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REPRESENTATIONS OF ORIENTIFOLD KHOVANOV–LAUDA–ROUQUIER ALGEBRAS AND THE ENOMOTO–KASHIWARA ALGEBRA

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We consider an "orientifold" generalization of Khovanov–Lauda–Rouquier algebras, depending on a quiver with an involution and a framing. Their representation theory is related, via a Schur–Weyl duality type functor, to Kac–Moody quantum symmetric pairs, and, via a categorification theorem, to highest weight modules over an algebra introduced by Enomoto and Kashiwara. Our first main result is a new shuffle realization of these highest weight modules and a combinatorial construction of their PBW and canonical bases in terms of Lyndon words. Our second main result is a classification of irreducible representations of orientifold KLR algebras and a computation of their global dimension in the case when the framing is trivial.

1. Introduction

Khovanov–Lauda–Rouquier (KLR) algebras were introduced in [Khovanov and Lauda 2009; Rouquier 2008] in the context of categorification of quantum groups. They have since played an increasingly important role in representation theory. Broadly speaking, KLR algebras can be regarded, via the Brundan–Kleshchev–Rouquier isomorphism [Brundan and Kleshchev 2009; Rouquier 2008], as a generalization of the affine Hecke algebra $\widehat{\mathcal{H}}(\mathtt{A}_m)$ of type $\mathtt{A}$. This generalization is twofold. Firstly, KLR algebras naturally possess a nontrivial grading, which is

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difficult to discern in the affine Hecke algebra. Secondly, KLR algebras constitute the correct replacement for $\widehat{\mathscr{H}}(\mathtt{A}_m)$ from the point of view of Schur–Weyl duality. Indeed, Kang, Kashiwara and Kim [Kang et al. 2018] have constructed functors relating categories of modules over KLR algebras and quantum affine algebras of any type, generalizing the relationship between $\widehat{\mathscr{H}}(\mathtt{A}_m)$ and $U_q(\widehat{\mathfrak{sl}}_n)$ established earlier by Chari and Pressley [1996].

It is natural to ask whether it is possible to construct a KLR-type generalization of affine Hecke algebras of other classical types. A positive answer to this question was given by Varagnolo and Vasserot [2011], as well as by Poulain d'Andecy and Walker [2020]. We will refer to the new graded algebras introduced there as *orientifold KLR algebras* (see Remark 2.5 for an explanation of the origin of this name). It must be stressed that orientifold KLR algebras are very different from the usual KLR algebras associated to Cartan data of other classical types. From the point of view of categorification, their representation theory is related to an algebra introduced by Enomoto and Kashiwara [2006], depending on a Dynkin diagram together with an involution. More precisely, it was shown in [Varagnolo and Vasserot 2011] that orientifold KLR algebras categorify irreducible highest weight modules ${}^{\theta}\mathbf{V}(\lambda)$ over the Enomoto–Kashiwara algebra. In analogy to $U_q(\mathfrak{n}_-)$, these modules also admit a geometric construction in terms of perverse sheaves on the stack of orthogonal representations of a quiver with a contravariant involution [Enomoto 2009], as well as a Ringel–Hall–type construction [Young 2016].

Our main motivation for studying orientifold KLR algebras is related to Schur–Weyl duality. In [Appel and Przeździecki 2022], we construct functors between categories of modules over orientifold KLR algebras and coideal subalgebras $\mathscr{B}_{\mathbf{c},\mathbf{s}}$ of quantum affine algebras $U_q(\hat{\mathfrak{g}})$ (see [Kolb 2014]), respectively. The parameter λ is related to the parameters $\mathbf{c}$ and $\mathbf{s}$ via an additional datum in the definition of an orientifold KLR algebra, given by a framing dimension vector. Our intention is to use these functors to develop the graded representation theory of Kac–Moody quantum symmetric pairs. The study of finite-dimensional representations of orientifold KLR algebras is the first step in this program.

Let us describe our results in more detail. In Section 2, we introduce a somewhat more general definition of orientifold KLR algebras (Definition 2.4) associated to hermitian matrices with an additional symmetry. We construct a faithful polynomial representation (Proposition 2.7) and prove a PBW theorem (Proposition 2.9). Section 3 is dedicated to the Enomoto–Kashiwara algebra. Inspired by the work of Leclerc [2004] and Kleshchev and Ram [2011], we construct a shuffle realization of the modules ${}^{\theta}\mathbf{V}(\lambda)$ (Definition 3.6 and Proposition 3.9). This allows us to apply the combinatorics of Lyndon words to obtain PBW and canonical bases for these modules, in the case $\lambda = 0$ (Theorem 3.28, Corollary 3.30), somewhat simplifying the original construction of these bases [Enomoto and Kashiwara 2008]. In Section 4, we apply these results to the representation theory of orientifold KLR algebras. A key ingredient is Varagnolo and Vasserot's categorification theorem [2011], identifying ${}^{\theta}\mathbf{V}(\lambda)$ with the Grothendieck group of the category of finite-dimensional representations of orientifold KLR algebras. In our main result (Theorem 4.10), we classify irreducible representations of orientifold KLR algebras in terms of θ-good Lyndon words, and construct them as heads (respectively, socles) of certain induced (respectively, coinduced) modules. As an application, we prove that orientifold KLR algebras have finite global dimension when $\lambda = 0$.

Future work. The present paper lays the foundations for a broader programme connecting the representation theory of quantum symmetric pairs with orientifold KLR algebras via generalized Schur–Weyl duality functors. In [Appel and Przeździecki 2022], the results of the present paper, together with a number of new techniques, including k-matrices for KLR algebras and localization for module categories, are used to construct Hernandez–Leclerc–type categories [2010; 2015] for coideal subalgebras $\mathcal{B}_{\mathbf{c},\mathbf{s}}$ in affine type $\mathtt{A.III}$ with generic parameters $\mathbf{c}$, $\mathbf{s}$.

In future work, we would like to generalize these results to nongeneric parameters and coideals of type $\mathtt{D.IV}$. This will, in turn, require the development of the representation theory of orientifold KLR algebras associated to nontrivial framings λ and quivers of affine type $\mathtt{D}$. To achieve this, we will combine the combinatorial techniques from the present paper with an in-depth study of the geometry of framed symplectic and orthogonal quiver representations.

We expect that further study of orientifold KLR algebras with nontrivial framings will also provide new information about the representation theory of (affine) Hecke algebras of types $\mathtt{B}$ and $\mathtt{C}$ with unequal parameters, including the so-called nonasymptotic case, which is still only partially understood.

In yet another direction, the connection to Hernandez–Leclerc categories suggests that the combinatorics of the dual canonical bases of the modules ${}^{\theta}\mathbf{V}(\lambda)$ should have an interesting interpretation in terms of cluster theory.

2. Orientifold KLR algebras

2A. ***Some combinatorics.*** Let $\Bbbk$ be a field. Let $\mathfrak{S}_n = \langle s_1, \dots, s_{n-1}\rangle$ denote the symmetric group on n letters, and let $\mathfrak{W}_n = \langle s_0, s_1, \dots, s_{n-1}\rangle$ denote the Weyl group of type $\mathtt{B}_n$, i.e., $(\mathbb{Z}/2\mathbb{Z})^n \rtimes \mathfrak{S}_n$. We regard them as Coxeter groups in the usual way. Given $0 \le m \le n$, let $\mathtt{D}_{m,n-m}$ (respectively, ${}^{\theta}\mathtt{D}_{m,n-m}$) denote the set of shortest left coset representatives with respect to the parabolic subgroup $\mathfrak{S}_m \times \mathfrak{S}_{n-m} \subset \mathfrak{S}_n$ (respectively, $\mathfrak{W}_m \times \mathfrak{S}_{n-m} \subset \mathfrak{W}_n$). Let $w_0 \in \mathfrak{S}_n$ (respectively, ${}^{\theta}w_0 \in \mathfrak{W}_n$) be the longest element, and let ${}^{\theta}w \in \mathfrak{W}_n$ be the longest element in ${}^{\theta}\mathtt{D}_{0,n}$, i.e., the signed permutation

$$ {}^{\theta}w(l) = -(n-l+1). $$

Let J be a set and $\theta\colon J \to J$ an involution. We denote by J^θ the subset of fixed points of θ. Let $\mathbb{N}[J]$ be the commutative semigroup freely generated by J. We call elements of $\mathbb{N}[J]$ *dimension vectors*. Given a dimension vector $\beta = \sum_{i\in J} \beta(i)\cdot i$, we set $\|\beta\| = \sum_{i\in J} \beta(i)$ and $\operatorname{supp}(\beta) = \{i \in J \mid \beta(i) \neq 0\}$. We call a sequence $\nu = \nu_1 \cdots \nu_n \in J^n$ a *composition* of β of length $\ell(\nu) = n$ if $|\nu| = \sum_{k=1}^n \nu_k = \beta$. We also set $\|\nu\| = n$. Let J^β denote the set of all compositions of β. There is a left action of $\mathfrak{S}_n$ on J^n by permutations

$$s_k \cdot \nu_1 \cdots \nu_n = \nu_1 \cdots \nu_{k+1}\nu_k \cdots \nu_n, \quad 1 \le k \le n-1, \tag{2-1}$$

whose orbits are the sets J^β for all β with $\|\beta\| = n$.

Let $J^\bullet = \bigcup_{\beta\in\mathbb{N}[J]} J^\beta$ be the set of compositions of all dimension vectors. We also refer to elements of $J^\bullet$ as *words* in J and denote the empty word by $\varnothing$. We consider $J^\bullet$ as a monoid with respect to concatenation: $\nu\mu = \nu_1 \cdots \nu_{\ell\nu}\mu_1 \cdots \mu_{\ell\mu}$, with $\varnothing$ as the identity.

The involution θ induces an involution $\theta\colon \mathbb{N}[J] \to \mathbb{N}[J]$. We call dimension vectors in $\mathbb{N}[J]^\theta$ *self-dual*. We will always assume, for any $\beta \in \mathbb{N}[J]^\theta$, that if $i \in J^\theta$, then $\beta(i)$ is even. Set $\|\beta\|_\theta = \|\beta\|/2$ and

$${}^\theta(-)\colon \mathbb{N}[J] \to \mathbb{N}[J]^\theta, \quad \beta \mapsto {}^\theta\beta = \beta + \theta(\beta).$$

We call a sequence $\nu = \nu_1 \cdots \nu_n \in J^n$ an *isotropic composition* of β if ${}^\theta|\nu| = \sum_{k=1}^n {}^\theta\nu_i = \beta$. We abbreviate $\nu_{-k} = \theta(\nu_k)$. Let ${}^\theta J^\beta$ denote the set of all isotropic compositions of β. There is a left action of $\mathfrak{W}_n$ on J^n extending (2-1), given by

$$s_0 \cdot \nu_1 \cdots \nu_n = \theta(\nu_1)\nu_2 \cdots \nu_n,$$

whose orbits are the sets ${}^\theta J^\beta$ for all self-dual β with $\|\beta\|_\theta = n$. Let ${}^\theta J^\bullet = \bigcup_{\beta\in\mathbb{N}[J]^\theta} {}^\theta J^\beta$ be the set of all isotropic compositions of all self-dual dimension vectors. The identity map defines a bijection $J^\bullet \cong {}^\theta J^\bullet$.

We will consider algebras depending on matrices and vectors with polynomial entries. Below we introduce some terminology for the latter.

Definition 2.1. We call a matrix $Q = (Q_{ij})_{i,j\in J}$ with entries in $\Bbbk[u, v]$ a *coefficient matrix*. We say that Q is:

(M1) *regular* if $Q_{ii} = 0$ for all $i \in J$,

(M2) *θ-symmetric* if $Q_{ij}(u, v) = Q_{\theta(j)\theta(i)}(-v, -u)$ for all $i, j \in J$,

(M3) *nonvanishing* if $Q_{ij} \neq 0$ for all $i \neq j \in J$,

(M4) *hermitian* if $Q_{ij}(u, v) = Q_{ji}(v, u)$ for each $i, j \in J$.

Moreover, we call a vector $Q' = (Q_i)_{i\in J}$ with entries in $\Bbbk[u]$ a *coefficient vector*. We say that Q' is:

(V1) *regular* if $Q_i = 0$ for all $i \in J^\theta$, (V2) *nonvanishing* if $Q_i \neq 0$ for all $i \notin J^\theta$,

(V3) *self-conjugate* if $Q_i(u) = Q_{\theta(i)}(-u)$.

If a coefficient matrix satisfies (M1)–(M4), respectively, if a coefficient vector satisfies (V1)–(V3), we call it *perfect*.

2B. *Reminder on KLR algebras.* Let $\beta \in \mathbb{N}[J]$ with $\|\beta\| = n$, and let Q be a regular coefficient matrix.

Definition 2.2. The *KLR algebra* $\mathcal{R}(\beta)$ associated to (J, Q, β) is the unital $\Bbbk$-algebra generated by $e(\nu)$ with $\nu \in J^\beta$, x_l with $1 \leq l \leq n$ and τ_k with $1 \leq k \leq n-1$, subject to the following relations:

- idempotent relations:

$$e(\nu)e(\nu') = \delta_{\nu,\nu'}e(\nu), \quad x_l e(\nu) = e(\nu)x_l, \quad \tau_k e(\nu) = e(s_k \cdot \nu)\tau_k,$$

- polynomial relations:

$$x_l x_{l'} = x_{l'} x_l,$$

- quadratic relations:

$$\tau_k^2 e(\nu) = Q_{\nu_k,\nu_{k+1}}(x_{k+1}, x_k)e(\nu),$$

- deformed braid relations:

$$\tau_k \tau_{k'} = \tau_{k'} \tau_k, \quad \text{if } k \neq k' \pm 1,$$

$$(\tau_{k+1}\tau_k\tau_{k+1} - \tau_k\tau_{k+1}\tau_k)e(\nu) = \delta_{\nu_k,\nu_{k+2}} \frac{Q_{\nu_k,\nu_{k+1}}(x_{k+1}, x_k) - Q_{\nu_k,\nu_{k+1}}(x_{k+1}, x_{k+2})}{x_k - x_{k+2}} e(\nu),$$

- mixed relations:

$$(\tau_k x_l - x_{s_k(l)}\tau_k)e(\nu) = \begin{cases} -e(\nu), & \text{if } l = k,\ \nu_k = \nu_{k+1}, \\ e(\nu), & \text{if } l = k+1,\ \nu_k = \nu_{k+1}, \\ 0, & \text{else.} \end{cases}$$

Whenever we want to emphasize the dependence of the KLR algebra on the full datum (J, Q, β), we will write $\mathcal{R}(J, Q, \beta)$.

Lemma 2.3. *If the coefficient matrix Q is hermitian, then there is an algebra isomorphism $\mathcal{R}(\beta) \to \mathcal{R}(\beta)$ sending*

$$e(\nu) \mapsto e(w_0(\nu)), \quad x_l \mapsto x_{n-l+1}, \quad \tau_k \mapsto -\tau_{n-k}. \tag{2-2}$$

If the coefficient matrix Q is hermitian and θ-symmetric, then there is an algebra isomorphism $\mathcal{R}(\beta) \to \mathcal{R}(\theta(\beta))$ sending

$$e(\nu) \mapsto e({}^\theta w(\nu)), \quad x_l \mapsto -x_{n-l+1}, \quad \tau_k \mapsto -\tau_{n-k}. \tag{2-3}$$

Proof. The first statement can be found in, e.g., [Rouquier 2008, §3.2.1]. The second statement follows from a direct check of the relations using θ-symmetry. □

If M is an $\mathscr{R}(\beta)$-module, we will denote by $M^\dagger$ the corresponding $\mathscr{R}(\theta(\beta))$-module with the action twisted by the inverse of the isomorphism given in (2-3).

2C. *Orientifold KLR algebras.* Let $\beta \in \mathbb{N}[J]^\theta$ with $\|\beta\|_\theta = n$, let Q be a regular θ-symmetric coefficient matrix and Q' a regular coefficient vector.

Definition 2.4. Associated to $(J, \theta, Q, Q', \beta)$, we define the *orientifold KLR algebra* ${}^\theta\mathscr{R}(\beta)$ to be the unital $\Bbbk$-algebra generated by $e(\nu)$ with $\nu \in {}^\theta J^\beta$, x_l with $1 \le l \le n$, τ_0 and τ_k with $1 \le k \le n-1$ subject to the following relations:

- idempotent relations:

$$e(\nu)e(\nu') = \delta_{\nu,\nu'}e(\nu), \qquad x_l e(\nu) = e(\nu)x_l,$$
$$\tau_k e(\nu) = e(s_k \cdot \nu)\tau_k, \quad \tau_0 e(\nu) = e(s_0 \cdot \nu)\tau_0,$$

- polynomial relations:

$$x_l x_{l'} = x_{l'} x_l,$$

- quadratic relations:

$$\tau_k^2 e(\nu) = Q_{\nu_k,\nu_{k+1}}(x_{k+1}, x_k)e(\nu), \quad \tau_0^2 e(\nu) = Q_{\nu_1}(-x_1)e(\nu),$$

- deformed braid relations:

$$\tau_k\tau_{k'} = \tau_{k'}\tau_k, \quad \text{if } k \neq k' \pm 1, \qquad \tau_0\tau_k = \tau_k\tau_0, \quad \text{if } k \neq 1,$$

$$(\tau_{k+1}\tau_k\tau_{k+1} - \tau_k\tau_{k+1}\tau_k)e(\nu) = \delta_{\nu_k,\nu_{k+2}} \frac{Q_{\nu_k,\nu_{k+1}}(x_{k+1}, x_k) - Q_{\nu_k,\nu_{k+1}}(x_{k+1}, x_{k+2})}{x_k - x_{k+2}} e(\nu),$$

$$\begin{aligned}
&\big((\tau_1\tau_0)^2 - (\tau_0\tau_1)^2\big)e(\nu) \\
&= \begin{cases}
\dfrac{Q_{\nu_2}(x_2) - Q_{\nu_1}(x_1)}{x_1 + x_2}\tau_1 e(\nu), & \text{if } \nu_1 \neq \nu_2,\ \nu_2 = \theta(\nu_1) \\
\dfrac{Q_{\nu_1,\nu_2}(x_2, -x_1) - Q_{\nu_1,\nu_2}(-x_2, -x_1)}{x_2}\tau_0 e(\nu), & \text{if } \nu_1 \neq \theta(\nu_1),\ \nu_2 = \theta(\nu_2), \\
\dfrac{Q_{\nu_1,\nu_2}(x_2, -x_1) - Q_{\nu_1,\nu_2}(x_2, x_1)}{x_1 x_2}(x_1\tau_0 + 1)e(\nu), & \text{if } \theta(\nu_1) = \nu_1 \neq \nu_2 = \theta(\nu_2), \\
0 & \text{else,}
\end{cases}
\end{aligned}$$

- mixed relations:

$$(\tau_k x_l - x_{s_k(l)}\tau_k)e(\nu) = \begin{cases} -e(\nu), & \text{if } l = k,\ \nu_k = \nu_{k+1}, \\ e(\nu), & \text{if } l = k+1,\ \nu_k = \nu_{k+1}, \\ 0, & \text{else,} \end{cases}$$

$$(\tau_0 x_1 + x_1\tau_0)e(\nu) = \begin{cases} 0, & \text{if } \nu_1 \neq \theta(\nu_1), \\ -2e(\nu), & \text{if } \nu_1 = \theta(\nu_1), \end{cases}$$

$$\tau_0 x_l = x_l\tau_0, \quad \text{if } l \neq 1.$$

By convention, we set ${}^\theta\mathscr{R}(0) = \Bbbk$. Whenever we want to emphasize the dependence of the orientifold KLR algebra on the full datum $(J, \theta, Q, Q', \beta)$, we will write ${}^\theta\mathscr{R}(J, Q, Q', \beta)$.

Remark 2.5. In the case when the matrices Q and Q' arise from a quiver with a contravariant involution and a framing (see Section 2F), under the assumption that the involution has no fixed points, the algebra ${}^\theta\mathscr{R}(\beta)$ was introduced by Varagnolo and Vasserot [2011]. The case of an involution with possible fixed points was first considered by Poulain d'Andecy and Walker [2020], and later by Poulain d'Andecy and Rostam [2021]. The latter paper takes a somewhat similar approach to ours — the definition of the algebra depends on polynomials Q_{ij}, but they are less general than ours, and the polynomials Q_i are absent.

In the literature, these algebras are typically referred to as "generalizations of KLR algebras for types BCD". However, we feel that this name may lead to confusion between, for example, the algebra ${}^\theta\mathscr{R}(\beta)$ and the KLR algebra $\mathscr{R}(\beta)$ associated to a quiver of type D. To avoid this confusion, we propose to introduce the name "orientifold KLR algebras" for ${}^\theta\mathscr{R}(\beta)$. The motivation comes from the connection with orientifold Donaldson–Thomas theory, see [Przeździecki 2019; Young 2020].

Proposition 2.6. *We list several isomorphisms between orientifold KLR algebras*:

(1) *If Q is hermitian and Q' self-conjugate, then there is an algebra automorphism*

$$ {}^\theta\mathscr{R}(\beta) \xrightarrow{\sim} {}^\theta\mathscr{R}(\beta), \quad e(\nu) \mapsto e({}^\theta w_0(\nu)),\ x_l \mapsto -x_l,\ \tau_k \mapsto -\tau_k, \tag{2-4} $$

with $\nu \in {}^\theta J^\beta$, $1 \le l \le n$ and $0 \le k \le n-1$.

(2) *If Q is hermitian and Q' self-conjugate, then there is an algebra isomorphism*

$$ \omega\colon {}^\theta\mathscr{R}(\beta) \xrightarrow{\sim} {}^\theta\mathscr{R}(\beta)^{\mathrm{op}}, \quad e(\nu) \mapsto e(\nu),\ x_l e(\nu) \mapsto x_l e(\nu),\ \tau_k e(\nu) \mapsto \tau_k e(s_k \cdot \nu). \tag{2-5} $$

(3) *Given $\{\zeta_i\}_{i\in J}$ in $\Bbbk$ satisfying $\zeta_i = -\zeta_{\theta(i)}$, as well as $\{\eta_{ij}\}_{i,j\in J}$ and $\{\eta_i\}_{i\in J}$ in $\Bbbk^\times$ satisfying: $\eta_{ij} = \eta_{\theta(j)\theta(i)}$ for all $i, j \in J$ and $\eta_i = \eta_{ii}$ for $i \in J^\theta$, let $\hat{Q}_{ij}(u, v) = \eta_{ij}\eta_{ji}(\eta_{jj}u + \zeta_j, \eta_{ii}v + \zeta_i)$ and $\hat{Q}_i(u) = \eta_i\eta_{\theta(i)}Q_i(\eta_{ii}u - \zeta_i)$. Then there is an algebra isomorphism ${}^\theta\mathscr{R}(J, \hat{Q}, \hat{Q}', \beta) \xrightarrow{\sim} {}^\theta\mathscr{R}(J, Q, Q', \beta)$ given by*

$$ e(\nu) \mapsto e(\nu), \qquad x_l e(\nu) \mapsto \eta^{-1}_{\nu_l,\nu_l}(x_l - \zeta_{\nu_l})e(\nu), $$
$$ \tau_k e(\nu) \mapsto \eta_{\nu_k,\nu_{k+1}}\tau_k e(\nu), \quad \tau_0 e(\nu) \mapsto \eta_{\nu_1}\tau_0 e(\nu). $$

Proof. The result follows by a direct computation from the defining relations. □

2D. *Polynomial representation.* Set

$$ \mathbb{P}_\nu = \Bbbk[x_1, \dots, x_n]e(\nu), \quad \widehat{\mathbb{P}}_\nu = \Bbbk[\![x_1, \dots, x_n]\!]e(\nu), \quad \widehat{\mathbb{K}}_\nu = \Bbbk((x_1, \dots, x_n))e(\nu), $$
$$ {}^\theta\mathbb{P}_\beta = \bigoplus_{\nu\in{}^\theta J^\beta} \mathbb{P}_\nu, \qquad {}^\theta\widehat{\mathbb{P}}_\beta = \bigoplus_{\nu\in{}^\theta J^\beta} \widehat{\mathbb{P}}_\nu, \qquad {}^\theta\widehat{\mathbb{K}}_\beta = \bigoplus_{\nu\in{}^\theta J^\beta} \widehat{\mathbb{K}}_\nu. $$

We abbreviate $x_{-l} = -x_l$ for $1 \leq l \leq n$. The group $\mathfrak{W}_n$ acts on $\Bbbk((x_1, \dots, x_n))$ from the left by $w \cdot x_l = x_{w(l)}$. This induces an action on ${}^\theta\widehat{\mathbb{K}}_\beta$ according to the rule

$$w \cdot fe(\nu) = w(f)e(w \cdot \nu), \tag{2-6}$$

for $w \in \mathfrak{W}_n$ and $f \in \Bbbk((x_1, \dots, x_n))$.

Let $P = (P_{ij})_{i,j \in J}$ be a coefficient matrix satisfying (M1)–(M3) and $P' = (P_i)_{i \in J}$ a coefficient vector satisfying (V1)–(V2). Set

$$\begin{aligned} Q_{ij}(u, v) &= P_{ij}(u, v)P_{ji}(v, u), \\ Q_i(u) &= P_i(u)P_{\theta(i)}(-u), \end{aligned} \tag{2-7}$$

with $i, j \in J$. Then $Q = (Q_{ij})$ is a perfect coefficient matrix and $Q' = (Q_i)$ a perfect coefficient vector.

Proposition 2.7. *The algebra ${}^\theta\mathfrak{R}(\beta)$ has a faithful polynomial representation on ${}^\theta\mathbb{P}_\beta$, given by*:

- $e(\nu)$, *where $\nu \in {}^\theta J^\beta$, acting as a projection onto $\mathbb{P}_\nu$*,
- $x_1, \dots, x_n$ *acting naturally by multiplication*,
- $\tau_1, \dots, \tau_{n-1}$ *acting via*

$$\tau_k \cdot fe(\nu) = \begin{cases} \dfrac{s_k(f) - f}{x_k - x_{k+1}} e(\nu), & \text{if } \nu_k = \nu_{k+1}, \\ P_{\nu_k, \nu_{k+1}}(x_k, x_{k+1}) s_k(f) e(s_k \cdot \nu), & \text{otherwise}, \end{cases}$$

- τ_0 *acting via*

$$\tau_0 \cdot fe(\nu) = \begin{cases} \dfrac{s_0(f) - f}{x_1} e(\nu), & \text{if } \theta(\nu_1) = \nu_1, \\ P_{\nu_1}(x_1) s_0(f) e(s_0 \cdot \nu), & \text{otherwise}. \end{cases}$$

Whenever we want to emphasize the dependence of the above representation on (P, P'), we will write ${}^\theta\mathbb{P}_\beta^{P,P'}$.

Proof. The proof that the operators defined above satisfy all the relations from Definition 2.4 not involving τ_0 is the same as in the case of the KLR algebra, and can be found in, e.g., the proof of [Rouquier 2008, Proposition 3.12]. The other relations are easy to check, with the exception of the deformed braid relations. We prove these explicitly below.

To simplify exposition, we omit the idempotents. We also abbreviate $i = \nu_1$ and $j = \nu_2$. First consider the case where $i \neq j$ and $j = \theta(i)$. Then:

$$\begin{aligned}\tau_1\tau_0\tau_1\tau_0(f) &= \tau_1\tau_0\tau_1 P_i(x_1)s_0(f) = \tau_1\tau_0\frac{P_i(x_2)s_1s_0(f)-P_i(x_1)s_0(f)}{x_1-x_2}\\ &= \tau_1 P_j(x_1)\frac{P_i(x_2)s_0s_1s_0(f)-P_i(-x_1)f}{-x_1-x_2}\\ &= P_{ij}(x_1,x_2)P_j(x_2)\frac{P_i(x_1)s_1s_0s_1s_0(f)-P_i(-x_2)s_1(f)}{-x_1-x_2},\end{aligned}$$

$$\begin{aligned}\tau_0\tau_1\tau_0\tau_1(f) &= \tau_0\tau_1\tau_0 P_{ij}(x_1,x_2)s_1(f) = \tau_0\tau_1 P_j(x_1)P_{ij}(-x_1,x_2)s_0s_1(f)\\ &= \tau_0\frac{P_j(x_2)P_{ij}(-x_2,x_1)s_1s_0s_1(f)-P_j(x_1)P_{ij}(-x_1,x_2)s_0s_1(f)}{x_1-x_2}\\ &= P_i(x_1)\frac{P_j(x_2)P_{ij}(-x_2,-x_1)s_0s_1s_0s_1(f)-P_j(-x_1)P_{ij}(x_1,x_2)s_1(f)}{-x_1-x_2}.\end{aligned}$$

Since, by θ-symmetry, we have $P_{ij}(x_1,x_2) = P_{ij}(-x_2,-x_1)$, it follows that

$$\begin{aligned}\big((\tau_1\tau_0)^2-(\tau_0\tau_1)^2\big)(f) &= \frac{P_j(x_2)P_i(-x_2)-P_i(x_1)P_j(-x_1)}{x_1+x_2}P_{ij}(x_1,x_2)s_1(f)\\ &= \frac{Q_j(x_2)-Q_i(x_1)}{x_1+x_2}\tau_1(f).\end{aligned}$$

Secondly, let $i \neq \theta(i)$ and $j = \theta(j)$. Then:

$$\begin{aligned}&\tau_1\tau_0\tau_1\tau_0(f)\\ &= \tau_1\tau_0\tau_1 P_i(x_1)s_0(f) = \tau_1\tau_0 P_{\theta(i),j}(x_1,x_2)P_i(x_2)s_1s_0(f)\\ &= \tau_1\frac{P_{\theta(i),j}(-x_1,x_2)P_i(x_2)s_0s_1s_0(f)-P_{\theta(i),j}(x_1,x_2)P_i(x_2)s_1s_0(f)}{x_1}\\ &= P_{j,\theta(i)}(x_1,x_2)\frac{P_{\theta(i),j}(-x_2,x_1)P_i(x_1)s_1s_0s_1s_0(f)-P_{\theta(i),j}(x_2,x_1)P_i(x_1)s_0(f)}{x_2},\end{aligned}$$

$$\begin{aligned}&\tau_0\tau_1\tau_0\tau_1(f)\\ &= \tau_0\tau_1\tau_0 P_{ij}(x_1,x_2)s_1(f) = \tau_0\tau_1\frac{P_{ij}(-x_1,x_2)s_0s_1(f)-P_{ij}(x_1,x_2)s_1(f)}{x_1}\\ &= \tau_0 P_{ji}(x_1,x_2)\frac{P_{ij}(-x_2,x_1)s_1s_0s_1(f)-P_{ij}(x_2,x_1)f}{x_2}\\ &= P_i(x_1)P_{ji}(-x_1,x_2)\frac{P_{ij}(-x_2,-x_1)s_0s_1s_0s_1(f)-P_{ij}(x_2,-x_1)s_0(f)}{x_2}.\end{aligned}$$

Again, θ-symmetry implies that

$$((\tau_1\tau_0)^2-(\tau_0\tau_1)^2)(f)$$
$$=\frac{-P_{j,\theta(i)}(x_1,x_2)P_{\theta(i),j}(x_2,x_1)+P_{ij}(x_2,-x_1)P_{j,i}(-x_1,x_2)}{x_2}P_i(x_1)s_0(f)$$
$$=\frac{Q_{ij}(x_2,-x_1)-Q_{ij}(-x_2,-x_1)}{x_2}\tau_0(f).$$

Thirdly, let $\theta(i)=i\neq j=\theta(j)$. Then:

$$\tau_1\tau_0\tau_1\tau_0(f)$$
$$=\tau_1\tau_0\tau_1\frac{s_0(f)-f}{x_1}=\tau_1\tau_0P_{ij}(x_1,x_2)\frac{s_1s_0(f)-s_1(f)}{x_2}$$
$$=\tau_1\frac{P_{ij}(-x_1,x_2)[s_0s_1s_0(f)-s_0s_1(f)]-P_{ij}(x_1,x_2)[s_1s_0(f)-s_1(f)]}{x_1x_2}$$
$$=P_{ji}(x_1,x_2)\frac{P_{ij}(-x_2,x_1)[s_1s_0s_1s_0(f)-s_1s_0s_1(f)]-P_{ij}(x_2,x_1)[s_0(f)-(f)]}{x_1x_2},$$

$$\tau_0\tau_1\tau_0\tau_1(f)$$
$$=\tau_0\tau_1\tau_0P_{ij}(x_1,x_2)s_1(f)=\tau_0\tau_1\frac{P_{ij}(-x_1,x_2)s_0s_1(f)-P_{ij}(x_1,x_2)s_1(f)}{x_1}$$
$$=\tau_0P_{ji}(x_1,x_2)\frac{P_{ij}(-x_2,x_1)s_1s_0s_1(f)-P_{ij}(x_2,x_1)f}{x_2}$$
$$=\frac{P_{ji}(-x_1,x_2)[P_{ij}(-x_2,-x_1)s_0s_1s_0s_1(f)-P_{ij}(x_2,-x_1)s_0(f)]}{x_1x_2}$$
$$-\frac{P_{ji}(x_1,x_2)[P_{ij}(-x_2,x_1)s_1s_0s_1(f)-P_{ij}(x_2,x_1)f]}{x_1x_2}.$$

By θ-symmetry, we conclude that

$$\big((\tau_1\tau_0)^2-(\tau_0\tau_1)^2\big)(f)=\frac{-P_{ji}(x_1,x_2)P_{ij}(x_2,x_1)+P_{ji}(-x_1,x_2)P_{ij}(x_2,-x_1)}{x_1x_2}s_0(f)$$
$$=\frac{Q_{i,j}(x_2,-x_1)-Q_{i,j}(x_2,x_1)}{x_1x_2}(x_1\tau_0+1)f.$$

Fourthly, let $i=\theta(i)$ and $j\neq\theta(j)$. One easily checks (using θ-symmetry) that $((\tau_1\tau_0)^2-(\tau_0\tau_1)^2)(f)=g\cdot s_1s_0s_1\Delta_0(f)-\Delta_0(g\cdot s_1s_0s_1(f))$, where g is an s_0-invariant polynomial and $\Delta_0=x_1^{-1}(s_0-1)$. It now follows from the properties of Demazure operators that

$$\big((\tau_1\tau_0)^2-(\tau_0\tau_1)^2\big)(f)$$
$$=g\cdot s_1s_0s_1\Delta_0(f)-\big(\Delta_0(g)\cdot s_1s_0s_1(f)+s_0(g)\Delta_0\big(s_1s_0s_1(f)\big)\big)=0.$$

Fifthly, let $i=j$ and $i\neq\theta(i)$. One checks, as above, that $((\tau_1\tau_0)^2-(\tau_0\tau_1)^2)(f)=\Delta_1(g\cdot s_0s_1s_0(f))-g\cdot s_0s_1s_0\Delta_1(f)$, where g is an s_1-invariant polynomial and $\Delta_1=(x_1-x_2)^{-1}(s_1-1)$. As above, it follows from the properties of Demazure operators that $((\tau_1\tau_0)^2-(\tau_0\tau_1)^2)(f) = 0$.

Finally, suppose that $i=j=\theta(j)$. Then each of τ_0 and τ_1 acts as a Demazure operator, but Demazure operators satisfy the braid relation. This completes the proof that ${}^\theta\mathbb{P}_\beta$ is a representation of ${}^\theta\mathcal{R}(\beta)$.

The proof of faithfulness is analogous to the case of KLR algebras, see, e.g., [Rouquier 2008, Proposition 3.12]. □

Next, for each $i, j\in J$, we choose holomorphic functions $c_{ij}(u,v)\in\Bbbk[\![u,v]\!]$ such that

(2-8) $$c_{ij}(u,v)c_{ji}(v,u)=1,\quad c_{ii}(u,v)=1,\quad c_{ij}(u,v)=c_{\theta(j)\theta(i)}(-v,-u).$$

Moreover, for each $i\in J$, we also choose holomorphic functions $c_i\in\Bbbk[[u]]$ such that

(2-9) $$c_i(u)=c_{\theta(i)}(-u),\qquad i=\theta(i)\;\Rightarrow\; c_i(u)=1.$$

Set
$$\widetilde{P}_{ij}(u,v)=P_{ij}(u,v)c_{ij}(u,v)\quad\text{and}\quad \widetilde{P}_i(u)=P_i(u)c_i(u).$$

Corollary 2.8. *There is an injective ${}^\theta\mathbb{P}_\beta$-algebra homomorphism*

(2-10) $${}^\theta\mathcal{R}(\beta)\hookrightarrow\Bbbk[\mathfrak{W}_n]\ltimes{}^\theta\widehat{\mathbb{K}}_\beta$$

given by
$$\tau_0e(\nu)=\begin{cases}x_1^{-1}(s_0-1)e(\nu), & \text{if } \nu_1=\theta(\nu_1),\\ \widetilde{P}_{\nu_1}(x_1)s_0e(\nu), & \text{otherwise},\end{cases}$$
$$\tau_ke(\nu)=\begin{cases}(x_k-x_{k+1})^{-1}(s_k-1)e(\nu), & \text{if } \nu_k=\nu_{k+1},\\ \widetilde{P}_{\nu_k,\nu_{k+1}}(x_k,x_{k+1})s_ke(\nu), & \text{otherwise},\end{cases}$$
for $1\le k\le n-1$.

Proof. This follows directly from Proposition 2.7. □

2E. *PBW theorem.* In this subsection, assume that Q is a coefficient matrix satisfying (M1)–(M3) and Q' a coefficient vector satisfying (V1)–(V2). The algebra ${}^\theta\mathcal{R}(\beta)$ is filtered with $\deg x_l$, $\deg e(\nu)=0$ and $\deg\tau_k=1$. We say that ${}^\theta\mathcal{R}(\beta)$ satisfies the *PBW property* if $\operatorname{gr}{}^\theta\mathcal{R}(\beta)\cong{}^0\mathcal{H}_n^f\ltimes{}^\theta\mathbb{P}_\beta$, where ${}^0\mathcal{H}_n^f$ is the (nonaffine) nil-Hecke algebra of type B_n (see, e.g., [Kostant and Kumar 1986]).

For any $w\in\mathfrak{W}_n$, choose a reduced expression $w=s_{k_1}\cdots s_{k_l}$ and set $\tau_w=\tau_{s_{k_1}}\cdots\tau_{s_{k_l}}$. The definition of τ_w depends on the choice of reduced expression.

Proposition 2.9. *Let $n\ge 1$. The following are equivalent*:

(1) *${}^\theta\mathcal{R}(\beta)$ satisfies the PBW property*,

(2) ${}^{\theta}\mathcal{R}(\beta)$ *is a free* $\Bbbk$*-module with basis*

$$\{\tau_w x_1^{a_1} \dots x_n^{a_n} e(\nu) \mid w \in \mathfrak{W}_n,\ (a_1, \dots, a_n) \in \mathbb{Z}^n_{\geq 0},\ \nu \in {}^{\theta}J^{\beta}\},$$

(3) *Q and Q' are perfect.*

Proof. The proof is a straightforward generalization of the proof of [Rouquier 2008, Theorem 3.7]. Let us briefly comment on the new features. Suppose that (2) holds, and let $\nu_1 \neq \theta(\nu_1)$. The quadratic relation then implies that

$$Q_{\theta(\nu_1)}(-x_1)\tau_0 e(\nu) = \tau_0^3 e(\nu) = \tau_0 Q_{\nu_1}(-x_1) e(\nu) = Q_{\nu_1}(x_1)\tau_0 e(\nu).$$

It follows that

$$\big(Q_{\theta(\nu_1)}(-x_1) - Q_{\nu_1}(x_1)\big)\tau_0 e(\nu) = 0.$$

Now (2) implies that $Q_{\theta(\nu_1)}(-x_1) - Q_{\nu_1}(x_1) = 0$, i.e., Q' is self-conjugate. Conversely, if both Q and Q' are perfect, then we can use Proposition 2.7, with $P_{ij} = Q_{ij}$, $P_{ji} = 1$ with $i < j$, $P_i = Q_i$ and $P_{\theta(i)} = 1$ with $i < \theta(i)$ for some ordering of J, to deduce (2). □

2F. *Orientifold KLR algebras associated to quivers.* Let $\Gamma = (J, \Omega)$ be a quiver with vertices J and arrows Ω. We assume that Γ does not have loops. Given an arrow $a \in \Omega$, let $s(a)$ be its source, and $t(a)$ its target. If $i, j \in J$, let $\Omega_{ij} \subset \Omega$ be the subset of arrows a such that $s(a) = i$ and $t(a) = j$. Let $a_{ij} = |\Omega_{ij}|$ and abbreviate $\overleftrightarrow{a}_{ij} = a_{ij} + a_{ji}$. We assume that $a_{ij} < \infty$ for all $i, j \in J$.

Definition 2.10. A (contravariant) *involution* of the quiver Γ is a pair of involutions $\theta\colon J \to J$ and $\theta\colon \Omega \to \Omega$ such that:

(1) $s(\theta(a)) = \theta(t(a))$ and $t(\theta(a)) = \theta(s(a))$ for all $a \in \Omega$,

(2) if $t(a) = \theta(s(a))$, then $a = \theta(a)$.

Fix a quiver Γ with an involution θ and two dimension vectors $\beta \in \mathbb{N}[J]^{\theta}$, $\lambda \in \mathbb{N}[J]$ such that $\|\beta\|_{\theta} = n$ and $\lambda(i) = 0$ if $i \in J^{\theta}$. We call λ the *framing dimension vector*. Note that λ need not be self-dual.

Set

$$P_{ij}(u, v) = \delta_{i \neq j}(v - u)^{a_{ij}} \quad \text{and} \quad P_i(u) = \delta_{i \neq \theta(i)}(-u)^{\lambda(i)}$$

for $i, j \in J$, and define (Q, Q') as in (2-7). Since, by Definition 2.10, $a_{ij} = a_{\theta(j)\theta(i)}$, the coefficient matrix P is θ-symmetric and, therefore, (Q, Q') is perfect.

Definition 2.11. The KLR algebra associated to (Γ, β) and the orientifold KLR algebra associated to $(\Gamma, \theta, \beta, \lambda)$ are, respectively,

$$\mathcal{R}^{\Gamma}(\beta) = \mathcal{R}(J, Q, \beta) \quad \text{and} \quad {}^{\theta}\mathcal{R}^{\Gamma}(\beta; \lambda) = {}^{\theta}\mathcal{R}(J, Q, Q', \beta).$$

We endow these algebras with the following grading:

$$\begin{aligned}\deg e(\nu) &= 0,\\ \deg x_k &= 2,\\ \deg \tau_k e(\nu) &= \begin{cases} -2, & \text{if } \nu_k = \nu_{k+1},\\ \overleftrightarrow{a}_{\nu_k,\nu_{k+1}}, & \text{otherwise,}\end{cases}\\ \deg \tau_0 e(\nu) &= \begin{cases} -2, & \text{if } \theta(\nu_1) = \nu_1,\\ {}^{\theta}\lambda(\nu_1), & \text{otherwise.}\end{cases}\end{aligned}$$

Most of the time we will omit Γ from the notation, as the choice of quiver is clear from the context. Also note that, by Proposition 2.7, the algebra ${}^{\theta}\mathcal{R}(\beta;\lambda)$ has a faithful polynomial representation on ${}^{\theta}\mathbb{P}_{\beta}^{P,P'}$.

3. Enomoto–Kashiwara algebra, quantum shuffle modules and Lyndon words

3A. ***Notation.*** Let $J = \{\alpha_k \mid k \in \mathbb{Z}_{\text{odd}}\}$ and equip $\mathsf{Q} = \mathbb{Z}[J]$ with the symmetric bilinear form

$$\alpha_k \cdot \alpha_l = \begin{cases} 2, & \text{if } k = l,\\ -1, & \text{if } k = l \pm 2,\\ 0, & \text{otherwise.}\end{cases} \tag{3-1}$$

Then $(J,\cdot)$ is the Cartan datum associated to $\mathfrak{g} = \mathfrak{sl}_\infty$. We identify J with the set of simple roots of the root system Φ of type $\mathtt{A}_\infty$. Then $\Phi^+ = \{\beta_{k,l} \mid k \le l \in \mathbb{Z}_{\text{odd}}\}$, where $\beta_{k,l} = \alpha_k + \alpha_{k+2} + \cdots + \alpha_l$, is a set of positive roots. Let $\mathsf{P} = \{\lambda \in \mathbb{Q} \otimes_{\mathbb{Z}} \mathsf{Q} \mid \lambda \cdot i \in \mathbb{Z} \text{ for all } i \in J\}$ be the weight lattice, $\mathsf{P}_+ = \{\lambda \in \mathsf{P} \mid \lambda \cdot i \in \mathbb{Z}_{\ge 0} \text{ for all } i \in J\}$ be the set of dominant integral weights, and $\mathsf{Q}_+ = \mathbb{N}[J]$. Given $\beta = \sum_{i \in J} c_i i \in \mathsf{Q}_+$, let $N(\beta) = \frac{1}{2}\big(\beta\cdot\beta - \sum_{i\in J} c_i i \cdot i\big)$.

Let $\theta\colon \mathsf{Q} \to \mathsf{Q}$ be the involution sending $\alpha_k \mapsto \alpha_{-k}$. The bilinear form (3-1) restricts to Q^θ. The image of Φ under the symmetrization map

$$\mathsf{Q} \to \mathsf{Q}^\theta, \quad \alpha_k \mapsto \alpha_k + \alpha_{-k}$$

is isomorphic to the unreduced root system ${}^{\theta}\Phi$ of type $\mathtt{BC}_\infty$, and the image ${}^{\theta}\Phi^+$ of Φ^+ is a set of positive roots for ${}^{\theta}\Phi$.

Let q be an indeterminate and set $\mathcal{K} = \mathbb{Q}(q)$ and $\mathcal{A} = \mathbb{Z}[q^{\pm 1}]$. Let $\bar{\ }\colon \mathcal{K} \to \mathcal{K}$ be the *bar involution*, i.e., the $\mathbb{Q}$-algebra involution with $\bar{q} = q^{-1}$. Set

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad [n]! = [n][n-1]\cdots[1], \quad [2n]!! = [2n][2n-2]\cdots[2].$$

If A is a $\mathcal{K}$-algebra, $a \in A$ and $n \in \mathbb{N}$, then $a^{(n)} = a^n/[n]!$. For $\nu = \nu_1^{a_1}\cdots\nu_k^{a_k} \in J^\bullet$ with $\nu_j \ne \nu_{j+1}$, set $[\nu]! = [a_1]!\cdots[a_k]!$.

3B. ***The algebras* f *and* f*.** Let $\mathbf{f}$ be the $\mathcal{K}$-algebra generated by the elements f_i, where $i \in J$, subject to the q-Serre relations:

$$\sum_{k+l=1-i\cdot j} (-1)^k f_i^{(k)} f_j f_i^{(l)} = 0, \quad \text{where } i \neq j.$$

The algebra $\mathbf{f}$ is Q-graded with f_i in degree $-i$. We denote by $-|u|$ the Q-degree of a homogeneous element $u \in \mathbf{f}$. Given $\nu = \nu_1 \cdots \nu_n \in J^\bullet$, let $f_\nu = f_{\nu_1} \cdots f_{\nu_n}$. We will use notation of this form more generally, i.e., given any collection of elements y_i labeled by $i \in J$, we write $y_\nu = y_{\nu_1} \cdots y_{\nu_n}$.

Kashiwara [1991] introduced q-derivations $e'_i, e^*_i \in \mathrm{End}_{\mathcal{K}}(\mathbf{f})$ characterized by

$$e'_i(f_j) = \delta_{ij}, \quad e'_i(uv) = e'_i(u)v + q^{-i\cdot|u|}ue'_i(v),$$
$$e^*_i(f_j) = \delta_{ij}, \quad e^*_i(uv) = q^{-i\cdot|v|}e^*_i(u)v + ue^*_i(v),$$

for all homogeneous elements $u, v \in \mathbf{f}$. Both $\{e'_i \mid i \in J\}$ and $\{e^*_i \mid i \in J\}$ satisfy the q-Serre relations.

There is a unique nondegenerate symmetric bilinear form $(\cdot\,,\cdot)$ on $\mathbf{f}$ such that $(1, 1) = 1$ and $(e'_i(u), v) = (u, f_i v)$ for $u, v \in \mathbf{f}$ and $i \in J$. This form differs slightly from the form $(\cdot\,,\cdot)_L$ introduced by Lusztig [1993, Proposition 1.2.3] — see [Leclerc 2004, §2.2] for the precise relationship. Let $\mathbf{f}_{\mathcal{A}}$ be the integral form of $\mathbf{f}$, i.e., the $\mathcal{A}$-subalgebra generated by the $f_i^{(k)}$, with $i \in J$ and $k \in \mathbb{N}$, and let

$$\mathbf{f}^*_{\mathcal{A}} = \{u \in \mathbf{f} \mid (u, v) \in \mathcal{A} \text{ for all } v \in \mathbf{f}_{\mathcal{A}}\}$$

be its dual.

3C. ***Enomoto–Kashiwara algebra.*** The subalgebra of $\mathrm{End}_{\mathcal{K}}(\mathbf{f})$ generated by the e'_i and left multiplication by f_i is called the *reduced* q*-analogue* of $U(\mathfrak{g})$. The generators satisfy the relation

$$e'_i f_j = q^{-\alpha_i\cdot\alpha_j} f_j e'_i + \delta_{ij}.$$

Enomoto and Kashiwara [2006] defined a related algebra, which also depends on the involution θ. As it appears, this algebra does not have a distinctive name in the literature, so we call it the Enomoto–Kashiwara algebra.

Definition 3.1. The *Enomoto–Kashiwara* algebra ${}^\theta\mathcal{B}(\mathfrak{g})$ is the $\mathcal{K}$-algebra generated by E_i, F_i and the invertible elements T_i, with $i \in J$, subject to the following relations:

- the T_i commute,
- $T_{\theta(i)} = T_i$ for any i,
- $T_i E_j T_i^{-1} = q^{(i+\theta(i))\cdot j} E_j$ and $T_i F_j T_i^{-1} = q^{-(i+\theta(i))\cdot j} F_j$ for $i, j \in J$,
- $E_i F_j = q^{-i\cdot j} F_j E_i + \delta_{ij} + \delta_{\theta(i)j} T_i$ for all $i, j \in J$,
- the E_i and the F_i satisfy the q-Serre relations.

Proposition 3.2. *Let* $\lambda \in \mathsf{P}_+$.

(1) *There exists a* ${}^\theta\mathcal{B}(\mathfrak{g})$*-module* ${}^\theta V(\lambda)$ *generated by a nonzero vector* v_λ *such that*:

  (a) $E_i v_\lambda = 0$ *for any* $i \in J$,

  (b) $T_i v_\lambda = q^{{}^\theta\lambda\cdot i} v_\lambda$ *for any* $i \in J$,

  (c) $\{u \in {}^\theta V(\lambda) \mid E_i u = 0 \text{ for any } i \in J\} = \mathcal{K} v_\lambda$.

(2) ${}^\theta V(\lambda)$ *is irreducible and unique up to isomorphism.*

(3) *There exists a unique symmetric bilinear form* $(\cdot\,,\cdot)$ *on* ${}^\theta V(\lambda)$ *such that* $(v_\lambda, v_\lambda) = 1$ *and* $(E_i u, v) = (u, F_i v)$ *for any* $i \in J$ *and* $u, v \in {}^\theta V(\lambda)$. *It is nondegenerate.*

(4) *There is a unique endomorphism* $\bar{\cdot}$ *of* ${}^\theta V(\lambda)$, *called the bar involution*, *such that* $\overline{v_\lambda} = v_\lambda$ *and* $\overline{av} = \bar{a}\bar{v}$, $\overline{F_i v} = F_i \bar{v}$ *for* $a \in \mathcal{K}$ *and* $v \in {}^\theta V(\lambda)$.

(5) *Let* ${}^\theta\widetilde{V}(\lambda)$ *be the free* $\mathbf{f}$*-module with generator* $\tilde{v}_\lambda$ *and a* ${}^\theta\mathcal{B}(\mathfrak{g})$*-module structure given by*

$$T_i(u\tilde{v}_\lambda) = q^{{}^\theta\lambda\cdot i - (i+\theta(i))\cdot|u|} u\tilde{v}_\lambda, \tag{3-2}$$

$$E_i(u\tilde{v}_\lambda) = e_i'(u)\tilde{v}_\lambda, \tag{3-3}$$

$$F_i(u\tilde{v}_\lambda) = (f_i u + q^{{}^\theta\lambda\cdot i - i\cdot|u|} u f_{\theta(i)})\tilde{v}_\lambda, \tag{3-4}$$

for any $i \in J$ *and* $u \in \mathbf{f}$. *Then the subspace of* ${}^\theta\widetilde{V}(\lambda)$ *spanned by the vectors* $F_\nu \cdot \tilde{v}_\lambda$ *is a* ${}^\theta\mathcal{B}(\mathfrak{g})$*-submodule isomorphic to* ${}^\theta V(\lambda)$.

Proof. See [Enomoto and Kashiwara 2008, Proposition 2.11, Lemma 2.15]. □

From now on, let us identify $\mathbf{f}$ with the subalgebra of ${}^\theta\mathcal{B}(\mathfrak{g})$ generated by the F_i. Note that it follows from Proposition 3.2 that ${}^\theta V(\lambda) = \mathbf{f}\cdot v_\lambda$. The module ${}^\theta V(\lambda)$ has a P^θ-grading:

$${}^\theta V(\lambda) = \bigoplus_{\mu\in P^\theta} {}^\theta V(\lambda)_\mu,$$

where ${}^\theta V(\lambda)_\mu = \{v \in {}^\theta V(\lambda) \mid T_i \cdot v = q^{\mu\cdot i} u\}$. If $v \in {}^\theta V(\lambda)_\mu$, write $\mu_v := \mu$ and ${}^\theta|v| = \mu_v$. The integral and dual integral forms are defined as ${}^\theta V(\lambda)^{\mathrm{low}}_{\mathcal{A}} = \mathbf{f}_{\mathcal{A}} v_\lambda$ and ${}^\theta V(\lambda)^{\mathrm{up}}_{\mathcal{A}} = \left\{v \in {}^\theta V(\lambda) \mid \left({}^\theta V(\lambda)^{\mathrm{low}}_{\mathcal{A}}, v\right) \in \mathcal{A}\right\}$, respectively.

The operators E_i satisfy a kind of “twisted derivation” property.

Lemma 3.3. *We have*

$$E_i y \cdot v = q^{-i\cdot|y|} y E_i \cdot v + (e_i'(y) + q^{-i\cdot|e^*_{\theta(i)}(y)|} e^*_{\theta(i)}(y) T_i) \cdot v$$

for any $y \in \mathbf{f}$ *and* $v \in {}^\theta V(\lambda)$.

Proof. This is [Enomoto and Kashiwara 2008, Lemma 2.9]. □

3D. *Quantum shuffle algebra.* The *quantum shuffle algebra* $\mathcal{F}$ is the Q-graded $\mathcal{K}$-algebra with basis $J^\bullet$, where $\deg_{\mathsf{Q}} \nu = -|\nu|$, and multiplication given by

$$\nu \circ \nu' = \sum_{w \in \mathrm{D}_{\|\beta\|,\|\beta'\|}} q^{-d(\nu,\nu',w)} w \cdot \nu\nu' \tag{3-5}$$

for $\nu \in J^\beta$ and $\nu' \in J^{\beta'}$, where $\nu\nu' = i_1 \cdots i_{\|\beta+\beta'\|}$ and

$$d(\nu, \nu', w) = \sum_{\substack{k \leq \|\beta\| < l, \\ w(k) > w(l)}} i_{w^{-1}(k)} \cdot i_{w^{-1}(l)}. \tag{3-6}$$

To $\nu = i_1 \cdots i_k \in J^\bullet$ one associates the q-derivation $\partial_\nu = e^*_{i_1} \cdots e^*_{i_k} \in \mathrm{End}_{\mathcal{K}}(\mathbf{f})$. There is a $\mathcal{K}$-linear map

$$\Psi : \mathbf{f} \longrightarrow \mathcal{F}, \quad \Psi(u) = \sum_{\substack{\nu \in J^\bullet, \\ |\nu| = |u|}} \partial_\nu(u) \cdot \nu \tag{3-7}$$

for a homogeneous element $u \in \mathbf{f}$.

Let $\mathbf{e}'_i, \mathbf{e}^*_i \in \mathrm{End}_{\mathcal{K}}(\mathcal{F})$ be the left and right deletion operators:

$$\mathbf{e}'_i(i_1 \cdots i_k) = \delta_{i,i_1} i_2 \cdots i_k, \quad \mathbf{e}^*_i(i_1 \cdots i_k) = \delta_{i,i_k} i_1 \cdots i_{k-1}, \quad \mathbf{e}'_i(\varnothing) = \mathbf{e}^*_i(\varnothing) = 0,$$

respectively.

Proposition 3.4. *The map (3-7) is an injective* Q*-graded algebra homomorphism satisfying*

$$\mathbf{e}'_i \circ \Psi = \Psi \circ e'_i \quad \textit{and} \quad \mathbf{e}^*_i \circ \Psi = \Psi \circ e^*_i.$$

Proof. This follows directly from [Leclerc 2004, Lemma 3 and Theorem 4]. The proof for left deletions is analogous. □

We will now consider some antiautomorphisms of $\mathbf{f}$ and $\mathcal{F}$. Set

$$\sigma : J^\bullet \to J^\bullet, \quad \nu \mapsto w_0(\nu), \qquad {}^\theta\sigma : J^\bullet \to J^\bullet, \quad \nu \mapsto {}^\theta w(\nu). \tag{3-8}$$

We extend these maps to $\mathcal{K}$-linear maps $\sigma : \mathcal{F} \to \mathcal{F}$ and ${}^\theta\sigma : \mathcal{F} \to \mathcal{F}$. We use the same symbols to denote the $\mathcal{K}$-linear maps

$$\sigma : \mathbf{f} \to \mathbf{f}, \quad f_\nu \mapsto f_{\sigma(\nu)}, \qquad {}^\theta\sigma : \mathbf{f} \to \mathbf{f}, \quad f_\nu \mapsto f_{{}^\theta\sigma(\nu)},$$

respectively.

Lemma 3.5. *The maps σ and ${}^\theta\sigma$ are algebra antiautomorphisms satisfying $\sigma \circ \Psi = \Psi \circ \sigma$ and ${}^\theta\sigma \circ \Psi = \Psi \circ {}^\theta\sigma$, respectively.*

Proof. The case of σ is [Leclerc 2004, Proposition 6]. The case of ${}^\theta\sigma$ follows easily from (3-5) and (3-6). □

3E. ***Quantum shuffle module.*** We will now realize the modules ${}^{\theta}V(\lambda)$ in terms of modules over the shuffle algebra.

Definition 3.6. We define the *quantum shuffle module* ${}^{\theta}\mathscr{F}(\lambda)$ to be the P^{θ}-graded $\mathscr{K}$-vector space with basis ${}^{\theta}J^{\bullet}$, where $\deg_{\mathsf{P}^{\theta}} \nu = {}^{\theta}\lambda - {}^{\theta}|\nu|$, and a right $\mathscr{F}$-action given by

$$\nu \circledcirc \nu' = \sum_{w \in {}^{\theta}\mathsf{D}_{\|\beta\|_{\theta}, \|\beta'\|}} q^{-d(\nu,\nu',w)} w \cdot \nu\nu' \tag{3-9}$$

for $\nu \in {}^{\theta}J^{\beta}$ and $\nu' \in J^{\beta'}$, where

$$d(\nu, \nu', w) = \sum_{\substack{1 \le k < l \le N,\\ w(k) > w(l)}} i_{w^{-1}(k)} \cdot i_{w^{-1}(l)} + \sum_{\substack{1 \le k < l \le N,\\ w(-k) > w(l)}} i_{w^{-1}(-k)} \cdot i_{w^{-1}(l)} - \sum_{\substack{\|\beta\|_{\theta} < l,\\ w(l) < w(-l)}} {}^{\theta}\lambda \cdot i_l,$$

with $N = \|\beta\|_{\theta} + \|\beta'\|$.

Remark 3.7. We have chosen to define ${}^{\theta}V(\lambda)$ as a left ${}^{\theta}\mathscr{B}(\mathfrak{g})$-module, but ${}^{\theta}\mathscr{F}(\lambda)$ as a right $\mathscr{F}$-module. This choice is a compromise. On the one hand, we wanted to be consistent with the conventions of [Enomoto and Kashiwara 2006; 2008]. On the other hand, as shown in [Appel and Przeździecki 2022], ${}^{\theta}V(\lambda)$ can be categorified via quantum symmetric pairs, which are, by convention (see, e.g., [Kolb 2014]), right coideal subalgebras.

Let $\mathbf{E}_i \in \mathrm{End}_{\mathscr{K}}({}^{\theta}\mathscr{F}(\lambda))$ be the right deletion operator:

$$\mathbf{E}_i(i_1 \cdots i_k) = \delta_{i,i_k} i_1 \cdots i_{k-1}, \quad \mathbf{E}_i(\varnothing) = 0.$$

Lemma 3.8. *Formula (3-9) defines a right $\mathscr{F}$-action on ${}^{\theta}\mathscr{F}(\lambda)$. Moreover, the endomorphisms $\mathbf{E}_i$ satisfy*

$$\mathbf{E}_i(v \circledcirc z) = q^{-i\cdot|z|}\mathbf{E}_i(v) \circledcirc z + v \circledcirc \mathbf{e}_i^*(z) + q^{-i\cdot|\mathbf{e}'_{\theta(i)}(z)|+\mu_v\cdot i} v \circledcirc \mathbf{e}'_{\theta(i)}(z).$$

Proof. The first statement follows easily from the definitions, so we omit a proof. Let us prove the second statement. It suffices to consider v and z of the form $v = \nu j$ and $z = k\mu l$, for $\nu \in {}^{\theta}J^{\bullet}$, $\mu \in J^{\bullet}$ and $j, k, l \in J$. Then (3-9) implies

$$v \circledcirc z = \nu j \circledcirc k\mu l = (v \circledcirc k\mu)l + q^{-d(v,z,w)}(\nu \circledcirc z)j + q^{-d(v,z,w')}(v \circledcirc \mu l)\theta(k),$$

where w transposes j and z while w' sends k to $\theta(k)$ and transposes it with μl. One easily sees that $d(v, z, w) = j \cdot |z|$ and $d(v, z, w') = \theta(k) \cdot |\mathbf{e}'_k(z)| - \mu_v(\theta(k))$. Hence,

$$\mathbf{E}_i(v \circledcirc z) = \delta_{i,l}(v \circledcirc k\mu) + \delta_{i,j} q^{-i\cdot|z|}(\nu \circledcirc z) + \delta_{i,\theta(k)} q^{-i\cdot|\mathbf{e}'_{\theta(i)}(z)|+\mu_v\cdot i}(v \circledcirc \mu l).$$

The statement follows. □

To $\nu = \nu_1 \cdots \nu_k \in J^\bullet$ one associates the operator ${}^\theta\partial_\nu = E_{\nu_1} \cdots E_{\nu_k} \in \mathrm{End}({}^\theta V(\lambda))$. There is a $\mathcal{K}$-linear map

$$
{}^\theta\Psi\colon {}^\theta V(\lambda) \to {}^\theta\mathcal{F}(\lambda), \quad {}^\theta\Psi(u) = \sum_{\substack{\nu \in {}^\theta J^\bullet, \\ {}^\theta|\nu| = {}^\theta|u|}} {}^\theta\partial_\nu(u) \cdot \sigma(\nu) \tag{3-10}
$$

for a homogeneous element $u \in {}^\theta V(\lambda)$. Let us abbreviate

$$
\mathbf{U} = \Psi(\mathbf{f}) \quad \text{and} \quad {}^\theta\mathbf{V}(\lambda) = {}^\theta\Psi({}^\theta V(\lambda)).
$$

Proposition 3.9. *The map* (3-10) *is injective*, $\mathbf{E}_i \circ {}^\theta\Psi = {}^\theta\Psi \circ E_i$ *and the diagram*

$$
\begin{array}{ccc}
\mathbf{f} & \xrightarrow{\ \Psi\ } & \mathcal{F} \\
\curvearrowright & & \curvearrowleft \\
{}^\theta V(\lambda) & \xrightarrow{\ {}^\theta\Psi\ } & {}^\theta\mathcal{F}(\lambda)
\end{array}
$$

commutes.

Proof. The injectivity of ${}^\theta\Psi$ follows directly from Proposition 3.2 (1c). Let ${}^\theta\Psi'\colon {}^\theta V(\lambda) \to {}^\theta\mathcal{F}(\lambda)$ be the map sending $y \cdot v_\lambda \mapsto \varnothing \circledcirc \Phi(\sigma(y))$ for $y \in \mathbf{f}$. Note that ${}^\theta\Psi'$ is defined on all of ${}^\theta V(\lambda)$ since ${}^\theta V(\lambda) = \mathbf{f} \cdot v_\lambda$. We claim that ${}^\theta\Psi'$ intertwines the actions of $\mathbf{f}$ and $\mathcal{F}$, and that ${}^\theta\Psi = {}^\theta\Psi'$. For the first claim, note that (3-9) implies that $\nu \circledcirc i = \nu \circ i + q^{{}^\theta\lambda(i) - i \cdot |\nu|}\theta(i) \circ \nu$, for $i \in J$ and $\nu \in J^\bullet$. Hence, by Proposition 3.2 (5) and (3-4), the first claim follows. Lemma 3.3 and Lemma 3.8 imply that $\mathbf{E}_i \circ {}^\theta\Psi' = {}^\theta\Psi' \circ E_i$. Let $v \in {}^\theta V(\lambda)$ be homogeneous, and let $\nu \in {}^\theta J^\bullet$ with ${}^\theta|v| = {}^\theta|\nu|$. Let $\gamma_\nu(v)$ be the coefficient of $\sigma(\nu)$ in ${}^\theta\Psi'(v)$. Then $\gamma_\nu(v) = \mathbf{E}_{\sigma(\nu)} \circ {}^\theta\Psi'(v) = {}^\theta\partial_\nu(v)$. Hence ${}^\theta\Psi = {}^\theta\Psi'$, which completes the proof. □

3F. *θ-good words.* We fix a total order on the set J and equip $J^\bullet$ with the corresponding antilexicographic order. Both are denoted by $\leq$. Given a linear combination u of words, let $\max(u)$ be the largest word appearing in u.

Lemma 3.10. *If* $\mu' \leq \mu$, $\nu' \leq \nu$ *and* ${}^\theta w(\nu') \leq {}^\theta w(\nu)$, *for* $\mu, \mu' \in {}^\theta J^\bullet$ *and* $\nu, \nu' \in J^\bullet$ (*with* $\|\mu\| = \|\mu'\|$ *and* $\|\nu\| = \|\nu'\|$), *then* $\max(\mu' \circledcirc \nu') \leq \max(\mu \circledcirc \nu)$. *If any of the former three inequalities is strict*, *then the last inequality is strict*, *too.*

Proof. If $w \in {}^\theta\mathtt{D}_{\|\mu\|_\theta, \|\nu\|}$, then the condition in the hypothesis forces $w \cdot \mu'\nu'$ to be smaller than or equal to $w \cdot \mu\nu$. □

A word $\nu \in J^\bullet$ is called *good* if $\nu = \max(\Psi(x))$ for some homogeneous $x \in \mathbf{f}$. Let $J^\bullet_+$ denote the set of good words and $J^\beta_+ = J^\bullet_+ \cap J^\beta$. We now define the analogue of good words for quantum shuffle modules.

Definition 3.11. A word $\nu \in {}^\theta J^\bullet$ is called θ-*good* if $\nu = \max({}^\theta\Psi(u))$ for some homogeneous $u \in {}^\theta V(\lambda)$. Let ${}^\theta J^\bullet_+$ denote the set of all θ-good words, and let ${}^\theta J^\beta_+ = {}^\theta J^\bullet_+ \cap {}^\theta J^\beta$.

In [Leclerc 2004], a *monomial* basis $\{\mathbf{m}_\nu = \Psi(f_{\sigma(\nu)}) \mid \nu \in J_+^\bullet\}$ of $\mathbf{U}$ was constructed. An analogous basis exists for ${}^\theta\mathbf{V}(\lambda)$.

Lemma 3.12. *There is a unique basis of homogeneous vectors* $\{{}^\theta\mathbf{m}_\nu^* \mid \nu \in {}^\theta J_+^\bullet\}$ *of* ${}^\theta\mathbf{V}(\lambda)$ *such that* $\mathbf{E}_\mu({}^\theta\mathbf{m}_\nu) = \delta_{\mu,\nu}$ *for any* μ *with* ${}^\theta|\mu| = {}^\theta|\nu|$. *The adjoint basis is* $\{{}^\theta\mathbf{m}_\nu = {}^\theta\Psi(F_{\sigma(\nu)} \cdot v_\lambda)\}$.

Proof. The proof is analogous to the proof of [Leclerc 2004, Proposition 12]. □

Let $\mathcal{F}^{\mathrm{fr}}$ be the free associative $\mathcal{K}$-algebra generated by J (with multiplication given by concatenation of letters), and let V^{fr} be its right regular representation. There is an algebra homomorphism

$$\Xi\colon \mathcal{F}^{\mathrm{fr}} \to \mathcal{F}, \quad \nu = \nu_1 \cdots \nu_k \mapsto \nu_1 \circ \cdots \circ \nu_k = \Psi(f_\nu)$$

and a linear map

$${}^\theta\Xi_\lambda\colon V^{\mathrm{fr}} \to {}^\theta\mathbf{V}(\lambda), \quad \nu \mapsto \varnothing \circledcirc \Xi(\nu) = {}^\theta\mathbf{m}_\nu.$$

intertwining the actions of $\mathcal{F}^{\mathrm{fr}}$ and $\mathcal{F}$. We have the following characterization of θ-good words:

Lemma 3.13. *The following are equivalent*:

(1) $\nu \in {}^\theta J^\bullet$ *is* θ*-good*,

(2) ν *cannot be expressed modulo* $\ker {}^\theta\Xi_\lambda$ *as a linear combination of words* $\mu > \nu$.

Proof. Let $u \in {}^\theta\mathbf{V}(\lambda)$ and $\nu \in {}^\theta J^\bullet$ satisfy ${}^\theta|u| = {}^\theta|\nu|$ and $\mathbf{E}_\nu(u) \neq 0$. Proposition 3.2 (3) implies that $0 \neq (\mathbf{E}_\nu(u), \varnothing) = (u, {}^\theta\mathbf{m}_\nu)$. If ν could be expressed modulo $\ker {}^\theta\Xi_\lambda$ as a linear combination of words $\mu > \nu$, then there would exist a relation of the form

$$ {}^\theta\mathbf{m}_\nu = \sum_{\mu > \nu} c_\mu {}^\theta\mathbf{m}_\mu \tag{3-11}$$

for some $c_\nu \in \mathcal{K}$. Hence,

$$0 \neq \mathbf{E}_\nu(u) = \sum_{\mu > \nu} c_\mu \mathbf{E}_\mu(u).$$

Therefore, $\mathbf{E}_\mu(u) \neq 0$ for some $\mu > \nu$, which implies that μ is not θ-good. This proves the implication (1) $\Longrightarrow$ (2).

Conversely, let ${}^\theta\tilde{J}_+^\bullet$ be the set of words in ${}^\theta J^\bullet$ satisfying (2). We have shown that ${}^\theta J_+^\bullet \subseteq {}^\theta\tilde{J}_+^\bullet$. Lemma 3.12 implies that the set $\{{}^\theta\mathbf{m}_\nu \mid \nu \in {}^\theta\tilde{J}_+^\bullet\}$ contains a basis of ${}^\theta\mathbf{V}(\lambda)$. Moreover, it is linearly independent. Indeed, if there was a linear relation between words of ${}^\theta\tilde{J}_+^\bullet$, one could express the smallest one in terms of the others and it would not belong to ${}^\theta\tilde{J}_+^\bullet$. □

Lemma 3.14. *The* θ*-good words have the following properties*:

(1) *If* ν *is* θ*-good and* $\nu = \mu_1\mu_2$, *then* μ_1 *is* θ*-good.*

(2) *If* ν *is* θ*-good, then* ν *is good.*

Proof. By Proposition 3.9, ${}^{\theta}\mathbf{V}(\lambda)$ is stable under the operators $\mathbf{E}_i$. Pick $u \in {}^{\theta}\mathbf{V}(\lambda)$ with $\max(u) = \nu$. Then $\max(\mathbf{E}_{\mu_2}(u)) = \mathbf{E}_{\mu_2}(\max(u)) = \mu_1$. This proves the first part. Next, suppose that ν is not good. Then, by [Leclerc 2004, Lemma 21], we have a relation of the form $\mathbf{m}_\nu = \sum_{\mu > \nu} c_\mu \mathbf{m}_\mu$. Applying both sides to $\varnothing$, we get (3-11). Hence, by Lemma 3.13, ν is not θ-good. This proves the second part. □

3G. *Lyndon words.* A nontrivial word $\nu \in J^\bullet$ is called *Lyndon* if it is smaller than all its proper left factors. Note that our definition uses the opposite of the convention of [Leclerc 2004; Kleshchev and Ram 2011], where right factors are used instead. Let $\mathscr{L}$ denote the set of Lyndon words and $\mathscr{L}_+ = \mathscr{L} \cap J^\bullet_+$ the set of good Lyndon words.

Proposition 3.15. *Lyndon words have the following properties*:

(1) *Every word $\nu \in J^\bullet$ has a unique factorization $\nu = \nu^{\langle k \rangle} \cdots \nu^{\langle 1 \rangle}$ into Lyndon words such that $\nu^{\langle 1 \rangle} \geq \cdots \geq \nu^{\langle k \rangle}$.*

(2) *The word ν is good if and only if each $\nu^{\langle m \rangle}$ is good.*

(3) *The map $\nu \mapsto |\nu|$ yields a bijection $\mathscr{L}_+ \xrightarrow{\sim} \Phi^+$. The induced order on Φ^+ is convex.*

(4) *Let $\mu \in \mathscr{L} \backslash J$ and write $\mu = \mu_{(1)}\mu_{(2)}$ with $\mu_{(2)}$ a proper Lyndon subword of maximal length. Then $\mu_{(1)} \in \mathscr{L}$.*

Proof. For part (1), see, e.g., [Lothaire 2002, Theorem 11.5.1]. For parts (2) and (3), see [Leclerc 2004, Propositions 17, 18 and 26]. For part (4), see [Leclerc 2004, Lemma 14]. □

We call the factorization from Proposition 3.15 (1) the *Lyndon factorization* and the Lyndon words in this factorization *Lyndon factors*. We will write it in two ways: $\nu = \nu^{\langle k \rangle} \cdots \nu^{\langle 1 \rangle}$ for $\nu^{\langle 1 \rangle} \geq \cdots \geq \nu^{\langle k \rangle}$ or $\nu = (\nu^{\langle l \rangle})^{n_l} \cdots (\nu^{\langle 1 \rangle})^{n_1}$ for $\nu^{\langle 1 \rangle} > \cdots > \nu^{\langle l \rangle}$. The factorization from Proposition 3.15 (4) is called the *standard factorization* of a Lyndon word.

Given $x, y \in \mathscr{F}$, let $[x, y]_q = xy - q^{|x| \cdot |y|} yx$. One defines a map $[\ ] \colon \mathscr{L} \to J^\bullet$ by induction on the standard factorization: $[i] = i$ for $i \in J$, and $[\nu] = [\nu_{(2)}, \nu_{(1)}]_q$ if $\nu = \nu_{(1)}\nu_{(2)}$ is the standard factorization of ν. Next, given $\nu = \nu^{\langle k \rangle} \cdots \nu^{\langle 1 \rangle} \in J^\bullet$, let $[\nu] = [\nu^{\langle k \rangle}] \cdots [\nu^{\langle 1 \rangle}]$. For $\nu \in J^\bullet_+$, set

$$\mathbf{l}_\nu = \Xi([\nu]), \quad \nu \in J^\bullet_+, \qquad {}^{\theta}\mathbf{l}_\nu = {}^{\theta}\Xi_\lambda([\nu]), \quad \nu \in {}^{\theta}J^\bullet_+.$$

Proposition 3.16. *For any $\nu \in J^\bullet$, we have* $\min([\nu]) = \nu$. *Moreover, the set $\{\mathbf{l}_\nu \mid \nu \in J^\bullet_+\}$ is a basis of* $\mathbf{U}$.

Proof. See [Leclerc 2004, Propositions 19 and 22]. □

The basis from Proposition 3.16 is called the *Lyndon basis*.

Lemma 3.17. *The set* $\{{}^\theta\mathbf{l}_\nu \mid \nu \in {}^\theta J^\bullet_+\}$ *is a basis of* ${}^\theta\mathbf{V}(\lambda)$. *Moreover, the transition matrix* $(c_{\nu\mu})$ *from* $\{{}^\theta\mathbf{l}_\nu \mid \nu \in {}^\theta J^\bullet_+\}$ *to* $\{{}^\theta\mathbf{m}_\mu \mid \mu \in {}^\theta J^\bullet_+\}$ *is triangular with* $c_{\nu\nu} = \prod_{i=1}^k (-1)^{\ell(\nu^{\langle k\rangle})-1} q^{-N(|\nu^{\langle k\rangle}|)}$.

Proof. By Proposition 3.16, we can write $[\nu] = c_{\nu\nu}\nu + \sum_{\nu<\mu} c_{\nu\mu}\mu$, for some $c_{\nu\mu} \in \mathcal{K}$. Applying ${}^\theta\Xi_\lambda$ to both sides, we get ${}^\theta\mathbf{l}_\nu = c_{\nu\nu}{}^\theta\mathbf{m}_\nu + \sum_{\mu>\nu} c_{\nu\mu}{}^\theta\mathbf{m}_\mu$. By Lemma 3.13, this can be rewritten as ${}^\theta\mathbf{l}_\nu = c_{\nu\nu}{}^\theta\mathbf{m}_\nu + \sum_{\nu<\mu\in{}^\theta J^\bullet_+} c'_{\nu\mu}{}^\theta\mathbf{m}_\mu$. Hence the transition matrix is triangular. To show the last statement of the lemma, one uses the same calculation as in [Leclerc 2004, Proposition 30]. □

Assumption 1. From now on, we assume that we are working with the standard ordering of J, i.e., $\alpha_k \le \alpha_l$ if and only if $k \le l$. In this case, the map ${}^\theta\sigma$ in (3-8) preserves $\mathcal{L}_+$.

Before stating the next lemma, we need to introduce some notation. Given $\mu, \mu' \in \mathcal{L}_+$ with $|\mu| = \beta_{k,l}$, $|\mu'| = \beta_{m,n}$, we write

$$\mu \subset \mu' \quad\iff\quad m < k \text{ and } l < n.$$

Lemma 3.18. *The following hold*:

(1) *If* $\nu \in \mathcal{L}_+$, *then* $\mathbf{l}_\nu$ *is a multiple of* ν.

(2) *If* $\nu, \mu \in \mathcal{L}_+$ *and* $\mu \subset \nu$, *then* $\nu \circ \mu = \mu \circ \nu$.

Proof. It suffices to prove the first statement for $\nu = \nu_1 \cdots \nu_l \in \mathcal{L}_+$. We proceed by induction on l. The base case $l = 1$ is clear. Let $\nu = \nu_{(1)}\nu_{(2)}$ be the standard factorization of ν. Since we are working with the standard ordering on J, $\nu_{(1)} = i$ for some $i \in J$. By induction, we get that $\mathbf{l}_\nu = \Xi([\nu]) = \Xi([\nu_{(2)}]) \circ i - q^{-1} i \circ \Xi([\nu_{(2)}])$ is a multiple of $\nu_{(2)} \circ i - q^{-1} i \circ \nu_{(2)}$. Write $\nu_{(2)} = j\nu'_{(2)}$ with $j \in J$. Then (3-5) implies that $\nu_{(2)} \circ i - q^{-1} i \circ \nu_{(2)} = \big(j(\nu'_{(2)} \circ i) + qi\nu_{(2)}\big) - q^{-1}\big(i\nu_{(2)} + qj(i \circ \nu'_{(2)})\big) = [2]\nu$. This completes the proof of the first statement. The second statement now follows directly from [Leclerc 2004, Proposition 30] and [Enomoto and Kashiwara 2008, Proposition 3.14 (3)]. □

Definition 3.19. We say that $\nu \in \mathcal{L}$ is *θ-Lyndon* if $\nu \ge {}^\theta w(\nu)$. Let ${}^\theta\mathcal{L}$ be the set of θ-Lyndon words, and ${}^\theta\mathcal{L}_+ = J^\bullet_+ \cap {}^\theta\mathcal{L}$. Let ${}^\theta J^\bullet_{+,0}$ denote the set of all θ-good words $\mu = \nu^{\langle k\rangle} \cdots \nu^{\langle 1\rangle}$, with $\nu^{\langle k\rangle}, \cdots, \nu^{\langle 1\rangle} \in {}^\theta\mathcal{L}_+$. Moreover, if $\mu = \nu^{\langle k\rangle} \cdots \nu^{\langle 1\rangle} \in {}^\theta J^\bullet_+$ and $\nu^{\langle k\rangle}, \cdots, \nu^{\langle 1\rangle} \notin {}^\theta\mathcal{L}_+$, then μ is called *θ-cuspidal*. Let ${}^\theta J^\bullet_{+,c}$ denote the set of all θ-cuspidal words.

Lemma 3.20. *The θ-good Lyndon words have the following properties*:

(1) *If* $\nu \in \mathcal{L}_+$, *then* $\nu \in \mathbf{U}$.

(2) *Let* $\mu \in {}^\theta J^\bullet$ *and* $\nu \in {}^\theta\mathcal{L}$ *with* $\nu \ge \mu$. *Then* $\mu\nu = \max(\mu \circledcirc \nu)$.

(3) ${}^\theta\mathcal{L}_+ \subseteq \mathcal{L} \cap {}^\theta J^\bullet_+$.

(4) *Let* $\mu \in {}^{\theta}J^{\bullet}_{+}$ *and* $\nu \in {}^{\theta}\mathscr{L}_{+}$ *with* $\nu \geq \mu$. *Then* $\mu\nu \in {}^{\theta}J^{\bullet}_{+}$.

(5) *If all of the Lyndon factors of* ν *are in* ${}^{\theta}\mathscr{L}_{+}$, *then* $\nu \in {}^{\theta}J^{\bullet}_{+}$.

(6) *The map* $\nu \mapsto {}^{\theta}|\nu|$ *yields a bijection* ${}^{\theta}\mathscr{L}_{+} \xrightarrow{\sim} {}^{\theta}\Phi^{+}$.

Proof. Since ν is good, there exists some homogeneous $x \in \mathbf{U}$ such that $x = \nu + y$ with ν greater than any word μ in y. By Assumption 1 and [Leclerc 2004, §8.1], ν is of the form $\alpha_k\alpha_{k-2}\cdots\alpha_{k-2l}$, which implies that ν is the smallest word of weight $|\nu|$, so $x = \nu$. The proof of (2) is similar to the proof of [Leclerc 2004, Lemma 15]. If $\nu \in {}^{\theta}\mathscr{L}_{+}$, then, by definition, $\nu \in \mathscr{L}_{+}$ and $\nu \geq {}^{\theta}w(\nu)$. Hence, $\max(\varnothing \circledcirc \nu) = \nu$. By part (1), $\nu \in \mathbf{U}$, so $\nu \in {}^{\theta}J^{\bullet}_{+}$. This proves (3).

Let us prove (4). If $\mu = \varnothing$, then the statement reduces to (3). Otherwise, choose a homogeneous element $\varnothing \neq x \in {}^{\theta}\mathbf{V}(\lambda)$ such that $\mu = \max(x)$. Then, after possible rescaling, $x = \mu + r$, where r is a linear combination of words $< \mu$. We have $x \circledcirc \nu = \mu \circledcirc \nu + r \circledcirc \nu$. Part (2) implies that $\max(\mu \circledcirc \nu) = \mu\nu$. It follows from Lemma 3.10 that $\max(\mu \circledcirc \nu) > \max(r \circledcirc \nu)$.

Next, we prove (5). Suppose that each factor of $\nu = \nu^{\langle k\rangle} \cdots \nu^{\langle 1\rangle}$ is θ-Lyndon. If $k = 1$, then ν is θ-good by (3). By induction on the number of Lyndon factors, we can assume that $\nu' = \nu^{\langle k\rangle} \cdots \nu^{\langle 2\rangle}$ is θ-good. The statement now follows from (4). Part (6) is clear from the definitions. □

Given $\nu = \nu^{\langle s\rangle} \cdots \nu^{\langle 1\rangle}$, $\nu' = \nu^{\langle t\rangle} \cdots \nu^{\langle s+1\rangle} \in J^{\bullet}_{+}$, let $\mathrm{sh}(\nu, \nu') = \mu^{\langle t\rangle} \cdots \mu^{\langle 1\rangle}$ be the good word obtained by shuffling the Lyndon factors of ν and ν' in such a way that $\mu^{\langle t\rangle} \leq \cdots \leq \mu^{\langle 1\rangle}$.

Lemma 3.21. *The map*

$$ {}^{\theta}J^{\bullet}_{+,c} \times {}^{\theta}J^{\bullet}_{+,0} \to {}^{\theta}J^{\bullet}_{+}, \quad (\nu, \nu') \mapsto \mathrm{sh}(\nu, \nu'), $$

is a well-defined injection.

Proof. It is clear the map is injective, so we only have to show that $\mathrm{sh}(\nu, \nu')$ is θ-good. We argue by induction on the number k of Lyndon factors in $\nu' = \nu^{\langle k\rangle} \cdots \nu^{\langle 1\rangle}$. If $k = 0$, then ν is θ-good by assumption. Otherwise, letting $\nu'' = \nu^{\langle k\rangle} \cdots \nu^{\langle 2\rangle}$, we can assume that $\mathrm{sh}(\nu, \nu'')$ is θ-good. If $\nu^{\langle 1\rangle} \geq \mathrm{sh}(\nu, \nu')$, then $\mathrm{sh}(\nu, \nu') = \mathrm{sh}(\nu, \nu'')\nu^{\langle 1\rangle}$, and we conclude that $\mathrm{sh}(\nu, \nu') \in {}^{\theta}J^{\bullet}_{+}$ from Lemma 3.20 (4).

If $\nu^{\langle 1\rangle} < \mathrm{sh}(\nu, \nu')$, then we require the following generalization of Lemma 3.20 (4): given $a \in {}^{\theta}J^{\bullet}_{+}$ and $b \in {}^{\theta}\mathscr{L}_{+}$ with $b < a$, we have $\mathrm{sh}(a, b) \in {}^{\theta}J^{\bullet}_{+}$. The old proof carries over except that instead of invoking Lemma 3.20 (2), we need to show that $\max(a \circledcirc b) = \mathrm{sh}(a, b)$. Without loss of generality, we may assume a is Lyndon. Since $b \geq {}^{\theta}w(b)$, we have $\max(a \circledcirc b) = \max(a \circ b)$. Let us write $a = a_n \cdots a_1$ and $b = b_m \cdots b_1$. Since $a_n \geq \cdots \geq a_1 > b_1$, it follows that $\max(a \circ b) = ba$. □

Given $\beta \in \mathrm{Q}^{\theta}_{+}$, let ${}^{\theta}\mathrm{kpf}(\beta)$ denote the number of ways to write β as a sum of roots in ${}^{\theta}\Phi^{+}$.

Proposition 3.22. *If* $\lambda = 0$, *then*: (*i*) ${}^\theta\mathscr{L}_+ = \mathscr{L} \cap {}^\theta J_+^\bullet$, *and* (*ii*) ${}^\theta J_+^\bullet = {}^\theta J_{+,0}^\bullet$. *Hence*, $\dim_q {}^\theta\mathbf{V}_\beta = {}^\theta\mathrm{kpf}(\beta)$.

Proof. Let S be the set of all words $\nu = \nu^{\langle k\rangle} \cdots \nu^{\langle 1\rangle}$ with $\nu^{\langle 1\rangle} \geq \cdots \geq \nu^{\langle k\rangle}$ and each $\nu^{\langle i\rangle} \in {}^\theta J_+^\bullet$. Lemma 3.12 and Lemma 3.20 (5) imply that $\{{}^\theta\mathbf{m}_\nu \mid \nu \in S\}$ is contained in the monomial basis $\{{}^\theta\mathbf{m}_\nu \mid \nu \in {}^\theta J_+^\bullet\}$ of ${}^\theta\mathbf{V}$. Let ${}^\theta\mathbf{V}' \subseteq {}^\theta\mathbf{V}$ be the span of the former. By construction, the generating series of the dimensions of the homogeneous components of ${}^\theta\mathbf{V}'$ is equal to $\prod_{\beta \in {}^\theta\Phi^+} 1/(1 - \exp\beta)$. On the other hand, it follows from [Enomoto and Kashiwara 2008, Theorem 4.15] that this is also the generating series of the dimensions of the homogeneous components of ${}^\theta\mathbf{V}$. Hence, ${}^\theta\mathbf{V}' = {}^\theta\mathbf{V}$. The statement follows. □

Remark 3.23. Instead of appealing to [Enomoto and Kashiwara 2008, Theorem 4.15] in the proof of Proposition 3.22, one could alternatively use the categorification theorem [Varagnolo and Vasserot 2011, Theorem 8.31] (cited as Theorem 4.5 below), together with the geometric realization of orientifold KLR algebras from [Varagnolo and Vasserot 2011] and the classification of isomorphism classes of symplectic/orthogonal representations of symmetric quivers from [Derksen and Weyman 2002]. Indeed, this approach appears promising in generalizing the construction of bases for ${}^\theta\mathbf{V}(\lambda)$ to the $\lambda \neq 0$ case.

3H. *Symmetric words.* A word $\nu \in {}^\theta\mathscr{L}_+$ is called *symmetric* if ${}^\theta w(\nu) = \nu$ and *nonsymmetric* otherwise. Given $\nu \in {}^\theta J_+^\bullet$, let ν_θ be the word obtained from ν by deleting its symmetric Lyndon factors and ν^θ the word obtained by deleting the nonsymmetric ones. We say that $\nu \in {}^\theta J_+^\bullet$ is *symmetric* if $\nu = \nu^\theta$. For each $k \geq 1$, let ξ_k be the unique symmetric word in ${}^\theta\mathscr{L}_+$ with $|\xi_k| = \beta_{-2k+1,2k-1}$.

Lemma 3.24. *Let* $\nu \in {}^\theta\mathscr{L}_+$. *If* $\nu < \xi_k$, *then* ξ_{k+1} *is a subword of* ν. *Hence*, $\xi_k > \xi_l$ *if and only if* $k < l$.

Proof. The statement follows immediately from Lemma 3.20 (6). □

Assumption 2. From now until the end of Section 3, we assume that $\lambda = 0$. We abbreviate ${}^\theta\mathscr{F} = {}^\theta\mathscr{F}(0)$ and ${}^\theta V = {}^\theta V(0)$.

Lemma 3.25. *Suppose that* $\nu \in {}^\theta J_+^\bullet$ *is symmetric or* $\nu \in {}^\theta\mathscr{L}_+$. *Then* ν *is the smallest word in* ${}^\theta J_+^{{}^\theta|\nu|}$.

Proof. Abbreviate $\beta = {}^\theta|\nu|$. First assume that $\nu \in {}^\theta J_+^\bullet$ is symmetric. Let $\nu = \nu^{\langle k\rangle} \cdots \nu^{\langle 1\rangle}$ be its Lyndon factorization. Suppose that there exists a word $\mu = \mu^{\langle l\rangle} \cdots \mu^{\langle 1\rangle} \in {}^\theta J_+^{{}^\theta|\nu|}$ with $\mu < \nu$. Then, as explained before Lemma 4.1 in [Melançon 1992], there is an a such that $\mu^{\langle b\rangle} = \nu^{\langle b\rangle}$ for $b < a$ and $\mu^{\langle a\rangle} < \nu^{\langle a\rangle}$. Hence, $\nu^{\langle a\rangle} > \mu^{\langle a\rangle} \geq \cdots \geq \mu^{\langle l\rangle}$. Write $\bar\nu = \nu^{\langle k\rangle} \cdots \nu^{\langle a\rangle}$ and $\bar\mu = \mu^{\langle l\rangle} \cdots \mu^{\langle a\rangle}$.

Since $\nu^{\langle a\rangle}$ is symmetric, we have $\nu^{\langle a\rangle} = \xi_d$ for some $d \geq 1$. By Proposition 3.22 and Lemma 3.24, ξ_{d+1} is a subword of each $\mu^{\langle i\rangle}$, where $i \geq a$. In particular, each $\mu^{\langle i\rangle}$ contains $\alpha_{\pm(2d-1)}$ and $\alpha_{\pm(2d+1)}$. Hence, if we write ${}^\theta|\bar{\nu}| = {}^\theta|\bar{\mu}| = \sum_{i\in\mathbb{N}_{\mathrm{odd}}} c_i(\alpha_i + \alpha_{-i})$, then $c_{2d+1} = c_{2d-1}$. On the other hand, since each $\nu^{\langle i\rangle}$, where $i \geq a$, is a symmetric good Lyndon word smaller than $\nu^{\langle a\rangle}$, Lemma 3.24 implies that each $\nu^{\langle i\rangle}$ contains $\nu^{\langle a\rangle}$ as a subword. Hence $c_{2d+1} < c_{2d-1}$, which is a contradiction.

Secondly, assume that $\nu \in {}^\theta\mathscr{L}_+$. We may assume ν is not symmetric. In that case, observe that if ${}^\theta|\mu| = {}^\theta|\nu|$ for some $\mu \in {}^\theta J_+^\bullet$, then $|\mu| = |\nu|$. The result now follows from [Kleshchev and Ram 2011, Lemma 5.9]. □

3I. ***PBW and canonical bases.*** Let us first recall some basic facts about PBW bases. For the moment let us restrict $(J, \cdot)$ to a finite Cartan subdatum of type A_m. By [Leclerc 2004, Proposition 26], the antilexicographic order $\nu^{\langle 1\rangle} > \cdots > \nu^{\langle N\rangle}$ on the set of good Lyndon words induces, via the bijection from Proposition 3.15 (3), a convex order $\beta_1 > \cdots > \beta_N$ on the set of positive roots. This convex order arises from a unique reduced decomposition $w_0 = s_{i_N} \cdots s_{i_1}$ in the usual way: $\beta_N = \alpha_{i_N}$, $\beta_{N-1} = s_{i_N}(\alpha_{i_{N-1}})$, …, $\beta_1 = s_{i_N} \cdots s_{i_2}(\alpha_1)$. Let $P_{\nu^{\langle k\rangle}} = T''_{i_N,1} \cdots T''_{i_{k+1},1}(f_{i_k})$, where $T''_{i,1}$ is the braid group operation from [Lusztig 1993, §37.1] with $e = -1$ and $v_i = q$. Set $P^{(l)}_{\nu^{\langle k\rangle}} = (1/[l]!)P^l_\nu$ and, given $\nu = (\nu^{\langle N\rangle})^{l_N} \cdots (\nu^{\langle 1\rangle})^{l_1} \in J_+^\bullet$, let $P_\nu = P^{(l_N)}_{\nu^{\langle N\rangle}} \cdots P^{(l_1)}_{\nu^{\langle 1\rangle}}$ and $\mathbf{P}_\nu = \Psi(P_\nu)$. Taking an appropriate limit $m \to \infty$, [Lusztig 1993, Proposition 41.1.4] implies that $\{P_\nu \mid \nu \in J_+^\bullet\}$ is an $\mathscr{A}$-basis of $\mathbf{f}_\mathscr{A}$.

Next, given $\nu \in {}^\theta\mathscr{L}_+$, let

$$P^{[n]}_\nu = \begin{cases} P^{(n)}_\nu, & \text{if } \nu \text{ is not symmetric,} \\ \dfrac{1}{[2n]!!} P^n_\nu, & \text{if } \nu \text{ is symmetric.} \end{cases}$$

Given $\nu = (\nu^{\langle l\rangle})^{n_l} \cdots (\nu^{\langle 1\rangle})^{n_1} \in {}^\theta J_+^\bullet$, define

$${}^\theta P_\nu = \sigma\Big(\prod_{1\leq i\leq l} P^{[n_i]}_{\nu^{\langle i\rangle}}\Big) \cdot v_0 \quad \text{and} \quad {}^\theta\mathbf{P}_\nu = {}^\theta\Psi(P_\nu).$$

Proposition 3.26. *The set $\{{}^\theta P_\nu \mid \nu \in {}^\theta J_+^\bullet\}$ is an $\mathscr{A}$-basis of ${}^\theta V^{\mathrm{low}}_\mathscr{A}$.*

Proof. See [Enomoto and Kashiwara 2008, Lemma 5.1]. Note that the weaker statement that $\{{}^\theta P_\mu\}$ is a $\mathscr{K}$-basis of ${}^\theta V^{\mathrm{low}}_\mathscr{A}$ follows from Lemma 3.17 and Lemma 3.27 (1) below. □

We call $\{{}^\theta\mathbf{P}_\nu \mid \nu \in {}^\theta J_+^\bullet\}$ the PBW basis of ${}^\theta\mathbf{V}^{\mathrm{low}}_\mathscr{A}$. By [Leclerc 2004, Proposition 30], for any $\nu \in J_+^\bullet$, there exists $\kappa_\nu = \overline{\kappa_\nu} \in \mathscr{A}$ with $\mathbf{l}_\nu = \kappa_\nu\mathbf{P}_\nu$. Since we are working with the standard ordering of J, [Leclerc 2004, Proposition 56] implies that $\kappa_\nu = 1$ for any $\nu \in \mathscr{L}_+$. If $\nu = (\nu^{\langle l\rangle})^{n_l} \cdots (\nu^{\langle 1\rangle})^{n_1} \in {}^\theta J_+^\bullet$, then $\kappa_\nu = \prod_{i=1}^l [n_i]!$. Set

$${}^\theta\kappa_\nu = \kappa_\nu \cdot \prod_{\substack{i=1 \\ \nu^{\langle i\rangle}\,\mathrm{symm}}}^{l} \prod_{j=1}^{n_i} (q^j + q^{-j}) = \prod_{\substack{i=1, \\ \nu^{\langle i\rangle}\,\mathrm{symm}}}^{l} [n_i]!! \cdot \prod_{\substack{i=1, \\ \nu^{\langle i\rangle}\,\mathrm{nonsymm}}}^{l} [n_i]!.$$

Lemma 3.27. *Let* $\nu \in {}^\theta J_+^\bullet$.

(1) ${}^\theta \mathbf{l}_\nu = {}^\theta\kappa_\nu {}^\theta\mathbf{P}_\nu$ *and* ${}^\theta\kappa_\nu = \overline{{}^\theta\kappa_\nu} \in \mathscr{A}$.

(2) *We have*

$$\overline{{}^\theta\mathbf{P}_\nu} = {}^\theta\mathbf{P}_\nu + \sum_{\mu>\nu} d_{\nu\mu} {}^\theta\mathbf{P}_\mu$$

for some $d_{\nu\mu} \in \mathscr{A}$.

Proof. The first part follows directly from the definitions. Let $A_{\mathbf{P}}$, $A_{\mathbf{m}}$ and A be the transition matrices between $\{{}^\theta\mathbf{P}_\nu\}$ and $\{\overline{{}^\theta\mathbf{P}_\nu}\}$, $\{\mathbf{m}_\nu\}$ and $\{\overline{\mathbf{m}_\nu}\}$, as well as $\{{}^\theta\mathbf{P}_\nu\}$ and $\{\mathbf{m}_\nu\}$, respectively. By definition, $A_{\mathbf{m}} = \mathrm{id}$. Hence, $A_{\mathbf{P}} = \bar{A}A^{-1}$. Lemma 3.17 implies that $\bar{A}$ and A^{-1} are both lower triangular, with eigenvalues $\overline{{}^\theta\kappa_\nu}$ and ${}^\theta\kappa_\nu^{-1}$. Part (1) now implies that $A_{\mathbf{P}}$ is indeed lower unitriangular. Since $\{{}^\theta\mathbf{P}_\nu\}$ forms an $\mathscr{A}$-basis of ${}^\theta\mathbf{V}_{\mathscr{A}}^{\mathrm{low}}$ and $\overline{{}^\theta\mathbf{V}_{\mathscr{A}}^{\mathrm{low}}} = {}^\theta\mathbf{V}_{\mathscr{A}}^{\mathrm{low}}$, we have $d_{\nu\mu} \in \mathscr{A}$. □

Theorem 3.28. *There is a unique* $\mathscr{A}$*-basis* $\{{}^\theta\mathbf{b}_\nu \mid \nu \in {}^\theta J_+^\bullet\}$ *of* ${}^\theta\mathbf{V}_{\mathscr{A}}^{\mathrm{low}}$, *called the* ***canonical basis***, *such that*

$${}^\theta\mathbf{b}_\nu = {}^\theta\mathbf{P}_\nu + \sum_{\mu>\nu} c_{\nu\mu} {}^\theta\mathbf{P}_\mu,$$

$c_{\nu\mu} \in q\mathbb{Z}[q]$ *and* $\overline{{}^\theta\mathbf{b}_\nu} = {}^\theta\mathbf{b}_\nu$. *Moreover,*

$$({}^\theta\mathbf{b}_\nu, {}^\theta\mathbf{b}_\mu)_{q=0} = \delta_{\nu,\mu}.$$

Proof. The proof is an application of a standard argument, see, e.g., [Lusztig 1990, §7.10]. □

Remark 3.29. Theorem 3.28 also appears in [Enomoto and Kashiwara 2008] as Theorem 5.5. The proof in *loc. cit.* is somewhat different from ours, in particular, it does not involve shuffle modules.

Let $\{{}^\theta\mathbf{P}_\nu^* \mid \nu \in {}^\theta J_+^\bullet\}$ and $\{{}^\theta\mathbf{b}_\nu^* \mid \nu \in {}^\theta J_+^\bullet\}$ be the bases of ${}^\theta\mathbf{V}_{\mathscr{A}}^{\mathrm{up}}$ dual, with respect to the bilinear form $(\cdot\,,\cdot)$, to the PBW and the canonical bases of ${}^\theta\mathbf{V}_{\mathscr{A}}^{\mathrm{low}}$, respectively.

Corollary 3.30. *We have*

$$ {}^\theta\mathbf{b}_\nu^* = {}^\theta\mathbf{P}_\nu^* + \sum_{\mu<\nu} \big({}^\theta\mathbf{b}_\nu^*, {}^\theta\mathbf{P}_\mu\big) {}^\theta\mathbf{P}_\mu^*. \tag{3-12}$$

Hence, $\max({}^\theta\mathbf{b}_\nu^*) = \nu$ *and the coefficient of* ν *in* ${}^\theta\mathbf{b}_\nu^*$ *is* ${}^\theta\kappa_\nu$. *In particular, if* $\nu \in {}^\theta\mathscr{L}_+$ *or* ν *is symmetric, then* ${}^\theta\mathbf{b}_\nu^* = {}^\theta\mathbf{P}_\nu^*$.

Proof. The proof is analogous to [Leclerc 2004, Proposition 40]. The last statement follows from Lemma 3.25. □

3J. ***Standard and costandard basis.*** Given $\nu = (\nu^{\langle l\rangle})^{n_l} \cdots (\nu^{\langle 1\rangle})^{n_1} \in {}^\theta J^\bullet_+$, let

$$\Delta_\nu = q^{-s(\nu)}(\nu^{\langle l\rangle})^{\circ n_l} \circ \cdots \circ (\nu^{\langle 1\rangle})^{\circ n_1} \quad \text{and} \quad {}^\theta\Delta_\nu = q^{-{}^\theta s(\nu)} \varnothing \circledcirc \Delta_\nu,$$

where

$$s(\nu) = \sum_{i=1}^{l} \frac{n_i(n_i-1)}{2} \quad \text{and} \quad {}^\theta s(\nu) = \sum_{\substack{i=1,\\ \nu^{\langle i\rangle}\,\text{symm}}}^{l} n_i. \tag{3-13}$$

Lemma 3.31. *If* $\nu \in {}^\theta J^\bullet_+$, *then*: $\Delta_\nu = \Delta_{\nu^\theta} \circ \Delta_{\nu_\theta}$, $\max({}^\theta\Delta_\nu) = \nu$ *and the coefficient of the word* ν *in* ${}^\theta\Delta_\nu$ *equals* ${}^\theta\kappa_\nu$.

Proof. We prove the first statement by induction on the number k of Lyndon factors in the Lyndon factorization of ν^θ. If $k=0$, the claim is obvious. Next, suppose that there are $k+1$ Lyndon factors in ν^θ, and let ξ_m be the smallest. If ξ_m is also the smallest word in the standard factorization of ν, then, by induction, we are done. Otherwise, let μ be a Lyndon factor of ν with $\mu < \xi_m$. Since $\mu \in {}^\theta\mathscr{L}_+$, Lemma 3.24 implies that $\xi_m \subset \mu$. By Lemma 3.18, we conclude that $\mu \circ \xi_m = \xi_m \circ \mu$. It now follows by induction that $\Delta_\nu = \Delta_{\nu^\theta} \circ \Delta_{\nu_\theta}$.

We now prove the last two statements by induction on the number k of Lyndon factors in ν. The base case $k=0$ is trivial. Let $\nu' = \nu^{\langle k\rangle} \cdots \nu^{\langle 2\rangle}$. Lemma 3.14 implies that $\nu' \in {}^\theta J^\bullet_+$. Hence, by induction, $\max({}^\theta\Delta_{\nu'}) = \nu'$. Since $\lambda = 0$, we have $\nu^{\langle 1\rangle} \in {}^\theta\mathscr{L}_+$, and so $\nu^{\langle 1\rangle} \geq {}^\theta w(\nu^{\langle 1\rangle})$. It follows from Lemma 3.10 and Lemma 3.20 (2) that $\max({}^\theta\Delta_\nu) = \max(\nu' \circledcirc \nu^{\langle 1\rangle}) = \nu$. By induction, we may also assume that $\dim_q({}^\theta\Delta_{\nu'})_{\nu'} = {}^\theta\kappa_{\nu'}$. Let us call the result of applying $w \in {}^\theta\mathtt{D}_{\|\nu'\|_\theta, \|\nu^{\langle 1\rangle}\|}$ to ν a θ-shuffle. It is easy to see that the θ-shuffles equal to ν are precisely those arising from one of the n_1 (respectively, $2n_1$) standard insertions of $\nu^{\langle 1\rangle}$ between words equal to $\nu^{\langle 1\rangle}$ in ν' if $\nu^{\langle 1\rangle}$ is not symmetric (respectively, is symmetric). We conclude that $\dim_q({}^\theta\Delta_\nu)_\nu = {}^\theta\kappa_\nu$ from the fact that the transposition of two words equal to $\nu^{\langle 1\rangle}$ appears in the shuffle action with the coefficient q^{-2}. □

Given $\nu \in {}^\theta J^\bullet_+$ with $\nu = \nu^{\langle k\rangle} \cdots \nu^{\langle 1\rangle}$, let

$${}^\theta\nabla_\nu = q^{-{}^\theta s(\nu) - t(\nu)} \varnothing \circledcirc \big({}^\theta w(\nu^{\langle k\rangle}) \circ \cdots \circ {}^\theta w(\nu^{\langle 1\rangle})\big),$$

where $t(\nu)$ is the degree of an element τ_w, with w the longest minimal length coset representative with respect to the parabolic subgroup of $\mathfrak{W}_n$ defined by the decomposition of ν into Lyndon words (see [Lauda and Vazirani 2011, §2.3]).

Recall that we have fixed the standard order $\leq$ on J and equipped $J^\bullet$ with the antilexicographic order $\leq$. Let $\leq'$ denote both the opposite order on J and the induced lexicographic order on $J^\bullet$. Given a linear combination u of words, let $\max'(u)$ be the largest word appearing in u with respect to $\leq'$.

Lemma 3.32. *We have* $\max'({}^\theta\nabla_\nu) = \nu$ *and the coefficient of* ν *in* ${}^\theta\nabla_\nu$ *equals* ${}^\theta\kappa_\nu$.

Proof. It is an easy modification of the last paragraph in the proof of Lemma 3.31. □

4. Finite-dimensional representation theory of orientifold KLR algebras

We again let λ be arbitrary until Section 4D, where we make the restriction $\lambda = 0$.

If A is a graded algebra, let A-Mod be the category of all graded left A-modules, with degree-preserving module homomorphisms as morphisms. If M and N are graded A-modules, let $\mathrm{Hom}_A(M, N)_n$ denote the space of all homogeneous homomorphisms of degree n, and $\mathrm{HOM}_A(M, N) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_A(M, N)_n$. Let $M\{n\}$ denote the module obtained from M by shifting the grading by n. Let A-pMod denote the full subcategory of finitely generated graded projective modules, and A-fMod the full subcategory of graded finite dimensional modules. Given any of these abelian categories $\mathscr{C}$, we denote its Grothendieck group by $[\mathscr{C}]$.

We consider (orientifold) KLR algebras associated to the $\mathtt{A}_\infty$ quiver $\Gamma = (J, \Omega)$, with J as in Section 3A and Ω the standard linear orientation, as well as the involution θ from Section 3A. Let $\mathbb{1}\mathbb{1}$ and ${}^\theta\mathbb{1}$ denote the regular representations (in degree zero) of the trivial algebras $\mathscr{R}(0)$ and ${}^\theta\mathscr{R}(0;\lambda)$, respectively. For a fixed $\lambda \in \mathbb{N}[J]$, set

$$\mathscr{R}\text{-Mod} = \bigoplus_{\beta\in\mathbb{N}[J]} \mathscr{R}(\beta)\text{-Mod} \quad \text{and} \quad {}^\theta\mathscr{R}(\lambda)\text{-Mod} = \bigoplus_{\beta\in\mathbb{N}[J]^\theta} {}^\theta\mathscr{R}(\beta;\lambda)\text{-Mod}.$$

We use analogous notation for direct sums of categories of finite dimensional and finitely generated projective modules.

4A. ***Reminder on categorification via KLR algebras.*** Basic information about the representation theory of KLR algebras, including the definitions of the Khovanov–Lauda pairing $(\cdot\,,\cdot)\colon \mathscr{R}(\beta)\text{-pMod} \times \mathscr{R}(\beta)\text{-fMod} \to \mathscr{A}$ and the dualities $P \mapsto P^\sharp$ on $\mathscr{R}$-pMod and $M \mapsto M^\flat$ on $\mathscr{R}$-fMod, can be found in, e.g., [Khovanov and Lauda 2009], [Kleshchev and Ram 2011, §3] or [Varagnolo and Vasserot 2011, §7]. Since these definitions and the notations are standard, we will not explicitly recall them. If $M \in \mathscr{R}(\beta)$-Mod and $\nu \in J^\theta$, we call $M_\nu = e(\nu)M$ the ν-weight space of M.

Let us recall the definition of the convolution product of modules over KLR algebras. Let $\beta, \beta' \in \mathbb{N}[J]$, with $\|\beta\| = n$ and $\|\beta'\| = n'$. Set

$$e_{\beta,\beta'} = \sum_{\substack{\nu\in J^{\beta+\beta'},\\ \nu_1\cdots\nu_n\in J^\beta}} e(\nu) \in \mathscr{R}(\beta+\beta').$$

There is a nonunital algebra homomorphism

$$\iota_{\beta,\beta'}\colon \mathscr{R}(\beta,\beta') := \mathscr{R}(\beta)\otimes\mathscr{R}(\beta') \to \mathscr{R}(\beta+\beta') \tag{4-1}$$

given by $e(\nu)\otimes e(\mu) \mapsto e(\nu\mu)$ for $\nu \in J^\beta$, $\mu \in J^{\beta'}$ and

$$x_l \otimes 1 \mapsto x_l e_{\beta,\beta'}, \quad 1\otimes x_{l'} \mapsto x_{m+l'} e_{\beta,\beta'}, \qquad \text{with } 1 \le l^{(\prime)} \le n^{(\prime)}, \tag{4-2}$$

$$\tau_k \otimes 1 \mapsto \tau_k e_{\beta,\beta'}, \quad 1\otimes \tau_{k'} \mapsto \tau_{m+k'} e_{\beta,\beta'}, \qquad \text{with } 1 \le k^{(\prime)} < n^{(\prime)}. \tag{4-3}$$

Let M be a graded $\mathcal{R}(\beta)$-module and N be a graded $\mathcal{R}(\beta')$-module. Their *convolution product* is defined as

$$M \circ N = \mathcal{R}(\beta+\beta')e_{\beta,\beta'} \otimes_{\mathcal{R}(\beta,\beta')} (M \otimes N).$$

It descends to a product on [$\mathcal{R}$-pMod] and [$\mathcal{R}$-fMod].

The embedding (4-1) generalizes to an embedding

$$\iota_{\underline{\beta}} \colon \mathcal{R}(\underline{\beta}) := \mathcal{R}(\beta_1) \otimes \cdots \otimes \mathcal{R}(\beta_m) \to \mathcal{R}(|\underline{\beta}|) \tag{4-4}$$

for any $\underline{\beta} \in (\mathbb{N}[J])^m$. The embedding (4-4) gives rise to a triple of adjoint functors $(\mathrm{Ind}_{\underline{\beta}}, \mathrm{Res}_{\underline{\beta}}, \mathrm{Coind}_{\underline{\beta}})$ between categories of graded modules.

As explained in [Khovanov and Lauda 2009, §2.2] and [Kleshchev and Ram 2011, §3.6], convolution with the class of (an appropriate graded shift of) the polynomial representation $P(i^{(n)})$ of the nil-Hecke algebra $\mathcal{R}(ni)$ yields an $\mathcal{A}$-module homomorphism

$$\theta_i^{(n)} = - \circ [P(i^{(n)})] \colon [\mathcal{R}(\beta)\text{-pMod}] \to [\mathcal{R}(\beta+ni)\text{-pMod}].$$

Let us recall the fundamental categorification theorem from [Khovanov and Lauda 2009, §3], see also [Kleshchev and Ram 2011, Theorem 4.4].

Theorem 4.1 (Khovanov–Lauda). *There exists a unique pair of adjoint* (*with respect to Lusztig's form on* $\mathbf{f}$ *and the Khovanov–Lauda pairing*) Q*-graded* $\mathcal{A}$*-linear isomorphisms*

$$\gamma \colon \mathbf{f}_{\mathcal{A}} \xrightarrow{\sim} [\mathcal{R}\text{-pMod}] \quad \textit{and} \quad \gamma^* \colon [\mathcal{R}\text{-fMod}] \xrightarrow{\sim} \mathbf{f}^*_{\mathcal{A}}$$

such that $\gamma(1) = [\mathbb{1}]$ *and* $\gamma(xf_i^{(n)}) = \theta_i^{(n)}(\gamma(x))$ *for all* $x \in \mathbf{f}_{\mathcal{A}}$. *These isomorphisms intertwine*: (*i*) *multiplication in* $\mathbf{f}$ *with the convolution product*, (*ii*) *comultiplication in* $\mathbf{f}$ *with restriction functors*, *and* (*iii*) *the bar involution on* $\mathbf{f}$ *with the involutions* $-^\sharp$ *and* $-^\flat$.

4B. ***Categorification via orientifold KLR algebras.*** We recall some fundamental definitions and results concerning orientifold KLR algebras from [Varagnolo and Vasserot 2011, §8]. We refer the reader to *loc. cit.* for a detailed exposition.

Let $\beta \in \mathbb{N}[J]^\theta$ and $\beta' \in \mathbb{N}[J]$, with $\|\beta\|_\theta = n$ and $\|\beta'\| = n'$. Set

$$^\theta e_{\beta,\beta'} = \sum_{\substack{\nu \in {}^\theta J^{\beta+{}^\theta\beta'}, \\ \nu_1 \dots \nu_n \in {}^\theta J^{\beta}, \\ \nu_{n+1} \dots \nu_{n+n'} \in J^{\beta'}}} e(\nu) \in {}^\theta\mathcal{R}(\beta + {}^\theta\beta'; \lambda).$$

There is an injective nonunital algebra homomorphism

$$^\theta\iota_{\beta,\beta'} \colon {}^\theta\mathcal{R}(\beta,\beta';\lambda) := {}^\theta\mathcal{R}(\beta;\lambda) \otimes \mathcal{R}(\beta') \to {}^\theta\mathcal{R}(\beta+{}^\theta\beta';\lambda) \tag{4-5}$$

given by formulae (4-2)–(4-3), with $\nu \in {}^{\theta}J^{\beta}$ and $e_{\beta,\beta'}$ replaced by ${}^{\theta}e_{\beta,\beta'}$, and $\tau_0 \otimes 1 \mapsto \tau_0{}^{\theta}e_{\beta,\beta'}$. The *convolution action* of $N \in \mathcal{R}(\beta')$-Mod on $M \in {}^{\theta}\mathcal{R}(\beta;\lambda)$-Mod is defined as

$$M \mathbin{\circledcirc} N = {}^{\theta}\mathcal{R}(\beta + {}^{\theta}\beta';\lambda)\,{}^{\theta}e(\beta,\beta') \otimes_{{}^{\theta}\mathcal{R}(\beta,\beta';\lambda)} (M \otimes N).$$

Proposition 4.2. *The category* $\mathcal{R}$-Mod *is monoidal with product* $\circ$ *and unit* $\mathbb{1}$. *Moreover, there is a right monoidal action* (*see, e.g.,* [Davydov 1998]) *of* $\mathcal{R}$-Mod *on* ${}^{\theta}\mathcal{R}(\lambda)$-Mod *via* $\circledcirc$.

Proof. It is routine to check that the conditions in the definition of a monoidal action are satisfied. □

The embedding (4-5) generalizes to an embedding

$$(4\text{-}6)\quad {}^{\theta}\iota_{\underline{\beta}} : {}^{\theta}\mathcal{R}(\beta_0,\underline{\beta};\lambda) := {}^{\theta}\mathcal{R}(\beta_0;\lambda) \otimes \mathcal{R}(\beta_1) \otimes \cdots \otimes \mathcal{R}(\beta_m) \to {}^{\theta}\mathcal{R}(\beta_0 + {}^{\theta}|\underline{\beta}|;\lambda)$$

for any $\beta_0 \in \mathbb{N}[J]^{\theta}$ and $\underline{\beta} \in (\mathbb{N}[J])^m$. The embedding (4-6) gives rise to a triple of adjoint functors $({}^{\theta}\mathrm{Ind}_{\beta_0,\underline{\beta}}, {}^{\theta}\mathrm{Res}_{\beta_0,\underline{\beta}}, {}^{\theta}\mathrm{Coind}_{\beta_0,\underline{\beta}})$ between categories of graded modules.

Lemma 4.3. *Let* $M_0 \in {}^{\theta}\mathcal{R}(\beta;\lambda)$-fMod *and* $M_i \in \mathcal{R}(\beta_i)$-fMod. *Then, up to a grading shift, we have*

$$\begin{aligned} {}^{\theta}\mathrm{Coind}_{\beta_0,\underline{\beta}}\big(M_0 \otimes (\otimes M_i)\big) &\cong {}^{\theta}\mathrm{Ind}_{\beta_0,\theta(\underline{\beta})}\big(M_0 \otimes (\otimes M_i^{\dagger})\big) \\ &\cong {}^{\theta}\mathrm{Coind}_{\beta_0,|\underline{\beta}|}\big(M_0 \otimes \big(\mathrm{Coind}_{\underline{\beta}}(\otimes M_i)\big)\big), \end{aligned}$$

where $\theta(\underline{\beta}) = (\theta(\beta_1),\dots,\theta(\beta_m))$ *and* $-^{\dagger}$ *is the twist defined below Lemma 2.3.*

Proof. The proof is analogous to that of [Lauda and Vazirani 2011, Theorem 2.2]. □

Let $\beta_0 \in \mathbb{N}[J]^{\theta}$ and $\beta_1, \beta_2 \in \mathbb{N}[J]$. Define

$$M_1 \mathbin{\hat{\circ}} M_2 = \mathrm{Coind}_{\beta_1,\beta_2}(M_1 \otimes M_2) \quad \text{and} \quad M_0 \mathbin{\hat{\circledcirc}} M_1 = {}^{\theta}\mathrm{Coind}_{\beta_0,\beta_1}(M_0 \otimes M_1),$$

for M_i as in Lemma 4.3.

Corollary 4.4. *The category* $\mathcal{R}$-Mod *is also monoidal with product* $\hat{\circ}$ *and unit* $\mathbb{1}$. *Moreover, there is a monoidal action of* $\mathcal{R}$-Mod *on* ${}^{\theta}\mathcal{R}(\lambda)$-Mod *via* $\hat{\circledcirc}$.

The functors $P \mapsto P^{\sharp} = \mathrm{HOM}_{{}^{\theta}\mathcal{R}_m(\lambda)}(P, {}^{\theta}\mathcal{R}_m(\lambda))$ and $M \mapsto M^{\flat} = \mathrm{HOM}_{\Bbbk}(P,\Bbbk)$ on ${}^{\theta}\mathcal{R}_m(\lambda)$-pMod and ${}^{\theta}\mathcal{R}_m(\lambda)$-fMod, respectively, descend to $\mathcal{A}$-antilinear involutions on the corresponding Grothendieck groups. We also have an analogue of the Khovanov–Lauda pairing

$$\begin{aligned} (\cdot\,,\cdot) : [{}^{\theta}\mathcal{R}(\beta;\lambda)\text{-pMod}] \times [{}^{\theta}\mathcal{R}(\beta;\lambda)\text{-fMod}] &\to \mathcal{A}, \\ ([P],[M]) &\mapsto \dim_q(P^{\omega} \otimes_{{}^{\theta}\mathcal{R}(\beta;\lambda)} M), \end{aligned}$$

where P^{ω} is the twist of P by the antiinvolution (2-5).

Moreover, set ${}^\theta\mathcal{R}_m(\lambda) = \bigoplus_{\|\beta\|_\theta = n} {}^\theta\mathcal{R}(\beta;\lambda)$ and ${}^\theta e_{m,\beta'} = \oplus_{\|\beta\|_\theta = m} {}^\theta e_{\beta,\beta'}$. Abbreviate ${}^\theta\mathrm{Ind}_{m,i}^{m+1} = {}^\theta\mathcal{R}_{m+1}(\lambda) \otimes_{{}^\theta\mathcal{R}_{m,i}(\lambda)} -$ and ${}^\theta\mathrm{Coind}_{m,i}^{m+1} = \mathrm{HOM}_{{}^\theta\mathcal{R}_{m,i}(\lambda)}({}^\theta\mathcal{R}_{m+1}(\lambda), -)$, with ${}^\theta\mathcal{R}_{m,i}(\lambda) = {}^\theta\mathcal{R}_m(\lambda) \otimes \mathcal{R}(i)$. Setting

$$F_i(P) = {}^\theta\mathrm{Ind}_{m,i}^{m+1}(P \otimes P(i)), \qquad E_i(P) = L(i) \otimes_{\mathcal{R}(i)} {}^\theta e_{m-1,i} P,$$
$$F_i^*(M) = {}^\theta\mathrm{Coind}_{m,i}^{m+1}(M \otimes L(i)), \quad E_i^*(M) = {}^\theta e_{m-1,i} M,$$

defines exact functors

$${}^\theta\mathcal{R}_m(\lambda)\text{-pMod} \underset{E_i}{\overset{F_i}{\rightleftarrows}} {}^\theta\mathcal{R}_{m+1}(\lambda)\text{-pMod}, \qquad {}^\theta\mathcal{R}_m(\lambda)\text{-fMod} \underset{E_i^*}{\overset{F_i^*}{\rightleftarrows}} {}^\theta\mathcal{R}_{m+1}(\lambda)\text{-fMod}$$

commuting with the dualities $-^\sharp$ and $-^\flat$. We will use the same notation for the induced operators on the corresponding Grothendieck groups.

We now recall the main theorem [Varagnolo and Vasserot 2011, Theorem 8.31] on the categorification of modules over the Enomoto–Kashiwara algebra.

Theorem 4.5 (Varagnolo–Vasserot). *The operators F_i and E_i (respectively, F_i^* and E_i^*) define a representation of ${}^\theta\mathcal{B}(\mathfrak{g})$ on $\mathcal{K} \otimes_{\mathcal{A}} [{}^\theta\mathcal{R}(\lambda)\text{-pMod}]$ (respectively, $\mathcal{K} \otimes_{\mathcal{A}} [{}^\theta\mathcal{R}(\lambda)\text{-fMod}]$). Moreover, there exists a unique pair of adjoint P^θ-graded $\mathcal{A}$-linear isomorphisms*

$${}^\theta\gamma : {}^\theta V(\lambda)_{\mathcal{A}}^{\mathrm{low}} \xrightarrow{\sim} [{}^\theta\mathcal{R}(\lambda)\text{-pMod}], \quad {}^\theta\gamma^* : [{}^\theta\mathcal{R}(\lambda)\text{-fMod}] \xrightarrow{\sim} {}^\theta V(\lambda)_{\mathcal{A}}^{\mathrm{up}}$$

which, upon base change to $\mathcal{K}$, become isomorphisms of ${}^\theta\mathcal{B}(\mathfrak{g})$-modules. They intertwine the bar involution on ${}^\theta V(\lambda)$ with the involutions $-^\sharp$ and $-^\flat$.

If $M \in {}^\theta\mathcal{R}(\beta;\lambda)$-Mod and $\nu \in {}^\theta J^\beta$, we call $M_\nu = e(\nu)M$ the ν-weight space of M. The *character* of a ${}^\theta\mathcal{R}(\beta;\lambda)$-module M is ${}^\theta\mathrm{ch}_q(M) = \sum_\nu \dim_q(e(\nu)M)\cdot\nu \in {}^\theta\mathcal{F}(\lambda)$. This gives rise to an $\mathcal{A}$-linear map ${}^\theta\mathrm{ch}_q : [{}^\theta\mathcal{R}(\lambda)\text{-fMod}] \to {}^\theta\mathcal{F}(\lambda)$. We then call $\max({}^\theta\mathrm{ch}_q(M))$, if it exists, the highest weight of M.

Corollary 4.6. *The following triangle commutes*:

$$\begin{array}{ccc} & \left[{}^\theta\mathcal{R}(\lambda)\text{-fMod}\right] & \\ {}^{{}^\theta\gamma^*}\swarrow & & \searrow^{{}^\theta\mathrm{ch}_q} \\ {}^\theta V(\lambda)_{\mathcal{A}}^{\mathrm{up}} & \xrightarrow{\quad {}^\theta\Psi \quad} & {}^\theta\mathcal{F}(\lambda) \end{array}$$

The map ${}^\theta\mathrm{ch}_q$ is injective and ${}^\theta\mathrm{ch}_q(M \circledcirc N) = {}^\theta\mathrm{ch}_q(M) \circledcirc \mathrm{ch}_q(N)$.

Proof. The proof is analogous to [Kleshchev and Ram 2011, Theorem 4.4 (3)]. □

4C. ***Reminder on KLR representation theory.*** An irreducible $\mathscr{R}(\beta)$-module L is called *cuspidal* if $\max(\mathrm{ch}_q(L)) \in \mathscr{L}_+$, i.e., its highest weight is a good Lyndon word. By [Kleshchev and Ram 2011, Proposition 8.4], for each $\nu \in \mathscr{L}_+$, there exists a unique cuspidal irreducible $\mathscr{R}(|\nu|)$-module $L(\nu)$.

Let $\nu = (\nu^{\langle l\rangle})^{n_l} \cdots (\nu^{\langle 1\rangle})^{n_1} \in J_+^\beta$. The corresponding standard and costandard modules are, respectively,

$$\Delta(\nu) = L(\nu^{\langle l\rangle})^{\circ n_l} \circ \cdots \circ L(\nu^{\langle 1\rangle})^{\circ n_1}\{s(\nu)\}, \quad \nabla(\nu) = L(\nu^{\langle l\rangle})^{\circ n_l} \mathbin{\hat\circ} \cdots \mathbin{\hat\circ} L(\nu^{\langle 1\rangle})^{\circ n_1}\{s(\nu)\},$$

with $s(\nu)$ as in (3-13).

Theorem 4.7 (Kleshchev–Ram, McNamara). *Let $\nu \in J_+^\beta$. Then*:

(1) *The standard $\mathscr{R}(\beta)$-module $\Delta(\nu)$ has an irreducible head $L(\nu)$, and the costandard module $\nabla(\nu)$ has $L(\nu)$ as its socle.*

(2) *The highest weight of $L(\nu)$ is ν, and* $\dim_q L(\nu)_\nu = \kappa_\nu$.

(3) $L(\nu) = L(\nu)^\flat$.

(4) $\{L(\nu) \mid \nu \in J_+^\beta\}$ *is a complete and irredundant set of irreducible graded $\mathscr{R}(\beta)$-modules up to isomorphism and degree shift.*

(5) *If $L(\mu)$ is a composition factor of $\Delta(\nu)$ (respectively, $\nabla(\nu)$), then $\mu \le \nu$ (respectively, $\mu \le' \nu$). Moreover, $L(\nu)$ appears in $\Delta(\nu)$ and $\nabla(\nu)$ with multiplicity one.*

(6) *If $\nu = \mu^n$ for a good Lyndon word μ, then $\Delta(\nu) = L(\nu)$.*

Proof. See [Kleshchev and Ram 2011, Theorem 7.2] and [McNamara 2015, Theorem 3.1]. □

4D. ***Orientifold KLR: irreducibles and global dimension.*** Now assume $\lambda = 0$.

Lemma 4.8. *If $\nu \in {}^\theta J_+^\bullet$ is symmetric, then ${}^\theta L(\nu) = {}^\theta\mathbb{1} \circledcirc L(\nu)\{{}^\theta s(\nu)\}$ is irreducible. The highest weight of ${}^\theta L(\nu)$ is ν, ${}^\theta\mathrm{ch}_q\, {}^\theta L(\nu) = {}^\theta\mathbf{b}_\nu^*$, and* $\dim_q {}^\theta L(\nu)_\nu = {}^\theta\kappa_\nu$.

Proof. It follows from Lemma 3.10, Lemma 3.25, and Corollary 4.6 that all composition factors of ${}^\theta L(\nu)$ have highest weight ν. We know from Theorem 4.7 (2) that $\max(\mathrm{ch}_q(L(\nu))) = \nu$ and $\dim_q L(\nu)_\nu = \kappa_\nu$. The last part of Corollary 4.6, together with an argument analogous to that in the last paragraph of the proof of Lemma 3.31, then shows that the highest weight of ${}^\theta L(\nu)$ is ν and $\dim_q {}^\theta L(\nu)_\nu = {}^\theta\kappa_\nu$.

Let $\beta = {}^\theta|\nu|$. By Theorem 4.5, ${}^\theta\mathrm{ch}_q\, {}^\theta L(\nu) \in {}^\theta\mathbf{V}^{\mathrm{up}}_{\mathscr{A},\beta}$. Since $\{{}^\theta\mathbf{b}_\mu^* \mid \mu \in {}^\theta J_+^\beta\}$ is an $\mathscr{A}$-basis of ${}^\theta\mathbf{V}^{\mathrm{up}}_{\mathscr{A},\beta}$, we have ${}^\theta\mathrm{ch}_q\, {}^\theta L(\nu) = \sum_{\mu \in {}^\theta J_+^\beta} c_\mu\, {}^\theta\mathbf{b}_\mu^*$ for some $c_\mu \in \mathscr{A}$. By Corollary 3.30, $\max({}^\theta\mathbf{b}_\mu^*) = \mu$, and, by Lemma 3.25, ν is the smallest word in ${}^\theta J_+^\beta$. Hence, $c_\mu = 0$ unless $\mu = \nu$. Comparing the coefficients of ν in $\mathrm{ch}_q L(\nu)$ and ${}^\theta\mathbf{b}_\nu^*$, we conclude that $c_\nu = 1$. The irreducibility of ${}^\theta L(\nu)$ follows directly from the equality ${}^\theta\mathrm{ch}_q {}^\theta L(\nu) = {}^\theta\mathbf{b}_\nu^*$. □

For $\nu \in {}^\theta J_+^\beta$, let

$$ {}^\theta\Delta(\nu) = {}^\theta\mathbb{1} \circledcirc \Delta(\nu) \quad \text{and} \quad {}^\theta\nabla(\nu) = {}^\theta\mathbb{1} \mathbin{\hat{\circledcirc}} \nabla(\nu). $$

Lemma 4.9. *Let $\nu \in {}^\theta J_+^\bullet$. Then $\Delta(\nu) = \Delta(\nu^\theta) \circ \Delta(\nu_\theta)$, $\max({}^\theta\mathrm{ch}_q\, {}^\theta\Delta(\nu)) = \nu$, and $\dim_q({}^\theta\Delta(\nu))_\nu = {}^\theta\kappa_\nu$.*

Proof. The proof of the first statement is analogous to the proof of the first statement of Lemma 3.31. Using the inductive argument and the notation from that proof, one observes that $\mu\xi_m$ is the lowest good word of weight $|\mu\xi_m|$. Theorem 4.7 (5) then implies that $L(\mu) \circ L(\xi_m) = \Delta(\mu\xi_m) = L(\mu\xi_m) = \nabla(\mu\xi_m) = L(\xi_m) \circ L(\mu)$, allowing the induction to proceed.

Since $\dim_q L(\mu) = 1$, for all $\mu \in \mathscr{L}_+$ (see [Kleshchev and Ram 2011, §8.4]), we have ${}^\theta\mathrm{ch}_q({}^\theta\Delta(\nu)) = {}^\theta\Delta_\nu$. The second and third statements now follow from the second and third statements of Lemma 3.31. □

Theorem 4.10. *Let $\nu \in {}^\theta J_+^\beta$. Then:*

(1) *The standard ${}^\theta\mathscr{R}(\beta)$-module ${}^\theta\Delta(\nu)$ has an irreducible head ${}^\theta L(\nu)$, and the costandard ${}^\theta\mathscr{R}(\beta)$-module ${}^\theta\nabla(\nu)$ has ${}^\theta L(\nu)$ as its socle.*

(2) *The highest weight of ${}^\theta L(\nu)$ is ν, and $\dim_q {}^\theta L(\nu) = {}^\theta\kappa_\nu$.*

(3) ${}^\theta L(\nu) = {}^\theta L(\nu)^\flat$.

(4) *$\{{}^\theta L(\nu) \mid \nu \in {}^\theta J_+^\beta\}$ is a complete and irredundant set of irreducible graded ${}^\theta\mathscr{R}(\beta)$-modules up to isomorphism and degree shift.*

(5) *If ${}^\theta L(\mu)$ is a composition factor of ${}^\theta\Delta(\nu)$ (respectively, ${}^\theta\nabla(\nu)$), then $\mu \le \nu$ (respectively, $\mu \le' \nu$). Moreover, ${}^\theta L(\nu)$ appears in ${}^\theta\Delta(\nu)$ and ${}^\theta\nabla(\nu)$ with multiplicity one.*

(6) *If ν is a Lyndon word or $\nu = \nu^\theta$, then ${}^\theta\Delta(\nu) = {}^\theta L(\nu)$ is irreducible.*

Proof. The structure of the proof is similar to [Kleshchev and Ram 2011, Theorem 7.2], see also [McNamara 2015, Theorem 3.1]. Let us explain the main points. If $\nu_\theta = (\nu^{\langle l\rangle})^{n_l} \cdots (\nu^{\langle 1\rangle})^{n_1}$, let $\beta_0 = {}^\theta|\nu^\theta|$, $\underline{\beta} = (n_l|\nu^{\langle l\rangle}|, \cdots, n_1|\nu^{\langle 1\rangle}|)$, and abbreviate

$$ {}^\theta\mathrm{Res}_\nu = {}^\theta\mathrm{Res}_{\beta_0,\underline{\beta}} \quad \text{and} \quad {}^\theta\mathscr{R}_\nu = {}^\theta\mathscr{R}(\beta_0, \underline{\beta}). $$

Also, abbreviate

$$ {}^\theta L(\vec{\nu}) = {}^\theta L(\nu^\theta) \otimes L(\nu^{\langle l\rangle})^{\circ n_l} \otimes \cdots \otimes L(\nu^{\langle 1\rangle})^{\circ n_1}\{s(\nu_\theta)\}. $$

Let L be an irreducible ${}^\theta\mathscr{R}(\beta)$-module in the head of ${}^\theta\Delta(\nu)$. By adjunction and the first part of Lemma 4.9, $\mathrm{HOM}_{{}^\theta\mathscr{R}(\beta)}({}^\theta\Delta(\nu), {}^\theta\Delta(\nu)) = \mathrm{HOM}_{{}^\theta\mathscr{R}_\nu}({}^\theta L(\vec{\nu}), {}^\theta\mathrm{Res}_\nu\, {}^\theta\Delta(\nu))$ and $0 \ne \mathrm{HOM}_{{}^\theta\mathscr{R}(\beta)}({}^\theta\Delta(\nu), L) = \mathrm{HOM}_{{}^\theta\mathscr{R}_\nu}({}^\theta L(\vec{\nu}), {}^\theta\mathrm{Res}_\nu\, L)$. Hence, we get the commutative diagram

$$\begin{array}{ccccc} {}^{\theta}L(\vec{\nu}) & \hookrightarrow & {}^{\theta}\mathrm{Res}_{\nu}\, {}^{\theta}\Delta(\nu) & \hookrightarrow & {}^{\theta}\Delta(\nu) \\ \| & & \twoheaddownarrow & & \twoheaddownarrow \\ {}^{\theta}L(\vec{\nu}) & \hookrightarrow & {}^{\theta}\mathrm{Res}_{\nu}\, L & \hookrightarrow & L \end{array}$$

The injectivity of the two arrows on the left follows from the ${}^{\theta}\mathfrak{R}_{\nu}$-module ${}^{\theta}L(\vec{\nu})$ being irreducible, which is implied by Theorem 4.7 (5) and Lemma 4.8. Further, Theorem 4.7 (2), Lemma 4.8, and Lemma 4.9 also imply that

$$\dim_q {}^{\theta}L(\vec{\nu})_{\nu} = {}^{\theta}\kappa_{\nu} = \dim_q {}^{\theta}\Delta(\nu)_{\nu}.$$

Hence, $\dim_q L_{\nu} = {}^{\theta}\kappa_{\nu}$ as well, implying that the head of ${}^{\theta}\Delta(\nu)$ is irreducible. This proves (1) in the case of standard modules, as well as (2). Note that the modules ${}^{\theta}L(\nu)$ we have thus constructed are pairwise nonisomorphic since they have different highest weights.

Next, (3) follows from [Varagnolo and Vasserot 2011, Proposition 2] and the fact that ${}^{\theta}\kappa_{\nu}$ is bar-invariant (Lemma 3.27). Part (4) follows from Proposition 3.22, Theorem 4.5, and the fact that we have constructed ${}^{\theta}\mathrm{kpf}(\beta)$ nonisomorphic irreducible graded ${}^{\theta}\mathfrak{R}(\beta)$-modules $\{{}^{\theta}L(\nu) \mid \nu \in {}^{\theta}J_{+}^{\beta}\}$. Next, we return to (1) in the case of costandard modules. An analogous argument to that in the case of standard modules, using Lemma 3.32 and the adjunction between restriction and coinduction now shows that ${}^{\theta}\nabla(\nu)$ has an irreducible socle with highest weight ν, which, by (4), must be isomorphic to ${}^{\theta}L(\nu)$. Part (5) follows immediately from the facts that $\nu = \max\bigl({}^{\theta}\mathrm{ch}_q({}^{\theta}\Delta(\nu))\bigr) = \max'\bigl({}^{\theta}\mathrm{ch}_q({}^{\theta}\nabla(\nu))\bigr)$ and $\dim_q {}^{\theta}\Delta(\nu)_{\nu} = \dim_q {}^{\theta}\nabla(\nu)_{\nu} = \dim_q {}^{\theta}L(\nu)_{\nu}$. Next, part (6) follows from Lemma 3.25 and (5). □

Corollary 4.11. *As a graded algebra, ${}^{\theta}\mathfrak{R}(\beta)$ has global dimension $\|\beta\|_{\theta}$.*

Proof. The proof is analogous to [McNamara 2015, Theorem 4.7]. For the sake of simplicity, we ignore the grading shifts. Since $\lambda = 0$, the set ${}^{\theta}J_{+}^{\bullet}$ contains no θ-cuspidal words. Let $\nu, \mu \in {}^{\theta}J_{+}^{\beta}$. If $\nu_{\theta} = (\nu^{\langle l\rangle})^{n_l} \cdots (\nu^{\langle 1\rangle})^{n_1}$, we let $L(\vec{\nu}) = L(\nu^{\theta}) \otimes L(\nu^{\langle l\rangle})^{\circ n_l} \otimes \cdots \otimes L(\nu^{\langle 1\rangle})^{\circ n_1}$. Also let $\underline{\beta} = (|\nu^{\theta}|, n_l|\nu^{\langle l\rangle}|, \cdots, n_1|\nu^{\langle 1\rangle}|)$. Then, Lemma 4.3 and adjunction between induction and restriction imply that

$$\mathrm{Ext}^{i}_{{}^{\theta}\mathfrak{R}(\beta)}\bigl({}^{\theta}\nabla(\nu), {}^{\theta}\Delta(\mu)\bigr) = \mathrm{Ext}^{i}_{\mathfrak{R}(\underline{\beta})}\bigl(L(\vec{\nu}), \mathrm{Res}_{\underline{\beta}}\, {}^{\theta}\Delta(\mu)\bigr),$$

which, by [McNamara 2015, Theorem 4.7] is zero for $i > \|\beta\|_{\theta}$. The rest of the proof is exactly the same as in [McNamara 2015]. □

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