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LOCAL EXTERIOR SQUARE AND ASAI *L*-FUNCTIONS FOR GL(n) IN ODD CHARACTERISTIC

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LOCAL EXTERIOR SQUARE AND ASAI L-FUNCTIONS FOR GL(n) IN ODD CHARACTERISTIC

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Let F be a nonarchimedean local field of odd characteristic $p > 0$. We consider local exterior square L-functions $L(s, \pi, \wedge^2)$, Bump–Friedberg L-functions $L(s, \pi, \mathrm{BF})$, and Asai L-functions $L(s, \pi, \mathrm{As})$ of an irreducible admissible representation π of $\mathrm{GL}_m(F)$. In particular, we establish that those L-functions, via the theory of integral representations, are equal to their corresponding Artin L-functions $L\big(s, \wedge^2(\phi(\pi))\big)$, $L\big(s+\frac{1}{2}, \phi(\pi)\big)L\big(2s, \wedge^2(\phi(\pi))\big)$, and $L\big(s, \mathrm{As}(\phi(\pi))\big)$ of the associated Langlands parameter $\phi(\pi)$ under the local Langlands correspondence. These are achieved by proving the identity for irreducible supercuspidal representations, exploiting the local-to-global argument due to Henniart and Lomelí.

1. Introduction

Let F be a nonarchimedean local field of positive characteristic $p \neq 2$. Let π be an irreducible admissible representation of $\mathrm{GL}_m(F)$, where m is a positive integer. The local Langlands correspondence provides a bijection between the set of equivalence classes of irreducible admissible (complex-valued) representations of $\mathrm{GL}_m(F)$ and the set of equivalence classes of m-dimensional Weil–Deligne representations of the Weil–Deligne group W'_F of F. Let r denote either the exterior square representation $\wedge^2 : \mathrm{GL}_m(\mathbb{C}) \to \mathrm{GL}_{m(m-1)/2}(\mathbb{C})$ or the Asai representation (the twisted tensor induction) $\mathrm{As} : \mathrm{GL}_m(\mathbb{C}) \to \mathrm{GL}_{m^2}(\mathbb{C})$ (see [Anandavardhanan and Rajan 2005, §2.1] and [Shankman 2018, §1.2]) of the Langlands dual group $\mathrm{GL}_m(\mathbb{C})$ of $\mathrm{GL}_m(F)$. Let $L(s, (r\circ\phi)(\pi))$, where $s \in \mathbb{C}$, be the Artin local L-function defined by Deligne and Langlands [Tate 1979], where $\phi(\pi) : W'_F \to \mathrm{GL}_m(\mathbb{C})$ is the Weil–Deligne representation corresponding to π under the local Langlands correspondence. In this paper, we address that L-factors, in the cases of exterior square, Bump–Friedberg, and Asai local L-functions, are compatible with the local Langlands correspondence, and establish a series of equalities of local L-functions:

- Jacquet–Shalika cases, Theorem 3.8:

$$L(s, \pi, \wedge^2) = L\big(s, \wedge^2(\phi(\pi))\big);$$

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- Bump–Friedberg cases, Theorem 4.6:

$$L(s,\pi,\mathrm{BF}) = L\big(s+\tfrac{1}{2},\phi(\pi)\big)L\big(2s,\wedge^2(\phi(\pi))\big);$$

- Flicker cases, Theorem 4.7:

$$L(s,\pi,\mathrm{As}) = L\big(s,\mathrm{As}(\phi(\pi))\big);$$

where L-factors on the left-hand sides are defined by the theory of integral representations in positive characteristic.

In the late 1980s, global zeta integrals for (partial) Asai L-functions for GL_m appeared in the work of Flicker [1988; 1993]. Around that time, Jacquet and Shalika [1990] and Bump and Friedberg [1990] independently constructed two different integral representations for an (incomplete) exterior square L-function associated to a cuspidal automorphic representation on GL_m over a global field. Recently there has been renewed interest in the local theory of Asai and exterior square L-functions via Rankin–Selberg methods. In characteristic 0, the identities were shown for Jacquet–Shalika integrals by the author [Jo 2020a], and by Matringe for Bump–Friedberg integrals [Matringe 2015] and Flicker integrals [Matringe 2009; 2011]. As a matter of fact, these results improve discrete series cases of Kewat and Raghunathan [2012] for Jacquet–Shalika integrals and of Anandavardhanan and Rajan [2005] for Flicker integrals. In the positive characteristic $p>0$, Artin L-factors $L\big(s,\wedge^2(\phi(\pi))\big)$ and $L\big(s,\mathrm{As}(\phi(\pi))\big)$ coincide with $\mathcal{L}(s,\pi,\wedge^2)$ and $\mathcal{L}(s,\pi,\mathrm{As})$, respectively, via the Langlands–Shahidi method by a sequence of work by Henniart and Lomelí [2011; 2013a; 2013b]. Similar problems have been worked out by Henniart [2010] in the characteristic zero cases.

The method to prove the matching was developed by Cogdell and Piatetski-Shapiro [2017] in the framework of local L-functions of pairs of irreducible generic representations (π_1,π_2). The computation of local Rankin–Selberg L-functions boils down to decomposing it as the product of what is called the exceptional L-functions (in the sense of [Cogdell and Piatetski-Shapiro 2017]) $L_{\mathrm{ex}}(s,\pi_1^{(k_1)}\times\pi_2^{(k_2)})$ for pairs of Bernstein–Zelevinsky's derivatives $(\pi_1^{(k_1)},\pi_2^{(k_2)})$. The "derivative" in the sense of Bernstein–Zelevinsky $\pi^{(k)}$ is given by representations of smaller groups $\mathrm{GL}_{m-k}(F)$. The advantage of adapting such derivatives enables us to proceed by induction on the rank $m-k$ of general linear groups $\mathrm{GL}_{m-k}(F)$.

Each pole of exceptional L-functions, which we often refer to as an *exceptional pole*, is astonishingly characterized by local distinctness or existence of certain models. The classification of irreducible generic distinguished representations has been widely explored in various works. Indeed, topics of the classification are brought to light for $(S_{2n}(F),\Theta)$-distinguished representations (Shalika models) in [Matringe 2017], for $H_m(F)$-distinguished representation (linear and Friedberg–Jacquet models) in [Matringe 2015], for $\mathrm{GL}_m(F)$-distinguished representations

(Flicker–Rallis models) in [Matringe 2011], and for $(\mathrm{GL}_m(F), \theta)$-distinguished representations (Bump–Ginzburg models) in [Kaplan 2017]. In particular, the main results are summarized as, so to speak, "the hereditary property of models" motivated from the classification of irreducible admissible generic representations of $\mathrm{GL}_m(F)$ (Whittaker models) due to Rodier [1973, p. 427].

When we combine these two ingredients, the factorization of local L-functions and the classification of distinguished representations, we obtain major applications: the inductive relation of local L-functions for irreducible generic representations $\pi = \mathrm{Ind}_{\mathrm{Q}}^{\mathrm{GL}_m}(\Delta_1 \otimes \Delta_2 \otimes \cdots \otimes \Delta_t)$ (Corollary 2.14):

$$L(s, \pi, \wedge^2) = \prod_{1 \le k \le t} L(s, \Delta_k, \wedge^2) \prod_{1 \le i < j \le t} L(s, \Delta_i \times \Delta_j), \tag{1-1}$$

and the weak multiplicativity of γ-factors for parabolically normalized induced (not necessarily irreducible) representations $\pi = \mathrm{Ind}_{\mathrm{Q}}^{\mathrm{GL}_m}(\Delta_1 \otimes \Delta_2 \otimes \cdots \otimes \Delta_t)$ (Theorem 2.12):

$$\gamma(s, \pi, \wedge^2, \psi) \sim \prod_{1 \le k \le t} \gamma(s, \Delta_k, \wedge^2, \psi) \prod_{1 \le i < j \le t} \gamma(s, \Delta_i \times \Delta_j, \psi), \tag{1-2}$$

where $\sim$ means the equality up to a unit in $\mathbb{C}[q^{\pm s}]$ and the Δ_i are discrete series representations. Building upon (1-1), we can incorporate the Langlands classification of irreducible admissible representations in terms of discrete series ones into the theory of local L-functions. In turn, (1-2) allows us to compute local L-functions further in accordance with the Bernstein–Zelevinsky classification of discrete series representations in terms of irreducible supercuspidal ones. As a consequence, we express all exterior square L-factors for irreducible admissible representations in terms of L-factors for irreducible supercuspidal ones in a purely local mean. This comes down to reducing our main questions to all irreducible supercuspidal representations, which eventually serve as building blocks.

We emphasize that the third identity in the Flicker cases is not new and can be found in [Anandavardhanan et al. 2021, Appendix A]. However, we discovered that our technique seems to carry out uniformly to other L-functions for GL_m. As an application of our approach, the main result of [Anandavardhanan et al. 2021] immediately implies the agreement of local Asai L-factors for irreducible supercuspidal representations, which is sufficient to extend it to all irreducible admissible representations, reflecting on the local Langlands correspondence. At this point, unlike [Anandavardhanan et al. 2021, Appendix A], additional globalizations are not required to generalize the equality unconditionally. In the course of following the direction taken in [Anandavardhanan et al. 2021, Appendix A], we encountered a few stumbling blocks. In contrast to characteristic 0 cases, we could not find a good way to adjust the globalization of discrete series representations in [Gan and Lomelí 2018, Proposition 8.2] to our circumstance. As seen in several other's work [Anandavardhanan and Rajan 2005; Kewat and Raghunathan 2012; Kable 2004; Yamana 2017], there might not be a guarantee that the different places v_1 and v_2 are coprime in order to conclude that $\log(q_{v_1})/\log(q_{v_2})$ is irrational, and this coprimality condition may prompt an issue in characteristic $p > 0$. We propose to resolve all these difficulties by globalizing irreducible supercuspidal representations in [Henniart and Lomelí 2011; 2013b, Theorem 3.1] and controlling all but one place in which we are interested.

In practice, we demonstrate the identity sequentially for irreducible supercuspidal representations and eventually for discrete series representations under the working hypothesis analogous to Kaplan's inquiry [2017, Remark 4.18] that $(\mathrm{GL}_m(F), \theta)$-distinguished discrete series representations in positive characteristic are self-dual. Thankfully, we remove the hypothesis by investigating irreducible generic subquotients of principal series representations. We expect to overcome Kaplan's issue beyond the principal series cases by reconciling the different definitions of local symmetric square L-functions possessing their own insights about representations. The poles of $L(s, \pi, \mathrm{Sym}^2)$ can be determined, by means of the Rankin–Selberg method, using the occurrence of $(\mathrm{GL}_m(F), \theta)$-distinguished representations [Yamana 2017], whereas the symmetric square L-functions through the Langlands–Shahidi method $\mathcal{L}(s, \pi, \mathrm{Sym}^2)$ can be related to the presence of the self-duality $\pi \simeq \tilde{\pi}$, using the Rankin–Selberg L-factor $L(s, \pi \times \pi)$ as a product of $\mathcal{L}(s, \pi, \wedge^2)$ and $\mathcal{L}(s, \pi, \mathrm{Sym}^2)$ [Henniart and Lomelí 2011; 2013b]. Taking it for granted that $L(s, \pi, \mathrm{Sym}^2)$ can be factored in terms of exceptional L-factors for derivatives (see [Jo 2021, Theorem 3.15]), our discourse sheds light on some impetus toward systematic development of symmetric square L-factors via integral representations [Yamana 2017] in number theoretic aspects and the classification of $(\mathrm{GL}_m(F), \theta)$-distinguished representations over local function fields [Kaplan 2017] in representation theoretic perspectives. We will return to these matters in the near future.

Finally, it is worth pointing out that the main result of this paper will be used to prove the claim in the preprint by Chen and Gan [2021, Theorem 1.1], that *the exterior square L-function can be equivalently defined by the Langlands–Shahidi method or the local zeta integrals of Jacquet and Shalika* [1990] in positive characteristic.

Let us overview the content of this paper. Section 2A begins with a summary of the theory of derivatives of Bernstein and Zelevinsky and the basic existence theorem of Jacquet–Shalika integrals. Section 2B is devoted to classifying all irreducible generic distinguished representations with respect to given closed algebraic subgroups, especially H_{2n} and S_{2n}, due to Matringe. By combining the factorization of Section 2A, the classification of Section 2B, and the method of deformations and specializations, we prove a weak version of multiplicativity of γ-factors and the inductive relation of L-factors. Using the globalization of irreducible supercuspidal representations presented in Section 3B, if necessary, we complete computing local exterior square L-functions at the end of Section 3B, local Bump–Friedberg L-functions in Section 4A, and local Asai L-functions in Section 4B.

2. Jacquet–Shalika zeta integrals

2A. ***Derivatives and exceptional poles.*** Let F be a nonarchimedean local field of characteristic $p \neq 0, 2$. We let $\mathcal{O}$ denote its ring of integers, $\mathfrak{p}$ its maximal ideal, and q the cardinality of its residual field. We will let ϖ denote a uniformizer, so $\mathfrak{p} = (\varpi)$. We normalize the absolute value by $|\varpi|^{-1} = |\mathcal{O}/\mathfrak{p}|$. The character of GL_m given by $g \mapsto |\det(g)|$ is denoted by ν.

For the group $\mathrm{GL}_m := \mathrm{GL}_m(F)$, we often confront the two cases: m is even and m is odd. For the former, we let $m = 2n$, and for the latter $m = 2n+1$. Let σ_m be the permutation matrix given by

$$\sigma_{2n} = \left(\begin{array}{cccc|cccc} 1 & 2 & \cdots & n & n+1 & n+2 & \cdots & 2n \\ 1 & 3 & \cdots & 2n-1 & 2 & 4 & \cdots & 2n \end{array}\right)$$

when $m = 2n$ is even, and by

$$\sigma_{2n+1} = \left(\begin{array}{cccc|ccccc} 1 & 2 & \cdots & n & n+1 & n+2 & \cdots & 2n & 2n+1 \\ 1 & 3 & \cdots & 2n-1 & 2 & 4 & \cdots & 2n & 2n+1 \end{array}\right)$$

when $m = 2n+1$ is odd. Let B_m be the Borel subgroup consisting of the upper triangular matrices with Levi subgroup A_m of diagonal matrices and unipotent radical N_m. We let Z_m denote the center consisting of scalar matrices. We define P_m to be the mirabolic subgroup given by

$$P_m = \left\{ \begin{pmatrix} g & {}^t u \\ & 1 \end{pmatrix} \;\middle|\; g \in \mathrm{GL}_{m-1},\ u \in F^{m-1} \right\}.$$

We denote by U_m the unipotent radical of P_m. As a group, P_m has a structure of a semidirect product $P_m = \mathrm{GL}_{m-1} \ltimes U_m$. We let $\mathcal{M}_m$ be the set of $m \times m$ matrices and $\mathcal{N}_m$ be the subspace of upper triangular matrices of $\mathcal{M}_m$. Let $\{e_i \mid 1 \leq i \leq m\}$ be the standard low basis of F^m.

We let ψ_F denote a nontrivial additive character of F. We let ψ denote the character of N_m defined by

$$\psi(n) = \psi_F\left(\sum_{i=1}^{n-1} n_{i,i+1}\right), \qquad n = (n_{i,j}) \in N_m.$$

We denote by $\mathcal{A}_F(m)$ the set of equivalence classes of all admissible representations of GL_m on complex vector spaces. Furthermore, we say that a representation $\pi \in \mathcal{A}_F(m)$ is called *generic* if $\mathrm{Hom}_{N_m}(\pi, \psi) \neq \{0\}$. We say that a representation $\pi \in \mathcal{A}_F(m)$ is of *Whittaker type* if

$$\dim_{\mathbb{C}} \mathrm{Hom}_{N_m}(\pi, \psi) = 1.$$

For any character χ of $F^\times$, χ can be uniquely decomposed as $\chi = \chi_0 \nu^{s_0}$, where χ_0 is a unitary character and s_0 is a real number. We use the notation $s_0 = \mathrm{Re}(\chi)$ for the real part of the exponent of the character χ.

If $\pi \in \mathcal{A}_F(m)$ is irreducible and generic, it is known that π is of Whittaker type [Gelfand and Kajdan 1975]. By Frobenius reciprocity, there exists a unique embedding of π into $\mathrm{Ind}_{N_m}^{\mathrm{GL}_m}(\psi)$ up to scalar. The image $\mathcal{W}(\pi, \psi)$ of V_π is called the *Whittaker model* of π. For a nonzero functional $\lambda \in \mathrm{Hom}_{N_m}(\pi, \psi)$, we define the *Whittaker function* $W_v \in \mathcal{W}(\pi, \psi)$ associated to $v \in V_\pi$ by

$$W_v(g) = \lambda(\pi(g)v), \quad g \in \mathrm{GL}_m .$$

We set $W := W_v$. It follows from [Bernstein and Zelevinsky 1976, Lemma 4.5] and [Zelevinsky 1980, §9] that if $\Delta_1, \Delta_2, \ldots, \Delta_t$ are irreducible *essentially square integrable*, which we call *discrete series representations*, then the representation of the form $\mathrm{Ind}_{\mathrm{Q}}^{\mathrm{GL}_m}(\Delta_1 \otimes \Delta_2 \otimes \cdots \otimes \Delta_t)$ is a representation of Whittaker type, where the induction is the normalized parabolic induction from the standard parabolic subgroup Q attached to the partition $(m_1, m_2, \ldots, m_t)$ of m and $\Delta_i \in \mathcal{A}_F(m_i)$. Also, whenever the parabolic subgroup Q and ambient group GL_m are clear from the context, we simply write $\mathrm{Ind}(\Delta_1 \otimes \Delta_2 \otimes \cdots \otimes \Delta_t)$.

Let $\mathrm{Rep}(G)$ denote the category of smooth representations of an l-group G. There are four functors Ψ^-, Ψ^+, Φ^-, and Φ^+. The functor Ψ^- is a normalized Jacquet functor and Φ^- is a normalized ψ-twisted Jacquet functor from $\mathrm{Rep}(P_m)$ to $\mathrm{Rep}(\mathrm{GL}_{m-1})$ and $\mathrm{Rep}(P_{m-1})$, respectively. Given $\tau \in \mathrm{Rep}(P_m)$ on the space V_τ, $\Psi^-(\tau)$ is realized on the space $V_\tau / V_\tau(U_m, \mathbf{1})$ with the action

$$\Psi^-(\tau)(g)(v + V_\tau(U_m, \mathbf{1})) = |\det(g)|^{-1/2}\big(\tau(g)v + V_\tau(U_m, \mathbf{1})\big)$$

and the subspace $V_\tau(U_m, \mathbf{1}) = \langle \tau(u)v - v \mid v \in V_\tau,\ u \in U_m \rangle$. Likewise $\Phi^-(\tau)$ is realized on the space $V_\tau / V_\tau(U_m, \psi)$ with the action

$$\Phi^-(\tau)(p)(v + V_\tau(U_m, \psi)) = |\det(p)|^{-1/2}\big(\tau(p)v + V_\tau(U_m, \psi)\big)$$

and the subspace $V_\tau(U_m, \psi) = \langle \tau(u)v - \psi(u)v \mid v \in V_\tau, u \in U_m \rangle$. The functors Ψ^+ and Φ^+ are normalized and compactly supported inductions from $\mathrm{Rep}(\mathrm{GL}_{m-1})$ and $\mathrm{Rep}(P_{m-1})$, respectively, to $\mathrm{Rep}(P_m)$. Given $\sigma \in \mathrm{Rep}(\mathrm{GL}_{m-1})$,

$$\Psi^+(\sigma) = \mathrm{ind}_{\mathrm{GL}_{m-1}\, U_m}^{P_m}\big(|\det(g)|^{1/2}\sigma \otimes \mathbf{1}\big) = |\det(g)|^{1/2}\sigma \otimes \mathbf{1}$$

is realized on the space V_σ, where ind denotes a compactly supported induction. If $\sigma \in \mathrm{Rep}(P_{m-1})$, then $\Phi^+(\sigma) = \mathrm{ind}_{P_{m-1}U_m}^{P_m}\big(|\det(g)|^{1/2}\sigma \otimes \psi\big)$.

For $\tau \in \mathrm{Rep}(P_m)$, four functors are utilized to define what is called the *Bernstein–Zelevinsky k-th derivatives* $\tau^{(k)}$. Let $\tau^{(k)} \in \mathrm{Rep}(\mathrm{GL}_{m-k})$ be $\tau^{(k)} = \Psi^-(\Phi^-)^{k-1}(\tau)$

for $1 \le k \le m$. The smooth representation τ affords a natural filtration by P_m-modules

$$0 \subseteq \tau_m \subseteq \tau_{m-1} \subseteq \cdots \subseteq \tau_1 = \tau$$

such that $\tau_k/\tau_{k+1} = (\Phi^+)^{k-1}\Psi^+(\tau^{(k)})$ and $\tau_k = (\Phi^+)^{k-1}(\Phi^-)^{k-1}(\tau)$. Let $\pi = \mathrm{Ind}_{\mathrm{Q}}^{\mathrm{GL}_m}(\Delta_1 \otimes \cdots \otimes \Delta_t)$ be a parabolically induced representation, where Δ_i is an irreducible essentially square integrable representation of GL_{m_i} so that $m = m_1 + \cdots + m_t$. Then $\pi^{(k)}$ has a filtration whose successive quotients are isomorphic to $\mathrm{Ind}(\Delta_1^{(k_1)} \otimes \cdots \otimes \Delta_t^{(k_t)})$, with $k = k_1 + \cdots + k_t$ [Bernstein and Zelevinsky 1977, Theorem 4.4 and Lemma 4.5]. For every $0 \le k \le m-1$, let $(\omega_{\pi_{i_k}^{(k)}})_{i_k = 1,2,\ldots,r_k}$ be the family of the central characters of nonzero successive quotient of the form $\pi_{i_k}^{(k)} = \mathrm{Ind}(\Delta_1^{(k_1)} \otimes \cdots \otimes \Delta_t^{(k_t)})$.

Let $\mathcal{S}(F^n)$ be the space of smooth locally constant compactly supported functions on F^n. For each Whittaker function $W \in \mathcal{W}(\pi, \psi)$ and Schwartz–Bruhat function $\Phi \in \mathcal{S}(F^n)$, we define the Jacquet–Shalika integrals:

$$\begin{aligned} &J(s, W, \Phi) \\ &\quad = \int_{N_n \backslash \mathrm{GL}_n} \int_{\mathcal{N}_n \backslash \mathcal{M}_n} W\left(\sigma_{2n} \begin{pmatrix} I_n & X \\ & I_n \end{pmatrix} \begin{pmatrix} g & \\ & g \end{pmatrix}\right) \psi^{-1}(\mathrm{Tr}(X)) \Phi(e_n g) |\det(g)|^s \, dX \, dg \end{aligned}$$

in the even case $m = 2n$ and

$$\begin{aligned} J(s, W, \Phi) = \int_{N_n \backslash \mathrm{GL}_n} \int_{\mathcal{N}_n \backslash \mathcal{M}_n} \int_{F^n} W\left(\sigma_{2n+1} \begin{pmatrix} I_n & X & \\ & I_n & \\ & & 1 \end{pmatrix} \begin{pmatrix} g & & \\ & g & \\ & & 1 \end{pmatrix} \begin{pmatrix} I_n & & \\ & I_n & \\ & y & 1 \end{pmatrix}\right) \\ \times \psi^{-1}(\mathrm{Tr}(X)) \Phi(y) |\det(g)|^{s-1} \, dy \, dX \, dg \end{aligned}$$

in the odd case $m = 2n+1$. Several nice consequences follow from inserting an asymptotic formula over the torus for W into the local zeta integral $J(s, W, \Phi)$ [Jo 2020b, Theorem 3.3 and Lemma 3.10].

Theorem 2.1. *Let $\pi = \mathrm{Ind}_{\mathrm{Q}}^{\mathrm{GL}_m}(\Delta_1 \otimes \Delta_2 \otimes \cdots \otimes \Delta_t)$ be a parabolically induced representation. Let $W \in \mathcal{W}(\pi, \psi)$ and $\Phi \in \mathcal{S}(F^n)$.*

(i)-(1) *(Even case, $m = 2n$) If we have,* $\mathrm{Re}(s) > -\frac{1}{k}\omega_{\pi_{i_{2n-2k}}^{(2n-2k)}}$, *for all $1 \le k \le n$ and all $1 \le i_{2k} \le r_{2k}$, then each local integral $J(s, W, \Phi)$ converges absolutely.*

(i)-(2) *(Odd case, $m = 2n+1$) If we have* $\mathrm{Re}(s) > -\frac{1}{k}\omega_{\pi_{i_{2n+1-2k}}^{(2n+1-2k)}}$, *for all $1 \le k \le n$ and all $1 \le i_{2k-1} \le r_{2k-1}$, then each local integral $J(s, W, \Phi)$ converges absolutely.*

(ii) *Each $J(s, W, \Phi)$ is a rational function in $\mathbb{C}(q^{-s})$, hence $J(s, W, \Phi)$ as a function of s extends meromorphically to all $\mathbb{C}$.*

(iii) *Each $J(s, W, \Phi)$ can be written with a common denominator determined by π. Hence the family has "bounded denominators".*

Let $\mathcal{J}(\pi)$ be the complex linear space of the local integrals $J(s, W, \Phi)$. The family of local integrals $\mathcal{J}(\pi)$ is a $\mathbb{C}[q^{\pm s}]$-fractional ideal of $\mathbb{C}(q^{-s})$ containing 1 [Jo 2020b, Theorems 3.6 and 3.9]. Since the ring $\mathbb{C}[q^s, q^{-s}]$ is a principal ideal domain, the fractional ideal $\mathcal{J}(\pi)$ has a generator. Since $1 \in \mathcal{J}(\pi)$, we can take a generator having numerator 1 and normalized (up to units) to be of the form $P(q^{-s})^{-1}$ with $P(X) \in \mathbb{C}[X]$ having $P(0) = 1$. The *local exterior square L-function*, or simply the *exterior square L-factor*,

$$L(s, \pi, \wedge^2) = \frac{1}{P(q^{-s})}$$

is defined to be the normalized generator of the fractional ideal $\mathcal{J}(\pi)$ spanned by the local zeta integrals $J(s, W, \Phi)$.

We define the Fourier transform on $\mathcal{S}(F^m)$ by

$$\hat{\Phi}(y) = \int_{F^n} \Phi(x)\psi(x\,{}^t y)\,dx.$$

We assume that the measure on F^m is the self-dual measure. Then the Fourier inversion takes the form $\hat{\hat{\Phi}}(x) = \Phi(-x)$. Let

$$w_m := \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}$$

denote the long Weyl element in GL_m. For $(\pi, V_\pi) \in \mathrm{Rep}(\mathrm{GL}_m)$, let π^ι denote the representation of GL_m on the same space V_π given by $\pi^\iota(g) = \pi({}^t g^{-1})$. If π is irreducible, it is known that $\pi^\iota \simeq \tilde{\pi}$, the contragredient representation of π. The parabolically induced representation $\pi^\iota = \mathrm{Ind}(\tilde{\Delta}_t \otimes \tilde{\Delta}_{t-1} \otimes \cdots \otimes \tilde{\Delta}_1)$ is, again, of Whittaker type. If $W \in \mathcal{W}(\pi, \psi)$, then $\widetilde{W}(g) := W(w_m\,{}^t g^{-1})$ belongs to $\mathcal{W}(\pi^\iota, \psi^{-1})$. We let τ_m be a matrix given by

$$\begin{pmatrix} & I_n \\ I_n & \end{pmatrix}, \text{ when } m = 2n, \qquad \begin{pmatrix} & I_n & \\ I_n & & \\ & & 1 \end{pmatrix}, \text{ when } m = 2n+1.$$

As a consequence of the uniqueness of bilinear forms on $\mathcal{W}(\pi, \psi) \times \mathcal{S}(F^n)$, we can define the local γ-factor, which gives rise to the local functional equation for our integrals $J(s, W, \Phi)$ [Cogdell and Matringe 2015; Matringe 2014] (see [Jo 2020a, Theorem 2.10, (2.1)]).

Theorem 2.2. *Let $\pi = \mathrm{Ind}_{\mathrm{Q}}^{\mathrm{GL}_m}(\Delta_1 \otimes \Delta_2 \otimes \cdots \otimes \Delta_t)$ be a parabolically induced representation of GL_m. Then there is a rational function $\gamma(s, \pi, \wedge^2, \psi) \in \mathbb{C}(q^{-s})$ such that for every W in $\mathcal{W}(\pi, \psi)$, and every Φ in $\mathcal{S}(F^n)$, we have*

$$J(1-s, \varrho(\tau_m)\widetilde{W}, \hat{\Phi}) = \gamma(s, \pi, \wedge^2, \psi) J(s, W, \Phi),$$

where ϱ denotes right translation.

An equally important local factor is the local ε-factor

$$\varepsilon(s,\pi,\wedge^2,\psi)=\gamma(s,\pi,\wedge^2,\psi)\frac{L(s,\pi,\wedge^2)}{L(1-s,\pi^\iota,\wedge^2)},$$

which is an invertible element $\varepsilon(s,\pi,\wedge^2,\psi)$ in $\mathbb{C}[q^{\pm s}]$. With the local ε-factor, the functional equation becomes

$$\frac{J(1-s,\varrho(\tau_m)\widetilde{W},\hat{\Phi})}{L(1-s,\pi^\iota,\wedge^2)}=\varepsilon(s,\pi,\wedge^2,\psi)\frac{J(s,W,\Phi)}{L(s,\pi,\wedge^2)}.$$

Let $\pi=\mathrm{Ind}_{\mathrm{Q}}^{\mathrm{GL}_{2n}}(\Delta_1\otimes\Delta_2\otimes\cdots\otimes\Delta_t)$ be a parabolically induced representation. Let $\mathcal{S}_0(F^n)$ be the subspace of $\Phi\in\mathcal{S}(F^n)$ for which $\Phi(0,0,\dots,0)=0$. Suppose there exists a function in $\mathcal{J}(\pi)$ having a pole of order d_{s_0} at $s=s_0$. We investigate the rational function defined by an individual zeta integral $J(s,W,\Phi)$. Then the Laurent expansion about $s=s_0$ will take the form

$$J(s,W,\Phi)=\frac{B_{s_0}(W,\Phi)}{(q^s-q^{s_0})^{d_{s_0}}}+(\text{higher order terms}).$$

We define the Shalika subgroup S_{2n} of GL_{2n} by

$$S_{2n}=\left\{\begin{pmatrix}I_n & Z\\ & I_n\end{pmatrix}\begin{pmatrix}h & \\ & h\end{pmatrix}\,\middle|\, Z\in\mathcal{M}_n,\ h\in GL_n\right\}.$$

Let us denote an action of the Shalika subgroup S_{2n} on $\mathcal{S}(F^n)$ by

$$R\left(\begin{pmatrix}I_n & Z\\ & I_n\end{pmatrix}\begin{pmatrix}h & \\ & h\end{pmatrix}\right)\Phi(x)=\Phi(xh)$$

for $\Phi\in\mathcal{S}(F^n)$. The coefficient of the leading term, $B_{s_0}(W,\Phi)$, will define a nontrivial bilinear form on $\mathcal{W}(\pi,\psi)\times\mathcal{S}(F^n)$ enjoying the quasiinvariance

$$B_{s_0}\big(\varrho(g)W,R(g)\Phi\big)=|\det(h)|^{-s_0}\psi(\mathrm{Tr}(Z))B_{s_0}(W,\Phi)$$

for $g=\left(\begin{smallmatrix}I_n & Z\\ & I_n\end{smallmatrix}\right)\left(\begin{smallmatrix}h & \\ & h\end{smallmatrix}\right)\in S_{2n}$. The pole at $s=s_0$ of the family $\mathcal{J}(\pi)$ is called *exceptional* if the associated bilinear form $B_{s_0}(W,\Phi)$ vanishes identically on $\mathcal{W}(\pi,\psi)\times\mathcal{S}_0(F^n)$. If $s=s_0$ is an exceptional pole of $\mathcal{J}(\pi)$, then the bilinear form B_{s_0} factors to a nonzero bilinear form on $\mathcal{W}(\pi,\psi)\times\mathcal{S}(F^n)/\mathcal{S}_0(F^n)$. The quotient $\mathcal{S}(F^n)/\mathcal{S}_0(F^n)$ is isomorphic to $\mathbb{C}$ via the map $\Phi\mapsto\Phi(0)$. Let Θ be the character of S_{2n} given by

$$\Theta\left(\begin{pmatrix}I_n & Z\\ & I_n\end{pmatrix}\begin{pmatrix}h & \\ & h\end{pmatrix}\right)=\psi(\mathrm{Tr}(Z)).$$

We say that $\pi\in\mathcal{A}_F(2n)$ is (S_{2n},Θ)-*distinguished* if $\mathrm{Hom}_{S_{2n}}(\pi,\Theta)\neq\{0\}$. A nonzero linear functional Λ in $\mathrm{Hom}_{S_{2n}}(\pi,\Theta)$ (respectively, Λ_s in $\mathrm{Hom}_{S_{2n}}(\pi,\nu^{-s/2}\Theta)$) is called a *Shalika functional* (respectively, a *twisted Shalika functional*). If $s=s_0$ is an exceptional pole, then the form B_{s_0} can be written as $B_{s_0}(W, \Phi) = \Lambda_{s_0}(W)\Phi(0)$ with Λ_{s_0} the Shalika functional on $\mathcal{W}(\pi, \psi)$. Using the notation, we let

$$L_{\mathrm{ex}}(s, \pi, \wedge^2) = \prod_{s_0} (1 - q^{s_0} q^{-s})^{d_{s_0}},$$

where s_0 runs through all the exceptional poles of $\mathcal{J}(\pi)$ with d_{s_0} the maximal order of the pole at $s = s_0$. The factorization of local exterior square L-functions proposed by Cogdell and Piatetski-Shapiro asserts that it can be expressed in terms of the exceptional exterior square L-factors of the derivatives of π [Jo 2020b].

Theorem 2.3. *Let $\pi = \mathrm{Ind}_{\mathrm{Q}}^{\mathrm{GL}_m}(\Delta_1 \otimes \Delta_2 \otimes \cdots \otimes \Delta_t)$ be an irreducible generic representation of GL_m such that all the derivatives $\pi^{(k)}$ of π are completely reducible with irreducible generic constituents of the form $\pi_i^{(k)} = \mathrm{Ind}(\Delta_1^{(k_1)} \otimes \cdots \otimes \Delta_t^{(k_t)})$ with $k = k_1 + \cdots + k_t$. For each k, i is indexing the partition of k. Then*:

(i) $m = 2n$: $L(s, \pi, \wedge^2) = \mathrm{lcm}_{k,i} \left\{ L_{\mathrm{ex}}(s, \pi_i^{(2k)}, \wedge^2)^{-1} \right\}$,

(ii) $m = 2n+1$: $L(s, \pi, \wedge^2) = \mathrm{lcm}_{k,i} \left\{ L_{\mathrm{ex}}(s, \pi_i^{(2k+1)}, \wedge^2)^{-1} \right\}$,

where the least common multiple is with respect to divisibility in $\mathbb{C}[q^{\pm s}]$ and is taken over all k with $k = 0, 1, \ldots, n-1$ and for each k all constituents $\pi_i^{(2k)}$ (respectively, $\pi_i^{(2k+1)}$) of $\pi^{(2k)}$ (respectively, $\pi^{(2k+1)}$).

A similar definition for $L_{\mathrm{ex}}(s, \pi \times \sigma)$ and a factorization formula has been constructed by Cogdell and Piatetski-Shapiro in the context of local Rankin–Selberg L-functions for a pair of representations (π, σ) of GL_m [Cogdell and Piatetski-Shapiro 2017; Matringe 2015, §4.1].

2B. ***Classifications of distinguished representations.*** For $m = 2n$, we let M_{2n} denote the standard Levi subgroup of GL_{2n} associated with the partition (n, n) of $2n$. Let $w_{2n} = \sigma_{2n}$, and then we set $H_{2n} = w_{2n} M_{2n} w_{2n}^{-1}$. Let $w_{2n+1} = w_{2n+2}|_{\mathrm{GL}_{2n+1}}$ so that

$$w_{2n+1} = \left(\begin{array}{cccc|ccccc} 1 & 2 & \cdots & n+1 & n+2 & n+3 & \cdots & 2n & 2n+1 \\ 1 & 3 & \cdots & 2n+1 & 2 & 4 & \cdots & 2n-2 & 2n \end{array} \right).$$

In the odd case, $w_{2n+1} \neq \sigma_{2n+1}$, and we denote by M_{2n+1} the standard Levi subgroup attached to the partition $(n+1, n)$ of $2n+1$. We set $H_{2n+1} = w_{2n+1} M_{2n+1} w_{2n+1}^{-1}$. We observe that H_m is compatible in the sense that $H_m \cap \mathrm{GL}_{m-1} = H_{m-1}$. If α is a character of $F^\times$ and $\mathrm{diag}(g, g') \in M_m$, we denote by χ_α the character

$$\chi_\alpha : w_m \begin{pmatrix} g & \\ & g' \end{pmatrix} w_m^{-1} \mapsto \alpha\left(\frac{\det(g)}{\det(g')} \right)$$

of H_m. Let χ be a character of H_m. We say that $\pi \in \mathcal{A}_F(m)$ is (H_m, χ)-*distinguished* if $\mathrm{Hom}_{H_m}(\pi, \chi) \neq 0$. If χ is trivial, it is customary to say that π is H_m-*distinguished*. In order to classify all irreducible generic distinguished representations, we need to know that the induced representations of the form $\mathrm{Ind}_Q^{\mathrm{GL}_{2n}}(\Delta\otimes\tilde{\Delta})$ are distinguished. These types of properties over non-Archimedean local fields in characteristic zero were originally investigated by Cogdell and Piatetski-Shapiro [1994]. Afterwards the conjecture was settled by Matringe [2015; 2017]. Parts of the proof of [Matringe 2015, Proposition 3.8] contain inaccuracies, and subsequently it is clarified in [Matringe 2017, Proposition 5.3].

Proposition 2.4 (N. Matringe). *Let Δ be discrete series representations of* GL_n *and α a character of $F^\times$. Then irreducible generic representations of the form* $\mathrm{Ind}_Q^{\mathrm{GL}_{2n}}(\Delta\otimes\tilde{\Delta})$ *are both (H_{2n},χ_α)- and (S_{2n},Θ)-distinguished.*

Proof. We consider parabolically induced representations of the form

$$\Pi_s := \mathrm{Ind}_Q^{\mathrm{GL}_{2n}}(\Delta_0\nu^s\otimes\tilde{\Delta}_0\nu^{-s}),$$

with Δ_0 a unitary discrete series representations of GL_n and s a complex parameter. The proof in [Matringe 2015, Proposition 3.8] relies on Bernstein's analytic continuation principle for invariant linear forms. In order to apply it to positive characteristic, we need to explain that the space $\mathrm{Hom}_{S_{2n}}(\Pi_s,\Theta)$ is of dimension at most one for all s except the finite number for which Π_s is irreducible. However, if this is the case, $\mathrm{Hom}_{S_{2n}}(\Pi_s,\Theta)$ embeds as a subspace of $\mathrm{Hom}_{H_{2n}\cap P_{2n}}(\Pi_s,\mathbf{1}_{H_{2n}})$ via [Matringe 2014, Proposition 4.3] along with $\mathrm{Hom}_{S_{2n}}(\Pi_s,\Theta)\subseteq\mathrm{Hom}_{S_{2n}\cap P_{2n}}(\Pi_s,\Theta)$. Thanks to an auxiliary deformation parameter s, the proof of [Matringe 2015, Proposition 5.1-8] asserts that except for a finite number of s, the space $\mathrm{Hom}_{H_{2n}\cap P_{2n}}(\Pi_s,\mathbf{1}_{H_{2n}})$ is of dimension at most 1, as desired.

Alternatively, the quickest way is to use the equivalence between (H_{2n},χ_α)-distinctions and (S_{2n},Θ)-distinctions [Yang 2022, Corollary 3.6], which only depends on Gan's approach of theta correspondence [2019, Theorem 3.1]. This allows us to reduce to the case for $\alpha=0$, where the result is immediate from Blanc and Delorme [2008], as described in [Matringe 2014, §5]. We refer the interested reader to [Offen 2018, Proposition 3.2.15] for an expository construction of this open orbit contribution. □

We are now ready to introduce the classification of (H_{2n},χ_α)-distinguished representations that was established by Matringe [2015, Theorem 3.1]. The classification result holds in positive characteristic $p\neq 2$, though written in characteristic 0 only. Indeed, the proof relies crucially on Bernstein and Zelevinsky's version of Mackey's theorem [1977, Theorem 5.2], the explicit description of discrete series representations and their Jacquet modules [Zelevinsky 1980, Proposition 9.5], and the fact that a discrete series representation of GL_{2n+1} cannot be H_{2n+1}-distinguished [Matringe 2014, Theorem 3.1]. All the aforementioned properties are true in positive characteristic (see [Anandavardhanan et al. 2021, Appendix A] and [Gan 2019, §4]).

Theorem 2.5 (N. Matringe, $m = 2n$). *Let $\pi = \mathrm{Ind}_{\mathrm{Q}}^{\mathrm{GL}_{2n}}(\Delta_1 \otimes \Delta_2 \otimes \cdots \otimes \Delta_t)$ be an irreducible generic representation of GL_{2n}. Let α be a character of $F^\times$ with $0 \leq \mathrm{Re}(\alpha) \leq \frac{1}{2}$. Then π is (H_{2n}, χ_α)-distinguished if and only if there is a reordering of the Δ_i and an integer r between 1 and $[t/2]$, such that $\Delta_{i+1} = \tilde{\Delta}_i$ for $i = 1, 3, \ldots, 2r-1$, and Δ_i is (H_{2n_i}, χ_α)-distinguished for $i > 2r$.*

For a discrete series representation Δ of GL_{2n}, Δ is H_{2n}-distinguished if and only if it is (S_{2n}, Θ)-distinguished. Matringe [2014, §5], using an analytic approach, and Gan [2019, Theorem 4.2], using the theta correspondence, individually settled this connection. Combining this with [Matringe 2017, Theorem 1.1 and Proposition 5.3], we classify the (S_{2n}, Θ)-distinguished generic representation of GL_{2n} in terms of (S_{2n_i}, Θ)-distinguished discrete series representations Δ_i [Matringe 2017, Corollary 1.1]. We refer the reader to [Matringe 2017] for further details of the proof.

Theorem 2.6 (N. Matringe). *Let $\pi = \mathrm{Ind}_{\mathrm{Q}}^{\mathrm{GL}_{2n}}(\Delta_1 \otimes \Delta_2 \otimes \cdots \otimes \Delta_t)$ be an irreducible generic representation of GL_{2n}. Then π is (S_{2n}, Θ)-distinguished if and only if there is a reordering of the Δ_i and an integer r between 1 and $[t/2]$, such that $\Delta_{i+1} = \tilde{\Delta}_i$ for $i = 1, 3, \ldots, 2r-1$, and Δ_i is (S_{2n_i}, Θ)-distinguished for $i > 2r$.*

In the light of Theorem 2.5, Theorem 2.6, and [Gan 2019, Theorem 4.2], Matringe and Gan's equivalence is valid in more general setting of irreducible generic representations of GL_{2n}.

2C. ***Deformations and specializations.*** Let $\pi = \mathrm{Ind}_{\mathrm{Q}}^{\mathrm{GL}_m}(\Delta_1 \otimes \Delta_2 \otimes \cdots \otimes \Delta_t)$ be a parabolically induced representation of GL_m. Let $\mathcal{D}_\pi$ denote the complex manifold $\big(\mathbb{C}/(2\pi i/\log(q))\mathbb{Z}\big)^t$. The isomorphism $\mathcal{D}_\pi \to (\mathbb{C}^\times)^t$ is defined by

$$u := (u_1, u_2, \ldots, u_t) \mapsto q^u := (q^{u_1}, q^{u_2}, \ldots, q^{u_t}).$$

We use $q^{\pm u}$ as short for $(q^{\pm u_1}, q^{\pm u_2}, \ldots, q^{\pm u_t})$. For $u \in \mathcal{D}_\pi$, we set

$$\pi_u = \mathrm{Ind}_{\mathrm{Q}}^{\mathrm{GL}_m}(\Delta_1 \nu^{u_1} \otimes \Delta_2 \nu^{u_2} \otimes \cdots \otimes \Delta_t \nu^{u_t}).$$

Let us set

$$\pi_u^{(k_1,k_2,\ldots,k_t)} = \mathrm{Ind}_{\mathrm{Q}}^{\mathrm{GL}_m}(\Delta_1^{(k_1)} \nu^{u_1} \otimes \Delta_2^{(k_2)} \nu^{u_2} \otimes \cdots \otimes \Delta_t^{(k_t)} \nu^{u_t}).$$

Section 2C is indebted to Cogdell and Piatetski-Shapiro [2017], and we closely follow the path of the adaptation that was used in [Matringe 2009; 2015; Jo 2020a] to study the characteristic zero case. In particular, the deformation and specialization argument is widely available in the literature [Cogdell and Piatetski-Shapiro 2017; Matringe 2009; 2015; Jo 2020a]. Henceforth, we only remark on the nature of the difference but the reader should consult [Cogdell and Piatetski-Shapiro 2017; Matringe 2009] for complete details.

Definition 2.7. We say that $u=(u_1,u_2,\ldots,u_t)\in\mathcal{D}_\pi$ is in *general position* if it satisfies the following properties:

(i) For every sequences of nonnegative integers $(k_1,k_2,\ldots,k_t)$, a nonzero representation

$$\pi_u^{(k_1,k_2,\ldots,k_t)}=\operatorname{Ind}\big(\Delta_1^{(k_1)}\nu^{u_1}\otimes\Delta_2^{(k_2)}\nu^{u_2}\otimes\cdots\otimes\Delta_t^{(k_t)}\nu^{u_t}\big)$$

is irreducible;

(ii) If $(a_1r_1,a_2r_2,\ldots,a_tr_t)$ and $(b_1r_1,b_2r_2,\ldots,b_tr_t)$ are two different sequences such that

$$\sum_{i=1}^t a_ir_i=\sum_{i=1}^t b_ir_i,$$

then two representations

$$\operatorname{Ind}\big(\Delta_1^{(a_1r_1)}\nu^{u_1}\otimes\Delta_2^{(a_2r_2)}\nu^{u_2}\otimes\cdots\otimes\Delta_t^{(a_tr_t)}\nu^{u_t}\big)$$
$$\operatorname{Ind}\big(\Delta_1^{(b_1r_1)}\nu^{u_1}\otimes\Delta_2^{(b_2r_2)}\nu^{u_2}\otimes\cdots\otimes\Delta_t^{(b_tr_t)}\nu^{u_t}\big)$$

possess distinct central characters;

(iii) If $(i,j,k,\ell)\in\{1,2,\ldots,t\}$, with $\{i,j\}\neq\{k,\ell\}$, then $L(s,\Delta_i\nu^{u_i}\times\Delta_j\nu^{u_j})$ and $L(s,\Delta_k\nu^{u_k}\times\Delta_\ell\nu^{u_\ell})$ have no common poles;

(iv) If $(i,j)\in\{1,2,\ldots,t\}$, with $i\neq j$, then $L(s,\Delta_i\nu^{u_i},\wedge^2)$ and $L(s,\Delta_j\nu^{u_j},\wedge^2)$ have no common poles;

(v) If $(i,j,k)\in\{1,2,\ldots,t\}$, with $i\neq j$, then $L(s,\Delta_i\nu^{u_i}\times\Delta_j\nu^{u_j})$ and $L(s,\Delta_k\nu^{u_k},\wedge^2)$ have no common poles;

(vi) If $1\le i\neq j\le t$ and $(\Delta_i^{(a_ir_i)})^\sim\simeq\Delta_j^{(a_jr_j)}\nu^e$ for some complex number e, then the dimension of the space

$$\operatorname{Hom}_{P_{2(n_i-a_ir_i)}\cap S_{2(n_i-a_ir_i)}}\big(\operatorname{Ind}\big(\Delta_i^{(a_ir_i)}\nu^{(u_i+u_j+e)/2}\otimes(\Delta_i^{(a_ir_i)}\nu^{(u_i+u_j+e)/2})^\sim\big),\Theta\big)$$

is at most 1.

We confirm that off a finite number of hyperplanes in u, the deformed representation π_u is in general position [Jo 2020a, Proposition 4.1]. The important point is that $u\in\mathcal{D}_\pi$ in general position depends only on the representation π, not $s\in\mathbb{C}$. The purpose of (ii) is that outside a finite number of hyperplanes, the central character of $\pi_u^{(a_1r_1,a_2r_2,\ldots,a_tr_t)}$ are distinct and therefore there are only trivial extensions among these representation. As a result, off these hyperplanes, the derivatives $\pi_u^{(k)}=\oplus\pi_u^{(a_1r_1,a_2r_2,\ldots,a_tr_t)}$ are completely reducible, where $k=\sum_{i=1}^t a_ir_i$ and each $\pi_u^{(a_1r_1,a_2r_2,\ldots,a_tr_t)}$ are irreducible. Conditions (i) and (ii) ensure that Theorem 2.3 is applicable. The purpose of Condition (vi) is that the occurrence of the exceptional pole of $L(s,\pi,\wedge^2)$ at $s=0$ can be determined by the existence of Shalika functional from [Jo 2020a, Lemma 3.2]. Throughout Section 2C, we assume the working hypothesis proposed by E. Kaplan [2017, Remark 4.18] for fields of odd characteristic.

Working Hypothesis. Let Δ be an (S_{2n}, Θ)-distinguished discrete series representation of GL_{2n}. Then Δ is self-dual. Namely, $\tilde{\Delta} \simeq \Delta$.

The following statement is a consequence of the working hypothesis along with Theorem 2.6:

Corollary 2.8. *Assume the working hypothesis. Let $\pi = \mathrm{Ind}_{\mathrm{Q}}^{\mathrm{GL}_{2n}}(\Delta_1 \otimes \Delta_2 \otimes \cdots \otimes \Delta_t)$ be an irreducible generic representation of GL_{2n}. If π is (S_{2n}, Θ)-distinguished, then π is self-dual. Namely, $\tilde{\pi} \simeq \pi$.*

The working hypothesis needs not be considered for the subclass of irreducible principal series representations induced from Borel subgroups due to Theorem 2.6, and we shall verify the presumption case-by-case in Section 2.

Proposition 2.9. *Let $\pi = \mathrm{Ind}_{B_{2n}}^{\mathrm{GL}_{2n}}(\chi_1 \otimes \chi_2 \otimes \cdots \otimes \chi_{2n})$ be a (S_{2n}, Θ)-distinguished irreducible principal series representation of GL_{2n}. Then π is self-dual. Namely, $\tilde{\pi} \simeq \pi$.*

Now we provide an interpretation of Theorem 2.6 in terms of local L-functions, which is analogous to [Matringe 2015, Proposition 4.13].

Proposition 2.10. *Let $\pi = \mathrm{Ind}_{\mathrm{Q}}^{\mathrm{GL}_{2n}}(\Delta_1 \otimes \Delta_2 \otimes \cdots \otimes \Delta_t)$ be an irreducible generic representation of GL_{2n}, where each Δ_i is a discrete series representation of GL_{n_i} with $2n = \sum_{i=1}^{t} n_i$ and $t \geq 2$. Suppose that $L_{\mathrm{ex}}(s, \pi, \wedge^2)$ has a pole at $s = s_0$. Then we are in one of the following cases*:

(i) *There are $(i, j) \in \{1, 2, \ldots, t\}$, with $i \neq j$, such that n_i and n_j are even, and $L_{\mathrm{ex}}(s, \Delta_i, \wedge^2)$ and $L_{\mathrm{ex}}(s, \Delta_j, \wedge^2)$ have $s = s_0$ as a common pole.*

(ii) *There are $(i, j, k, \ell) \in \{1, 2, \ldots, t\}$, with $\{i, j\} \neq \{k, \ell\}$, such that $L_{\mathrm{ex}}(s, \Delta_i \times \Delta_j)$ and $L_{\mathrm{ex}}(s, \Delta_k \times \Delta_\ell)$ have $s = s_0$ as a common pole.*

(iii) *There are $(i, j, k) \in \{1, 2, \ldots, t\}$, with $i \neq j$, such that n_k is even and $L_{\mathrm{ex}}(s, \Delta_i \times \Delta_j)$ and $L_{\mathrm{ex}}(s, \Delta_k, \wedge^2)$ have $s = s_0$ as a common pole.*

Proof. Suppose that $L_{\mathrm{ex}}(s, \pi, \wedge^2)$ has a pole at $s = s_0$. Since $L(s, \pi, \wedge^2) = L(s - s_0, \pi \nu^{s_0/2}, \wedge^2)$, the representation $\pi \nu^{s_0/2}$ admits a nontrivial Shalika functional. We know from Theorem 2.6 that $\pi \nu^{s_0/2}$ is isomorphic to

$$\mathrm{Ind}\Big(\big(\Delta_{i_1} \nu^{s_0/2} \otimes (\Delta_{i_1} \nu^{s_0/2})^{\sim}\big) \otimes \cdots \otimes \big(\Delta_{i_r} \nu^{s_0/2} \otimes (\Delta_{i_r} \nu^{s_0/2})^{\sim}\big) \otimes \Delta_{i_{r+1}} \nu^{s_0/2} \otimes \cdots \otimes \Delta_{i_t} \nu^{s_0/2}\Big),$$

with $0 \leq r \leq [t/2]$, where $\Delta_{i_j} \nu^{s_0/2}$ affords a Shalika functional and each n_{i_j} is even for all $j > r$. Putting it in a different way, $(\Delta_i \nu^{s_0/2})^{\sim} \simeq \Delta_j \nu^{s_0/2}$ with $i \neq j$, or $\Delta_k \nu^{s_0/2}$ owns a Shalika functional, where n_k is an even number.

According to [Matringe 2015, Proposition 4.6], $(\Delta_i \nu^{s_0/2})^{\sim} \simeq \Delta_j \nu^{s_0/2}$ or equivalently $\tilde{\Delta}_i \simeq \Delta_j \nu^{s_0}$ if and only if $L_{\mathrm{ex}}(s, \Delta_i \times \Delta_j)$ has a pole at $s = s_0$.

If $\Delta_k \nu^{s_0/2}$ has the Shalika functional, the space $\mathrm{Hom}_{S_{n_k}}(\Delta_k \nu^{s_0/2}, \Theta)$ is nontrivial and its central character $\omega_{\Delta_k \nu^{s_0/2}}$ is trivial. Since $\Delta_k \nu^{s_0/2}$ is the irreducible square integrable representation, we obtain from [Jo 2020a, Proposition 3.4] that $L_{\mathrm{ex}}(s, \Delta_k \nu^{s_0/2}, \wedge^2)$ has a pole at $s = 0$, or equivalently, $L_{\mathrm{ex}}(s, \Delta_k, \wedge^2)$ has a pole at $s = s_0$. Therefore $s = s_0$ is the common pole for either of three cases in Proposition 2.10. □

Let Δ be a discrete series representation. Such a representation Δ is the unique irreducible quotient of the form: $\mathrm{Ind}_{\mathrm{Q}}^{\mathrm{GL}_m}(\rho \otimes \rho\nu \otimes \cdots \otimes \rho\nu^{\ell-1})$, where the induction is a normalized parabolic induction from the standard parabolic subgroup Q attached to the partition $(r, r, \ldots, r)$ of $m = r\ell$ and $\rho \in \mathcal{A}_F(r)$ is irreducible and supercuspidal [Zelevinsky 1980]. We denote by $\Delta = [\rho, \rho\nu, \ldots, \rho\nu^{\ell-1}]$ such a quotient. Using Hartogs' theorem [Jo 2020a] is closer to the original spirit of the direction in [Cogdell and Piatetski-Shapiro 2017]. Nevertheless, we present an alternative approach employing Proposition 2.10 to keep uniformity with [Matringe 2009; 2015].

Proposition 2.11. *Assume the working hypothesis, and let us denote by* $\pi = \mathrm{Ind}_{\mathrm{Q}}^{\mathrm{GL}_m}(\Delta_1 \otimes \Delta_2 \otimes \cdots \otimes \Delta_t)$ *a parabolically induced representation of* GL_m. *Let* $u = (u_1, u_2, \ldots, u_t) \in \mathcal{D}_\pi$ *be in general position, and let*

$$\pi_u = \mathrm{Ind}_{\mathrm{Q}}^{\mathrm{GL}_m}(\Delta_1 \nu^{u_1} \otimes \Delta_2 \nu^{u_2} \otimes \cdots \otimes \Delta_t \nu^{u_t})$$

be the deformed representation. Then we have the following:

(i) $\displaystyle L(s, \pi_u, \wedge^2) = \prod_{1 \le k \le t} L(s + 2u_k, \Delta_k, \wedge^2) \prod_{1 \le i < j \le t} L(s + u_i + u_j, \Delta_i \times \Delta_j).$

(ii) *There is a polynomial* $Q(X) \in \mathbb{C}[X]$ *such that*

$$L(s, \pi, \wedge^2) = Q(q^{-s}) \prod_{1 \le k \le t} L(s, \Delta_k, \wedge^2) \prod_{1 \le i < j \le t} L(s, \Delta_i \times \Delta_j).$$

Proof. Let us take Δ_i to be associated to the segment $[\rho_i, \rho_i \nu, \ldots, \rho_i \nu^{\ell_i - 1}]$, with ρ_i an irreducible supercuspidal representation of GL_{r_i}, $m_i = r_i \ell_i$, and $m = \sum_{i=1}^t r_i \ell_i$. Keeping Theorem 2.3 in mind, we set

$$L(s, \pi_u, \wedge^2)^{-1} = \mathrm{lcm}\big\{L_{\mathrm{ex}}(s, \pi_u^{(a_1 r_1, a_2 r_2, \ldots, a_t r_t)}, \wedge^2)^{-1}\big\},$$

where $0 \le a_i \le \ell_i$, $m - \sum_{i=1}^t a_i r_i$ is an even number, and the least common multiple is taken in terms of divisibility in $\mathbb{C}[q^{\pm s}]$. Suppose that $L_{\mathrm{ex}}(s, \pi_u^{(a_1 r_1, a_2 r_2, \ldots, a_t r_t)}, \wedge^2)$ has a pole at $s = s_0$. If the number of indices i such that $r_i \ne \ell_i$ is more than 3, we deduce from Proposition 2.10 that:

(i) There are $(i, j) \in \{1, 2, \ldots, t\}$, with $i \ne j$, such that $m_i - a_i r_i$ and $m_j - a_j r_j$ are even, and $L\big(s, \Delta_i^{(a_i r_i)} \nu^{u_i}, \wedge^2\big)$ and $L\big(s, \Delta_j^{(a_j r_j)} \nu^{u_j}, \wedge^2\big)$ have $s = s_0$ as a common pole.

(ii) There are $(i, j, k, \ell) \in \{1, 2, \dots, t\}$, with $\{i, j\} \neq \{k, \ell\}$, such that the functions $L\big(s, \Delta_i^{(a_i r_i)} \nu^{u_i} \times \Delta_j^{(a_j r_j)} \nu^{u_j}\big)$ and $L\big(s, \Delta_k^{(a_k r_k)} \nu^{u_k} \times \Delta_\ell^{(a_\ell r_\ell)} \nu^{u_\ell}\big)$ have $s = s_0$ as a common pole.

(iii) There are $(i, j, k) \in \{1, 2, \dots, t\}$, with $i \neq j$, such that $m_k - a_k r_k$ is even, and $L\big(s, \Delta_i^{(a_i r_i)} \nu^{u_i} \times \Delta_j^{(a_j r_j)} \nu^{u_j}\big)$ and $L\big(s, \Delta_k^{(a_k r_k)} \nu^{u_k}, \wedge^2\big)$ have $s = s_0$ as a common pole.

However, Conditions (iii), (iv), and (v) of general positions ensure that the above scenario cannot happen as long as u is in general position, because exceptional poles are poles of original L-factors $L(s, \Delta_i \times \Delta_j)$ and $L(s, \Delta_k, \wedge^2)$. Owing to [Jo 2020a, Corollary 4.11], when there exists exactly one pair (i, j) of indices $i \neq j$ such that $r_i \neq \ell_i$ and $r_j \neq \ell_j$, we have

$$L_{\mathrm{ex}}\big(s, \mathrm{Ind}(\Delta_i^{(a_i r_i)} \nu^{u_i} \otimes \Delta_j^{(a_j r_j)} \nu^{u_j}), \wedge^2\big) = L_{\mathrm{ex}}\big(s, \Delta_i^{(a_i r_i)} \nu^{u_i} \times \Delta_j^{(a_j r_j)} \nu^{u_j}\big).$$

If i is the only index such that $r_i \neq \ell_i$, it is nothing but $L_{\mathrm{ex}}(s, \Delta_i^{(a_i r_i)} \nu^{u_i}, \wedge^2)$.

To summarize, $L_{\mathrm{ex}}(s, \pi_u^{(a_1 r_1, a_2 r_2, \dots, a_t r_t)}, \wedge^2)$ is equal to $L_{\mathrm{ex}}(s, \Delta_i^{(a_i r_i)} \nu^{u_i} \times \Delta_j^{(a_j r_j)} \nu^{u_j})$ for $i < j$ or $L_{\mathrm{ex}}(s, \Delta_i^{(a_i r_i)} \nu^{u_i}, \wedge^2)$. Following the rest of the proof in [Jo 2020a, Theorem 5.1], we arrive at

$$\begin{aligned} L(s, \pi_u, \wedge^2) &= \prod_{1 \le k \le t} L(s, \Delta_k \nu^{u_k}, \wedge^2) \prod_{1 \le i < j \le t} L(s, \Delta_i \nu^{u_i} \times \Delta_j \nu^{u_j}) \\ &= \prod_{1 \le k \le t} L(s + 2u_k, \Delta_k, \wedge^2) \prod_{1 \le i < j \le t} L(s + u_i + u_j, \Delta_i \times \Delta_j). \end{aligned}$$

Concerning the second part, let $\mathcal{W}_\pi^{(0)}$ be the Whittaker model associated to π_u [Cogdell and Piatetski-Shapiro 2017, §3.1]. For $W_u \in \mathcal{W}_\pi^{(0)}$, it follows from the standard Bernstein's principle of meromorphic continuation and rationality [Jo 2020a, Propositions 4.2 and 4.4] that $J(s, W_u, \Phi)$ defines a rational function in $\mathbb{C}(q^{-s}, q^{-u})$. We conclude (i), that the rational function

$$\frac{J(s, W_u, \Phi)}{\prod_{1 \le k \le t} L(s + 2u_k, \Delta_k, \wedge^2) \prod_{1 \le i < j \le t} L(s + u_i + u_j, \Delta_i \times \Delta_j)}$$

has no poles on the Zariski open set of u in general position. We can take one step further to assert that the ratio lies in $\mathbb{C}[q^{\pm s}, q^{\pm u}]$ by the proof of [Matringe 2015, Lemma 5.1] and [Jo 2020a, Proposition 5.3]. The statement is now an immediate consequence of specialization to $u = 0$. □

We denote by $P \sim Q$ that the ratio is a unit in $\mathbb{C}[q^{\pm s}]$ for two rational functions $P(q^{-s})$ and $Q(q^{-s})$ in $\mathbb{C}(q^{-s})$. As alluded in the Langlands–Shahidi method [Ganapathy and Lomelí 2015; Henniart and Lomelí 2011; 2013b; Lomelí 2016], the unit emerging in Theorem 2.12 (ii) will be presumably 1. This is so-called the *multiplicativity of γ-factors*. However, demonstrating the multiplicativity property requires manipulating integrals in a delicate manner. Nonetheless, it seems likely that the weaker one that is relevant to us is enough for the application therein.

Theorem 2.12. *Assume the working hypothesis. Let* $\pi = \mathrm{Ind}_{\mathrm{Q}}^{\mathrm{GL}_m}(\Delta_1 \otimes \Delta_2 \otimes \cdots \otimes \Delta_t)$ *be a parabolically induced representation of* GL_m. *Let* $u = (u_1, u_2, \ldots, u_t) \in \mathcal{D}_\pi$ *be in general position and* $\pi_u = \mathrm{Ind}_{\mathrm{Q}}^{\mathrm{GL}_m}(\Delta_1 \nu^{u_1} \otimes \Delta_2 \nu^{u_2} \otimes \cdots \otimes \Delta_t \nu^{u_t})$ *be the deformed representation. Then we have the following*:

(i) $\gamma(s, \pi_u, \wedge^2, \psi) \sim \prod_{1 \le k \le t} \gamma(s+2u_k, \Delta_k, \wedge^2, \psi) \prod_{1 \le i < j \le t} \gamma(s+u_i+u_j, \Delta_i \times \Delta_j, \psi)$,

(ii) $\gamma(s, \pi, \wedge^2, \psi) \sim \prod_{1 \le k \le t} \gamma(s, \Delta_k, \wedge^2, \psi) \prod_{1 \le i < j \le t} \gamma(s, \Delta_i \times \Delta_j, \psi)$.

Proof. The proof proceeds along the line of [Jo 2020a, Proposition 5.4] and [Matringe 2015, Proposition 5.5] by applying Theorem 2.2 and Proposition 2.11 to our framework, and this idea originated from Cogdell and Piatetski-Shapiro [2017, Proposition 4.3]. Statement (ii) can be shown by specializing to $u = 0$. □

To proceed further, we adopt the terminology from [Cogdell and Piatetski-Shapiro 2017; Matringe 2015]. We say that $\pi \in \mathcal{A}_F(m)$ is a representation of *Langlands type* if Ξ has the form $\mathrm{Ind}_{\mathrm{Q}}^{\mathrm{GL}_m}(\Delta_{\circ 1}\nu^{u_1} \otimes \Delta_{\circ 2}\nu^{u_2} \otimes \cdots \otimes \Delta_{\circ t}\nu^{u_t})$, where each $\Delta_{\circ i}$ is the irreducible square integrable representation of GL_{m_i}, $m_1 + m_2 + \cdots + m_t = m$, each u_i is real, and they are ordered so that $u_1 \ge u_2 \ge \cdots \ge u_t$. Let π be an irreducible admissible representation of GL_m. Regardless of being generic, π can be realized as the unique Langlands quotient of Langlands type $\Xi = \mathrm{Ind}_{\mathrm{Q}}^{\mathrm{GL}_m}(\Delta_{\circ 1}\nu^{u_1} \otimes \Delta_{\circ 2}\nu^{u_2} \otimes \cdots \otimes \Delta_{\circ t}\nu^{u_t})$ which is of Whittaker type. The exterior square L-factor is defined to be

$$L(s, \Xi, \wedge^2) = L(s, \pi, \wedge^2).$$

Theorem 2.13. *Assume the working hypothesis. Consider a representation of Langlands type of* GL_m, $\pi = \mathrm{Ind}_{\mathrm{Q}}^{\mathrm{GL}_m}(\Delta_{\circ 1}\nu^{u_1} \otimes \Delta_{\circ 2}\nu^{u_2} \otimes \cdots \otimes \Delta_{\circ t}\nu^{u_t})$. *Then we have*

$$L(s, \pi, \wedge^2) = \prod_{1 \le k \le t} L(s + 2u_k, \Delta_{\circ k}, \wedge^2) \prod_{1 \le i < j \le t} L(s + u_i + u_j, \Delta_{\circ i} \times \Delta_{\circ j}).$$

Proof. The proof is akin to those of [Cogdell and Piatetski-Shapiro 2017, Theorem 4.1], [Jo 2020a, Theorem 5.7], and [Matringe 2009, Theorem 4.26]. In order to be concise, we do not include the complete details. □

We pass to the case of irreducible generic representations.

Corollary 2.14. *Assume the working hypothesis. Let* $\pi = \mathrm{Ind}_{\mathrm{Q}}^{\mathrm{GL}_m}(\Delta_1 \otimes \Delta_2 \otimes \cdots \otimes \Delta_t)$ *be an irreducible generic representation of* GL_m. *Then we have*

$$L(s, \pi, \wedge^2) = \prod_{1 \le k \le t} L(s, \Delta_k, \wedge^2) \prod_{1 \le i < j \le t} L(s, \Delta_i \times \Delta_j).$$

Proof. Since π is irreducible, essentially square integrable representations Δ_i can be rearranged to be in Langlands order without changing π. $\square$

We define the symmetric square L-factor to be the ratio of Rankin–Selberg L-factors for $\mathrm{GL}_m \times \mathrm{GL}_m$ by exterior square L-factors for GL_m:

$$L(s, \pi, \mathrm{Sym}^2) = \frac{L(s, \pi \times \pi)}{L(s, \pi, \wedge^2)}. \tag{2-1}$$

In comparison to [Matringe 2009; 2015], we pursue purely local means more to express a local exterior square L-function in terms of local L-functions for supercuspidal representations. Performing this step has the benefit of making the globalization result of Henniart and Lomelí [2011; 2013b] feasible, instead of globalizing discrete series representations [Kaplan 2017; Kewat and Raghunathan 2012; Matringe 2009] as a black box.

Theorem 2.15. *Assume that the working hypothesis holds for the subclass of all irreducible supercuspidal representations. Let* $\Delta_\circ = \left[\rho_\circ \nu^{-(\ell-1)/2}, \dots, \rho_\circ \nu^{(\ell-1)/2}\right]$ *be an irreducible square integrable representation of* $\mathrm{GL}_{\ell r}$, *with* $\rho_\circ$ *an irreducible unitary supercuspidal representation of* GL_r.

(i) *Suppose that* ℓ *is even. Then we have*

$$L(s, \Delta_\circ, \wedge^2) = \prod_{i=1}^{\ell/2} L\big(s, \rho_\circ \nu^{(\ell+1)/2-i}, \wedge^2\big) L\big(s, \rho_\circ \nu^{\ell/2-i}, \mathrm{Sym}^2\big);$$

$$L(s, \Delta_\circ, \mathrm{Sym}^2) = \prod_{i=1}^{\ell/2} L\big(s, \rho_\circ \nu^{(\ell+1)/2-i}, \mathrm{Sym}^2\big) L\big(s, \rho_\circ \nu^{\ell/2-i}, \wedge^2\big).$$

(ii) *Suppose that* ℓ *is odd. Then we have*

$$L(s, \Delta_\circ, \wedge^2) = \prod_{i=1}^{(\ell+1)/2} L\big(s, \rho_\circ \nu^{(\ell+1)/2-i}, \wedge^2\big) \prod_{i=1}^{(\ell-1)/2} L\big(s, \rho_\circ \nu^{\ell/2-i}, \mathrm{Sym}^2\big);$$

$$L(s, \Delta_\circ, \mathrm{Sym}^2) = \prod_{i=1}^{(\ell+1)/2} L\big(s, \rho_\circ \nu^{(\ell+1)/2-i}, \mathrm{Sym}^2\big) \prod_{i=1}^{(\ell-1)/2} L\big(s, \rho_\circ \nu^{\ell/2-i}, \wedge^2\big);$$

Proof. Our proof is truly influenced by Shahidi [1992, Proposition 8.1]. By the uniqueness of the Whittaker functional, the Whittaker model for $\Delta_\circ$ agrees with that for $\xi = \mathrm{Ind}\big(\rho_\circ \nu^{-(\ell-1)/2} \otimes \cdots \otimes \rho_\circ \nu^{(\ell-1)/2}\big)$. Likewise the same feature holds for $\xi^\iota := \mathrm{Ind}\big(\tilde{\rho}_\circ \nu^{-(\ell-1)/2} \otimes \cdots \otimes \tilde{\rho}_\circ \nu^{(\ell-1)/2}\big)$ and $\tilde{\Delta}_\circ$. This puts us in a position to manifest that

$$\gamma(s, \Delta_\circ, \wedge^2, \psi) = \gamma\big(s, \mathrm{Ind}\big(\rho_\circ \nu^{-(\ell-1)/2} \otimes \cdots \otimes \rho_\circ \nu^{(\ell-1)/2}\big), \wedge^2, \psi\big).$$

Let u be in general position and $\xi_u = \mathrm{Ind}\left(\rho_\circ \nu^{u_1-(\ell-1)/2} \otimes \cdots \otimes \rho_\circ \nu^{u_\ell+(\ell-1)/2}\right)$ its associated deformed representation. Upon noting the assumption that any (S_{2n}, Θ)-distinguished irreducible supercuspidal representation ρ is self-dual, we see that Proposition 2.10 to Theorem 2.12 can be completely carried over verbatim to the triple $(\Delta_\circ, \xi, \xi_u)$. The remainder of the proof is parallel to that of [Jo 2020a, Theorem 5.12] (cf. proof of Proposition 4.3), and we find

$$L(s, \Delta_\circ, \wedge^2) = \begin{cases} \displaystyle\prod_{i=1}^{\ell/2} L\left(s, \rho_\circ \nu^{(\ell+1)/2-i}, \wedge^2\right) L\left(s, \rho_\circ \nu^{\ell/2-i}, \mathrm{Sym}^2\right), & \ell \text{ even}, \\ \displaystyle\prod_{i=1}^{(\ell+1)/2} L\left(s, \rho_\circ \nu^{(\ell+1)/2-i}, \wedge^2\right) \prod_{i=1}^{(\ell-1)/2} L\left(s, \rho_\circ \nu^{\ell/2-i}, \mathrm{Sym}^2\right), & \ell \text{ odd}. \end{cases}$$

The expression of the local symmetric square L-function $L(s, \Delta_\circ, \mathrm{Sym}^2)$ is a direct consequence of the factorization $L(s, \Delta_\circ \times \Delta_\circ) = L(s, \Delta_\circ, \wedge^2) L(s, \Delta_\circ, \mathrm{Sym}^2)$, just as in (2-1). □

2D. ***The equality for principal series representations.*** We briefly review the Langlands–Shahidi method for the local exterior square L-function [Ganapathy and Lomelí 2015; Henniart and Lomelí 2011]. Let $\boldsymbol{G} = Sp_{2m}$ be a symplectic group over F in $2m$ variables. The group $\boldsymbol{M} \simeq \mathrm{GL}_m$ can be embedded as a Levi component of a maximal Siegel parabolic subgroup $\boldsymbol{P} = \boldsymbol{M}\boldsymbol{N}$ with unipotent radical $\boldsymbol{N}$. Let r be the adjoint representation of the L-group of $\boldsymbol{M}$ on ${}^L\mathfrak{n}$, the Lie algebra of the L-group of $\boldsymbol{N}$. We can check that $r = r_1 \oplus r_2$. The irreducible representation r_1 gives the standard γ-factor of GL_n and r_2 gives the *Langlands–Shahidi exterior square γ-factor*,

$$\gamma(s, \pi, r_2, \psi) = \gamma_{LS}(s, \pi, \wedge^2, \psi).$$

The γ-factor $\gamma_{LS}(s, \pi, \wedge^2, \psi)$ defined in [Henniart and Lomelí 2011] is a rational function in $\mathbb{C}(q^{-s})$. Let $P(X)$ be the unique polynomial in $\mathbb{C}[X]$ satisfying $P(0) = 1$ and such that $P(q^{-s})$ is the numerator of $\gamma_{LS}(s, \pi, \wedge^2, \psi)$. Whenever π is tempered, the *local Langlands–Shahidi exterior square L-function* is defined by

$$\mathcal{L}(s, \rho, \wedge^2) := P(q^{-s})^{-1}.$$

We observe that π tempered implies that $\mathcal{L}(s, \rho, \wedge^2)$ is holomorphic for $\mathrm{Re}(s) > 0$ [Henniart and Lomelí 2011, §4.6]. The *Langlands–Shahidi exterior square ε-factor* is defined to satisfy the relation

$$\varepsilon_{LS}(s, \pi, \wedge^2, \psi) = \gamma_{LS}(s, \pi, \wedge^2, \psi) \frac{\mathcal{L}(1-s, \tilde{\pi}, \wedge^2)}{\mathcal{L}(s, \pi, \wedge^2)}.$$

Besides, various types of L-factors $\mathcal{L}(s, \pi, \mathrm{Sym}^2)$ for $\boldsymbol{G} = SO_{2m+1}$, $\mathcal{L}(s, \pi \times \pi)$ for $\boldsymbol{G} = \mathrm{GL}_{2m}$, and $\mathcal{L}(s, \pi, \mathrm{As})$ for $\boldsymbol{G} = U_m$, can be extracted from [Henniart and Lomelí 2013b; Lomelí 2016].

Proposition 2.16. *Let Δ be a discrete series representation of the form*

$$[\chi, \chi\nu, \ldots, \chi\nu^{\ell-1}], \tag{2-2}$$

where χ is a character of $F^\times$. Then we have

$$L(s, \Delta, \wedge^2) = \mathcal{L}(s, \Delta, \wedge^2).$$

As a consequence, if $\ell = 2n$ is even and Δ is (S_{2n}, Θ)-distinguished, then Δ is self-dual.

Proof. Just as observed in Proposition 2.9, the working hypothesis does not need to be checked for the character χ of $F^\times$, and Δ is automatically self-dual. As in the proof of Theorem 4.4, we can easily reduce it to the case where Δ is a unitary representation. We are then left with applying Theorem 2.15 to Δ, from which the equality shall follow by comparing it with the work of Shahidi [1992, Proposition 8.1]. □

Let us turn our attention to the subclass of irreducible generic subquotients of principal series representations. This class is not necessarily spherical.

Proposition 2.17. *Let π be an irreducible generic subquotient of a principal series representation of GL_m. Then we have*

$$L(s, \pi, \wedge^2) = \mathcal{L}(s, \pi, \wedge^2).$$

Proof. From [Bernstein and Zelevinsky 1977; Zelevinsky 1980], π is of the form $\mathrm{Ind}(\Delta_1 \otimes \Delta_2 \otimes \cdots \otimes \Delta_t)$, where each Δ_i is either a character χ_i of $F^\times$ or a discrete series representation given by the segment of the form (2-2). In considering Proposition 2.16, any (S_{2n_i}, Θ)-distinguished representations Δ_i satisfy the working hypothesis. The inductive relation formula, Corollary 2.14, is applicable, and it can be shown that

$$L(s, \pi, \wedge^2) = \prod_{1 \le k \le t} L(s, \Delta_k, \wedge^2) \prod_{1 \le i < j \le t} L(s, \Delta_i \times \Delta_j).$$

In the aspect of Proposition 2.16, we only need to compare it with [Ganapathy and Lomelí 2015, Theorem 3.1 (xi)]. □

The unramified character χ means that it is invariant under the maximal compact subgroup $\mathcal{O}^\times$ of $F^\times$. As before, the working hypothesis is no longer needed for the set of irreducible unramified representations. Hence, Corollary 2.14 in the preceding section Section 2C, has the following result:

Corollary 2.18. *Let $\pi = \mathrm{Ind}_{B_m}^{\mathrm{GL}_m}(\chi_1 \otimes \chi_2 \otimes \cdots \otimes \chi_m)$ be an irreducible full induced representation from the Borel subgroup of unramified character χ_i of $F^\times$. Then*

$$L(s, \pi, \wedge^2) = \prod_{1 \le i < j \le m} \frac{1}{1 - \chi_i(\varpi)\chi_j(\varpi) q^{-s}}.$$

3. Local to global argument

3A. ***Eulerian integral representations.*** We denote by $\mathbb{F}_q$ the residue field of F, and let $k = \mathbb{F}_q(t)$ be a (global) function field of the projective line $\mathbb{P}^1$ over $\mathbb{F}_q$. Let $\mathbb{A}$ denote its ring of adèles. Let (Π, V_Π) be a cuspidal automorphic representation of $\mathrm{GL}_m(\mathbb{A})$. We denote by $|\mathbb{P}^1|$ the set of closed points of $\mathbb{P}^1$. The set $|\mathbb{P}^1|$ is in bijection with the set of places of k. Hence we write by abuse of notation $|\mathbb{P}^1|$ for the set of places of k. Since Π is irreducible, we have restricted tensor product decomposition $\Pi = \bigotimes_v' \Pi_v$ with (Π_v, V_{Π_v}) irreducible admissible generic representations of $\mathrm{GL}_m(k_v)$ [Flath 1979], see [Cogdell 2003, §4]. Let its central character be ω_Π. We let $P_{n-1,1} = Z_n P_n$ be the standard parabolic subgroup associated to the partition $(n-1, 1)$ of n. Each $\Phi \in \mathcal{S}(\mathbb{A}^n)$ defines a smooth function on $\mathrm{GL}_n(\mathbb{A})$, left invariant by $P_n(\mathbb{A})$, by $g \mapsto \Phi(e_n g)$ for $g \in \mathrm{GL}_n(\mathbb{A})$. We consider the function

$$f(s, g; \Phi, \omega_\Pi) = |\det(g)|^s \int_{\mathbb{A}^\times} \omega_\Pi(z)\Phi(z e_n g)|z|^{ns}\, d^\times z,$$

with the absolute convergence of the integral [Jacquet and Shalika 1981, (4.1)]. We extend ω_π to a character of $P_{n-1,1}$ by $\omega_\Pi(p) = \omega_\Pi(a)$ for $p = \left(\begin{smallmatrix} h & u \\ & a \end{smallmatrix}\right) \in P_{n-1,1}$. We construct the Eisenstein series by

$$E(s, g; \Phi, \omega_\Pi) = \sum_{\gamma \in P_{n-1,1}(k)\backslash \mathrm{GL}_n(k)} F(s, \gamma g; \Phi, \omega_\Pi)$$

This series is convergent absolutely for $\mathrm{Re}(s) > 1$ [Jacquet and Shalika 1981, (4.1)]. The mirabolic (Godement–Jacquet) Eisenstein series $E(s, g; \Phi, \omega_\Pi)$ has a meromorphic continuation to all of $\mathbb{C}$ and satisfies the following functional equation [Jacquet and Shalika 1981, §4]:

$$E(s, g; \Phi, \omega_\Pi) = E(1-s, {}^\iota g; \hat{\Phi}, \omega_\Pi^{-1}), \tag{3-1}$$

where ${}^\iota g = {}^t g^{-1}$ and the Fourier transform on $\mathcal{S}(\mathbb{A}^n)$ is defined by

$$\hat{\Phi}(y) = \int_{\mathbb{A}^n} \Phi(x)\psi(x\, {}^t y)\, dx.$$

For $m = 2n$, $\Phi \in \mathcal{S}(\mathbb{A}^n)$, and $\varphi \in V_\Pi$, we let

$$I_\psi(s, \varphi, \Phi) = \int_{Z_n(k)\,\mathrm{GL}_n(k)\backslash \mathrm{GL}_n(\mathbb{A})} \int_{\mathcal{M}_n(k)\backslash \mathcal{M}_n(\mathbb{A})} \varphi\left(\begin{pmatrix} I_n & X \\ & I_n \end{pmatrix}\begin{pmatrix} g & \\ & g \end{pmatrix}\right) \times \psi^{-1}(\mathrm{Tr}(X)) E(s, g : \Phi, \omega_\Pi)\, dX\, dg.$$

For $m = 2n+1$, $\Phi \in \mathcal{S}(\mathbb{A}^n)$, and $\varphi \in V_\Pi$, we define a global integral as

$$\begin{aligned} I_\psi(s, \varphi, \Phi) &= \int_{\mathbb{A}^n} \int_{\mathrm{GL}_n(k)\backslash \mathrm{GL}_n(\mathbb{A})} \int_{\mathcal{M}_n(k)\backslash\mathcal{M}_n(\mathbb{A})} \int_{k^n\backslash\mathbb{A}^n} \varphi\left(\begin{pmatrix} I_n & X & Z \\ & I_n & \\ & & 1 \end{pmatrix}\begin{pmatrix} g & & \\ & g & \\ & & 1 \end{pmatrix}\begin{pmatrix} I_n & & \\ & I_n & \\ & y & 1 \end{pmatrix}\right) \\ &\qquad\qquad \times \psi^{-1}(\mathrm{Tr}(X))\Phi(y)\,|\det(g)|^{s-1}\,dZ\,dX\,dg\,dy. \end{aligned}$$

The following theorem gives a meaning to these global integrals:

Theorem 3.1. *The integral $I_\psi(s, \varphi, \Phi)$ is convergent for* $\mathrm{Re}(s)$ *large enough, represents a meromorphic function on the entire plane, and satisfies the functional equation*

$$I_\psi(s, \varphi, \Phi) = I_{\psi^{-1}}(1-s, \varrho(\tau_m)\tilde{\varphi}, \hat{\Phi}),$$

where ϱ denotes right translation and $\tilde{\varphi}(g) = \varphi({}^\iota g)$.

Proof. The analytic properties have been established for the even case $m = 2n$ in [Jacquet and Shalika 1990, §5] and the odd case $m = 2n+1$ in [Jacquet and Shalika 1990, §9]. The functional equation for $m = 2n$ follows immediately from that of the Eisenstein series $E(s, g : \Phi, \omega_\Pi)$ (3-1). See also [Kewat and Raghunathan 2012, Theorem 3.11]. We take this occasion to refine the elaboration for $m = 2n+1$ in [Cogdell and Matringe 2015, §3.5] thoroughly. If $\varphi \in V_\Pi$, then φ_1 and φ_2 are defined in [Jacquet and Shalika 1990, p. 219]:

$$\varphi_1(g) = \int_{\mathbb{A}^n} \varphi\left(g\begin{pmatrix} I_n & & \\ & I_n & \\ & y & 1 \end{pmatrix}\right)\Phi(y)\,dy; \quad \varphi_2(g) = \int_{\mathbb{A}^n} \varphi\left(g\begin{pmatrix} I_n & & y \\ & I_n & \\ & & 1 \end{pmatrix}\right)\hat{\Phi}(-{}^t y)\,dy,$$

where $\Phi \in \mathcal{S}(\mathbb{A}^n)$. We begin to deal with the equation on the bottom of page 219 in [Jacquet and Shalika 1990]:

$$\begin{aligned} &\int_{k^n\backslash\mathbb{A}^n} \int_{\mathcal{M}_n(k)\backslash\mathcal{M}_n(\mathbb{A})} \varphi_1\left(\begin{pmatrix} I_n & X & \\ & I_n & \\ & & 1 \end{pmatrix}\begin{pmatrix} I_n & & Z \\ & I_n & \\ & & 1 \end{pmatrix}\begin{pmatrix} g & & \\ & g & \\ & & 1 \end{pmatrix}\right)\psi^{-1}(\mathrm{Tr}(X))\,dX\,dZ \\ &\quad = \int_{k^n\backslash\mathbb{A}^n} \int_{\mathcal{M}_n(k)\backslash\mathcal{M}_n(\mathbb{A})} \varphi_2\left(\begin{pmatrix} I_n & X & \\ & I_n & \\ & & 1 \end{pmatrix}\begin{pmatrix} I_n & & \\ & I_n & \\ & Z & 1 \end{pmatrix}\begin{pmatrix} g & & \\ & g & \\ & & 1 \end{pmatrix}\right) \\ &\qquad\qquad \times \psi^{-1}(\mathrm{Tr}(X))\,dX\,dZ\,|\det(g)|. \end{aligned}$$

(Here, φ in the corresponding formula in [Jacquet and Shalika 1990, p. 219] seems to be φ_2). As opposed to Jacquet and Shalika who conjugate them with the permutation matrix

$$\begin{pmatrix} & w_n & \\ w_n & & \\ & & 1 \end{pmatrix},$$

we exploit τ_{2n+1}. This articulation is consistent with the shape of the local functional equation in [Cogdell and Matringe 2015, Theorem 3.1]. By applying $g \mapsto \tau_{2n+1}{}^{\iota}g\tau_{2n+1}^{-1}$, and then changing the variables $X \mapsto -X$ and $Z \mapsto -Z$, the above integral is written as

$$\int_{k^n\backslash\mathbb{A}^n}\int_{\mathcal{M}_n(k)\backslash\mathcal{M}_n(\mathbb{A})}\tilde{\varphi}_2\left(\begin{pmatrix} I_n & X & \\ & I_n & \\ & & 1\end{pmatrix}\begin{pmatrix} I_n & & Z \\ & I_n & \\ & & 1\end{pmatrix}\begin{pmatrix} {}^tg^{-1} & & \\ & {}^tg^{-1} & \\ & & 1\end{pmatrix}\tau_{2n+1}\right) \\ \times \psi(\mathrm{Tr}(X))\,dX\,dZ\,|\det(g)|.$$

We insert the definitions of φ_1 and φ_2 and utilize the assignment $g \mapsto \tau_{2n+1}{}^{\iota}g\tau_{2n+1}^{-1}$ on the last matrix. After the change of variables $y \mapsto -y$, the identity becomes

$$\int_{\mathbb{A}^n}\int_{k^n\backslash\mathbb{A}^n}\int_{\mathcal{M}_n(k)\backslash\mathcal{M}_n(\mathbb{A})}\varphi\left(\begin{pmatrix} I_n & X & Z \\ & I_n & \\ & & 1\end{pmatrix}\begin{pmatrix} g & & \\ & g & \\ & & 1\end{pmatrix}\begin{pmatrix} I_n & & \\ & I_n & \\ & y & 1\end{pmatrix}\right) \\ \times \psi^{-1}(\mathrm{Tr}(X))\Phi(y)\,dX\,dZ\,dy \\ = \int_{\mathbb{A}^n}\int_{k^n\backslash\mathbb{A}^n}\int_{\mathcal{M}_n(k)\backslash\mathcal{M}_n(\mathbb{A})}\tilde{\varphi}\left(\begin{pmatrix} I_n & X & Z \\ & I_n & \\ & & 1\end{pmatrix}\begin{pmatrix} {}^tg^{-1} & & \\ & {}^tg^{-1} & \\ & & 1\end{pmatrix}\begin{pmatrix} I_n & & \\ & I_n & \\ & y & 1\end{pmatrix}\tau_{2n+1}\right) \\ \times \psi(\mathrm{Tr}(X))\hat{\Phi}(y)\,dX\,dZ\,dy\,|\det(g)|$$

from which the desired global functional equation for integrals follows. $\square$

Let

$$W_\varphi(g) = \int_{N_m(k)\backslash N_m(\mathbb{A})}\varphi(ng)\psi^{-1}(n)\,dn,$$
$$\widetilde{W}_\varphi(g) = \int_{N_m(k)\backslash N_m(\mathbb{A})}\tilde{\varphi}(w_m ng)\psi(n)\,dn$$

be the associated Whittaker function of φ and $\tilde{\varphi}$, respectively. We have yet to check that our integrals are Eulerian.

Proposition 3.2 (Jacquet–Shalika). *For $\varphi \in V_\Pi$ and $\Phi \in \mathcal{S}(F^n)$, global Jacquet–Shalika integrals*

$$J_\psi(s, W_\varphi, \Phi) = \int_{N_n(\mathbb{A})\backslash\,\mathrm{GL}_n(\mathbb{A})}\int_{\mathcal{N}_n(\mathbb{A})\backslash\mathcal{M}_n(\mathbb{A})} W_\varphi\left(\begin{pmatrix} I_n & X \\ & I_n\end{pmatrix}\begin{pmatrix} g & \\ & g\end{pmatrix}\right) \\ \times \psi^{-1}(\mathrm{Tr}(X))\Phi(e_n g)\,|\det(g)|^s\,dX\,dg$$

in the even case $m = 2n$ and

$$J_\psi(s, W_\varphi, \Phi) = \int_{N_n(\mathbb{A})\backslash \mathrm{GL}_n(\mathbb{A})} \int_{\mathcal{N}_n(\mathbb{A})\backslash \mathcal{M}_n(\mathbb{A})} \int_{\mathbb{A}^n} W_\varphi\left(\begin{pmatrix} I_n & X & \\ & I_n & \\ & & 1 \end{pmatrix}\begin{pmatrix} g & & \\ & g & \\ & & 1\end{pmatrix}\begin{pmatrix} I_n & & \\ & I_n & \\ & y & 1\end{pmatrix}\right) \times \psi^{-1}(\mathrm{Tr}(X))\Phi(y)|\det(g)|^{s-1}\,dy\,dX\,dg$$

in the odd case $m = 2n+1$ converge when $\mathrm{Re}(s)$ *is sufficiently large and, when this is the case, we have*

$$I_\psi(s, \varphi, \Phi) = J_\psi(s, W_\varphi, \Phi).$$

We suppose, in addition, that $W_\varphi(g) = \prod_{v\in|\mathbb{P}^1|} W_{\varphi_v}(g_v)$, $\psi(n) = \prod_{v\in|\mathbb{P}^1|}\psi(n_v)$, and $\Phi(g) = \prod_{v\in|\mathbb{P}^1|}\Phi_v(g_v)$. Then, when $\mathrm{Re}(s)$ *is sufficiently large,*

$$J_\psi(s, W_\varphi, \Phi) = \prod_{v\in|\mathbb{P}^1|} J_{\psi_v}(s, W_{\varphi_v}, \Phi_v).$$

Likewise, the right-hand side of the functional equation is also unfold and can be factored as

$$\begin{aligned} I_{\psi^{-1}}(1-s, \varrho(\tau_m)\tilde{\varphi}, \hat{\Phi}) &= J_{\psi^{-1}}(1-s, \varrho(\tau_m)\widetilde{W}_\varphi, \hat{\Phi}) \\ &= \prod_{v\in|\mathbb{P}^1|} J_{\psi^{-1}}(1-s, \varrho(\tau_m)\widetilde{W}_{\varphi_v}, \hat{\Phi}_v), \end{aligned}$$

with the convergence for $\mathrm{Re}(s) \ll 0$.

Proof. All these statements are drawn, with some minor changes of notation, from [Jacquet and Shalika 1990, Proposition 5 in §6] for $m = 2n$ and [Jacquet and Shalika 1990, §9.2] for $m = 2n+1$. □

Throughout, we will take $S \subset |\mathbb{P}^1|$ to be a finite set of places such that for all $v \notin S$, Π_v and ψ_v are all unramified and ψ_v normalized. The partial L-function is a product of local factors

$$L^S(s, \Pi, \wedge^2) = \prod_{v\notin S} L(s, \Pi_v, \wedge^2).$$

More precisely, this product converges for $\mathrm{Re}(s)$ large enough (see [Jacquet and Shalika 1990, §8–9]). The global L-function and ε-factors for Π are

$$L(s, \Pi, \wedge^2, S) = \prod_{v\in|\mathbb{P}^1|} L(s, \Pi_v, \wedge^2) = L^S(s, \Pi, \wedge^2)\prod_{v\in S} L(s, \Pi_v, \wedge^2)$$

and

$$\varepsilon(s, \Pi, \wedge^2, S) = \prod_{v\in|\mathbb{P}^1|} \varepsilon(s, \Pi_v, \wedge^2, \psi_v) = \prod_{v\in S}\varepsilon(s, \Pi_v, \wedge^2, \psi_v).$$

As for the ε-factor, we know that $\varepsilon(s, \Pi_v, \wedge^2, \psi_v) \equiv 1$ for $v \notin S$. The independence of $\varepsilon(s, \Pi, \wedge^2, S)$ from the choice of ψ can be seen as a consequence of the global functional equation below.

Theorem 3.3. *The global L-function $L(s, \Pi, \wedge^2, S)$ has a meromorphic continuation to the entire plane, and it satisfies the global functional equation*

$$L(s, \Pi, \wedge^2, S) = \varepsilon(s, \Pi, \wedge^2, S) L(1-s, \widetilde{\Pi}, \wedge^2, S),$$

where $\varepsilon(s, \Pi, \wedge^2, S)$ is entire and nonvanishing. This identity further implies that $\varepsilon(s, \Pi, \wedge^2, S)$ is independent of ψ as well.

Proof. From the unfolding in Proposition 3.2, and the local calculation of [Jacquet and Shalika 1990, §7.2 and §9.4] together with Corollary 2.18, we know that for $\mathrm{Re}(s)$ large and for appropriate choice of φ, we have

$$\begin{aligned} I_\psi(s, \varphi, \Phi) = J_\psi(s, W_\varphi, \Phi) &= \prod_{v \in |\mathbb{P}^1|} J_{\psi_v}(s, W_{\varphi_v}, \Phi_v) \\ &= \biggl(\prod_{v \in S} J_{\psi_v}(s, W_{\varphi_v}, \Phi_v)\biggr) L^S(s, \Pi, \wedge^2) \\ &= \biggl(\prod_{v \in S} \frac{J_{\psi_v}(s, W_{\varphi_v}, \Phi_v)}{L(s, \Pi_v, \wedge^2)}\biggr) L(s, \Pi, \wedge^2, S) \\ &= \biggl(\prod_{v \in S} e_v(s, W_{\varphi_v}, \Phi_v)\biggr) L(s, \Pi, \wedge^2, S), \end{aligned}$$

where $e_v(s, W_{\varphi_v}, \Phi_v) = J_{\psi_v}(s, W_{\varphi_v}, \Phi_v)/L(s, \Pi_v, \wedge^2)$. It follows from Theorem 2.1, $e_v(s, W_{\varphi_v}, \Phi_v)$ is entire. Therefore $L(s, \Pi, \wedge^2, S)$ has a meromorphic continuation, as the integral $I_\psi(s, \varphi, \Phi)$ is a meromorphic function on the entire plane from Theorem 3.1. While on the other side, we obtain

$$\begin{aligned} I_{\psi^{-1}}(1-s, \varrho(\tau_m)\tilde{\varphi}, \hat{\Phi}) &= J_{\psi^{-1}}(1-s, \varrho(\tau_m)\widetilde{W}_\varphi, \hat{\Phi}) \\ &= \biggl(\prod_{v \in S} \tilde{e}_v(1-s, \varrho(\tau_m)\widetilde{W}_{\varphi_v}, \hat{\Phi}_v)\biggr) L(1-s, \widetilde{\Pi}, \wedge^2, S), \end{aligned}$$

with $\tilde{e}_v(1-s, \varrho(\tau_m)\widetilde{W}_{\varphi_v}, \hat{\Phi}_v) = J_{\psi^{-1}}\bigl(1-s, \varrho(\tau_m)\widetilde{W}_{\varphi_v}, \hat{\Phi}_v\bigr)/L(1-s, \widetilde{\Pi}_v, \wedge^2)$. However we derive from the local functional equation, Theorem 2.2, that

$$\begin{aligned} \tilde{e}_v(1-s, \varrho(\tau_m)\widetilde{W}_{\varphi_v}, \hat{\Phi}_v) &= \frac{J_{\psi^{-1}}(1-s, \varrho(\tau_m)\widetilde{W}_{\varphi_v}, \hat{\Phi}_v)}{L(1-s, \widetilde{\Pi}_v, \wedge^2)} \\ &= \varepsilon(s, \Pi_v, \wedge^2, \psi_v) \frac{J_{\psi_v}(s, W_{\varphi_v}, \Phi_v)}{L(s, \Pi_v, \wedge^2)} \\ &= \varepsilon(s, \Pi_v, \wedge^2, \psi_v) e_v(s, W_{\varphi_v}, \Phi_v). \end{aligned}$$

Combining these all together, we get

$$L(s,\Pi,\wedge^2,S)=\Big(\prod_{v\in S}\varepsilon(s,\Pi_v,\wedge^2,\psi_v)\Big)L(1-s,\widetilde{\Pi},\wedge^2,S)$$
$$=\varepsilon(s,\Pi,\wedge^2,S)L(1-s,\widetilde{\Pi},\wedge^2,S),$$

since for $v\notin S$ we know Π_v and ψ_v are unramified so that $\varepsilon(s,\Pi_v,\wedge^2,\psi_v)\equiv 1$. □

3B. ***The equality for discrete series representations.*** Let $k_0=\mathbb{F}_q((t))$ be the completion of k at the point $0\in|\mathbb{P}^1|$. We start with a local irreducible unitary supercuspidal representation $\rho_\circ$ and globalize it according to the result of Henniart and Lomelí [2011; 2013b, Theorem 3.1].

Theorem 3.4 (Henniart–Lomelí). *Let $\rho_\circ$ be an irreducible unitary supercuspidal representation of $\mathrm{GL}_m(F)$. We choose an isomorphism $\xi: F\xrightarrow{\sim}k_0$. Then there exists a cuspidal unitary automorphic representation $\Pi=\bigotimes'_v\Pi_v$ whose local components Π_v satisfy*:

- *$\rho_\circ$ corresponds to Π_0 via ξ*;
- *at the places $v\in|\mathbb{P}^1|$ away from 0, 1, and ∞, Π_v is irreducible and unramified*;
- *Π_1 is an irreducible generic subquotient of an unramified principal series representation*;
- *Π_∞ is an irreducible generic subquotient of a tamely ramified principal series representation.*

We have control at all places outside 0, which makes it possible to deduce the identity for irreducible supercuspidal representations.

Theorem 3.5 (supercuspidal cases). *Let ρ be an irreducible supercuspidal representation of GL_r. Then we have*

$$L(s,\rho,\wedge^2)=\mathcal{L}(s,\rho,\wedge^2).$$

As a consequence, if ρ is (S_{2n},Θ)-distinguished, then ρ is self-dual.

Proof. Twisting by an unramified character does not affect the conclusion, so we can assume that $\rho=\rho_\circ$ is unitary. (See the proof of Theorem 4.4 for details, cf. [Lomelí 2016, §6.6]). We define the Langlands–Shahidi global L-function and ε-factors for Π by

$$\mathcal{L}(s,\Pi,\wedge^2,S)=\prod_{v\in|\mathbb{P}^1|}\mathcal{L}(s,\Pi_v,\wedge^2),$$
$$\varepsilon_{LS}(s,\Pi,\wedge^2,\psi,S)=\prod_{v\in|\mathbb{P}^1|}\varepsilon_{LS}(s,\Pi_v,\wedge^2,\psi_v).$$

We choose a finite set $S = \{0, 1, \infty\}$ of places. Applying Theorem 3.4 to the irreducible unitary supercuspidal representation $\rho_\circ$, we obtain a cuspidal unitary automorphic representation Π. For our convenience, we rewrite the global functional equation in [Henniart and Lomelí 2011, §4.1 (vi)] as

$$\mathcal{L}(s, \Pi, \wedge^2, S) = \varepsilon_{LS}(s, \Pi, \wedge^2, S)\mathcal{L}(1-s, \widetilde{\Pi}, \wedge^2, S). \tag{3-2}$$

The function $\varepsilon_{LS}(s, \Pi, \wedge^2, S)$ is entire and nonvanishing. From the global functional equation given by Theorem 3.3 and (3-2), this means that the ratio of L-function satisfies

$$\frac{L(s, \Pi, \wedge^2, S)}{\mathcal{L}(s, \Pi, \wedge^2, S)} = \eta(s, \Pi, S)\frac{L(1-s, \widetilde{\Pi}, \wedge^2, S)}{\mathcal{L}(1-s, \widetilde{\Pi}, \wedge^2, S)},$$

where $\eta(s, \Pi, S) = \varepsilon(s, \Pi, \wedge^2, S)\varepsilon_{LS}(s, \Pi, \wedge^2, S)^{-1}$ is entire and nonvanishing. Applying the already established principal series representations in Corollary 2.18, along with Proposition 2.17, at the places $|\mathbb{P}^1| - \{0\}$ yields:

$$\prod_{v\notin\{0\}} L(s, \Pi_v, \wedge^2) = \prod_{v\notin\{0\}} \mathcal{L}(s, \Pi_v, \wedge^2),$$

$$\prod_{v\notin\{0\}} L(1-s, \widetilde{\Pi}_v, \wedge^2) = \prod_{v\notin\{0\}} \mathcal{L}(1-s, \widetilde{\Pi}_v, \wedge^2).$$

Therefore, at the remaining one place, we have

$$\frac{L(s, \rho_\circ, \wedge^2)}{\mathcal{L}(s, \rho_\circ, \wedge^2)} = \eta(s, \Pi, S)\frac{L(1-s, \tilde{\rho}_\circ, \wedge^2)}{\mathcal{L}(1-s, \tilde{\rho}_\circ, \wedge^2)}.$$

In view of [Ganapathy and Lomelí 2015] and [Kewat and Raghunathan 2012, Theorem 3.7], $L(s, \rho_\circ, \wedge^2)$ and $\mathcal{L}(s, \rho_\circ, \wedge^2)$ are regular and nonvanishing in the region $\mathrm{Re}(s) > 0$, whereas similar analytic properties for $L(1-s, \tilde{\rho}_\circ, \wedge^2)$ and $\mathcal{L}(1-s, \tilde{\rho}_\circ, \wedge^2)$ are valid in the half plane $\mathrm{Re}(s) < 1$. This forces that the ratio $L(s, \rho_\circ, \wedge^2)/\mathcal{L}(s, \rho_\circ, \wedge^2)$ is an entire and nonvanishing function, and hence it is a constant. Since these L-factors are normalized, these must be equal.

We now gain the full strength of flexibility to transport L-factors in the Langlands–Shahidi side to the Rankin–Selberg side. The L-factor $\mathcal{L}(s, \rho \times \rho)$ is decomposed as the product of $\mathcal{L}(s, \rho, \wedge^2)$ and $\mathcal{L}(s, \rho, \mathrm{Sym}^2)$ (see [Ganapathy and Lomelí 2015; Henniart and Lomelí 2011; Shahidi 1992, Corollary 8.2]). Then the pole of $\mathcal{L}(s, \rho, \wedge^2)$ at $s = 0$ detected by the existence of the Shalika functional [Jo 2020a, Theorem 3.6 (ii)] contributes the pole of $\mathcal{L}(s, \rho \times \rho)$. This is amount to saying that ρ is self-dual [Matringe 2015, Proposition 4.6]. □

Once we know the inductivity of ε-factors, we expect that $\eta(s, \Pi, S) \equiv 1$, independent of the choice of S. We now come to the case of discrete series representations.

Theorem 3.6 (discrete series cases). *Let Δ be a discrete series representation of* GL_m. *Then we have*

$$L(s, \Delta, \wedge^2) = \mathcal{L}(s, \Delta, \wedge^2).$$

As a consequence, if Δ is (S_{2n}, Θ)-distinguished, then Δ is self-dual.

Proof. As indicated in the proof of Theorem 4.4, after proper unramified twisting of Δ, we can easily reduce the equality to the case when Δ is a unitary representation of the form $\left[\rho_\circ \nu^{-(\ell-1)/2}, \dots, \rho_\circ \nu^{(\ell-1)/2}\right]$ with $\rho_\circ$ a unitary irreducible supercuspidal representation of GL_r (cf. [Lomelí 2016, §6.6]). Taking advantage of Theorems 2.15 and 3.5, this finally matches with the expression in [Shahidi 1992, Proposition 8.1]. Concerning the second assertion, we literally reiterate the second part of the proof of Theorem 3.5 line-by-line, and therefore we omit thorough arguments entirely. □

The identity can be extended to the class of all irreducible admissible representations of GL_m.

Theorem 3.7. *Let π be an irreducible admissible representation of* GL_m. *Then*

$$L(s, \pi, \wedge^2) = \mathcal{L}(s, \pi, \wedge^2).$$

Proof. We realize π as the unique Langlands quotient of Langlands type $\Xi = \mathrm{Ind}_{\mathrm{Q}}^{\mathrm{GL}_m}(\Delta_{\circ 1}\nu^{u_1} \otimes \Delta_{\circ 2}\nu^{u_2} \otimes \cdots \otimes \Delta_{\circ t}\nu^{u_t})$, which is again of Whittaker type. Thanks to Theorem 3.6, the working hypothesis is not required to be checked for discrete series representations. Then Theorem 2.13 gives us that

$$L(s, \Xi, \wedge^2) = \prod_{1 \le k \le t} L(s + 2u_k, \Delta_{\circ k}, \wedge^2) \prod_{1 \le i < j \le t} L(s + u_i + u_j, \Delta_{\circ i} \times \Delta_{\circ j}),$$

which coincides with corresponding decompositions in Langlands–Shahidi theory [Ganapathy and Lomelí 2015, §3.1 (xi)]. □

By exploiting the main result of Henniart and Lomelí [2011], it can be summarized that the definition of local exterior square L-function via the theory of integral representations is compatible with the local Langlands correspondence. In what follows, we let W_F' denote the Weil–Deligne group of F, and let ϕ an m-dimensional (complex-valued) Frobenius semisimple representation of W_F'. We call this the *Weil–Deligne representation* of W_F'. Let $\wedge^2$ denote the exterior representation of $\mathrm{GL}_m(\mathbb{C})$. We then denote by $L(s, \wedge^2(\phi))$ the *Artin exterior square L-factor* attached to ϕ.

Theorem 3.8. *Let π be an irreducible admissible representation of* $\mathrm{GL}_m(F)$ *and $\phi(\pi)$ the Weil–Deligne representation of W_F' corresponding to π under the local Langlands correspondence. Then*

$$L(s, \pi, \wedge^2) = \mathcal{L}(s, \pi, \wedge^2) = L\big(s, \wedge^2(\phi(\pi))\big).$$

4. Bump–Friedberg and Flicker zeta integrals

4A. ***Bump–Friedberg L-factors.*** Define the embedding $J : \mathrm{GL}_n \times \mathrm{GL}_n \to \mathrm{GL}_m$ by

$$J(g, g')_{k,l} = \begin{cases} g_{i,j}, & \text{if } k = 2i-1,\ l = 2j-1, \\ g'_{i,j}, & \text{if } k = 2i,\ l = 2j, \\ 0, & \text{otherwise,} \end{cases}$$

for $m = 2n$ and $J : \mathrm{GL}_{n+1} \times \mathrm{GL}_n \to \mathrm{GL}_m$ by

$$J(g, g')_{k,l} = \begin{cases} g_{i,j}, & \text{if } k = 2i-1,\ l = 2j-1, \\ g'_{i,j}, & \text{if } k = 2i,\ l = 2j, \\ 0, & \text{otherwise,} \end{cases}$$

for $m = 2n+1$. As for the intention of holding onto coherent terminology with [Matringe 2015], interested readers may perceive that we interchange the role of g and g' in [Bump and Friedberg 1990]. Let $\pi = \mathrm{Ind}_{\mathrm{Q}}^{\mathrm{GL}_m}(\Delta_1 \otimes \cdots \otimes \Delta_t)$ be a parabolically induced representation. For each Whittaker function $W \in \mathcal{W}(\pi, \psi)$ and Schwartz–Bruhat function $\Phi \in \mathcal{S}(F^n)$, we define Bump–Friedberg integrals:

$$Z(s_1, s_2, W, \Phi) = \int_{N_n \backslash \mathrm{GL}_n} \int_{N_n \backslash \mathrm{GL}_n} W(J(g, g'))\Phi(e_m J(g, g'))|\det(g)|^{s_1 - 1/2}|\det(g')|^{1/2+s_2-s_1}\, dg\, dg'$$

when $m = 2n$ and

$$Z(s_1, s_2, W, \Phi) = \int_{N_n \backslash \mathrm{GL}_n} \int_{N_{n+1} \backslash \mathrm{GL}_{n+1}} W(J(g, g'))\Phi(e_m J(g, g'))|\det(g)|^{s_1}|\det(g')|^{s_2-s_1}\, dg\, dg'$$

when $m = 2n+1$. If r is a real number, we denote by δ_r the character

$$\delta_r : J(g, g') \mapsto \left|\frac{\det(g)}{\det(g')}\right|^r.$$

We denote by χ_m and μ_m the characters of H_m:

$$\chi_m\left(w_m \begin{pmatrix} g & \\ & g' \end{pmatrix} w_m^{-1}\right) = \begin{cases} \mathbf{1}_{H_m}, & \text{for } m = 2n; \\ \left|\dfrac{\det(g)}{\det(g')}\right|, & \text{for } m = 2n+1; \end{cases}$$

$$\mu_m\left(w_m \begin{pmatrix} g & \\ & g' \end{pmatrix} w_m^{-1}\right) = \begin{cases} \left|\dfrac{\det(g)}{\det(g')}\right|, & \text{for } m = 2n; \\ \mathbf{1}_{H_m}, & \text{for } m = 2n+1. \end{cases}$$

We turn toward the case for $s_1 = s + \frac{1}{2}$ and $s_2 = 2s$. We unify Bump–Friedberg zeta integrals as one single integral of the form

$$Z(s, W, \Phi) = \int_{(N_m \cap H_m) \backslash H_m} W(h)\chi_m^{1/2}(h)\Phi(e_m h)|\det(h)|^s\, dh.$$

The twisted analogue of Bump–Friedberg zeta integrals attached to χ_α is defined by

$$Z(s, W, \Phi, \chi_\alpha) = \int_{(N_m \cap H_m)\backslash H_m} W(h)\chi_\alpha(h)\chi_m^{1/2}(h)\Phi(e_m h)|\det(h)|^s\, dh.$$

The integral $Z(s, W, \Phi, \chi_\alpha)$ converges absolutely for s of real part large enough. The $\mathbb{C}$-vector space generated by Bump–Friedberg zeta integrals

$$\big\langle Z(s, W, \Phi, \chi_\alpha) \mid W \in \mathcal{W}(\pi, \psi),\, \Phi \in \mathcal{S}(F^m)\big\rangle$$

is a $\mathbb{C}[q^{\pm s}]$-fractional ideal $\mathcal{I}(\pi, \chi_\alpha, \mathrm{BF})$ of $\mathbb{C}(q^{-s})$. The ideal $\mathcal{I}(\pi, \chi_\alpha, \mathrm{BF})$ is principal and has a unique generator of the form $P(q^{-s})^{-1}$, where $P(X)$ is a polynomial in $\mathbb{C}[X]$ with $P(0) = 1$. The *Bump–Friedberg L-factor* associated to π is defined by the unique normalized generator [Matringe 2015, Proposition 4.8]

$$L(s, \pi, \chi_\alpha, \mathrm{BF}) = \frac{1}{P(q^{-s})}.$$

If $\alpha = \mathbf{1}_{F^\times}$ is a trivial character, we write $L(s, \pi, \mathrm{BF})$ for $L(s, \pi, \chi_{\mathbf{1}_{F^\times}}, \mathrm{BF})$. The *Bump–Friedberg γ-factor*

$$\gamma(s, \pi, \mathrm{BF}, \psi) = \varepsilon(s, \pi, \mathrm{BF}, \psi)\frac{L(1/2 - s, \pi^\iota, \delta_{-1/2}, \mathrm{BF})}{L(s, \pi, \mathrm{BF})}$$

is a rational function in $\mathbb{C}(q^{-s})$ that depends on a choice of a nontrivial character ψ (see [Matringe 2015, Proposition 4.11]). While the proof of [Matringe 2014, Proposition 6.2] reflects the structure of Weil–Deligne representations, our aim is to show the factorization of $L(s, \pi, \chi_\alpha, \mathrm{BF})$ as a product of the standard L-factor $L(s + 1/2, \pi)$ and the exterior square L-factor $L(2s, \pi, \wedge^2)$ within the framework of the Rankin–Selberg method. Our approach here is more direct and concise.

Theorem 4.1 (supercuspidal cases). *Let ρ be an irreducible supercuspidal representation of* GL_r. *Then*

$$L(s, \rho, \mathrm{BF}) = L\big(s + \tfrac{1}{2}, \rho\big)L(2s, \rho, \wedge^2).$$

Proof. If $r = 1$, then ρ is a character of $F^\times$. The integral is just the Tate integral of the form $\int_{F^\times} \rho(z)\Phi(z)|z|^{s+1/2}\, d^\times z$, hence

$$L(s, \rho, \mathrm{BF}) = L\big(s + \tfrac{1}{2}, \rho\big) = L\big(s + \tfrac{1}{2}, \rho\big)L(2s, \rho, \wedge^2),$$

where the last equality comes from $L(2s, \rho, \wedge^2) = 1$ (see [Jo 2020a, Theorem 2.13]).

We deduce from Theorem 2.3 aligned with [Matringe 2015, Proposition 4.14] that all the poles of $L(s, \rho, \mathrm{BF})$ and $L(s, \rho, \wedge^2)$ are necessarily simple. Given $r = 2n+1$ with $n \geq 1$, the result of Matringe [2014, Theorem 3.1] (see Theorem 4.2) tells us that ρ cannot be H_{2n+1}-distinguished. According to [Jacquet 1979, §3.1] coupled with [Jo 2020a, Theorem 3.6 (ii)] and [Matringe 2015, Corollary 4.3], we have

$$L(s, \rho, \mathrm{BF}) = L(s, \rho) = L(s, \rho, \wedge^2) = 1.$$

Now we turn to the case when $r = 2n$. Analyzing poles of local L-functions is just a question of certain distinctions of representations. To be precise, [Jo 2020a, Theorem 3.6 (i)] together with [Matringe 2015, Corollary 4.3] and Section 2B lead us to the following equivalent statements:

(i) $L(2s, \rho, \wedge^2)$ has a pole at $s = s_0$;

(ii) $L(s, \rho, \mathrm{BF})$ has a pole at $s = s_0$;

(iii) $\rho\nu^{s_0}$ is (S_{2n}, Θ)-distinguished;

(iv) $\rho\nu^{s_0}$ is H_{2n}-distinguished.

The above characterization of poles of L-factors can be reinterpreted as

$$L(s, \rho, \mathrm{BF}) = L(2s, \rho, \wedge^2) = L\big(s + \tfrac{1}{2}, \rho\big) L(2s, \rho, \wedge^2),$$

where the last identity follows from $L\big(s + \frac{1}{2}, \rho\big) = 1$ (see [Jacquet 1979, §3.1]). □

Unlike the case of Jacquet and Shalika's zeta integrals Sections 2 and 3, it is necessary to additionally use the hereditary property of H_{2m+1}-distinguished representations due to Matringe [2015, Theorem 3.1].

Theorem 4.2 (N. Matringe, $m = 2n+1$). *Let $\pi = \mathrm{Ind}_{\mathrm{Q}}^{\mathrm{GL}_{2n+1}}(\Delta_1 \otimes \Delta_2 \otimes \cdots \otimes \Delta_t)$ be an irreducible generic representation of GL_{2n+1}. Let α be a character of $F^\times$ with $0 \le \mathrm{Re}(\alpha) \le \frac{1}{2}$. Then π is $(H_{2n+1}, \chi_\alpha \delta_{-1/2})$-distinguished if and only if π is a parabolically induced representation of the form $\mathrm{Ind}_{P_{2n,1}}^{\mathrm{GL}_{2n+1}}(\pi' \otimes \alpha\nu^{-1/2})$, for π' an irreducible generic (H_{2n}, χ_α)-distinguished representation of GL_{2n} such that $\mathrm{Ind}_{P_{2n,1}}^{\mathrm{GL}_{2n+1}}(\pi' \otimes \alpha\nu^{-1/2})$ is still irreducible and generic.*

Throughout the rest of Section 4A, a variant of the systematic machinery developed in Section 2C (in particular, Proposition 2.10 to Theorem 2.12) should continue to work out in the context of Bump–Friedberg zeta integrals, and it is dealt with in [Matringe 2015, §4] in great detail and clarity. By doing so, Bump–Friedberg local L-functions are compatible with the classification of discrete series representation in terms of supercuspidal ones owing to Bernstein and Zelevinsky [1977] and [Zelevinsky 1980].

Proposition 4.3. *Let $\Delta_\circ = [\rho_\circ\nu^{-(\ell-1)/2}, \ldots, \rho_\circ\nu^{(\ell-1)/2}]$ be an irreducible square integrable representation of $\mathrm{GL}_{\ell r}$, with $\rho_\circ$ an irreducible unitary supercuspidal representation of GL_r.*

(i) *Suppose that ℓ is even. Then we have*

$$\begin{aligned} &L(s, \Delta_\circ, \mathrm{BF}) \\ &\quad = L(s + \ell/2, \rho_\circ) \prod_{i=1}^{\ell/2} L(2s, \rho_\circ\nu^{(\ell+1)/2-i}, \wedge^2) L(2s, \rho_\circ\nu^{\ell/2-i}, \mathrm{Sym}^2). \end{aligned}$$

(ii) *Suppose that ℓ is odd. Then we have*

$$L(s,\Delta_\circ,\mathrm{BF}) = L\Big(s+\frac{\ell}{2},\rho_\circ\Big)\prod_{i=1}^{(\ell+1)/2} L(2s,\rho_\circ\nu^{(\ell+1)/2-i},\wedge^2)\prod_{i=1}^{(\ell-1)/2} L(2s,\rho_\circ\nu^{\ell/2-i},\mathrm{Sym}^2).$$

Proof. By the uniqueness of the Whittaker functional, the Whittaker model for $\Delta_\circ$ coincides with that for $\mathrm{Ind}(\rho_\circ\nu^{-(\ell-1)/2}\otimes\cdots\otimes\rho_\circ\nu^{(\ell-1)/2})$. Likewise the same trait holds for dual objects provided by $\mathrm{Ind}(\tilde{\rho}_\circ\nu^{-(\ell-1)/2}\otimes\cdots\otimes\tilde{\rho}_\circ\nu^{(\ell-1)/2})$ and $\tilde{\Delta}_\circ$. According to [Matringe 2015, Proposition 5.5],

$$\gamma(s,\Delta_\circ,\mathrm{BF},\psi) \sim \prod_{i=0}^{\ell-1}\gamma\Big(s+\frac{1-\ell}{2}+i,\rho_\circ,\mathrm{BF},\psi\Big)\prod_{0\le i<j\le\ell-1}\gamma(2s+1-\ell+i+j,\rho_\circ\times\rho_\circ,\psi).$$

With the help of Theorem 4.1, the expression can be reformulated in terms of L-factors as

$$\gamma(s,\Delta_\circ,\mathrm{BF},\psi)\sim\prod_{i=0}^{\ell-1}\frac{L(-s-i+\ell/2,\tilde{\rho}_\circ)}{L(s+i+1-\ell/2,\rho_\circ)}\prod_{i=0}^{\ell-1}\frac{L(-2s+\ell-2i,\tilde{\rho}_\circ,\wedge^2)}{L(2s+1-\ell+2i,\rho_\circ,\wedge^2)} \times\prod_{0\le i<j\le\ell-1}\frac{L(-2s+\ell-i-j,\tilde{\rho}_\circ\times\tilde{\rho}_\circ)}{L(2s+1-\ell+i+j,\rho_\circ\times\rho_\circ)}.$$

By virtue of [Jo 2020a, Lemma 5.11] combined with $L(-s,\rho_\circ)\sim L(s,\tilde{\rho}_\circ)$ (see [Jacquet et al. 1983, §8.2 (15)–(16)]), it may be written as

$$\gamma(s,\Delta_\circ,\mathrm{BF},\psi)\sim\prod_{i=0}^{\ell-1}\frac{L(s+i-\ell/2,\rho_\circ)}{L(s+i+1-\ell/2,\rho_\circ)}\prod_{i=0}^{\ell-1}\frac{L(2s-\ell+2i,\rho_\circ,\wedge^2)}{L(2s+1-\ell+2i,\rho_\circ,\wedge^2)} \times\prod_{0\le i<j\le\ell-1}\frac{L(2s-\ell+i+j,\rho_\circ\times\rho_\circ)}{L(2s+1-\ell+i+j,\rho_\circ\times\rho_\circ)}.$$

We do the case where ℓ is even, the case where ℓ is odd is treated similarly. At this moment, we repeat the proof given in [Jo 2020a, Theorem 5.12] with adjusting s to $2s$. After canceling common factors, our quotient is simplified to

$$\gamma(s,\Delta_\circ,\mathrm{BF},\psi) \sim\frac{L(s-\ell/2,\rho_\circ)}{L(s+\ell/2,\rho_\circ)}\prod_{i=0}^{(\ell/2)-1}\frac{L(2s-\ell+2i,\rho_\circ,\wedge^2)L(2s-\ell+2i+1,\rho_\circ,\mathrm{Sym}^2)}{L(2s+2i+1,\rho_\circ,\wedge^2)L(2s+2i,\rho_\circ,\mathrm{Sym}^2)}. \tag{4-1}$$

Using [Matringe 2015, Corollary 4.1], $L\big(\frac{1}{2}-s,\tilde{\Delta}_\circ,\delta_{-1/2},\mathrm{BF}\big)^{-1}$ has zeros in the half plane $\mathrm{Re}(s)\ge\frac{1}{2}$, while $L(s,\Delta_\circ,\mathrm{BF})^{-1}$ has its zeros contained in the region $\mathrm{Re}(s)\le 0$. Since the half planes $\mathrm{Re}(s)\ge\frac{1}{2}$ and $\mathrm{Re}(s)\le 0$ are disjoint, they do not share factors in $\mathbb{C}[q^{\pm s}]$. As ρ is unitary, the poles of the product of L-factors in the numerator must lie on the line $\mathrm{Re}(s) = (\ell - i)/2$ for $i = 0, \dots, \ell - 2, \ell - 1$, while those in the denominator are located on the line $\mathrm{Re}(s) = -i/2$ for $i = 0, \dots, \ell - 2, \ell - 1, \ell$. Therefore, they do not have common factors at all. We establish the identity from the observation that the ratios (4-1) and $\gamma(s, \Delta_\circ, \mathrm{BF}, \psi) \sim L\big(\frac{1}{2} - s, \tilde{\Delta}_\circ, \delta_{-1/2}, \mathrm{BF}\big)/L(s, \Delta_\circ, \mathrm{BF})$ are all reduced and the indices i are rearranged. $\square$

Theorem 4.4 is the key step to improve the factorization to the set of discrete series representations. If we can do this, then the application of the Langlands classification theorem allows us to extend it to all irreducible admissible representations.

Theorem 4.4 (discrete series cases). *Let Δ be an irreducible essentially square integrable representation of* GL_m. *Then*

$$L(s, \Delta, \mathrm{BF}) = L\big(s + \tfrac{1}{2}, \Delta\big) L(2s, \Delta, \wedge^2).$$

Proof. We choose an unramified quasicharacter ν^{s_1}, $s_1 \in \mathbb{C}$, so that $\Delta = \Delta_\circ \nu^{s_1}$, where $\Delta_\circ$ is an irreducible square integrable representation of GL_m. We can easily verify that $L(s, \Delta_\circ \nu^{s_1}, \mathrm{BF}) = L(s + s_1, \Delta_\circ, \mathrm{BF})$, $L\big(s + \frac{1}{2}, \Delta_\circ \nu^{s_1}\big) = L\big(s + s_1 + \frac{1}{2}, \Delta_\circ\big)$, and $L(2s, \Delta_\circ \nu^{s_1}, \wedge^2) = L(2s + 2s_1, \Delta_\circ, \wedge^2)$. Hence for the calculation, we may assume that $\Delta = \Delta_\circ$ is unitary. The representation Δ is the segment consisting of supercuspidal representations of the form $\Delta_\circ = [\rho_\circ \nu^{-(\ell-1)/2}, \dots, \rho_\circ \nu^{(\ell-1)/2}]$, where $\rho_\circ$ is an irreducible unitary supercuspidal representation of GL_r with $m = \ell r$. We replace σ by $\mathbf{1}_{F^\times}$ in [Cogdell and Piatetski-Shapiro 2017, Corollary in §2.6.2]. Then the formula becomes

$$L\big(s + \tfrac{1}{2}, \Delta_\circ\big) = L\big(s + \tfrac{1}{2}, \Delta_\circ \times \mathbf{1}_{F^\times}\big) = L\big(s + \tfrac{\ell}{2}, \rho_\circ\big).$$

This may also be seen from [Jacquet 1979, Proposition 3.1.3]. Now we are left with invoking Theorem 2.15. $\square$

Finally, Theorem 4.5 renders the factorization result unconditional.

Theorem 4.5. *Let π be an irreducible admissible representation of* GL_m. *Then we have*

$$L(s, \pi, \mathrm{BF}) = L\Big(s + \frac{1}{2}, \pi\Big) L(2s, \pi, \wedge^2).$$

Proof. We realize π as the unique Langlands quotient of Langlands type $\Xi = \mathrm{Ind}_{\mathrm{Q}}^{\mathrm{GL}_m}(\Delta_{\circ 1} \nu^{u_1} \otimes \Delta_{\circ 2} \nu^{u_2} \otimes \cdots \otimes \Delta_{\circ t} \nu^{u_t})$, which is again of Whittaker type. The local Bump–Friedberg L-function is defined to be $L(s, \pi, \mathrm{BF}) = L(s, \Xi, \mathrm{BF})$. By [Matringe 2015, Theorem 5.2], we have the equality

$$L(s, \Xi, \mathrm{BF}) = \prod_{1 \le k \le t} L(s + u_k, \Delta_{\circ k}, \mathrm{BF}) \prod_{1 \le i < j \le t} L(2s + u_i + u_j, \Delta_{\circ i} \times \Delta_{\circ j}).$$

Applying Theorem 4.4, the product can be further decomposed as

$$L(s,\,\Xi,\,\mathrm{BF}) = \prod_{1\le k\le t} L\big(s+u_k+\tfrac{1}{2},\,\Delta_{\circ k}\big) \prod_{1\le k\le t} L(2s+2u_k,\,\Delta_{\circ k},\,\wedge^2) \times \prod_{1\le i<j\le t} L(2s+u_i+u_j,\,\Delta_{\circ i}\times\Delta_{\circ j}).$$

Collecting the contributions for the first product $\prod L\big(s+u_k+\frac{1}{2},\,\Delta_{\circ k}\big)$ gives the standard L-factor $L\big(s+\frac{1}{2},\,\Xi\big) = L\big(s+\frac{1}{2},\pi\big)$ by [Jacquet 1979, Theorem 3.4], while gathering those for the rest of the product

$$\prod L(2s+2u_k,\,\Delta_{\circ k},\,\wedge^2)\prod L(2s+u_i+u_j,\,\Delta_{\circ i}\times\Delta_{\circ j})$$

yields the exterior square L-factor $L(2s,\,\Xi,\,\wedge^2) = L(2s,\pi,\,\wedge^2)$ by Theorem 2.13. □

We end this section with relating Bump–Friedberg L-factors to Galois theoretic counterparts. In conclusion, it is a consequence of the local Langlands correspondence that $L\big(s+\frac{1}{2},\pi\big) = L\big(s+\frac{1}{2},\phi(\pi)\big)$ combined with Theorem 3.8 and Theorem 4.5.

Theorem 4.6. *Let π be an irreducible admissible representation of $\mathrm{GL}_m(F)$ and $\phi(\pi)$ its associated Weil–Deligne representation of W'_F under the local Langlands correspondence. Then we have*

$$L(s,\pi,\mathrm{BF}) = L\big(s+\tfrac{1}{2},\,\phi(\pi)\big)L\big(2s,\,\wedge^2(\phi(\pi))\big).$$

4B. ***Asai L-factors.*** Let E be a quadratic extension of F. We denote by $x\mapsto\bar{x}$ the nontrivial associated Galois action. We fix an element $z\in E^\times$ such that $\bar{z}=-z$ and a nontrivial character ψ_F of F. Let

$$\psi_E(x) = \psi_F\Big(\frac{x-\bar{x}}{z-\bar{z}}\Big), \quad x\in E.$$

Then the additive character ψ_E of E is trivial on F and defines a character of $N_m(E)$, which by abuse of notation we again denote by ψ_E. We shall use the Fourier transform induced by the additive character ψ on the space of Schwartz–Bruhat space $\mathcal{S}(F^m)$. Let $\pi = \mathrm{Ind}_{\mathrm{Q}}^{\mathrm{GL}_m}(\Delta_1\otimes\cdots\otimes\Delta_t)\in\mathcal{A}_E(m)$ be a parabolically induced representation with an associated Whittaker model $\mathcal{W}(\pi,\,\psi_E)$. For each Whittaker function $W\in\mathcal{W}(\pi,\,\psi_E)$ and each Schwartz–Bruhat function $\Phi\in\mathcal{S}(F^m)$, we define the local Flicker integral [1988; 1993] by

$$\mathcal{Z}(s,\,W,\,\Phi) = \int_{N_m\backslash\,\mathrm{GL}_m} W(g)\Phi(e_m g)\,|\det(g)|^s\,dg,$$

which is absolutely convergent when the real part of s is sufficiently large. Each $\mathcal{Z}(s,\,W,\,\Phi)$ is a rational function of q^{-s}, and hence extends meromorphically to all of $\mathbb{C}$. These integrals $\mathcal{Z}(s,\,W,\,\Phi)$ span a fractional ideal $\mathcal{I}(\pi,\,\mathrm{As})$ of $\mathbb{C}[q^{\pm s}]$

generated by a normalized generator of the form $P(q^{-s})^{-1}$, where the polynomial $P(X) \in \mathbb{C}[X]$ satisfies $P(0) = 1$. The *local Asai L-function* attached to π is defined by such a unique normalized generator [Matringe 2009, Definition 3.1]

$$L(s, \pi, \mathrm{As}) = \frac{1}{P(q^{-s})}.$$

Let us define the *local Asai ε-factor*, as usual [Matringe 2015, §3] (see [Anandavardhanan et al. 2021, §8]), by

$$\varepsilon(s, \pi, \psi, \mathrm{As}) = \gamma(s, \pi, \psi, \mathrm{As}) \frac{L(s, \pi, \mathrm{As})}{L(1-s, \pi^{\iota}, \mathrm{As})}.$$

The Weil–Deligne group W'_E of E is of index two in the Weil–Deligne group W'_F of F, and the quotient W'_F/W'_E is naturally identified with $\mathrm{Gal}(E/F)$. We fix an element σ in W'_F which does not belong to W'_E once and for all. The image of σ in W'_F/W'_E is the nontrivial element of $\mathrm{Gal}(E/F)$, which by abuse of notation is also denoted by σ. Given an m-dimensional (complex) Frobenius semisimple representation ϕ of W'_E, the *Asai representation* $\mathrm{As}(\phi) : W'_F \to \mathrm{GL}(\mathbb{C}^m \otimes \mathbb{C}^m) \simeq \mathrm{GL}_{m^2}(\mathbb{C})$ given by (twisted) tensor induction of ϕ is defined as (see [Anandavardhanan and Rajan 2005, §2.1], [Krishnamurthy 2012, §2], and [Shankman 2018, §1.2]):

$$\mathrm{As}(\phi)(\tau)(v \otimes w) = \begin{cases} \phi(\tau)(v) \otimes \phi(\sigma\tau\sigma^{-1})(w), & \text{if } \tau \in W'_E, \\ \phi(\tau\sigma^{-1})(w) \otimes \phi(\sigma\tau)(v), & \text{if } \tau \notin W'_E. \end{cases}$$

We then denote by $L(s, \mathrm{As}(\phi))$ the *Artin L-factor* attached to the Asai representation.

The conjugation σ extends naturally to an automorphism of $\mathrm{GL}_m(E)$, which we also denote by σ. If $\pi \in \mathcal{A}_E(m)$, we denote by π^σ the representation $g \mapsto \pi(\sigma(g))$.

Theorem 4.7. *Let π be an irreducible admissible representation of* $\mathrm{GL}_m(E)$ *and $\phi(\pi)$ its associated Weil–Deligne representation of W'_E under the local Langlands correspondence. Then we have*

$$L(s, \pi, \mathrm{As}) = \mathcal{L}(s, \pi, \mathrm{As}) = L\big(s, \mathrm{As}(\phi(\pi))\big).$$

Proof. We first consider the case of irreducible unitary supercuspidal representations $\rho_\circ$ of GL_r. As a consequence of [Anandavardhanan et al. 2021] joined with [Anandavardhanan and Rajan 2005, Proposition 6] and [Henniart and Lomelí 2013a], we have

$$L(s, \rho_\circ, \mathrm{As}) = \mathcal{L}(s, \rho_\circ, \mathrm{As}).$$

Let Δ be a discrete series representation. In the spirit of twists by unramified characters for Langlands–Shahidi theoretic L-factors [Henniart and Lomelí 2013a, §3.1 (vi)] and Rankin–Selberg theoretic L-factors [Matringe 2009, Theorem 2.3], there is no harm to assume that $\Delta = \Delta_\circ$ is an irreducible square integrable representation of $\mathrm{GL}_{r\ell}$ associated to the segment $[\rho_\circ \nu^{-(\ell-1)/2}, \dots, \rho_\circ \nu^{(\ell-1)/2}]$ with $\rho_\circ$ an irreducible unitary supercuspidal representation of GL_r. Let $\chi_{E/F}$ be an extension to $E^\times$ of the character $F^\times$ associated to E/F by the local class field theory. As explained in [Anandavardhanan et al. 2021, Appendix A], [Matringe 2009, Corollary 4.24] and [Matringe 2009, Theorem 4.26] driven from the Cogdell and Piatetski-Shapiro method similar to Section 2C depend on the complete classification of $\mathrm{GL}_m(F)$-distinguished representations [Matringe 2011]. Looking at the proof of this proposition, we need to check that the $\mathrm{GL}_m(F)$-distinguished representation, namely, $\mathrm{Hom}_{\mathrm{GL}_m(F)}(\pi, \mathbf{1}_{\mathrm{GL}_m(F)}) \neq \{0\}$, is still Galois self-dual, $\pi^\sigma \simeq \tilde{\pi}$, for any nonarchimedean local field of odd residual characteristic. It is presently written in this generality in the literature, see [Offen 2018, §3.2.12]. Counting on the weak multiplicativity of $\gamma(s, \pi, \mathrm{As}, \psi)$ [Matringe 2009, Corollary 4.24], we get the results below using arguments parallel to the one employed in the proof of Goldberg [1994, Theorem 5.6]:

$$L(s, \Delta_\circ, \mathrm{As}) = \prod_{i=1}^{\ell/2} L(s, \rho_\circ \nu^{(\ell+1)/2-i}, \mathrm{As}) L(s, \chi_{E/F} \otimes \rho_\circ \nu^{\ell/2-i}, \mathrm{As}),$$

when ℓ is even, and

$$L(s, \Delta_\circ, \mathrm{As}) = \prod_{i=1}^{(\ell+1)/2} L(s, \rho_\circ \nu^{(\ell+1)/2-i}, \mathrm{As}) \prod_{i=1}^{(\ell-1)/2} L(s, \chi_{E/F} \otimes \rho_\circ \nu^{\ell/2-i}, \mathrm{As}),$$

when ℓ is odd. The expression is similar to that in Theorem 2.15. This places us in a position to deduce

$$L(s, \Delta, \mathrm{As}) = \mathcal{L}(s, \Delta, \mathrm{As}) \tag{4-2}$$

for any discrete series representations Δ of $\mathrm{GL}_{r\ell}$.

In general, we realize π as the unique Langlands quotient of Langlands type $\Xi = \mathrm{Ind}_{\mathrm{Q}}^{\mathrm{GL}_m}(\Delta_{\circ 1} \nu^{u_1} \otimes \Delta_{\circ 2} \nu^{u_2} \otimes \cdots \otimes \Delta_{\circ t} \nu^{u_t})$. As such, by the inductive relation of $L(s, \pi, \mathrm{As})$ [Matringe 2009, Theorem 4.26], one has the equality

$$L(s, \pi, \mathrm{As}) = \prod_{1 \le k \le t} L(s + 2u_k, \Delta_{\circ k}, \wedge^2) \prod_{1 \le i < j \le t} L(s + u_i + u_j, \Delta_{\circ i} \times \Delta_{\circ j}^\sigma).$$

Consequently, the equality

$$L(s, \pi, \mathrm{As}) = \mathcal{L}(s, \pi, \mathrm{As})$$

follows from [Henniart and Lomelí 2013a, §4.2], along with (4-2) for all irreducible admissible representations π of $\mathrm{GL}_m(E)$. The remaining part is simply to quote the main theorem of Henniart and Lomelí [2013a, Theorem 4.3]. □

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References

[Anandavardhanan and Rajan 2005] U. K. Anandavardhanan and C. S. Rajan, "Distinguished representations, base change, and reducibility for unitary groups", *Int. Math. Res. Not.* **2005**:14 (2005), 841–854. MR Zbl

[Anandavardhanan et al. 2021] U. K. Anandavardhanan, R. Kurinczuk, N. Matringe, V. Sécherre, and S. Stevens, "Galois self-dual cuspidal types and Asai local factors", *J. Eur. Math. Soc.* **23**:9 (2021), 3129–3191. MR Zbl

[Bernstein and Zelevinsky 1976] J. N. Bernstein and A. V. Zelevinsky, "Representations of the group GL(n, F), where F is a local non-Archimedean field", *Uspehi Mat. Nauk* **31**:3(189) (1976), 5–70. In Russian; translated in *Russian Math. Surveys* **31**:3 (1976), 1–68. MR Zbl

[Bernstein and Zelevinsky 1977] I. N. Bernstein and A. V. Zelevinsky, "Induced representations of reductive $\mathfrak{p}$-adic groups, I", *Ann. Sci. École Norm. Sup.* (4) **10**:4 (1977), 441–472. MR Zbl

[Blanc and Delorme 2008] P. Blanc and P. Delorme, "Vecteurs distributions H-invariants de représentations induites, pour un espace symétrique réductif p-adique G/H", *Ann. Inst. Fourier* (*Grenoble*) **58**:1 (2008), 213–261. MR Zbl

[Bump and Friedberg 1990] D. Bump and S. Friedberg, "The exterior square automorphic L-functions on GL(n)", pp. 47–65 in *Festschrift in honor of I. I. Piatetski-Shapiro on the occasion of his sixtieth birthday*, *II* (Tel Aviv, 1989), edited by S. Gelbart et al., Israel Math. Conf. Proc. **3**, Weizmann, Jerusalem, 1990. MR Zbl

[Chen and Gan 2021] R. Chen and W. T. Gan, "Unitary Friedberg–Jacquet periods", preprint, 2021. arXiv 2108.04064

[Cogdell 2003] J. W. Cogdell, "Analytic theory of L-functions for GL$_n$", pp. 197–228 in *An introduction to the Langlands program* (Jerusalem, 2001), edited by J. Bernstein and S. Gelbart, Birkhäuser, Boston, 2003. MR Zbl

[Cogdell and Matringe 2015] J. W. Cogdell and N. Matringe, "The functional equation of the Jacquet–Shalika integral representation of the local exterior-square L-function", *Math. Res. Lett.* **22**:3 (2015), 697–717. MR Zbl

[Cogdell and Piatetski-Shapiro 1994] J. W. Cogdell and I. I. Piatetski-Shapiro, "Exterior square L-function for $\mathrm{GL}(n)$", lecture notes, Fields Inst., 1994, available at https://tinyurl.com/cogdellnotes.

[Cogdell and Piatetski-Shapiro 2017] J. W. Cogdell and I. I. Piatetski-Shapiro, "Derivatives and L-functions for GL_n", pp. 115–173 in *Representation theory, number theory, and invariant theory* (New Haven, CT, 2015), edited by J. Cogdell et al., Progr. Math. **323**, Birkhäuser, Cham, 2017. MR Zbl

[Flath 1979] D. Flath, "Decomposition of representations into tensor products", pp. 179–183 in *Automorphic forms, representations and L-functions, I* (Corvallis, OR, 1977), edited by A. Borel and W. Casselman, Proc. Sympos. Pure Math. **33**, Amer. Math. Soc., Providence, RI, 1979. MR Zbl

[Flicker 1988] Y. Z. Flicker, "Twisted tensors and Euler products", *Bull. Soc. Math. France* **116**:3 (1988), 295–313. MR Zbl

[Flicker 1993] Y. Z. Flicker, "On zeroes of the twisted tensor L-function", *Math. Ann.* **297**:2 (1993), 199–219. MR Zbl

[Gan 2019] W. T. Gan, "Periods and theta correspondence", pp. 113–132 in *Representations of reductive groups* (Jerusalem/Rehovot, Israel, 2017), edited by A. Aizenbud et al., Proc. Sympos. Pure Math. **101**, Amer. Math. Soc., Providence, RI, 2019. MR Zbl

[Gan and Lomelí 2018] W. T. Gan and L. Lomelí, "Globalization of supercuspidal representations over function fields and applications", *J. Eur. Math. Soc.* **20**:11 (2018), 2813–2858. MR Zbl

[Ganapathy and Lomelí 2015] R. Ganapathy and L. Lomelí, "On twisted exterior and symmetric square γ-factors", *Ann. Inst. Fourier (Grenoble)* **65**:3 (2015), 1105–1132. MR Zbl

[Gelfand and Kajdan 1975] I. M. Gelfand and D. A. Kajdan, "Representations of the group $\mathrm{GL}(n, K)$ where K is a local field", pp. 95–118 in *Lie groups and their representations* (Budapest, 1971), edited by I. M. Gelfand, Halsted, New York, 1975. MR Zbl

[Goldberg 1994] D. Goldberg, "Some results on reducibility for unitary groups and local Asai L-functions", *J. Reine Angew. Math.* **448** (1994), 65–95. MR Zbl

[Henniart 2010] G. Henniart, "Correspondance de Langlands et fonctions L des carrés extérieur et symétrique", *Int. Math. Res. Not.* **2010**:4 (2010), 633–673. MR Zbl

[Henniart and Lomelí 2011] G. Henniart and L. Lomelí, "Local-to-global extensions for GL_n in non-zero characteristic: a characterization of $\gamma_F(s, \pi, \mathrm{Sym}^2, \psi)$ and $\gamma_F(s, \pi, \wedge^2, \psi)$", *Amer. J. Math.* **133**:1 (2011), 187–196. MR Zbl

[Henniart and Lomelí 2013a] G. Henniart and L. Lomelí, "Characterization of γ-factors: the Asai case", *Int. Math. Res. Not.* **2013**:17 (2013), 4085–4099. MR Zbl

[Henniart and Lomelí 2013b] G. Henniart and L. Lomelí, "Uniqueness of Rankin–Selberg products", *J. Number Theory* **133**:12 (2013), 4024–4035. MR Zbl

[Jacquet 1979] H. Jacquet, "Principal L-functions of the linear group", pp. 63–86 in *Automorphic forms, representations and L-functions, II* (Corvallis, OR, 1977), edited by A. Borel and W. Casselman, Proc. Sympos. Pure Math. **33**, Amer. Math. Soc., Providence, RI, 1979. MR Zbl

[Jacquet and Shalika 1981] H. Jacquet and J. A. Shalika, "On Euler products and the classification of automorphic representations, I", *Amer. J. Math.* **103**:3 (1981), 499–558. MR Zbl

[Jacquet and Shalika 1990] H. Jacquet and J. A. Shalika, "Exterior square L-functions", pp. 143–226 in *Automorphic forms, Shimura varieties, and L-functions, II* (Ann Arbor, MI, 1988), edited by L. Clozel and J. S. Milne, Perspect. Math. **11**, Academic Press, Boston, 1990. MR Zbl

[Jacquet et al. 1983] H. Jacquet, I. I. Piatetskii-Shapiro, and J. A. Shalika, "Rankin–Selberg convolutions", *Amer. J. Math.* **105**:2 (1983), 367–464. MR Zbl

[Jo 2020a] Y. Jo, "Derivatives and exceptional poles of the local exterior square L-function for GL_m", *Math. Z.* **294**:3-4 (2020), 1687–1725. MR Zbl

[Jo 2020b] Y. Jo, "Factorization of the local exterior square L-function of GL_m", *Manuscripta Math.* **162**:3-4 (2020), 493–536. MR Zbl

[Jo 2021] Y. Jo, "Rankin–Selberg integrals for local symmetric square factors on GL(2)", *Mathematika* **67**:2 (2021), 388–421. MR

[Kable 2004] A. C. Kable, "Asai L-functions and Jacquet's conjecture", *Amer. J. Math.* **126**:4 (2004), 789–820. MR Zbl

[Kaplan 2017] E. Kaplan, "The characterization of theta-distinguished representations of GL(n)", *Israel J. Math.* **222**:2 (2017), 551–598. MR Zbl

[Kewat and Raghunathan 2012] P. K. Kewat and R. Raghunathan, "On the local and global exterior square L-functions of GL_n", *Math. Res. Lett.* **19**:4 (2012), 785–804. MR Zbl

[Krishnamurthy 2012] M. Krishnamurthy, "Determination of cusp forms on GL(2) by coefficients restricted to quadratic subfields", *J. Number Theory* **132**:6 (2012), 1359–1384. MR Zbl

[Lomelí 2016] L. A. Lomelí, "On automorphic L-functions in positive characteristic", *Ann. Inst. Fourier* (*Grenoble*) **66**:5 (2016), 1733–1771. MR Zbl

[Matringe 2009] N. Matringe, "Conjectures about distinction and local Asai L-functions", *Int. Math. Res. Not.* **2009**:9 (2009), 1699–1741. MR Zbl

[Matringe 2011] N. Matringe, "Distinguished generic representations of GL(n) over p-adic fields", *Int. Math. Res. Not.* **2011**:1 (2011), 74–95. MR Zbl

[Matringe 2014] N. Matringe, "Linear and Shalika local periods for the mirabolic group, and some consequences", *J. Number Theory* **138** (2014), 1–19. MR Zbl

[Matringe 2015] N. Matringe, "On the local Bump–Friedberg L-function", *J. Reine Angew. Math.* **709** (2015), 119–170. MR Zbl

[Matringe 2017] N. Matringe, "Shalika periods and parabolic induction for GL(n) over a non-Archimedean local field", *Bull. Lond. Math. Soc.* **49**:3 (2017), 417–427. MR Zbl

[Offen 2018] O. Offen, "Period integrals of automorphic forms and local distinction", pp. 159–195 in *Relative aspects in representation theory*, *Langlands functoriality and automorphic forms* (Marseille, 2016), edited by V. Heiermann and D. Prasad, Lecture Notes in Math. **2221**, Springer, 2018. MR Zbl

[Rodier 1973] F. Rodier, "Whittaker models for admissible representations of reductive p-adic split groups", pp. 425–430 in *Harmonic analysis on homogeneous spaces* (Williamstown, MA, 1972), edited by C. C. Moore, Proc. Sympos. Pure Math. **26**, Amer. Math. Soc., Providence, RI, 1973. MR Zbl

[Shahidi 1992] F. Shahidi, "Twisted endoscopy and reducibility of induced representations for p-adic groups", *Duke Math. J.* **66**:1 (1992), 1–41. MR Zbl

[Shankman 2018] D. Shankman, "Local Langlands correspondence for Asai L-functions and ϵ-factors", preprint, 2018. arXiv 1810.11852

[Tate 1979] J. Tate, "Number theoretic background", pp. 3–26 in *Automorphic forms*, *representations and L-functions*, *II* (Corvallis, OR, 1977), edited by A. Borel and W. Casselman, Proc. Sympos. Pure Math. **33**, Amer. Math. Soc., Providence, RI, 1979. MR Zbl

[Yamana 2017] S. Yamana, "Local symmetric square L-factors of representations of general linear groups", *Pacific J. Math.* **286**:1 (2017), 215–256. MR Zbl

[Yang 2022] C. Yang, "Linear periods for unitary representations", *Math. Z.* **302**:4 (2022), 2253–2284. MR Zbl

[Zelevinsky 1980] A. V. Zelevinsky, "Induced representations of reductive $\mathfrak{p}$-adic groups, II: On irreducible representations of GL(n)", *Ann. Sci. École Norm. Sup.* (4) **13**:2 (1980), 165–210. MR Zbl

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