# Pacific Journal of Mathematics


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Volume 323 No. 1 March 2023

PACIFIC JOURNAL OF MATHEMATICS
Vol. 323, No. 1, 2023
https://doi.org/10.2140/pjm.2023.323.1

# COMBINATORIAL PROPERTIES OF NONARCHIMEDEAN CONVEX SETS

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**We study combinatorial properties of convex sets over arbitrary valued fields. We demonstrate analogs of some classical results for convex sets over the reals (for example, the fractional Helly theorem and Bárány's theorem on points in many simplices), along with some additional properties not satisfied by convex sets over the reals, including finite breadth and VC dimension. These results are deduced from a simple combinatorial description of modules over the valuation ring in a spherically complete valued field.**

## 1. Introduction

Convexity in the context of nonarchimedean valued fields was introduced in a series of papers by Monna [1946], and has been extensively studied since then in nonarchimedean functional analysis (see for instance the monographs [Perez-Garcia and Schikhof 2010; Schneider 2002] on the subject). Convexity here is defined analogously to the real case, with the role of the unit interval played instead by a valuational unit ball (see Definition 2.1). Convex subsets of $\mathbb{R}^d$ admit rich combinatorial structure, including many classical results around the theorems of Helly, Radon, Carathéodory, Tverberg, etc. — we refer to [De Loera et al. 2019] for a recent survey of the subject. In the case of $\mathbb{R}$, or more generally a real closed field, there is a remarkable parallel between the combinatorial properties of convex and semialgebraic sets (which correspond to definable sets from the point of view of model theory). They share many (but not all) properties in the form of various restrictions on the possible intersection patterns, including the fractional Helly theorem and existence of (weak) $\varepsilon$-nets. A well-studied phenomenon in model theory establishes strong parallels between definable sets in $\mathbb{R}$ and in many nonarchimedean valued fields such as the $p$-adics $\mathbb{Q}_p$ or various fields of power series (see for instance [van den Dries 2014]). In this paper we focus on the combinatorial study of convex sets over general valued fields, trying to understand

*MSC2020:* 12J25, 52A01, 52A20, 52A35.
*Keywords:* nonarchimedean fields, valued fields, combinatorial convexity, Helly theorem, Bárány theorem, VC dimension, breadth.

if there is similarly a parallel theory. On the one hand, we demonstrate valued field analogs of some classical results for convex sets over the reals (e.g., the fractional Helly theorem and Bárány's theorem on points in many simplices). On the other hand, we establish some additional properties not satisfied by convex sets over the reals, including finite breadth and VC dimension. This suggests that in a sense convex sets over valued fields are the best of both worlds combinatorially, and satisfy various properties enjoyed either by convex or by semialgebraic sets over the reals.

We give a quick outline of the paper. Section 2 covers some basics concerning convexity for subsets of $K^d$ over an arbitrary valued field $K$, in particular discussing the connection to modules over the valuation ring. These results are mostly standard (or small variations of standard results), and can be found in [Perez-Garcia and Schikhof 2010; Schneider 2002] under the unnecessary assumption that $K$ is spherically complete and $(\Gamma, +) \subseteq (\mathbb{R}_{>0}, \times)$; we provide some proofs for completeness. In Section 3 we give a simple combinatorial description of the submodules of $K^d$ over the valuation ring $\mathcal{O}_K$ in the case of a spherically complete field $K$ (Theorem 3.6 and Corollary 3.12), and an analog for finitely generated modules over arbitrary valued fields (Corollary 3.14). We also give an example of a convex set over the field of Puiseux series demonstrating that the assumption of spherical completeness is necessary for our presentation in the nonfinitely generated case (Example 3.11). In Section 4 we use this description of modules to deduce various combinatorial properties of the family of convex subsets $\mathrm{Conv}_{K^d}$ of $K^d$ over an arbitrary valued field $K$. First we show that $\mathrm{Conv}_{K^d}$ has breadth $d$ (Theorem 4.3), VC dimension $d+1$ (Theorem 4.8), dual VC dimension $d$ (Theorem 4.10) — in stark contrast, all of these are infinite for the family of convex subsets of $\mathbb{R}^d$ for $d \geq 2$. On the other hand, we obtain valued field analogs of the following classical results: the family $\mathrm{Conv}_{K^d}$ has Helly number $d+1$ (Theorem 4.5), fractional Helly number $d+1$ (Theorem 4.14), satisfies a strong form of Tverberg's theorem (Theorem 4.15) and the Boros–Füredi/Bárány theorem on the existence of a common point in a positive fraction of all geometric simplices generated by an arbitrary finite set of points in $K^d$ (Theorem 4.16). Some of the proofs here are adaptations of the classical arguments, and some rely crucially on the finite breadth property specific to the valued field context. Finally, in Section 5A we point out some further applications, for example a valued field analog of the celebrated $(p, q)$-theorem of Alon and Kleitman [1992] (Corollary 5.1), and that all convex sets over a spherically complete field are externally definable in the sense of model theory (Remark 5.7); as well as pose some questions and conjectures. We also discuss some other notions of convexity over nonarchimedean fields appearing in the literature in Section 5B, and place our work in the context of the study of abstract convexity spaces in discrete geometry and combinatorics in Section 5C.

## 2. Preliminaries on convexity over valued fields

***Notation.*** For $n \in \mathbb{N}_{\geq 1}$, we write $[n] = \{1, \dots, n\}$ and $\langle \cdot \rangle$ denotes the span in vector spaces. Throughout the paper, $K$ will denote a valued field, with value group $\Gamma = \Gamma_K$, and valuation $\nu = \nu_K : K \to \Gamma_\infty := \Gamma \sqcup \{\infty\}$, valuation ring $\mathcal{O} = \mathcal{O}_K = \nu^{-1}([0, \infty])$, maximal ideal $\mathfrak{m} = \mathfrak{m}_K = \nu^{-1}((0, \infty])$, and residue (class) field $k = \mathcal{O}/\mathfrak{m}$. The residue map $\mathcal{O} \to k$ will be denoted $\alpha \mapsto \bar{\alpha}$. For a ring $R$, $R^\times$ denotes its group of units.

The following definition of convexity is analogous to the usual one over $\mathbb{R}$, with the unit interval replaced by the (valuational) unit ball.

**Definition 2.1.** (1) For $d \in \mathbb{N}_{\geq 1}$, a set $X \subseteq K^d$ is *convex* if, for any $n \in \mathbb{N}_{\geq 1}$, $x_1, \dots, x_n \in X$, and $\alpha_1, \dots, \alpha_n \in \mathcal{O}$ such that $\alpha_1 + \cdots + \alpha_n = 1$ we have $\alpha_1 x_1 + \cdots + \alpha_n x_n \in X$ (in the vector space $K^d$).

(2) The family of convex subsets of $K^d$ will be denoted $\mathrm{Conv}_{K^d}$.

It is immediate from the definition that the intersection of any collection of convex subsets of $K^d$ is convex.

**Definition 2.2.** Given an arbitrary set $X \subseteq K^d$, its *convex hull* $\mathrm{conv}(X)$ is the convex set given by the intersection of all convex sets containing $X$, equivalently

$$\mathrm{conv}(X) = \left\{ \sum_{i=1}^{n} \alpha_i x_i : n \in \mathbb{N},\ \alpha_i \in \mathcal{O},\ x_i \in X,\ \sum_{i=1}^{n} \alpha_i = 1 \right\}.$$

**Definition 2.3.** A (valuational) *quasiball* is a set $B = \{x \in K : \nu(x - c) \in \Delta\}$ for some $c \in K$ and an upwards closed subset $\Delta$ of $\Gamma_\infty$. In this case we say that $B$ is *around* $c$, and refer to $\Delta$ as the quasiradius of $B$. We say that $B$ is a *closed* (respectively, *open*) *ball* if additionally $\Delta = \{\gamma \in \Gamma : \gamma \geq r\}$ (respectively, $\Delta = \{\gamma \in \Gamma : \gamma > r\}$) for some $r \in \Gamma$, and just *ball* if $B$ is either an open or a closed ball (in which case we refer to $r$ as its *radius*).

**Remark 2.4.** (1) If the value group $\Gamma$ is Dedekind complete, then every quasiball is a ball (except for $K$ itself, which is a quasiball of quasiradius $\Gamma_\infty$).

(2) If $B$ is a quasiball of quasiradius $\Delta$ around $c$ and $c' \in B$ is arbitrary, then $B$ is also a quasiball of quasiradius $\Delta$ around $c'$.

(3) Thus, any two quasiballs are either disjoint, or one of them contains the other.

**Example 2.5.** (1) The convex subsets of $K = K^1$ are exactly $\varnothing$ and the quasiballs (see Proposition 2.10 and Example 2.11).

(2) If $e_1, \dots, e_d$ is the standard basis of the vector space $K^d$, then

$$\mathrm{conv}(\{0, e_1, \dots, e_d\}) = \mathcal{O}^d.$$

(3) The image and the preimage of a convex set under an affine map are convex. In particular, a translate of a convex set is convex, and a projection of a convex set is convex. (Recall that given two vector spaces $V$, $W$ over the same field $K$, a map $f : V \to W$ is *affine* if $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$ for all $x, y \in V, \alpha, \beta \in K, \alpha + \beta = 1$.)

One might expect, by analogy with real convexity, that the definition of a convex set could be simplified to: if $x, y \in X$, $\alpha, \beta \in \mathcal{O}$ such that $\alpha+\beta=1$, then $\alpha x+\beta y \in X$. The following two propositions show that this is the case if and only if the residue field is not isomorphic to $\mathbb{F}_2$, and that in general we have to require closure under 3-element convex combinations.

**Proposition 2.6.** *Let $K$ be a valued field and $X \subseteq K^d$. If $X$ is closed under* 3-*element convex combinations* (*in the sense that if $x, y, z \in X$ and $\alpha, \beta, \gamma \in \mathcal{O}$ such that $\alpha + \beta + \gamma = 1$, then $\alpha x + \beta y + \gamma z \in X$*), *then $X$ is convex.*

*Proof.* Suppose $X$ is closed under 3-element convex combinations. We will show by induction on $n$ that then $X$ is closed under $n$-element convex combinations. Let $n \geq 3$, $x_1, \ldots, x_n \in X$ and $\alpha_1, \ldots, \alpha_n \in \mathcal{O}$ such that $\alpha_1 + \cdots + \alpha_n = 1$ be given. Then one of the following two cases holds.

*Case* 1. $\alpha_1 + \alpha_2 \in \mathcal{O}^\times$. Then $\alpha_1/(\alpha_1 + \alpha_2)$ and $\alpha_2/(\alpha_1 + \alpha_2)$ are elements of $\mathcal{O}$ that sum to 1, so

$$\frac{\alpha_1}{\alpha_1+\alpha_2}x_1 + \frac{\alpha_2}{\alpha_1+\alpha_2}x_2 \in X$$

by assumption. But then

$$\alpha_1 x_1 + \cdots + \alpha_n x_n = (\alpha_1 + \alpha_2)\Big(\frac{\alpha_1}{\alpha_1+\alpha_2}x_1 + \frac{\alpha_2}{\alpha_1+\alpha_2}x_2\Big) + \alpha_3 x_3 + \cdots + \alpha_n x_n \in X$$

by the induction hypothesis, as it is a convex combination of $n - 1$ elements of $X$.

*Case* 2. $\alpha_1 + \alpha_2 \in \mathfrak{m}$. Then, as $v\big(\sum_{i=1}^n \alpha_i\big) = 0$, there must exist some $i$ with $3 \leq i \leq n$ such that $\alpha_i \in \mathcal{O}^\times$. Hence $\alpha_1 + \alpha_2 + \alpha_i \in \mathcal{O}^\times$, so $\alpha_1/(\alpha_1 + \alpha_2 + \alpha_i)$, $\alpha_2/(\alpha_1 + \alpha_2 + \alpha_i)$, and $\alpha_i/(\alpha_1 + \alpha_2 + \alpha_i)$ are elements of $\mathcal{O}$ that sum to 1. Thus

$$\frac{\alpha_1}{\alpha_1+\alpha_2+\alpha_i}x_1 + \frac{\alpha_2}{\alpha_1+\alpha_2+\alpha_i}x_2 + \frac{\alpha_i}{\alpha_1+\alpha_2+\alpha_i}x_i \in X$$

by assumption, and so

$$\begin{aligned}\alpha_1 x_1+\cdots+\alpha_n x_n = (\alpha_1+\alpha_2+\alpha_i)\Big(\frac{\alpha_1}{\alpha_1+\alpha_2+\alpha_i}x_1+\frac{\alpha_2}{\alpha_1+\alpha_2+\alpha_i}x_2+\frac{\alpha_i}{\alpha_1+\alpha_2+\alpha_i}x_i\Big)\\ + \alpha_3 x_3 + \cdots + \alpha_{i-1}x_{i-1} + \alpha_{i+1}x_{i+1} + \cdots + \alpha_n x_n \in X\end{aligned}$$

by the induction hypothesis, as it is a convex combination of $n-2$ elements of $X$. $\square$

**Proposition 2.7.** *For any valued field $K$, the following are equivalent*:

(1) *For every $d \geq 1$, every set in $K^d$ that is closed under* 2-*element convex combinations is convex.*

(2) *The residue field $k$ is not isomorphic to $\mathbb{F}_2$.*

*Proof.* (1) *implies* (2). If $k = \mathbb{F}_2$, consider the set

$$X := \{(a_1, a_2, a_3) \mid a_1, a_2, a_3 \in \mathcal{O},\ \exists i \text{ such that } a_i \in \mathfrak{m}\} \subseteq K^3.$$

We claim that $X$ is closed under 2-element convex combinations. That is, given arbitrary $(a_1, a_2, a_3), (b_1, b_2, b_3) \in X$ and $\alpha, \beta \in \mathcal{O}$ with $\alpha + \beta = 1$, we must show that $\alpha(a_1, a_2, a_3) + \beta(b_1, b_2, b_3) \in X$. We have $\bar{\alpha} + \bar{\beta} = 1$ in $k = \mathbb{F}_2$, so necessarily one of $\bar{\alpha}$ and $\bar{\beta}$ is 1 and the other is 0. Without loss of generality $\bar{\alpha} = 1$ and $\bar{\beta} = 0$. Then $\beta \in \mathfrak{m}$. By definition of $X$, $a_i \in \mathfrak{m}$ for some $i$. Then $\alpha a_i \in \mathfrak{m}$, and $\beta b_i \in \mathfrak{m}$ as $b_i \in \mathcal{O}$, so $\alpha a_i + \beta b_i \in \mathfrak{m}$. Thus $(\alpha a_1 + \beta b_1, \alpha a_2 + \beta b_2, \alpha a_3 + \beta b_3) \in X$. However $X$ is not convex: for an arbitrary $a \in \mathfrak{m}$ we have $(0, 0, 0), (1, 0, 0), (0, 1, 1) \in X$, $1, -1 \in \mathcal{O}$, but $(-1)(0, 0, 0) + 1(1, 0, 0) + 1(0, 1, 1) = (1, 1, 1) \notin X$. (This example can be modified to work in $K^2$.)

(2) *implies* (1). If $k \ncong \mathbb{F}_2$, suppose $X$ is closed under 2-element convex combinations. By Proposition 2.6, we only need to check that it is then closed under 3-element convex combinations. Let $x, y, z \in X$, and $\alpha, \beta, \gamma \in \mathcal{O}$ such that $\alpha + \beta + \gamma = 1$. Then one of the following two cases holds.

*Case* 1. At least one of $\alpha + \beta, \beta + \gamma, \alpha + \gamma$ is an element of $\mathcal{O}^\times$. Without loss of generality, $\alpha + \beta \in \mathcal{O}^\times$. Then $(\alpha/(\alpha + \beta))x + (\beta/(\alpha + \beta))y \in X$ by assumption, and thus

$$\alpha x + \beta y + \gamma z = (\alpha + \beta)\Big(\frac{\alpha}{\alpha+\beta}x + \frac{\beta}{\alpha+\beta}y\Big) + \gamma z \in X.$$

*Case* 2. $\alpha + \beta, \beta + \gamma, \alpha + \gamma \in \mathfrak{m}$. In the residue field, $\bar{\alpha} + \bar{\beta} = \bar{\beta} + \bar{\gamma} = \bar{\alpha} + \bar{\gamma} = 0$, and $\bar{\alpha} + \bar{\beta} + \bar{\gamma} = 1$, hence necessarily $\bar{\alpha} = \bar{\beta} = \bar{\gamma} = 1$, and $\mathrm{char}(k) = 2$. Since $k \ncong \mathbb{F}_2$, there is $\delta \in \mathcal{O}$ such that $\bar{\delta} \notin \{0, 1\}$. Then $\bar{\alpha} + \bar{\delta} = 1 + \bar{\delta} \neq 0$ and $\bar{\beta} - \bar{\delta} + \bar{\gamma} = \bar{\delta} \neq 0$, so

$$\alpha x + \beta y + \gamma z =$$
$$(\alpha + \delta)\Big(\frac{\alpha}{\alpha+\delta}x + \frac{\delta}{\alpha+\delta}y\Big) + (\beta - \delta + \gamma)\Big(\frac{\beta-\delta}{\beta-\delta+\gamma}y + \frac{\gamma}{\beta-\delta+\gamma}z\Big) \in X. \quad \square$$

The following proposition gives a very strong form of Radon's theorem (not only do we obtain a partition into two sets with intersecting convex hulls, but moreover one of the points is in the convex hull of the other ones).

**Proposition 2.8.** *Let $K$ be a valued field. For any $d + 2$ points $x_1, \dots, x_{d+2} \in K^d$, one of them is in the convex hull of the others.*

*Proof.* There exist $a_1, \dots, a_{d+2} \in K$, *not all* 0, such that $\sum_{i=1}^{d+2} a_i x_i = 0$ and $\sum_{i=1}^{d+2} a_i = 0$ (because those are $d + 1$ linear equations on $d + 2$ variables, as we are working in $K^d$). Let $i \in [d + 2]$ be such that $\nu(a_i)$ is minimal among $\nu(a_1), \dots, \nu(a_{d+2})$, in particular $a_i \neq 0$. Then $x_i = \sum_{j \neq i}(-a_j/a_i)x_j$, and this is a convex combination: for $i \neq j$ we have $-a_j/a_i \in \mathcal{O}$ (as $\nu(-a_j/a_i) = \nu(a_j) - \nu(a_i) \geq 0$ by the choice of $i$) and $\sum_{j\neq i}(-a_j/a_i) = \left(-\sum_{j\neq i} a_j\right)/a_i = a_i/a_i = 1$. □

By a repeated application of Proposition 2.8 we immediately get a very strong form of Carathéodory's theorem:

**Corollary 2.9.** *Let $K$ be a valued field. Then the convex hull of any finite set in $K^d$ is already given by the convex hull of at most $d+1$ points from it.*

Convex sets over valued fields have a natural algebraic characterization.

**Proposition 2.10.** (1) *A subset $C \subseteq K^d$ is an $\mathcal{O}$-submodule of $K^d$ if and only if it is convex and contains* $0$.

(2) *Nonempty convex subsets of $K^d$ are precisely the translates of $\mathcal{O}$-submodules of $K^d$.*

*Proof.* (1) $\mathcal{O}$-submodules of $K^d$ are clearly convex and contain 0. Now suppose $C \subseteq K^d$ is convex and $0 \in C$. Then for any $\alpha \in \mathcal{O}$ and $x \in C$, $\alpha x = \alpha x + (1-\alpha)0 \in C$. And for any $x, y \in C$, $x + y = 1 \cdot x + 1 \cdot y - 1 \cdot 0 \in C$. Therefore $C$ is an $\mathcal{O}$-submodule.

(2) Given a nonempty convex $C \subseteq K^d$, we can choose $a \in K^d$ such that the translate $C+a$ contains 0 and is still convex, hence $C+a$ is an $\mathcal{O}$-submodule of $K^d$ by (1). □

**Example 2.11.** Let $C$ be an $\mathcal{O}$-submodule of $K$, and take $\Delta := \nu(C)$. Then $\Delta$ is nonempty because it contains $\infty = \nu(0)$, and upward-closed because for $\gamma \in \Delta$ and $\delta > \gamma$, there is $x \in C$ with $\nu(x) = \gamma$, and $\alpha \in K$ with $\nu(\alpha) = \delta - \gamma$; then $\alpha x \in C$ and $\nu(\alpha x) = \delta$. Clearly $C \subseteq \{x \in K \mid \nu(x) \in \Delta\}$ by definition of $\Delta$. To show $C \supseteq \{x \in K \mid \nu(x) \in \Delta\}$, given any $x \in K$ with $\nu(x) \in \Delta$, there is $y \neq 0 \in C$ with $\nu(y) = \nu(x)$, and $x/y \in \mathcal{O}$, so $x = (x/y)y \in C$. Thus $C = \{x \in K \mid \nu(x) \in \Delta\}$ is a quasiball around 0.

**Corollary 2.12.** *The convex hull of any finite set in $K^d$ is the image of $\mathcal{O}^d$ under an affine map.*

*Proof.* By Corollary 2.9, the convex hull of a finite subset of $K^d$ is the convex hull of some $d+1$ points $x_0, \ldots, x_d$ from it (possibly with $x_i = x_j$ for some $i, j$). Let $e_1, \ldots, e_d$ be the standard basis for $K^d$, and let $f$ be an affine map $f : K^d \to K^d$ such that $f(0) = x_0$ and $f(e_i) = x_i$ for $1 \leq i \leq d$ (we can take $f$ to be the composition of two affine maps: the linear map sending $e_i$ to $x_i - x_0$ for $1 \leq i \leq d$, and translation by $x_0$). Then we have $\operatorname{conv}(\{x_0, \ldots, x_d\}) = f(\operatorname{conv}\{0, e_1, \ldots, e_d\}) = f(\mathcal{O}^d)$, by Example 2.5(2). □

**Proposition 2.13.** *For any convex $C \subseteq K^d$ and $a \in K^d$, the translate $C + a := \{x + a \mid x \in C\}$ is either equal to or disjoint from $C$.*

*Proof.* If $x \in C \cap (C + a)$, then $y + a = y + x - (x - a) \in C$ for all $y \in C$ since that is a convex combination, and conversely $y = (y + a) - x + (x - a) \in C$ if $y + a \in C$. □

**Definition 2.14.** Given a valued field $K$, by a *valued $K$-vector space* we mean a $K$-vector space $V$ equipped with a surjective map $\nu = \nu_V : V \to \Gamma_\infty = \Gamma \cup \{\infty\}$ such that $\nu(x) = \infty$ if and only if $x = 0$, $\nu(x+y) \geq \min\{\nu(x), \nu(y)\}$ and $\nu(\alpha x) = \nu_K(\alpha) + \nu(x)$ for all $x, y \in V$ and $\alpha \in K$.

**Remark 2.15.** Here we restrict to the case when $V$ has the same value group as $K$, and refer to [Fuchs 1975] for a more general treatment (see also [Johnson 2016, Section 6.1.3; Hrushovski 2014, Section 2.5; Aschenbrenner et al. 2017, Section 2.3]).

By a morphism of valued $K$-vector spaces we mean a morphism of vector spaces preserving valuation. If $V$ and $W$ are valued $K$-vector spaces, their direct sum $V \oplus W$ is the direct sum of the underlying vector spaces equipped with the valuation $\nu(x, y) := \min\{\nu_V(x), \nu_W(y)\}$. In particular, the vector space $K^d$ is a valued $K$-vector space with respect to the valuation $\nu_{K^d} : K^d \to \Gamma_\infty$ given by

$$\nu_{K^d}(x_1, \ldots, x_d) := \min\{\nu_K(x_1), \ldots, \nu_K(x_d)\}.$$

Note that for any scalar $\alpha \in K$ and vector $v \in K^d$ we have $\nu_{K^d}(\alpha v) = \nu_K(\alpha) + \nu_{K^d}(v)$. By a (*valuational*) *ball* in $K^d$ we mean a set of the form $\{x \in K^d : \nu_{K^d}(x - c) \square r\}$ for some center $c \in K^d$, radius $r \in \Gamma \cup \{\infty\}$ and $\square \in \{>, \geq\}$ (corresponding to open or closed ball, respectively). The collection of all open balls forms a basis for the *valuation topology* on $K^d$ turning it into a topological vector space. Note that due to the "ultrametric" property of valuations, every open ball is also a closed ball, and vice versa. Equivalently, this topology on $K^d$ is just the product topology induced from the valuation topology on $K$.

Recall that the *affine span* $\operatorname{aff}(X)$ of a set $X \subseteq K^d$ is the intersection of all affine sets (i.e., translates of vector subspaces of $K^d$) containing $X$, equivalently

$$\operatorname{aff}(X) = \left\{ \sum_{i=1}^{n} \alpha_i x_i : n \in \mathbb{N}_{\geq 1},\ \alpha_i \in K,\ x_i \in X,\ \sum_{i=1}^{n} \alpha_i = 1 \right\}.$$

We have $\operatorname{conv}(X) \subseteq \operatorname{aff}(X)$ for any $X$.

**Proposition 2.16.** *Any convex set in $K^d$ is open in its affine span.*

*Proof.* For $x \in C \subseteq K^d$, $C$ convex, let $d' \leq d$ be the dimension of the affine span of $C$, and let $y_1, \ldots, y_{d'} \in C$ be such that $x, y_1, \ldots, y_{d'}$ are affinely independent, and thus have the same affine span as $C$. Then the map $(\alpha_1, \ldots, \alpha_{d'}) \mapsto x + \alpha_1(y_1 - x) + \cdots + \alpha_{d'}(y_{d'} - x)$ is a homeomorphism from $K^{d'}$ to the affine span of $C$, and sends $\mathcal{O}^{d'}$ (which is open in $K^{d'}$) to a neighborhood of $x$ contained in $C$. $\square$

**Corollary 2.17.** *Convex sets in $K^d$ are closed.*

*Proof.* For convex $C \subseteq K^d$ and $x \in \operatorname{aff}(C) \setminus C$, $C + x$ is an open subset of $\operatorname{aff}(C)$ that is disjoint from $C$, so $C$ is a closed subset of its affine span, and hence closed in $K^d$, since affine subspaces are closed. $\square$

## 3. Classification of $\mathcal{O}$-submodules of $K^d$

In this section we provide a simple description for the $\mathcal{O}$-submodules of $K^d$ over a spherically complete valued field $K$ (and over an arbitrary valued field $K$ in the finitely generated case). Combined with the description of convex sets in terms of $\mathcal{O}$-submodules from Section 2, this will allow us to establish various combinatorial properties of convex sets over valued fields in the next section. In the following lemma, the construction of the valuation $\nu$ is a special case of the standard construction of the quotient norm, when modding out a normed space by a closed subspace, while the second part is more specific to our situation.

**Lemma 3.1.** *Let $K$ be a valued field, and $V \subseteq K^d$ a subspace. Then the quotient vector space $K^d/V$ is a valued $K$-vector space equipped with the valuation*

$$\nu(u) := \max\{\nu_{K^d}(v) \mid \pi(v) = u,\ v \in K^d\},$$

*for $u \in K^d/V$, where $\pi : K^d \to K^d/V$ is the projection map (and the maximum is taken in $\Gamma_\infty$). If $\dim(V) = n$, then $K^d/V \cong K^{d-n}$ as valued $K$-vector spaces, and there is a valuation preserving embedding of $K$-vector spaces $f : K^d/V \hookrightarrow K^d$ so that $\pi \circ f = \mathrm{id}_{K^d/V}$.*

*Proof.* First we prove the lemma for $n = 1$. Let $V \subseteq K^d$ be one-dimensional. There exists $i \in [d]$ such that $\nu_{K^d}((x_1, \ldots, x_d)) = \nu_K(x_i)$ for all $(x_1, \ldots, x_d) \in V$ (indeed, if $\nu_K(x_i) = \min\{\nu_K(x_1), \ldots, \nu_K(x_d)\}$ for some $(x_1, \ldots, x_d) \in V$, then we also have $\nu_K(\alpha x_i) = \nu_K(\alpha) + \nu_K(x_i) = \nu_K(\alpha) + \min\{\nu_K(x_1), \ldots, \nu_K(x_d)\} = \min\{\nu_K(\alpha x_1), \ldots, \nu_K(\alpha x_d)\}$ for any $\alpha \in K$). Given any $(x_1, \ldots, x_d) \in K^d$ with $x_i = 0$ and $(y_1, \ldots, y_d) \in V$, we have

$$\begin{aligned}
\nu_{K^d}(x_1 + y_1, \ldots, x_d + y_d) &= \min_{j \in [d]}\{\nu_K(x_j + y_j)\} \\
&= \min\{\nu_K(y_i), \min_{j \neq i}\{\nu_K(x_j + y_j)\}\} \leq \nu_K(y_i) \\
&= \nu_{K^d}(y_1, \ldots, y_d).
\end{aligned} \tag{3-1}$$

Now consider an arbitrary affine translate $x + V$ of $V$, $x = (x_1, \ldots, x_d) \in K^d$. Then there exists $x' = (x'_1, \ldots, x'_d) \in x + V$ so that $x'_i = 0$. Indeed, fix any $0 \neq y' \in V$, then $V = \{\alpha y' : \alpha \in K\}$. Take $\alpha' := -x_i/y'_i$ (note that, by the choice of $i$, $y' \neq 0 \Rightarrow \nu_{K^d}(y') \neq \infty \Rightarrow \nu_K(y'_i) \neq \infty \Rightarrow y'_i \neq 0$), and let $x' := x + \alpha' y'$. We claim that $\nu_{K^d}(x') = \max\{\nu_{K^d}(z) : z \in x + V\}$, thus the valuation $\nu$ on $K^d/V$ is well defined. Indeed, $x + V = x' + V$, so fix any $y \in V$. If $\nu_{K^d}(x') < \nu_{K^d}(x' + y)$, we must necessarily have $\nu_{K^d}(x') = \nu_{K^d}(y)$, but by (3-1) we have $\nu_{K^d}(x' + y) \leq \nu_{K^d}(y)$, so $\nu_{K^d}(y) < \nu_{K^d}(y)$ — a contradiction; thus $\nu_{K^d}(x') \geq \nu_{K^d}(x' + y)$.

Let $K' := \{(x_1, \ldots, x_d) \in K^d \mid x_i = 0\}$, then we have $K^d = V \oplus K'$ as vector spaces, hence the projection of $K^d$ onto $K'$ along $V$ induces an isomorphism between $K^d/V$

and $K'$, which in turn is naturally isomorphic to $K^{d-1}$, and these isomorphisms preserve the valuation and give the desired embedding $f : K^d/V \to K^d$. The general case follows by induction on $n$ using the vector space isomorphism theorems. □

We recall an appropriate notion of completeness for valued fields. Recall that a family $\{C_i : i \in I\}$ of subsets of a set $X$ is *nested* if for any $i, j \in I$, either $C_i \subseteq C_j$ or $C_j \subseteq C_i$.

**Definition 3.2.** A valued field $K$ is *spherically complete* if every nested family of (closed or open) valuational balls has nonempty intersection.

For the following standard fact, see for example Theorem 5 in Section II.3 and Theorem 8 in Section II.6 of [Schilling 1950].

**Fact 3.3.** *Every valued field $K$ (with valuation $\nu_K$, value group $\Gamma_K$ and residue field $k_K$) admits a* ***spherical completion****, i.e., a valued field $\widetilde{K}$ (with valuation $\nu_{\widetilde{K}}$, value group $\Gamma_{\widetilde{K}}$ and residue field $k_{\widetilde{K}}$), so that:*

1. *$\widetilde{K}$ is an* ***immediate extension*** *of $K$, i.e., $\widetilde{K}$ is a field extension of $K$, $\nu_{\widetilde{K}} \restriction_K = \nu_K$, $\Gamma_{\widetilde{K}} = \Gamma_K$ and $k_{\widetilde{K}} = k_K$.*
2. *$\widetilde{K}$ is spherically complete.*

We remark that in general a valued field might have multiple nonisomorphic spherical completions.

**Lemma 3.4.** *If $K$ is spherically complete, then every nested family of nonempty convex subsets of $K^d$ has a nonempty intersection.*

*Proof.* By induction on $d$. For $d = 1$, let $\{C_i\}_{i\in I}$ be a nested family of nonempty convex sets, so each $C_i$ is a quasiball; see Example 2.5(1). If there exists some $i \in I$ such that $C_i$ is the smallest of these under inclusion, then any element of $C_i$ is in the intersection of the whole family. Hence we may assume that for each $i \in I$ there exists some $i' \in I$ such that $C_{i'} \subsetneq C_i$. Let $\Delta_i$ and $\Delta_{i'}$ be the quasiradii of $C_i$ and $C_{i'}$, respectively. We may assume that both quasiballs are around the same point $x_i \in C_{i'}$ (by Remark 2.4), hence necessarily $\Delta_{i'} \subsetneq \Delta_i$. Let $r_i \in \Delta_i \setminus \Delta_{i'}$, and let $C'_i$ be a (open or closed) ball of radius $r_i$ around $x_i$. We have $C'_i \subseteq C_i$, so if $\bigcap_{i\in I} C'_i$ is nonempty, then so is $\bigcap_{i\in I} C_i$. Hence it is sufficient to show that $\{C'_i\}_{i\in I}$ is nested, and then the intersection is nonempty by spherical completeness of $K$. By construction for any $i, j \in I$ there exists some $\ell \in I$ such that $C_\ell \subseteq C'_i \cap C'_j$, so $C'_i$ and $C'_j$ have nonempty intersection, and are thus nested as they are balls.

For $d \geq 2$, let $\{C_i\}_{i\in I}$ be a nested family of nonempty convex sets, and let $\pi_1 : K^d \to K$ be the projection onto the first coordinate. Then $\{\pi_1(C_i)\}_{i\in I}$ is a nested family of nonempty convex sets in $K$, hence has an intersection point $x$. Then $\{\pi_1^{-1}(x) \cap C_i\}_{i\in I}$ is a nested family of nonempty convex sets in $\pi_1^{-1}(x) \cong K^{d-1}$, which is nonempty by the induction hypothesis. □

**Lemma 3.5.** *If $C \subseteq K^d$ is an $\mathcal{O}$-module, and $\gamma \in \Gamma_\infty$, then the set*

$$X_\gamma = \{(x_1, \ldots, x_{d-1}) \in \mathcal{O}^{d-1} \mid \exists \alpha \in K,\ \nu(\alpha) = \gamma,\ (\alpha, \alpha x_1, \ldots, \alpha x_{d-1}) \in C\}$$

*is convex.*

*Proof.* Let $x = (x_1, \ldots, x_{d-1}), y = (y_1, \ldots, y_{d-1}), z = (z_1, \ldots, z_{d-1}) \in X_\gamma$ and $\beta_1, \beta_2, \beta_3 \in \mathcal{O}$ with $\beta_1 + \beta_2 + \beta_3 = 1$ be arbitrary. Then there exist some $\alpha_1, \alpha_2, \alpha_3 \in K$ with $\nu(\alpha_i) = \gamma$, so that

$$(\alpha_1, \alpha_1 x_1, \ldots, \alpha_1 x_{d-1}), (\alpha_2, \alpha_2 y_1, \ldots, \alpha_2 y_{d-1}), (\alpha_3, \alpha_3 z_1, \ldots, \alpha_3 z_{d-1}) \in C.$$

Taking $\alpha := \alpha_1$, we have

$$x' := (\alpha, \alpha x_1, \ldots, \alpha x_{d-1}), \quad y' := (\alpha, \alpha y_1, \ldots, \alpha y_{d-1}),$$
$$z' := (\alpha, \alpha z_1, \ldots, \alpha z_{d-1}) \in C,$$

as for every $i \in [3]$, $\alpha/\alpha_i \in \mathcal{O}$, and hence $(\alpha/\alpha_i)v \in C$ for any $v \in C$ as $C$ is an $\mathcal{O}$-module. Using this and convexity of $C$ we thus have

$$\begin{aligned}(\alpha, \alpha(\beta_1 x_1 + \beta_2 y_1 + \beta_3 z_1), \ldots, \alpha(\beta_1 x_{d-1} + \beta_2 y_{d-1} + \beta_3 z_{d-1}))\\ = \beta_1(\alpha, \alpha x_1, \ldots, \alpha x_{d-1}) + \beta_2(\alpha, \alpha y_1, \ldots, \alpha y_{d-1}) + \beta_3(\alpha, \alpha z_1, \ldots, \alpha z_{d-1})\\ = \beta_1 x' + \beta_2 y' + \beta_3 z' \in C.\end{aligned}$$

This shows that $\beta_1 x + \beta_2 y + \beta_3 z \in X_\gamma$, and hence that $X_\gamma$ is convex using Proposition 2.6. $\square$

Combining the lemmas, we obtain a description of the $\mathcal{O}_K$-submodules of $K^d$ for spherically complete $K$:

**Theorem 3.6.** *Suppose $K$ is a spherically complete valued field, $d \in \mathbb{N}_{\geq 1}$, and let $C \subseteq K^d$ be an $\mathcal{O}$-submodule. Then there exists a complete flag of vector subspaces $\{0\} \subsetneq F_1 \subsetneq \cdots \subsetneq F_d = K^d$ and a decreasing sequence of nonempty, upwards closed subsets $\Delta_1 \supseteq \Delta_2 \supseteq \cdots \supseteq \Delta_d$ of $\Gamma_\infty$ such that*

$$C = \{v_1 + \cdots + v_d \mid v_i \in F_i,\ \nu(v_i) \in \Delta_i\}.$$

**Remark 3.7.** If $F_i$ and $\Delta_i$ satisfy the conclusion of Theorem 3.6 for $C$, then $\nu_{K^d}(C \cap F_1) = \nu_{K^d}(C) = \Delta_1$.

Indeed, any $v \in C$ is of the form $v = v_1 + \cdots + v_d$ with $v_i \in F_i,\ \nu(v_i) \in \Delta_i$ and $\Delta_1 \supseteq \Delta_i$ for all $i \in [d]$, hence $\nu(v) \geq \min\{\nu(v_i) : i \in [d]\} \in \Delta_1$, hence $\nu(v) \in \Delta_1$ as $\Delta_1$ is upwards closed, so $\nu(C) \subseteq \Delta_1$. Conversely, assume $\gamma \in \Delta_1$. If $\gamma = \infty$, then $\nu(0) = \infty$ and $0 \in F_1$. So assume $\gamma \in \Gamma$ and let $v$ be any nonzero vector in $F_1$, and define $\delta := \nu(v) \in \Gamma$. Taking $\alpha \in K$ so that $\nu_K(\alpha) = \gamma - \delta$, we have $\alpha v \in F_1$ and $\nu_{K^d}(\alpha v) = \nu_K(\alpha) + \nu_{K^d}(v) = \gamma$. Note also that $\alpha v = v_1 + \cdots + v_d$

with $v_1 := \alpha v,\ v_i := 0$ for $2 \le i \le d$; in particular $v_i \in F_i$ and $\nu(v_i) \in \Delta_i$, so $\alpha v \in C$, hence $\Delta_1 \subseteq \nu(F_1 \cap C)$.

*Proof of Theorem 3.6.* By induction on $d$. For $d = 1$, every $\mathcal{O}$-submodule of $K$ is a quasiball $C = \{x \in K : \nu(x) \in \Delta\}$ for some upwards closed $\Delta \subseteq \Gamma \cup \{\infty\}$ (see Example 2.11), hence we take $F_1 := K$ and $\Delta_1 := \Delta$.

For $d > 1$, let $\Delta_1 := \{\gamma \in \Gamma_\infty \mid \exists v \in C,\ \nu_{K^d}(v) = \gamma\}$. Note that $\Delta_1$ is nonempty because it contains $\infty = \nu(0)$. Then there is some $i \in [d]$ such that every $\gamma \in \Delta_1$ is the valuation of the $i$-th coordinate of some element of $C$. To see this, note that for each $i \in [d]$, the set

$$S_i := \{\gamma \in \Gamma_\infty \mid \exists v = (v_1, \dots, v_d) \in C \text{ such that } \nu_{K^d}(v) = \nu(v_i) = \gamma\}$$

is upwards closed in $\Gamma_\infty$. Indeed, assume $v = (v_1, \dots, v_d) \in C,\ \gamma = \nu(v_i) = \min\{\nu(v_j) : j \in [d]\}$ and $\delta \ge \gamma$ in $\Gamma_\infty$. Let $\alpha \in K$ be arbitrary with $\nu(\alpha) = \delta - \gamma$, then $\alpha \in \mathcal{O}$, hence $\alpha v \in C$, and so $\nu_{K^d}(\alpha v) = \min\{\nu(\alpha v_j) : j \in [d]\} = \nu(\alpha v_j) = \delta$. As we also have $\Delta_1 = \bigcup_{i \in [d]} S_i$, it follows that $\Delta_1 = S_i$ for some $i \in [d]$ as wanted (and thus $\Delta_1$ is upwards closed in $\Gamma_\infty$).

Without loss of generality we may assume $i = 1$. Then, given any $\gamma \in \Delta_1$, there is some $(\alpha, y_1, \dots, y_{d-1}) \in C$ such that $\gamma = \nu(\alpha) \le \min\{\nu(y_j) : j \in [d-1]\}$. Taking $x_j := y_j/\alpha \in \mathcal{O}$, we thus have $(\alpha, \alpha x_1, \dots, \alpha x_{d-1}) \in C$. Hence for any $\gamma \in \Delta_1$, the set

$$X_\gamma := \{(x_1, \dots, x_{d-1}) \in \mathcal{O}^{d-1} \mid \exists \alpha \in K,\ \nu(\alpha) = \gamma \wedge (\alpha, \alpha x_1, \dots, \alpha x_{d-1}) \in C\}$$

is nonempty and convex (by Lemma 3.5). Note that for $\gamma < \delta \in \Gamma_\infty$ we have $X_\gamma \subseteq X_\delta$, hence $\bigcap_{\gamma \in \Delta_1} X_\gamma \neq \varnothing$ by Lemma 3.4. That is, there exists $(x_1, \dots, x_{d-1}) \in \mathcal{O}^{d-1}$ such that for all $\gamma \in \Delta_1$, there exists $\alpha \in K$ with $\nu(\alpha) = \gamma \wedge (\alpha, \alpha x_1, \dots, \alpha x_{d-1}) \in C$. Hence

$$\forall \alpha \in K,\ \nu(\alpha) \in \Delta_1 \Longrightarrow (\alpha, \alpha x_1, \dots, \alpha x_{d-1}) \in C, \tag{3-2}$$

since there exists $\beta \in K$ such that $\nu(\beta) = \nu(\alpha) \wedge (\beta, \beta x_1, \dots, \beta x_{d-1}) \in C$, so $\alpha/\beta \in \mathcal{O}$ and multiplying by it we get $(\alpha, \alpha x_1, \dots, \alpha x_{d-1}) \in C$.

Let $F_1 := \langle (1, x_1, \dots, x_{d-1}) \rangle$. Let $\pi : K^d \twoheadrightarrow K^d/F_1$ be the projection map, $f : K^d/F_1 \hookrightarrow K^d$ the valuation preserving embedding given by Lemma 3.1, and $\pi' := f \circ \pi : K^d \to K^d$. Note that $K^d/F_1 \cong K^{d-1}$ as a valued $K$-vector space by Lemma 3.1, and that $\widetilde{C} := \pi(C)$ is still an $\mathcal{O}$-submodule of $K^d/F_1$. By the induction hypothesis there is a full flag $\{0\} \subsetneq \widetilde{F}_2 \subsetneq \cdots \subsetneq \widetilde{F}_d = K^d/F_1$ and upwards closed subsets $\nu_{K^d/F_1}(\widetilde{C}) = \Delta_2 \supseteq \cdots \supseteq \Delta_d$ of $\Gamma_\infty$ satisfying the conclusion of the theorem with respect to $\widetilde{C}$ (the equality $\nu_{K^d/F_1}(\widetilde{C}) = \Delta_2$ is by Remark 3.7). Note that

$$\forall v \in K^d,\ \nu_{K^d}(\pi'(v)) = \nu_{K^d/F_1}(\pi(v)) \ge \nu_{K^d}(v). \tag{3-3}$$

In particular we have $\Delta_1 \supseteq \Delta_2$.

Let the subspaces $F_2, \dots, F_d$ be the preimages of $\widetilde{F}_2, \dots, \widetilde{F}_d$ in $K^d$. We let $W := f(K^d/F_1) \subseteq K^d$ be the image of the valuation preserving embedding $f : K^d/F_1 \hookrightarrow K^d$. Then we have

$$C = \{v_1 + w \mid v_1 \in F_1,\ \nu_{K^d}(v_1) \in \Delta_1,\ w \in C \cap W\}. \tag{3-4}$$

To see this, given an arbitrary $v \in C$, let $w := \pi'(v)$ and $v_1 := v - w$. As $\pi \circ f = \mathrm{id}_{K^d/F_1}$ by assumption, we have $\pi(w) = \pi(\pi'(v)) = \pi(f(\pi(v))) = \pi(v)$, hence $v_1 \in F_1$. By (3-3) we have $\nu_{K^d}(w) \geq \nu_{K^d}(v)$, and thus $\nu_{K^d}(v_1) \geq \min\{\nu_{K^d}(v), \nu_{K^d}(w)\} \geq \nu_{K^d}(v)$ as well. Thus $\nu_{K^d}(v_1) \in \Delta_1$, and $v_1 \in F_1$, which together with (3-2) and the definition of $F_1$ implies $v_1 \in C$; hence $w = v - v_1 \in C$ as well. The opposite inclusion is obvious.

Furthermore, applying the isomorphism $f : K^d/F_1 \to W$ to

$$\widetilde{C} = C/F_1 = \{v_2 + \cdots + v_d \mid v_i \in \widetilde{F}_i,\ \nu_{K^d/F_1}(v_i) \in \Delta_i\},$$

we get

$$C \cap W = \{v_2 + \cdots + v_d \mid v_i \in F_i \cap W,\ \nu_{K^d}(v_i) \in \Delta_i\},$$

which together with (3-4) implies

$$C = \{v_1 + \cdots + v_d \mid v_i \in F_i,\ \nu(v_i) \in \Delta_i,\ v_i \in W \text{ for } i \geq 2\}.$$

Now $C = \{v_1 + \cdots + v_d \mid v_i \in F_i,\ \nu(v_i) \in \Delta_i\}$ follows because for any such vectors $v_1, \dots, v_d$, the vector $v_i$ (for $i \geq 2$) can be moved into $W$ by subtracting an element of $F_1$ with valuation in $\Delta_1$, and collecting the differences in with $v_1$. That is, given arbitrary $v_i \in F_i$ with $\nu(v_i) \in \Delta_i$, let $w_i := \pi'(v_i) \in W$ for $i \geq 2$, and let $w_1 := v_1 + (v_2 - \pi'(v_2)) + \cdots + (v_d - \pi'(v_d))$. As above, using (3-3), for each $i \geq 2$ we have $\nu_{K^d}(v_i - \pi'(v_i)) \geq \min\{\nu_{K^d}(v_i), \nu_{K^d}(\pi'(v_i))\} \geq \nu_{K^d}(v_i) \in \Delta_i \subseteq \Delta_1$. Hence $\nu_{K^d}(w_1) \geq \min\{v_1, v_2 - \pi'(v_2), \dots, v_d - \pi'(v_d)\} \in \Delta_1$. We also have $\nu_{K^d}(w_i) \geq \nu_{K^d}(v_i) \in \Delta_i$ for $i \geq 2$ by (3-3). Using that $f$ is a one-sided inverse of $\pi$ as above, we also have $v_i - \pi'(v_i) \in F_1 \subseteq F_i$ for $i \geq 2$. It follows that $w_i \in F_i$ for all $i \in [d]$. Putting all of this together, we get $w_1 + \cdots + w_d = v_1 + \cdots + v_d$, $w_i \in F_i,\ \nu(w_i) \in \Delta_i$, and $w_i \in W$ for $i \geq 2$. □

**Remark 3.8.** Note that as $F_d = K^d$ in Theorem 3.6, we have

$$\Delta_d = \{\gamma \in \Gamma_\infty \mid \forall v \in K^d,\ \nu(v) = \gamma \Longrightarrow v \in C\}.$$

That is, $\Delta_d$ is the quasiradius of the largest quasiball around 0 contained in $C$.

**Remark 3.9.** Given a convex set $0 \in C \subseteq K^d$ and any $F_i$ and $\Delta_i,\ i \in [d]$ satisfying the conclusion of Theorem 3.6 with respect to it, for every $j \in [d]$ we have

$$C \cap F_j = \{v_1 + \cdots + v_j \mid v_i \in F_i,\ \nu(v_i) \in \Delta_i \text{ for all } j \in [i]\}.$$

Indeed, if $x \in C \cap F_j$, then $x = v_1 + \cdots + v_d \in F_j$ for some $v_i \in F_i$ with $\nu(v_i) \in \Delta_i$ for $i \in [d]$. Then, using that the $F_i$ are increasing under inclusion and $\Delta_i$ are increasing under inclusion and upwards closed, $v_{j+1} + \cdots + v_d \in F_j$ and taking $v'_j := v_j + \cdots + v_d$ we have $v'_j \in F_j$, $\nu(v'_j) \geq \min\{\nu(v_i) : j \leq i \leq d\} \in \Delta_j$ and $x = v_1 + \cdots + v_{j-1} + v'_j$. Conversely, any element $x = v_1 + \cdots + v_j$ with $v_i \in F_i$, $\nu(v_i) \in \Delta_i$ for $i \in [j]$ can be written as $x = v_1 + \cdots + v_d$ with $v_i := 0 \in F_i$ and $\nu(v_i) = \infty \in \Delta_i$ for $j+1 \leq i \leq d$. So $x \in C \cap F_j$.

**Remark 3.10.** (1) It follows from the conclusion of Theorem 3.6 that the subspace $F_{d-1}$ is a linear hyperplane in $K^d$, and every element of $C$ differs from an element of $F_{d-1}$ (and hence of $F_{d-1} \cap C$ in view of Remark 3.9) by a vector in $K^d$ with valuation in $\Delta_d$ (with $\Delta_d$ as in Remark 3.8).

(2) Conversely, $F_{d-1}$ can be chosen to be *any* linear hyperplane $H$ in $K^d$ such that every element of $C$ differs from an element of $H$ by a vector in $K^d$ with valuation in $\Delta_d$. To see this, let $H$ be such a hyperplane in $K^d$. Then $C \cap H$ is a convex subset of $H \cong K^{d-1}$ containing 0, hence an $\mathcal{O}$-submodule of $H$ by Proposition 2.10. Applying Theorem 3.6 to $C \cap H$ in $H$ (with the induced valuation on $H$), there are $\Delta_1 \supseteq \Delta_2 \supseteq \cdots \supseteq \Delta_{d-1}$ and a full flag $\{0\} \subsetneq F_1 \subsetneq \cdots \subsetneq F_{d-1} = H$, such that $C \cap H = \{v_1 + \cdots + v_{d-1} \mid v_i \in F_i,\ \nu(v_i) \in \Delta_i\}$. Then

$$\{v_1 + \cdots + v_d \mid v_i \in F_i,\ \nu(v_i) \in \Delta_i\} = \{w + v_d \mid w \in C \cap H,\ \nu(v_d) \in \Delta_d\} = C.$$

**Example 3.11.** The assumption of spherical completeness of $K$ is necessary in Theorem 3.6. For example, let $K := \bigcup_{n \geq 1} k((t^{1/n}))$ be the field of Puiseux series over a field $k$, and let $\widetilde{K} := k[[t^{\mathbb{Q}}]]$ be the field of Hahn series over $k$ with rational exponents. The field $\widetilde{K}$ is the spherical completion of $K$ (both fields have value group $\mathbb{Q}$ and valuation $\nu(x) = q$ where $x$ has leading term $t^q$; see [Aschenbrenner et al. 2017, Example 3.3.23] for instance). In particular $\sum_{n \geq 1} t^{1-1/n} \in \widetilde{K} \setminus K$, and let

$$\widetilde{C} := \Big\{\alpha\Big(1, \sum_{n \geq 1} t^{1-1/n}\Big) + v \;\Big|\; \alpha \in \widetilde{K},\ v \in \widetilde{K}^2,\ \nu_{\widetilde{K}}(\alpha) \geq 0,\ \nu_{\widetilde{K}^2}(v) \geq 1\Big\} \subseteq \widetilde{K}^2,$$

as well as $C := \widetilde{C} \cap K^2$. Then $\widetilde{C}$ is convex in $\widetilde{K}^2$, and hence $C$ is also convex as a subset of $K^2$. The basic idea behind why $C$ is not of the form described in Theorem 3.6 is that $C$ is close enough to $\widetilde{C}$, and the subspace $F_1$ appearing in the conclusion of Theorem 3.6 for $\widetilde{C}$ must be close to $\big\langle\big(1, \sum_{n \geq 1} t^{1-1/n}\big)\big\rangle$; specifically, it must be $\big\langle\big(1, x + \sum_{n \geq 1} t^{1-1/n}\big)\big\rangle$ for some $x \in K^2$ with $\nu(x) \geq 1$, but $K^2$ contains no such subspaces.

Indeed, by Remark 3.7, given any $F_i$ and $\Delta_i$ satisfying the conclusion of Theorem 3.6 with respect to $C$, the valuation of every element of $C$ must also be the valuation of some element of $F_1 \cap C$. So, to show that $C$ is not of the form described in Theorem 3.6, it suffices to show that $C$ contains elements of valuation arbitrarily close to 0, but that for every 1-dimensional subspace $F_1 \subset K^2$, there is some $q > 0$ in $\Gamma$ such that every element of $F_1 \cap C$ has valuation at least $q$ (and note that from the definition of $C$, every element in it has positive valuation).

**Claim.** *For every $n \in \mathbb{N}_{\geq 1}$, there is some $v \in C$ with $\nu_{K^2}(v) = 1/n$.*

*Proof.* To see this, note that

$$t^{1/n}\left(1, \sum_{m=1}^{n-1} t^{1-1/m}\right) = t^{1/n}\left(1, \sum_{m \geq 1} t^{1-1/m}\right) - t^{1/n}\left(0, \sum_{m \geq n} t^{1-1/m}\right) \in C$$

as $\nu_K(t^{1/n}) = 1/n \geq 0$ and $\nu_{K^2}\left(t^{1/n}\left(0, \sum_{m \geq n} t^{1-1/m}\right)\right) = 1/n + (1 - 1/n) \geq 1$. □

**Claim.** *For every 1-dimensional subspace $F_1 \subset K^2$, there is some $n \in \mathbb{N}_{n \geq 1}$ such that every element of $F_1 \cap C$ has valuation at least $1/n$.*

*Proof.* We prove this by breaking into two cases.

*Case* 1. $F_1 = \langle (0, 1) \rangle$. Assume $x \in F_1 \cap C$, then $x = (x_1, x_2) = \alpha\left(1, \sum_{n \geq 1} t^{1-1/n}\right) + v$ for some $\alpha \in K$, $v = (v_1, v_2) \in \widetilde{K}^2$ with $\nu_{\widetilde{K}}(\alpha) \geq 0$, $\nu_{\widetilde{K}^2}(v) \geq 1$, and $x_1 = 0$, so $\alpha = -v_1$. But $1 \leq \nu_{\widetilde{K}^2}(v) = \min\{\nu_{\widetilde{K}}(v_1), \nu_{\widetilde{K}}(v_2)\}$, hence $\nu_{\widetilde{K}}(\alpha) \geq 1$ as well. Since $\nu_{\widetilde{K}}\left(\sum_{n \geq 1} t^{1-1/n}\right) = 0$, it follows that

$$\nu_{\widetilde{K}^2}(x) = \min\left\{\nu_{\widetilde{K}}(0), \nu_{\widetilde{K}}\left(\alpha\left(\sum_{n \geq 1} t^{1-1/n}\right)\right)\right\} \geq 1.$$

Thus every element of $F_1 \cap C$ has valuation at least 1.

*Case* 2. $F_1 = \langle (1, x) \rangle$ for some $x \in K$. Given any $x \in K$, there must exist some $n \in \mathbb{N}$ such that $\nu_{\widetilde{K}}\left(x - \sum_{m \geq 1} t^{1-1/m}\right) \leq 1 - 1/n$. Given any $v \in F_1 \cap C$, we have

$$v = \alpha(1, x) = \beta\left(1, \sum_{m \geq 1} t^{1-1/m}\right) + w$$

for some $\alpha \in K$, some $\beta \in \widetilde{K}$ with $\nu_{\widetilde{K}}(\beta) \geq 0$ and $w = (w_1, w_2) \in \widetilde{K}^2$ with $\nu_{\widetilde{K}^2}(w) \geq 1$. Without loss of generality, $\alpha \neq 0$, so we have

$$x = \frac{\alpha x}{\alpha} = \left(w_2 + \beta \sum_{m \geq 1} t^{1-1/m}\right)(w_1 + \beta)^{-1} = \left(\frac{w_2}{\beta} + \sum_{m \geq 1} t^{1-1/m}\right)\left(1 + \frac{w_1}{\beta}\right)^{-1}.$$

If $\nu_{\widetilde{K}}(\beta) < 1/n$, then

$$\nu_{\widetilde{K}}\left(\frac{w_1}{\beta}\right) > 1 - \frac{1}{n}, \quad \nu_{\widetilde{K}}\left(\frac{w_2}{\beta}\right) > 1 - \frac{1}{n},$$
$$\nu_{\widetilde{K}}\left(\left(1 + \frac{w_1}{\beta}\right)^{-1}\right) = 0, \quad \nu_{\widetilde{K}}\left(\left(1 + \frac{w_1}{\beta}\right)^{-1} - 1\right) > 1 - \frac{1}{n},$$

so

$$\nu\left(x - \sum_{m \geq 1} t^{1-1/m}\right) = \nu\left(\frac{w_2}{\beta}(w_1 + \beta)^{-1} + \left(\sum_{m \geq 1} t^{1-1/m}\right)\left(\left(1 + \frac{w_1}{\beta}\right)^{-1} - 1\right)\right) > 1 - \frac{1}{n},$$

a contradiction to the choice of $n$. Thus $\nu(\beta) \geq 1/n$, and hence $\nu(v) \geq 1/n$. □

Thus no 1-dimensional subspace $F_1$ of $K^2$ can fill its desired role in the presentation for $C$.

Theorem 3.6 implies the following simple description of convex sets over spherically complete valued fields.

**Corollary 3.12.** *If $K$ is a spherically complete valued field and $d \in \mathbb{N}_{\geq 1}$, then the nonempty convex subsets of $K^d$ are precisely the affine images of $\nu^{-1}(\Delta_1) \times \cdots \times \nu^{-1}(\Delta_d)$ for some upwards closed $\Delta_1, \ldots, \Delta_d \subseteq \Gamma_\infty$.*

*Proof.* Let $C \subseteq K^d$ be an affine image of $\nu^{-1}(\Delta_1) \times \cdots \times \nu^{-1}(\Delta_d)$ for some upwards closed $\Delta_1, \ldots, \Delta_d \subseteq \Gamma_\infty$. Note that $\nu^{-1}(\Delta_1) \times \cdots \times \nu^{-1}(\Delta_d)$ is convex, and an image of a convex set under an affine map is convex (Example 2.5), hence $C$ is convex.

Conversely, let $\varnothing \neq C \subseteq K^d$ be convex. Since the affine images of $\mathcal{O}$-submodules of $K^d$ give us all nonempty convex sets by Proposition 2.10, without loss of generality $0 \in C$ and $C$ is an $\mathcal{O}$-submodule of $K^d$. Let $\{0\} \subsetneq F_1 \subsetneq \cdots \subsetneq F_d = K^d$ and $\nu_{K^d}(C) = \Delta_1 \supseteq \Delta_2 \supseteq \cdots \supseteq \Delta_d$ be as given by Theorem 3.6 for $C$. Using Lemma 3.1 we can choose $v_1, \ldots, v_d \in K^d$ such that for every $i \in [d]$ we have:

(1) $v_1, \ldots, v_i$ is a basis for $F_i$.

(2) $\nu(v_i) = 0$.

(3) $\nu(v_i + x) \leq 0$ for all $x \in F_{i-1}$.

Then $C$ is the image of $\nu^{-1}(\Delta_1) \times \cdots \times \nu^{-1}(\Delta_d)$ under the linear map $f : K^d \to K^d$ such that $f(e_i) = v_i$, where $e_i$ is the $i$-th standard basis vector. Indeed, if $x \in f(\nu^{-1}(\Delta_1) \times \cdots \times \nu^{-1}(\Delta_d))$ then $x = \sum_{i=1}^d c_i v_i$ for some $c_i$ with $\nu(c_i) \in \Delta_i$. Using (2) this implies $\nu(c_i v_i) = \nu(c_i) \in \Delta_i$, and $c_i v_i \in F_i$, hence $x \in C$. Conversely, let $x$ be an arbitrary element of $C$, then $x = w_1 + \cdots + w_d$ for some $w_i \in F_i$ with $\nu(w_i) \in \Delta_i$. Each $w_i$ is a linear combination of $v_1, \ldots, v_i$, say $w_i = \sum_{j=1}^i c_{i,j} v_j$.

Now we claim that for any $i \in [d]$, $\alpha \in K$ and $v \in F_{i-1}$ we have $\nu(\alpha v_i + v) = \min\{\nu(\alpha v_i), \nu(v)\}$. Indeed, replacing $v$ and $\alpha$ by $\alpha^{-1} v \in F_{i-1}$ and $\alpha^{-1}\alpha \in K$, respectively, changes both sides of the claimed equality by the same amount, hence we may assume that $\alpha = 0$ or $\alpha = 1$. The first case holds trivially, in the second case we need to show that $\nu(v_i + v) = \min\{\nu(v_i), \nu(v)\}$. If $\nu(v_i) \neq \nu(v)$ this holds by the ultrametric inequality, so we assume $\nu(v_i) = \nu(v) = 0$ (using (2)). Then, using (3), $0 \geq \nu(v_i + v) \geq \min\{\nu(v_i), \nu(v)\} = 0$, so $\nu(v_i + v) = 0$ as well.

Applying this claim by induction on $i \in [d]$, we get

$$\nu\left(\sum_{j=1}^{i} c_{i,j} v_j\right) = \min_j\{\nu(c_{i,j} v_j)\},$$

which using (2) implies $\nu(w_i) = \nu\big(\sum_{j=1}^i c_{i,j} v_j\big) = \min_j\{\nu(c_{i,j})\}$ for each $i \in [d]$. As for each $i \in [d]$, we have $\nu(w_i) \in \Delta_i$ and $\Delta_i$ is upwards closed, it follows that

$\nu(c_{i,j}) \in \Delta_i$ for all $i \in [d]$, $j \in [i]$. Regrouping the summands $c_{i,j}v_i$, it follows that $x = w_1 + \cdots + w_d$ is a linear combination of $v_1, \ldots, v_d$ where the coefficient of $v_i$ has valuation in $\Delta_i$, hence $x$ belongs to $f(\nu^{-1}(\Delta_1) \times \cdots \times \nu^{-1}(\Delta_d))$. □

We can eliminate the assumption of spherical completeness of the field when only considering convex hulls of finite sets. We will say that a convex set is *finitely generated* if it is the convex hull of a finite set of points.

**Lemma 3.13.** *A subset $C \subseteq K^d$ is a finitely generated $\mathcal{O}$-module if and only if it is a finitely generated convex set and contains* 0.

*Proof.* If an $\mathcal{O}$-module $C \subseteq K^d$ is generated as an $\mathcal{O}$-module by some finite set $X$, then it is the convex hull of $X \cup \{0\}$. If a set $C$ is the convex hull of some finite set $X$ and contains 0, then it is an $\mathcal{O}$-module by Proposition 2.10, clearly generated as an $\mathcal{O}$-module by $X$. □

We have the following analog of Theorem 3.6 in the finitely generated case over an arbitrary valued field.

**Corollary 3.14.** *Let $K$ be an **arbitrary** valued field and $C$ a **finitely generated** convex set containing* 0. *Then there is a full flag $\{0\} \subsetneq F_1 \subsetneq \cdots \subsetneq F_d = K^d$ and an increasing sequence $\gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_d \in \Gamma_\infty$ such that*

$$C = \{v_1 + \cdots + v_d \mid v_i \in F_i,\ \nu(v_i) \geq \gamma_i\}.$$

*Proof.* Let $C \ni 0$ be the convex hull of some finite set $X \subseteq K^d$. By a repeated application of Proposition 2.8, $C$ is the convex hull of some $d+1$ elements $v_0, \ldots, v_d$ from $X$ (possibly with $x_i = x_j$ for some $i, j$). As $0 \in C$, we have $0 = \sum_{i=0}^d \alpha_i v_i$ for some $\alpha_i \in \mathcal{O}$ with $\sum_{i=0}^d \alpha_i = 1$. Let $j$ be such that $\nu(\alpha_j)$ is minimal among $\{\nu(\alpha_i) : 0 \leq i \leq d\}$. In particular $\alpha_j \neq 0$, hence $v_j = \left(1 - \sum_{i \neq j} \alpha_i/\alpha_j\right)0 + \sum_{i \neq j}(\alpha_i/\alpha_j)v_i$. By the choice of $j$ we have $\alpha_i/\alpha_j \in \mathcal{O}$ for all $i \neq j$, hence also $1 - \sum_{i \neq j} \alpha_i/\alpha_j \in \mathcal{O}$, thus $v_j \in \operatorname{conv}(\{0\} \cup \{v_i : i \neq j\})$, and so also $C = \operatorname{conv}(\{0\} \cup \{v_i : i \neq j\})$. Reordering if necessary, we can thus assume that $C$ is the convex hull of some $\{0, v_1, \ldots, v_d\} \subseteq C$ with $\nu(v_1) \leq \nu(v_i)$ for each $i \in [d]$.

Let $F_1 := \langle v_1 \rangle$ and $\gamma_1 := \nu(v_1)$. Let $\pi_1 : K^d \twoheadrightarrow K^d/F_1$ be the projection map, $f_1 : K^d/F_1 \hookrightarrow K^d$ the valuation preserving embedding given by Lemma 3.1, $V_1 := f_1(K^d/F_1)$ and $\pi_1' := f_1 \circ \pi_1 : K^d \to K^d$.

For $i \geq 2$, as we explained after (3-4) in the proof of Theorem 3.6, we have $v_i - \pi_1'(v_i) \in F_1$; and by (3-3) and our assumption we have $\nu(\pi_1'(v_i)) \geq \nu(v_i) \geq \nu(v_1)$. So $v_i - \pi_1'(v_i) \in \mathcal{O}v_1$ for all $i \geq 2$, which implies

$$\operatorname{conv}(\{0, v_1, \pi_1'(v_2), \ldots, \pi_1'(v_d)\}) = \operatorname{conv}(\{0, v_1, \ldots, v_d\}) = C.$$

Without loss of generality we suppose $\nu(\pi_1'(v_2)) \leq \nu(\pi_1'(v_i))$ for $i \geq 3$, and let $F_2 := \langle v_1, \pi_1'(v_2) \rangle$ and $\gamma_2 := \nu(\pi_1'(v_2)) \geq \nu(v_1) = \gamma_1$ by assumption. By definition of the valuation on the quotient space, using the properties of $f$, we have

$$\nu_K(\pi_1'(v_i)) = \nu_{K^d/F_1}(\pi_1(v_i)) = \nu_{K^d/F_1}(\pi_1(\pi_1'(v_i))) \geq \nu_{K^d}(\pi_1'(v_i) + \alpha v_1)$$

for all $\alpha \in K$. As in the proof of Corollary 3.12, this implies $\nu(\beta\pi_1'(v_i) + \alpha v_1) = \min\{\beta\nu(\pi_1'(v_i)), \nu(\alpha v_1))\}$ for all $i \geq 2$ and $\alpha, \beta \in K$. It follows that

$$\{nv_1 + m\pi_1'(v_2) \mid n, m \in \mathcal{O}\} = \{w_1 + w_2 \mid w_i \in F_i,\ \nu(w_i) \geq \gamma_i\}.$$

To see that the set on the right is contained in the set on the left, assume $x = w_1 + w_2$ for some $w_i \in F_i$, $\nu(w_i) \geq \gamma_i$. Then $w_1 = \alpha_1 v_1$ and $w_2 = \alpha_2 v_1 + \beta\pi_1'(v_2)$ for some $\alpha_1, \alpha_2, \beta \in K$, and by the observation above $\gamma_2 \leq \nu(w_2) = \min\{\nu(\alpha_2 v_1), \nu(\beta\pi_1'(v_2))\}$. So $x = (\alpha_1 + \alpha_2)v_1 + \beta\pi_1'(v_2)$, $\nu((\alpha_1 + \alpha_2)v_1) \geq \gamma_1 = \nu(v_1)$, so $(\alpha_1 + \alpha_2) \in \mathcal{O}$, and $\nu(\beta) \geq \gamma_2$, as wanted.

Now we replace $v_i$ by $\pi_1'(v_i)$ for $i \geq 2$, and let $\pi_2 : K^d \twoheadrightarrow K^d/F_2$ be the projection map, $f_2 : K^d/F_2 \hookrightarrow K^d$ the valuation preserving embedding given by Lemma 3.1, $V_2 := f_2(K^d/F_2)$ and $\pi_2' := f_2 \circ \pi_2 : K^d \to K^d$. For $i \geq 3$, $v_i - \pi_2'(v_i) \in F_2$ and $v_i - \pi_2'(v_i) \in \mathcal{O}v_1 + \mathcal{O}v_2$, so again replacing $v_i$ with $\pi_2'(v_i)$ for $i \geq 3$ does not change the convex hull. Again we may assume $\nu(\pi_2'(v_3)) \leq \nu(\pi_2'(v_i))$ for $i \geq 4$, and let $F_3 := \langle v_1, v_2, v_3\rangle$ and $\gamma_3 := \nu(\pi_2'(v_3))$. Repeating this argument as above $d$ times, we have chosen vectors $v_i$, increasing spaces $F_i = \langle v_1, \dots, v_i\rangle$ and increasing $\gamma_i = \nu(v_i) \in \Gamma$ for $i \in [d]$, so that

$$\begin{aligned} C = \operatorname{conv}(\{0, v_1, \dots, v_d\}) &= \{n_1 v_1 + \cdots + n_d v_d \mid n_i \in \mathcal{O}\} \\ &= \{w_1 + \cdots + w_d \mid w_i \in F_i,\ \nu(w_i) \geq \gamma_i\}. \qquad \square \end{aligned}$$

## 4. Combinatorial properties of convex sets

The following definition is from [Aschenbrenner et al. 2016, Section 2.4].

**Definition 4.1.** Given a set $X$ and $d \in \mathbb{N}_{\geq 1}$, a family of subsets $\mathcal{F} \subseteq \mathcal{P}(X)$ has *breadth $d$* if any nonempty intersection of finitely many sets in $\mathcal{F}$ is the intersection of at most $d$ of them, and $d$ is minimal with this property.

**Lemma 4.2.** *Let $K$ be a valued field and $S$ a convex subset of $K^d$.*

(1) *If $0 \in S$ and $S$ is finitely generated, then it is generated as an $\mathcal{O}$-module by a finite linearly independent set of vectors.*

(2) *Let $\widetilde{K}$ be a valued field extension of $K$ and $\widetilde{S} := \operatorname{conv}_{\widetilde{K}^d}(S) \subseteq \widetilde{K}^d$. Then $\widetilde{S} \cap K^d = S$.*

*Proof.* (1) By Lemma 3.13, $S$ is generated as an $\mathcal{O}$-module by some finite set $v_1, \dots, v_n \in S$. Assume these vectors are not linearly independent, then $0 = \sum_{i\in[n]} \alpha_i v_i$ for $\alpha_i \in K$ not all 0. Let $i \in [n]$ be such that $\nu(\alpha_i) \leq \nu(\alpha_j)$ for all $j \in [n]$, and $\alpha_i \neq 0$. Then $v_i = \sum_{j\neq i}(\alpha_j/(-\alpha_i))v_j$ and $\nu(\alpha_j/(-\alpha_i)) = \nu(\alpha_j) - \nu(\alpha_i) \geq 0$, hence $\alpha_j/(-\alpha_i) \in \mathcal{O}$ for all $j \neq i$, and $S$ is still generated as an $\mathcal{O}$-module by the set $\{v_j : j \neq i\}$. Repeating this finitely many times, we arrive at a linearly independent set of generators.

(2) Since convexity is invariant under translates, we may assume $0 \in S$. Since every element in the convex hull of a set is in the convex hull of some finite subset, we may also assume that $S$ is finitely generated as an $\mathcal{O}$-module, and by (1) let $v_1, \ldots, v_n \in S$ be a linearly independent (in the vector space $K^d$, so $n \leq d$) set of its generators. Let $v_{n+1}, \ldots, v_d \in K^d$ be such that $\{v_i : i \in [d]\}$ is a basis of $K^d$, and say $v_i = (v_{i,j} : j \in [d])$ with $v_{i,j} \in K$. Then the square matrix $A := (v_{i,j} : i, j \in [d]) \in M_{d\times d}(K)$ is invertible, so $A^{-1} \in M_{d\times d}(K) \subseteq M_{d\times d}(\widetilde{K})$, so $A$ is also invertible in $M_{d\times d}(\widetilde{K})$, hence $\{v_i : i \in [d]\}$ are linearly independent vectors in $\widetilde{K}^d$ as well. But now if $\sum_{i\in[n]} \alpha_i v_i = u$ for some $\alpha_i \in \widetilde{K}$ and $u \in K^d$, then necessarily $\alpha_i \in K$ for all $i$ (otherwise we would get a nontrivial linear combination of $v_1, \ldots, v_d$ in $\widetilde{K}^d$). Thus, any element of the $\mathcal{O}_{\widetilde{K}}$-module generated by $v_1, \ldots, v_n$ which is in $K^d$ already belongs to the $\mathcal{O}_K$-module generated by $v_1, \ldots, v_n$, hence $\widetilde{S} \cap K^d = S$. □

We can now demonstrate an (optimal) finite bound on the breadth of the family of convex sets over valued fields. In sharp contrast, over the reals there is no such finite bound already for convex subsets of $\mathbb{R}^2$ (for any $n$, a convex $n$-gon in $\mathbb{R}^2$ is the intersection of $n$ half-planes, but not the intersection of any fewer of them).

**Theorem 4.3.** *Let $K$ be a valued field and $d \geq 1$. Then the family* $\mathrm{Conv}_{K^d}$ *has breadth $d$. That is, any nonempty intersection of finitely many convex subsets of $K^d$ is the intersection of at most $d$ of them.*

*Proof.* The family $\mathrm{Conv}_{K^d}$ cannot have breadth less than $d$ because the $d$ coordinate-aligned hyperplanes are convex, have common intersection $\{0\}$, but any $d-1$ of them intersect in a line.

We now show that $\mathrm{Conv}_{K^d}$ has breadth at most $d$, by induction on $d$. The case $d = 1$ is clear by Example 2.5(1) since for any two quasiballs, they are either disjoint or one is contained in the other. For $d > 1$, assume $C_1, \ldots, C_n \in \mathrm{Conv}_{K^d}$ with $n \geq d$ are convex and satisfy $\bigcap_{i\in[n]} C_i \neq \varnothing$. Translating, we may assume $0 \in \bigcap_{i\in[n]} C_i$.

We may also assume that $K$ is spherically complete. Indeed, let $\widetilde{K}$ be a spherical completion of $K$ as in Fact 3.3, and let $\widetilde{C}_i := \mathrm{conv}_{\widetilde{K}^d}(C_i) \in \mathrm{Conv}_{\widetilde{K}^d}$. By Lemma 4.2(2), $\widetilde{C}_i \cap K^d = C_i$ for each $i \in [n]$. Hence $\bigcap_{i\in[n]} \widetilde{C}_i \neq \varnothing$, and if $\bigcap_{i\in[n]} \widetilde{C}_i = \bigcap_{i\in S} \widetilde{C}_i$ for some $S \subseteq [n]$ with $|S| \leq d$, then also $\bigcap_{i\in[n]} C_i = \bigcap_{i\in S} C_i$.

Then let the vector subspaces $\{0\} \subsetneq F_1 \subsetneq \cdots \subsetneq F_d = K^d$ and the upwards closed subsets $\Delta_1 \supseteq \Delta_2 \supseteq \cdots \supseteq \Delta_d$ of $\Gamma_\infty$ be as given by Theorem 3.6 for the convex set

$C := C_1 \cap \cdots \cap C_n$. By Remark 3.8 we have

$$\Delta_d = \{\gamma \in \Gamma_\infty \mid \forall v \in K^d,\ \nu(v) = \gamma \Longrightarrow v \in C_1 \cap \cdots \cap C_n\}.$$

It follows that there is some $i_d \in [n]$ such that in fact

$$\Delta_d = \{\gamma \in \Gamma_\infty \mid \ \forall v \in K^d,\ \nu(v) = \gamma \Longrightarrow v \in C_{i_d}\}. \tag{4-1}$$

(Since these are finitely many upwards closed sets in $\Gamma$, their intersection is already given by one of them.)

Let $\{0\} \subsetneq F'_1 \subsetneq \cdots \subsetneq F'_d = K^d$ and $\Delta'_1 \supseteq \Delta'_2 \supseteq \cdots \supseteq \Delta'_d$ be as given by Theorem 3.6 for $C_{i_d}$. By Remark 3.10(1), $F'_{d-1}$ is a linear hyperplane so that every element of $C_{i_d}$ differs from an element of $F'_{d-1} \cap C_{i_d}$ by a vector with valuation in $\Delta'_d$. As $\Delta_d = \Delta'_d$ by (4-1) and $C \subseteq C_{i_d}$, by Remark 3.10(1) we may assume that $F_{d-1} = F'_{d-1}$, hence every element in $C_{i_d}$ differs from an element of $F_{d-1} \cap C_{i_d}$ by a vector with valuation in $\Delta_d$.

Consider $C \cap F_{d-1} = C_1 \cap \cdots \cap C_n \cap F_{d-1} = (C_1 \cap F_{d-1}) \cap \cdots \cap (C_n \cap F_{d-1})$. Note that each $C_i \cap F_{d-1}$ is a convex subset of $F_{d-1} \cong K^{d-1}$, so by induction hypothesis there exist $i_1, \ldots, i_{d-1} \in [n]$ such that

$$C_{i_1} \cap \cdots \cap C_{i_{d-1}} \cap F_{d-1} = C_1 \cap \cdots \cap C_n \cap F_{d-1} = C \cap F_{d-1}. \tag{4-2}$$

Let $x \in C_{i_1} \cap \cdots \cap C_{i_d}$ be arbitrary. As $x \in C_{i_d}$, by the choice of $F_{d-1}$, $x = w + v_d$ for some $w \in F_{d-1}$ and $v_d \in K^d$ with $\nu(v_d) \in \Delta_d$. By the choice of $\Delta_d$ we have $v_d \in C_{i_1} \cap \cdots \cap C_{i_d}$. And as each $C_i$ is a module, it follows that also $w \in C_{i_1} \cap \cdots \cap C_{i_d}$. Combining this with (4-2) and using Remark 3.9 (for $j = d-1$) we thus have

$$\begin{aligned} C_{i_1} \cap \cdots \cap C_{i_d} &= \{w + v_d \mid w \in C_{i_1} \cap \cdots \cap C_{i_d} \cap F_{d-1},\ \nu(v_d) \in \Delta_d\} \\ &= \{w + v_d \mid w \in C \cap F_{d-1},\ \nu(v_d) \in \Delta_d\} = \{v_1 + \cdots + v_d \mid v_i \in F_i,\ \nu(v_i) \in \Delta_i\} \\ &= C_1 \cap \cdots \cap C_n. \end{aligned}$$ □

**Definition 4.4.** (1) A family of sets $\mathcal{F} \subseteq \mathcal{P}(X)$ has *Helly number* $k \in \mathbb{N}_{\geq 1}$ if given any $n \in \mathbb{N}$ and any sets $S_1, \ldots, S_n \in \mathcal{F}$, if every $k$-subset of $\{S_1, \ldots, S_n\}$ has nonempty intersection, then $\bigcap_{i \in [n]} S_i \neq \varnothing$.

(2) The *Helly number of* $\mathcal{F}$ refers to the minimal $k$ with this property (or $\infty$ if it does not exist).

(3) We say that $\mathcal{F}$ has the *Helly property* if it has a finite Helly number.

**Theorem 4.5.** *Let $K$ be a valued field and $d \geq 1$. Then the Helly number of* $\mathrm{Conv}_{K^d}$ *is $d+1$.*

*Proof.* The Helly number is bounded by the Radon number minus 1 in an arbitrary convexity space (see Section 5C), but we include a proof for completeness. Let $n$

be arbitrary, and let $S_1, \dots, S_n \subseteq K^d$ be convex sets so that any $d+1$ of them have a nonempty intersection. We will show by induction on $n$ that $S_1 \cap \dots \cap S_n \neq \varnothing$.

*Base case*: $n = d+2$. By assumption for each $i \in [d+2]$ there exists some $x_i \in K^d$ so that $x_i \in \bigcap_{j \in [d+2] \setminus \{i\}} S_j$. By Proposition 2.8 there exists some $i^* \in [d+2]$ so that $x_{i^*} \in \operatorname{conv}(\{x_i \mid i \neq i^*\})$. By the choice of the $x_i$ we have $x_{i^*} \in S_i$ for all $i \neq i^*$. We also have $x_i \in S_{i^*}$ for all $i \neq i^*$, $S_{i^*}$ is convex and $x_{i^*} \in \operatorname{conv}(\{x_i \mid i \neq i^*\})$, hence $x_{i^*} \in S_{i^*}$. Thus $x_{i^*} \in \bigcap_{i \in [d+2]} S_i$, as wanted.

*Inductive step*: $n > d+2$. Let $\widetilde{S}_{n-1} := S_{n-1} \cap S_n$; in particular $\widetilde{S}_{n-1}$ is convex. By induction hypothesis, any $n-1$ sets from $\{S_1, \dots, S_n\}$ have a nonempty intersection. Hence any $n-2$ sets from $\{S_1, \dots, S_{n-2}, \widetilde{S}_{n-1}\}$ have a nonempty intersection. As $n-2 \geq d+1$ by assumption, applying the induction hypothesis again we get

$$S_1 \cap \dots \cap S_n = S_1 \cap \dots \cap S_{n-2} \cap \widetilde{S}_{n-1} \neq \varnothing.$$

This completes the induction, and shows that $\operatorname{Conv}_{K^d}$ has Helly number $d+1$.

It remains to show that $\operatorname{Conv}_{K^d}$ does not have Helly number $d$. Let $e_i \in K^d$ be the $i$-th standard basis vector. The set $E := \{0, e_1, \dots, e_d\}$ is affinely independent, hence the intersection of the affine spans of its $d+1$ maximal proper subsets is empty. The convex hull of a subset of $K^d$ is contained in its affine hull, hence the intersection of the $d+1$ convex hulls of its maximal proper subsets is also empty. But for any $d$ among the $(d+1)$ maximal proper subsets of $E$, some element of $E$ belongs to their intersection, and hence in particular the intersection of their convex hulls is nonempty. ☐

We recall some terminology around the *Vapnik–Chervonenkis dimension* (and refer to [Aschenbrenner et al. 2016, Sections 1 and 2] for further details).

**Definition 4.6.** Let $\mathcal{F} \subseteq \mathcal{P}(X)$ be a family of subsets of $X$.

(1) For a subset $Y \subseteq X$, we let $\mathcal{F} \cap Y := \{S \cap Y : S \in Y\} \subseteq \mathcal{P}(Y)$.

(2) We say that $\mathcal{F}$ *shatters* a subset $Y \subseteq X$ if $\mathcal{F} \cap Y = \mathcal{P}(Y)$.

(3) The *VC dimension* of $\mathcal{F}$, or $\mathrm{VC}(\mathcal{F})$, is the largest $k \in \mathbb{N}$ (if one exists) such that $\mathcal{F}$ shatters some subset of $X$ size $k$. If $\mathcal{F}$ shatters arbitrarily large finite subsets of $X$, then it is said to have infinite VC dimension.

(4) The *dual family* $\mathcal{F}^* \subseteq \mathcal{P}(\mathcal{F})$ is given by $\mathcal{F}^* = \{S_x \mid x \in X\}$, where $S_x = \{A \in \mathcal{F} \mid x \in A\}$.

(5) The *dual VC dimension* of $\mathcal{F}$, or $\mathrm{VC}^*(\mathcal{F})$, is the VC dimension of $\mathcal{F}^*$. Equivalently, it is the largest $k \in \mathbb{N}$ (or $\infty$ if no such $k$ exists) such that there are sets $S_1, \dots, S_k \in \mathcal{F}$ that generate a Boolean algebra with $2^k$ atoms, i.e., for any distinct $I, J \subseteq [k]$, $\bigcap_{i \in I} S_i \cap \bigcap_{i \in [k] \setminus I} (X \setminus S_i) \neq \bigcap_{i \in J} S_i \cap \bigcap_{i \in [k] \setminus J} (X \setminus S_i)$.

(6) The *shatter function* $\pi_{\mathcal{F}} : \mathbb{N} \to \mathbb{N}$ of $\mathcal{F}$ is

$$\pi_{\mathcal{F}}(n) := \max\{|\mathcal{F} \cap Y| : Y \subseteq X, |Y| = n\}.$$

(7) By the Sauer–Shelah lemma (see for instance [Aschenbrenner et al. 2016, Lemma 2.1]), if $\mathrm{VC}(\mathcal{F}) \leq d$, then $\pi_{\mathcal{F}}(n) \leq (e/d)^d n^d$ for all $n \geq d$ (and $\pi_{\mathcal{F}}(n) = 2^n$ for all $n$ if $\mathrm{VC}(\mathcal{F}) = \infty$).

(8) The *VC density* of $\mathcal{F}$, or $\mathrm{vc}(\mathcal{F})$, is the infimum of all $r \in \mathbb{R}_{>0}$ such that $\pi_{\mathcal{F}}(n) = O(n^r)$, and $\infty$ if there is no such $r$. (In particular $\mathrm{vc}(\mathcal{F}) \leq \mathrm{VC}(\mathcal{F})$.)

(9) Finally, we define the *dual shatter function* $\pi^*_{\mathcal{F}} := \pi_{\mathcal{F}^*}$ and the *dual* VC-*density* $\mathrm{vc}^*(\mathcal{F}) := \mathrm{vc}(\mathcal{F}^*)$ of the family $\mathcal{F}$.

**Remark 4.7.** Note that if $\mathcal{F} \subseteq \mathcal{P}(X)$ and $Y \subseteq X$, then $\mathrm{VC}(\mathcal{F} \cap Y) \leq \mathrm{VC}(\mathcal{F})$ and $\mathrm{VC}^*(\mathcal{F} \cap Y) \leq \mathrm{VC}^*(\mathcal{F})$.

The following results are in stark contrast with the situation for the family of convex sets over the reals, where already the family of convex subsets of $\mathbb{R}^2$ has infinite VC dimension (e.g., any set of points on a circle is shattered by the family of convex hulls of its subsets).

**Theorem 4.8.** *Let $K$ be a valued field and $d \geq 1$. Then the family* $\mathrm{Conv}_{K^d}$ *has VC dimension $d+1$.*

*Proof.* We have $\mathrm{VC}(\mathrm{Conv}_{K^d}) \geq d+1$ since the set $E := \{0, e_1, \dots, e_d\} \subseteq K^d$, with $e_i$ the $i$-th vector of the standard basis, is shattered by $\mathrm{Conv}_{K^d}$. Indeed, the convex hull of any subset is contained in its affine span, and for any $S \subseteq E$, $\mathrm{aff}(S)$ does not contain any of the points in $E \setminus S$.

On the other hand, $\mathrm{VC}(\mathrm{Conv}_{K^d}) \leq d+1$ as no subset $Y$ of $K^d$ with $|Y| \geq d+2$ can be shattered by $\mathrm{Conv}_{K^d}$. Indeed, by Proposition 2.8, at least one of the points of $Y$ belongs to every convex set containing all the other points of $Y$. □

The dual VC dimension of a family of sets is bounded by its breadth.

**Fact 4.9** [Aschenbrenner et al. 2016, Lemma 2.9]. *Let $\mathcal{F} \subseteq \mathcal{P}(X)$ be a family of subsets of $X$ of breadth at most $d$. Then also* $\mathrm{VC}^*(\mathcal{F}) \leq d$.

Using this fact, we get the following:

**Theorem 4.10.** *For any valued field $K$ and $d \geq 1$, the family* $\mathrm{Conv}_{K^d}$ *has dual VC dimension $d$.*

*Proof.* The dual VC dimension of $\mathrm{Conv}_{K^d}$ is at least $d$ because the $d$ coordinate-aligned (convex) hyperplanes in $K^d$ generate a Boolean algebra with $2^d$ atoms.

Conversely, the breadth of $\mathrm{Conv}_{K^d}$ is $d$ by Theorem 4.3, hence by Fact 4.9 its dual VC dimension is also at most $d$. □

**Definition 4.11.** (1) A family of sets $\mathcal{F} \subseteq \mathcal{P}(X)$ has *fractional Helly number* $k \in \mathbb{N}_{\geq 1}$ if for every $\alpha \in \mathbb{R}_{>0}$ there exists $\beta \in \mathbb{R}_{>0}$, so that for any $n \in \mathbb{N}$ and any sets $S_1, \ldots, S_n \in \mathcal{F}$ (possibly with repetitions), if there are at least $\alpha\binom{n}{k}$ $k$-element subsets of the multiset $\{S_1, \ldots, S_n\}$ with a nonempty intersection, then there are at least $\beta n$ sets from $\{S_1, \ldots, S_n\}$ with a nonempty intersection.

(2) The *fractional Helly number* of $\mathcal{F}$ refers to the minimal $k$ with this property. We say that $\mathcal{F}$ has the *fractional Helly property* if it has a fractional Helly number.

Note that any finite family of sets trivially has fractional Helly number 1 by choosing $\beta$ sufficiently small with respect to the size of $\mathcal{F}$. We will use the following theorem of Matoušek.

**Fact 4.12** [Matoušek 2004, Theorem 2]. *Let $\mathcal{F} \subseteq \mathcal{P}(X)$ be a set system whose dual shatter function satisfies $\pi^*_{\mathcal{F}}(n) = o(n^k)$, i.e., $\lim_{n\to\infty} \pi^*_{\mathcal{F}}(n)/n^k = 0$, where $k$ is a fixed integer. Then $\mathcal{F}$ has fractional Helly number $k$.*

**Remark 4.13.** Moreover, if $\mathrm{VC}^*(\mathcal{F}) = d < \infty$, then the fractional Helly number is at most $d+1$, and the $\beta$ witnessing this can be chosen depending only on $d$ and $\alpha$ (and not on the family $\mathcal{F}$).

Indeed, by Definition 4.6, if $\mathrm{VC}^*(\mathcal{F}) \leq d$, then $\pi^*_{\mathcal{F}}(n) \leq (e/d)^d n^d$ for all $n \geq d$, hence $\pi^*_{\mathcal{F}}(n) \leq cn^d$ for all $n \in \mathbb{N}$, where $c = c(d) := (e/d)^d + 2^d$. We can choose $m = m(d, \alpha)$, so that $\pi^*_{\mathcal{F}}(m) < \frac{1}{4}\alpha\binom{m}{d+1}$. Then it follows from the proof of [Matoušek 2004, Theorem 2] that $\beta = 1/(2m)$ works for all $n \geq m/\beta = 2m^2$, and trivially $\beta = 1/(2m^2)$ works for all $n \leq 2m^2$, hence $\beta := \beta(\alpha, d) := 1/(2m^2)$ works for all $n \in \mathbb{N}$.

Using this, we get the following:

**Theorem 4.14.** *If $K$ is a valued field, $d \geq 1$, and $X \subseteq K^d$ is an arbitrary subset, then the fractional Helly number of the family*

$$\mathrm{Conv}_{K^d} \cap X = \{C \cap X : C \in \mathrm{Conv}_{K^d}\} \subseteq \mathcal{P}(X)$$

*is at most $d+1$. Moreover, $\beta$ in Definition 4.11 can be chosen depending only on $d$ and $\alpha$ (and not on the field $K$ or set $X$). And if $K$ is infinite, then the fractional Helly number of the family $\mathrm{Conv}_{K^d}$ is exactly $d+1$.*

*Proof.* By Fact 4.12 we have that the fractional Helly number of a set system is at most the smallest integer larger than its dual VC density. Dual VC density is, in turn, at most its dual VC dimension. Also for any set $X \subseteq K^d$ we have $\mathrm{VC}^*(\mathrm{Conv}_{K^d} \cap X) \leq \mathrm{VC}^*(\mathrm{Conv}_{K^d})$ by Remark 4.7. So $\mathrm{Conv}_{K^d} \cap X$ has dual VC density at most $d$ by Theorem 4.10, hence its fractional Helly number is at most $d+1$ by Fact 4.12. And an appropriate $\beta$ can be chosen depending only on $d$ and $\alpha$ by Remark 4.13.

To show that the fractional Helly number of $\mathrm{Conv}_{K^d}$ is at least $d+1$ when $K$ is infinite, we can use the standard example with affine hyperplanes in general position. We include the details for completeness. First note that as the field $K$ is infinite, for any $K$-vector space $V$ of dimension $k$ and $v \in V \setminus \{0\}$ there exists an infinite set $S \subseteq V$ so that $v \in S$ and any $k$ vectors from $S$ are linearly independent. This is clear for $k = 1$ by taking any infinite set of nonzero vectors, so assume that $k > 1$. By induction on $i \in \mathbb{N}_{\geq k}$ we can find sets $S_i$ such that $v \in S_i$, $|S_i| \geq i$ and every $k$ vectors from $S_i$ are linearly independent, for all $i$. Let $S_k$ be any basis of $V$ containing $v$. Assume $i > k$ and $S_i$ satisfies the assumption. Since $K$ is infinite, $V$ is not a union of finitely many proper subspaces; in particular there exists some

$$w \in V \setminus \bigcup_{s \subseteq S_i, |s|=k-1} \langle s \rangle.$$

Let $S_{i+1} := S_i \cup \{w\}$. Since any $s \subseteq S_i$ with $|s| = k-1$ is linearly independent by the inductive assumption, it follows that $s \cup \{w\}$ is also linearly independent, hence $S_{i+1}$ satisfies the assumption. Finally, $S := \bigcup_{i \in \mathbb{N}_{\geq k}} S_i$ is as wanted.

In particular, we can find an infinite set of vectors $S$ in $K^d \times K$ so that any $d+1$ of them are linearly independent and the standard basis vector $e_{d+1} \in S$. Then

$$X := \{\langle v, - \rangle : v \in S\} \subseteq (K^d \times K)^*$$

is an infinite set of dual vectors such that any $d+1$ of them are linearly independent, and it contains the projection map onto the last coordinate $\pi_{d+1} := \langle e_{d+1}, - \rangle : (x_1, \ldots, x_{d+1}) \mapsto x_{d+1}$. Consider the family

$$\mathcal{H} := \{\ker(f) \mid f \in X \setminus \{\pi_{d+1}\}\} \subseteq \mathcal{P}(K^d \times K)$$

of kernels of these dual vectors (excluding the projection map onto the last coordinate), and let

$$\mathcal{H}' := \{\{v \in K^d \mid (v, 1) \in H\} \mid H \in \mathcal{H}\} \subseteq \mathcal{P}(K^d).$$

Then $\mathcal{H}'$ is an infinite family of affine hyperplanes in $K^d$, and we wish to show that any $d$ elements of $\mathcal{H}'$ intersect in a point, and any $d+1$ elements of $\mathcal{H}'$ have empty intersection. For any pairwise distinct $f_1, \ldots, f_d \in X \setminus \{\pi_{d+1}\}$, by linear independence,

$$\dim(\ker(f_1) \cap \cdots \cap \ker(f_d)) = d + 1 - \dim(\langle f_1, \ldots, f_d \rangle) = 1.$$

And by their linear independence with $\pi_{d+1}$,

$$\dim(\ker(f_1) \cap \cdots \cap \ker(f_d) \cap \ker(\pi_{d+1})) = 0.$$

That is, $\ker(f_1) \cap \cdots \cap \ker(f_d)$ is a line in $K^d \times K$ that intersects $\ker(\pi_{d+1}) = K^d \times \{0\}$ only at the origin, and thus must also intersect $K^d \times \{1\}$ in a single point;

this shows that every $d$ elements of $\mathcal{H}'$ intersect in a point. And any pairwise distinct $f_1, \ldots, f_{d+1} \in X \setminus \{\pi_{d+1}\}$ span $(K^d \times K)^*$ by linear independence, so $\ker(f_1) \cap \cdots \cap \ker(f_{d+1}) = \{0\}$, and thus has empty intersection with $K^d \times \{1\}$. This shows that every $d+1$ elements of $\mathcal{H}'$ have empty intersection.

Using $\alpha = 1$, for any $\beta > 0$, take an arbitrary $n \geq (d+1)/\beta$. Let $H_1, \ldots, H_n \in \mathcal{H}'$ be any distinct hyperplanes from this collection. All $d$-subsets, $\alpha\binom{n}{d}$ of them, of $\{H_1, \ldots, H_n\}$ have an intersection point, but there are no $\beta n \geq d+1$ of them with a common intersection point. Therefore $\mathrm{Conv}_{K^d}$ does not have fractional Helly number $d$. □

Note that Theorems 4.5 and 4.14 replicate results for real convex sets, while Theorems 4.3, 4.8, and 4.10 do not: as we have already remarked, $\mathrm{Conv}_{\mathbb{R}^2}$ has infinite breadth, VC dimension, and dual VC dimension. The following result is somewhere in between. The classical Tverberg theorem says that for any $X \subseteq \mathbb{R}^d$ with $|X| \geq (d+1)(r-1)+1$, $X$ can be partitioned into $r$ disjoint subsets $X_1, \ldots, X_r$ whose convex hulls intersect: $\mathrm{conv}(X_1) \cap \cdots \cap \mathrm{conv}(X_r) \neq \varnothing$. Over valued fields, we obtain a much stronger version (any element of the nonempty set $X_r$ in the statement of Theorem 4.15 belongs to the convex hulls of each of the sets $X_i$, $i \in [r]$ — which gives the usual conclusion of Tverberg's theorem over the reals):

**Theorem 4.15.** *Let $K$ be a valued field and $d, r \in \mathbb{N}_{\geq 1}$. Then any set $X \subseteq K^d$ with*

$$|X| \geq (d+1)(r-1)+1$$

*points in $K^d$ can be partitioned into subsets $X_1, \ldots, X_r$ such that $|X_i| = d+1$ for $i < r$, $|X_r| = |X| - (d+1)(r-1)$, and $\mathrm{conv}(X_i) \supseteq \mathrm{conv}(X_j)$ for all $i \leq j \in [r]$.*

*Proof.* Since any finitely generated convex set is the convex hull of some $d+1$ points from it by Corollary 2.9, we can find $X_1 \subseteq X$ with $|X_1| = d+1$ and $\mathrm{conv}(X_1) = \mathrm{conv}(X)$, $X_2 \subseteq X \setminus X_1$ with $|X_2| = d+1$ and $\mathrm{conv}(X_2) = \mathrm{conv}(X \setminus X_1)$, and so on: once $X_1, \ldots, X_{i-1}$ have been chosen, pick $X_i \subseteq X \setminus \left(\bigcup_{j=1}^{i-1} X_j\right)$ such that $|X_i| = d+1$, $\mathrm{conv}(X_i) = \mathrm{conv}\left(X \setminus \bigcup_{j=1}^{i-1} X_j\right)$, and then let $X_r$ consist of everything left over at the end. □

From this strong Tverberg theorem and the fractional Helly property, we finally get an analog of the result due to Boros and Füredi [1984] and Bárány [1982] on the common points in the intersections of many "simplices" over valued fields. Note that the conclusion is actually stronger than over the reals: the common point comes from the set $X$ itself. This answers a question asked by Kobi Peterzil and Itay Kaplan. Our argument is an adaptation of the second proof in [Matoušek 2002, Theorem 9.1.1].

**Theorem 4.16.** *For each $d \geq 1$ there is a constant $c = c(d) > 0$ such that for any valued field $K$ and any finite $X \subseteq K^d$ (say $n := |X|$), there is some $a \in X$ contained in the convex hulls of at least $c\binom{n}{d+1}$ of the $\binom{n}{d+1}$ subsets of $X$ of size $d+1$.*

*Proof.* Let $X \subseteq K^d$ with $|X| = n$ be given, and let

$$\mathcal{F} := \mathrm{Conv}_{K^d} \cap X = \{C \cap X : C \in \mathrm{Conv}_{K^d}\}$$

be the family of all subsets of $X$ cut out by the convex subsets of $K^d$. Let $(S_i)_{i \in [N]}$ with $S_i \in \mathrm{Conv}_{K^d}$ be the sequence listing all convex hulls of subsets of $X$ of size $d+1$ in an arbitrary order (possibly with repetitions). Then $N = \binom{n}{d+1}$, and for a $(d+1)$-element subset $Y \subseteq X$ we let $g(Y) \in [N]$ be the index at which $\mathrm{conv}(Y)$ appears in this sequence. For each $i \in [N]$ let $S_i' := S_i \cap X \in \mathcal{F}$. It is thus sufficient to show that there exists some $\alpha > 0$, depending only on $d$, such that at least $\alpha\binom{N}{d+1}$ of the $(d+1)$-element subsets $I \subseteq [N]$ satisfy $\bigcap_{i \in I} S_i' \neq \varnothing$ — as then Theorem 4.14 applied to $\mathcal{F} \subseteq \mathcal{P}(X)$ shows the existence of $c > 0$ depending only on $\alpha$ and $d$, and hence only on $d$, so that for some $I \subseteq [N]$ with $|I| \geq cN = c\binom{n}{d+1}$ there exists some $a \in \bigcap_{i \in I} S_i' \subseteq \bigcap_{i \in I} S_i$ (in particular $a \in X$).

Now we find an appropriate $\alpha$. For any $(d+1)^2$-element subset $Y \subseteq X$, by Theorem 4.15 (with $r := d+1$), we can fix a partition of $Y$ into $d+1$ disjoint parts $Y_1, \ldots, Y_{d+1}$, each of which having $d+1$ elements, and so that $\mathrm{conv}(Y_i) \supseteq \mathrm{conv}(Y_j)$ for all $i \leq j \in [d+1]$. In particular any element of the nonempty set $Y_{[d+1]} \subseteq X$ belongs to $\bigcap_{i \in [d+1]} (\mathrm{conv}(Y_i) \cap X) = \bigcap_{i \in [d+1]} (S'_{g(Y_i)})$. As $g$ is a bijection, $Y \mapsto \{g(Y_i) : i \in [d+1]\}$ gives a function $f$ from $(d+1)^2$-element subsets of $X$ to $(d+1)$-element subsets $I \subseteq [N]$, so that $\bigcap_{i \in I} S_i' \neq \varnothing$. Moreover, $f$ is an injection. Indeed, given a set $\{j_i : i \in [d+1]\}$ in the image of $f$, as $g$ is a bijection, there is a unique set $\{Y_1, \ldots, Y_{d+1}\}$ with $Y_i \subseteq X$ disjoint of size $d+1$, so that $g(Y_i) = j_i$ for all $i \in [d+1]$, and there can be only one set $Y \subseteq X$ of size $(d+1)^2$ for which it is a partition. If follows that the number of sets $I \subseteq [N]$ with $\bigcap_{i \in I} S_i' \neq \varnothing$ is at least

$$\binom{n}{(d+1)^2} = \Omega(n^{(d+1)^2}) \geq \alpha\binom{N}{d+1}$$

for some sufficiently small $\alpha$ depending only on $d$. □

## 5. Final remarks and questions

**5A.** ***Some further results and future directions.*** The results of Section 4 imply the following analog of the celebrated $(p, q)$-*theorem* of Alon and Kleitman [1992] for convex sets over valued fields.

**Corollary 5.1.** *For any $d, p, q \in \mathbb{N}_{\geq 1}$ with $p \geq q \geq d+1$ there exists $T = T(p, q, d) \in \mathbb{N}$ such that if $K$ is a valued field and $\mathcal{F}$ is a family of convex subsets of $K^d$ such that among every $p$ sets of $\mathcal{F}$, some $q$ have a nonempty intersection, then there exists a $T$-element set $Y \subseteq K^d$ intersecting all sets of $\mathcal{F}$.*

Corollary 5.1 follows formally by applying [Alon et al. 2002, Theorem 8] since the family $\mathrm{Conv}_{K^d}$ has fractional Helly property (Theorem 4.14) and is closed under intersections. Alternatively, it follows with a slightly better bound on $T$ by combining the fractional Helly property with the existence of $\varepsilon$-nets for families of bounded VC dimension (Theorem 4.8), as outlined at the end of [Matoušek 2004, Section 1]. The problem of determining the optimal bound on $T(p, q, d)$ is widely open over the reals (see [Bárány and Kalai 2022, Section 2.6]), and we expect that it might be easier in the valued fields setting.

Kalai [1984] and Eckhoff [1985] proved that in the fractional Helly property for convex sets over the reals, one can take $\beta(d, \alpha) = 1 - (1 - \alpha)^{1/(d+1)}$ (and this bound is sharp).

**Problem 5.2.** What is the optimal dependence of $\beta$ on $d, \alpha$ in Theorem 4.14?

Over $\mathbb{R}$, Sierksma's Dutch cheese conjecture predicts a lower bound for the number of Tverberg partitions (see for instance [De Loera et al. 2019, Conjecture 3.12]). We expect the same bound to hold over valued fields:

**Conjecture 5.3.** For any valued field $K$ and $X \subset K^d$ with $|X| = (r-1)(d+1)+1$, there are at least $((r-1)!)^d$ partitions of $X$ into parts whose convex hulls intersect.

**Remark 5.4.** In Theorem 4.15, we showed the existence of Tverberg partitions satisfying the stronger property that the convex hulls of the parts are linearly ordered by inclusion. It is not true that for $X \subseteq K^d$ with $|X| = (d+1)(r-1)+1$, there are at least $((r-1)!)^d$ different ways of partitioning $X$ into $X_1, \ldots, X_r$ such that $\operatorname{conv}(X_1) \supseteq \cdots \supseteq \operatorname{conv}(X_r)$. Thus any attempt to prove Conjecture 5.3 would have to involve other Tverberg partitions that do not have this property. For an example in $K^2$ where this bound fails, let $x \in K$ with $\nu(X) \neq 0$, and let $X := \{(x^n, x^{-n}) \mid n \in [3(r-1)+1]\}$. For any partition of $X$ into $X_1, \ldots, X_r$ such that $\operatorname{conv}(X_1) \supseteq \cdots \supseteq \operatorname{conv}(X_r)$, for each $i < r$, $X_i$ must consist of the points corresponding to the lowest and highest values of $n$ among all points not already in $X_1 \cup \cdots \cup X_{i-1}$, together with one of the other $3(r-i)-1$ remaining points, and $X_r$ must consist of whatever point is left over. So the number of partitions of $X$ of this form is $\prod_{i=1}^{r-1}(3(r-i)-1) < \prod_{i=1}^{r-1} 3(r-i) = 3^{r-1}(r-1)! < ((r-1)!)^2$ for large enough $r$.

We expect that the *colorful* Tverberg theorem also holds over valued fields, however the proofs for convex sets over $\mathbb{R}$ rely on topological arguments not readily available in the valued field context:

**Conjecture 5.5.** For any integers $r, d \geq 2$ there exists $t \geq r$ such that for any valued field $K$ and $X \subseteq K^d$ with $|X| = t(d+1)$, partitioned into $d+1$ color classes $C_1, \ldots, C_{d+1}$ each of size $t$, there exist pairwise disjoint $X_1, \ldots, X_r \subseteq X$ with $|X_i \cap C_j| = 1$ for $i \in [r]$ and $j \in [d+1]$, and $\bigcap_{i \in [r]} \operatorname{conv}(X_i) \neq \varnothing$.

It would formally imply (see [Matoušek 2002, Section 9.2]) the "second selection lemma" over valued fields generalizing Theorem 4.16:

**Conjecture 5.6.** For each $d \in \mathbb{N}_{\geq 1}$ there exist $c, s > 0$ such that for any valued field $K$, $\alpha \in (0, 1]$ and $n \in \mathbb{N}$, for every $X \subseteq K^d$ with $|X| = n$, and every family $\mathcal{F}$ of $(d+1)$-element subsets of $X$ with $|\mathcal{F}| \geq \alpha\binom{n}{d+1}$, there is a point contained in the convex hulls of at least $c\alpha^s\binom{n}{d+1}$ of the elements of $\mathcal{F}$.

Corollary 3.12 has the following immediate model-theoretic application.

**Remark 5.7.** If $K$ is a spherically complete valued field, then every convex subset of $K^d$ is definable in the expansion of the field $K$ by a predicate for each Dedekind cut of the value group (so in particular definable in *Shelah expansion of $K$* by all externally definable sets [Shelah 2009; Chernikov and Simon 2013]). And conversely, every Dedekind cut of the value group is definable in the expansion of $K$ by a predicate for each $\mathcal{O}$-submodule of $K$. In particular, if $K$ has value group $\mathbb{Z}$, then all convex subsets of $K^d$ form a definable family.

**Example 5.8.** In contrast, naming a single (bounded) convex subset of $\mathbb{R}^2$ in the field of reals allows to define the set of integers. Indeed, we can define a continuous and piecewise linear function $f : [0, 1] \to [0, 1]$ such that

$$C := \{(x, y) : x \in [0, 1],\ 0 \leq y \leq f(x)\}$$

is convex but the set of points where $f$ is not differentiable is exactly $\{1/n : n \in \mathbb{N}_{\geq 2}\}$. Now in the field of reals with a predicate for $C$ we can define $f$ and the set of points where it is not differentiable, hence $\mathbb{N}$ is also definable.

**5B. *Other notions of convexity over nonarchimedean fields.*** We briefly overview several other kinds of convexities over nonarchimedean fields considered in the literature. The extension of Hilbert (projective) geometry to convex sets in a generalized sense is a topic of high current interest, see for instance [Guilloux 2016]. In a different spirit, in tropical geometry, convex sets over real closed nonarchimedean fields have been considered (unlike what is done here, this leads to a combinatorial convexity similar to the classical one, since by Tarski's completeness theorem, polyhedral properties of a combinatorial nature are the same over all real closed fields). Moreover, tropical polyhedra are obtained as images of such polyhedra by the nonarchimedean valuation, see for instance [Develin and Yu 2007]. Polytopes and simplexes in $p$-adic fields are introduced in [Darnière 2017; 2019], and demonstrated to play in $p$-adically closed fields the role played by real simplexes in the classical results of triangulation of semialgebraic sets over real closed fields. Although we are not aware of any direct link of these results with the present work, we hope for some connections to be found in the future.

**5C. *Abstract convexity spaces.*** Our results here can be naturally placed in the context of abstract convexity spaces; we refer to [van de Vel 1993] for an introduction to the subject. A *convexity space* is a pair $(X, \mathcal{C})$, where $X$ is a set and $\mathcal{C} \subseteq 2^X$ is a family of subsets of $X$ closed under intersection with $\varnothing, X \in \mathcal{C}$. The sets in $\mathcal{C}$ are called *convex*. Given a subset $Y \subseteq X$, the *convex hull* of $Y$, denoted $\operatorname{conv}(Y)$, is the smallest set in $\mathcal{C}$ containing $Y$ (equivalently, the intersection of all sets in $\mathcal{C}$ containing $Y$). A convex set $C \in \mathcal{C}$ is called a *half-space* if its complement is also convex. The convexity space $(X, \mathcal{C})$ is *separable* if for every $C \in \mathcal{C}$ and $x \in X \setminus C$, there exists a half-space $H \in \mathcal{C}$ such that $C \subseteq H$ and $x \notin H$ (equivalently, if every convex set is the intersection of all half-spaces containing it). Separability is an abstraction of the hyperplane separation (and more generally Hahn–Banach) theorem. In particular, $(\mathbb{R}^d, \operatorname{Conv}_{\mathbb{R}^d})$ is a separable convexity space (see [Moran and Yehudayoff 2019, Section 1.1] or [van de Vel 1993] for many other examples). The *Radon number*[1] of a convexity space $(X, \mathcal{C})$ is the smallest $k \in \mathbb{N}_{\geq 1}$ (if it exists) such that every $Y \subseteq X$ with $|Y| > k$ can be partitioned into two parts $Y_1, Y_2$ such that $\operatorname{conv}(Y_1) \cap \operatorname{conv}(Y_2) \neq \varnothing$. The classical Radon's theorem states that the Radon number of $(\mathbb{R}^d, \operatorname{Conv}_{\mathbb{R}^d})$ equals $d+1$. Given $\varnothing \neq Y \subseteq X$, a partition $Y_1, \ldots, Y_r$ of $Y$ is *Tverberg* if $\bigcap_{i=1}^r \operatorname{conv}(Y_i) \neq \varnothing$. The *$r$-th Tverberg number* of $(X, \mathcal{C})$ is the smallest $k$ such that every $Y \subseteq X$ with $|Y| > k$ has a Tverberg partition in $r+1$ parts. Note that the first Tverberg number is the Radon number, and the classical theorem of Tverberg says that the $r$-th Tverberg number of $(\mathbb{R}^d, \operatorname{Conv}_{\mathbb{R}^d})$ is $r(d+1)$.

Now let $K$ be a valued field and $d \in \mathbb{N}_{\geq 1}$. Then $(K^d, \operatorname{Conv}_{K^d})$ is a convexity space, but we stress that it is *not separable*; in fact, $\varnothing$ and $K^d$ are the only half-spaces. This is because for any nonempty proper convex set $C$, if we let $x \in C$, $y \in K^d \setminus C$, and $\alpha \in K \setminus \mathcal{O}$, then $z := x + \alpha(y-x) \notin C$, since $y = \alpha^{-1} z + (1-\alpha^{-1})x$ is a convex combination. But then $x = (1-\alpha)^{-1}(z - \alpha y)$ is a convex combination of elements of $K^d \setminus C$, so $K^d \setminus C$ is not convex.

Proposition 2.8 implies that the Radon number of $(K^d, \operatorname{Conv}_{K^d})$ is $d+1$. By the Levi inequality in an arbitrary convexity space [van de Vel 1993, Chapter II(1.9)], it follows that the Helly number of $\operatorname{Conv}_{K^d}$ (Definition 4.4) is at most $d+1$ (we included a proof in Theorem 4.5 for completeness). It was also recently shown in [Holmsen and Lee 2021] that in any convexity space $(X, \mathcal{C})$ with Radon number $k$, $\mathcal{C}$ has a fractional Helly number (Definition 4.11) bounded by some function of $k$. In the case of $(K^d, \operatorname{Conv}_{K^d})$ this general bound is much weaker than the optimal bound $d+1$ given in Theorem 4.14. Corollary 2.9 implies that the Carathéodory number of $(K^d, \operatorname{Conv}_{K^d})$ is $d+1$ (see [van de Vel 1993, Chapter II(1.5)] for the definition). Finally, Theorem 4.15 implies that the $r$-th Tverberg number of $(K^d, \operatorname{Conv}_{K^d})$ is $r(d+1)$; finiteness of the $r$-th Tverberg numbers for all $r$ follows from the finiteness of the Radon number in an arbitrary convexity space, with a much weaker bound [van de Vel 1993, Chapter II(5.2)].

[1] An alternative definition uses $\geq$ instead of $>$, leading to a value higher by 1. The definition here follows [van de Vel 1993, Chapter II].

## Acknowledgements

We thank the referees for many very helpful literature pointers and suggestions on improving the paper. Sections 5B and 5C were added following their suggestions. We thank Lou van den Dries for pointing out Monna's work to us, Dave Marker for pointing out Example 5.8, and Matthias Aschenbrenner for a helpful conversation. Both authors were partially supported by the NSF CAREER grant DMS-1651321, and Chernikov was additionally supported by a Simons fellowship.

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Received September 16, 2021. Revised January 13, 2023.

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The Pacific Journal of Mathematics (ISSN 1945-5844 electronic, 0030-8730 printed) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

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# PACIFIC JOURNAL OF MATHEMATICS

Volume 323 No. 1 March 2023


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