}}$ is said to be *relatively compact* if every its subsequence contains a further subsequence that weakly converges to a distribution function. It is clear that a weakly convergent sequence of distribution functions is relatively compact. In general, the property of relative compactness is not difficult for checking due to Prokhorov's theorem and various probability inequalities. Also some criteria of relative compactness are known for particular important sequences of distribution functions (for example, see [15; 16; 17]).

Theorem 14. *Suppose that $(F_n)_{n\in\mathbb{N}}$ satisfies* (14). *If $(F_n)_{n\in\mathbb{N}}$ is relatively compact and $\gamma_n \to \gamma$, $G_n \Rightarrow G$, $n \to \infty$, with some $\gamma \in \mathbb{R}$ and $G \in \boldsymbol{V}$, then (γ, G) is the spectral pair for a quasi-infinitely divisible distribution function F and the sequence $(F_n)_{n\in\mathbb{N}}$ weakly converges to F.*

This theorem yields sufficient conditions for the weak convergence within the class of quasi-infinitely divisible distribution functions under the assumption (14).

Corollary 15. *Suppose that $(F_n)_{n\in\mathbb{N}}$ satisfies* (14). *Let F be a quasi-infinitely divisible distribution function F with spectral pair (γ, G), where $\gamma \in \mathbb{R}$ and $G \in \boldsymbol{V}$. If $(F_n)_{n\in\mathbb{N}}$ is relatively compact and $\gamma_n \to \gamma$, $G_n \Rightarrow G$, $n \to \infty$, then $(F_n)_{n\in\mathbb{N}}$ weakly converges to F.*

Corollaries 12 and 15 directly yield the following criterion.

Theorem 16. *Suppose that $(F_n)_{n\in\mathbb{N}}$ satisfies* (14). *Let F be a quasi-infinitely divisible distribution function F with spectral pair (γ, G), where $\gamma \in \mathbb{R}$ and $G \in \boldsymbol{V}$. The sequence $(F_n)_{n\in\mathbb{N}}$ weakly converges to F if and only if $(F_n)_{n\in\mathbb{N}}$ is relatively compact and $\gamma_n \to \gamma$, $G_n \Rightarrow G$, $n \to \infty$. Moreover, the convergence $G_n(+\infty) \to G(+\infty)$, $n \to \infty$, can be added to the necessary conditions.*

We now formulate the analogs of Theorems 14 and 16, and of Corollary 15 under the assumption (15). They are directly stated due to the following simple note.

Suppose that a sequence $(G_n)_{n\in\mathbb{N}}$ from $\boldsymbol{V}$ satisfies (15) for some decompositions (12). If $\overline{\lim}_{n\to\infty} G_n(+\infty)$ is finite, then (14) holds. Indeed, according to (12), it follows from the inequalities

$$\|G_n\| \leqslant \|G_n^+\| + \|G_n^-\| = G_n^+(+\infty) + G_n^-(+\infty) = G_n(+\infty) + 2G_n^-(+\infty), \quad n \in \mathbb{N}.$$

So we obtain the following results.

Theorem 17. *Suppose that $(F_n)_{n\in\mathbb{N}}$ satisfies* (15) *for some decompositions* (12). *If $(F_n)_{n\in\mathbb{N}}$ is relatively compact and $\gamma_n \to \gamma$, $G_n \Rightarrow G$, and $G_n(+\infty) \to G(+\infty)$, $n \to \infty$, with some $\gamma \in \mathbb{R}$ and $G \in \boldsymbol{V}$, then all assertions of Theorem 14 hold.*

So Theorems 14 and 17 complement Theorem 8: we use weaker convergence for the spectral functions $(G_n)_{n\in\mathbb{N}}$, but we additionally assume the relative compactness of $(F_n)_{n\in\mathbb{N}}$.

Corollary 18. *Suppose that* $(F_n)_{n\in\mathbb{N}}$ *satisfies* (15) *for some decompositions* (12). *Let* F *be a quasi-infinitely divisible distribution function* F *with the spectral pair* (γ, G), *where* $\gamma \in \mathbb{R}$ *and* $G \in \boldsymbol{V}$. *If* $(F_n)_{n\in\mathbb{N}}$ *is relatively compact and* $\gamma_n \to \gamma$, $G_n \Rightarrow G$ *and* $G_n(+\infty) \to G(+\infty)$, $n \to \infty$, *then* $(F_n)_{n\in\mathbb{N}}$ *weakly converges to* F.

Corollaries 12 and 18 directly yield the following criterion.

Theorem 19. *Suppose that* $(F_n)_{n\in\mathbb{N}}$ *satisfies* (15) *for some decompositions* (12). *Let* F *be a quasi-infinitely divisible distribution function* F *with spectral pair* (γ, G), *where* $\gamma \in \mathbb{R}$ *and* $G \in \boldsymbol{V}$. *The sequence* $(F_n)_{n\in\mathbb{N}}$ *weakly converges to* F *if and only if* $(F_n)_{n\in\mathbb{N}}$ *is relatively compact and* $\gamma_n \to \gamma$, $G_n \Rightarrow G$, *and* $G_n(+\infty) \to G(+\infty)$, $n \to \infty$.

On account of comments before Example 1 in Section 2, Theorems 16 and 19 complement similar well-known results for the weak convergence of infinitely divisible distribution functions (see [13, p. 87]).

4. Proofs

Proof of Theorem 5. First, observe that the function g is measurable, because it is an almost everywhere limit of continuous (hence measurable) functions g_n, $n \in \mathbb{N}$. So we have

$$\int_{\mathbb{R}} g_n(t)\rho(t)\,dt \to \int_{\mathbb{R}} g(t)\rho(t)\,dt, \quad n \to \infty \tag{16}$$

for any function $\rho \in L_1(\mathbb{R})$. Indeed, due to (7), there exists a constant $B_0 > 0$ such that $|g_n(t)| \leqslant \|G_n\| \leqslant B_0$ for all $n \in \mathbb{N}$, and convergence (16) holds by the Lebesgue dominated convergence theorem.

Let us define the function

$$\varphi(x) := \int_{\mathbb{R}} e^{itx}\rho(t)\,dt, \quad x \in \mathbb{R}.$$

Observe that for every $n \in \mathbb{N}$ we have

$$\begin{aligned}\int_{\mathbb{R}} g_n(t)\rho(t)\,dt &= \int_{\mathbb{R}}\Big(\int_{\mathbb{R}} e^{itx}\,dG_n(x)\Big)\rho(t)\,dt \\ &= \int_{\mathbb{R}}\Big(\int_{\mathbb{R}} e^{itx}\rho(t)\,dt\Big)\,dG_n(x) = \int_{\mathbb{R}}\varphi(x)\,dG_n(x).\end{aligned} \tag{17}$$

Let us consider the last integral. Due to (4) and (7), by Helly's first theorem (see [27, pp. 222 and 240]), there exists a subsequence $(G_{n_k})_{k\in\mathbb{N}}$ in $(G_n)_{n\in\mathbb{N}}$ and a function of bounded variation $G_* : \mathbb{R} \to \mathbb{R}$ such that $G_{n_k}(x) \to G_*(x)$ as $k \to \infty$ for all $x \in \mathbb{R}$. Note that, in general, G_* may not be right-continuous (see Example 3). But φ is bounded and continuous on $\mathbb{R}$ and hence there exists the Riemann–Stieltjes integral $\int_{\mathbb{R}}\varphi(x)\,dG_*(x)$. Also the (Lebesgue–Stieltjes) integrals $\int_{\mathbb{R}}\varphi(x)\,dG_n(x)$ coincide with the corresponding Riemann–Stieltjes integrals. Next, due to the well-known fact that $\varphi(x) \to 0$ as $x \to \pm\infty$, by Helly's second theorem (see [27, p. 240]), we have the following convergence for the Riemann–Stieltjes integrals:

$$\int_{\mathbb{R}} \varphi(x)\, dG_{n_k}(x) \to \int_{\mathbb{R}} \varphi(x)\, dG_*(x), \quad k \to \infty.$$

Let us define $G(x) := G_*(x+) - G_*(-\infty)$, $x \in \mathbb{R}$ (note that $G_*(-\infty) \neq 0$ in general, see Example 4). So G is right-continuous on $\mathbb{R}$ and $G(-\infty) = 0$, i.e., $G \in \boldsymbol{V}$. Since $G(x)$ equals $G_*(x) - G_*(-\infty)$ for all $x \in \mathbb{R}$ except at most countable set, due to the continuity of φ, we have

$$\int_{\mathbb{R}} \varphi(x)\, dG_*(x) = \int_{\mathbb{R}} \varphi(x)\, dG(x),$$

where the integral on the right-hand side can be considered as Lebesgue–Stieltjes integral. Thus we have the following convergence with the Lebesgue–Stieltjes integrals:

$$\int_{\mathbb{R}} \varphi(x)\, dG_{n_k}(x) \to \int_{\mathbb{R}} \varphi(x)\, dG(x), \quad k \to \infty.$$

The integral on the right-hand side admits the following representation analogously to (17):

$$\int_{\mathbb{R}} \varphi(x)\, dG(x) = \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} e^{itx}\, dG(x) \Big) \rho(t)\, dt.$$

Due to (16) and (17), we also have

$$\int_{\mathbb{R}} \varphi(x)\, dG_{n_k}(x) \to \int_{\mathbb{R}} g(t)\rho(t)\, dt, \quad k \to \infty.$$

Thus we obtain

$$\int_{\mathbb{R}} g(t)\rho(t)\, dt = \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} e^{itx}\, dG(x) \Big) \rho(t)\, dt \tag{18}$$

for any function $\rho \in L_1(\mathbb{R})$. This implies that

$$g(t) = \int_{\mathbb{R}} e^{itx}\, dG(x) \quad \text{for almost every } t \in \mathbb{R}. \tag{19}$$

Indeed, conversely, suppose that there exists a bounded set E of nonzero Lebesgue measure such that $\Delta(t) := g(t) - \int_{\mathbb{R}} e^{itx}\, dG(x) \neq 0$, $t \in E$. Let us introduce the sets

$$E_1 := \{t \in E : \operatorname{Re} \Delta(t) > 0\}, \quad E_2 := \{t \in E : \operatorname{Re} \Delta(t) < 0\},$$

$$E_3 := \{t \in E : \operatorname{Im} \Delta(t) > 0\}, \quad E_4 := \{t \in E : \operatorname{Im} \Delta(t) < 0\}.$$

It is easily seen that $E = E_1 \cup E_2 \cup E_3 \cup E_4$. Hence at least one E_j has nonzero Lebesgue measure. We denote any such set by E_*. Next, according to the property of strict positivity of integral, we obtain

$$\Big| \int_{E_*} \Delta(t)\, dt \Big| \geqslant \Big| \int_{E_*} \operatorname{Re} \Delta(t)\, dt \Big| = \int_{E_*} |\operatorname{Re} \Delta(t)|\, dt > 0 \quad \text{for } E_* = E_1 \text{ or } E_* = E_2,$$

and

$$\left|\int_{E_*}\Delta(t)\,dt\right| \geqslant \left|\int_{E_*}\operatorname{Im}\Delta(t)\,dt\right| = \int_{E_*}|\operatorname{Im}\Delta(t)|\,dt > 0 \quad \text{for } E_* = E_3 \text{ or } E_* = E_4.$$

Thus we have

$$\left|\int_{E_*} g(t)\,dt - \int_{E_*}\Big(\int_{\mathbb{R}} e^{itx}\,dG(x)\Big)\,dt\right| = \left|\int_{E_*}\Delta(t)\,dt\right| > 0.$$

This contradicts (18) when we choose ρ as follows: $\rho(t) = 1,\ t \in E_*$ and $\rho(t) = 0$, $t \notin E_*$. It is valid since $\rho \in L_1(\mathbb{R})$ due to the boundedness $E_* \subset E$. Thus (19) is true.

Let us show that (19) holds for every continuity point of the function g. Let $\mathcal{T}$ be the set of all $t \in \mathbb{R}$ for which (19) holds. Hence the Lebesgue measure of $\mathbb{R}\setminus\mathcal{T}$ equals zero. Let g be continuous at the fixed point t_0. So we can choose $t_m \in \mathcal{T},\ m \in \mathbb{N}$ such that $t_m \to t_0,\ m \to \infty$. Then $g(t_m) \to g(t_0),\ m \to \infty$, and at the same time

$$g(t_m) = \int_{\mathbb{R}} e^{it_m x}\,dG(x) \to \int_{\mathbb{R}} e^{it_0 x}\,dG(x), \quad m \to \infty$$

due to continuity of the function $t \mapsto \int_{\mathbb{R}} e^{itx}\,dG(x)$ on $\mathbb{R}$. Thus we have

$$g(t_0) = \int_{\mathbb{R}} e^{it_0 x}\,dG(x).$$

According to (19), the function g almost everywhere coincides with the continuous function $t \mapsto \int_{\mathbb{R}} e^{itx}\,dG(x),\ t \in \mathbb{R}$. So the latter function is uniquely determined by g within the class of all continuous complex-valued functions on $\mathbb{R}$. Next, it is well known that $t \mapsto \int_{\mathbb{R}} e^{itx}\,dG(x),\ t \in \mathbb{R}$, uniquely determines G within the class $\boldsymbol{V}$. Therefore g uniquely determines G in the class $\boldsymbol{V}$.

Let's return to the sequence $(G_{n_k})_{k\in\mathbb{N}}$. From the above we know that $G_{n_k}(x) \to G_*(x)$ for all $x \in \mathbb{R}$ and $G(x) = G_*(x) - G_*(-\infty)$ for all $x \in \mathbb{R}$ except at most a countable set D where G_* is not right-continuous. Then for all $x_1, x_2 \in \mathbb{R}\setminus D$ we have

$$\begin{aligned}(20)\quad & G_{n_k}(x_2) - G_{n_k}(x_1) \\ & \to (G(x_2) + G_*(-\infty)) - (G(x_1) + G_*(-\infty)) = G(x_2) - G(x_1), \quad k \to \infty.\end{aligned}$$

Let $(G_{m_l})_{l\in\mathbb{N}}$ be an arbitrary subsequence of $(G_n)_{n\in\mathbb{N}}$. Analogously to the above, there exists a further subsequence $(G_{m'_k})_{k\in\mathbb{N}}$ in $(G_{m_k})_{k\in\mathbb{N}}$, which pointwise converges to some function of bounded variation $H_*:\mathbb{R}\to\mathbb{R}$, i.e., $G_{m'_k}(x) \to H_*(x)$, $k \to \infty$ for all $x \in \mathbb{R}$. Defining $H(x) := H_*(x+) - H_*(-\infty),\ x \in \mathbb{R}$, we as before will obtain $g(t) = \int_{\mathbb{R}} e^{itx}\,dH(x)$ for almost all $t \in \mathbb{R}$, with $H \in \boldsymbol{V}$. Since G is a unique function within $\boldsymbol{V}$, which represents g by (8), we have $H(x) = G(x),\ x \in \mathbb{R}$. We also have

$$G_{m'_k}(x_2) - G_{m'_k}(x_1) \to G(x_2) - G(x_1), \quad k \to \infty$$

for all $x_1, x_2 \in \mathbb{R}$ except at most countable set D' where H_* is not right-continuous (in general $D' \neq D$). So we proved that $G_n \Rightarrow G,\ n \to \infty$.

Let us consider the numbers $g_n(0) = \int_{\mathbb{R}} dG_n(x) = G_n(+\infty),\ n \in \mathbb{N}$. If we suppose that g is continuous at $t = 0$, then, by the above remarks, we will have $g(0) = \int_{\mathbb{R}} dG(x) = G(+\infty)$. Therefore, assuming to hold $g_n(0) \to g(0),\ n \to \infty$, we will obtain $G_n(+\infty) \to G(+\infty),\ n \to \infty$.

It remains to prove that $\|G\| \leqslant B$. On the contrary, suppose that this is false. Then we can find $y_0, y_1, \dots, y_N \in \mathbb{R}$ such that

$$B < \sum_{j=1}^{N} |G(y_j) - G(y_{j-1})| \leqslant \|G\|. \tag{21}$$

Let us take our sequence $(G_{n_k})_{k\in\mathbb{N}}$ and the set D, which is at most countable. Since G is right-continuous and the set $\mathbb{R} \setminus D$ is dense, we can assume that $y_0, y_1, \dots, y_N$ are chosen from $\mathbb{R} \setminus D$. Next, due to the convergence (20) and assumption (7), we have

$$\sum_{j=1}^{N} |G(y_j) - G(y_{j-1})| = \lim_{k\to\infty} \sum_{j=1}^{N} |G_{n_k}(y_j) - G_{n_k}(y_{j-1})| \leqslant \varlimsup_{n\to\infty} \|G_n\| \leqslant B,$$

which contradicts (21). □

Proof of Proposition 6. By the assumption $g_n(0) \to c \in \mathbb{R},\ n \to \infty$. Since $g_n(0) = \int_{\mathbb{R}} dG_n(x) = G_n(+\infty),\ n \in \mathbb{N}$, we have the convergence $G_n(+\infty) \to c,\ n \to \infty$. Let us consider decompositions (9). We have $G_n(+\infty) = G_n^+(+\infty) - G_n^-(+\infty)$, $n \in \mathbb{N}$. Also observe that

$$\|G_n\| \leqslant \|G_n^+\| + \|G_n^-\| = G_n^+(+\infty) + G_n^-(+\infty) = G_n(+\infty) + 2G_n^-(+\infty), \quad n \in \mathbb{N}.$$

Therefore

$$B = \varlimsup_{n\to\infty} \|G_n\| \leqslant \lim_{n\to\infty} G_n(+\infty) + 2 \varlimsup_{n\to\infty} G_n^-(+\infty) = c + 2M.$$

Thus we have (7) with $B \leqslant g(0) + 2M$. □

Proof of Theorem 7. By the assumption $g_n(0) \to g(0),\ n \to \infty$. Since $g_n(0) = \int_{\mathbb{R}} dG_n(x) = G_n(+\infty),\ n \in \mathbb{N}$, we have the convergence $G_n(+\infty) \to g(0),\ n \to \infty$. So the sequence $G_n(+\infty) \in \mathbb{R},\ n \in \mathbb{N}$, has a finite limit $g(0)$ that must be real. According to Proposition 6, condition (7) holds with some $B \leqslant g(0) + 2M$. Using Theorem 5, we get all its assertions. So $g(t) = \int_{\mathbb{R}} e^{itx}\, dG(x)$ holds for some $G \in \boldsymbol{V}$ and for all $t \in \mathbb{R}$ that are continuity points of the function g. Under the assumption, g is continuous at $t = 0$, and we have $g(0) = \int_{\mathbb{R}} dG(x) = G(+\infty)$. Since $G_n(+\infty) \to g(0),\ n \to \infty$, we obtain that

$$G_n(+\infty) \to G(+\infty), \quad n \to \infty.$$ □

We need the following lemma for proving Theorem 10.

Lemma 20. *For any $t \in \mathbb{R}$ and $\tau > 0$ the following representations hold*:

$$e^{itx} - 1 - \frac{it}{\tau}\sin(\tau x) = \int_{A_{t,\tau}} e^{isx}\,dU_{t,\tau}(s),$$

(22)

$$\left(e^{itx} - 1 - \frac{it}{\tau}\sin(\tau x)\right)\frac{1}{x^2} = \int_{A_{t,\tau}} e^{isx}\,dV_{t,\tau}(s),$$

$$\left(e^{itx} - 1 - \frac{it}{\tau}\sin(\tau x)\right)\frac{1+x^2}{x^2} = \int_{A_{t,\tau}} e^{isx}\,dW_{t,\tau}(s), \quad x \in \mathbb{R}, \tag{23}$$

where $A_{t,\tau} := \big\{s \in \mathbb{R} : |s| \leqslant \max\{|t|, \tau\}\big\}$, and

$$U_{t,\tau}(s) := \mathbb{1}_t(s) - \mathbb{1}_0(s) - \tfrac{t}{2\tau}(\mathbb{1}_\tau(s) - \mathbb{1}_{-\tau}(s)), \quad s \in \mathbb{R}, \tag{24}$$

$$V_{t,\tau}(s) := \int_{-\infty}^{s} \rho_{t,\tau}(y)\,dy,$$

(25)

$$\rho_{t,\tau}(s) := -\tfrac{1}{2}\big(|s-t| - |s| - \tfrac{t}{2\tau}(|s-\tau| - |s+\tau|)\big), \quad s \in \mathbb{R},$$

$$W_{t,\tau}(s) := U_{t,\tau}(s) + V_{t,\tau}(s), \quad s \in \mathbb{R}. \tag{26}$$

For any $t \in \mathbb{R}$ and $\tau > 0$ it is true that $U_{t,\tau}(s) = 0$ and $\rho_{t,\tau}(s) = 0$ for all $s \notin A_{t,\tau}$, $\rho_{t,\tau}$ is a continuous function on $\mathbb{R}$ with a broken-line graph, and, in particular, $\rho_{t,\tau} \in L_1(\mathbb{R})$, the functions $U_{t,\tau}$, $V_{t,\tau}$, and $W_{t,\tau}$ belong to the class $\boldsymbol{V}$.

Proof of Lemma 20. Let us fix $t \in \mathbb{R}$, $\tau > 0$, and define $A_{t,\tau}$ as in the formulation. We write

$$e^{itx} - 1 - \frac{it}{\tau}\sin(\tau x) = e^{itx} - 1 - \frac{t}{2\tau}(e^{i\tau x} - e^{-i\tau x}) = \int_{\mathbb{R}} e^{isx}\,dU_{t,\tau}(s), \quad x \in \mathbb{R}, \tag{27}$$

where $U_{t,\tau}$ is defined by (24). Using the definition of the function $\mathbb{1}_a(\,\cdot\,)$, $a \in \mathbb{R}$, it is easily seen that $U_{t,\tau}$ is an right-continuous function on $\mathbb{R}$, $U_{t,\tau}(s) = 0$ for all $s \notin A_{t,\tau}$, and in particular, $U_{t,\tau} \in \boldsymbol{V}$. Therefore the set $\mathbb{R}$ can be changed by $A_{t,\tau}$ in the integral (27).

Let us consider the function

$$\varphi_{t,\tau}(x) := \left(e^{itx} - 1 - \frac{it}{\tau}\sin(\tau x)\right)\frac{1}{x^2}, \quad x \in \mathbb{R}.$$

Observe that $\varphi_{t,\tau} \in L_1(\mathbb{R})$. So we define

$$\rho_{t,\tau}(s) := \frac{1}{2\pi}\int_{\mathbb{R}} e^{-isx}\varphi_{t,\tau}(x)\,dx, \quad s \in \mathbb{R}. \tag{28}$$

Let us find an explicit formula for $\rho_{t,\tau}(s)$ for every $s \in \mathbb{R}$. Observe that $x \mapsto \operatorname{Re}\varphi_{t,\tau}(x)$, $x \in \mathbb{R}$, is an even function and $x \mapsto \operatorname{Im}\varphi_{t,\tau}(x)$, $x \in \mathbb{R}$, is an odd function. Therefore

$$\begin{aligned}\rho_{t,\tau}(s) &= \frac{1}{2\pi}\int_{\mathbb{R}}\big(\operatorname{Re}\varphi_{t,\tau}(x)\cos(sx)+\operatorname{Im}\varphi_{t,\tau}(x)\ \sin(sx)\big)\,dx\\ &= \frac{1}{\pi}\int_0^\infty\big(\operatorname{Re}\varphi_{t,\tau}(x)\cos(sx)+\operatorname{Im}\varphi_{t,\tau}(x)\ \sin(sx)\big)\,dx\\ &= \frac{1}{\pi}\int_0^\infty\left(\frac{\cos(tx)-1}{x^2}\cos(sx)+\frac{\sin(tx)-\frac{t}{\tau}\sin(\tau x)}{x^2}\sin(sx)\right)dx,\quad s\in\mathbb{R}.\end{aligned}$$

Next, using the known trigonometric formulas, we write

$$\begin{aligned}&\rho_{t,\tau}(s)\\ &= \frac{1}{\pi}\int_0^\infty\left(\frac{\cos(tx)\cos(sx)+\sin(tx)\sin(sx)-\cos(sx)}{x^2}-\frac{\frac{t}{\tau}\sin(\tau x)\sin(sx)}{x^2}\right)dx\\ &= \frac{1}{\pi}\int_0^\infty\left(\frac{\cos((s-t)x)-\cos(sx)}{x^2}-\frac{t}{\tau}\,\frac{\cos((s-\tau)x)-\cos((s+\tau)x)}{2x^2}\right)dx\\ &= \frac{1}{\pi}\int_0^\infty\frac{\cos(|s-t|x)-\cos(|s|x)}{x^2}\,dx\\ &\qquad\qquad -\frac{t}{2\tau}\cdot\frac{1}{\pi}\int_0^\infty\frac{\cos(|s-\tau|x)-\cos(|s+\tau|x)}{x^2}\,dx,\quad s\in\mathbb{R}.\end{aligned}$$

It is known (see [14, p. 447, formula 3.782(2)]) that

$$\int_0^\infty\frac{1-\cos(ax)}{x^2}\,dx=\frac{a\pi}{2},\quad a\geqslant 0.$$

Hence

$$\rho_{t,\tau}(s)=-\tfrac{1}{2}\big(|s-t|-|s|-\tfrac{t}{2\tau}(|s-\tau|-|s+\tau|)\big),\quad s\in\mathbb{R},$$

as in (25). We see that $\rho_{t,\tau}$ is a continuous function with a broken-line graph. Also observe that $\rho_{t,\tau}(s)=0$ for all $s\notin A_{t,\tau}$. Indeed, if $s>\max\{|t|,\tau\}$, then

$$\rho_{t,\tau}(s)=-\tfrac{1}{2}\big(s-t-s-\tfrac{t}{2\tau}(s-\tau-(s+\tau))\big)=-\tfrac{1}{2}\big(-t-\tfrac{t}{2\tau}\cdot(-2\tau)\big)=-\tfrac{1}{2}(-t+t)=0,$$

and if $s<-\max\{|t|,\tau\}$, then

$$\rho_{t,\tau}(s)=-\tfrac{1}{2}\big(-(s-t)+s-\tfrac{t}{2\tau}(-(s-\tau)+s+\tau)\big)=-\tfrac{1}{2}\big(t-\tfrac{t}{2\tau}\cdot 2\tau\big)=-\tfrac{1}{2}(t-t)=0.$$

Thus $\rho_{t,\tau}\in L_1(\mathbb{R})$. By the way, observe that $V_{t,\tau}$, which is defined by (25), is a continuous function on $\mathbb{R}$ and it vanishes at $-\infty$, i.e., $V_{t,\tau}\in\boldsymbol{V}$. Then, according to these remarks and (28), we have

$$\varphi_{t,\tau}(x)=\int_{\mathbb{R}}e^{isx}\rho_{t,\tau}(s)\,ds=\int_{A_{t,\tau}}e^{isx}\rho_{t,\tau}(s)\,ds=\int_{A_{t,\tau}}e^{isx}\,dV_{t,\tau}(s),\quad x\in\mathbb{R}.$$

Next, summing the proved equalities in (22), we get (23) with $W_{t,\tau}$ defined by (26). Since $U_{t,\tau}$ and $V_{t,\tau}$ belong to $\boldsymbol{V}$, we conclude that $W_{t,\tau}\in\boldsymbol{V}$. $\square$

Proof of Theorem 10. Let f be a characteristic function of the limit distribution function F. By the continuity theorem, we have

$$f_n(t) \to f(t), \quad n \to \infty \qquad \text{for every } t \in \mathbb{R}. \tag{29}$$

Moreover, it is well known (see [25]) that

$$\sup_{t\in[-T,T]} |f_n(t) - f(t)| \to 0, \quad n \to \infty \qquad \text{for any } T > 0. \tag{30}$$

First let us recall that characteristic functions of quasi-infinitely divisible distributions have no zeroes on the real line (see [23] or (32) below). So, in particular, $f_n(t) \neq 0,\ t \in \mathbb{R},\ n \in \mathbb{N}$. We now show that $f(t) \neq 0$ for all $t \in \mathbb{R}$. For any fixed $n \in \mathbb{N}$ and $t \in \mathbb{R}$ we consider

$$\begin{aligned} |f_n(t)| &= \exp\left\{\int_{\mathbb{R}} \big(\cos(tx) - 1\big)\frac{1+x^2}{x^2}\, dG_n(x)\right\} \\ &\geqslant \exp\left\{-\left|\int_{\mathbb{R}} \big(\cos(tx) - 1\big)\frac{1+x^2}{x^2}\, dG_n(x)\right|\right\} \\ &\geqslant \exp\left\{-\int_{\mathbb{R}} \big(1-\cos(tx)\big)\frac{1+x^2}{x^2}\, d|G_n|(x)\right\}. \end{aligned}$$

Let us estimate the inner function $x \mapsto (1 - \cos(tx))\frac{1+x^2}{x^2},\ x \in \mathbb{R}$, which is equal to $\frac{t^2}{2}$ at $x = 0$ for the continuity by the well known convention. Due to the inequality $1 - \cos y \leqslant \frac{y^2}{2},\ y \in \mathbb{R}$, for the case $|tx| \leqslant 2$ we have

$$(1-\cos(tx))\frac{1+x^2}{x^2} \leqslant \frac{t^2x^2}{2}\cdot\frac{1+x^2}{x^2} = \frac{t^2+t^2x^2}{2} \leqslant \frac{t^2}{2} + 2.$$

Using the simple inequality $1 - \cos y \leqslant 2,\ y \in \mathbb{R}$, for the case $|tx| > 2$ we obtain

$$(1-\cos(tx))\frac{1+x^2}{x^2} \leqslant 2\cdot\frac{1+x^2}{x^2} = 2\cdot\left(\frac{1}{x^2}+1\right) \leqslant 2\cdot\left(\frac{t^2}{4}+1\right) = \frac{t^2}{2}+2.$$

Thus

$$(1-\cos(tx))\frac{1+x^2}{x^2} \leqslant \frac{t^2}{2}+2 \quad \text{for any } x \in \mathbb{R},\ t \in \mathbb{R}. \tag{31}$$

Thus for any $n \in \mathbb{N}$ and $t \in \mathbb{R}$ we obtain

$$|f_n(t)| \geqslant \exp\left\{-\int_{\mathbb{R}} \big(\tfrac{t^2}{2}+2\big)\, d|G_n|(x)\right\} = \exp\big\{-\big(\tfrac{t^2}{2}+2\big)\|G_n\|\big\} > 0. \tag{32}$$

Hence, due to (14) and (29), we have

$$|f(t)| = \lim_{n\to\infty} |f_n(t)| \geqslant \exp\big\{-\big(\tfrac{t^2}{2}+2\big)\varlimsup_{n\to\infty}\|G_n\|\big\} = \exp\big\{-\big(\tfrac{t^2}{2}+2\big)B\big\} > 0, \quad t \in \mathbb{R},$$

i.e., $f(t) \neq 0$ for any $t \in \mathbb{R}$.

Due to the above remarks, the distinguished logarithms $t \mapsto \mathrm{Ln}\, f(t)$ and $t \mapsto \mathrm{Ln}\, f_n(t)$, $n \in \mathbb{N}$, are defined for all $t \in \mathbb{R}$. According to (11), we have

$$\text{(33)}\quad \mathrm{Ln}\, f_n(t) = it\gamma_n + \int_{\mathbb{R}} \left(e^{itx} - 1 - \frac{it}{\tau} \sin(\tau x) \right) \frac{1+x^2}{x^2}\, dG_n(x), \quad t \in \mathbb{R},\ n \in \mathbb{N}.$$

Due to the convergence (29), we have that

$$\text{(34)}\qquad \mathrm{Ln}\, f_n(t) \to \mathrm{Ln}\, f(t), \quad n \to \infty \qquad \text{for every } t \in \mathbb{R}.$$

Hence, in particular,

$$\gamma_n = \frac{\mathrm{Im}(\mathrm{Ln}\, f_n(\tau))}{\tau} \to \frac{\mathrm{Im}(\mathrm{Ln}\, f(\tau))}{\tau} \in \mathbb{R}, \quad n \to \infty.$$

We denote this limit by γ. So we have

$$\text{(35)}\qquad \gamma_n \to \gamma, \quad n \to \infty.$$

We next introduce the following functions

$$\psi(t,s) := \mathrm{Ln}\, f(t) - \tfrac{1}{2}(\mathrm{Ln}\, f(t-s) + \mathrm{Ln}\, f(t+s)), \quad t \in \mathbb{R},\ s \geqslant 0,$$

and analogously

$$\text{(36)}\quad \psi_n(t,s) := \mathrm{Ln}\, f_n(t) - \tfrac{1}{2}(\mathrm{Ln}\, f_n(t-s) + \mathrm{Ln}\, f_n(t+s)), \quad t \in \mathbb{R},\ s \geqslant 0,\ n \in \mathbb{N}.$$

From (34) we conclude that

$$\text{(37)}\qquad \psi_n(t,s) \to \psi(t,s), \quad n \to \infty \qquad \text{for any } t \in \mathbb{R},\ s \geqslant 0.$$

Moreover, since (30) implies the convergence (see [20, p. 15], or [31, p. 34])

$$\sup_{t \in [-T,T]} |\mathrm{Ln} f_n(t) - \mathrm{Ln}\, f(t)| \to 0, \quad n \to \infty \qquad \text{for any } T > 0,$$

it is clear that

$$\text{(38)}\qquad \sup_{t,s \in [-T,T]} |\psi_n(t,s) - \psi(t,s)| \to 0, \quad n \to \infty \qquad \text{for any } T > 0.$$

We next show that ψ_n, $n \in \mathbb{N}$, are uniformly bounded over $t \in \mathbb{R}$ and $n \in \mathbb{N}$ for any fixed $s \geqslant 0$. Using (33) in (36), we have

$$\begin{aligned}
\text{(39)}\quad & \psi_n(t,s) \\
&= it\gamma_n + \int_{\mathbb{R}} \left(e^{itx} - 1 - \frac{it}{\tau} \sin(\tau x) \right) \frac{1+x^2}{x^2}\, dG_n(x) \\
&\quad - \tfrac{1}{2} \left(i2t\gamma_n + \int_{\mathbb{R}} \left(e^{itx}(e^{-isx} + e^{isx}) - 2 - \frac{i2t}{\tau} \sin(\tau x) \right) \frac{1+x^2}{x^2}\, dG_n(x) \right) \\
&= \int_{\mathbb{R}} e^{itx}(1 - \cos(sx)) \frac{1+x^2}{x^2}\, dG_n(x), \quad t \in \mathbb{R},\ s \geqslant 0,\ n \in \mathbb{N}.
\end{aligned}$$

The estimate (31) yields

$$\sup_{t\in\mathbb{R}}|\psi_n(t,s)| \leqslant \sup_{t\in\mathbb{R}}\int_{\mathbb{R}}\left|e^{itx}(1-\cos(sx))\frac{1+x^2}{x^2}\right| d|G_n|(x)$$
$$=\int_{\mathbb{R}}(1-\cos(sx))\frac{1+x^2}{x^2}\,d|G_n|(x) \leqslant \left(\frac{s^2}{2}+1\right)\|G_n\|,\quad s\geqslant 0,\ n\in\mathbb{N}.$$

According to (14), there exists a constant $B_0 \geqslant 0$ such that $\|G_n\| \leqslant B_0$ for all $n\in\mathbb{N}$. Then we conclude

$$\sup_{n\in\mathbb{N}}\sup_{t\in\mathbb{R}}|\psi_n(t,s)| \leqslant B_0\cdot\left(\frac{s^2}{2}+1\right),\quad s\geqslant 0. \tag{40}$$

Additionally, in view of (37), we obtain

$$\sup_{t\in\mathbb{R}}|\psi(t,s)| \leqslant B_0\cdot\left(\frac{s^2}{2}+1\right),\quad s\geqslant 0. \tag{41}$$

Next, since $\int_0^\infty (s^2+1)\,e^{-s}\,ds<\infty$, we can define the functions

$$g_n(t):=\int_0^\infty \psi_n(t,s)\,e^{-s}\,ds,\quad n\in\mathbb{N}\qquad\text{and}\qquad g(t):=\int_0^\infty \psi(t,s)\,e^{-s}\,ds,\quad t\in\mathbb{R},$$

and, due to (37), conclude at once by the Lebesgue dominated convergence theorem that

$$g_n(t)\to g(t),\quad n\to\infty\qquad\text{for every } t\in\mathbb{R}.$$

Let us prove that

$$\sup_{t\in[-T,T]}|g_n(t)-g(t)|\to 0,\quad n\to\infty\qquad\text{for any } T>0. \tag{42}$$

We fix any $T>0$ and $\varepsilon>0$. It is clear that for every $n\in\mathbb{N}$

$$\sup_{t\in[-T,T]}|g_n(t)-g(t)| \leqslant \int_0^\infty \sup_{t\in[-T,T]}|\psi_n(t,s)-\psi(t,s)|\,e^{-s}\,ds. \tag{43}$$

We denote by $J_n(T)$ the last integral for every $n\in\mathbb{N}$. Let us choose a constant $h_\varepsilon>0$ such that

$$B_0\int_{h_\varepsilon}^\infty (s^2+2)\,e^{-s}\,ds<\varepsilon. \tag{44}$$

Then we write $J_n(T)=J_{n,1}(T)+J_{n,2}(T)$, $n\in\mathbb{N}$, where

$$J_{n,1}(T):=\int_0^{h_\varepsilon}\sup_{t\in[-T,T]}|\psi_n(t,s)-\psi(t,s)|\,e^{-s}\,ds,$$
$$J_{n,2}(T):=\int_{h_\varepsilon}^\infty\sup_{t\in[-T,T]}|\psi_n(t,s)-\psi(t,s)|\,e^{-s}\,ds.$$

All the integrals $J_n(T),\ J_{n,1}(T)$, and $J_{n,2}(T)$ are nonnegative. Observe that

$$\begin{aligned} J_{n,1}(T) &\leqslant \sup_{\substack{t\in[-T,T]\\ s\in[0,h_\varepsilon]}} |\psi_n(t,s)-\psi(t,s)| \int_0^{h_\varepsilon} e^{-s}\,ds \\ &\leqslant \sup_{t,s\in[-T_\varepsilon,T_\varepsilon]} |\psi_n(t,s)-\psi(t,s)|, \quad n\in\mathbb{N}, \end{aligned}$$

where $T_\varepsilon := \max\{T, h_\varepsilon\}$. Due to (38), the last supremum vanishes as $n\to\infty$. So there exists $n_\varepsilon\in\mathbb{N}$ such that $J_{n,1}(T)<\varepsilon$ for any $n\geqslant n_\varepsilon$. Let us turn to $J_{n,2}(T)$. According to (40), (41), and (44), we have

$$J_{n,2}(T) \leqslant \int_{h_\varepsilon}^\infty \Big(\sup_{t\in\mathbb{R}}|\psi_n(t,s)| + \sup_{t\in\mathbb{R}}|\psi(t,s)|\Big)\, e^{-s}\,ds \leqslant \int_{h_\varepsilon}^\infty B_0(s^2+2)\,e^{-s}\,ds < \varepsilon.$$

Then $J_n(T) = J_{n,1}(T)+J_{n,2}(T) < 2\varepsilon$ for any $n\geqslant n_\varepsilon$. Since $\varepsilon>0$ was chosen arbitrarily, $J_n(T)\to 0$ as $n\to\infty$. Thus, according to (43), we obtain (42). Since g is a uniform limit of continuous functions g_n on any segment $[-T,T]$ as $n\to\infty$, the function g is continuous on $\mathbb{R}$.

Let us consider the functions $g_n,\ n\in\mathbb{N}$. Using (39), we write

$$\begin{aligned} g_n(t) &= \int_0^\infty \psi_n(t,s)\,e^{-s}\,ds \\ &= \int_0^\infty \Big(\int_{\mathbb{R}} e^{itx}(1-\cos(sx))\frac{1+x^2}{x^2}\,dG_n(x)\Big)\, e^{-s}\,ds \\ &= \int_{\mathbb{R}} \Big(\int_0^\infty (1-\cos(sx))\,e^{-s}\,ds\Big)\, e^{itx}\frac{1+x^2}{x^2}\,dG_n(x), \quad t\in\mathbb{R},\ n\in\mathbb{N}. \end{aligned}$$

The inner integral is calculated (see [14, p. 486, formula 3.893(2)]):

$$\int_0^\infty (1-\cos(sx))\,e^{-s}\,ds = 1-\int_0^\infty \cos(sx)\,e^{-s}\,ds = 1-\frac{1}{1+x^2} = \frac{x^2}{1+x^2}, \quad x\in\mathbb{R}.$$

Therefore we have

$$g_n(t) = \int_{\mathbb{R}} e^{itx}\,dG_n(x), \quad t\in\mathbb{R},\ n\in\mathbb{N}.$$

We now use Theorem 5. So there exists a unique function $G\in \boldsymbol{V}$ such that $\|G\|\leqslant B$ and the equality

$$g(t) = \int_{\mathbb{R}} e^{itx}\,dG(x)$$

holds *for all* $t\in\mathbb{R}$, because g is continuous on $\mathbb{R}$. Moreover, due to the theorem, we have $G_n \Rightarrow G$ and also $G_n(+\infty)\to G(+\infty),\ n\to\infty$.

We now prove that for any $t \in \mathbb{R}$ and $\tau > 0$

$$\int_{\mathbb{R}} \Big(e^{itx} - 1 - \frac{it}{\tau}\sin(\tau x)\Big)\frac{1+x^2}{x^2}\,dG_n(x) \to \int_{\mathbb{R}} \Big(e^{itx} - 1 - \frac{it}{\tau}\sin(\tau x)\Big)\frac{1+x^2}{x^2}\,dG(x), \quad n \to \infty. \tag{45}$$

Let us fix $t \in \mathbb{R}$ and $\tau > 0$. From Lemma 20 we know that

$$\Big(e^{itx} - 1 - \frac{it}{\tau}\sin(\tau x)\Big)\frac{1+x^2}{x^2} = \int_{A_{t,\tau}} e^{isx}\,dW_{t,\tau}(s), \quad x \in \mathbb{R},$$

where $A_{t,\tau} = \big\{s \in \mathbb{R} : |s| \leqslant \max\{|t|, \tau\}\big\}$ and $W_{t,\tau} \in \boldsymbol{V}$. Hence for every $n \in \mathbb{N}$

$$\begin{aligned}\int_{\mathbb{R}} \Big(e^{itx} - 1 - \frac{it}{\tau}\sin(\tau x)\Big)\frac{1+x^2}{x^2}\,dG_n(x) &= \int_{\mathbb{R}}\Big(\int_{A_{t,\tau}} e^{isx}\,dW_{t,\tau}(s)\Big)\,dG_n(x)\\ &= \int_{A_{t,\tau}}\Big(\int_{\mathbb{R}} e^{isx}\,dG_n(x)\Big)\,dW_{t,\tau}(s) = \int_{A_{t,\tau}} g_n(s)\,dW_{t,\tau}(s).\end{aligned}$$

Also we have analogously that

$$\int_{\mathbb{R}} \Big(e^{itx} - 1 - \frac{it}{\tau}\sin(\tau x)\Big)\frac{1+x^2}{x^2}\,dG(x) = \int_{A_{t,\tau}} g(s)\,dW_{t,\tau}(s).$$

Thus (45) takes the form

$$\int_{A_{t,\tau}} g_n(s)\,dW_{t,\tau}(s) \to \int_{A_{t,\tau}} g(s)\,dW_{t,\tau}(s), \quad n \to \infty.$$

This convergence holds. Indeed, for every $n \in \mathbb{N}$

$$\begin{aligned}\Big|\int_{A_{t,\tau}} g_n(s)\,dW_{t,\tau}(s) - \int_{A_{t,\tau}} g(s)\,dW_{t,\tau}(s)\Big| &\leqslant \int_{A_{t,\tau}} |g_n(s) - g(s)|\,d|W_{t,\tau}|(s)\\ &\leqslant \sup_{s \in A_{t,\tau}} |g_n(s) - g(s)| \cdot \|W_{t,\tau}\|,\end{aligned}$$

where, due to (42), the supremum vanishes as $n \to \infty$. Thus we proved (45).

From (33), (35), and (45), for any $t \in \mathbb{R}$ we have

$$\begin{aligned}\operatorname{Ln} f_n(t) &= it\gamma_n + \int_{\mathbb{R}} \Big(e^{itx} - 1 - \frac{it}{\tau}\sin(\tau x)\Big)\frac{1+x^2}{x^2}\,dG_n(x)\\ &\to it\gamma + \int_{\mathbb{R}} \Big(e^{itx} - 1 - \frac{it}{\tau}\sin(\tau x)\Big)\frac{1+x^2}{x^2}\,dG(x), \quad n \to \infty.\end{aligned}$$

According to (34), we conclude that

$$\operatorname{Ln} f(t) = it\gamma + \int_{\mathbb{R}} \left(e^{itx} - 1 - \frac{it}{\tau} \sin(\tau x) \right) \frac{1+x^2}{x^2} \, dG(x), \quad t \in \mathbb{R},$$

where, as we have already proved, $\gamma \in \mathbb{R}$ and $G \in \boldsymbol{V}$. Thus f has the Lévy–Khinchin type representation with (γ, G), i.e., the distribution function F corresponding to f is quasi-infinitely divisible. □

Proof of Theorem 11. Let f be a characteristic function of the limit distribution function F. So we have (29) and also (30) (see comments in the proof of Theorem 10).

Recall that $f_n(t) \neq 0$, $t \in \mathbb{R}$, $n \in \mathbb{N}$. Let us choose $\delta > 0$ such that $f(t) \neq 0$, $|t| \leqslant \delta$ (it is possible because f is continuous on $\mathbb{R}$ and $f(0) = 1$). Let us consider values of the Khinchin functional $\chi_\delta(\,\cdot\,)$ (see [24, p. 79]) with parameter δ on f and f_n, $n \in \mathbb{N}$:

$$\chi_\delta(f) = -\frac{1}{\delta} \int_0^\delta \ln |f(s)| \, ds, \qquad \chi_\delta(f_n) = -\frac{1}{\delta} \int_0^\delta \ln |f_n(s)| \, ds, \quad n \in \mathbb{N}.$$

These quantities are finite and nonnegative. Due to (30), we have

$$\chi_\delta(f_n) \to \chi_\delta(f), \quad n \to \infty. \tag{46}$$

Observe that

$$\begin{aligned} \chi_\delta(f_n) &= -\frac{1}{\delta} \int_0^\delta \left(\int_{\mathbb{R}} (\cos(sx) - 1) \frac{1+x^2}{x^2} \, dG_n(x) \right) ds \\ &= \int_{\mathbb{R}} \left(\frac{1}{\delta} \int_0^\delta (1 - \cos(sx)) \, ds \right) \frac{1+x^2}{x^2} \, dG_n(x) \\ &= \int_{\mathbb{R}} \left(1 - \frac{\sin(\delta x)}{\delta x} \right) \frac{1+x^2}{x^2} \, dG_n(x), \quad n \in \mathbb{N}. \end{aligned} \tag{47}$$

where we set

$$(\cos(sx) - 1) \frac{1+x^2}{x^2} \Big|_{x=0} = -\frac{s^2}{2}, \quad \left(1 - \frac{\sin(\delta x)}{\delta x} \right) \frac{1+x^2}{x^2} \Big|_{x=0} = \frac{\delta^2}{3!}, \tag{48}$$

according to known expansions $\cos y = 1 - \frac{y^2}{2} + o(y^2)$ and $\sin y = y - \frac{y^3}{3!} + o(y^3)$, $y \to 0$. Let us consider the inner function of the integral in (47):

$$x \mapsto \left(1 - \frac{\sin(\delta x)}{\delta x} \right) \frac{1+x^2}{x^2}, \quad x \in \mathbb{R}. \tag{49}$$

By convention (48), it is continuous at the point $x = 0$. We see that this function is continuous and strictly positive on $\mathbb{R}$. Also observe that it tends to 1 as $x \to \pm\infty$.

Hence it is clear that there exist positive constants c_δ and C_δ such that

$$0 < c_\delta \leqslant \left(1 - \frac{\sin(\delta x)}{\delta x}\right)\frac{1+x^2}{x^2} \leqslant C_\delta < \infty, \quad x \in \mathbb{R}. \tag{50}$$

Let us take some decompositions (12) for $G_n,\ n \in \mathbb{N}$. According to (47):

$$\chi_\delta(f_n) = \int_{\mathbb{R}}\left(1 - \frac{\sin(\delta x)}{\delta x}\right)\frac{1+x^2}{x^2}\, dG_n^+(x) - \int_{\mathbb{R}}\left(1 - \frac{\sin(\delta x)}{\delta x}\right)\frac{1+x^2}{x^2}\, dG_n^-(x), \quad n \in \mathbb{N}.$$

Due to (50), we obtain

$$\chi_\delta(f_n) \geqslant c_\delta \int_{\mathbb{R}} dG_n^+(x) - C_\delta \int_{\mathbb{R}} dG_n^-(x) = c_\delta\, G_n^+(+\infty) - C_\delta\, G_n^-(+\infty), \quad n \in \mathbb{N}.$$

From this we have

$$G_n^+(+\infty) \leqslant \frac{\chi_\delta(f_n) + C_\delta\, G_n^-(+\infty)}{c_\delta}, \quad n \in \mathbb{N}.$$

Hence, due to (15) and (46), we get

$$\begin{aligned}\overline{\lim_{n\to\infty}}\, G_n^+(+\infty) &\leqslant \frac{1}{c_\delta}\Big(\overline{\lim_{n\to\infty}}\, \chi_\delta(f_n) + C_\delta\, \overline{\lim_{n\to\infty}}\, G_n^-(+\infty)\Big)\\ &= \frac{1}{c_\delta}(\chi_\delta(f) + C_\delta M) < \infty.\end{aligned} \tag{51}$$

According to (12) and conventions there, it is true that

$$\|G_n\| \leqslant \|G_n^+\| + \|G_n^-\| = G_n^+(+\infty) + G_n^-(+\infty), \quad n \in \mathbb{N}. \tag{52}$$

So we conclude from (15) and (51) that (14) holds for some $B < \infty$.

Thus all assertions of Theorem 10 hold. In particular, $G_n \Rightarrow G$ and $G_n(+\infty) \to G(+\infty),\ n \to \infty$, where G is some function from $\boldsymbol{V}$. It remains to prove that $B \leqslant G(+\infty) + 2M$. Using inequality (52), we write

$$\begin{aligned}B = \overline{\lim_{n\to\infty}}\, \|G_n\| &\leqslant \overline{\lim_{n\to\infty}}\, (G_n^+(+\infty) + G_n^-(+\infty))\\ &\leqslant \overline{\lim_{n\to\infty}}\, (G_n^+(+\infty) - G_n^-(+\infty)) + 2\, \overline{\lim_{n\to\infty}}\, G_n^-(+\infty),\end{aligned}$$

but $G_n(+\infty) = G_n^+(+\infty) - G_n^-(+\infty),\ n \in \mathbb{N}$, and we obtain

$$B \leqslant \overline{\lim_{n\to\infty}}\, G_n(+\infty) + 2\, \overline{\lim_{n\to\infty}}\, G_n^-(+\infty) = G(+\infty) + 2M,$$

as required. □

Proof of Theorem 14. Let $(F_{n_k})_{k\in\mathbb{N}}$ be an arbitrary subsequence of $(F_n)_{n\in\mathbb{N}}$, which weakly converges to some distribution function F_*. Due to the assumption of relative compactness of $(F_n)_{n\in\mathbb{N}}$, such subsequence exists. By Theorem 10, F_* is quasi-infinitely divisible with some spectral pair (γ_*, G_*), where $\gamma_* \in \mathbb{R}$ and $G_* \in \boldsymbol{V}$.

Moreover, $\gamma_{n_k} \to \gamma_*$ and $G_{n_k} \Rightarrow G_*,\ k \to \infty$. According to the assumption that $\gamma_n \to \gamma,\ n \to \infty$, we conclude that $\gamma_* = \gamma$. Let us show that $G_* = G$. By definition, the convergence $G_{n_k} \Rightarrow G_*,\ k \to \infty$, implies the existence of a subsequence $(G_{n'_l})_{l\in\mathbb{N}}$ in $(G_{n_k})_{k\in\mathbb{N}}$ such that

$$G_{n'_l}(x_2) - G_{n'_l}(x_1) \to G_*(x_2) - G_*(x_1), \quad l \to \infty$$

for all $x_1, x_2 \in \mathbb{R}$ except at most countable set D'. Due to the assumption that $G_n \Rightarrow G,\ n \to \infty$, we can choose a further subsequence $(G_{n''_l})_{l\in\mathbb{N}}$ in $(G_{n'_l})_{l\in\mathbb{N}}$ such that

$$G_{n''_l}(x_2) - G_{n''_l}(x_1) \to G(x_2) - G(x_1), \quad l \to \infty$$

for all $x_1, x_2 \in \mathbb{R}$ except at most countable set D'' (and let $D' \neq D''$ in general). Therefore

$$G_*(x_2) - G_*(x_1) = G(x_2) - G(x_1)$$

for all $x_1, x_2 \in \mathbb{R}$ except at most countable set $D' \cup D''$. Letting $x_1 \to -\infty$ over $x_1 \in \mathbb{R} \setminus (D' \cup D'')$ we have $G_*(x_2) = G(x_2)$ for every $x_2 \in \mathbb{R} \setminus (D' \cup D'')$ and, consequently for all $x_2 \in \mathbb{R}$, because $G_*, G \in \boldsymbol{V}$, i.e., they are right-continuous and

$$G_*(-\infty) = G(-\infty) = 0.$$

Thus we proved that $\gamma_* = \gamma$ and $G_* = G$.

The previous remark means that (γ, G) is the spectral pair for some quasi-infinitely divisible distribution function F. We also saw that every subsequence $(F_{n_k})_{k\in\mathbb{N}}$, which weakly converges to some distribution function, converges exactly to F, because a spectral pair uniquely determines a distribution function. Therefore, since $(F_n)_{n\in\mathbb{N}}$ is relatively compact, we conclude that whole sequence $(F_n)_{n\in\mathbb{N}}$ weakly converges to F (this is a known fact, see [8, p. 337]). $\square$

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C^*-IRREDUCIBILITY OF COMMENSURATED SUBGROUPS

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Given a commensurated subgroup Λ of a group Γ, we completely characterize when the inclusion $\Lambda \leq \Gamma$ is C^*-irreducible and provide new examples of such inclusions. In particular, we obtain that $\mathrm{PSL}(n, \mathbb{Z}) \leq \mathrm{PGL}(n, \mathbb{Q})$ is C^*-irreducible for any $n \in \mathbb{N}$, and that the inclusion of a C^*-simple group into its abstract commensurator is C^*-irreducible.

The main ingredient that we use is the fact that the action of a commensurated subgroup $\Lambda \leq \Gamma$ on its Furstenberg boundary $\partial_F \Lambda$ can be extended in a unique way to an action of Γ on $\partial_F \Lambda$. Finally, we also investigate the counterpart of this extension result for the universal minimal proximal space of a group.

1. Introduction

A group Γ is said to be *C^*-simple* if its reduced C^*-algebra $C_r^*(\Gamma)$ is simple. After the breakthrough characterizations of C^*-simplicity in [Kalantar and Kennedy 2017; Breuillard et al. 2017], several directions of research applying the new methods in different settings arose.

One of the recent interesting directions is investigating when inclusions of groups $\Lambda \leq \Gamma$ are *C^*-irreducible*, in the sense that every intermediate C^*-algebra B in $C_r^*(\Lambda) \subset B \subset C_r^*(\Gamma)$ is simple. Rørdam [2021] started a systematic study of this property and provided a dynamical criterion for an inclusion of groups to be C^*-irreducible. Together with results in [Amrutam 2021; Ursu 2022; Bédos and Omland 2023], this has provided a complete characterization of C^*-irreducibility of an inclusion in the case that Λ is a normal subgroup of Γ.

Recall that a subgroup Λ of a group Γ is said to be *commensurated* if, for any $g \in \Gamma$, $\Lambda \cap g\Lambda g^{-1}$ has finite index in Λ. This is a much more flexible generalization of normal subgroups and finite-index subgroups. For example, for every $n \geq 2$, $\mathrm{PSL}(n, \mathbb{Z})$ is an infinite-index commensurated subgroup of the simple group $\mathrm{PSL}(n, \mathbb{Q})$.

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In this work, we generalize the above characterization of C^*-irreducibility to commensurated subgroups (see Theorem 3.5). The main ingredient in our proof is the fact that the action of Λ on its Furstenberg boundary $\partial_F\Lambda$ can be uniquely extended to an action of Γ on $\partial_F\Lambda$ if Λ is a commensurated subgroup in Γ (see Theorem 3.1).

As one of the applications, we show that if Γ is a C^*-simple group, then the inclusion of Γ in its abstract commensurator $\mathrm{Comm}(\Gamma)$ is C^*-irreducible (see Corollary 3.14). To our best knowledge, this is also the first observation of the fact that if Γ is a C^*-simple group, then $\mathrm{Comm}(\Gamma)$ is C^*-simple as well.

Given a subgroup Λ of a group Γ, Ursu [2022] introduced a universal Λ-strongly proximal Γ-boundary $B(\Gamma, \Lambda)$ and showed that if $\Lambda \trianglelefteq \Gamma$, then $B(\Gamma, \Lambda) = \partial_F\Lambda$. In Section 4, we generalize this fact to commensurated subgroups and also observe that, in general, $B(\Gamma, \Lambda)$ is not extremally disconnected.

Finally, we also show that, given a commensurated subgroup Λ of a group Γ, the action of Λ on its universal minimal proximal space $\partial_p\Lambda$ can also be extended in a unique way to an action of Γ on $\partial_p\Lambda$ (see Theorem 5.1). We use this fact for concluding that, for a certain locally finite commensurated subgroup G of Thompson's group V, the resulting action of V on ∂_pG is free (see Example 5.4).

2. Preliminaries

Given a compact Hausdorff space X, we denote by $\mathrm{Prob}(X)$ the space of regular probability measures on X. An action of a group Γ on X by homeomorphisms is said to be *minimal* if X does not contain any nontrivial closed invariant subset, and to be *topologically free* if, for any $g \in \Gamma \setminus \{e\}$, the set $\{x \in X : gx = x\}$ has empty interior (if Γ is countable, then $\Gamma \curvearrowright X$ is topologically free if and only if the set of points in X which are not fixed by any nontrivial element of Γ is dense in X). The action is said to be *proximal* if, given $x, y \in X$, there is a net $(g_i) \subset \Gamma$ such that the nets (g_ix) and (g_iy) converge and $\lim g_ix = \lim g_iy$. We say that the action is *strongly proximal* if the induced action $\Gamma \curvearrowright \mathrm{Prob}(X)$ is proximal. The action is called a *boundary action* (or X is a Γ-*boundary*) if it is both minimal and strongly proximal. We denote by $\partial_F\Gamma$ the *Furstenberg boundary* of Γ, i.e., the universal Γ-boundary (see [Glasner 1976, Section III.1]). The group Γ is C^*-simple if and only if $\Gamma \curvearrowright \partial_F\Gamma$ is free [Breuillard et al. 2017, Theorem 3.1].

Given Γ-boundaries X and Y, if there exists $\varphi\colon X \to Y$ a homeomorphism which is Γ-equivariant (Γ-*isomorphism*), then it follows from [Glasner 1976, Lemma II.4.1] that φ is the unique Γ-isomorphism between X and Y.

Let $\Lambda \leq \Gamma$ be a finite-index subgroup. Then any strongly proximal Γ-action is also Λ-strongly proximal [Glasner 1976, Lemma II.3.1] and any Γ-boundary is also a Λ-boundary [Glasner 1976, Lemma II.3.2]. Furthermore, by [Glasner 1976, Theorem II.4.4], which is stated for the universal minimal proximal space but whose proof also works for the Furstenberg boundary, the action $\Lambda \curvearrowright \partial_F \Lambda$ can be extended to $\Gamma \curvearrowright \partial_F \Lambda$ and $\partial_F \Lambda$ is Γ-isomorphic to $\partial_F \Gamma$. In particular, $\partial_F \Lambda$ and $\partial_F \Gamma$ are also Λ-isomorphic.

Given a group isomorphism $\psi\colon \Gamma_1 \to \Gamma_2$, by universality there is a unique homeomorphism $\tilde{\psi}\colon \partial_F \Gamma_1 \to \partial_F \Gamma_2$ such that $\tilde{\psi}(gx) = \psi(g)\tilde{\psi}(x)$ for any $g \in \Gamma_1$ and $x \in \partial_F \Gamma_1$.

Given a group Γ, let $\mathrm{Sub}(\Gamma)$ be the space of subgroups of Γ endowed with the pointwise convergence topology and with the Γ-action given by conjugation. Given a subgroup $\Lambda \leq \Gamma$, a Λ-*uniformly recurrent subgroup* (URS) is a nonempty closed Λ-invariant minimal set $\mathcal{U} \subset \mathrm{Sub}(\Gamma)$. Moreover, we say that $\mathcal{U}$ is *amenable* if one (equivalently all) of its elements is amenable. By [Kennedy 2020, Theorem 4.1], a group Γ is C^*-simple if and only if it does not admit any nontrivial amenable Γ-uniformly recurrent subgroup.

An inclusion of groups $\Lambda \leq \Gamma$ is said to be C^*-*irreducible* if every intermediate C^*-algebra of $C_r^*(\Lambda) \subset C_r^*(\Gamma)$ is simple.

Given $\Lambda \leq \Gamma$ and $g \in \Gamma$, let $g^\Lambda := \{hgh^{-1} : h \in \Lambda\}$. We say that Γ *is icc relatively to* Λ if, for any $g \in \Gamma \setminus \{e\}$, $|g^\Lambda| < \infty$. The group Γ is said to be *icc* if it is icc relatively to itself.

3. C*-irreducibility of commensurated subgroups

Let Γ be a group. Two subgroups $\Lambda_1, \Lambda_2 \leq \Gamma$ are said to be *commensurable* if $[\Lambda_1 : \Lambda_1 \cap \Lambda_2] < \infty$ and $[\Lambda_2 : \Lambda_1 \cap \Lambda_2] < \infty$. Notice that this is an equivalence relation.

A subgroup $\Lambda \leq \Gamma$ is said to be *commensurated* if, for any $g \in \Gamma$, Λ is commensurable with $g\Lambda g^{-1}$. Equivalently, for any $g \in \Gamma$, $[\Lambda : \Lambda \cap g\Lambda g^{-1}] < \infty$. In this case, we write $\Lambda \leq_c \Gamma$. In the literature, this notion is also referred to by saying that Λ is an *almost normal subgroup* of Γ or that (Γ, Λ) is a *Hecke pair*.

The following result generalizes [Glasner 1976, Theorem II.4.4] and [Ozawa 2014, Lemma 20]:

Theorem 3.1. *Let $\Lambda \leq_c \Gamma$. Then $\Lambda \curvearrowright \partial_F \Lambda$ extends in a unique way to an action of Γ on $\partial_F \Lambda$.*

Proof. Given $g \in \Gamma$, let $\varphi_g\colon \partial_F \Lambda \to \partial_F(\Lambda \cap g\Lambda g^{-1})$ be the $(\Lambda \cap g\Lambda g^{-1})$-isomorphism. Also, let $\psi_g\colon \partial_F(\Lambda \cap g^{-1}\Lambda g) \to \partial_F(\Lambda \cap g\Lambda g^{-1})$ be the homeomorphism such that for all $h \in \Lambda \cap g^{-1}\Lambda g$ and $x \in \partial_F(\Lambda \cap g^{-1}\Lambda g)$ we have $\psi_g(hx) = ghg^{-1}\psi_g(x)$. Let $T_g := (\varphi_g)^{-1}\psi_g \varphi_{g^{-1}}\colon \partial_F \Lambda \to \partial_F \Lambda$. We claim that $g \mapsto T_g$ is a Γ-action which extends $\Lambda \curvearrowright \partial_F \Lambda$.

Given $h \in \Lambda \cap g^{-1}\Lambda g$ and $x \in \partial_F \Lambda$, one can readily check that $T_g(hx) = ghg^{-1}T_g(x)$.

Given $g, h \in \Gamma$, we have that $[\Lambda : \Lambda \cap h^{-1}\Lambda h \cap (gh)^{-1}\Lambda(gh)] < \infty$. Furthermore, given $k \in \Lambda \cap h^{-1}\Lambda h \cap (gh)^{-1}\Lambda(gh)$ and $x \in \partial_F \Lambda$, we have $T_{gh}(kx) = (gh)k(gh)^{-1}T_{gh}(x)$. On the other hand, $T_g T_h(kx) = (gh)k(gh)^{-1}T_g T_h(x)$. In particular, $(T_g T_h)^{-1}T_{gh}$ is a $(\Lambda \cap h^{-1}\Lambda h \cap (gh)^{-1}\Lambda(gh))$-automorphism, hence $T_{gh} = T_g T_h$.

Finally, given $g \in \Lambda$, we have that $x \mapsto g^{-1}T_g(x)$ is a $(\Lambda \cap g^{-1}\Lambda g)$-automorphism, so that $g^{-1}T_g = \mathrm{Id}_{\partial_F \Lambda}$. □

Remark 3.2. The existence part of Theorem 3.1 was shown by Dai and Glasner [2019, Theorem 6.1] using a different method and assuming that Γ is countable.

Given a subset S of a group Γ, let $C_\Gamma(S)$ be the *centralizer* of S in Γ. In the next result, we follow the argument of [Breuillard et al. 2017, Lemma 5.3].

Lemma 3.3. *Let $\Lambda \leq_c \Gamma$ and consider $\Gamma \curvearrowright \partial_F \Lambda$. Given $s \in \Gamma$, if $s \in C_\Gamma(\Lambda \cap s^{-1}\Lambda s)$, then $\mathrm{Fix}(s) = \partial_F \Lambda$. Conversely, if $\Lambda \curvearrowright \partial_F \Lambda$ is free and $\mathrm{Fix}(s) \neq \varnothing$, then $s \in C_\Gamma(\Lambda \cap s^{-1}\Lambda s)$.*

Proof. If $s \in C_\Gamma(\Lambda \cap s^{-1}\Lambda s)$, then, given $h \in \Lambda \cap s^{-1}\Lambda s$ and $x \in \partial_F \Lambda$, we have $s(hx) = hs(x)$. Since $[\Lambda : \Lambda \cap s^{-1}\Lambda s] < \infty$, we conclude that s acts trivially on $\partial_F \Lambda$.

Suppose now that $\Lambda \curvearrowright \partial_F \Lambda$ is free and $\mathrm{Fix}(s) \neq \varnothing$. Given $t \in A$, with

$$A := \{t \in \Lambda \cap s^{-1}\Lambda s : t\,\mathrm{Fix}(s) \cap \mathrm{Fix}(s) \neq \varnothing\},$$

the actions of sts^{-1} and t coincide on $\mathrm{Fix}(s) \cap t^{-1}\,\mathrm{Fix}(s)$. Since $sts^{-1}, t \in \Lambda$ and $\Lambda \curvearrowright \partial_F \Lambda$ is free, we obtain that $t = sts^{-1}$. Since, by [Breuillard et al. 2017, Lemma 5.1], A generates $\Lambda \cap s^{-1}\Lambda s$, we conclude that $s \in C_\Gamma(\Lambda \cap s^{-1}\Lambda s)$. □

The proof of the following result is an adaptation of the argument in [Kennedy 2020, Remark 4.2] and its hypothesis is the same as in [Rørdam 2021, Theorem 5.3 (ii)]:

Proposition 3.4. *Let $\Lambda \leq \Gamma$. Suppose that there exists a Γ-boundary X such that, for any $\mu \in \mathrm{Prob}(X)$, there exists a net $(g_i) \subset \Lambda$ such that $g_i \mu$ converges to δ_x, for some $x \in X$, on which Γ acts freely. Then Γ does not admit any nontrivial amenable Λ-URS.*

Proof. Suppose $\mathcal{U}$ is a nontrivial amenable Λ-URS, and take $K \in \mathcal{U}$. Since K is amenable, there exists $\mu \in \mathrm{Prob}(X)$ fixed by K. Let $(g_i) \subset \Lambda$ be a net such that $g_i \mu \to \delta_x$, for some $x \in X$, on which Γ acts freely. By taking a subnet, we may assume that $g_i K g_i^{-1} \to L \in \mathrm{Sub}(\Gamma)$. Take $g \in L \setminus \{e\}$ and $(k_i) \subset K$ such that $g_i k_i g_i^{-1} = g$ for i sufficiently big. Then

$$\delta_x = \lim g_i \mu = \lim g_i k_i \mu = \lim g_i k_i g_i^{-1} g_i \mu = g\delta_x,$$

contradicting the fact that Γ acts freely on x. □

The following result generalizes [Ursu 2022, Theorems 1.3 and 1.9] and [Bédos and Omland 2023, Theorem 6.4], as well as the claim about finite-index subgroups in [Rørdam 2021, Theorem 5.3]:

Theorem 3.5. *Let $\Lambda \leq_c \Gamma$. The following conditions are equivalent*:

(1) $\Lambda \leq \Gamma$ *is C^*-irreducible*;

(2) Λ *is C^*-simple and* Γ *is icc relatively to* Λ;

(3) Λ *is C^*-simple and, for any $s \in \Gamma \setminus \{e\}$, we have that $s \notin C_\Gamma(\Lambda \cap s^{-1}\Lambda s)$*;

(4) $\Gamma \curvearrowright \partial_F \Lambda$ *is free*;

(5) *There is no nontrivial amenable Λ-URS of* Γ;

(6) Λ *is C^*-simple and $\Gamma \curvearrowright \partial_F \Lambda$ is faithful.*

Proof. (1) $\Longrightarrow$ (2): Follows from [Rørdam 2021, Remark 3.8 and Proposition 5.1].

(2) $\Longrightarrow$ (3): Suppose that there is $s \in \Gamma \setminus \{e\}$ such that $s \in C_\Gamma(\Lambda \cap s^{-1}\Lambda s)$. Take $g_1, \ldots, g_n \in \Lambda$ left coset representatives for $\Lambda/(\Lambda \cap s^{-1}\Lambda s)$. Then

$$s^\Lambda = \{g_i k s k^{-1} g_i^{-1} : 1 \leq i \leq n, k \in \Lambda \cap s^{-1}\Lambda s\} = \{g_i s g_i^{-1} : 1 \leq i \leq n\}$$

is finite.

(3) $\Longrightarrow$ (4): Follows from Lemma 3.3.

(4) $\Longrightarrow$ (1): Follows from [Rørdam 2021, Theorem 5.3].

(5) $\Longrightarrow$ (2): If Λ is not C^*-simple, then it contains a nontrivial amenable Λ-uniformly recurrent subgroup. If Γ is not icc relatively to Λ, there exists $s \in \Gamma \setminus \{e\}$ such that s^Λ is finite. Hence, the Λ-orbit of $\langle s \rangle$ is a finite nontrivial amenable Λ-uniformly recurrent subgroup.

(4) $\Longrightarrow$ (5): Follows from Proposition 3.4.

(3) $\Longleftrightarrow$ (6): Follows from Lemma 3.3. □

Remark 3.6. Rørdam [2021, Theorem 5.3] showed that an inclusion $\Lambda \leq \Gamma$ satisfying the hypothesis of Proposition 3.4 is C^*-irreducible, and asked whether the converse holds. We do not know whether the converse of Proposition 3.4 holds and whether the absence of nontrivial amenable Λ-URS of Γ is equivalent to $\Lambda \leq \Gamma$ being C^*-irreducible in general.

Corollary 3.7. *Given $n \in \mathbb{N}$, the inclusion*

$$\mathrm{PSL}(n, \mathbb{Z}) \leq \mathrm{PGL}(n, \mathbb{Q})$$

is C^*-irreducible.

Proof. It was shown in [Bekka et al. 1994] that $\mathrm{PSL}(n,\mathbb{Z})$ is C^*-simple.

Let $U(n,\mathbb{Z})$ be the group of units of the ring $M_n(\mathbb{Z})$. By [Krieg 1990, Corollary V.5.3], $U(n,\mathbb{Z}) \leq_c \mathrm{GL}(n,\mathbb{Q})$. Since $[U(n,\mathbb{Z}) : \mathrm{SL}(n,\mathbb{Z})] = 2$, we conclude that $\mathrm{SL}(n,\mathbb{Z}) \leq_c \mathrm{GL}(n,\mathbb{Q})$ as well. Since taking quotients preserves being commensurated, it follows that $\mathrm{PSL}(n,\mathbb{Z}) \leq_c \mathrm{PGL}(n,\mathbb{Q})$.

Let $(e_{ij})_{1\leq i,j\leq n} \in M_n(\mathbb{Z})$ be the matrix units and fix $[a] \in \mathrm{PGL}(n,\mathbb{Q}) \setminus \{[\mathrm{Id}]\}$. By taking conjugates of $[a]$ by elements of the form $[\mathrm{Id} + m \cdot e_{ij}] \in \mathrm{PSL}(n,\mathbb{Z})$, $m \in \mathbb{Z}$, $1 \leq i \neq j \leq n$, it is easy to see that $[a]^{\mathrm{PSL}(n,\mathbb{Z})}$ is infinite, so that $\mathrm{PGL}(n,\mathbb{Q})$ is icc relatively to $\mathrm{PSL}(n,\mathbb{Z})$.

The conclusion then follows from Theorem 3.5. □

Remark 3.8. Let us sketch a different proof of Corollary 3.7 which gives the stronger statement that $\mathrm{PSL}(n,\mathbb{Z}) \leq \mathrm{PGL}(n,\mathbb{R})$ is C^*-irreducible, where $\mathrm{PGL}(n,\mathbb{R})$ is seen as a discrete group.

Clearly, it suffices to show that, for any countable group Γ such that $\mathrm{PSL}(n,\mathbb{Z}) \leq \Gamma \leq \mathrm{PGL}(n,\mathbb{R})$, the inclusion $\mathrm{PSL}(n,\mathbb{Z}) \leq \Gamma$ is C^*-irreducible. By the argument in [Bryder 2017, Example 3.4.3], the action of $\mathrm{PGL}(n,\mathbb{R})$ on the projective space $P^{n-1}(\mathbb{R})$ is topologically free. Since $\mathrm{PSL}(n,\mathbb{Z}) \curvearrowright P^{n-1}(\mathbb{R})$ is a boundary action, the result follows from [Rørdam 2021, Theorem 5.3].

Corollary 3.9. *Let Λ be a finite-index subgroup of a group Γ. If Γ is C^*-simple, then $\Lambda \leq \Gamma$ is C^*-irreducible. Conversely, if Λ is C^*-simple, then Γ is icc if and only if $\Lambda \leq \Gamma$ is C^*-irreducible.*

Proof. If Γ is C^*-simple, then $\Gamma \curvearrowright \partial_F \Gamma$ is free. Since $\partial_F \Gamma$ is Γ-isomorphic to $\partial_F \Lambda$, it follows that $\Lambda \leq \Gamma$ is C^*-irreducible.

If Γ is icc, then, since $[\Gamma : \Lambda] < \infty$, it is also icc relatively to Λ, hence $\Lambda \leq \Gamma$ is C^*-irreducible by Theorem 3.5. The last implication is immediate. □

Example 3.10. The inclusion given by the Sanov subgroup $\mathbb{F}_2 \leq \mathrm{PSL}(2,\mathbb{Z})$ is finite-index, hence it is C^*-irreducible by Corollary 3.9.

Free groups. Fix $m, n \in \mathbb{N}$ such that $2 \leq m < n$ and consider the free groups $\mathbb{F}_m = \langle a_1, \dots, a_m \rangle \leq \langle a_1, \dots, a_n \rangle = \mathbb{F}_n$. Rørdam [2021, Example 5.4] observed that $\mathbb{F}_m \leq \mathbb{F}_n$ is C^*-irreducible. Notice that $\mathbb{F}_m$ is far from being commensurated in $\mathbb{F}_n$. In fact, given $g \in \mathbb{F}_n \setminus \mathbb{F}_m$, we have that $\mathbb{F}_m \cap g\mathbb{F}_m g^{-1} = \{e\}$ (i.e., $\mathbb{F}_m$ is *malnormal* in $\mathbb{F}_n$). In particular, this example is not covered by Theorems 3.1 and 3.5. Nonetheless, there does exist an extension to $\mathbb{F}_n$ of the action $\mathbb{F}_m \curvearrowright \partial_F \mathbb{F}_m$, but it is far from being unique, since the generators $a_{m+1}, \dots, a_n$ can be mapped into any homeomorphisms on $\partial_F \mathbb{F}_m$.

Furthermore, we claim that $\mathbb{F}_m \leq \mathbb{F}_n$ satisfies condition (5) in Theorem 3.5. We will prove this by using Proposition 3.4.

Let

$$\partial\mathbb{F}_n := \Big\{(x_i) \in \prod_{\mathbb{N}}\{a_1, a_1^{-1}, \dots, a_n, a_n^{-1}\} : \forall i \in \mathbb{N},\ x_{i+1} \neq x_i^{-1}\Big\}$$

be the Gromov boundary of $\mathbb{F}_n$, and consider the action of $\mathbb{F}_n$ on $\partial\mathbb{F}_n$ by left multiplication. Fix $\mu \in \mathrm{Prob}(\partial\mathbb{F}_n)$, and we will show that there is $w \in \partial\mathbb{F}_n$ on which $\mathbb{F}_n$ acts freely and such that $\delta_w \in \overline{\mathbb{F}_m\mu}$.

Let $z_+ := (a_1)_{i\in\mathbb{N}} \in \partial\mathbb{F}_n$, and let $z_- := (a_1^{-1})_{i\in\mathbb{N}} \in \partial\mathbb{F}_n$. Notice that, for all $y \in \partial\mathbb{F}_n \setminus \{z_-\}$, we have that, as $k \to +\infty$, $a_1^k y \to z_+$. Furthermore, a_1 fixes z_-.

It follows from the dominated convergence theorem that

$$a_1^k\mu \to \mu(\{z_-\})\delta_{z_-} + (1-\mu(\{z_-\}))\delta_{z_+},$$

as $k \to +\infty$. In particular, $\nu := \mu(\{z_-\})\delta_{z_-} + (1-\mu(\{z_-\}))\delta_{z_+} \in \overline{\mathbb{F}_n\mu}$.

Let $w := a_1a_2^1a_1a_2^2a_1a_2^3\cdots a_1a_2^la_1a_2^{l+1}\cdots \in \partial\mathbb{F}_n$. Since w is not eventually periodic, we have that $\mathbb{F}_n$ acts freely on w. Given $k \in \mathbb{N}$, let $g_k := w_1\cdots w_k a_2 \in \mathbb{F}_m$. We have that $g_k z_\pm = w_1\cdots w_k a_2 z_\pm \to w$, as $k \to +\infty$. Therefore, $\delta_w \in \overline{\mathbb{F}_m\nu} \subset \overline{\mathbb{F}_m\mu}$, thus showing the claim.

Abstract commensurator. Let Γ be a group and Ω be the set of isomorphisms between finite-index subgroups of Γ. Given $\alpha, \beta \in \Omega$, we say that $\alpha \sim \beta$ if there exists a finite-index subgroup $H \leq \mathrm{dom}(\alpha)\cap\mathrm{dom}(\beta)$ such that $\alpha|_H = \beta|_H$. Recall that the *abstract commensurator* of Γ, denoted by $\mathrm{Comm}(\Gamma)$, is the group whose underlying set is $\Omega/\!\sim$, with product given by composition (defined up to finite-index subgroup).

Let Λ be a commensurated subgroup of Γ. Given $g \in \Gamma$, let

$$\beta_g : \Lambda \cap g^{-1}\Lambda g \to \Lambda \cap g\Lambda g^{-1}, \quad h \mapsto ghg^{-1},$$

and $j_\Lambda^\Gamma : \Gamma \to \mathrm{Comm}(\Lambda)$ be the homomorphism given by $j_\Lambda^\Gamma(g) := [\beta_g]$. In order to ease the notation, we will sometimes denote j_Λ^Γ simply by j, and it will always be clear from the context what the involved groups are. Let us now collect a few elementary facts about j.

Lemma 3.11. *Let Γ be a group. Then $j_\Gamma^\Gamma(\Gamma) \leq_c \mathrm{Comm}(\Gamma)$.*

Proof. Fix $[\alpha] \in \mathrm{Comm}(\Gamma)$. Given $g \in \mathrm{dom}(\alpha)$, we have that $[\alpha]j(g)[\alpha]^{-1} = j(\alpha(g))$. In particular, $j(\Gamma)\cap[\alpha]j(\Gamma)[\alpha]^{-1} \supset j(\mathrm{Im}(\alpha))$. Since $[\Gamma : \mathrm{Im}(\alpha)] < \infty$, we conclude that $\big[j(\Gamma) : j(\Gamma)\cap[\alpha]j(\Gamma)[\alpha]^{-1}\big] < \infty$. □

Lemma 3.12. *Let $\Lambda \leq_c \Gamma$. Then* $\ker j_\Lambda^\Gamma = \{g \in \Gamma : |g^\Lambda| < \infty\}$.

Proof. Given $g \in \ker j$, there exists a finite-index subgroup $H \leq \Lambda \cap g^{-1}\Lambda g$ such that, for all $h \in H$, $ghg^{-1} = h$, which implies that $|g^\Lambda| < \infty$. Conversely, if $|g^\Lambda| < \infty$, then $H := \{k \in \Lambda : kg = gk\}$ is a finite-index subgroup of Λ and $g \in \ker j$. □

As a consequence of Lemma 3.12, if Γ is an icc group, then $j : \Gamma \to \mathrm{Comm}(\Gamma)$ is injective [Kida 2011, Lemma 3.8 (i)]. The next result is known [Kida 2011, Lemma 3.8 (iii)]. For the convenience of the reader, we provide the proof here.

Lemma 3.13. *If* Γ *is an icc group, then* $\mathrm{Comm}(\Gamma)$ *is icc relatively to* Γ.

Proof. Given $[\alpha] \in \mathrm{Comm}(\Gamma)$ and $g \in \mathrm{dom}(\alpha)$, we have

$$j(g)[\alpha]j(g^{-1}) = j(g\alpha(g^{-1}))[\alpha].$$

If $[\alpha] \neq e$, then $H := \{g \in \mathrm{dom}(\alpha) : g = \alpha(g)\}$ has infinite-index in $\mathrm{dom}(\alpha)$. Given $g_1, g_2 \in \mathrm{dom}(\alpha)$ such that $g_1 H \neq g_2 H$, one can readily check that $g_1\alpha(g_1)^{-1} \neq g_2\alpha(g_2)^{-1}$. From this, it follows immediately that $[\alpha]^\Gamma$ is infinite. □

Bédos and Omland [2023, Corollary 6.6] showed that if Γ is a C^*-simple group, then $\Gamma \leq \mathrm{Aut}(\Gamma)$ is C^*-irreducible. The same conclusion holds when we consider the abstract commensurator:

Corollary 3.14. *Given a* C^**-simple group* Γ, *we have that* $\Gamma \leq \mathrm{Comm}(\Gamma)$ *is* C^**-irreducible.*

Proof. Recall that any C^*-simple group is icc (this follows, e.g., from Theorem 3.5). The result is then a consequence of Theorem 3.5 and Lemma 3.13. □

Remark 3.15. Corollary 3.14 generalizes the fact proven in [Le Boudec and Matte Bon 2018, Corollary 4.4] that, if Thompson's group F is C^*-simple, then $\mathrm{Comm}(F)$ is C^*-simple.

Remark 3.16. Let $\mathbb{F}_n$ be a nonabelian free group of finite rank. Then Corollary 3.14 implies that $\mathrm{Comm}(\mathbb{F}_n)$ is C^*-simple. In particular, it does not admit any nontrivial amenable normal subgroup. It is an open problem whether $\mathrm{Comm}(\mathbb{F}_n)$ is a simple group [Caprace and Monod 2018, Problem 7.2].

4. Relative boundaries

Given groups $\Lambda \leq \Gamma$, Ursu [2022, Proposition 4.1] introduced a Λ-strongly proximal Γ-boundary $B(\Gamma, \Lambda)$ which is universal with these properties.

Consider $\Gamma := \mathrm{PSL}(2, \mathbb{Z})$ and the boundary action $\Gamma \curvearrowright \mathbb{R} \cup \{\infty\}$. The stabilizer Γ_∞ of ∞ is isomorphic to $\mathbb{Z}$ and consists of the translations $g_n(x) := x + n$, $n \in \mathbb{Z}$, $x \in \mathbb{R}$.

Proposition 4.1. *The action of* $\Gamma = \mathrm{PSL}(2, \mathbb{Z})$ *on* $B(\Gamma, \Gamma_\infty)$ *is topologically free but nonfree. In particular,* $B(\Gamma, \Gamma_\infty)$ *is not extremally disconnected.*

Proof. For any $x \in \mathbb{R} \cup \{\infty\}$, we have $g_n(x) \to \infty$ as $n \to +\infty$. As a consequence of the dominated convergence theorem, it follows easily that $\Gamma_\infty \curvearrowright \mathbb{R} \cup \{\infty\}$ is strongly proximal. Hence, there is a Γ-equivariant map $B(\Gamma, \Gamma_\infty) \to \mathbb{R} \cup \{\infty\}$. Since $\Gamma_\infty \curvearrowright B(\Gamma, \Gamma_\infty)$ is strongly proximal, it follows from amenability of Γ_∞ that

Γ_∞ fixes some point in $B(\Gamma, \Gamma_\infty)$. In particular, $\Gamma \curvearrowright B(\Gamma, \Gamma_\infty)$ is not free. On the other hand, since $\Gamma \curvearrowright \mathbb{R} \cup \{\infty\}$ is topologically free, it follows from [Breuillard et al. 2017, Lemma 3.2] that $\Gamma \curvearrowright B(\Gamma, \Gamma_\infty)$ is topologically free. As a consequence of [Frolík 1971, Theorem 3.1], $B(\Gamma, \Gamma_\infty)$ is not extremally disconnected. $\square$

Remark 4.2. Let Γ be a group. One of the key properties in the applications of $\partial_F \Gamma$ to C^*-simplicity of Γ is the fact that $C(\partial_F \Gamma)$ is injective, shown in [Kalantar and Kennedy 2017, Theorem 3.12]. Proposition 4.1 implies that $C(B(\Gamma, \Lambda))$ is not injective, in general. We believe that this is evidence that $B(\Gamma, \Lambda)$ is not likely to play the same role as the Furstenberg boundary in C^*-algebraic applications.

Our next aim is to show that, given $\Lambda \leq_c \Gamma$, it holds that $B(\Gamma, \Lambda) = \partial_F \Lambda$. We start with a result which we believe has its own interest.

Theorem 4.3. *Let $\Lambda \leq_c \Gamma$ and $\Gamma \curvearrowright X$ be a minimal action on a compact space such that $\Lambda \curvearrowright X$ is proximal. Then $\Lambda \curvearrowright X$ is minimal as well.*

Proof. Let $M \subset X$ be a closed nonempty Λ-invariant set. For any $g \in \Gamma$, we have that gM is $g\Lambda g^{-1}$-invariant.

Fix $g_1, \dots, g_n \in \Gamma$. We have that $H := \Lambda \cap g_1 \Lambda g_1^{-1} \cap \cdots \cap g_n \Lambda g_n^{-1}$ has finite index in Λ. In particular, $H \curvearrowright X$ is proximal and admits a unique minimal component K. Since each $g_i M$ is $g_i \Lambda g_i^{-1}$-invariant, we conclude that $K \subset \bigcap_{i=1}^n g_i M$.

By compactness of X, we obtain that $L := \bigcap_{g \in \Gamma} gM \neq \varnothing$. Since L is Γ-invariant, we have $X = L \subset M$. $\square$

The following is an immediate consequence of the previous theorem:

Corollary 4.4. *Let $\Lambda \leq_c \Gamma$. If X is a Γ-boundary which is also Λ-strongly proximal, then X is a Λ-boundary.*

By arguing as in [Ursu 2022, Corollary 4.3], we conclude the following:

Corollary 4.5. *If $\Lambda \leq_c \Gamma$, then $B(\Gamma, \Lambda) = \partial_F \Lambda$.*

5. Commensurated subgroups and proximal actions

Given a group Γ, there exists a universal minimal proximal Γ-space $\partial_p \Gamma$ [Glasner 1976, Theorem II.4.2]. It was shown in [Frisch et al. 2019, Proposition 2.12] and [Glasner et al. 2021, Theorem 1.5] that a countable group Γ is icc if and only if $\Gamma \curvearrowright \partial_p \Gamma$ is faithful if and only if $\Gamma \curvearrowright \partial_p \Gamma$ is free.

One can easily check that the statements of Theorem 3.1 and Lemma 3.3 hold with $\partial_p \Lambda$ instead of $\partial_F \Lambda$, with the exact same proofs (in particular, [Breuillard et al. 2017, Lemma 5.1], which is needed in the proof of Lemma 3.3, uses only proximality). Thus, we obtain:

Theorem 5.1. *Let $\Lambda \leq_c \Gamma$. Then $\Lambda \curvearrowright \partial_p \Lambda$ extends in a unique way to an action of Γ on $\partial_p \Lambda$. Furthermore, given $s \in \Gamma$, if $s \in \mathrm{C}_\Gamma(\Lambda \cap s^{-1}\Lambda s)$, then* $\mathrm{Fix}(s) = \partial_p \Lambda$. *Conversely, if $\Lambda \curvearrowright \partial_p \Lambda$ is free and* $\mathrm{Fix}(s) \neq \varnothing$, *then $s \in \mathrm{C}_\Gamma(\Lambda \cap s^{-1}\Lambda s)$.*

As a consequence, we obtain the following:

Theorem 5.2. *Let $\Lambda \leq_c \Gamma$ and suppose that $\Lambda \curvearrowright \partial_p \Lambda$ is free. The following conditions are equivalent*:

(1) *Γ is icc relatively with Λ;*

(2) *for any $s \in \Gamma \setminus \{e\}$, we have that $s \notin C_\Gamma(\Lambda \cap s^{-1}\Lambda s)$;*

(3) *$\Gamma \curvearrowright \partial_p \Lambda$ is free;*

(4) *$\Gamma \curvearrowright \partial_p \Lambda$ is faithful.*

Proof. The implications (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) are proven as in Theorem 3.5.

(4) $\Longrightarrow$ (1): Suppose that there is $g \in \Gamma \setminus \{e\}$ such that $|g^\Lambda| < \infty$. Then it follows that $H := \{h \in \Lambda : gh = hg\}$ is a finite-index subgroup of Λ, hence $H \curvearrowright \partial_p \Lambda$ is also minimal and proximal. Since the homeomorphism on $\partial_p \Lambda$ given by g is H-equivariant, we conclude that g acts trivially on $\partial_p \Lambda$. $\square$

Remark 5.3. Given a group Γ, let $L(\Gamma)$ be its *group von Neumann algebra*. Given $\Lambda \leq \Gamma$, it follows from [Rørdam 2021, Proposition 5.1] and [Bédos and Omland 2023, Corollary 4.3] that Γ is icc relatively to Λ if and only if any intermediate von Neumann algebra of $L(\Lambda) \subset L(\Gamma)$ is a factor if and only if any intermediate C^*-algebra of $C_r^*(\Lambda) \subset C_r^*(\Gamma)$ is prime.

Let us now apply Theorem 5.2 to a certain locally finite commensurated subgroup of Thompson's group V.

Example 5.4. Let $X := \{0, 1\}$ and, given $n \geq 0$, let X^n be the set of words in X of length n. Given $w \in X^n$, let $\mathcal{C}(w) := \{(s_n) \in X^{\mathbb{N}} : s_{[1,n]} = w\}$. Recall that Thompson's group V is the group of homeomorphisms on $X^{\mathbb{N}}$ consisting of elements g for which there exist two partitions $\{\mathcal{C}(w_1), \dots, \mathcal{C}(w_m)\}$ and $\{\mathcal{C}(z_1), \dots, \mathcal{C}(z_m)\}$ of $\{0, 1\}^{\mathbb{N}}$ such that $g(w_i s) = z_i s$ for every $1 \leq i \leq m$ and $s \in X^{\mathbb{N}}$.

Let us define inductively groups G_n acting by permutations on X^n. Let $G_1 := \mathbb{Z}_2$ acting nontrivially on X and, for $n \in \mathbb{N}$,

$$G_{n+1} := \Big(\bigoplus_{w \in X^n} \mathbb{Z}_2 \Big) \rtimes G_n,$$

where the action of G_{n+1} on X^{n+1} is defined as follows: given $v \in X^n$, $x \in X$, $\sigma \in G_n$ and $f \in \bigoplus_{X^n} \mathbb{Z}_2$,

$$(f, \sigma)(vx) := \sigma(v) f_{\sigma(v)}(x).$$

Let $G := \lim_{n\in\mathbb{N}} G_n$. Then G acts faithfully on $X^{\mathbb{N}}$ and, as observed in [Le Boudec 2017, Proposition 7.11], $G \leq_c V$.

We claim that V is icc relatively with G. Given $u \in X^n$, let the *rigid stabilizer* of u, denoted by $\mathrm{rist}_G(u)$, be the subgroup of G consisting of the elements which, for every $v \in X^n \setminus \{u\}$, act as the identity on $\mathcal{C}(v)$. Given $g \in G$, there is $\tilde{g} \in \mathrm{rist}_G(u)$ such that $\tilde{g}(us) = ug(s)$ for any $s \in X^{\mathbb{N}}$. Clearly, the map $g \mapsto \tilde{g}$ is an isomorphism from G to $\mathrm{rist}_G(u)$. Fix $h \in V \setminus \{e\}$ and take $w \in X^n$ and $z \in X^m$ such that $w \neq z$, $n \geq m$ and $h(ws) = zs$ for any $s \in X^{\mathbb{N}}$. Furthermore, take $v \in X^{n-m}$ such that $zv \neq w$. Given $s \in X^{\mathbb{N}}$, we have that

$$\{\tilde{g}h\tilde{g}^{-1}(wvs) : \tilde{g} \in \mathrm{rist}_G(zv)\} = \{zvg(s) : g \in G\}. \tag{1}$$

Since $G \curvearrowright X^{\mathbb{N}}$ is faithful, it follows from (1) that $|h^G| = \infty$, thus proving the claim.

From [Glasner et al. 2021, Theorem 1.5], we obtain that $G \curvearrowright \partial_p G$ is free and from Theorem 5.2, we conclude that $V \curvearrowright \partial_p G$ is free.

Remark 5.5. Le Boudec and Matte Bon [2018, Theorem 1.5] showed that Thompson's group V is C^*-simple, hence $V \curvearrowright \partial_F V$ is free. However, their proof is done by showing that V does not admit nontrivial amenable URS, not by exhibiting a concrete topologically free V-boundary. It seems as an interesting problem to determine whether $V \curvearrowright \partial_p G$ is strongly proximal, thus providing an alternative proof of C^*-simplicity of V.

Remark 5.6. In [Breuillard et al. 2017, Theorem 1.4], it was shown that the class of C^*-simple groups is closed by taking normal subgroups. Obviously, this class is not closed by taking commensurated subgroups, since any finite subgroup is commensurated. Moreover, Example 5.4 shows that, given $\Lambda \leq_c \Gamma$ such that Γ is icc relatively to Λ, C^*-simplicity of Γ does not pass to Λ in general.

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LOCAL MAASS FORMS AND EICHLER–SELBERG RELATIONS FOR NEGATIVE-WEIGHT VECTOR-VALUED MOCK MODULAR FORMS

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By comparing two different evaluations of a modified (à la Borcherds) higher Siegel theta lift on even lattices of signature (r, s), we prove Eichler–Selberg relations for a wide class of negative-weight vector-valued mock modular forms. In doing so, we detail several properties of the lift, as well as showing that it produces an infinite family of local (and locally harmonic) Maaß forms on Grassmanians in certain signatures.

1. Introduction

Theta lifts have a storied history in the literature, receiving a vast amount of attention in the past few decades with applications throughout mathematics. We are concerned with generalizations of the Siegel theta lift originally studied by Borcherds in the celebrated paper [2]. The classical Siegel lift maps half-integral weight modular forms to those of integral weight, and has seen a wide number of important applications. For example, in arithmetic geometry [14; 21], deep results in number theory [10], fundamental work of Bruinier and Funke [9], among many others.

More recently, Bruinier and Schwagenscheidt [12] investigated the Siegel theta lift on Lorentzian lattices (that is, even lattices of signature $(1, n)$), and in doing so provided a construction of recurrence relations for mock modular forms of weight $\frac{3}{2}$, as well as commenting as to how one could provide a similar structure for those of weight $\frac{1}{2}$, thereby including Ramanujan's classical mock theta functions.

In the last few years, several authors have also considered so-called "higher" Siegel theta lifts of the shape ($k := \frac{1}{2}(1-n),\ j \in \mathbb{N}_0$)

$$\int_{\mathcal{F}}^{\mathrm{reg}} \langle R_{k-2j}^{j} f, \overline{\Theta_L(\tau, z)} \rangle v^k \, d\mu(\tau),$$

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where $R_\kappa^n := R_{n-2} \circ R_{n-4} \circ \cdots \circ R_\kappa$ is an iterated version of the Maaß raising operator $R_\kappa := 2i\frac{\partial}{\partial\tau} + \frac{\kappa}{v}$, f is weight $k-2j$ harmonic Maaß form, and Θ_L is the standard Siegel theta function associated to an even lattice L of signature $(1, n)$. Here and throughout, $\tau = u + iv \in \mathbb{H}$ and $z \in \mathrm{Gr}(L)$, the Grassmanian of L. Furthermore, $\langle\,\cdot\,,\cdot\,\rangle$ denotes the natural bilinear pairing. For example, they were considered by Bruinier and Ono (for $k = 0$, $j = 1$) in the influential work [11], by Bruinier, Ehlen and Yang in the breakthrough paper [8] in relation to the Gross–Zagier conjecture, and by Alfes-Neumann, Bringmann, Males and Schwagenscheidt in [1] for $n = 2$ and generic j.

In [32], Mertens investigated the classical Hurwitz class numbers, denoted by $H(n)$ for $n \in \mathbb{N}$. Using techniques in (scalar-valued) mock modular forms, he gave an infinite family of class number relations for odd n, two of which are

$$\sum_{s\in\mathbb{Z}} H(n-s^2) + \lambda_1(n) = \tfrac{1}{3}\sigma_1(n), \qquad \sum_{s\in\mathbb{Z}} (4s^2-n)H(n-s^2) + \lambda_3(n) = 0, \tag{1-1}$$

where $\lambda_k(n) = \frac{1}{2}\sum_{d|n} \min\bigl(d, \frac{n}{d}\bigr)^k$ and σ_k is the usual k-th power divisor function. Because of their close similarity to the classical formula of Kronecker [28] and Hurwitz [24; 25]

$$\sum_{s\in\mathbb{Z}} H(n-s^2) - 2\lambda_1(n) = 2\sigma_1(n),$$

and those arising from the Eichler–Selberg trace formula, Mertens referred to the relationships (1-1) as *Eichler–Selberg relations*. More generally, let $[\,\cdot\,,\cdot\,]_\nu$ denote the ν-th Rankin–Cohen bracket (see Section 2). In general, the Rankin–Cohen bracket $[f, g]$ is a mixed mock modular form of degree ν. It is of inherent interest to determine its natural completion, say Λ, to a holomorphic modular form. Then following Mertens [33], we say that a (mock-) modular form f satisfies an Eichler–Selberg relation if there exists some holomorphic modular form g and some form Λ such that

$$[f, g]_\nu + \Lambda$$

is a holomorphic modular form. In the influential paper [33], Mertens showed the beautiful result that all mock-modular forms of weight $\frac{3}{2}$ with holomorphic shadow satisfy Eichler–Selberg relations, using the powerful theory of holomorphic projection and the Serre–Stark theorem stating that unary theta series form a basis for the spaces of holomorphic modular forms of the dual weight $\frac{1}{2}$.[1] In particular, Mertens explicitly describes the form Λ which completes the Rankin–Cohen brackets.

Following previous examples, to demonstrate the statement, let $\mathcal{H}$ denote the generating function of Hurwitz class numbers, let $\vartheta = \sum_{n\in\mathbb{Z}} q^{n^2}$, where $\tau \in \mathbb{H}$, and

[1]Mertens also provides results for mock theta functions in weight $\frac{1}{2}$, but since there is no analogue of Serre–Stark in the dual weight $\frac{3}{2}$ this is a real restriction.

$q^n = e^{2\pi i n\tau}$ throughout. Then Mertens' results show that [33, p. 377]

$$[\mathcal{H}, \vartheta]_\nu + 2^{-2\nu-1}\binom{2\nu}{\nu}\Bigg(\sum_{r\geq 1} 2 \sum_{\substack{m^2-n^2=r\\ m,n\geq 1}} (m-n)^{2\nu-1} q^r + \sum_{r\geq 1} r^{2\nu+1} q^r\Bigg)$$

is a holomorphic modular form of weight $2\nu+2$ for all $\nu \geq 1$, and a quasimodular form of weight 2 if $\nu = 0$.

In [31], Males combined techniques of [1; 12] during a further investigation of the higher Siegel lift on Lorentzian lattices. This lift was shown to be central in producing certain Eichler–Selberg relations in the vector-valued case, providing an analogue of the scalar-valued weight $\frac{3}{2}$ case of Mertens. We remark that the shape of the form Λ in the case of signature (1, 1) is very close to that of Mertens (see [31, Theorem 1.1]), though we do not recall it here to save on complicated definitions in the introduction.

In the current paper, we develop the theory for even generic signature (r, s) lattices L and more general modified Siegel theta functions as in Borcherds [2], and consider the lift

$$\Psi_j^{\mathrm{reg}}(f, z) := \int_{\mathcal{F}}^{\mathrm{reg}} \langle R_{k-2j}^j(f)(\tau), \overline{\Theta_L(\tau, \psi, p)}\rangle v^k \, d\mu(\tau),$$

where Θ_L is a modified Siegel theta function as in Borcherds [2], essentially obtained by including a certain polynomial p in the summand of the usual vector-valued Siegel theta function. We require p to be homogenous and spherical of degree $d^+ \in \mathbb{N}_0$ in the first r variables, and $d^- \in \mathbb{N}_0$ in the last s variables (see (2-2) for precise definitions). Here, ψ is an isometry which in turn defines z; see (2-3). Modifying the theta function in this way preserves modular properties of Θ_L, while allowing us to obtain different weights of output functions. Furthermore, since the case $j = 0$ is well-understood in the literature, we assume throughout that $j > 0$. We remark that the signature (1, 2) with $j = 0$ case has also been studied in [16; 17].

In particular, we evaluate the higher lift in the now-standard ways of unfolding in Corollary 3.2, as well as recognizing it as a constant term in the Fourier expansion of the Rankin–Cohen bracket of a holomorphic modular form and a theta function (up to a boundary integral that vanishes for a certain class of input functions) in Theorem 3.3. For the second of these theorems, we use that at special points w, one may define positive- and negative-definite sublattices $P := L \cap w$ and $N := L \cap w^\perp$. In the simplest case, which we assume for the introduction, we have that $L = P \oplus N$. Then the theta series splits as $\Theta_L = \Theta_P \otimes \Theta_N$, where Θ_P is a positive-definite theta series and Θ_N a negative-definite one. Then we let $\mathcal{G}_P^+$ be the holomorphic part of a preimage of Θ_P under $\xi_\kappa := 2iv^\kappa \overline{\frac{\partial}{\partial \bar{\tau}}}$. For the sake of simplicity, we assume that $\mathcal{G}_P^+ + g$ in the statement of Theorem 1.1 is bounded at $i\infty$ in the introduction; we overcome this assumption in Theorem 3.4 and offer a precise relation there.

Following the ideas of [31], by comparing these two evaluations of our lift and invoking Serre duality, we obtain the following theorem.

Theorem 1.1. *Let L be an even lattice of signature (r, s), with associated Weil representation ρ_L. Let g be any holomorphic vector-valued modular form of weight $2-\left(\frac{r}{2}+d^+\right)$ for ρ_L. Suppose that $\mathcal{G}_P^+ + g$ is bounded at $i\infty$. Then $\mathcal{G}_P^+ + g$ satisfies an explicit Eichler–Selberg relation. In particular, the form Λ is explicitly determined.*

The concept of so-called locally harmonic Maaß forms was introduced by Bringmann, Kane and Kohnen in [4]. These are functions that behave like classical harmonic Maaß forms, except for an exceptional set of density zero, where they have jump singularities. Since their inception, locally harmonic Maaß forms have seen applications throughout number theory, for example, in relation to central values of L-functions of elliptic curves [20], as well as traces of cycle integrals and periods of meromorphic modular forms [1; 30] among many others. Examples of such locally harmonic Maaß forms are usually achieved in the literature via similar theta lift machinery to that studied here. In addition to the direction of Theorem 1.1, we also discuss the action of the Laplace–Beltrami operator on the lift Ψ_j^{reg} in Theorem 4.2. In doing so, we prove the following theorem, thereby providing an infinite family of local Maaß forms (and locally harmonic Maaß forms) in signatures $(2, s)$. To state the result, we let $F_{m,k-2j,\mathfrak{s}}$ be a Maaß–Poincaré series as defined in (2-1).

Theorem 1.2. *Let L be an even isotropic lattice of signature $(2, s)$. Then the lift $\Psi_j^{\mathrm{reg}}(F_{m,k-2j,\mathfrak{s}}, z)$ is a local Maaß form on $\mathrm{Gr}(L)$ with eigenvalue $\left(\mathfrak{s}-\frac{k}{2}\right)\left(1-\mathfrak{s}-\frac{k}{2}\right)$ under the Laplace–Beltrami operator.*

We provide an example of an input function to our lift. To this end, we specialize our setting to signature $(1, 2)$, in which case vector-valued modular forms can be identified with the usual scalar-valued framework on the complex upper half-plane, and in particular $\mathrm{Gr}(L) \cong \mathbb{H}$. (We explain the required choices in Section 5.) In 1975, Cohen [15] defined the generalized class numbers

$$H(\ell-1, |D|) := \begin{cases} 0 & \text{if } D \neq 0, 1 \pmod 4, \\ \zeta(3-2\ell) & \text{if } D = 0, \\ L\left(2-\ell, \left(\frac{D_0}{\cdot}\right)\right) \sum_{d \mid j} \mu(d) \left(\frac{D_0}{d}\right) d^{\ell-2}\, \sigma_{2\ell-3}\left(\frac{j}{d}\right), & \text{else,} \end{cases}$$

where $D = D_0 j^2$, as well as their generating functions

$$\mathcal{H}_\ell(\tau) := \sum_{n \geq 0} H(\ell, n)\, q^n, \quad \ell \in \mathbb{N} \setminus \{1\}.$$

Here, ζ refers to the Riemann zeta function, $L(s, \chi)$ to the Dirichlet L-function twisted by a Dirichlet character χ, and μ is the Möbius function. The functions $\mathcal{H}_\ell$

are known as Cohen–Eisenstein series today, and can be viewed as half integral weight analogues of the classical integral weight Eisenstein series. Note that the numbers $H(2,n)$ are precisely the Hurwitz class numbers introduced above, and $\mathcal{H}_2 = \mathcal{H}$. Cohen proved that $\mathcal{H}_\ell \in M_{\ell-(1/2)}(\Gamma_0(4))$, the space of scalar-valued modular forms of weight $\frac{1}{2}$ on the usual congruence subgroup $\Gamma_0(4)$, and the coefficients satisfy Kohnen's plus space condition by definition. (See [6, (2.13)–(2.15), Corollary 2.25] for more details on this.)

However, evaluating our lift requires negative weight and a nonconstant principal part of the input function. To overcome both obstructions, we let

$$f_{-2\ell,N}(\tau) = q^{-N} + \sum_{n>m} c_{-2\ell}(N,n)\, q^n, \quad N \geq -m,$$

$$m := \begin{cases} \left\lfloor \frac{-2\ell}{12} \right\rfloor - 1 & \text{if } -2\ell \equiv 2 \pmod{12}, \\ \left\lfloor \frac{-2\ell}{12} \right\rfloor, & \text{else} \end{cases}$$

be the unique weakly holomorphic modular form of weight -2ℓ for $\mathrm{SL}_2(\mathbb{Z})$ with such a Fourier expansion, an explicit description of $f_{-2\ell,N}$ was given by Duke and Jenkins [18], and by Duke, Imamoḡlu and Tóth [19, Theorem 1]. Our machinery now enables us to obtain Eichler–Selberg relations for the weakly holomorphic function $f_{-2\ell,N}(\tau)\mathcal{H}_\ell(\tau)$ along the lines of [15, Section 6], as well as the following variant of Theorem 1.2.

Theorem 1.3. *The lift $\Psi_j^{\mathrm{reg}}(f_{-2\ell,N}\mathcal{H}_\ell, z)$ is a local Maaß form on $\mathbb{H}$ for every $j \in \mathbb{N}$, $\ell \in \mathbb{N} \setminus \{1\}$, and $-m \leq N \in \mathbb{N}$ with exceptional set given by the net of Heegner geodesics*

$$\bigcup_{D=1}^{N} \{z = x + iy \in \mathbb{H} : \exists a, b, c \in \mathbb{Z},\ b^2 - 4ac = D,\ a|z|^2 + bx + c = 0\}.$$

Remarks. (1) Theorem 1.3 generalizes immediately to any weakly holomorphic modular form g. The exceptional set is given by the union of geodesics of discriminant $D > 0$, for which the coefficient of g at q^{-D} is nonzero.

(2) Recently, Wagner [37] constructed a pullback of $\mathcal{H}_\ell$ under the ξ-operator, namely a harmonic Maaß form $\mathscr{H}_\ell$ of weight $-\ell + \frac{1}{2}$ on $\Gamma_0(4)$ that satisfies $\xi_{(1/2)-\ell}\mathscr{H}_\ell = \mathcal{H}_{\ell+2}$. An explicit definition of $\mathscr{H}_\ell$ can be found in [37, (1.5), (1.6)]. However, $\mathscr{H}_\ell$ is a harmonic Maaß form with noncuspidal image under ξ, and we restrict ourselves to a more restrictive growth condition in the discussion of Maaß forms (see Section 2) to ensure convergence of our lift. It would be interesting to investigate different regularizations of our lift, and in particular, lift the function $\mathscr{H}_\ell$.

The paper is organized as follows. We establish the overall framework in Section 2. Section 3 is devoted to two evaluations of our theta lift and to the proof of Theorem 1.1. In Section 4, we compute the action of the Laplace–Beltrami operator on our theta lift and prove Theorem 1.2. Lastly, Section 5 offers more details on the specialization to signature $(1, 2)$, a proof of Theorem 1.3, and an indication on Eichler–Selberg relations for Cohen–Eisenstein series at the very end.

2. Preliminaries

We summarize some facts, which we require throughout.

The Weil representation. We recall the metaplectic double cover

$$\widetilde{\Gamma} := \mathrm{Mp}_2(\mathbb{Z}) := \left\{ (\gamma, \phi) : \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}),\ \phi : \mathbb{H} \to \mathbb{C} \text{ holomorphic},\ \phi^2(\tau) = c\tau + d \right\}$$

of $\mathrm{SL}_2(\mathbb{Z})$, which is generated by the pairs

$$\widetilde{T} := \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1 \right), \quad \widetilde{S} := \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{\tau} \right),$$

where we fix a suitable branch of the complex square root throughout. Furthermore, we define $\widetilde{\Gamma}_\infty$ as the subgroup generated by $\widetilde{T}$.

We let L be an even lattice of signature (r, s), and Q be a quadratic form on L with associated bilinear form $(\,\cdot\,,\,\cdot\,)_Q$. Moreover, we denote the dual lattice of L by L', and the group ring of L'/L by $\mathbb{C}[L'/L]$. The group ring $\mathbb{C}[L'/L]$ has a standard basis, whose elements will be called $\mathfrak{e}_\mu$ for $\mu \in L'/L$. We recall that there is a natural bilinear form $\langle\,\cdot\,,\,\cdot\,\rangle$ on $\mathbb{C}[L'/L]$ defined by $\langle \mathfrak{e}_\mu, \mathfrak{e}_\nu \rangle = \delta_{\mu,\nu}$.

Equipped with this structure, the Weil representation ρ_L of $\widetilde{\Gamma}$ associated to L is defined on the generators by

$$\rho_L(\widetilde{T})(\mathfrak{e}_\mu) := e(Q(\mu))\, \mathfrak{e}_\mu, \quad \rho_L(\widetilde{S})(\mathfrak{e}_\mu) := \frac{e\left(\frac{1}{8}(s-r)\right)}{\sqrt{|L'/L|}} \sum_{\nu \in L'/L} e(-(\nu, \mu)_Q)\, \mathfrak{e}_\nu,$$

where we stipulate $e(x) := e^{2\pi i x}$ throughout. We let $L^- := (L, -Q)$ and call ρ_{L^-} the dual Weil representation of L.

The generalized upper half-plane and the invariant Laplacian. We follow the introduction in [7, Sections 3.2, 4.1], and let the signature of L be $(2, s)$ here. We assume that L is isotropic, i.e., it contains a nontrivial vector x of norm 0, and by rescaling we may assume that it is primitive, that is if $x = cy$ for some $y \in L$ and $c \in \mathbb{Z}$ then $c = \pm 1$. Note that for $s \geq 3$ all lattices contain such an isotropic vector (see [2, Section 8]).

Let $z \in L$ be a primitive norm 0 vector and $z' \in L'$ with $(z, z')_Q = 1$. Let $K := L \cap z^\perp \cap z'^\perp$. Let $d \in K$ be a primitive norm 0 vector, and $d' \in K'$ with

$(d, d')_Q = 1$. It follows that $D := K \cap d^{\perp} \cap d'^{\perp}$ is a negative-definite lattice, and we write

$$Z = (d' - Q(d')\, d)\, z_1 + z_2\, d + z_3\, d_3 + \cdots + z_\ell\, d_\ell =: (z_1, z_2, \ldots, z_\ell) \in K \otimes \mathbb{C},$$

since $z_3\, d_3 + \cdots + z_\ell\, d \in D \otimes \mathbb{C}$. Each z_j has a real part x_j and a imaginary part y_j, and we note that

$$Q(Y) := Q(y_1, \ldots, y_\ell) = y_1 y_2 - y_3^2 - y_4^2 - \cdots - y_\ell^2.$$

This gives rise to the generalized upper half-plane

$$\mathbb{H}_\ell := \{Z \in K \otimes \mathbb{C} : y_1 > 0,\ Q(Y) > 0\} \cong \mathrm{Gr}(L).$$

Letting

$$\partial_\mu := \frac{\partial}{\partial z_\mu} = \tfrac{1}{2}\left(\frac{\partial}{\partial x_\mu} - i\,\frac{\partial}{\partial y_\mu}\right), \quad \bar{\partial}_\mu := \frac{\partial}{\partial \bar{z}_\mu} = \tfrac{1}{2}\left(\frac{\partial}{\partial x_\mu} + i\,\frac{\partial}{\partial y_\mu}\right),$$

it can be shown that the invariant Laplacian on $\mathbb{H}_\ell$ has the coordinate representation [34]

$$\Omega := \sum_{\mu,\nu=1}^{\ell} y_\mu\, y_\nu\, \partial_\mu\, \bar{\partial}_\nu - Q(Y)\left(\partial_1 \bar{\partial}_2 + \bar{\partial}_1 \partial_2 - \tfrac{1}{2}\sum_{\mu=3}^{\ell} \partial_\mu\, \bar{\partial}_\mu\right).$$

Maaß forms. Let $\kappa \in \frac{1}{2}\mathbb{Z}$, $(\gamma, \phi) \in \widetilde{\Gamma}$ and consider a function $f : \mathbb{H} \to \mathbb{C}[L'/L]$. The modular transformation in this setting is captured by the slash-operator

$$f|_{\kappa,\rho_L}(\gamma, \phi)(\tau) := \phi(\tau)^{-2\kappa} \rho_L^{-1}(\gamma, \phi)\, f(\gamma\tau),$$

which leads to vector-valued Maaß forms as follows [9].

Definition. Let $f : \mathbb{H} \to \mathbb{C}[L'/L]$ be smooth. Then f is a Maaß form of weight κ with respect to ρ_L if it satisfies the following three conditions.

(1) We have $f|_{\kappa,\rho_L}(\gamma, \phi)(\tau) = f(\tau)$ for every $\tau \in \mathbb{H}$ and every $(\gamma, \phi) \in \widetilde{\Gamma}$.

(2) The function f is an eigenfunction of the weight κ hyperbolic Laplace operator, which is explicitly given by

$$\Delta_\kappa := -v^2\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) + i\kappa v\left(\frac{\partial}{\partial u} + i\,\frac{\partial}{\partial v}\right).$$

(3) There exists a polynomial[2] in q denoted by $P_f : \{0 < |w| < 1\} \to \mathbb{C}[L'/L]$ such that $f(\tau) - P_f(q) \in O(e^{-\varepsilon v})$ as $v \to \infty$ for some $\varepsilon > 0$.

We call f a harmonic Maaß form if the eigenvalue equals 0.

[2]Such a polynomial is called the principal part of f.

We write $H_{\kappa,L}$ for the vector space of harmonic Maaß forms of weight κ with respect to ρ_L, and $M^!_{\kappa,L} \subseteq H_{\kappa,L}$ for the subspace of weakly holomorphic vector valued modular forms. The subspace $S^!_{\kappa,L} \subseteq M^!_{\kappa,L}$ collects all forms that vanish at all cusps, and such forms are referred to as weakly holomorphic cusp forms.

Bruinier and Funke [9] proved that a harmonic Maaß form f of weight $\kappa \neq 1$ decomposes as a sum $f = f^+ + f^-$ of a holomorphic and a nonholomorphic part, whose Fourier expansions are of the shape

$$f^+(\tau) = \sum_{\mu \in L'/L} \sum_{\substack{n \in \mathbb{Q} \\ n \gg -\infty}} c_f^+(\mu, n)\, q^n \mathfrak{e}_\mu,$$
$$f^-(\tau) = \sum_{\mu \in L'/L} \sum_{\substack{n \in \mathbb{Q} \\ n < 0}} c_f^-(\mu, n)\, \Gamma(1-\kappa, 4\pi |n| v)\, q^n \mathfrak{e}_\mu,$$

where $\Gamma(t, x) := \int_x^\infty u^{t-1} e^{-u}\, du$ denotes the incomplete gamma function.

Harmonic Maaß forms can be inspected via the action of various differential operators. We require the antiholomorphic operator

$$\xi_\kappa := 2i v^\kappa \overline{\frac{\partial}{\partial \bar{\tau}}},$$

as well as the Maaß raising and lowering operators

$$R_\kappa := 2i \frac{\partial}{\partial \tau} + \frac{\kappa}{v}, \qquad L_\kappa := -2i v^2 \frac{\partial}{\partial \bar{\tau}}.$$

The operator ξ_κ defines a surjective map from $H_{\kappa,L}$ to $S^!_{2-\kappa,L^-}$ [9]. In particular, it intertwines with the slash operator introduced above, and the space $M^!_{\kappa,L}$ is precisely the kernel of ξ_κ when restricted to $H_{\kappa,L}$. Hence, every $f \in H_{\kappa,L}$ has a cuspidal shadow in our case.

The operators R_κ and L_κ increase and decrease the weight κ by 2 respectively, but do not preserve the eigenvalue under Δ_κ. For any $n \in \mathbb{N}_0$, we let

$$R_\kappa^0 := \mathrm{id}, \quad R_\kappa^n := R_{\kappa+2n-2} \circ \cdots \circ R_{\kappa+2} \circ R_\kappa,$$
$$L_\kappa^0 := \mathrm{id}, \quad L_\kappa^n := L_{\kappa-2n+2} \circ \cdots \circ L_{\kappa-2} \circ L_\kappa$$

be the iterated Maaß raising and lowering operators, which increase or decrease the weight κ by $2n$.

Remark. If one relaxes the growth condition (iii) to linear exponential growth, that is, $f(\tau) \in O(e^{\varepsilon v})$ as $v \to \infty$ for some $\varepsilon > 0$, then f^- is permitted to have an additional (constant) term of the form $c_f^-(\mu, 0)\, v^{1-\kappa} \mathfrak{e}_\mu$. In this case, ξ_κ maps such a form to a weakly holomorphic modular form instead of a weakly holomorphic cusp form.

Local Maaß forms. Locally harmonic Maaß forms were introduced by Bringmann, Kane and Kohnen [4] for negative weights, and independently by Hövel [23] for weight 0. We generalize the exposition due to Bringmann, Kane and Kohnen here and provide a definition in our setting on Grassmannians and for arbitrary eigenvalues.

Definition. A local Maaß form of weight κ with closed exceptional set $X \subsetneq \mathbb{H}_\ell$ of measure zero is a function $f : \mathbb{H}_\ell \to \mathbb{C}[L'/L]$, which satisfies four properties:

(1) For all $(\gamma, \phi) \in \widetilde{\Gamma}$ and all $Z \in \mathbb{H}_\ell$ it holds that $f|_{\kappa,\rho_L}(\gamma,\phi)(Z) = f(Z)$.

(2) For every $Z \in \mathbb{H}_\ell \setminus X$, there exists a neighborhood of Z, in which f is real analytic and an eigenfunction of Ω.

(3) We have

$$f(Z) = \tfrac{1}{2} \lim_{\varepsilon \searrow 0} \big(f(Z + (i\varepsilon, 0, \dots, 0)^t) + f(Z - (i\varepsilon, 0, \dots, 0)^t) \big)$$

for every $Z \in X$.

(4) The function f is of at most polynomial growth towards all cusps.

Paralleling the definition of harmonic Maaß forms, we call a local Maaß form locally harmonic if the eigenvalue from the second condition is 0.

Poincaré series.

Weakly holomorphic Poincaré series. Following Knopp and Mason [27, Section 3], we let $m \in \mathbb{Z}$, $\kappa \in \frac{1}{2}\mathbb{N}$ satisfying $\kappa > 2$, $\mu \in L'/L$, and define

$$\mathbb{F}_{\mu,m,\kappa}(\tau) := \tfrac{1}{2} \sum_{(\gamma,\phi) \in \widetilde{\Gamma}_\infty \backslash \widetilde{\Gamma}} \big(e((m+1)\tau)\, \mathfrak{e}_\mu \big)|_{\kappa,\rho_L}(\gamma,\phi).$$

Knopp and Mason [27] prove that $\mathbb{F}_{\mu,m,\kappa}$ converges absolutely, and that it defines a weakly holomorphic modular form of weight κ for ρ_L. In addition, they computed the Fourier expansion of $\mathbb{F}_{\mu,m,\kappa}$, which is of the shape

$$\mathbb{F}_{\mu,m,\kappa}(\tau) = \sum_{\nu \in L'/L} \left(\delta_{\mu,\nu} q^{m+1} + \sum_{n \geq 0} c(n)\, q^{n+1} \right) \mathfrak{e}_\nu .$$

The Fourier coefficients $c(n)$ can be found in [27, Theorem 3.2] explicitly.

Maaß–Poincaré series. We recall an important example of harmonic Maaß forms. To this end, let $\kappa \in -\frac{1}{2}\mathbb{N}$, let $M_{\mu,\nu}$ be the usual M-Whittaker function (see [35, Section 13.14]), and define the auxiliary function

$$\mathcal{M}_{\kappa,\mathfrak{s}}(y) := |y|^{-\frac{\kappa}{2}} M_{\operatorname{sgn}(y)\frac{\kappa}{2}, \mathfrak{s}-\frac{1}{2}}(|y|), \qquad y \in \mathbb{R} \setminus \{0\}.$$

We average $\mathcal{M}_\kappa$ over $\widetilde{\Gamma}$ with respect to the parameters $\mu \in L'/L$, $m \in \mathbb{N} \setminus \{Q(\mu)\}$, and κ, $\mathfrak{s}$. This yields the vector-valued Maaß–Poincaré series [7]

$$(2\text{-}1) \qquad F_{\mu,m,\kappa,\mathfrak{s}}(\tau) := \frac{1}{2\Gamma(2\mathfrak{s})} \sum_{(\gamma,\phi)\in\widetilde{\Gamma}_\infty\backslash\widetilde{\Gamma}} (\mathcal{M}_{\kappa,\mathfrak{s}}(4\pi m v)\, e(-mu)\, \mathfrak{e}_\mu)|_{\kappa,\rho_L}(\gamma,\phi).$$

By our choice of parameters and taking cosets, the series converges absolutely. The eigenvalue under Δ_κ is given by $\left(\mathfrak{s} - \frac{\kappa}{2}\right)\left(1 - \mathfrak{s} - \frac{\kappa}{2}\right)$. Hence if $\mathfrak{s} = \frac{\kappa}{2}$ or $\mathfrak{s} = 1 - \frac{\kappa}{2}$, then we have $F_{\mu,m,\kappa,\mathfrak{s}} \in H_{\kappa,L}$. The principal part of $F_{\mu,m,\kappa,\mathfrak{s}}$ is given by $e(-m\tau)(\mathfrak{e}_\mu + \mathfrak{e}_{-\mu})$ in this case, and $\xi_\kappa F_{\mu,-m,\kappa,\mathfrak{s}}$ is a weight $2-\kappa$ cusp form.

Furthermore, the Maaß–Poincaré series have the following useful property thanks to their simple principal part.

Lemma 2.1. *Let $f \in H_{\kappa,L}$ with $\kappa \in -\frac{1}{2}\mathbb{N}$, and principal part*

$$P_f(\tau) = \sum_{\mu\in L'/L} \sum_{n<0} c_f^+(\mu,n)\, e(n\tau)\, \mathfrak{e}_\mu \in \mathbb{C}[L'/L][e(-\tau)].$$

Then, we have

$$f(\tau) = \tfrac{1}{2} \sum_{\mu\in L'/L} \sum_{m>0} c_f^+(\mu,-m) F_{\mu,m,\kappa,1-\frac{\kappa}{2}}(\tau).$$

Additionally, we require the following computational lemma, which is taken from [1, Lemma 2.1], and follows inductively from [8, Proposition 3.4].

Lemma 2.2. *For any $n \in \mathbb{N}_0$ it holds that*

$$R_\kappa^n(F_{\mu,m,\kappa,\mathfrak{s}})(\tau) = (4\pi m)^n \frac{\Gamma\left(\mathfrak{s}+n+\frac{\kappa}{2}\right)}{\Gamma\left(\mathfrak{s}+\frac{\kappa}{2}\right)} F_{\mu,m,\kappa+2n,\mathfrak{s}}(\tau).$$

Restriction, trace maps, and Rankin–Cohen brackets. As before, we fix an even lattice L. We let $A_{\kappa,L}$ be the space of smooth functions $f : \mathbb{H} \to \mathbb{C}[L'/L]$, which are invariant under the weight κ slash operator with respect to the representation ρ_L. Moreover, let $K \subseteq L$ be a finite index sublattice. Hence, we have $L' \subseteq K'$, and thus

$$L/K \subseteq L'/K \subseteq K'/K.$$

This induces a map

$$L'/K \to L'/L, \quad \mu \mapsto \bar{\mu}.$$

If $\mu \in K'/K$, $f \in A_{\kappa,L}$, $g \in A_{\kappa,K}$, and μ is a fixed preimage of $\bar{\mu}$ in L'/K, we define

$$(f_K)_\mu := \begin{cases} f_{\bar{\mu}} & \text{if } \mu \in L'/K, \\ 0 & \text{if } \mu \notin L'/K, \end{cases} \qquad (g^L)_{\bar{\mu}} = \sum_{\alpha\in L/K} g_{\alpha+\mu}.$$

Lemma 2.3 [13, Section 3]. *In the notation above, there are two natural maps*

$$\begin{aligned} \mathrm{res}_{L/K} : A_{\kappa,L} &\to A_{\kappa,K}, & \mathrm{tr}_{L/K} : A_{\kappa,K} &\to A_{\kappa,L}, \\ f &\mapsto f_K, & g &\mapsto g^L, \end{aligned}$$

satisfying

$$\langle f, \bar{g}^L \rangle = \langle f_K, \bar{g} \rangle$$

for any $f \in A_{\kappa,L},\ g \in A_{\kappa,K}$.

Let $\kappa, \ell \in \frac{1}{2}\mathbb{Z},\ f \in A_{\kappa,K},\ g \in A_{\ell,L}$. Writing

$$f = \sum_{\mu} f_\mu \mathfrak{e}_\mu, \quad g = \sum_{\nu} g_\nu \mathfrak{e}_\nu$$

and letting $n \in \mathbb{N}_0$, we define the tensor product of f and g as well as the n-th Rankin–Cohen bracket of f and g as

$$\begin{aligned} f \otimes g &:= \sum_{\mu,\nu} f_\mu g_\nu \mathfrak{e}_{\mu+\nu} \in A_{\kappa+\ell, K \oplus L}, \\ [f, g]_n &:= \frac{1}{(2\pi i)^n} \sum_{\substack{r,s \geq 0 \\ r+s=n}} \frac{(-1)^r \Gamma(\kappa+n)\Gamma(\ell+n)}{\Gamma(s+1)\Gamma(\kappa+n-s)\Gamma(r+1)\Gamma(\ell+n-r)} f^{(r)} \otimes g^{(s)}, \end{aligned}$$

where $f^{(r)}$ and $g^{(s)}$ are usual higher derivatives of f and g. Then we have the following vector-valued analogue of [8, Proposition 3.6].

Lemma 2.4. *Let* $f \in H_{\kappa,L_1}$ *and* $g \in H_{\ell,L_2}$. *For* $n \in \mathbb{N}_0$ *it holds that*

$$\begin{aligned} &(-4\pi)^n L_{\kappa+\ell+2n}([f, g]_n) \\ &\qquad = \frac{\Gamma(\kappa+n)}{n!\,\Gamma(\kappa)} L_\kappa(f) \otimes R_\ell^n(g) + (-1)^n \frac{\Gamma(\ell+n)}{n!\,\Gamma(\ell)} R_\kappa^n(f) \otimes L_\ell(g). \end{aligned}$$

Finally, we have the following lemma, which can be verified straightforwardly (see [1, Proof of Theorem 4.1]).

Lemma 2.5. *Let h be a smooth function*, *g be holomorphic*, *and* $\kappa, \ell \in \mathbb{R}$. *Then it holds that*

$$R_{\ell-\kappa}(v^\kappa \bar{g} \otimes h) = v^k \bar{g} \otimes R_\ell h.$$

Theta functions and special points. We fix an even lattice L of signature (r, s) and extend the quadratic form on L to $L \otimes \mathbb{R}$ in the natural way. We denote the orthogonal projection of $\lambda \in L + \mu$ onto the linear subspaces spanned by z and its orthogonal complement with respect to $(\,\cdot\,,\,\cdot\,)_Q$ by λ_z and $\lambda_{z^\perp}$ respectively. In other words, we have

$$L \otimes \mathbb{R} = z \oplus z^\perp, \quad \lambda = \lambda_z + \lambda_{z^\perp}.$$

Let $\mathrm{Gr}(L)$ be the Grassmannian of r-dimensional subspaces of $L\otimes\mathbb{R}$. Let $Z\subseteq\mathrm{Gr}(L)$ be the set of all such subspaces on which Q is positive definite. One can endow Z with the structure of a smooth manifold.

Let $p_r : \mathbb{R}^{r,0} \to \mathbb{C}$ and $p_s : \mathbb{R}^{0,s} \to \mathbb{C}$ be spherical polynomials, which are homogeneous of degree $d^+, d^- \in \mathbb{N}_0$ respectively. Define

$$p := p_r \otimes p_s \tag{2-2}$$

and let $\psi : L \otimes \mathbb{R} \to \mathbb{R}^{r,s}$ be an isometry. We set

$$z := \psi^{-1}(\mathbb{R}^{r,0}) \in Z, \quad z^{\perp} = \psi^{-1}(\mathbb{R}^{0,s}). \tag{2-3}$$

For a positive-definite lattice (K, Q) of rank n and a homogeneous spherical polynomial p of degree d, we define the usual theta function

$$\Theta_K(\tau, \psi_K, p) := \sum_{\lambda\in K'} p(\psi_K(\lambda))\, e(Q(\lambda)\tau),$$

where ψ_K is the isometry associated to K. It is a holomorphic modular form of weight $\frac{n}{2}+d$ for ρ_K. If the isometry is trivial, we write $\Theta_K(\tau, p)$.

Following Borcherds [2] and Hövel [23], we define the general Siegel theta function as follows.[3]

Definition. Let $\tau \in \mathbb{H}$ and assume the notation above. Then we put

$$\Theta_L(\tau, \psi, p) := v^{\frac{s}{2}+d^-} \sum_{\mu\in L'/L} \sum_{\lambda\in L+\mu} p(\psi(\lambda))\, e(Q(\lambda_z)\tau + Q(\lambda_{z^\perp})\bar{\tau})\, \mathfrak{e}_\mu.$$

One can check that the function Θ_L converges absolutely on $\mathbb{H}\times Z$. The following result is [23, Satz 1.55], which follows directly from [2, Theorem 4.1].

Lemma 2.6. *Let $(\gamma, \phi) \in \widetilde{\Gamma}$. Then we have*

$$\Theta_L(\gamma\tau, \psi, p) = \phi(\tau)^{r+2d^+-(s+2d^-)} \rho_L(\gamma, \phi)\, \Theta_L(\tau, \psi, p).$$

Thus, we define

$$k := \tfrac{r-s}{2} + d^+ - d^-.$$

The following terminology is borrowed from [12].

Definition. An element $w \in \mathrm{Gr}(L)$ is called a special point if it is defined over $\mathbb{Q}$, that is, $w \in L \otimes \mathbb{Q}$.

[3] In fact, Borcherds considered a slightly more general theta function, where the polynomial p does not necessarily vanish under Δ_K. For us however, this more general case would not yield spherical theta functions as we desire.

We observe that if w is a special point, then $w^\perp$ is a special point as well. This yields the splitting

$$L \otimes \mathbb{Q} = w \oplus w^\perp,$$

which in turn yields the positive and negative-definite lattices

$$P := L \cap w, \quad N := L \cap w^\perp.$$

Clearly, $P \oplus N$ is a sublattice of L of finite index, and according to Lemma 2.3, the theta functions associated to both lattices are related by

$$\Theta_L = (\Theta_{P\oplus N})^L.$$

We identify $\mathbb{C}[(P \oplus N)'/(P \oplus N)]$ with $\mathbb{C}[P'/P] \otimes \mathbb{C}[N'/N]$, and let ψ_P, ψ_N be the restrictions of ψ onto P, N respectively. Consequently, we have the splitting

$$\Theta_{P\oplus N}(\tau, \psi, p) = \Theta_P(\tau, \psi_P, p_r) \otimes v^{\frac{s}{2}+d^-}\, \overline{\Theta_{N^-}(\tau, \psi_N, p_s)}$$

at a special point w, which can be verified straightforwardly. Furthermore, we observe that $\Theta_P(\tau, \psi_P, p_r)$ is holomorphic and of weight $\frac{r}{2} + d^+$ as a function of τ, while $v^{\frac{s}{2}+d^-}\, \overline{\Theta_{N^-}(\tau, \psi_N, p_s)}$ is of weight $-\frac{s}{2} - d^-$ with respect to τ.

Serre duality.

Proposition 2.7 [29, Proposition 2.5, Serre duality]. *Let L be an even lattice and $\kappa \in \frac{1}{2}\mathbb{Z}$. Assume that*

$$g(\tau) = \sum_{h\in L'/L} \sum_{n\geq 0} c_g(h, n)\, e(n\tau)\, \mathfrak{e}_h$$

is bounded at the cusp $i\infty$. Then g is a holomorphic modular form of weight κ for the Weil representation ρ_L if and only if we have

$$\sum_{h\in L'/L} \sum_{n\geq 0} c_g(h, n)\, c_f(h, -n) = 0$$

for every weakly holomorphic modular form f of weight $2 - \kappa$ for $\bar{\rho}_L$.

3. The theta lift

We consider the theta lift $\Psi_j^{\mathrm{reg}}(f, z)$ and evaluate it in two different ways. Using Serre duality goes back to Borcherds [3].

Evaluation in terms of ${}_2F_1$. We begin by evaluating the higher modified lift as a series involving Gauß hypergeometric functions as follows.

Evaluating the theta lift of Maaß–Poincaré series for general spectral parameters. Let $\mathfrak{s} \in \mathbb{C}$ be such that

$$F_{m,\kappa,\mathfrak{s}}(\tau) := \sum_{\mu \in L'/L} F_{\mu,m,\kappa,\mathfrak{s}}(\tau)$$

converges absolutely, that is, $\mathrm{Re}(\mathfrak{s}) > 1 - \frac{\kappa}{2}$.

Theorem 3.1. *We have*

$$\begin{aligned}\Psi_j^{\mathrm{reg}}(F_{m,k-2j,\mathfrak{s}}, z) = (4\pi m)^{j+1-k-\frac{s}{2}-d^-} &\frac{\Gamma\big(\mathfrak{s}+\frac{k}{2}\big)\Gamma\big(\frac{k+s}{2}+d^- - 1 + \mathfrak{s}\big)}{2\Gamma(2-k+2j)\Gamma\big(\mathfrak{s}+\frac{k}{2}-j\big)} \\ &\times \sum_{\mu \in L'/L} \sum_{\substack{\lambda \in L+\mu \\ Q(\lambda)=-m}} \overline{p(\psi(\lambda))} \left(\frac{Q(\lambda)}{Q(\lambda_{z^\perp})}\right)^{\frac{k+s}{2}+d^- -1+\mathfrak{s}} \\ &\times {}_2F_1\left(k+\mathfrak{s}, \frac{k+s}{2}+d^- - 1 + \mathfrak{s}; 2\mathfrak{s}; \frac{Q(\lambda)}{Q(\lambda_{z^\perp})}\right).\end{aligned}$$

Remark. Choosing the homogeneous polynomial in the theta kernel function to be the constant function 1 and computing the action of R_{k-2j}^j on $F_{m,k-2j,\mathfrak{s}}$ by Lemma 2.2, this result becomes [7, Theorem 2.14].

Proof. We summarize the argument from [7, Theorem 2.14] for convenience of the reader. We need to evaluate

$$\Psi_j^{\mathrm{reg}}(F_{m,k-2j,\mathfrak{s}}, z) = \int_{\mathcal{F}}^{\mathrm{reg}} \langle R_{k-2j}^j(F_{m,k-2j,\mathfrak{s}})(\tau), \overline{\Theta_L(\tau,\psi,p)}\rangle v^k \, d\mu(\tau).$$

Consequently, we compute the action of the raising operator first, and have

$$\Psi_j^{\mathrm{reg}}(F_{m,k-2j,\mathfrak{s}}, z) = (4\pi m)^j \frac{\Gamma\big(\mathfrak{s}+\frac{k}{2}\big)}{\Gamma\big(\mathfrak{s}+\frac{k}{2}-j\big)} \int_{\mathcal{F}}^{\mathrm{reg}} \langle (F_{m,k,\mathfrak{s}})(\tau), \overline{\Theta_L(\tau,\psi,p)}\rangle v^k \, d\mu(\tau)$$

by Lemma 2.2. Secondly, we insert the definitions of both functions and unfold the integral, obtaining

$$\begin{aligned}\Psi_j^{\mathrm{reg}}(F_{m,k-2j,\mathfrak{s}}, z) = \frac{(4\pi m)^j\Gamma\big(\mathfrak{s}+\frac{k}{2}\big)}{2\Gamma(2-k+2j)\Gamma\big(\mathfrak{s}+\frac{k}{2}-j\big)} &\sum_{\mu \in L'/L} \sum_{\lambda \in L+\mu} \overline{p(\psi(\lambda))} \\ \times \int_0^1 \int_0^\infty (4\pi m v)^{-\frac{k}{2}} &M_{-\frac{k}{2},\mathfrak{s}-\frac{1}{2}}(4\pi m v)\, e(-mu) \\ &\times \overline{e(Q(\lambda_z)\tau + Q(\lambda_{z^\perp})\overline{\tau})}\, v^{\frac{s}{2}+d^- + k-2} \, dv\, du.\end{aligned}$$

Third, we compute the integral over u using that $\overline{e(w)} = e(-\overline{w})$ and that

$$\int_0^1 e(-mu)\, e(-Q(\lambda_z)u - Q(\lambda_{z^\perp})u)\, du = \begin{cases} 1 & \text{if } Q(\lambda_z) + Q(\lambda_{z^\perp}) = -m, \\ 0, & \text{else.} \end{cases}$$

Hence, we obtain

$$\Psi_j^{\mathrm{reg}}(F_{m,k-2j,\mathfrak{s}}, z) = \frac{(4\pi m)^{j-\frac{k}{2}}\Gamma\left(\mathfrak{s}+\frac{k}{2}\right)}{2\Gamma(2-k+2j)\,\Gamma\left(\mathfrak{s}+\frac{k}{2}-j\right)} \sum_{\mu\in L'/L} \sum_{\substack{\lambda\in L+\mu \\ Q(\lambda)=-m}} \overline{p(\psi(\lambda))} \\ \times \int_0^\infty M_{-\frac{k}{2},\mathfrak{s}-\frac{1}{2}}(4\pi m v)\, e^{-2\pi v(Q(\lambda_z)-Q(\lambda_{z^\perp}))}\, v^{\frac{s+k}{2}+d^- -2}\, dv.$$

The integral is a Laplace transform. Using that

$$\frac{m}{2m} + \frac{Q(\lambda_z) - Q(\lambda_{z^\perp})}{2m} = \frac{Q(\lambda_{z^\perp})}{Q(\lambda)}$$

along with [35, (13.23.1)], it evaluates

$$\int_0^\infty M_{-\frac{k}{2},\mathfrak{s}-\frac{1}{2}}(4\pi m v)\, e^{-2\pi v(Q(\lambda_z)-Q(\lambda_{z^\perp}))}\, v^{\frac{k+s}{2}+d^- -2}\, dv \\ = \frac{(4\pi m)^{1-\frac{k+s}{2}-d^-}\,\Gamma\left(\frac{k+s}{2}+d^- -1+\mathfrak{s}\right)}{\left(\frac{Q(\lambda_z)-Q(\lambda_{z^\perp})}{2m}+\frac{1}{2}\right)^{\frac{k+s}{2}+d^- -1+\mathfrak{s}}} \\ \times {}_2F_1\left(k+\mathfrak{s},\, \frac{k+s}{2}+d^- -1+\mathfrak{s};\, 2\mathfrak{s};\, \frac{1}{\frac{1}{2}+\frac{Q(\lambda_z)-Q(\lambda_{z^\perp})}{2m}}\right).$$

We recall $Q(\lambda) = Q(\lambda_z) + Q(\lambda_{z^\perp}) = -m$ and rewrite the argument of the hypergeometric function to

$$\frac{m}{2m} + \frac{Q(\lambda_z) - Q(\lambda_{z^\perp})}{2m} = \frac{Q(\lambda_{z^\perp})}{Q(\lambda)}.$$

Thus, we arrive at

$$\Psi_j^{\mathrm{reg}}(F_{m,k-2j,\mathfrak{s}}, z) = (4\pi m)^{j+1-k-\frac{s}{2}-d^-} \frac{\Gamma\left(\mathfrak{s}+\frac{k}{2}\right)\Gamma\left(\frac{k+s}{2}+d^- -1+\mathfrak{s}\right)}{2\Gamma(2-k+2j)\,\Gamma\left(\mathfrak{s}+\frac{k}{2}-j\right)} \\ \times \sum_{\mu\in L'/L} \sum_{\substack{\lambda\in L+\mu \\ Q(\lambda)=-m}} \overline{p(\psi(\lambda))} \left(\frac{Q(\lambda)}{Q(\lambda_{z^\perp})}\right)^{\frac{k+s}{2}+d^- -1+\mathfrak{s}} \\ \times {}_2F_1\left(k+\mathfrak{s},\, \frac{k+s}{2}+d^- -1+\mathfrak{s};\, 2\mathfrak{s};\, \frac{Q(\lambda)}{Q(\lambda_{z^\perp})}\right),$$

as claimed. $\square$

Combining the previous result with Lemma 2.1 yields the following consequence.

Corollary 3.2. *Let $j \in \mathbb{N}_0$ and $f \in H_{k-2j,L}$. Assume that $k-2j<0$. Then we have*

$$\Psi_j^{\mathrm{reg}}(f,z) = \frac{(4\pi)^{j+1-k-\frac{s}{2}-d^-}\, j!\,\Gamma\left(\frac{s}{2}+d^-+j\right)}{4\Gamma(2-k+2j)} \sum_{\substack{\lambda\in L' \\ Q(\lambda)<0}} c_f^+(\lambda, Q(\lambda))\,\overline{p(\psi(\lambda))}$$
$$\times \frac{|Q(\lambda)|^{2j+1-k}}{|Q(\lambda_{z^\perp})|^{\frac{s}{2}+j+d^-}}\, {}_2F_1\left(1+j, \frac{s}{2}+d^-+j; 2-k+2j; \frac{Q(\lambda)}{Q(\lambda_{z^\perp})}\right).$$

Proof. Since the weight of f is negative, we have

$$f(\tau) = \frac{1}{2}\sum_{h\in L'/L}\sum_{m\geq 0} c_f^+(h,-m)\,F_{h,m,k-2j,1-\frac{k}{2}+j}(\tau)$$

according to Lemma 2.1. We observe that the term corresponding to $m=0$ will vanish due to $c_f^+(h,0)=0$ by our more restrictive growth condition on Maaß forms. Consequently, we have

$$\Psi_j^{\mathrm{reg}}(f,z) = \frac{1}{2}\sum_{\mu\in L'/L}\sum_{m>0} c_f^+(\mu,-m)\,\Psi_j^{\mathrm{reg}}(F_{\mu,m,k-2j,1-\frac{k}{2}+j}, z).$$

We insert the spectral parameter $\mathfrak{s} = 1-\frac{k-2j}{2}$ into Theorem 3.1, which yields the claim. □

Evaluation in terms of the constant term in a Fourier expansion. Next we determine the lift as a constant term in a Fourier expansion plus a certain boundary integral that vanishes for a certain class of input function.

Theorem 3.3. *Let $f\in H_{k-2j,L}$ and w be a special point, and $\mathcal{G}_P^+$ be the holomorphic part of a preimage of Θ_P under $\xi_{2-(\frac{r}{2}+d^+)}$. Then we have*

$$\Psi_j^{\mathrm{reg}}(f,w) = \frac{j!\,(4\pi)^j\,\Gamma\left(2-\frac{r}{2}-d^+\right)}{\Gamma\left(2-\frac{r}{2}-d^++j\right)}$$
$$\times\left(\mathrm{CT}(\langle f_{P\oplus N}(\tau), [\mathcal{G}_P^+(\tau), \Theta_{N^-}(\tau)]_j\rangle)\right.$$
$$\left. - \int_{\mathcal{F}}^{\mathrm{reg}} \langle L_{k-2j}(f_{P\oplus N})(\tau), [\mathcal{G}_P^+(\tau), \Theta_{N^-}(\tau)]_j\rangle\, v^{-2}\, d\tau\right).$$

Remark. In general, the coefficients of $\mathcal{G}_P^+$ are expected to be transcendental. However, in weights $\frac{1}{2}$ and $\frac{3}{2}$ the function $\mathcal{G}_P^+$ may be chosen to have rational coefficients — a situation which is expected to also hold for ξ-preimages of CM modular forms. It is therefore expected that one obtains rationality (up to powers of π) of the modified higher lift only in these cases, and stipulating that f is weakly holomorphic meaning that the final integral vanishes.

By a slight abuse of notation, we write $\Theta_L(\tau, w, p)$ for the theta function evaluated at an isometry ψ that produces a special point w.

Proof of Theorem 3.3. We restrict to special points $w \in \mathrm{Gr}(L)$. So we can write

$$\langle R^j_{k-2j}(f)(\tau), \overline{\Theta_L(\tau, w, p)}\rangle = \langle R^j_{k-2j}(f_{P\oplus N})(\tau), \overline{\Theta_{P\oplus N}(\tau, w, p)}\rangle.$$

Next, we use that the raising and lowering operator are adjoint to each other (see [7, Lemma 4.2]), which gives

$$\Psi^{\mathrm{reg}}_j(f, w) = \int^{\mathrm{reg}}_{\mathcal{F}} \langle f_{P\oplus N}(\tau), L^{j-1}_k(\overline{\Theta_{P\oplus N}(\tau, w, p)})\rangle\, v^{k-2}\, d\tau.$$

We observe that the boundary terms disappear in the same fashion as during the proof of [7, Lemma 4.4]. Next, we rewrite

$$\Psi^{\mathrm{reg}}_j(f, w) = (-1)^j \int^{\mathrm{reg}}_{\mathcal{F}} \langle f_{P\oplus N}(\tau), R^j_{-k}(\overline{\Theta_{P\oplus N}(\tau, w, p)}v^k)\rangle\, v^{-2}\, d\tau$$

and recall that

$$\Theta_{P\oplus N}(\tau, w, p) = \Theta_P(\tau, p_r)\otimes v^{\frac{s}{2}+d^-}\overline{\Theta_{N^-}(\tau, p_s)} = v^{\frac{s}{2}+d^-}\Theta_P(\tau, p_r)\otimes\overline{\Theta_{N^-}(\tau, p_s)}.$$

Consequently, we obtain

$$\begin{aligned} R^j_{-k}(\overline{\Theta_{P\oplus N}(\tau, w, p)}v^k) &= R^j_{-k}(v^{k+\frac{s}{2}+d^-}\overline{\Theta_P(\tau, p_r)}\otimes\Theta_{N^-}(\tau, p_s)) \\ &= v^{k+\frac{s}{2}+d^-}\overline{\Theta_P(\tau, p_r)}\otimes(R^j_{\frac{s}{2}+d^-}\Theta_{N^-}(\tau, p_s)) \end{aligned}$$

by Lemma 2.5. In particular, we note that $v^{k+\frac{s}{2}+d^-}\overline{\Theta_P(\tau, p_r)}$ has weight

$$-k - \tfrac{s}{2} - d^- = -\tfrac{r}{2} - d^+.$$

We choose a preimage $\mathcal{G}_P$ of $\Theta_P(\tau, p_r)$ under $\xi_{2-(\frac{r}{2}+d^+)}$, namely

$$\Theta_P(\tau, p_r) = \xi_{2-\frac{r}{2}-d^+}\,\mathcal{G}_P(\tau) = v^{-\frac{r}{2}-d^+}\,\overline{L_{2-\frac{r}{2}-d^+}\,\mathcal{G}_P},$$

which yields

$$R^j_{-k}(\overline{\Theta_{P\oplus N}(\tau, w, p)}v^k) = L_{2-\frac{r}{2}-d^+}\,\mathcal{G}_P(\tau)\otimes(R^j_{\frac{s}{2}+d^-}\Theta_{N^-}(\tau, p_s)).$$

We apply the computation of the Rankin–Cohen brackets given in Lemma 2.4 noting that $L_\ell\Theta_{N^-} = 0$, and that it suffices to deal with the holomorphic part $\mathcal{G}^+_P$ of $\mathcal{G}_P$ (both by virtue of holomorphicity in computing the Rankin–Cohen bracket). Thus,

$$\begin{aligned} &R^j_{-k}(\overline{\Theta_{P\oplus N}(\tau, w, p)}v^k) \\ &\qquad = \frac{j!(-4\pi)^j\Gamma(2-k)}{\Gamma(2-k+j)} v^{-\frac{s}{2}-d^-} L_{2-k+\frac{s}{2}+d^-+2j}[\mathcal{G}^+_P(\tau), \Theta_{N^-}(\tau, p_s)]_j. \end{aligned}$$

Hence, the theta lift becomes

$$\Psi_j^{\mathrm{reg}}(f,w) = \frac{j!(4\pi)^j\Gamma\big(2-\frac{r}{2}-d^+\big)}{\Gamma\big(2-\frac{r}{2}-d^++j\big)}\int_{\mathcal{F}}^{\mathrm{reg}}\langle f_{P\oplus N}(\tau), L_{2-k+2j}[\mathcal{G}_P^+(\tau),\Theta_{N^-}(\tau,p_s)]_j\rangle v^{-2}\,d\tau.$$

The last step is to apply Stokes' theorem, compare the proof of [7, Lemma 4.2] for example, which yields

$$\Psi_j^{\mathrm{reg}}(f,w) = \frac{j!(4\pi)^j\Gamma\big(2-\frac{r}{2}-d^+\big)}{\Gamma\big(2-\frac{r}{2}-d^++j\big)} \times\Bigg(\lim_{T\to\infty}\int_{iT}^{1+iT}\langle f_{P\oplus N}(\tau),[\mathcal{G}_P^+(\tau),\Theta_{N^-}(\tau,p_s)]_j\rangle v^{-2}\,d\tau - \int_{\mathcal{F}}^{\mathrm{reg}}\langle L_{k-2j}(f_{P\oplus N})(\tau),[\mathcal{G}_P^+(\tau),\Theta_{N^-}(\tau,p_s)]_j\rangle v^{-2}\,d\tau\Bigg),$$

utilizing again that boundary terms vanish. We observe that the left integral can be regarded as the Fourier coefficient of index 0 in the Fourier expansion of the integrand, see the bottom of page 14 in [12]. This proves the claim. □

We end this section by noting that to obtain recurrence relations, as in [12], one would need to compute the Fourier expansion of the lift. In general, this is a lengthy but straightforward process, and since we do not require it in this paper we omit the details. In essence, one follows the calculations of Borcherds [2] by using Lemma 2.2. A resulting technicality is to then take care of the different spectral parameter. One may overcome this by relating the coefficients of Maaß–Poincaré series to those with other spectral parameters, again using the action of the iterated Maaß raising operator as in Lemma 2.2.

Eichler–Selberg relations. We now prove a refined version of Theorem 1.1. To this end, we define the function

$$\Lambda_L(\psi,p,j) := \frac{(4\pi)^{1-\frac{r}{2}-d^+}\Gamma\big(\frac{s}{2}+j+d^-\big)\Gamma\big(2-\frac{r}{2}-d^++j\big)}{4\Gamma(2-k+2j)\Gamma\big(2-\frac{r}{2}-d^+\big)} \times\sum_{\substack{m\geq 1,\,\lambda\in L'\\ Q(\lambda)=-m}}\overline{p(\psi(\lambda))}\frac{|Q(\lambda)|^{2j+1-k}}{|Q(\lambda_{z^\perp})|^{\frac{s}{2}+j+d^-}} \times{}_2F_1\bigg(1+j,\frac{s}{2}+j+d^-;2-k+2j;\frac{Q(\lambda)}{Q(\lambda_{z^\perp})}\bigg)q^m \tag{3-1}$$

for $j>0$. We write

$$\mathcal{G}_P^+(\tau) = \sum_{\mu\in L'/L}\sum_{n\gg-\infty}a(n)\,q^n\,\mathfrak{e}_\mu$$

and furthermore define

$$\mathscr{G}_P^+(\tau) := \mathcal{G}_P^+(\tau) - \sum_{\mu\in L'/L}\sum_{n<0} a(n)\mathbb{F}_{\mu,n-1,2j+2-k}(\tau).$$

Since one may add any weakly holomorphic modular form of appropriate weight for ρ_L to $\mathcal{G}_P^+$, Theorem 1.1 follows directly from the following result (noting that the linear combination of Maaß–Poincaré series may change).

Theorem 3.4. *Let L be an even lattice of signature (r, s), let p be as before, and w be a special point defined by the isometry ψ. Let $j > 0$ and k be such that $2j+2-k > 2$. Then the function*

$$[\mathscr{G}_P^+(\tau), \Theta_{N^-}(\tau, p_s)]_j^L - \Lambda_L(\psi, p, j)$$

is a holomorphic vector-valued modular form of weight $2j+2-k$ for ρ_L.

Remarks. (1) This provides the general vector-valued analogue, assuming that the lattice is chosen such that $2j+2-k > 2$, of Mertens' scalar-valued results in weights $\frac{1}{2}$ and $\frac{3}{2}$ [33].

(2) Note that the slight correction of $\mathcal{G}_P^+$ by Poincaré series was missing in [31].

(3) In certain cases the hypergeometric function may be simplified (for example, the $n = 1$ case as in [12; 31], which yields a form very similar to Mertens' scalar-valued result). It appears to be possible that one should be able to prove the same results via holomorphic projection acting on vector-valued modular forms (see [26]) in much the same way as Mertens' original scalar valued proofs in [33].

Proof of Theorem 3.4. Let f be a weakly holomorphic form of weight $k-2j$ with Fourier coefficients c_f^+. By construction, the form $\mathscr{G}_P^+$ is holomorphic at $i\infty$, and hence

$$\mathrm{CT}(\langle f_{P\oplus N}(\tau), [\mathscr{G}_P^+(\tau), \Theta_{N^-}(\tau, p_s)]_j^L\rangle)$$

contains only the Fourier coefficients of nonpositive index of f. We note that $L_{k-2j} f = 0$, and subtract the resulting expressions of the lift from Corollary 3.2 and Theorem 3.3. We obtain

$$\begin{aligned}
0 &= \mathrm{CT}(\langle f_{P\oplus N}(\tau), [\mathscr{G}_P^+(\tau), \Theta_{N^-}(\tau, p_s)]_j^L\rangle)\\
&\quad - \frac{(4\pi)^{1-\frac{r}{2}-d^+}\Gamma\big(\frac{s}{2}+j+d^-\big)\Gamma\big(2-\frac{r}{2}-d^++j\big)}{4\Gamma(2-k+2j)\Gamma\big(2-\frac{r}{2}-d^+\big)}\\
&\qquad \times \sum_{\substack{m\geq 1,\,\lambda\in L'\\ Q(\lambda)=-m}} c_f^+(\lambda, -m)\,\overline{p(\psi(\lambda))}\frac{|Q(\lambda)|^{2j+1-k}}{|Q(\lambda_{z^\perp})|^{\frac{s}{2}+j+d^-}}\\
&\qquad \times {}_2F_1\bigg(1+j, \frac{s}{2}+j+d^-; 2-k+2j; \frac{Q(\lambda)}{Q(\lambda_{z^\perp})}\bigg).
\end{aligned}$$

The Rankin–Cohen bracket is bilinear and a linear combination of vector-valued Poincaré series is modular itself. We apply Proposition 2.7 and the claim follows. □

In a similar way to [33, Corollary 5.4], we obtain the following structural corollary by rewriting Theorem 3.4, keeping the same notation as throughout this paper.

Corollary 3.5. *Let θ denote the space generated by all Θ_{N^-} functions of weight $\frac{s}{2}+d^-$ for ρ_{N^-}. Then the equivalence classes $\Lambda_L(\psi, p, j)+M^!_{2j+2-k,L}$ generate the $\mathbb{C}$-vector space*

$$[\mathcal{M}^{mock}_{2j+2-k,P}, \theta]^L_j / M^!_{2j+2-k,L}.$$

4. The action of the Laplace–Beltrami operator

In this section, we prove Theorem 1.3. To this end, we compute the action of the Laplace–Beltrami operator on the lift, and show that for certain spectral parameters, we obtain a local Maaß form. We recall that the signature of L is assumed here to be $(2, s)$. Moreover, we observe that our Siegel theta function Θ_L and the Siegel theta function inspected by Bruinier depend in the same way on Z, and thus the following result applies.

Proposition 4.1 [7, Proposition 4.5]. *The Siegel theta function $\Theta_L(\tau, Z, p)$ considered as a function on $\mathbb{H}\times\mathbb{H}_\ell$ satisfies the differential equation*

$$\Omega\Theta_L(\tau, Z, p)v^{\frac{\ell}{2}} = -\tfrac{1}{2}\Delta_k\Theta_L(\tau, Z, p)v^{\frac{\ell}{2}}.$$

Our next step is to inspect the action of Ω on our theta lift. By Lemma 2.1 it suffices to investigate

$$\Psi^{\mathrm{reg}}_j(F_{m,k-2j,\mathfrak{s}}, Z) = \int^{\mathrm{reg}}_{\mathcal{F}} \langle R^j_{k-2j}(F_{m,k-2j,\mathfrak{s}})(\tau), \overline{\Theta_L(\tau, Z, p)}\rangle v^k\, d\mu(\tau).$$

Let

$$H(m) := \bigcup_{\mu\in L'/L}\ \bigcup_{\substack{\lambda\in\mu+L\\ Q(\lambda)=-m}} \lambda^{\perp} \subseteq \mathrm{Gr}(L),$$

which collects the singularities of $\Psi^{\mathrm{reg}}_j(F_{m,k-2j,\mathfrak{s}}, Z)$ as a function of Z. We apply the previous proposition to our theta lift, which yields a variant of [7, Theorem 4.6].

Theorem 4.2. *Let $Z\in\mathbb{H}_\ell\setminus H(m)$ and $\mathrm{Re}(\mathfrak{s}) > 1-\frac{k}{2}$. Then it holds that*

$$\Omega\Psi^{\mathrm{reg}}_j(F_{m,k-2j,\mathfrak{s}}, Z) = \left(\mathfrak{s}-\tfrac{k}{2}\right)\left(1-\mathfrak{s}-\tfrac{k}{2}\right)\Psi^{\mathrm{reg}}_j(F_{m,k-2j,\mathfrak{s}}, Z).$$

Proof. First, we note that

$$\Omega\Psi^{\mathrm{reg}}_j(F_{m,k-2j,\mathfrak{s}}, Z) = \int^{\mathrm{reg}}_{\mathcal{F}} \langle R^j_{k-2j}(F_{m,k-2j,\mathfrak{s}})(\tau), \Omega\overline{\Theta_L(\tau, Z, p)}v^{\frac{\ell}{2}}\rangle v^{k-\frac{\ell}{2}}\, d\mu(\tau),$$

because all partial derivatives with respect to Z converge locally uniformly in Z as $T \to \infty$ (see [7, p. 99]). By the previous proposition, we infer that

$$\Omega\Psi_j^{\mathrm{reg}}(F_{m,k-2j,\mathfrak{s}}, Z) = -\frac{1}{2}\int_{\mathcal{F}}^{\mathrm{reg}} \langle R_{k-2j}^{j}(F_{m,k-2j,\mathfrak{s}})(\tau), \Delta_k \overline{\Theta_L(\tau, Z, p)} v^{\frac{\ell}{2}}\rangle v^{k-\frac{\ell}{2}}\, d\mu(\tau).$$

By the splitting $\Delta_k = R_{k-2} L_k$ and the adjointness of both operators (see [7, Lemmas 4.2–4.4]), we obtain

$$\Omega\Psi_j^{\mathrm{reg}}(F_{m,k-2j,\mathfrak{s}}, Z) = -\frac{1}{2}\int_{\mathcal{F}}^{\mathrm{reg}} \langle \Delta_k R_{k-2j}^{j}(F_{m,k-2j,\mathfrak{s}})(\tau), \overline{\Theta_L(\tau, Z, p)} v^{\frac{\ell}{2}}\rangle v^{k-\frac{\ell}{2}}\, d\mu(\tau).$$

Lastly, we observe that Δ_k and R_{k-2j}^{j} commute by virtue of Lemma 2.2, namely

$$\Delta_k R_{k-2j}^{j}(F_{m,k-2j,\mathfrak{s}})(\tau) = \left(\mathfrak{s} - \tfrac{k}{2}\right)\left(1 - \mathfrak{s} - \tfrac{k}{2}\right) R_{k-2j}^{j}(F_{m,k-2j,\mathfrak{s}})(\tau),$$

and this establishes the claim by rewriting

$$\langle R_{k-2j}^{j}(F_{m,k-2j,\mathfrak{s}})(\tau), \overline{\Theta_L(\tau, Z, p)} v^{\frac{\ell}{2}}\rangle v^{k-\frac{\ell}{2}} = \langle R_{k-2j}^{j}(F_{m,k-2j,\mathfrak{s}})(\tau), \overline{\Theta_L(\tau, Z, p)}\rangle v^{k}$$

again. □

Proof of Theorem 1.2. By Theorem 4.2, the lift is an eigenfunction of the Laplace–Beltrami operator with the quoted eigenvalue. Since $\Psi_j^{\mathrm{reg}}(F_{m,k-2j,\mathfrak{s}}, Z)$ is an eigenfunction of an elliptic differential operator, it is real-analytic in $\mathrm{Gr}(L)$ outside of $H(m)$. The other conditions for the lift to be a vector-valued local Maaß form can be easily seen by applying the proof of [5, Theorem 1.1] *mutatis mutandis*. When $\mathfrak{s} = \frac{k}{2}$ or $\mathfrak{s} = \frac{k}{2} - 1$, we obtain locally harmonic Maaß forms. □

5. Cohen–Eisenstein series

We specialize the framework from Section 2 following [12, Section 4.4] (or [36, Section 2.2]). We fix the signature $(1, 2)$ as mentioned in the introduction, and the rational quadratic space

$$V := \left\{ X = \begin{pmatrix} x_2 & x_1 \\ x_3 & -x_2 \end{pmatrix} \in \mathbb{Q}^{2\times 2} \right\},$$

with quadratic form $Q(X) = \det(X)$. The Grassmannian of positive lines in $V \otimes \mathbb{R}$ can be identified with $\mathbb{H}$ via

$$\lambda(x+iy) = \frac{1}{\sqrt{2}y}\begin{pmatrix} -x & x^2+y^2 \\ -1 & x \end{pmatrix}.$$

We choose the lattice
$$L := \left\{ \begin{pmatrix} b & c \\ -a & -b \end{pmatrix} : a, b, c \in \mathbb{Z} \right\},$$
with dual lattice
$$L' = \left\{ \begin{pmatrix} \frac{b}{2} & c \\ -a & -\frac{b}{2} \end{pmatrix} : a, b, c \in \mathbb{Z} \right\}.$$
We observe that L' can be identified with the set of integral binary quadratic forms of discriminant
$$\det\begin{pmatrix} \frac{b}{2} & c \\ -a & -\frac{b}{2} \end{pmatrix} = -\tfrac{1}{4}(b^2-4ac).$$

Furthermore, $L'/L \cong \mathbb{Z}/2\mathbb{Z}$ with quadratic form $x \mapsto -\frac{1}{4}x^2$. According to [12, p. 22], it holds that
$$Q\left(\begin{pmatrix} \frac{b}{2} & c \\ -a & -\frac{b}{2} \end{pmatrix}_{x+iy}\right) = \frac{1}{4y^2}(a(x^2+y^2)+bx+c)^2,$$
$$Q\left(\begin{pmatrix} \frac{b}{2} & c \\ -a & -\frac{b}{2} \end{pmatrix}_{(x+iy)^\perp}\right) = -\frac{1}{4y^2}|[a, b, c](x+iy, 1)|^2.$$
We remark that both are invariant under modular substitutions. By a result from Eichler and Zagier [22, Theorem 5.4], the space of vector-valued modular forms of weight k for ρ_L is isomorphic to the space $M_k^+(\Gamma_0(4))$ of scalar-valued modular forms satisfying the Kohnen plus space condition via the map
$$f_0(\tau)\mathfrak{e}_0 + f_1(\tau)\mathfrak{e}_1 \mapsto f_0(4\tau) + f_1(4\tau).$$
This enables us to use scalar-valued forms as inputs for our theta lift.

Proof of Theorem 1.3. As outlined in the introduction, the function $f(\tau) := f_{-2\ell,N}(\tau)\mathcal{H}_\ell(\tau)$ is of weight $-\ell-\frac{1}{2} < 0$ for $\Gamma_0(4)$, has nonconstant principal part at the cusp $i\infty$, and its image under ξ is trivial, and hence in particular cuspidal. This enables us to apply Corollary 3.2 to f. To this end, we have the parameters
$$k = -\tfrac{1}{2} + d^+ + d^-, \quad k-2j = -\ell - \tfrac{1}{2}.$$
Rewriting those yields
$$j = \frac{\ell+d^++d^-}{2},$$
and the hypergeometric function from Theorem 3.1 becomes
$${}_2F_1\left(\frac{\ell+2+d^++d^-}{2}, \frac{\ell+2+d^++3d^-}{2}, \frac{5}{2}+\ell, \frac{4my^2}{|[a, b, c](z, 1)|^2}\right).$$

Inspecting the parameters, we have the condition $\ell + d^+ + d^- \in 2\mathbb{N}$ by $j \in \mathbb{N}$, and combining with $d^+,\ d^- \in \mathbb{N}_0,\ \ell \in \mathbb{N}\setminus\{1\}$, the smallest possible values are

$(\ell, d^+, d^-) = (2,0,0), (2,2,0), (2,1,1), (2,0,2)$. For example, the corresponding hypergeometric functions for the cases $(\ell, d^+, d^-) = (2,0,0), (2,1,1)$ are

$$\begin{aligned} {}_2F_1\big(2,2,\tfrac{9}{2},\tilde{z}\big) &= -\frac{35(11\tilde{z}-15)}{12\tilde{z}^3} - \frac{35(2\tilde{z}^2-7\tilde{z}+5)\arcsin(\sqrt{\tilde{z}})}{4\tilde{z}^{\frac{7}{2}}\sqrt{1-\tilde{z}}}, \\ {}_2F_1\big(3,4,\tfrac{9}{2},\tilde{z}\big) &= -\frac{35(8\tilde{z}^2-26\tilde{z}+15)}{128\tilde{z}^3(\tilde{z}-1)^2} + \frac{105(8\tilde{z}^2-12\tilde{z}+5)\arcsin(\sqrt{\tilde{z}})}{128\tilde{z}^{\frac{7}{2}}\sqrt{1-\tilde{z}}(\tilde{z}-1)^2}, \end{aligned}$$

and the other cases are of similar shape. Analogous expressions can be obtained for higher integer parameters via Gauß' contiguous relations for the hypergeometric function, which can be found in [35, Section 15.5(ii)] for instance.

We infer a local behavior as sketched in the introduction by virtue of ($4m = D = b^2 - 4ac$)

$$\arcsin(\sqrt{\tilde{z}}) = \arcsin\left(\frac{\sqrt{D}y}{|az^2+bz+c|}\right) = \arctan\left|\frac{\sqrt{D}y}{a|z|^2+bx+c}\right|,$$

which in turn follows by

$$(b^2-4ac)y^2 + (a|z|^2+bx+c)^2 = |az^2+bz+c|^2,$$

compare [4, Section 3]. The denominator $a|z|^2+bx+c$ vanishes if and only if z is located on the Heegner geodesic associated to $Q = [a,b,c]$. Since the principal part of f is given by

$$\sum_{n=0}^{N} H(\ell,n)\,q^{n-N} + O(q^{m+1}), \quad m = \begin{cases} \left\lfloor \frac{-2\ell}{12} \right\rfloor - 1 & \text{if } -2\ell \equiv 2 \pmod{12}, \\ \left\lfloor \frac{-2\ell}{12} \right\rfloor, & \text{else,} \end{cases}$$

we conclude that f has the exceptional set

$$\bigcup_{D=1}^{N} \{z = x+iy \in \mathbb{H} : \exists a,b,c \in \mathbb{Z},\ b^2-4ac = D,\ a|z|^2+bx+c = 0\}.$$

In other words, the exceptional set of f is a finite union of nets of Heegner geodesics. Furthermore, we recall that the spectral parameter in Corollary 3.2 is $\mathfrak{s} = 1 - \frac{k-2j}{2}$, and hence the eigenvalue under $\Delta_{-\ell-(1/2)}$ is

$$\big(\mathfrak{s}-\tfrac{k}{2}\big)\big(1-\mathfrak{s}-\tfrac{k}{2}\big) = (1-k+j)(-j) = j\big(j-\ell-\tfrac{3}{2}\big). \qquad \square$$

Eichler–Selberg relations for Cohen–Eisenstein series. Eichler–Selberg relations for Cohen–Eisenstein series could be obtained as follows. On one hand, the input function $f(\tau) = f_{-2\ell,N}(\tau)\mathcal{H}_\ell(\tau)$ is weakly holomorphic, thus we do not need to deal with the additional term

$$\int_{\mathcal{F}}^{\mathrm{reg}} \langle L_{k-2j}(f_{P\oplus N})(\tau), [\mathcal{G}_P^+(\tau), \Theta_{N^-}(\tau)]_j\rangle\, v^{-2}\, d\tau$$

arising from Theorem 3.3. Further, the function Λ_L from (3-1) simplifies to

$$\Lambda_L(\psi, p, j) = \frac{4^{3d^-}\pi^{\frac{1}{2}-d^+}\Gamma(j+1+d^-)\Gamma\left(\frac{3}{2}-d^++j\right)}{\Gamma\left(\ell+\frac{1}{2}\right)\Gamma\left(\frac{3}{2}-d^+\right)} \times \sum_{D\geq 1}\sum_{Q\in\mathcal{Q}_D}\overline{p(\psi(Q))}\frac{D^{\ell+\frac{3}{2}}y^{2+2j+2d^-}}{|Q(z,1)|^{2+2j+2d^-}} \times {}_2F_1\left(\frac{\ell+2+d^++d^-}{2}, \frac{\ell+2+d^++3d^-}{2}, \frac{5}{2}+\ell, \frac{Dy^2}{|Q(z,1)|^2}\right)q^D,$$

where $\mathcal{Q}_D$ denotes the set of integral binary quadratic forms of discriminant D. After evaluating the hypergeometric function as in the previous proof, one may follow our proof of Theorem 3.4, namely subtract the two evaluations of $\Psi_j^{\mathrm{reg}}(f, z)$ from each other and apply Serre duality to the resulting expression. Computing the principal part of $\mathcal{G}_P^+$ in addition, this yields the desired result. However, we do not pursue this here explicitly as the resulting expression is rather lengthy.

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REPRESENTATIONS OF ORIENTIFOLD KHOVANOV–LAUDA–ROUQUIER ALGEBRAS AND THE ENOMOTO–KASHIWARA ALGEBRA

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We consider an "orientifold" generalization of Khovanov–Lauda–Rouquier algebras, depending on a quiver with an involution and a framing. Their representation theory is related, via a Schur–Weyl duality type functor, to Kac–Moody quantum symmetric pairs, and, via a categorification theorem, to highest weight modules over an algebra introduced by Enomoto and Kashiwara. Our first main result is a new shuffle realization of these highest weight modules and a combinatorial construction of their PBW and canonical bases in terms of Lyndon words. Our second main result is a classification of irreducible representations of orientifold KLR algebras and a computation of their global dimension in the case when the framing is trivial.

1. Introduction

Khovanov–Lauda–Rouquier (KLR) algebras were introduced in [Khovanov and Lauda 2009; Rouquier 2008] in the context of categorification of quantum groups. They have since played an increasingly important role in representation theory. Broadly speaking, KLR algebras can be regarded, via the Brundan–Kleshchev–Rouquier isomorphism [Brundan and Kleshchev 2009; Rouquier 2008], as a generalization of the affine Hecke algebra $\widehat{\mathscr{H}}(\mathtt{A}_m)$ of type $\mathtt{A}$. This generalization is twofold. Firstly, KLR algebras naturally possess a nontrivial grading, which is

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difficult to discern in the affine Hecke algebra. Secondly, KLR algebras constitute the correct replacement for $\widehat{\mathscr{H}}(\mathtt{A}_m)$ from the point of view of Schur–Weyl duality. Indeed, Kang, Kashiwara and Kim [Kang et al. 2018] have constructed functors relating categories of modules over KLR algebras and quantum affine algebras of any type, generalizing the relationship between $\widehat{\mathscr{H}}(\mathtt{A}_m)$ and $U_q(\widehat{\mathfrak{sl}}_n)$ established earlier by Chari and Pressley [1996].

It is natural to ask whether it is possible to construct a KLR-type generalization of affine Hecke algebras of other classical types. A positive answer to this question was given by Varagnolo and Vasserot [2011], as well as by Poulain d'Andecy and Walker [2020]. We will refer to the new graded algebras introduced there as *orientifold KLR algebras* (see Remark 2.5 for an explanation of the origin of this name). It must be stressed that orientifold KLR algebras are very different from the usual KLR algebras associated to Cartan data of other classical types. From the point of view of categorification, their representation theory is related to an algebra introduced by Enomoto and Kashiwara [2006], depending on a Dynkin diagram together with an involution. More precisely, it was shown in [Varagnolo and Vasserot 2011] that orientifold KLR algebras categorify irreducible highest weight modules ${}^{\theta}\mathbf{V}(\lambda)$ over the Enomoto–Kashiwara algebra. In analogy to $U_q(\mathfrak{n}_-)$, these modules also admit a geometric construction in terms of perverse sheaves on the stack of orthogonal representations of a quiver with a contravariant involution [Enomoto 2009], as well as a Ringel–Hall–type construction [Young 2016].

Our main motivation for studying orientifold KLR algebras is related to Schur–Weyl duality. In [Appel and Przeździecki 2022], we construct functors between categories of modules over orientifold KLR algebras and coideal subalgebras $\mathscr{B}_{\mathbf{c},\mathbf{s}}$ of quantum affine algebras $U_q(\hat{\mathfrak{g}})$ (see [Kolb 2014]), respectively. The parameter λ is related to the parameters $\mathbf{c}$ and $\mathbf{s}$ via an additional datum in the definition of an orientifold KLR algebra, given by a framing dimension vector. Our intention is to use these functors to develop the graded representation theory of Kac–Moody quantum symmetric pairs. The study of finite-dimensional representations of orientifold KLR algebras is the first step in this program.

Let us describe our results in more detail. In Section 2, we introduce a somewhat more general definition of orientifold KLR algebras (Definition 2.4) associated to hermitian matrices with an additional symmetry. We construct a faithful polynomial representation (Proposition 2.7) and prove a PBW theorem (Proposition 2.9). Section 3 is dedicated to the Enomoto–Kashiwara algebra. Inspired by the work of Leclerc [2004] and Kleshchev and Ram [2011], we construct a shuffle realization of the modules ${}^{\theta}\mathbf{V}(\lambda)$ (Definition 3.6 and Proposition 3.9). This allows us to apply the combinatorics of Lyndon words to obtain PBW and canonical bases for these modules, in the case $\lambda = 0$ (Theorem 3.28, Corollary 3.30), somewhat simplifying the original construction of these bases [Enomoto and Kashiwara 2008]. In Section 4, we apply these results to the representation theory of orientifold KLR algebras. A key ingredient is Varagnolo and Vasserot's categorification theorem [2011], identifying ${}^{\theta}\mathbf{V}(\lambda)$ with the Grothendieck group of the category of finite-dimensional representations of orientifold KLR algebras. In our main result (Theorem 4.10), we classify irreducible representations of orientifold KLR algebras in terms of θ-good Lyndon words, and construct them as heads (respectively, socles) of certain induced (respectively, coinduced) modules. As an application, we prove that orientifold KLR algebras have finite global dimension when $\lambda = 0$.

Future work. The present paper lays the foundations for a broader programme connecting the representation theory of quantum symmetric pairs with orientifold KLR algebras via generalized Schur–Weyl duality functors. In [Appel and Przeździecki 2022], the results of the present paper, together with a number of new techniques, including k-matrices for KLR algebras and localization for module categories, are used to construct Hernandez–Leclerc–type categories [2010; 2015] for coideal subalgebras $\mathscr{B}_{\mathbf{c},\mathbf{s}}$ in affine type $\mathtt{A.III}$ with generic parameters $\mathbf{c}$, $\mathbf{s}$.

In future work, we would like to generalize these results to nongeneric parameters and coideals of type $\mathtt{D.IV}$. This will, in turn, require the development of the representation theory of orientifold KLR algebras associated to nontrivial framings λ and quivers of affine type $\mathtt{D}$. To achieve this, we will combine the combinatorial techniques from the present paper with an in-depth study of the geometry of framed symplectic and orthogonal quiver representations.

We expect that further study of orientifold KLR algebras with nontrivial framings will also provide new information about the representation theory of (affine) Hecke algebras of types $\mathtt{B}$ and $\mathtt{C}$ with unequal parameters, including the so-called nonasymptotic case, which is still only partially understood.

In yet another direction, the connection to Hernandez–Leclerc categories suggests that the combinatorics of the dual canonical bases of the modules ${}^{\theta}\mathbf{V}(\lambda)$ should have an interesting interpretation in terms of cluster theory.

2. Orientifold KLR algebras

2A. ***Some combinatorics.*** Let $\Bbbk$ be a field. Let $\mathfrak{S}_n = \langle s_1, \dots, s_{n-1}\rangle$ denote the symmetric group on n letters, and let $\mathfrak{W}_n = \langle s_0, s_1, \dots, s_{n-1}\rangle$ denote the Weyl group of type $\mathtt{B}_n$, i.e., $(\mathbb{Z}/2\mathbb{Z})^n \rtimes \mathfrak{S}_n$. We regard them as Coxeter groups in the usual way. Given $0 \le m \le n$, let $\mathtt{D}_{m,n-m}$ (respectively, ${}^{\theta}\mathtt{D}_{m,n-m}$) denote the set of shortest left coset representatives with respect to the parabolic subgroup $\mathfrak{S}_m \times \mathfrak{S}_{n-m} \subset \mathfrak{S}_n$ (respectively, $\mathfrak{W}_m \times \mathfrak{S}_{n-m} \subset \mathfrak{W}_n$). Let $w_0 \in \mathfrak{S}_n$ (respectively, ${}^{\theta}w_0 \in \mathfrak{W}_n$) be the longest element, and let ${}^{\theta}w \in \mathfrak{W}_n$ be the longest element in ${}^{\theta}\mathtt{D}_{0,n}$, i.e., the signed permutation

$${}^{\theta}w(l) = -(n-l+1).$$

Let J be a set and $\theta\colon J \to J$ an involution. We denote by J^θ the subset of fixed points of θ. Let $\mathbb{N}[J]$ be the commutative semigroup freely generated by J. We call elements of $\mathbb{N}[J]$ *dimension vectors*. Given a dimension vector $\beta = \sum_{i\in J} \beta(i)\cdot i$, we set $\|\beta\| = \sum_{i\in J}\beta(i)$ and $\operatorname{supp}(\beta) = \{i \in J \mid \beta(i) \neq 0\}$. We call a sequence $\nu = \nu_1 \cdots \nu_n \in J^n$ a *composition* of β of length $\ell(\nu) = n$ if $|\nu| = \sum_{k=1}^n \nu_k = \beta$. We also set $\|\nu\| = n$. Let J^β denote the set of all compositions of β. There is a left action of $\mathfrak{S}_n$ on J^n by permutations

$$s_k \cdot \nu_1 \cdots \nu_n = \nu_1 \cdots \nu_{k+1}\nu_k \cdots \nu_n, \quad 1 \le k \le n-1, \tag{2-1}$$

whose orbits are the sets J^β for all β with $\|\beta\| = n$.

Let $J^\bullet = \bigcup_{\beta\in\mathbb{N}[J]} J^\beta$ be the set of compositions of all dimension vectors. We also refer to elements of $J^\bullet$ as *words* in J and denote the empty word by $\varnothing$. We consider $J^\bullet$ as a monoid with respect to concatenation: $\nu\mu = \nu_1 \cdots \nu_{\ell\nu}\mu_1 \cdots \mu_{\ell\mu}$, with $\varnothing$ as the identity.

The involution θ induces an involution $\theta\colon \mathbb{N}[J] \to \mathbb{N}[J]$. We call dimension vectors in $\mathbb{N}[J]^\theta$ *self-dual*. We will always assume, for any $\beta \in \mathbb{N}[J]^\theta$, that if $i \in J^\theta$, then $\beta(i)$ is even. Set $\|\beta\|_\theta = \|\beta\|/2$ and

$$^\theta(-)\colon \mathbb{N}[J] \to \mathbb{N}[J]^\theta, \quad \beta \mapsto {}^\theta\beta = \beta + \theta(\beta).$$

We call a sequence $\nu = \nu_1 \cdots \nu_n \in J^n$ an *isotropic composition* of β if ${}^\theta|\nu| = \sum_{k=1}^n {}^\theta\nu_i = \beta$. We abbreviate $\nu_{-k} = \theta(\nu_k)$. Let ${}^\theta J^\beta$ denote the set of all isotropic compositions of β. There is a left action of $\mathfrak{W}_n$ on J^n extending (2-1), given by

$$s_0 \cdot \nu_1 \cdots \nu_n = \theta(\nu_1)\nu_2 \cdots \nu_n,$$

whose orbits are the sets ${}^\theta J^\beta$ for all self-dual β with $\|\beta\|_\theta = n$. Let ${}^\theta J^\bullet = \bigcup_{\beta\in\mathbb{N}[J]^\theta} {}^\theta J^\beta$ be the set of all isotropic compositions of all self-dual dimension vectors. The identity map defines a bijection $J^\bullet \cong {}^\theta J^\bullet$.

We will consider algebras depending on matrices and vectors with polynomial entries. Below we introduce some terminology for the latter.

Definition 2.1. We call a matrix $Q = (Q_{ij})_{i,j\in J}$ with entries in $\Bbbk[u, v]$ a *coefficient matrix*. We say that Q is:

(M1) *regular* if $Q_{ii} = 0$ for all $i \in J$,

(M2) *θ-symmetric* if $Q_{ij}(u, v) = Q_{\theta(j)\theta(i)}(-v, -u)$ for all $i, j \in J$,

(M3) *nonvanishing* if $Q_{ij} \neq 0$ for all $i \neq j \in J$,

(M4) *hermitian* if $Q_{ij}(u, v) = Q_{ji}(v, u)$ for each $i, j \in J$.

Moreover, we call a vector $Q' = (Q_i)_{i\in J}$ with entries in $\Bbbk[u]$ a *coefficient vector*. We say that Q' is:

(V1) *regular* if $Q_i = 0$ for all $i \in J^\theta$, (V2) *nonvanishing* if $Q_i \neq 0$ for all $i \notin J^\theta$,

(V3) *self-conjugate* if $Q_i(u) = Q_{\theta(i)}(-u)$.

If a coefficient matrix satisfies (M1)–(M4), respectively, if a coefficient vector satisfies (V1)–(V3), we call it *perfect*.

2B. *Reminder on KLR algebras.* Let $\beta \in \mathbb{N}[J]$ with $\|\beta\| = n$, and let Q be a regular coefficient matrix.

Definition 2.2. The *KLR algebra* $\mathcal{R}(\beta)$ associated to (J, Q, β) is the unital $\Bbbk$-algebra generated by $e(\nu)$ with $\nu \in J^\beta$, x_l with $1 \leq l \leq n$ and τ_k with $1 \leq k \leq n-1$, subject to the following relations:

- idempotent relations:
$$e(\nu)e(\nu') = \delta_{\nu,\nu'}e(\nu), \quad x_l e(\nu) = e(\nu)x_l, \quad \tau_k e(\nu) = e(s_k \cdot \nu)\tau_k,$$
- polynomial relations:
$$x_l x_{l'} = x_{l'} x_l,$$
- quadratic relations:
$$\tau_k^2 e(\nu) = Q_{\nu_k,\nu_{k+1}}(x_{k+1}, x_k)e(\nu),$$
- deformed braid relations:
$$\tau_k \tau_{k'} = \tau_{k'}\tau_k, \quad \text{if } k \neq k' \pm 1,$$
$$(\tau_{k+1}\tau_k\tau_{k+1} - \tau_k\tau_{k+1}\tau_k)e(\nu) = \delta_{\nu_k,\nu_{k+2}} \frac{Q_{\nu_k,\nu_{k+1}}(x_{k+1}, x_k) - Q_{\nu_k,\nu_{k+1}}(x_{k+1}, x_{k+2})}{x_k - x_{k+2}} e(\nu),$$
- mixed relations:
$$(\tau_k x_l - x_{s_k(l)}\tau_k)e(\nu) = \begin{cases} -e(\nu), & \text{if } l = k,\ \nu_k = \nu_{k+1}, \\ e(\nu), & \text{if } l = k+1,\ \nu_k = \nu_{k+1}, \\ 0, & \text{else.} \end{cases}$$

Whenever we want to emphasize the dependence of the KLR algebra on the full datum (J, Q, β), we will write $\mathcal{R}(J, Q, \beta)$.

Lemma 2.3. *If the coefficient matrix Q is hermitian, then there is an algebra isomorphism $\mathcal{R}(\beta) \to \mathcal{R}(\beta)$ sending*

$$e(\nu) \mapsto e(w_0(\nu)), \quad x_l \mapsto x_{n-l+1}, \quad \tau_k \mapsto -\tau_{n-k}. \tag{2-2}$$

If the coefficient matrix Q is hermitian and θ-symmetric, then there is an algebra isomorphism $\mathcal{R}(\beta) \to \mathcal{R}(\theta(\beta))$ sending

$$e(\nu) \mapsto e({}^\theta w(\nu)), \quad x_l \mapsto -x_{n-l+1}, \quad \tau_k \mapsto -\tau_{n-k}. \tag{2-3}$$

Proof. The first statement can be found in, e.g., [Rouquier 2008, §3.2.1]. The second statement follows from a direct check of the relations using θ-symmetry. $\square$

If M is an $\mathscr{R}(\beta)$-module, we will denote by $M^\dagger$ the corresponding $\mathscr{R}(\theta(\beta))$-module with the action twisted by the inverse of the isomorphism given in (2-3).

2C. *Orientifold KLR algebras.* Let $\beta \in \mathbb{N}[J]^\theta$ with $\|\beta\|_\theta = n$, let Q be a regular θ-symmetric coefficient matrix and Q' a regular coefficient vector.

Definition 2.4. Associated to $(J, \theta, Q, Q', \beta)$, we define the *orientifold KLR algebra* ${}^\theta\mathscr{R}(\beta)$ to be the unital $\Bbbk$-algebra generated by $e(\nu)$ with $\nu \in {}^\theta J^\beta$, x_l with $1 \le l \le n$, τ_0 and τ_k with $1 \le k \le n-1$ subject to the following relations:

- idempotent relations:

$$e(\nu)e(\nu') = \delta_{\nu,\nu'}e(\nu), \qquad x_l e(\nu) = e(\nu)x_l,$$
$$\tau_k e(\nu) = e(s_k \cdot \nu)\tau_k, \quad \tau_0 e(\nu) = e(s_0 \cdot \nu)\tau_0,$$

- polynomial relations:

$$x_l x_{l'} = x_{l'} x_l,$$

- quadratic relations:

$$\tau_k^2 e(\nu) = Q_{\nu_k,\nu_{k+1}}(x_{k+1}, x_k)e(\nu), \quad \tau_0^2 e(\nu) = Q_{\nu_1}(-x_1)e(\nu),$$

- deformed braid relations:

$$\tau_k\tau_{k'} = \tau_{k'}\tau_k, \quad \text{if } k \neq k' \pm 1, \qquad \tau_0\tau_k = \tau_k\tau_0, \quad \text{if } k \neq 1,$$
$$(\tau_{k+1}\tau_k\tau_{k+1} - \tau_k\tau_{k+1}\tau_k)e(\nu) = \delta_{\nu_k,\nu_{k+2}} \frac{Q_{\nu_k,\nu_{k+1}}(x_{k+1}, x_k) - Q_{\nu_k,\nu_{k+1}}(x_{k+1}, x_{k+2})}{x_k - x_{k+2}} e(\nu),$$
$$\big((\tau_1\tau_0)^2 - (\tau_0\tau_1)^2\big)e(\nu) = \begin{cases} \dfrac{Q_{\nu_2}(x_2) - Q_{\nu_1}(x_1)}{x_1 + x_2}\tau_1 e(\nu), & \text{if } \nu_1 \neq \nu_2,\ \nu_2 = \theta(\nu_1) \\ \dfrac{Q_{\nu_1,\nu_2}(x_2, -x_1) - Q_{\nu_1,\nu_2}(-x_2, -x_1)}{x_2}\tau_0 e(\nu), & \text{if } \nu_1 \neq \theta(\nu_1),\ \nu_2 = \theta(\nu_2), \\ \dfrac{Q_{\nu_1,\nu_2}(x_2, -x_1) - Q_{\nu_1,\nu_2}(x_2, x_1)}{x_1 x_2}(x_1\tau_0 + 1)e(\nu), & \text{if } \theta(\nu_1) = \nu_1 \neq \nu_2 = \theta(\nu_2), \\ 0 & \text{else,} \end{cases}$$

- mixed relations:

$$(\tau_k x_l - x_{s_k(l)}\tau_k)e(\nu) = \begin{cases} -e(\nu), & \text{if } l = k,\ \nu_k = \nu_{k+1}, \\ e(\nu), & \text{if } l = k+1,\ \nu_k = \nu_{k+1}, \\ 0, & \text{else,} \end{cases}$$
$$(\tau_0 x_1 + x_1\tau_0)e(\nu) = \begin{cases} 0, & \text{if } \nu_1 \neq \theta(\nu_1), \\ -2e(\nu), & \text{if } \nu_1 = \theta(\nu_1), \end{cases}$$
$$\tau_0 x_l = x_l\tau_0, \quad \text{if } l \neq 1.$$

By convention, we set ${}^\theta\mathcal{R}(0) = \Bbbk$. Whenever we want to emphasize the dependence of the orientifold KLR algebra on the full datum $(J, \theta, Q, Q', \beta)$, we will write ${}^\theta\mathcal{R}(J, Q, Q', \beta)$.

Remark 2.5. In the case when the matrices Q and Q' arise from a quiver with a contravariant involution and a framing (see Section 2F), under the assumption that the involution has no fixed points, the algebra ${}^\theta\mathcal{R}(\beta)$ was introduced by Varagnolo and Vasserot [2011]. The case of an involution with possible fixed points was first considered by Poulain d'Andecy and Walker [2020], and later by Poulain d'Andecy and Rostam [2021]. The latter paper takes a somewhat similar approach to ours — the definition of the algebra depends on polynomials Q_{ij}, but they are less general than ours, and the polynomials Q_i are absent.

In the literature, these algebras are typically referred to as "generalizations of KLR algebras for types BCD". However, we feel that this name may lead to confusion between, for example, the algebra ${}^\theta\mathcal{R}(\beta)$ and the KLR algebra $\mathcal{R}(\beta)$ associated to a quiver of type D. To avoid this confusion, we propose to introduce the name "orientifold KLR algebras" for ${}^\theta\mathcal{R}(\beta)$. The motivation comes from the connection with orientifold Donaldson–Thomas theory, see [Przeździecki 2019; Young 2020].

Proposition 2.6. *We list several isomorphisms between orientifold KLR algebras*:

(1) *If Q is hermitian and Q' self-conjugate*, *then there is an algebra automorphism*

$$ {}^\theta\mathcal{R}(\beta) \xrightarrow{\sim} {}^\theta\mathcal{R}(\beta), \quad e(\nu) \mapsto e({}^\theta w_0(\nu)),\ x_l \mapsto -x_l,\ \tau_k \mapsto -\tau_k, \tag{2-4}$$

with $\nu \in {}^\theta J^\beta$, $1 \le l \le n$ and $0 \le k \le n-1$.

(2) *If Q is hermitian and Q' self-conjugate*, *then there is an algebra isomorphism*

$$ \omega\colon {}^\theta\mathcal{R}(\beta) \xrightarrow{\sim} {}^\theta\mathcal{R}(\beta)^{\mathrm{op}}, \quad e(\nu)\mapsto e(\nu),\ x_l e(\nu)\mapsto x_l e(\nu),\ \tau_k e(\nu)\mapsto \tau_k e(s_k\cdot\nu). \tag{2-5}$$

(3) *Given $\{\zeta_i\}_{i\in J}$ in $\Bbbk$ satisfying $\zeta_i = -\zeta_{\theta(i)}$, as well as $\{\eta_{ij}\}_{i,j\in J}$ and $\{\eta_i\}_{i\in J}$ in $\Bbbk^\times$ satisfying*: *$\eta_{ij} = \eta_{\theta(j)\theta(i)}$ for all $i, j \in J$ and $\eta_i = \eta_{ii}$ for $i \in J^\theta$, let $\hat{Q}_{ij}(u, v) = \eta_{ij}\eta_{ji}(\eta_{jj}u + \zeta_j, \eta_{ii}v + \zeta_i)$ and $\hat{Q}_i(u) = \eta_i\eta_{\theta(i)}Q_i(\eta_{ii}u - \zeta_i)$. Then there is an algebra isomorphism ${}^\theta\mathcal{R}(J, \hat{Q}, \hat{Q}', \beta) \xrightarrow{\sim} {}^\theta\mathcal{R}(J, Q, Q', \beta)$ given by*

$$\begin{aligned} e(\nu) &\mapsto e(\nu), & x_l e(\nu) &\mapsto \eta^{-1}_{\nu_l,\nu_l}(x_l - \zeta_{\nu_l})e(\nu),\\ \tau_k e(\nu) &\mapsto \eta_{\nu_k,\nu_{k+1}}\tau_k e(\nu), & \tau_0 e(\nu) &\mapsto \eta_{\nu_1}\tau_0 e(\nu). \end{aligned}$$

Proof. The result follows by a direct computation from the defining relations. □

2D. *Polynomial representation.* Set

$$\begin{aligned} \mathbb{P}_\nu &= \Bbbk[x_1, \dots, x_n]e(\nu), & \widehat{\mathbb{P}}_\nu &= \Bbbk[\![x_1, \dots, x_n]\!]e(\nu), & \widehat{\mathbb{K}}_\nu &= \Bbbk((x_1, \dots, x_n))e(\nu),\\ {}^\theta\mathbb{P}_\beta &= \bigoplus_{\nu\in{}^\theta J^\beta} \mathbb{P}_\nu, & {}^\theta\widehat{\mathbb{P}}_\beta &= \bigoplus_{\nu\in{}^\theta J^\beta} \widehat{\mathbb{P}}_\nu, & {}^\theta\widehat{\mathbb{K}}_\beta &= \bigoplus_{\nu\in{}^\theta J^\beta} \widehat{\mathbb{K}}_\nu. \end{aligned}$$

We abbreviate $x_{-l} = -x_l$ for $1 \le l \le n$. The group $\mathfrak{W}_n$ acts on $\Bbbk((x_1, \dots, x_n))$ from the left by $w \cdot x_l = x_{w(l)}$. This induces an action on ${}^\theta\widehat{\mathbb{K}}_\beta$ according to the rule

$$w \cdot fe(\nu) = w(f)e(w \cdot \nu), \tag{2-6}$$

for $w \in \mathfrak{W}_n$ and $f \in \Bbbk((x_1, \dots, x_n))$.

Let $P = (P_{ij})_{i,j \in J}$ be a coefficient matrix satisfying (M1)–(M3) and $P' = (P_i)_{i \in J}$ a coefficient vector satisfying (V1)–(V2). Set

$$\begin{aligned} Q_{ij}(u, v) &= P_{ij}(u, v)P_{ji}(v, u), \\ Q_i(u) &= P_i(u)P_{\theta(i)}(-u), \end{aligned} \tag{2-7}$$

with $i, j \in J$. Then $Q = (Q_{ij})$ is a perfect coefficient matrix and $Q' = (Q_i)$ a perfect coefficient vector.

Proposition 2.7. *The algebra ${}^\theta\mathfrak{R}(\beta)$ has a faithful polynomial representation on ${}^\theta\mathbb{P}_\beta$, given by*:

- $e(\nu)$, *where* $\nu \in {}^\theta J^\beta$, *acting as a projection onto* $\mathbb{P}_\nu$,
- $x_1, \dots, x_n$ *acting naturally by multiplication*,
- $\tau_1, \dots, \tau_{n-1}$ *acting via*

$$\tau_k \cdot fe(\nu) = \begin{cases} \dfrac{s_k(f) - f}{x_k - x_{k+1}} e(\nu), & \text{if } \nu_k = \nu_{k+1}, \\ P_{\nu_k, \nu_{k+1}}(x_k, x_{k+1}) s_k(f) e(s_k \cdot \nu), & \text{otherwise}, \end{cases}$$

- τ_0 *acting via*

$$\tau_0 \cdot fe(\nu) = \begin{cases} \dfrac{s_0(f) - f}{x_1} e(\nu), & \text{if } \theta(\nu_1) = \nu_1, \\ P_{\nu_1}(x_1) s_0(f) e(s_0 \cdot \nu), & \text{otherwise}. \end{cases}$$

Whenever we want to emphasize the dependence of the above representation on (P, P'), we will write ${}^\theta\mathbb{P}_\beta^{P,P'}$.

Proof. The proof that the operators defined above satisfy all the relations from Definition 2.4 not involving τ_0 is the same as in the case of the KLR algebra, and can be found in, e.g., the proof of [Rouquier 2008, Proposition 3.12]. The other relations are easy to check, with the exception of the deformed braid relations. We prove these explicitly below.

To simplify exposition, we omit the idempotents. We also abbreviate $i=\nu_1$ and $j=\nu_2$. First consider the case where $i\neq j$ and $j=\theta(i)$. Then:

$$\begin{aligned}
\tau_1\tau_0\tau_1\tau_0(f) &= \tau_1\tau_0\tau_1 P_i(x_1)s_0(f) = \tau_1\tau_0\frac{P_i(x_2)s_1s_0(f)-P_i(x_1)s_0(f)}{x_1-x_2}\\
&= \tau_1 P_j(x_1)\frac{P_i(x_2)s_0s_1s_0(f)-P_i(-x_1)f}{-x_1-x_2}\\
&= P_{ij}(x_1,x_2)P_j(x_2)\frac{P_i(x_1)s_1s_0s_1s_0(f)-P_i(-x_2)s_1(f)}{-x_1-x_2},\\
\tau_0\tau_1\tau_0\tau_1(f) &= \tau_0\tau_1\tau_0 P_{ij}(x_1,x_2)s_1(f) = \tau_0\tau_1 P_j(x_1)P_{ij}(-x_1,x_2)s_0s_1(f)\\
&= \tau_0\frac{P_j(x_2)P_{ij}(-x_2,x_1)s_1s_0s_1(f)-P_j(x_1)P_{ij}(-x_1,x_2)s_0s_1(f)}{x_1-x_2}\\
&= P_i(x_1)\frac{P_j(x_2)P_{ij}(-x_2,-x_1)s_0s_1s_0s_1(f)-P_j(-x_1)P_{ij}(x_1,x_2)s_1(f)}{-x_1-x_2}.
\end{aligned}$$

Since, by θ-symmetry, we have $P_{ij}(x_1,x_2)=P_{ij}(-x_2,-x_1)$, it follows that

$$\begin{aligned}
\big((\tau_1\tau_0)^2-(\tau_0\tau_1)^2\big)(f) &= \frac{P_j(x_2)P_i(-x_2)-P_i(x_1)P_j(-x_1)}{x_1+x_2}P_{ij}(x_1,x_2)s_1(f)\\
&= \frac{Q_j(x_2)-Q_i(x_1)}{x_1+x_2}\tau_1(f).
\end{aligned}$$

Secondly, let $i\neq\theta(i)$ and $j=\theta(j)$. Then:

$$\begin{aligned}
&\tau_1\tau_0\tau_1\tau_0(f)\\
&= \tau_1\tau_0\tau_1 P_i(x_1)s_0(f) = \tau_1\tau_0 P_{\theta(i),j}(x_1,x_2)P_i(x_2)s_1s_0(f)\\
&= \tau_1\frac{P_{\theta(i),j}(-x_1,x_2)P_i(x_2)s_0s_1s_0(f)-P_{\theta(i),j}(x_1,x_2)P_i(x_2)s_1s_0(f)}{x_1}\\
&= P_{j,\theta(i)}(x_1,x_2)\frac{P_{\theta(i),j}(-x_2,x_1)P_i(x_1)s_1s_0s_1s_0(f)-P_{\theta(i),j}(x_2,x_1)P_i(x_1)s_0(f)}{x_2},\\
&\tau_0\tau_1\tau_0\tau_1(f)\\
&= \tau_0\tau_1\tau_0 P_{ij}(x_1,x_2)s_1(f) = \tau_0\tau_1\frac{P_{ij}(-x_1,x_2)s_0s_1(f)-P_{ij}(x_1,x_2)s_1(f)}{x_1}\\
&= \tau_0 P_{ji}(x_1,x_2)\frac{P_{ij}(-x_2,x_1)s_1s_0s_1(f)-P_{ij}(x_2,x_1)f}{x_2}\\
&= P_i(x_1)P_{ji}(-x_1,x_2)\frac{P_{ij}(-x_2,-x_1)s_0s_1s_0s_1(f)-P_{ij}(x_2,-x_1)s_0(f)}{x_2}.
\end{aligned}$$

Again, θ-symmetry implies that

$$
\begin{aligned}
&((\tau_1\tau_0)^2-(\tau_0\tau_1)^2)(f)\\
&\quad=\frac{-P_{j,\theta(i)}(x_1,x_2)P_{\theta(i),j}(x_2,x_1)+P_{ij}(x_2,-x_1)P_{j,i}(-x_1,x_2)}{x_2}P_i(x_1)s_0(f)\\
&\quad=\frac{Q_{ij}(x_2,-x_1)-Q_{ij}(-x_2,-x_1)}{x_2}\tau_0(f).
\end{aligned}
$$

Thirdly, let $\theta(i)=i\neq j=\theta(j)$. Then:

$$
\begin{aligned}
&\tau_1\tau_0\tau_1\tau_0(f)\\
&\quad=\tau_1\tau_0\tau_1\frac{s_0(f)-f}{x_1}=\tau_1\tau_0P_{ij}(x_1,x_2)\frac{s_1s_0(f)-s_1(f)}{x_2}\\
&\quad=\tau_1\frac{P_{ij}(-x_1,x_2)[s_0s_1s_0(f)-s_0s_1(f)]-P_{ij}(x_1,x_2)[s_1s_0(f)-s_1(f)]}{x_1x_2}\\
&\quad=P_{ji}(x_1,x_2)\frac{P_{ij}(-x_2,x_1)[s_1s_0s_1s_0(f)-s_1s_0s_1(f)]-P_{ij}(x_2,x_1)[s_0(f)-(f)]}{x_1x_2},
\end{aligned}
$$

$$
\begin{aligned}
&\tau_0\tau_1\tau_0\tau_1(f)\\
&\quad=\tau_0\tau_1\tau_0P_{ij}(x_1,x_2)s_1(f)=\tau_0\tau_1\frac{P_{ij}(-x_1,x_2)s_0s_1(f)-P_{ij}(x_1,x_2)s_1(f)}{x_1}\\
&\quad=\tau_0P_{ji}(x_1,x_2)\frac{P_{ij}(-x_2,x_1)s_1s_0s_1(f)-P_{ij}(x_2,x_1)f}{x_2}\\
&\quad=\frac{P_{ji}(-x_1,x_2)[P_{ij}(-x_2,-x_1)s_0s_1s_0s_1(f)-P_{ij}(x_2,-x_1)s_0(f)]}{x_1x_2}\\
&\qquad\qquad-\frac{P_{ji}(x_1,x_2)[P_{ij}(-x_2,x_1)s_1s_0s_1(f)-P_{ij}(x_2,x_1)f]}{x_1x_2}.
\end{aligned}
$$

By θ-symmetry, we conclude that

$$
\begin{aligned}
\big((\tau_1\tau_0)^2-(\tau_0\tau_1)^2\big)(f)&=\frac{-P_{ji}(x_1,x_2)P_{ij}(x_2,x_1)+P_{ji}(-x_1,x_2)P_{ij}(x_2,-x_1)}{x_1x_2}s_0(f)\\
&=\frac{Q_{i,j}(x_2,-x_1)-Q_{i,j}(x_2,x_1)}{x_1x_2}(x_1\tau_0+1)f.
\end{aligned}
$$

Fourthly, let $i=\theta(i)$ and $j\neq\theta(j)$. One easily checks (using θ-symmetry) that $((\tau_1\tau_0)^2-(\tau_0\tau_1)^2)(f)=g\cdot s_1s_0s_1\Delta_0(f)-\Delta_0(g\cdot s_1s_0s_1(f))$, where g is an s_0-invariant polynomial and $\Delta_0=x_1^{-1}(s_0-1)$. It now follows from the properties of Demazure operators that

$$
\begin{aligned}
&\big((\tau_1\tau_0)^2-(\tau_0\tau_1)^2\big)(f)\\
&\qquad=g\cdot s_1s_0s_1\Delta_0(f)-\big(\Delta_0(g)\cdot s_1s_0s_1(f)+s_0(g)\Delta_0\big(s_1s_0s_1(f)\big)\big)=0.
\end{aligned}
$$

Fifthly, let $i = j$ and $i \neq \theta(i)$. One checks, as above, that $((\tau_1\tau_0)^2 - (\tau_0\tau_1)^2)(f) = \Delta_1(g \cdot s_0 s_1 s_0(f)) - g \cdot s_0 s_1 s_0 \Delta_1(f)$, where g is an s_1-invariant polynomial and $\Delta_1 = (x_1 - x_2)^{-1}(s_1 - 1)$. As above, it follows from the properties of Demazure operators that $((\tau_1\tau_0)^2 - (\tau_0\tau_1)^2)(f) = 0$.

Finally, suppose that $i = j = \theta(j)$. Then each of τ_0 and τ_1 acts as a Demazure operator, but Demazure operators satisfy the braid relation. This completes the proof that ${}^\theta\mathbb{P}_\beta$ is a representation of ${}^\theta\mathcal{R}(\beta)$.

The proof of faithfulness is analogous to the case of KLR algebras, see, e.g., [Rouquier 2008, Proposition 3.12]. □

Next, for each $i, j \in J$, we choose holomorphic functions $c_{ij}(u, v) \in \Bbbk[\![u, v]\!]$ such that

$$(2\text{-}8) \qquad c_{ij}(u, v)c_{ji}(v, u) = 1, \quad c_{ii}(u, v) = 1, \quad c_{ij}(u, v) = c_{\theta(j)\theta(i)}(-v, -u).$$

Moreover, for each $i \in J$, we also choose holomorphic functions $c_i \in \Bbbk[[u]]$ such that

$$(2\text{-}9) \qquad c_i(u) = c_{\theta(i)}(-u), \qquad i = \theta(i) \Rightarrow c_i(u) = 1.$$

Set

$$\widetilde{P}_{ij}(u, v) = P_{ij}(u, v)c_{ij}(u, v) \quad \text{and} \quad \widetilde{P}_i(u) = P_i(u)c_i(u).$$

Corollary 2.8. *There is an injective ${}^\theta\mathbb{P}_\beta$-algebra homomorphism*

$$(2\text{-}10) \qquad {}^\theta\mathcal{R}(\beta) \hookrightarrow \Bbbk[\mathfrak{W}_n] \ltimes {}^\theta\widehat{\mathbb{K}}_\beta$$

given by

$$\tau_0 e(\nu) = \begin{cases} x_1^{-1}(s_0 - 1)e(\nu), & \text{if } \nu_1 = \theta(\nu_1), \\ \widetilde{P}_{\nu_1}(x_1)s_0 e(\nu), & \text{otherwise}, \end{cases}$$

$$\tau_k e(\nu) = \begin{cases} (x_k - x_{k+1})^{-1}(s_k - 1)e(\nu), & \text{if } \nu_k = \nu_{k+1}, \\ \widetilde{P}_{\nu_k,\nu_{k+1}}(x_k, x_{k+1})s_k e(\nu), & \text{otherwise}, \end{cases}$$

for $1 \leq k \leq n - 1$.

Proof. This follows directly from Proposition 2.7. □

2E. *PBW theorem.* In this subsection, assume that Q is a coefficient matrix satisfying (M1)–(M3) and Q' a coefficient vector satisfying (V1)–(V2). The algebra ${}^\theta\mathcal{R}(\beta)$ is filtered with $\deg x_l$, $\deg e(\nu) = 0$ and $\deg \tau_k = 1$. We say that ${}^\theta\mathcal{R}(\beta)$ satisfies the *PBW property* if $\operatorname{gr}{}^\theta\mathcal{R}(\beta) \cong {}^0\mathcal{H}_n^f \ltimes {}^\theta\mathbb{P}_\beta$, where ${}^0\mathcal{H}_n^f$ is the (nonaffine) nil-Hecke algebra of type B_n (see, e.g., [Kostant and Kumar 1986]).

For any $w \in \mathfrak{W}_n$, choose a reduced expression $w = s_{k_1} \cdots s_{k_l}$ and set $\tau_w = \tau_{s_{k_1}} \cdots \tau_{s_{k_l}}$. The definition of τ_w depends on the choice of reduced expression.

Proposition 2.9. *Let $n \geq 1$. The following are equivalent*:

(1) *${}^\theta\mathcal{R}(\beta)$ satisfies the PBW property*,

(2) ${}^{\theta}\mathcal{R}(\beta)$ *is a free* $\Bbbk$*-module with basis*

$$\left\{\tau_w x_1^{a_1}\dots x_n^{a_n} e(\nu) \mid w \in \mathfrak{W}_n,\ (a_1,\dots,a_n) \in \mathbb{Z}^n_{\geq 0},\ \nu \in {}^{\theta}J^{\beta}\right\},$$

(3) Q *and* Q' *are perfect.*

Proof. The proof is a straightforward generalization of the proof of [Rouquier 2008, Theorem 3.7]. Let us briefly comment on the new features. Suppose that (2) holds, and let $\nu_1 \neq \theta(\nu_1)$. The quadratic relation then implies that

$$Q_{\theta(\nu_1)}(-x_1)\tau_0 e(\nu) = \tau_0^3 e(\nu) = \tau_0 Q_{\nu_1}(-x_1)e(\nu) = Q_{\nu_1}(x_1)\tau_0 e(\nu).$$

It follows that

$$\big(Q_{\theta(\nu_1)}(-x_1) - Q_{\nu_1}(x_1)\big)\tau_0 e(\nu) = 0.$$

Now (2) implies that $Q_{\theta(\nu_1)}(-x_1) - Q_{\nu_1}(x_1) = 0$, i.e., Q' is self-conjugate. Conversely, if both Q and Q' are perfect, then we can use Proposition 2.7, with $P_{ij} = Q_{ij}$, $P_{ji} = 1$ with $i < j$, $P_i = Q_i$ and $P_{\theta(i)} = 1$ with $i < \theta(i)$ for some ordering of J, to deduce (2). □

2F. *Orientifold KLR algebras associated to quivers.* Let $\Gamma = (J, \Omega)$ be a quiver with vertices J and arrows Ω. We assume that Γ does not have loops. Given an arrow $a \in \Omega$, let $s(a)$ be its source, and $t(a)$ its target. If $i, j \in J$, let $\Omega_{ij} \subset \Omega$ be the subset of arrows a such that $s(a) = i$ and $t(a) = j$. Let $a_{ij} = |\Omega_{ij}|$ and abbreviate $\overleftrightarrow{a}_{ij} = a_{ij} + a_{ji}$. We assume that $a_{ij} < \infty$ for all $i, j \in J$.

Definition 2.10. A (contravariant) *involution* of the quiver Γ is a pair of involutions $\theta\colon J \to J$ and $\theta\colon \Omega \to \Omega$ such that:

(1) $s(\theta(a)) = \theta(t(a))$ and $t(\theta(a)) = \theta(s(a))$ for all $a \in \Omega$,

(2) if $t(a) = \theta(s(a))$, then $a = \theta(a)$.

Fix a quiver Γ with an involution θ and two dimension vectors $\beta \in \mathbb{N}[J]^{\theta}$, $\lambda \in \mathbb{N}[J]$ such that $\|\beta\|_{\theta} = n$ and $\lambda(i) = 0$ if $i \in J^{\theta}$. We call λ the *framing dimension vector*. Note that λ need not be self-dual.

Set

$$P_{ij}(u, v) = \delta_{i\neq j}(v-u)^{a_{ij}} \quad\text{and}\quad P_i(u) = \delta_{i\neq\theta(i)}(-u)^{\lambda(i)}$$

for $i, j \in J$, and define (Q, Q') as in (2-7). Since, by Definition 2.10, $a_{ij} = a_{\theta(j)\theta(i)}$, the coefficient matrix P is θ-symmetric and, therefore, (Q, Q') is perfect.

Definition 2.11. The KLR algebra associated to (Γ, β) and the orientifold KLR algebra associated to $(\Gamma, \theta, \beta, \lambda)$ are, respectively,

$$\mathcal{R}^{\Gamma}(\beta) = \mathcal{R}(J, Q, \beta) \quad\text{and}\quad {}^{\theta}\mathcal{R}^{\Gamma}(\beta; \lambda) = {}^{\theta}\mathcal{R}(J, Q, Q', \beta).$$

We endow these algebras with the following grading:

$$\begin{aligned}
\deg e(\nu) &= 0,\\
\deg x_k &= 2,\\
\deg \tau_k e(\nu) &= \begin{cases} -2, & \text{if } \nu_k = \nu_{k+1},\\ \overleftrightarrow{a}_{\nu_k,\nu_{k+1}}, & \text{otherwise,}\end{cases}\\
\deg \tau_0 e(\nu) &= \begin{cases} -2, & \text{if } \theta(\nu_1) = \nu_1,\\ {}^{\theta}\lambda(\nu_1), & \text{otherwise.}\end{cases}
\end{aligned}$$

Most of the time we will omit Γ from the notation, as the choice of quiver is clear from the context. Also note that, by Proposition 2.7, the algebra ${}^{\theta}\mathfrak{R}(\beta;\lambda)$ has a faithful polynomial representation on ${}^{\theta}\mathbb{P}_{\beta}^{P,P'}$.

3. Enomoto–Kashiwara algebra, quantum shuffle modules and Lyndon words

3A. ***Notation.*** Let $J = \{\alpha_k \mid k \in \mathbb{Z}_{\text{odd}}\}$ and equip $\mathsf{Q} = \mathbb{Z}[J]$ with the symmetric bilinear form

$$\alpha_k \cdot \alpha_l = \begin{cases} 2, & \text{if } k = l,\\ -1, & \text{if } k = l \pm 2,\\ 0, & \text{otherwise.}\end{cases} \tag{3-1}$$

Then $(J,\cdot)$ is the Cartan datum associated to $\mathfrak{g} = \mathfrak{sl}_\infty$. We identify J with the set of simple roots of the root system Φ of type $\mathtt{A}_\infty$. Then $\Phi^+ = \{\beta_{k,l} \mid k \le l \in \mathbb{Z}_{\text{odd}}\}$, where $\beta_{k,l} = \alpha_k + \alpha_{k+2} + \cdots + \alpha_l$, is a set of positive roots. Let $\mathsf{P} = \{\lambda \in \mathbb{Q} \otimes_{\mathbb{Z}} \mathsf{Q} \mid \lambda \cdot i \in \mathbb{Z} \text{ for all } i \in J\}$ be the weight lattice, $\mathsf{P}_+ = \{\lambda \in \mathsf{P} \mid \lambda \cdot i \in \mathbb{Z}_{\ge 0} \text{ for all } i \in J\}$ be the set of dominant integral weights, and $\mathsf{Q}_+ = \mathbb{N}[J]$. Given $\beta = \sum_{i\in J} c_i i \in \mathsf{Q}_+$, let $N(\beta) = \frac{1}{2}\big(\beta\cdot\beta - \sum_{i\in J} c_i i \cdot i\big)$.

Let $\theta\colon \mathsf{Q} \to \mathsf{Q}$ be the involution sending $\alpha_k \mapsto \alpha_{-k}$. The bilinear form (3-1) restricts to Q^θ. The image of Φ under the symmetrization map

$$\mathsf{Q} \to \mathsf{Q}^\theta, \quad \alpha_k \mapsto \alpha_k + \alpha_{-k}$$

is isomorphic to the unreduced root system ${}^{\theta}\Phi$ of type $\mathtt{BC}_\infty$, and the image ${}^{\theta}\Phi^+$ of Φ^+ is a set of positive roots for ${}^{\theta}\Phi$.

Let q be an indeterminate and set $\mathcal{K} = \mathbb{Q}(q)$ and $\mathcal{A} = \mathbb{Z}[q^{\pm 1}]$. Let $\bar{\ }\colon \mathcal{K} \to \mathcal{K}$ be the *bar involution*, i.e., the $\mathbb{Q}$-algebra involution with $\bar{q} = q^{-1}$. Set

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad [n]! = [n][n-1]\cdots[1], \quad [2n]!! = [2n][2n-2]\cdots[2].$$

If A is a $\mathcal{K}$-algebra, $a \in A$ and $n \in \mathbb{N}$, then $a^{(n)} = a^n/[n]!$. For $\nu = \nu_1^{a_1}\cdots\nu_k^{a_k} \in J^\bullet$ with $\nu_j \ne \nu_{j+1}$, set $[\nu]! = [a_1]!\cdots[a_k]!$.

3B. ***The algebras* f *and* f*.** Let $\mathbf{f}$ be the $\mathcal{K}$-algebra generated by the elements f_i, where $i \in J$, subject to the q-Serre relations:

$$\sum_{k+l=1-i\cdot j} (-1)^k f_i^{(k)} f_j f_i^{(l)} = 0, \quad \text{where } i \neq j.$$

The algebra $\mathbf{f}$ is Q-graded with f_i in degree $-i$. We denote by $-|u|$ the Q-degree of a homogeneous element $u \in \mathbf{f}$. Given $\nu = \nu_1 \cdots \nu_n \in J^\bullet$, let $f_\nu = f_{\nu_1} \cdots f_{\nu_n}$. We will use notation of this form more generally, i.e., given any collection of elements y_i labeled by $i \in J$, we write $y_\nu = y_{\nu_1} \cdots y_{\nu_n}$.

Kashiwara [1991] introduced q-derivations e_i', $e_i^* \in \mathrm{End}_{\mathcal{K}}(\mathbf{f})$ characterized by

$$\begin{aligned} e_i'(f_j) &= \delta_{ij}, \quad e_i'(uv) = e_i'(u)v + q^{-i\cdot|u|} u e_i'(v), \\ e_i^*(f_j) &= \delta_{ij}, \quad e_i^*(uv) = \ q^{-i\cdot|v|} e_i^*(u)v + u e_i^*(v), \end{aligned}$$

for all homogeneous elements $u, v \in \mathbf{f}$. Both $\{e_i' \mid i \in J\}$ and $\{e_i^* \mid i \in J\}$ satisfy the q-Serre relations.

There is a unique nondegenerate symmetric bilinear form $(\cdot\,,\cdot)$ on $\mathbf{f}$ such that $(1, 1) = 1$ and $(e_i'(u), v) = (u, f_i v)$ for $u, v \in \mathbf{f}$ and $i \in J$. This form differs slightly from the form $(\cdot\,,\cdot)_L$ introduced by Lusztig [1993, Proposition 1.2.3] — see [Leclerc 2004, §2.2] for the precise relationship. Let $\mathbf{f}_{\mathcal{A}}$ be the integral form of $\mathbf{f}$, i.e., the $\mathcal{A}$-subalgebra generated by the $f_i^{(k)}$, with $i \in J$ and $k \in \mathbb{N}$, and let

$$\mathbf{f}^*_{\mathcal{A}} = \{u \in \mathbf{f} \mid (u, v) \in \mathcal{A} \text{ for all } v \in \mathbf{f}_{\mathcal{A}}\}$$

be its dual.

3C. ***Enomoto–Kashiwara algebra.*** The subalgebra of $\mathrm{End}_{\mathcal{K}}(\mathbf{f})$ generated by the e_i' and left multiplication by f_i is called the *reduced q-analogue* of $U(\mathfrak{g})$. The generators satisfy the relation

$$e_i' f_j = q^{-\alpha_i\cdot\alpha_j} f_j e_i' + \delta_{ij}.$$

Enomoto and Kashiwara [2006] defined a related algebra, which also depends on the involution θ. As it appears, this algebra does not have a distinctive name in the literature, so we call it the Enomoto–Kashiwara algebra.

Definition 3.1. The *Enomoto–Kashiwara* algebra ${}^\theta\mathcal{B}(\mathfrak{g})$ is the $\mathcal{K}$-algebra generated by E_i, F_i and the invertible elements T_i, with $i \in J$, subject to the following relations:

- the T_i commute,
- $T_{\theta(i)} = T_i$ for any i,
- $T_i E_j T_i^{-1} = q^{(i+\theta(i))\cdot j} E_j$ and $T_i F_j T_i^{-1} = q^{-(i+\theta(i))\cdot j} F_j$ for $i, j \in J$,
- $E_i F_j = q^{-i\cdot j} F_j E_i + \delta_{ij} + \delta_{\theta(i)j} T_i$ for all $i, j \in J$,
- the E_i and the F_i satisfy the q-Serre relations.

Proposition 3.2. *Let* $\lambda \in \mathsf{P}_+$.

(1) *There exists a* ${}^\theta\mathcal{B}(\mathfrak{g})$*-module* ${}^\theta V(\lambda)$ *generated by a nonzero vector* v_λ *such that*:
   (a) $E_i v_\lambda = 0$ *for any* $i \in J$,
   (b) $T_i v_\lambda = q^{{}^\theta\lambda \cdot i} v_\lambda$ *for any* $i \in J$,
   (c) $\{u \in {}^\theta V(\lambda) \mid E_i u = 0 \text{ for any } i \in J\} = \mathcal{K} v_\lambda$.

(2) ${}^\theta V(\lambda)$ *is irreducible and unique up to isomorphism.*

(3) *There exists a unique symmetric bilinear form* $(\cdot\,,\cdot)$ *on* ${}^\theta V(\lambda)$ *such that* $(v_\lambda, v_\lambda) = 1$ *and* $(E_i u, v) = (u, F_i v)$ *for any* $i \in J$ *and* $u, v \in {}^\theta V(\lambda)$. *It is nondegenerate.*

(4) *There is a unique endomorphism* $\bar{\cdot}$ *of* ${}^\theta V(\lambda)$, *called the bar involution, such that* $\overline{v_\lambda} = v_\lambda$ *and* $\overline{av} = \bar{a}\bar{v}$, $\overline{F_i v} = F_i \bar{v}$ *for* $a \in \mathcal{K}$ *and* $v \in {}^\theta V(\lambda)$.

(5) *Let* ${}^\theta \widetilde{V}(\lambda)$ *be the free* $\mathbf{f}$*-module with generator* $\tilde{v}_\lambda$ *and a* ${}^\theta\mathcal{B}(\mathfrak{g})$*-module structure given by*

$$T_i(u\tilde{v}_\lambda) = q^{{}^\theta\lambda\cdot i-(i+\theta(i))\cdot|u|} u\tilde{v}_\lambda, \tag{3-2}$$

$$E_i(u\tilde{v}_\lambda) = e_i'(u)\tilde{v}_\lambda, \tag{3-3}$$

$$F_i(u\tilde{v}_\lambda) = (f_i u + q^{{}^\theta\lambda\cdot i - i\cdot|u|} u f_{\theta(i)})\tilde{v}_\lambda, \tag{3-4}$$

for any $i \in J$ *and* $u \in \mathbf{f}$. *Then the subspace of* ${}^\theta \widetilde{V}(\lambda)$ *spanned by the vectors* $F_\nu \cdot \tilde{v}_\lambda$ *is a* ${}^\theta\mathcal{B}(\mathfrak{g})$*-submodule isomorphic to* ${}^\theta V(\lambda)$.

Proof. See [Enomoto and Kashiwara 2008, Proposition 2.11, Lemma 2.15]. □

From now on, let us identify $\mathbf{f}$ with the subalgebra of ${}^\theta\mathcal{B}(\mathfrak{g})$ generated by the F_i. Note that it follows from Proposition 3.2 that ${}^\theta V(\lambda) = \mathbf{f}\cdot v_\lambda$. The module ${}^\theta V(\lambda)$ has a P^θ-grading:

$${}^\theta V(\lambda) = \bigoplus_{\mu\in P^\theta} {}^\theta V(\lambda)_\mu,$$

where ${}^\theta V(\lambda)_\mu = \{v \in {}^\theta V(\lambda) \mid T_i \cdot v = q^{\mu\cdot i}u\}$. If $v \in {}^\theta V(\lambda)_\mu$, write $\mu_v := \mu$ and ${}^\theta|v| = \mu_v$. The integral and dual integral forms are defined as ${}^\theta V(\lambda)^{\mathrm{low}}_{\mathcal{A}} = \mathbf{f}_{\mathcal{A}} v_\lambda$ and ${}^\theta V(\lambda)^{\mathrm{up}}_{\mathcal{A}} = \{v \in {}^\theta V(\lambda) \mid ({}^\theta V(\lambda)^{\mathrm{low}}_{\mathcal{A}}, v) \in \mathcal{A}\}$, respectively.

The operators E_i satisfy a kind of "twisted derivation" property.

Lemma 3.3. *We have*

$$E_i y\cdot v = q^{-i\cdot|y|} y E_i\cdot v + (e_i'(y) + q^{-i\cdot|e^*_{\theta(i)}(y)|} e^*_{\theta(i)}(y) T_i)\cdot v$$

for any $y \in \mathbf{f}$ *and* $v \in {}^\theta V(\lambda)$.

Proof. This is [Enomoto and Kashiwara 2008, Lemma 2.9]. □

3D. *Quantum shuffle algebra.* The *quantum shuffle algebra* $\mathcal{F}$ is the Q-graded $\mathcal{K}$-algebra with basis $J^\bullet$, where $\deg_{\mathsf{Q}} \nu = -|\nu|$, and multiplication given by

$$\nu \circ \nu' = \sum_{w \in \mathsf{D}_{\|\beta\|,\|\beta'\|}} q^{-d(\nu,\nu',w)} w \cdot \nu\nu' \tag{3-5}$$

for $\nu \in J^\beta$ and $\nu' \in J^{\beta'}$, where $\nu\nu' = i_1 \cdots i_{\|\beta+\beta'\|}$ and

$$d(\nu, \nu', w) = \sum_{\substack{k \le \|\beta\| < l,\\ w(k) > w(l)}} i_{w^{-1}(k)} \cdot i_{w^{-1}(l)}. \tag{3-6}$$

To $\nu = i_1 \cdots i_k \in J^\bullet$ one associates the q-derivation $\partial_\nu = e^*_{i_1} \cdots e^*_{i_k} \in \mathrm{End}_{\mathcal{K}}(\mathbf{f})$. There is a $\mathcal{K}$-linear map

$$\Psi : \mathbf{f} \longrightarrow \mathcal{F}, \quad \Psi(u) = \sum_{\substack{\nu \in J^\bullet,\\ |\nu| = |u|}} \partial_\nu(u) \cdot \nu \tag{3-7}$$

for a homogeneous element $u \in \mathbf{f}$.

Let $\mathbf{e}'_i, \mathbf{e}^*_i \in \mathrm{End}_{\mathcal{K}}(\mathcal{F})$ be the left and right deletion operators:

$$\mathbf{e}'_i(i_1 \cdots i_k) = \delta_{i,i_1} i_2 \cdots i_k, \quad \mathbf{e}^*_i(i_1 \cdots i_k) = \delta_{i,i_k} i_1 \cdots i_{k-1}, \quad \mathbf{e}'_i(\varnothing) = \mathbf{e}^*_i(\varnothing) = 0,$$

respectively.

Proposition 3.4. *The map* (3-7) *is an injective* Q-*graded algebra homomorphism satisfying*

$$\mathbf{e}'_i \circ \Psi = \Psi \circ e'_i \quad \textit{and} \quad \mathbf{e}^*_i \circ \Psi = \Psi \circ e^*_i.$$

Proof. This follows directly from [Leclerc 2004, Lemma 3 and Theorem 4]. The proof for left deletions is analogous. □

We will now consider some antiautomorphisms of $\mathbf{f}$ and $\mathcal{F}$. Set

$$\sigma : J^\bullet \to J^\bullet, \quad \nu \mapsto w_0(\nu), \qquad {}^\theta\sigma : J^\bullet \to J^\bullet, \quad \nu \mapsto {}^\theta w(\nu). \tag{3-8}$$

We extend these maps to $\mathcal{K}$-linear maps $\sigma : \mathcal{F} \to \mathcal{F}$ and ${}^\theta\sigma : \mathcal{F} \to \mathcal{F}$. We use the same symbols to denote the $\mathcal{K}$-linear maps

$$\sigma : \mathbf{f} \to \mathbf{f}, \quad f_\nu \mapsto f_{\sigma(\nu)}, \qquad {}^\theta\sigma : \mathbf{f} \to \mathbf{f}, \quad f_\nu \mapsto f_{{}^\theta\sigma(\nu)},$$

respectively.

Lemma 3.5. *The maps* σ *and* ${}^\theta\sigma$ *are algebra antiautomorphisms satisfying* $\sigma \circ \Psi = \Psi \circ \sigma$ *and* ${}^\theta\sigma \circ \Psi = \Psi \circ {}^\theta\sigma$, *respectively.*

Proof. The case of σ is [Leclerc 2004, Proposition 6]. The case of ${}^\theta\sigma$ follows easily from (3-5) and (3-6). □

3E. ***Quantum shuffle module.*** We will now realize the modules ${}^{\theta}V(\lambda)$ in terms of modules over the shuffle algebra.

Definition 3.6. We define the *quantum shuffle module* ${}^{\theta}\mathscr{F}(\lambda)$ to be the P^{θ}-graded $\mathscr{K}$-vector space with basis ${}^{\theta}J^{\bullet}$, where $\deg_{\mathsf{P}^{\theta}} \nu = {}^{\theta}\lambda - {}^{\theta}|\nu|$, and a right $\mathscr{F}$-action given by

$$\nu \circledcirc \nu' = \sum_{w \in {}^{\theta}\mathsf{D}_{\|\beta\|_{\theta}, \|\beta'\|}} q^{-d(\nu,\nu',w)} w \cdot \nu\nu' \tag{3-9}$$

for $\nu \in {}^{\theta}J^{\beta}$ and $\nu' \in J^{\beta'}$, where

$$d(\nu, \nu', w) = \sum_{\substack{1 \le k < l \le N,\\ w(k) > w(l)}} i_{w^{-1}(k)} \cdot i_{w^{-1}(l)} + \sum_{\substack{1 \le k < l \le N,\\ w(-k) > w(l)}} i_{w^{-1}(-k)} \cdot i_{w^{-1}(l)} - \sum_{\substack{\|\beta\|_{\theta} < l,\\ w(l) < w(-l)}} {}^{\theta}\lambda \cdot i_l,$$

with $N = \|\beta\|_{\theta} + \|\beta'\|$.

Remark 3.7. We have chosen to define ${}^{\theta}V(\lambda)$ as a left ${}^{\theta}\mathscr{B}(\mathfrak{g})$-module, but ${}^{\theta}\mathscr{F}(\lambda)$ as a right $\mathscr{F}$-module. This choice is a compromise. On the one hand, we wanted to be consistent with the conventions of [Enomoto and Kashiwara 2006; 2008]. On the other hand, as shown in [Appel and Przeździecki 2022], ${}^{\theta}V(\lambda)$ can be categorified via quantum symmetric pairs, which are, by convention (see, e.g., [Kolb 2014]), right coideal subalgebras.

Let $\mathbf{E}_i \in \operatorname{End}_{\mathscr{K}}({}^{\theta}\mathscr{F}(\lambda))$ be the right deletion operator:

$$\mathbf{E}_i(i_1 \cdots i_k) = \delta_{i,i_k} i_1 \cdots i_{k-1}, \quad \mathbf{E}_i(\varnothing) = 0.$$

Lemma 3.8. *Formula* (3-9) *defines a right $\mathscr{F}$-action on ${}^{\theta}\mathscr{F}(\lambda)$. Moreover*, *the endomorphisms $\mathbf{E}_i$ satisfy*

$$\mathbf{E}_i(v \circledcirc z) = q^{-i\cdot|z|}\mathbf{E}_i(v) \circledcirc z + v \circledcirc \mathbf{e}_i^*(z) + q^{-i\cdot|\mathbf{e}'_{\theta(i)}(z)|+\mu_v\cdot i} v \circledcirc \mathbf{e}'_{\theta(i)}(z).$$

Proof. The first statement follows easily from the definitions, so we omit a proof. Let us prove the second statement. It suffices to consider v and z of the form $v = \nu j$ and $z = k\mu l$, for $\nu \in {}^{\theta}J^{\bullet}$, $\mu \in J^{\bullet}$ and $j, k, l \in J$. Then (3-9) implies

$$v \circledcirc z = \nu j \circledcirc k\mu l = (v \circledcirc k\mu)l + q^{-d(v,z,w)}(\nu \circledcirc z)j + q^{-d(v,z,w')}(v \circledcirc \mu l)\theta(k),$$

where w transposes j and z while w' sends k to $\theta(k)$ and transposes it with μl. One easily sees that $d(v, z, w) = j \cdot |z|$ and $d(v, z, w') = \theta(k) \cdot |\mathbf{e}'_k(z)| - \mu_v(\theta(k))$. Hence,

$$\mathbf{E}_i(v \circledcirc z) = \delta_{i,l}(v \circledcirc k\mu) + \delta_{i,j} q^{-i\cdot|z|}(\nu \circledcirc z) + \delta_{i,\theta(k)} q^{-i\cdot|\mathbf{e}'_{\theta(i)}(z)|+\mu_v\cdot i}(v \circledcirc \mu l).$$

The statement follows. $\square$

To $\nu = \nu_1 \cdots \nu_k \in J^\bullet$ one associates the operator ${}^\theta\partial_\nu = E_{\nu_1} \cdots E_{\nu_k} \in \mathrm{End}({}^\theta V(\lambda))$. There is a $\mathcal{K}$-linear map

$$(3\text{-}10) \qquad {}^\theta\Psi\colon {}^\theta V(\lambda) \to {}^\theta\mathcal{F}(\lambda), \quad {}^\theta\Psi(u) = \sum_{\substack{\nu \in {}^\theta J^\bullet, \\ {}^\theta|\nu| = {}^\theta|u|}} {}^\theta\partial_\nu(u) \cdot \sigma(\nu)$$

for a homogeneous element $u \in {}^\theta V(\lambda)$. Let us abbreviate

$$\mathbf{U} = \Psi(\mathbf{f}) \quad \text{and} \quad {}^\theta\mathbf{V}(\lambda) = {}^\theta\Psi({}^\theta V(\lambda)).$$

Proposition 3.9. *The map* (3-10) *is injective*, $\mathbf{E}_i \circ {}^\theta\Psi = {}^\theta\Psi \circ E_i$ *and the diagram*

$$\begin{array}{ccc} \mathbf{f} & \xrightarrow{\quad\Psi\quad} & \mathcal{F} \\ \curvearrowright & & \curvearrowleft \\ {}^\theta V(\lambda) & \xrightarrow{\ {}^\theta\Psi\ } & {}^\theta\mathcal{F}(\lambda) \end{array}$$

commutes.

Proof. The injectivity of ${}^\theta\Psi$ follows directly from Proposition 3.2 (1c). Let ${}^\theta\Psi'\colon {}^\theta V(\lambda) \to {}^\theta\mathcal{F}(\lambda)$ be the map sending $y \cdot v_\lambda \mapsto \varnothing \circledcirc \Phi(\sigma(y))$ for $y \in \mathbf{f}$. Note that ${}^\theta\Psi'$ is defined on all of ${}^\theta V(\lambda)$ since ${}^\theta V(\lambda) = \mathbf{f} \cdot v_\lambda$. We claim that ${}^\theta\Psi'$ intertwines the actions of $\mathbf{f}$ and $\mathcal{F}$, and that ${}^\theta\Psi = {}^\theta\Psi'$. For the first claim, note that (3-9) implies that $\nu \circledcirc i = \nu \circ i + q^{{}^\theta\lambda(i) - i\cdot|\nu|}\theta(i) \circ \nu$, for $i \in J$ and $\nu \in J^\bullet$. Hence, by Proposition 3.2 (5) and (3-4), the first claim follows. Lemma 3.3 and Lemma 3.8 imply that $\mathbf{E}_i \circ {}^\theta\Psi' = {}^\theta\Psi' \circ E_i$. Let $v \in {}^\theta V(\lambda)$ be homogeneous, and let $\nu \in {}^\theta J^\bullet$ with ${}^\theta|v| = {}^\theta|\nu|$. Let $\gamma_\nu(v)$ be the coefficient of $\sigma(\nu)$ in ${}^\theta\Psi'(v)$. Then $\gamma_\nu(v) = \mathbf{E}_{\sigma(\nu)} \circ {}^\theta\Psi'(v) = {}^\theta\partial_\nu(v)$. Hence ${}^\theta\Psi = {}^\theta\Psi'$, which completes the proof. ☐

3F. *θ-good words.* We fix a total order on the set J and equip $J^\bullet$ with the corresponding antilexicographic order. Both are denoted by $\leq$. Given a linear combination u of words, let $\max(u)$ be the largest word appearing in u.

Lemma 3.10. *If* $\mu' \leq \mu$, $\nu' \leq \nu$ *and* ${}^\theta w(\nu') \leq {}^\theta w(\nu)$, *for* $\mu, \mu' \in {}^\theta J^\bullet$ *and* $\nu, \nu' \in J^\bullet$ (*with* $\|\mu\| = \|\mu'\|$ *and* $\|\nu\| = \|\nu'\|$), *then* $\max(\mu' \circledcirc \nu') \leq \max(\mu \circledcirc \nu)$. *If any of the former three inequalities is strict*, *then the last inequality is strict*, *too.*

Proof. If $w \in {}^\theta \mathsf{D}_{\|\mu\|_\theta, \|\nu\|}$, then the condition in the hypothesis forces $w \cdot \mu'\nu'$ to be smaller than or equal to $w \cdot \mu\nu$. ☐

A word $\nu \in J^\bullet$ is called *good* if $\nu = \max(\Psi(x))$ for some homogeneous $x \in \mathbf{f}$. Let $J^\bullet_+$ denote the set of good words and $J^\beta_+ = J^\bullet_+ \cap J^\beta$. We now define the analogue of good words for quantum shuffle modules.

Definition 3.11. A word $\nu \in {}^\theta J^\bullet$ is called θ-*good* if $\nu = \max({}^\theta\Psi(u))$ for some homogeneous $u \in {}^\theta V(\lambda)$. Let ${}^\theta J^\bullet_+$ denote the set of all θ-good words, and let ${}^\theta J^\beta_+ = {}^\theta J^\bullet_+ \cap {}^\theta J^\beta$.

In [Leclerc 2004], a *monomial* basis $\{\mathbf{m}_\nu = \Psi(f_{\sigma(\nu)}) \mid \nu \in J_+^\bullet\}$ of $\mathbf{U}$ was constructed. An analogous basis exists for ${}^\theta\mathbf{V}(\lambda)$.

Lemma 3.12. *There is a unique basis of homogeneous vectors* $\{{}^\theta\mathbf{m}_\nu^* \mid \nu \in {}^\theta J_+^\bullet\}$ *of* ${}^\theta\mathbf{V}(\lambda)$ *such that* $\mathbf{E}_\mu({}^\theta\mathbf{m}_\nu) = \delta_{\mu,\nu}$ *for any* μ *with* ${}^\theta|\mu| = {}^\theta|\nu|$. *The adjoint basis is* $\{{}^\theta\mathbf{m}_\nu = {}^\theta\Psi(F_{\sigma(\nu)} \cdot v_\lambda)\}$.

Proof. The proof is analogous to the proof of [Leclerc 2004, Proposition 12]. □

Let $\mathcal{F}^{\mathrm{fr}}$ be the free associative $\mathcal{K}$-algebra generated by J (with multiplication given by concatenation of letters), and let V^{fr} be its right regular representation. There is an algebra homomorphism

$$\Xi\colon \mathcal{F}^{\mathrm{fr}} \to \mathcal{F}, \quad \nu = \nu_1 \cdots \nu_k \mapsto \nu_1 \circ \cdots \circ \nu_k = \Psi(f_\nu)$$

and a linear map

$${}^\theta\Xi_\lambda\colon V^{\mathrm{fr}} \to {}^\theta\mathbf{V}(\lambda), \quad \nu \mapsto \varnothing \circledcirc \Xi(\nu) = {}^\theta\mathbf{m}_\nu.$$

intertwining the actions of $\mathcal{F}^{\mathrm{fr}}$ and $\mathcal{F}$. We have the following characterization of θ-good words:

Lemma 3.13. *The following are equivalent*:

(1) $\nu \in {}^\theta J^\bullet$ *is* θ*-good*,

(2) ν *cannot be expressed modulo* $\ker {}^\theta\Xi_\lambda$ *as a linear combination of words* $\mu > \nu$.

Proof. Let $u \in {}^\theta\mathbf{V}(\lambda)$ and $\nu \in {}^\theta J^\bullet$ satisfy ${}^\theta|u| = {}^\theta|\nu|$ and $\mathbf{E}_\nu(u) \neq 0$. Proposition 3.2 (3) implies that $0 \neq (\mathbf{E}_\nu(u), \varnothing) = (u, {}^\theta\mathbf{m}_\nu)$. If ν could be expressed modulo $\ker {}^\theta\Xi_\lambda$ as a linear combination of words $\mu > \nu$, then there would exist a relation of the form

$$ {}^\theta\mathbf{m}_\nu = \sum_{\mu > \nu} c_\mu {}^\theta\mathbf{m}_\mu \tag{3-11}$$

for some $c_\nu \in \mathcal{K}$. Hence,

$$0 \neq \mathbf{E}_\nu(u) = \sum_{\mu > \nu} c_\mu \mathbf{E}_\mu(u).$$

Therefore, $\mathbf{E}_\mu(u) \neq 0$ for some $\mu > \nu$, which implies that μ is not θ-good. This proves the implication (1) $\Longrightarrow$ (2).

Conversely, let ${}^\theta\tilde{J}_+^\bullet$ be the set of words in ${}^\theta J^\bullet$ satisfying (2). We have shown that ${}^\theta J_+^\bullet \subseteq {}^\theta\tilde{J}_+^\bullet$. Lemma 3.12 implies that the set $\{{}^\theta\mathbf{m}_\nu \mid \nu \in {}^\theta\tilde{J}_+^\bullet\}$ contains a basis of ${}^\theta\mathbf{V}(\lambda)$. Moreover, it is linearly independent. Indeed, if there was a linear relation between words of ${}^\theta\tilde{J}_+^\bullet$, one could express the smallest one in terms of the others and it would not belong to ${}^\theta\tilde{J}_+^\bullet$. □

Lemma 3.14. *The* θ*-good words have the following properties*:

(1) *If* ν *is* θ*-good and* $\nu = \mu_1\mu_2$, *then* μ_1 *is* θ*-good.*

(2) *If* ν *is* θ*-good*, *then* ν *is good.*

Proof. By Proposition 3.9, ${}^{\theta}\mathbf{V}(\lambda)$ is stable under the operators $\mathbf{E}_i$. Pick $u \in {}^{\theta}\mathbf{V}(\lambda)$ with $\max(u) = \nu$. Then $\max(\mathbf{E}_{\mu_2}(u)) = \mathbf{E}_{\mu_2}(\max(u)) = \mu_1$. This proves the first part. Next, suppose that ν is not good. Then, by [Leclerc 2004, Lemma 21], we have a relation of the form $\mathbf{m}_\nu = \sum_{\mu>\nu} c_\mu \mathbf{m}_\mu$. Applying both sides to $\varnothing$, we get (3-11). Hence, by Lemma 3.13, ν is not θ-good. This proves the second part. □

3G. *Lyndon words.* A nontrivial word $\nu \in J^\bullet$ is called *Lyndon* if it is smaller than all its proper left factors. Note that our definition uses the opposite of the convention of [Leclerc 2004; Kleshchev and Ram 2011], where right factors are used instead. Let $\mathscr{L}$ denote the set of Lyndon words and $\mathscr{L}_+ = \mathscr{L} \cap J_+^\bullet$ the set of good Lyndon words.

Proposition 3.15. *Lyndon words have the following properties*:

(1) *Every word $\nu \in J^\bullet$ has a unique factorization $\nu = \nu^{\langle k\rangle} \cdots \nu^{\langle 1\rangle}$ into Lyndon words such that $\nu^{\langle 1\rangle} \geq \cdots \geq \nu^{\langle k\rangle}$.*

(2) *The word ν is good if and only if each $\nu^{\langle m\rangle}$ is good.*

(3) *The map $\nu \mapsto |\nu|$ yields a bijection $\mathscr{L}_+ \xrightarrow{\sim} \Phi^+$. The induced order on Φ^+ is convex.*

(4) *Let $\mu \in \mathscr{L}\backslash J$ and write $\mu = \mu_{(1)}\mu_{(2)}$ with $\mu_{(2)}$ a proper Lyndon subword of maximal length. Then $\mu_{(1)} \in \mathscr{L}$.*

Proof. For part (1), see, e.g., [Lothaire 2002, Theorem 11.5.1]. For parts (2) and (3), see [Leclerc 2004, Propositions 17, 18 and 26]. For part (4), see [Leclerc 2004, Lemma 14]. □

We call the factorization from Proposition 3.15 (1) the *Lyndon factorization* and the Lyndon words in this factorization *Lyndon factors*. We will write it in two ways: $\nu = \nu^{\langle k\rangle} \cdots \nu^{\langle 1\rangle}$ for $\nu^{\langle 1\rangle} \geq \cdots \geq \nu^{\langle k\rangle}$ or $\nu = (\nu^{\langle l\rangle})^{n_l} \cdots (\nu^{\langle 1\rangle})^{n_1}$ for $\nu^{\langle 1\rangle} > \cdots > \nu^{\langle l\rangle}$. The factorization from Proposition 3.15 (4) is called the *standard factorization* of a Lyndon word.

Given $x, y \in \mathscr{F}$, let $[x, y]_q = xy - q^{|x|\cdot|y|}yx$. One defines a map $[\ ]\colon \mathscr{L} \to J^\bullet$ by induction on the standard factorization: $[i] = i$ for $i \in J$, and $[\nu] = [\nu_{(2)}, \nu_{(1)}]_q$ if $\nu = \nu_{(1)}\nu_{(2)}$ is the standard factorization of ν. Next, given $\nu = \nu^{\langle k\rangle} \cdots \nu^{\langle 1\rangle} \in J^\bullet$, let $[\nu] = [\nu^{\langle k\rangle}] \cdots [\nu^{\langle 1\rangle}]$. For $\nu \in J_+^\bullet$, set

$$\mathbf{l}_\nu = \Xi([\nu]), \quad \nu \in J_+^\bullet, \qquad {}^{\theta}\mathbf{l}_\nu = {}^{\theta}\Xi_\lambda([\nu]), \quad \nu \in {}^{\theta}J_+^\bullet.$$

Proposition 3.16. *For any $\nu \in J^\bullet$, we have $\min([\nu]) = \nu$. Moreover, the set $\{\mathbf{l}_\nu \mid \nu \in J_+^\bullet\}$ is a basis of $\mathbf{U}$.*

Proof. See [Leclerc 2004, Propositions 19 and 22]. □

The basis from Proposition 3.16 is called the *Lyndon basis*.

Lemma 3.17. *The set* $\{{}^\theta\mathbf{l}_\nu \mid \nu \in {}^\theta J_+^\bullet\}$ *is a basis of* ${}^\theta\mathbf{V}(\lambda)$. *Moreover*, *the transition matrix* $(c_{\nu\mu})$ *from* $\{{}^\theta\mathbf{l}_\nu \mid \nu \in {}^\theta J_+^\bullet\}$ *to* $\{{}^\theta\mathbf{m}_\mu \mid \mu \in {}^\theta J_+^\bullet\}$ *is triangular with* $c_{\nu\nu} = \prod_{i=1}^k (-1)^{\ell(\nu^{\langle k\rangle})-1} q^{-N(|\nu^{\langle k\rangle}|)}$.

Proof. By Proposition 3.16, we can write $[\nu] = c_{\nu\nu}\nu + \sum_{\nu<\mu} c_{\nu\mu}\mu$, for some $c_{\nu\mu} \in \mathcal{K}$. Applying ${}^\theta\Xi_\lambda$ to both sides, we get ${}^\theta\mathbf{l}_\nu = c_{\nu\nu}{}^\theta\mathbf{m}_\nu + \sum_{\mu>\nu} c_{\nu\mu}{}^\theta\mathbf{m}_\mu$. By Lemma 3.13, this can be rewritten as ${}^\theta\mathbf{l}_\nu = c_{\nu\nu}{}^\theta\mathbf{m}_\nu + \sum_{\nu<\mu\in{}^\theta J_+^\bullet} c'_{\nu\mu}{}^\theta\mathbf{m}_\mu$. Hence the transition matrix is triangular. To show the last statement of the lemma, one uses the same calculation as in [Leclerc 2004, Proposition 30]. □

Assumption 1. From now on, we assume that we are working with the standard ordering of J, i.e., $\alpha_k \le \alpha_l$ if and only if $k \le l$. In this case, the map ${}^\theta\sigma$ in (3-8) preserves $\mathcal{L}_+$.

Before stating the next lemma, we need to introduce some notation. Given $\mu, \mu' \in \mathcal{L}_+$ with $|\mu| = \beta_{k,l}$, $|\mu'| = \beta_{m,n}$, we write

$$\mu \subset \mu' \quad \Longleftrightarrow \quad m < k \text{ and } l < n.$$

Lemma 3.18. *The following hold*:

(1) *If* $\nu \in \mathcal{L}_+$, *then* $\mathbf{l}_\nu$ *is a multiple of* ν.

(2) *If* $\nu, \mu \in \mathcal{L}_+$ *and* $\mu \subset \nu$, *then* $\nu \circ \mu = \mu \circ \nu$.

Proof. It suffices to prove the first statement for $\nu = \nu_1 \cdots \nu_l \in \mathcal{L}_+$. We proceed by induction on l. The base case $l = 1$ is clear. Let $\nu = \nu_{(1)}\nu_{(2)}$ be the standard factorization of ν. Since we are working with the standard ordering on J, $\nu_{(1)} = i$ for some $i \in J$. By induction, we get that $\mathbf{l}_\nu = \Xi([\nu]) = \Xi([\nu_{(2)}]) \circ i - q^{-1} i \circ \Xi([\nu_{(2)}])$ is a multiple of $\nu_{(2)} \circ i - q^{-1} i \circ \nu_{(2)}$. Write $\nu_{(2)} = j\nu'_{(2)}$ with $j \in J$. Then (3-5) implies that $\nu_{(2)} \circ i - q^{-1} i \circ \nu_{(2)} = \big(j(\nu'_{(2)} \circ i) + qi\nu_{(2)}\big) - q^{-1}\big(i\nu_{(2)} + qj(i \circ \nu'_{(2)})\big) = [2]\nu$. This completes the proof of the first statement. The second statement now follows directly from [Leclerc 2004, Proposition 30] and [Enomoto and Kashiwara 2008, Proposition 3.14 (3)]. □

Definition 3.19. We say that $\nu \in \mathcal{L}$ is *θ-Lyndon* if $\nu \ge {}^\theta w(\nu)$. Let ${}^\theta\mathcal{L}$ be the set of θ-Lyndon words, and ${}^\theta\mathcal{L}_+ = J_+^\bullet \cap {}^\theta\mathcal{L}$. Let ${}^\theta J_{+,0}^\bullet$ denote the set of all θ-good words $\mu = \nu^{\langle k\rangle} \cdots \nu^{\langle 1\rangle}$, with $\nu^{\langle k\rangle}, \cdots, \nu^{\langle 1\rangle} \in {}^\theta\mathcal{L}_+$. Moreover, if $\mu = \nu^{\langle k\rangle} \cdots \nu^{\langle 1\rangle} \in {}^\theta J_+^\bullet$ and $\nu^{\langle k\rangle}, \cdots, \nu^{\langle 1\rangle} \notin {}^\theta\mathcal{L}_+$, then μ is called *θ-cuspidal*. Let ${}^\theta J_{+,c}^\bullet$ denote the set of all θ-cuspidal words.

Lemma 3.20. *The θ-good Lyndon words have the following properties*:

(1) *If* $\nu \in \mathcal{L}_+$, *then* $\nu \in \mathbf{U}$.

(2) *Let* $\mu \in {}^\theta J^\bullet$ *and* $\nu \in {}^\theta\mathcal{L}$ *with* $\nu \ge \mu$. *Then* $\mu\nu = \max(\mu \circledcirc \nu)$.

(3) ${}^\theta\mathcal{L}_+ \subseteq \mathcal{L} \cap {}^\theta J_+^\bullet$.

(4) *Let* $\mu \in {}^{\theta}J_+^{\bullet}$ *and* $\nu \in {}^{\theta}\mathscr{L}_+$ *with* $\nu \geq \mu$. *Then* $\mu\nu \in {}^{\theta}J_+^{\bullet}$.

(5) *If all of the Lyndon factors of* ν *are in* ${}^{\theta}\mathscr{L}_+$, *then* $\nu \in {}^{\theta}J_+^{\bullet}$.

(6) *The map* $\nu \mapsto {}^{\theta}|\nu|$ *yields a bijection* ${}^{\theta}\mathscr{L}_+ \xrightarrow{\sim} {}^{\theta}\Phi^+$.

Proof. Since ν is good, there exists some homogeneous $x \in \mathbf{U}$ such that $x = \nu + y$ with ν greater than any word μ in y. By Assumption 1 and [Leclerc 2004, §8.1], ν is of the form $\alpha_k\alpha_{k-2}\cdots\alpha_{k-2l}$, which implies that ν is the smallest word of weight $|\nu|$, so $x = \nu$. The proof of (2) is similar to the proof of [Leclerc 2004, Lemma 15]. If $\nu \in {}^{\theta}\mathscr{L}_+$, then, by definition, $\nu \in \mathscr{L}_+$ and $\nu \geq {}^{\theta}w(\nu)$. Hence, $\max(\varnothing \circledcirc \nu) = \nu$. By part (1), $\nu \in \mathbf{U}$, so $\nu \in {}^{\theta}J_+^{\bullet}$. This proves (3).

Let us prove (4). If $\mu = \varnothing$, then the statement reduces to (3). Otherwise, choose a homogeneous element $\varnothing \neq x \in {}^{\theta}\mathbf{V}(\lambda)$ such that $\mu = \max(x)$. Then, after possible rescaling, $x = \mu + r$, where r is a linear combination of words $< \mu$. We have $x \circledcirc \nu = \mu \circledcirc \nu + r \circledcirc \nu$. Part (2) implies that $\max(\mu \circledcirc \nu) = \mu\nu$. It follows from Lemma 3.10 that $\max(\mu \circledcirc \nu) > \max(r \circledcirc \nu)$.

Next, we prove (5). Suppose that each factor of $\nu = \nu^{\langle k\rangle}\cdots\nu^{\langle 1\rangle}$ is θ-Lyndon. If $k = 1$, then ν is θ-good by (3). By induction on the number of Lyndon factors, we can assume that $\nu' = \nu^{\langle k\rangle}\cdots\nu^{\langle 2\rangle}$ is θ-good. The statement now follows from (4). Part (6) is clear from the definitions. $\square$

Given $\nu = \nu^{\langle s\rangle}\cdots\nu^{\langle 1\rangle},\ \nu' = \nu^{\langle t\rangle}\cdots\nu^{\langle s+1\rangle} \in J_+^{\bullet}$, let $\mathrm{sh}(\nu, \nu') = \mu^{\langle t\rangle}\cdots\mu^{\langle 1\rangle}$ be the good word obtained by shuffling the Lyndon factors of ν and ν' in such a way that $\mu^{\langle t\rangle} \leq \cdots \leq \mu^{\langle 1\rangle}$.

Lemma 3.21. *The map*

$$ {}^{\theta}J_{+,c}^{\bullet} \times {}^{\theta}J_{+,0}^{\bullet} \to {}^{\theta}J_+^{\bullet}, \quad (\nu, \nu') \mapsto \mathrm{sh}(\nu, \nu'), $$

is a well-defined injection.

Proof. It is clear the map is injective, so we only have to show that $\mathrm{sh}(\nu, \nu')$ is θ-good. We argue by induction on the number k of Lyndon factors in $\nu' = \nu^{\langle k\rangle}\cdots\nu^{\langle 1\rangle}$. If $k = 0$, then ν is θ-good by assumption. Otherwise, letting $\nu'' = \nu^{\langle k\rangle}\cdots\nu^{\langle 2\rangle}$, we can assume that $\mathrm{sh}(\nu, \nu'')$ is θ-good. If $\nu^{\langle 1\rangle} \geq \mathrm{sh}(\nu, \nu')$, then $\mathrm{sh}(\nu, \nu') = \mathrm{sh}(\nu, \nu'')\nu^{\langle 1\rangle}$, and we conclude that $\mathrm{sh}(\nu, \nu') \in {}^{\theta}J_+^{\bullet}$ from Lemma 3.20 (4).

If $\nu^{\langle 1\rangle} < \mathrm{sh}(\nu, \nu')$, then we require the following generalization of Lemma 3.20 (4): given $a \in {}^{\theta}J_+^{\bullet}$ and $b \in {}^{\theta}\mathscr{L}_+$ with $b < a$, we have $\mathrm{sh}(a, b) \in {}^{\theta}J_+^{\bullet}$. The old proof carries over except that instead of invoking Lemma 3.20 (2), we need to show that $\max(a \circledcirc b) = \mathrm{sh}(a, b)$. Without loss of generality, we may assume a is Lyndon. Since $b \geq {}^{\theta}w(b)$, we have $\max(a \circledcirc b) = \max(a \circ b)$. Let us write $a = a_n\cdots a_1$ and $b = b_m\cdots b_1$. Since $a_n \geq \cdots \geq a_1 > b_1$, it follows that $\max(a \circ b) = ba$. $\square$

Given $\beta \in \mathrm{Q}_+^{\theta}$, let ${}^{\theta}\mathrm{kpf}(\beta)$ denote the number of ways to write β as a sum of roots in ${}^{\theta}\Phi^+$.

Proposition 3.22. *If* $\lambda = 0$, *then*: (*i*) ${}^\theta\mathscr{L}_+ = \mathscr{L} \cap {}^\theta J_+^\bullet$, *and* (*ii*) ${}^\theta J_+^\bullet = {}^\theta J_{+,0}^\bullet$. *Hence*, $\dim_q {}^\theta\mathbf{V}_\beta = {}^\theta\mathrm{kpf}(\beta)$.

Proof. Let S be the set of all words $\nu = \nu^{\langle k\rangle} \cdots \nu^{\langle 1\rangle}$ with $\nu^{\langle 1\rangle} \geq \cdots \geq \nu^{\langle k\rangle}$ and each $\nu^{\langle i\rangle} \in {}^\theta J_+^\bullet$. Lemma 3.12 and Lemma 3.20 (5) imply that $\{{}^\theta\mathbf{m}_\nu \mid \nu \in S\}$ is contained in the monomial basis $\{{}^\theta\mathbf{m}_\nu \mid \nu \in {}^\theta J_+^\bullet\}$ of ${}^\theta\mathbf{V}$. Let ${}^\theta\mathbf{V}' \subseteq {}^\theta\mathbf{V}$ be the span of the former. By construction, the generating series of the dimensions of the homogeneous components of ${}^\theta\mathbf{V}'$ is equal to $\prod_{\beta\in{}^\theta\Phi^+} 1/(1-\exp\beta)$. On the other hand, it follows from [Enomoto and Kashiwara 2008, Theorem 4.15] that this is also the generating series of the dimensions of the homogeneous components of ${}^\theta\mathbf{V}$. Hence, ${}^\theta\mathbf{V}' = {}^\theta\mathbf{V}$. The statement follows. □

Remark 3.23. Instead of appealing to [Enomoto and Kashiwara 2008, Theorem 4.15] in the proof of Proposition 3.22, one could alternatively use the categorification theorem [Varagnolo and Vasserot 2011, Theorem 8.31] (cited as Theorem 4.5 below), together with the geometric realization of orientifold KLR algebras from [Varagnolo and Vasserot 2011] and the classification of isomorphism classes of symplectic/orthogonal representations of symmetric quivers from [Derksen and Weyman 2002]. Indeed, this approach appears promising in generalizing the construction of bases for ${}^\theta\mathbf{V}(\lambda)$ to the $\lambda \neq 0$ case.

3H. *Symmetric words.* A word $\nu \in {}^\theta\mathscr{L}_+$ is called *symmetric* if ${}^\theta w(\nu) = \nu$ and *nonsymmetric* otherwise. Given $\nu \in {}^\theta J_+^\bullet$, let ν_θ be the word obtained from ν by deleting its symmetric Lyndon factors and ν^θ the word obtained by deleting the nonsymmetric ones. We say that $\nu \in {}^\theta J_+^\bullet$ is *symmetric* if $\nu = \nu^\theta$. For each $k \geq 1$, let ξ_k be the unique symmetric word in ${}^\theta\mathscr{L}_+$ with $|\xi_k| = \beta_{-2k+1,2k-1}$.

Lemma 3.24. *Let* $\nu \in {}^\theta\mathscr{L}_+$. *If* $\nu < \xi_k$, *then* ξ_{k+1} *is a subword of* ν. *Hence*, $\xi_k > \xi_l$ *if and only if* $k < l$.

Proof. The statement follows immediately from Lemma 3.20 (6). □

Assumption 2. From now until the end of Section 3, we assume that $\lambda = 0$. We abbreviate ${}^\theta\mathscr{F} = {}^\theta\mathscr{F}(0)$ and ${}^\theta V = {}^\theta V(0)$.

Lemma 3.25. *Suppose that* $\nu \in {}^\theta J_+^\bullet$ *is symmetric or* $\nu \in {}^\theta\mathscr{L}_+$. *Then* ν *is the smallest word in* ${}^\theta J_+^{{}^\theta|\nu|}$.

Proof. Abbreviate $\beta = {}^\theta|\nu|$. First assume that $\nu \in {}^\theta J_+^\bullet$ is symmetric. Let $\nu = \nu^{\langle k\rangle} \cdots \nu^{\langle 1\rangle}$ be its Lyndon factorization. Suppose that there exists a word $\mu = \mu^{\langle l\rangle} \cdots \mu^{\langle 1\rangle} \in {}^\theta J_+^{{}^\theta|\nu|}$ with $\mu < \nu$. Then, as explained before Lemma 4.1 in [Melançon 1992], there is an a such that $\mu^{\langle b\rangle} = \nu^{\langle b\rangle}$ for $b < a$ and $\mu^{\langle a\rangle} < \nu^{\langle a\rangle}$. Hence, $\nu^{\langle a\rangle} > \mu^{\langle a\rangle} \geq \cdots \geq \mu^{\langle l\rangle}$. Write $\bar\nu = \nu^{\langle k\rangle} \cdots \nu^{\langle a\rangle}$ and $\bar\mu = \mu^{\langle l\rangle} \cdots \mu^{\langle a\rangle}$.

Since $\nu^{\langle a\rangle}$ is symmetric, we have $\nu^{\langle a\rangle} = \xi_d$ for some $d \geq 1$. By Proposition 3.22 and Lemma 3.24, ξ_{d+1} is a subword of each $\mu^{\langle i\rangle}$, where $i \geq a$. In particular, each $\mu^{\langle i\rangle}$ contains $\alpha_{\pm(2d-1)}$ and $\alpha_{\pm(2d+1)}$. Hence, if we write ${}^{\theta}|\bar{\nu}| = {}^{\theta}|\bar{\mu}| = \sum_{i\in\mathbb{N}_{\text{odd}}} c_i(\alpha_i + \alpha_{-i})$, then $c_{2d+1} = c_{2d-1}$. On the other hand, since each $\nu^{\langle i\rangle}$, where $i \geq a$, is a symmetric good Lyndon word smaller than $\nu^{\langle a\rangle}$, Lemma 3.24 implies that each $\nu^{\langle i\rangle}$ contains $\nu^{\langle a\rangle}$ as a subword. Hence $c_{2d+1} < c_{2d-1}$, which is a contradiction.

Secondly, assume that $\nu \in {}^{\theta}\mathscr{L}_+$. We may assume ν is not symmetric. In that case, observe that if ${}^{\theta}|\mu| = {}^{\theta}|\nu|$ for some $\mu \in {}^{\theta}J_+^{\bullet}$, then $|\mu| = |\nu|$. The result now follows from [Kleshchev and Ram 2011, Lemma 5.9]. □

3I. ***PBW and canonical bases.*** Let us first recall some basic facts about PBW bases. For the moment let us restrict $(J, \cdot)$ to a finite Cartan subdatum of type A_m. By [Leclerc 2004, Proposition 26], the antilexicographic order $\nu^{\langle 1\rangle} > \cdots > \nu^{\langle N\rangle}$ on the set of good Lyndon words induces, via the bijection from Proposition 3.15 (3), a convex order $\beta_1 > \cdots > \beta_N$ on the set of positive roots. This convex order arises from a unique reduced decomposition $w_0 = s_{i_N} \cdots s_{i_1}$ in the usual way: $\beta_N = \alpha_{i_N}$, $\beta_{N-1} = s_{i_N}(\alpha_{i_{N-1}})$, …, $\beta_1 = s_{i_N} \cdots s_{i_2}(\alpha_1)$. Let $P_{\nu^{\langle k\rangle}} = T''_{i_N,1} \cdots T''_{i_{k+1},1}(f_{i_k})$, where $T''_{i,1}$ is the braid group operation from [Lusztig 1993, §37.1] with $e = -1$ and $\upsilon_i = q$. Set $P^{(l)}_{\nu^{\langle k\rangle}} = (1/[l]!)P^l_\nu$ and, given $\nu = (\nu^{\langle N\rangle})^{l_N} \cdots (\nu^{\langle 1\rangle})^{l_1} \in J_+^{\bullet}$, let $P_\nu = P^{(l_N)}_{\nu^{\langle N\rangle}} \cdots P^{(l_1)}_{\nu^{\langle 1\rangle}}$ and $\mathbf{P}_\nu = \Psi(P_\nu)$. Taking an appropriate limit $m \to \infty$, [Lusztig 1993, Proposition 41.1.4] implies that $\{P_\nu \mid \nu \in J_+^{\bullet}\}$ is an $\mathscr{A}$-basis of $\mathbf{f}_{\mathscr{A}}$.

Next, given $\nu \in {}^{\theta}\mathscr{L}_+$, let

$$P_\nu^{[n]} = \begin{cases} P_\nu^{(n)}, & \text{if } \nu \text{ is not symmetric,} \\ \dfrac{1}{[2n]!!} P_\nu^n, & \text{if } \nu \text{ is symmetric.} \end{cases}$$

Given $\nu = (\nu^{\langle l\rangle})^{n_l} \cdots (\nu^{\langle 1\rangle})^{n_1} \in {}^{\theta}J_+^{\bullet}$, define

$${}^{\theta}P_\nu = \sigma\Big(\prod_{1\leq i\leq l} P^{[n_i]}_{\nu^{\langle i\rangle}}\Big) \cdot v_0 \quad \text{and} \quad {}^{\theta}\mathbf{P}_\nu = {}^{\theta}\Psi(P_\nu).$$

Proposition 3.26. *The set $\{{}^{\theta}P_\nu \mid \nu \in {}^{\theta}J_+^{\bullet}\}$ is an $\mathscr{A}$-basis of ${}^{\theta}V_{\mathscr{A}}^{\text{low}}$.*

Proof. See [Enomoto and Kashiwara 2008, Lemma 5.1]. Note that the weaker statement that $\{{}^{\theta}P_\mu\}$ is a $\mathscr{K}$-basis of ${}^{\theta}V_{\mathscr{A}}^{\text{low}}$ follows from Lemma 3.17 and Lemma 3.27 (1) below. □

We call $\{{}^{\theta}\mathbf{P}_\nu \mid \nu \in {}^{\theta}J_+^{\bullet}\}$ the PBW basis of ${}^{\theta}\mathbf{V}_{\mathscr{A}}^{\text{low}}$. By [Leclerc 2004, Proposition 30], for any $\nu \in J_+^{\bullet}$, there exists $\kappa_\nu = \overline{\kappa_\nu} \in \mathscr{A}$ with $\mathbf{l}_\nu = \kappa_\nu \mathbf{P}_\nu$. Since we are working with the standard ordering of J, [Leclerc 2004, Proposition 56] implies that $\kappa_\nu = 1$ for any $\nu \in \mathscr{L}_+$. If $\nu = (\nu^{\langle l\rangle})^{n_l} \cdots (\nu^{\langle 1\rangle})^{n_1} \in {}^{\theta}J_+^{\bullet}$, then $\kappa_\nu = \prod_{i=1}^{l}[n_i]!$. Set

$${}^{\theta}\kappa_\nu = \kappa_\nu \cdot \prod_{\substack{i=1 \\ \nu^{\langle i\rangle}\text{symm}}}^{l} \prod_{j=1}^{n_i} (q^j + q^{-j}) = \prod_{\substack{i=1, \\ \nu^{\langle i\rangle}\text{symm}}}^{l} [n_i]!! \cdot \prod_{\substack{i=1, \\ \nu^{\langle i\rangle}\text{nonsymm}}}^{l} [n_i]!.$$

Lemma 3.27. *Let* $\nu \in {}^\theta J^\bullet_+$.

(1) ${}^\theta\mathbf{l}_\nu = {}^\theta\kappa_\nu {}^\theta\mathbf{P}_\nu$ *and* ${}^\theta\kappa_\nu = \overline{{}^\theta\kappa_\nu} \in \mathscr{A}$.

(2) *We have*

$$\overline{{}^\theta\mathbf{P}_\nu} = {}^\theta\mathbf{P}_\nu + \sum_{\mu>\nu} d_{\nu\mu} {}^\theta\mathbf{P}_\mu$$

for some $d_{\nu\mu} \in \mathscr{A}$.

Proof. The first part follows directly from the definitions. Let $A_{\mathbf{P}}$, $A_{\mathbf{m}}$ and A be the transition matrices between $\{{}^\theta\mathbf{P}_\nu\}$ and $\{\overline{{}^\theta\mathbf{P}_\nu}\}$, $\{\mathbf{m}_\nu\}$ and $\{\overline{\mathbf{m}_\nu}\}$, as well as $\{{}^\theta\mathbf{P}_\nu\}$ and $\{\mathbf{m}_\nu\}$, respectively. By definition, $A_{\mathbf{m}} = \mathrm{id}$. Hence, $A_{\mathbf{P}} = \bar{A}A^{-1}$. Lemma 3.17 implies that $\bar{A}$ and A^{-1} are both lower triangular, with eigenvalues $\overline{{}^\theta\kappa_\nu}$ and ${}^\theta\kappa_\nu^{-1}$. Part (1) now implies that $A_{\mathbf{P}}$ is indeed lower unitriangular. Since $\{{}^\theta\mathbf{P}_\nu\}$ forms an $\mathscr{A}$-basis of ${}^\theta\mathbf{V}^{\mathrm{low}}_{\mathscr{A}}$ and $\overline{{}^\theta\mathbf{V}^{\mathrm{low}}_{\mathscr{A}}} = {}^\theta\mathbf{V}^{\mathrm{low}}_{\mathscr{A}}$, we have $d_{\nu\mu} \in \mathscr{A}$. □

Theorem 3.28. *There is a unique* $\mathscr{A}$*-basis* $\{{}^\theta\mathbf{b}_\nu \mid \nu \in {}^\theta J^\bullet_+\}$ *of* ${}^\theta\mathbf{V}^{\mathrm{low}}_{\mathscr{A}}$, *called the* ***canonical basis***, *such that*

$$ {}^\theta\mathbf{b}_\nu = {}^\theta\mathbf{P}_\nu + \sum_{\mu>\nu} c_{\nu\mu} {}^\theta\mathbf{P}_\mu, $$

$c_{\nu\mu} \in q\mathbb{Z}[q]$ *and* $\overline{{}^\theta\mathbf{b}_\nu} = {}^\theta\mathbf{b}_\nu$. *Moreover*,

$$({}^\theta\mathbf{b}_\nu, {}^\theta\mathbf{b}_\mu)_{q=0} = \delta_{\nu,\mu}.$$

Proof. The proof is an application of a standard argument, see, e.g., [Lusztig 1990, §7.10]. □

Remark 3.29. Theorem 3.28 also appears in [Enomoto and Kashiwara 2008] as Theorem 5.5. The proof in *loc. cit.* is somewhat different from ours, in particular, it does not involve shuffle modules.

Let $\{{}^\theta\mathbf{P}^*_\nu \mid \nu \in {}^\theta J^\bullet_+\}$ and $\{{}^\theta\mathbf{b}^*_\nu \mid \nu \in {}^\theta J^\bullet_+\}$ be the bases of ${}^\theta\mathbf{V}^{\mathrm{up}}_{\mathscr{A}}$ dual, with respect to the bilinear form $(\cdot\,,\cdot)$, to the PBW and the canonical bases of ${}^\theta\mathbf{V}^{\mathrm{low}}_{\mathscr{A}}$, respectively.

Corollary 3.30. *We have*

$$ {}^\theta\mathbf{b}^*_\nu = {}^\theta\mathbf{P}^*_\nu + \sum_{\mu<\nu} \big({}^\theta\mathbf{b}^*_\nu, {}^\theta\mathbf{P}_\mu\big) {}^\theta\mathbf{P}^*_\mu. \tag{3-12}$$

Hence, $\max({}^\theta\mathbf{b}^*_\nu) = \nu$ *and the coefficient of* ν *in* ${}^\theta\mathbf{b}^*_\nu$ *is* ${}^\theta\kappa_\nu$. *In particular*, *if* $\nu \in {}^\theta\mathscr{L}_+$ *or* ν *is symmetric*, *then* ${}^\theta\mathbf{b}^*_\nu = {}^\theta\mathbf{P}^*_\nu$.

Proof. The proof is analogous to [Leclerc 2004, Proposition 40]. The last statement follows from Lemma 3.25. □

3J. ***Standard and costandard basis.*** Given $\nu = (\nu^{\langle l\rangle})^{n_l} \cdots (\nu^{\langle 1\rangle})^{n_1} \in {}^\theta J^\bullet_+$, let

$$\Delta_\nu = q^{-s(\nu)} (\nu^{\langle l\rangle})^{\circ n_l} \circ \cdots \circ (\nu^{\langle 1\rangle})^{\circ n_1} \quad \text{and} \quad {}^\theta\Delta_\nu = q^{-{}^\theta s(\nu)} \varnothing \circledcirc \Delta_\nu,$$

where

$$s(\nu) = \sum_{i=1}^{l} \frac{n_i(n_i-1)}{2} \quad \text{and} \quad {}^\theta s(\nu) = \sum_{\substack{i=1,\\ \nu^{\langle i\rangle}\text{symm}}}^{l} n_i. \tag{3-13}$$

Lemma 3.31. *If* $\nu \in {}^\theta J^\bullet_+$, *then*: $\Delta_\nu = \Delta_{\nu^\theta} \circ \Delta_{\nu_\theta}$, $\max({}^\theta\Delta_\nu) = \nu$ *and the coefficient of the word* ν *in* ${}^\theta\Delta_\nu$ *equals* ${}^\theta\kappa_\nu$.

Proof. We prove the first statement by induction on the number k of Lyndon factors in the Lyndon factorization of ν^θ. If $k = 0$, the claim is obvious. Next, suppose that there are $k+1$ Lyndon factors in ν^θ, and let ξ_m be the smallest. If ξ_m is also the smallest word in the standard factorization of ν, then, by induction, we are done. Otherwise, let μ be a Lyndon factor of ν with $\mu < \xi_m$. Since $\mu \in {}^\theta\mathscr{L}_+$, Lemma 3.24 implies that $\xi_m \subset \mu$. By Lemma 3.18, we conclude that $\mu \circ \xi_m = \xi_m \circ \mu$. It now follows by induction that $\Delta_\nu = \Delta_{\nu^\theta} \circ \Delta_{\nu_\theta}$.

We now prove the last two statements by induction on the number k of Lyndon factors in ν. The base case $k = 0$ is trivial. Let $\nu' = \nu^{\langle k\rangle} \cdots \nu^{\langle 2\rangle}$. Lemma 3.14 implies that $\nu' \in {}^\theta J^\bullet_+$. Hence, by induction, $\max({}^\theta\Delta_{\nu'}) = \nu'$. Since $\lambda = 0$, we have $\nu^{\langle 1\rangle} \in {}^\theta\mathscr{L}_+$, and so $\nu^{\langle 1\rangle} \geq {}^\theta w(\nu^{\langle 1\rangle})$. It follows from Lemma 3.10 and Lemma 3.20 (2) that $\max({}^\theta\Delta_\nu) = \max(\nu' \circledcirc \nu^{\langle 1\rangle}) = \nu$. By induction, we may also assume that $\dim_q({}^\theta\Delta_{\nu'})_{\nu'} = {}^\theta\kappa_{\nu'}$. Let us call the result of applying $w \in {}^\theta\mathrm{D}_{\|\nu'\|_\theta, \|\nu^{\langle 1\rangle}\|}$ to ν a θ-shuffle. It is easy to see that the θ-shuffles equal to ν are precisely those arising from one of the n_1 (respectively, $2n_1$) standard insertions of $\nu^{\langle 1\rangle}$ between words equal to $\nu^{\langle 1\rangle}$ in ν' if $\nu^{\langle 1\rangle}$ is not symmetric (respectively, is symmetric). We conclude that $\dim_q({}^\theta\Delta_\nu)_\nu = {}^\theta\kappa_\nu$ from the fact that the transposition of two words equal to $\nu^{\langle 1\rangle}$ appears in the shuffle action with the coefficient q^{-2}. □

Given $\nu \in {}^\theta J^\bullet_+$ with $\nu = \nu^{\langle k\rangle} \cdots \nu^{\langle 1\rangle}$, let

$${}^\theta\nabla_\nu = q^{-{}^\theta s(\nu) - t(\nu)} \varnothing \circledcirc \big({}^\theta w(\nu^{\langle k\rangle}) \circ \cdots \circ {}^\theta w(\nu^{\langle 1\rangle})\big),$$

where $t(\nu)$ is the degree of an element τ_w, with w the longest minimal length coset representative with respect to the parabolic subgroup of $\mathfrak{W}_n$ defined by the decomposition of ν into Lyndon words (see [Lauda and Vazirani 2011, §2.3]).

Recall that we have fixed the standard order $\leq$ on J and equipped $J^\bullet$ with the antilexicographic order $\leq$. Let $\leq'$ denote both the opposite order on J and the induced lexicographic order on $J^\bullet$. Given a linear combination u of words, let $\max'(u)$ be the largest word appearing in u with respect to $\leq'$.

Lemma 3.32. *We have* $\max'({}^\theta\nabla_\nu) = \nu$ *and the coefficient of* ν *in* ${}^\theta\nabla_\nu$ *equals* ${}^\theta\kappa_\nu$.

Proof. It is an easy modification of the last paragraph in the proof of Lemma 3.31. □

4. Finite-dimensional representation theory of orientifold KLR algebras

We again let λ be arbitrary until Section 4D, where we make the restriction $\lambda = 0$.

If A is a graded algebra, let A-Mod be the category of all graded left A-modules, with degree-preserving module homomorphisms as morphisms. If M and N are graded A-modules, let $\mathrm{Hom}_A(M, N)_n$ denote the space of all homogeneous homomorphisms of degree n, and $\mathrm{HOM}_A(M, N) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_A(M, N)_n$. Let $M\{n\}$ denote the module obtained from M by shifting the grading by n. Let A-pMod denote the full subcategory of finitely generated graded projective modules, and A-fMod the full subcategory of graded finite dimensional modules. Given any of these abelian categories $\mathscr{C}$, we denote its Grothendieck group by $[\mathscr{C}]$.

We consider (orientifold) KLR algebras associated to the $\mathtt{A}_\infty$ quiver $\Gamma = (J, \Omega)$, with J as in Section 3A and Ω the standard linear orientation, as well as the involution θ from Section 3A. Let $\mathbb{1}\mathbb{1}$ and ${}^\theta\mathbb{1}$ denote the regular representations (in degree zero) of the trivial algebras $\mathscr{R}(0)$ and ${}^\theta\mathscr{R}(0; \lambda)$, respectively. For a fixed $\lambda \in \mathbb{N}[J]$, set

$$\mathscr{R}\text{-Mod} = \bigoplus_{\beta\in\mathbb{N}[J]} \mathscr{R}(\beta)\text{-Mod} \quad \text{and} \quad {}^\theta\mathscr{R}(\lambda)\text{-Mod} = \bigoplus_{\beta\in\mathbb{N}[J]^\theta} {}^\theta\mathscr{R}(\beta; \lambda)\text{-Mod}.$$

We use analogous notation for direct sums of categories of finite dimensional and finitely generated projective modules.

4A. ***Reminder on categorification via KLR algebras.*** Basic information about the representation theory of KLR algebras, including the definitions of the Khovanov–Lauda pairing $(\cdot\,,\cdot) : \mathscr{R}(\beta)\text{-pMod} \times \mathscr{R}(\beta)\text{-fMod} \to \mathscr{A}$ and the dualities $P \mapsto P^\sharp$ on $\mathscr{R}$-pMod and $M \mapsto M^\flat$ on $\mathscr{R}$-fMod, can be found in, e.g., [Khovanov and Lauda 2009], [Kleshchev and Ram 2011, §3] or [Varagnolo and Vasserot 2011, §7]. Since these definitions and the notations are standard, we will not explicitly recall them. If $M \in \mathscr{R}(\beta)$-Mod and $\nu \in J^\theta$, we call $M_\nu = e(\nu)M$ the ν-weight space of M.

Let us recall the definition of the convolution product of modules over KLR algebras. Let $\beta, \beta' \in \mathbb{N}[J]$, with $\|\beta\| = n$ and $\|\beta'\| = n'$. Set

$$e_{\beta,\beta'} = \sum_{\substack{\nu\in J^{\beta+\beta'},\\ \nu_1\cdots\nu_n\in J^\beta}} e(\nu) \in \mathscr{R}(\beta + \beta').$$

There is a nonunital algebra homomorphism

$$\iota_{\beta,\beta'} : \mathscr{R}(\beta, \beta') := \mathscr{R}(\beta) \otimes \mathscr{R}(\beta') \to \mathscr{R}(\beta + \beta') \tag{4-1}$$

given by $e(\nu) \otimes e(\mu) \mapsto e(\nu\mu)$ for $\nu \in J^\beta$, $\mu \in J^{\beta'}$ and

$$x_l \otimes 1 \mapsto x_l e_{\beta,\beta'}, \quad 1 \otimes x_{l'} \mapsto x_{m+l'} e_{\beta,\beta'}, \qquad \text{with } 1 \le l^{(\prime)} \le n^{(\prime)}, \tag{4-2}$$

$$\tau_k \otimes 1 \mapsto \tau_k e_{\beta,\beta'}, \quad 1 \otimes \tau_{k'} \mapsto \tau_{m+k'} e_{\beta,\beta'}, \qquad \text{with } 1 \le k^{(\prime)} < n^{(\prime)}. \tag{4-3}$$

Let M be a graded $\mathscr{R}(\beta)$-module and N be a graded $\mathscr{R}(\beta')$-module. Their *convolution product* is defined as

$$M \circ N = \mathscr{R}(\beta+\beta')e_{\beta,\beta'} \otimes_{\mathscr{R}(\beta,\beta')} (M \otimes N).$$

It descends to a product on [$\mathscr{R}$-pMod] and [$\mathscr{R}$-fMod].

The embedding (4-1) generalizes to an embedding

$$\iota_{\underline{\beta}} \colon \mathscr{R}(\underline{\beta}) := \mathscr{R}(\beta_1) \otimes \cdots \otimes \mathscr{R}(\beta_m) \to \mathscr{R}(|\underline{\beta}|) \tag{4-4}$$

for any $\underline{\beta} \in (\mathbb{N}[J])^m$. The embedding (4-4) gives rise to a triple of adjoint functors $(\mathrm{Ind}_{\underline{\beta}}, \mathrm{Res}_{\underline{\beta}}, \mathrm{Coind}_{\underline{\beta}})$ between categories of graded modules.

As explained in [Khovanov and Lauda 2009, §2.2] and [Kleshchev and Ram 2011, §3.6], convolution with the class of (an appropriate graded shift of) the polynomial representation $P(i^{(n)})$ of the nil-Hecke algebra $\mathscr{R}(ni)$ yields an $\mathscr{A}$-module homomorphism

$$\theta_i^{(n)} = - \circ [P(i^{(n)})] \colon [\mathscr{R}(\beta)\text{-pMod}] \to [\mathscr{R}(\beta+ni)\text{-pMod}].$$

Let us recall the fundamental categorification theorem from [Khovanov and Lauda 2009, §3], see also [Kleshchev and Ram 2011, Theorem 4.4].

Theorem 4.1 (Khovanov–Lauda). *There exists a unique pair of adjoint* (*with respect to Lusztig's form on* $\mathbf{f}$ *and the Khovanov–Lauda pairing*) Q-*graded* $\mathscr{A}$-*linear isomorphisms*

$$\gamma \colon \mathbf{f}_{\mathscr{A}} \xrightarrow{\sim} [\mathscr{R}\text{-pMod}] \quad \textit{and} \quad \gamma^* \colon [\mathscr{R}\text{-fMod}] \xrightarrow{\sim} \mathbf{f}^*_{\mathscr{A}}$$

such that $\gamma(1) = [\mathbb{1}]$ *and* $\gamma(xf_i^{(n)}) = \theta_i^{(n)}(\gamma(x))$ *for all* $x \in \mathbf{f}_{\mathscr{A}}$. *These isomorphisms intertwine*: (*i*) *multiplication in* $\mathbf{f}$ *with the convolution product*, (*ii*) *comultiplication in* $\mathbf{f}$ *with restriction functors*, *and* (*iii*) *the bar involution on* $\mathbf{f}$ *with the involutions* $-^{\sharp}$ *and* $-^{\flat}$.

4B. ***Categorification via orientifold KLR algebras.*** We recall some fundamental definitions and results concerning orientifold KLR algebras from [Varagnolo and Vasserot 2011, §8]. We refer the reader to *loc. cit.* for a detailed exposition.

Let $\beta \in \mathbb{N}[J]^{\theta}$ and $\beta' \in \mathbb{N}[J]$, with $\|\beta\|_{\theta} = n$ and $\|\beta'\| = n'$. Set

$$^{\theta}e_{\beta,\beta'} = \sum_{\substack{\nu \in {}^{\theta}J^{\beta+{}^{\theta}\beta'},\\ \nu_1\dots\nu_n \in {}^{\theta}J^{\beta},\\ \nu_{n+1}\dots\nu_{n+n'} \in J^{\beta'}}} e(\nu) \in {}^{\theta}\mathscr{R}(\beta+{}^{\theta}\beta';\lambda).$$

There is an injective nonunital algebra homomorphism

$$^{\theta}\iota_{\beta,\beta'} \colon {}^{\theta}\mathscr{R}(\beta,\beta';\lambda) := {}^{\theta}\mathscr{R}(\beta;\lambda) \otimes \mathscr{R}(\beta') \to {}^{\theta}\mathscr{R}(\beta+{}^{\theta}\beta';\lambda) \tag{4-5}$$

given by formulae (4-2)–(4-3), with $\nu \in {}^{\theta}J^{\beta}$ and $e_{\beta,\beta'}$ replaced by ${}^{\theta}e_{\beta,\beta'}$, and $\tau_0 \otimes 1 \mapsto \tau_0{}^{\theta}e_{\beta,\beta'}$. The *convolution action* of $N \in \mathcal{R}(\beta')$-Mod on $M \in {}^{\theta}\mathcal{R}(\beta;\lambda)$-Mod is defined as

$$M \circledcirc N = {}^{\theta}\mathcal{R}(\beta + {}^{\theta}\beta';\lambda)\,{}^{\theta}e(\beta,\beta') \otimes_{{}^{\theta}\mathcal{R}(\beta,\beta';\lambda)} (M \otimes N).$$

Proposition 4.2. *The category* $\mathcal{R}$-Mod *is monoidal with product* $\circ$ *and unit* $\mathbb{1}$. *Moreover, there is a right monoidal action* (*see, e.g.*, [Davydov 1998]) *of* $\mathcal{R}$-Mod *on* ${}^{\theta}\mathcal{R}(\lambda)$-Mod *via* $\circledcirc$.

Proof. It is routine to check that the conditions in the definition of a monoidal action are satisfied. □

The embedding (4-5) generalizes to an embedding

(4-6) $${}^{\theta}\iota_{\underline{\beta}} : {}^{\theta}\mathcal{R}(\beta_0,\underline{\beta};\lambda) := {}^{\theta}\mathcal{R}(\beta_0;\lambda) \otimes \mathcal{R}(\beta_1) \otimes \cdots \otimes \mathcal{R}(\beta_m) \to {}^{\theta}\mathcal{R}(\beta_0 + {}^{\theta}|\underline{\beta}|;\lambda)$$

for any $\beta_0 \in \mathbb{N}[J]^{\theta}$ and $\underline{\beta} \in (\mathbb{N}[J])^m$. The embedding (4-6) gives rise to a triple of adjoint functors $({}^{\theta}\mathrm{Ind}_{\beta_0,\underline{\beta}}, {}^{\theta}\mathrm{Res}_{\beta_0,\underline{\beta}}, {}^{\theta}\mathrm{Coind}_{\beta_0,\underline{\beta}})$ between categories of graded modules.

Lemma 4.3. *Let* $M_0 \in {}^{\theta}\mathcal{R}(\beta;\lambda)$-fMod *and* $M_i \in \mathcal{R}(\beta_i)$-fMod. *Then, up to a grading shift, we have*

$$\begin{aligned}{}^{\theta}\mathrm{Coind}_{\beta_0,\underline{\beta}}\big(M_0 \otimes (\otimes M_i)\big) &\cong {}^{\theta}\mathrm{Ind}_{\beta_0,\theta(\underline{\beta})}\big(M_0 \otimes (\otimes M_i^{\dagger})\big)\\ &\cong {}^{\theta}\mathrm{Coind}_{\beta_0,|\underline{\beta}|}\big(M_0 \otimes (\mathrm{Coind}_{\underline{\beta}}(\otimes M_i))\big),\end{aligned}$$

where $\theta(\underline{\beta}) = (\theta(\beta_1),\ldots,\theta(\beta_m))$ *and* $-^{\dagger}$ *is the twist defined below Lemma 2.3.*

Proof. The proof is analogous to that of [Lauda and Vazirani 2011, Theorem 2.2]. □

Let $\beta_0 \in \mathbb{N}[J]^{\theta}$ and $\beta_1, \beta_2 \in \mathbb{N}[J]$. Define

$$M_1 \mathbin{\hat{\circ}} M_2 = \mathrm{Coind}_{\beta_1,\beta_2}(M_1 \otimes M_2) \quad \text{and} \quad M_0 \mathbin{\hat{\circledcirc}} M_1 = {}^{\theta}\mathrm{Coind}_{\beta_0,\beta_1}(M_0 \otimes M_1),$$

for M_i as in Lemma 4.3.

Corollary 4.4. *The category* $\mathcal{R}$-Mod *is also monoidal with product* $\hat{\circ}$ *and unit* $\mathbb{1}$. *Moreover, there is a monoidal action of* $\mathcal{R}$-Mod *on* ${}^{\theta}\mathcal{R}(\lambda)$-Mod *via* $\hat{\circledcirc}$.

The functors $P \mapsto P^{\sharp} = \mathrm{HOM}_{{}^{\theta}\mathcal{R}_m(\lambda)}(P, {}^{\theta}\mathcal{R}_m(\lambda))$ and $M \mapsto M^{\flat} = \mathrm{HOM}_{\Bbbk}(P,\Bbbk)$ on ${}^{\theta}\mathcal{R}_m(\lambda)$-pMod and ${}^{\theta}\mathcal{R}_m(\lambda)$-fMod, respectively, descend to $\mathcal{A}$-antilinear involutions on the corresponding Grothendieck groups. We also have an analogue of the Khovanov–Lauda pairing

$$\begin{aligned}(\,\cdot\,,\cdot\,) : [{}^{\theta}\mathcal{R}(\beta;\lambda)\text{-pMod}] \times [{}^{\theta}\mathcal{R}(\beta;\lambda)\text{-fMod}] &\to \mathcal{A},\\ ([P],[M]) &\mapsto \dim_q(P^{\omega} \otimes_{{}^{\theta}\mathcal{R}(\beta;\lambda)} M),\end{aligned}$$

where P^{ω} is the twist of P by the antiinvolution (2-5).

Moreover, set ${}^\theta\mathcal{R}_m(\lambda) = \bigoplus_{\|\beta\|_\theta = n} {}^\theta\mathcal{R}(\beta;\lambda)$ and ${}^\theta e_{m,\beta'} = \oplus_{\|\beta\|_\theta = m} {}^\theta e_{\beta,\beta'}$. Abbreviate ${}^\theta\mathrm{Ind}_{m,i}^{m+1} = {}^\theta\mathcal{R}_{m+1}(\lambda)\otimes_{{}^\theta\mathcal{R}_{m,i}(\lambda)} -$ and ${}^\theta\mathrm{Coind}_{m,i}^{m+1} = \mathrm{HOM}_{{}^\theta\mathcal{R}_{m,i}(\lambda)}({}^\theta\mathcal{R}_{m+1}(\lambda), -)$, with ${}^\theta\mathcal{R}_{m,i}(\lambda) = {}^\theta\mathcal{R}_m(\lambda)\otimes\mathcal{R}(i)$. Setting

$$\begin{aligned} F_i(P) &= {}^\theta\mathrm{Ind}_{m,i}^{m+1}(P\otimes P(i)), & E_i(P) &= L(i)\otimes_{\mathcal{R}(i)} {}^\theta e_{m-1,i}P, \\ F_i^*(M) &= {}^\theta\mathrm{Coind}_{m,i}^{m+1}(M\otimes L(i)), & E_i^*(M) &= {}^\theta e_{m-1,i}M, \end{aligned}$$

defines exact functors

$$\begin{tikzcd} {}^\theta\mathcal{R}_m(\lambda)\text{-pMod} \arrow[r, bend left, "F_i"] & {}^\theta\mathcal{R}_{m+1}(\lambda)\text{-pMod}, \arrow[l, bend left, "E_i"] \end{tikzcd} \qquad \begin{tikzcd} {}^\theta\mathcal{R}_m(\lambda)\text{-fMod} \arrow[r, bend left, "F_i^*"] & {}^\theta\mathcal{R}_{m+1}(\lambda)\text{-fMod} \arrow[l, bend left, "E_i^*"] \end{tikzcd}$$

commuting with the dualities $-^\sharp$ and $-^\flat$. We will use the same notation for the induced operators on the corresponding Grothendieck groups.

We now recall the main theorem [Varagnolo and Vasserot 2011, Theorem 8.31] on the categorification of modules over the Enomoto–Kashiwara algebra.

Theorem 4.5 (Varagnolo–Vasserot). *The operators F_i and E_i (respectively, F_i^* and E_i^*) define a representation of ${}^\theta\mathcal{B}(\mathfrak{g})$ on $\mathcal{K}\otimes_{\mathcal{A}}[{}^\theta\mathcal{R}(\lambda)\text{-pMod}]$ (respectively, $\mathcal{K}\otimes_{\mathcal{A}}[{}^\theta\mathcal{R}(\lambda)\text{-fMod}]$). Moreover, there exists a unique pair of adjoint P^θ-graded $\mathcal{A}$-linear isomorphisms*

$${}^\theta\gamma : {}^\theta V(\lambda)_{\mathcal{A}}^{\mathrm{low}} \xrightarrow{\sim} [{}^\theta\mathcal{R}(\lambda)\text{-pMod}], \quad {}^\theta\gamma^* : [{}^\theta\mathcal{R}(\lambda)\text{-fMod}] \xrightarrow{\sim} {}^\theta V(\lambda)_{\mathcal{A}}^{\mathrm{up}}$$

which, upon base change to $\mathcal{K}$, become isomorphisms of ${}^\theta\mathcal{B}(\mathfrak{g})$-modules. They intertwine the bar involution on ${}^\theta V(\lambda)$ with the involutions $-^\sharp$ and $-^\flat$.

If $M\in{}^\theta\mathcal{R}(\beta;\lambda)$-Mod and $\nu\in{}^\theta J^\beta$, we call $M_\nu = e(\nu)M$ the ν-weight space of M. The *character* of a ${}^\theta\mathcal{R}(\beta;\lambda)$-module M is ${}^\theta\mathrm{ch}_q(M) = \sum_\nu \dim_q(e(\nu)M)\cdot\nu \in {}^\theta\mathcal{F}(\lambda)$. This gives rise to an $\mathcal{A}$-linear map ${}^\theta\mathrm{ch}_q : [{}^\theta\mathcal{R}(\lambda)\text{-fMod}] \to {}^\theta\mathcal{F}(\lambda)$. We then call $\max({}^\theta\mathrm{ch}_q(M))$, if it exists, the highest weight of M.

Corollary 4.6. *The following triangle commutes:*

$$\begin{tikzcd} & {[{}^\theta\mathcal{R}(\lambda)\text{-fMod}]} \arrow[dl, "{}^\theta\gamma^*"'] \arrow[dr, "{}^\theta\mathrm{ch}_q"] & \\ {}^\theta V(\lambda)_{\mathcal{A}}^{\mathrm{up}} \arrow[rr, "{}^\theta\Psi"] & & {}^\theta\mathcal{F}(\lambda) \end{tikzcd}$$

The map ${}^\theta\mathrm{ch}_q$ is injective and ${}^\theta\mathrm{ch}_q(M\circledcirc N) = {}^\theta\mathrm{ch}_q(M)\circledcirc\mathrm{ch}_q(N)$.

Proof. The proof is analogous to [Kleshchev and Ram 2011, Theorem 4.4 (3)]. $\square$

4C. *Reminder on KLR representation theory.* An irreducible $\mathcal{R}(\beta)$-module L is called *cuspidal* if $\max(\mathrm{ch}_q(L)) \in \mathcal{L}_+$, i.e., its highest weight is a good Lyndon word. By [Kleshchev and Ram 2011, Proposition 8.4], for each $\nu \in \mathcal{L}_+$, there exists a unique cuspidal irreducible $\mathcal{R}(|\nu|)$-module $L(\nu)$.

Let $\nu = (\nu^{\langle l\rangle})^{n_l} \cdots (\nu^{\langle 1\rangle})^{n_1} \in J_+^\beta$. The corresponding standard and costandard modules are, respectively,

$$\Delta(\nu) = L(\nu^{\langle l\rangle})^{\circ n_l} \circ \cdots \circ L(\nu^{\langle 1\rangle})^{\circ n_1}\{s(\nu)\}, \quad \nabla(\nu) = L(\nu^{\langle l\rangle})^{\circ n_l}\,\hat{\circ} \cdots \hat{\circ}\, L(\nu^{\langle 1\rangle})^{\circ n_1}\{s(\nu)\},$$

with $s(\nu)$ as in (3-13).

Theorem 4.7 (Kleshchev–Ram, McNamara). *Let* $\nu \in J_+^\beta$. *Then*:

(1) *The standard* $\mathcal{R}(\beta)$*-module* $\Delta(\nu)$ *has an irreducible head* $L(\nu)$, *and the costandard module* $\nabla(\nu)$ *has* $L(\nu)$ *as its socle.*

(2) *The highest weight of* $L(\nu)$ *is* ν, *and* $\dim_q L(\nu)_\nu = \kappa_\nu$.

(3) $L(\nu) = L(\nu)^\flat$.

(4) $\{L(\nu) \mid \nu \in J_+^\beta\}$ *is a complete and irredundant set of irreducible graded* $\mathcal{R}(\beta)$*-modules up to isomorphism and degree shift.*

(5) *If* $L(\mu)$ *is a composition factor of* $\Delta(\nu)$ (*respectively*, $\nabla(\nu)$), *then* $\mu \leq \nu$ (*respectively*, $\mu \leq' \nu$). *Moreover*, $L(\nu)$ *appears in* $\Delta(\nu)$ *and* $\nabla(\nu)$ *with multiplicity one.*

(6) *If* $\nu = \mu^n$ *for a good Lyndon word* μ, *then* $\Delta(\nu) = L(\nu)$.

Proof. See [Kleshchev and Ram 2011, Theorem 7.2] and [McNamara 2015, Theorem 3.1]. □

4D. *Orientifold KLR: irreducibles and global dimension.* Now assume $\lambda = 0$.

Lemma 4.8. *If* $\nu \in {}^\theta J_+^\bullet$ *is symmetric*, *then* ${}^\theta L(\nu) = {}^\theta\mathbb{1} \circledcirc L(\nu)\{{}^\theta s(\nu)\}$ *is irreducible. The highest weight of* ${}^\theta L(\nu)$ *is* ν, ${}^\theta\mathrm{ch}_q\, {}^\theta L(\nu) = {}^\theta\mathbf{b}_\nu^*$, *and* $\dim_q {}^\theta L(\nu)_\nu = {}^\theta\kappa_\nu$.

Proof. It follows from Lemma 3.10, Lemma 3.25, and Corollary 4.6 that all composition factors of ${}^\theta L(\nu)$ have highest weight ν. We know from Theorem 4.7 (2) that $\max(\mathrm{ch}_q(L(\nu))) = \nu$ and $\dim_q L(\nu)_\nu = \kappa_\nu$. The last part of Corollary 4.6, together with an argument analogous to that in the last paragraph of the proof of Lemma 3.31, then shows that the highest weight of ${}^\theta L(\nu)$ is ν and $\dim_q {}^\theta L(\nu)_\nu = {}^\theta\kappa_\nu$.

Let $\beta = {}^\theta|\nu|$. By Theorem 4.5, ${}^\theta\mathrm{ch}_q\, {}^\theta L(\nu) \in {}^\theta\mathbf{V}^{\mathrm{up}}_{\mathcal{A},\beta}$. Since $\{{}^\theta\mathbf{b}_\mu^* \mid \mu \in {}^\theta J_+^\beta\}$ is an $\mathcal{A}$-basis of ${}^\theta\mathbf{V}^{\mathrm{up}}_{\mathcal{A},\beta}$, we have ${}^\theta\mathrm{ch}_q\, {}^\theta L(\nu) = \sum_{\mu \in {}^\theta J_+^\beta} c_\mu {}^\theta\mathbf{b}_\mu^*$ for some $c_\mu \in \mathcal{A}$. By Corollary 3.30, $\max({}^\theta\mathbf{b}_\mu^*) = \mu$, and, by Lemma 3.25, ν is the smallest word in ${}^\theta J_+^\beta$. Hence, $c_\mu = 0$ unless $\mu = \nu$. Comparing the coefficients of ν in $\mathrm{ch}_q L(\nu)$ and ${}^\theta\mathbf{b}_\nu^*$, we conclude that $c_\nu = 1$. The irreducibility of ${}^\theta L(\nu)$ follows directly from the equality ${}^\theta\mathrm{ch}_q {}^\theta L(\nu) = {}^\theta\mathbf{b}_\nu^*$. □

For $\nu \in {}^\theta J_+^\beta$, let

$$ {}^\theta\Delta(\nu) = {}^\theta\mathbb{1} \circledcirc \Delta(\nu) \quad \text{and} \quad {}^\theta\nabla(\nu) = {}^\theta\mathbb{1} \mathbin{\hat{\circledcirc}} \nabla(\nu). $$

Lemma 4.9. *Let $\nu \in {}^\theta J_+^\bullet$. Then $\Delta(\nu) = \Delta(\nu^\theta) \circ \Delta(\nu_\theta)$, $\max({}^\theta\mathrm{ch}_q\, {}^\theta\Delta(\nu)) = \nu$, and $\dim_q({}^\theta\Delta(\nu))_\nu = {}^\theta\kappa_\nu$.*

Proof. The proof of the first statement is analogous to the proof of the first statement of Lemma 3.31. Using the inductive argument and the notation from that proof, one observes that $\mu\xi_m$ is the lowest good word of weight $|\mu\xi_m|$. Theorem 4.7 (5) then implies that $L(\mu) \circ L(\xi_m) = \Delta(\mu\xi_m) = L(\mu\xi_m) = \nabla(\mu\xi_m) = L(\xi_m) \circ L(\mu)$, allowing the induction to proceed.

Since $\dim_q L(\mu) = 1$, for all $\mu \in \mathscr{L}_+$ (see [Kleshchev and Ram 2011, §8.4]), we have ${}^\theta\mathrm{ch}_q({}^\theta\Delta(\nu)) = {}^\theta\Delta_\nu$. The second and third statements now follow from the second and third statements of Lemma 3.31. □

Theorem 4.10. *Let $\nu \in {}^\theta J_+^\beta$. Then:*

1. *The standard ${}^\theta\mathscr{R}(\beta)$-module ${}^\theta\Delta(\nu)$ has an irreducible head ${}^\theta L(\nu)$, and the costandard ${}^\theta\mathscr{R}(\beta)$-module ${}^\theta\nabla(\nu)$ has ${}^\theta L(\nu)$ as its socle.*
2. *The highest weight of ${}^\theta L(\nu)$ is ν, and $\dim_q {}^\theta L(\nu) = {}^\theta\kappa_\nu$.*
3. ${}^\theta L(\nu) = {}^\theta L(\nu)^\flat$.
4. *$\{{}^\theta L(\nu) \mid \nu \in {}^\theta J_+^\beta\}$ is a complete and irredundant set of irreducible graded ${}^\theta\mathscr{R}(\beta)$-modules up to isomorphism and degree shift.*
5. *If ${}^\theta L(\mu)$ is a composition factor of ${}^\theta\Delta(\nu)$ (respectively, ${}^\theta\nabla(\nu)$), then $\mu \le \nu$ (respectively, $\mu \le' \nu$). Moreover, ${}^\theta L(\nu)$ appears in ${}^\theta\Delta(\nu)$ and ${}^\theta\nabla(\nu)$ with multiplicity one.*
6. *If ν is a Lyndon word or $\nu = \nu^\theta$, then ${}^\theta\Delta(\nu) = {}^\theta L(\nu)$ is irreducible.*

Proof. The structure of the proof is similar to [Kleshchev and Ram 2011, Theorem 7.2], see also [McNamara 2015, Theorem 3.1]. Let us explain the main points. If $\nu_\theta = (\nu^{\langle l\rangle})^{n_l} \cdots (\nu^{\langle 1\rangle})^{n_1}$, let $\beta_0 = {}^\theta|\nu^\theta|$, $\underline{\beta} = (n_l|\nu^{\langle l\rangle}|, \cdots, n_1|\nu^{\langle 1\rangle}|)$, and abbreviate

$$ {}^\theta\mathrm{Res}_\nu = {}^\theta\mathrm{Res}_{\beta_0,\underline{\beta}} \quad \text{and} \quad {}^\theta\mathscr{R}_\nu = {}^\theta\mathscr{R}(\beta_0, \underline{\beta}). $$

Also, abbreviate

$$ {}^\theta L(\vec{\nu}) = {}^\theta L(\nu^\theta) \otimes L(\nu^{\langle l\rangle})^{\circ n_l} \otimes \cdots \otimes L(\nu^{\langle 1\rangle})^{\circ n_1}\{s(\nu_\theta)\}. $$

Let L be an irreducible ${}^\theta\mathscr{R}(\beta)$-module in the head of ${}^\theta\Delta(\nu)$. By adjunction and the first part of Lemma 4.9, $\mathrm{HOM}_{{}^\theta\mathscr{R}(\beta)}({}^\theta\Delta(\nu), {}^\theta\Delta(\nu)) = \mathrm{HOM}_{{}^\theta\mathscr{R}_\nu}({}^\theta L(\vec{\nu}), {}^\theta\mathrm{Res}_\nu\, {}^\theta\Delta(\nu))$ and $0 \neq \mathrm{HOM}_{{}^\theta\mathscr{R}(\beta)}({}^\theta\Delta(\nu), L) = \mathrm{HOM}_{{}^\theta\mathscr{R}_\nu}({}^\theta L(\vec{\nu}), {}^\theta\mathrm{Res}_\nu\, L)$. Hence, we get the commutative diagram

$$\begin{tikzcd} {}^\theta L(\vec{\nu}) \arrow[r, hook] \arrow[d, equal] & {}^\theta\mathrm{Res}_\nu {}^\theta\Delta(\nu) \arrow[r, hook] \arrow[d, two heads] & {}^\theta\Delta(\nu) \arrow[d, two heads] \\ {}^\theta L(\vec{\nu}) \arrow[r, hook] & {}^\theta\mathrm{Res}_\nu L \arrow[r, hook] & L \end{tikzcd}$$

The injectivity of the two arrows on the left follows from the ${}^\theta\mathscr{R}_\nu$-module ${}^\theta L(\vec{\nu})$ being irreducible, which is implied by Theorem 4.7 (5) and Lemma 4.8. Further, Theorem 4.7 (2), Lemma 4.8, and Lemma 4.9 also imply that

$$\dim_q {}^\theta L(\vec{\nu})_\nu = {}^\theta\kappa_\nu = \dim_q {}^\theta\Delta(\nu)_\nu.$$

Hence, $\dim_q L_\nu = {}^\theta\kappa_\nu$ as well, implying that the head of ${}^\theta\Delta(\nu)$ is irreducible. This proves (1) in the case of standard modules, as well as (2). Note that the modules ${}^\theta L(\nu)$ we have thus constructed are pairwise nonisomorphic since they have different highest weights.

Next, (3) follows from [Varagnolo and Vasserot 2011, Proposition 2] and the fact that ${}^\theta\kappa_\nu$ is bar-invariant (Lemma 3.27). Part (4) follows from Proposition 3.22, Theorem 4.5, and the fact that we have constructed ${}^\theta\mathrm{kpf}(\beta)$ nonisomorphic irreducible graded ${}^\theta\mathscr{R}(\beta)$-modules $\{{}^\theta L(\nu) \mid \nu \in {}^\theta J_+^\beta\}$. Next, we return to (1) in the case of costandard modules. An analogous argument to that in the case of standard modules, using Lemma 3.32 and the adjunction between restriction and coinduction now shows that ${}^\theta\nabla(\nu)$ has an irreducible socle with highest weight ν, which, by (4), must be isomorphic to ${}^\theta L(\nu)$. Part (5) follows immediately from the facts that $\nu = \max\big({}^\theta\mathrm{ch}_q({}^\theta\Delta(\nu))\big) = \max'\big({}^\theta\mathrm{ch}_q({}^\theta\nabla(\nu))\big)$ and $\dim_q {}^\theta\Delta(\nu)_\nu = \dim_q {}^\theta\nabla(\nu)_\nu = \dim_q {}^\theta L(\nu)_\nu$. Next, part (6) follows from Lemma 3.25 and (5). □

Corollary 4.11. *As a graded algebra*, ${}^\theta\mathscr{R}(\beta)$ *has global dimension* $\|\beta\|_\theta$.

Proof. The proof is analogous to [McNamara 2015, Theorem 4.7]. For the sake of simplicity, we ignore the grading shifts. Since $\lambda = 0$, the set ${}^\theta J_+^\bullet$ contains no θ-cuspidal words. Let $\nu, \mu \in {}^\theta J_+^\beta$. If $\nu_\theta = (\nu^{\langle l\rangle})^{n_l} \cdots (\nu^{\langle 1\rangle})^{n_1}$, we let $L(\vec{\nu}) = L(\nu^\theta) \otimes L(\nu^{\langle l\rangle})^{\circ n_l} \otimes \cdots \otimes L(\nu^{\langle 1\rangle})^{\circ n_1}$. Also let $\underline{\beta} = (|\nu^\theta|, n_l|\nu^{\langle l\rangle}|, \cdots, n_1|\nu^{\langle 1\rangle}|)$. Then, Lemma 4.3 and adjunction between induction and restriction imply that

$$\mathrm{Ext}^i_{{}^\theta\mathscr{R}(\beta)}\big({}^\theta\nabla(\nu), {}^\theta\Delta(\mu)\big) = \mathrm{Ext}^i_{\mathscr{R}(\underline{\beta})}\big(L(\vec{\nu}), \mathrm{Res}_{\underline{\beta}} {}^\theta\Delta(\mu)\big),$$

which, by [McNamara 2015, Theorem 4.7] is zero for $i > \|\beta\|_\theta$. The rest of the proof is exactly the same as in [McNamara 2015]. □

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