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# GENERALISATIONS OF HECKE ALGEBRAS FROM LOOP BRAID GROUPS

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**We introduce a generalisation $\mathrm{LH}_n$ of the ordinary Hecke algebras informed by the loop braid group $\mathrm{LB}_n$ and the extension of the Burau representation thereto. The ordinary Hecke algebra has many remarkable arithmetic and representation theoretic properties, and many applications. We show that $\mathrm{LH}_n$ has analogues of several of these properties. In particular we consider a class of local (tensor space/functor) representations of the braid group derived from a meld of the (nonfunctor) Burau representation (1935) and the (functor) Deguchi et al., Kauffman and Saleur, and Martin and Rittenberg representations here called Burau–Rittenberg representations. In its most supersymmetric case somewhat mystical cancellations of anomalies occur so that the Burau–Rittenberg representation extends to a loop Burau–Rittenberg representation. And this factors through $\mathrm{LH}_n$. Let $\mathrm{SP}_n$ denote the corresponding (not necessarily proper) quotient algebra, $k$ the ground ring, and $t \in k$ the loop-Hecke parameter. We prove the following:**

**(1) $\mathrm{LH}_n$ is finite dimensional over a field.**

**(2) The natural inclusion $\mathrm{LB}_n \hookrightarrow \mathrm{LB}_{n+1}$ passes to an inclusion $\mathrm{SP}_n \hookrightarrow \mathrm{SP}_{n+1}$.**

**(3) Over $k = \mathbb{C}$, $\mathrm{SP}_n$ / rad is generically the sum of simple matrix algebras of dimension (and Bratteli diagram) given by Pascal's triangle. (Specifically $\mathrm{SP}_n$ / rad $\cong \mathbb{C}S_n/e^1_{(2,2)}$ where $S_n$ is the symmetric group and $e^1_{(2,2)}$ is a $\lambda = (2,2)$ primitive idempotent.)**

**(4) We determine the other fundamental invariants of $\mathrm{SP}_n$ representation theory: the Cartan decomposition matrix; and the quiver, which is of type-A.**

**(5) The structure of $\mathrm{SP}_n$ is independent of the parameter $t$, except for $t = 1$.**

**(6) For $t^2 \neq 1$ then $\mathrm{LH}_n \cong \mathrm{SP}_n$ at least up to rank $n = 7$ (for $t = -1$ they are not isomorphic for $n > 2$; for $t = 1$ they are not isomorphic for $n > 1$).**

**Finally we discuss a number of other intriguing points arising from this construction in topology, representation theory and combinatorics.**

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*Keywords:* Loop braid group, Hecke algebra, charge conservation.

## 1. Introduction

Until the 1980s, methods to construct linear representations of the braid group $B_n$ were relatively scarce. We have those factoring through the symmetric group and the Burau representation [1935], and those factoring through the Hecke algebra [Hoefsmit 1974] and the Temperley–Lieb algebra [Temperley and Lieb 1971]; and, as for every group, the closure in the monoidal category $\mathrm{Rep}(B_n)$. These proceed essentially through "combinatorial" devices such as Artin's presentation. Then there are some more intrinsically "topological" constructions such as Artin's representation [1947] (and Burau can be recast in this light [Long and Paton 1993]).

In the 80s there were notable steps forward. Algebraic formulations of the Yang–Baxter equation began to yield representations; see e.g., [Baxter 1982]. Jones' discovery [1986] of link invariants from finite dimensional quotients of the group algebra $\mathbb{K}[B_n]$ inspired a revolution in braid group representations and topological invariants [Kauffman 1990; Birman and Wenzl 1989; Murakami 1987; Freyd et al. 1985; Wenzl 1988]. Work of Drinfeld [1987], Reshetikhin and Turaev [1991], Jimbo [1986] and others on quantum groups yielded yet further representations. Enriched through modern category theory [Turaev 1994; Kassel and Turaev 2008; Bakalov and Kirillov 2001; Damiani et al. 2021], constructions are now relatively abundant.

The connections among $B_n$ representations, (2+1)-dimensional topological quantum field theory (see e.g., [Witten 1989]) and statistical mechanics (see e.g., [Baxter 1982; Akutsu and Wadati 1987; Martin 1988; Deguchi 1989; Deguchi and Akutsu 1990]) were already well established in the 1980s. Even more recently, the importance of such representations in topological phases of matter [Freedman et al. 2003; Rowell and Wang 2018] in two spacial dimensions has led to an invigoration of interest, typically focused on unitary representations associated with the 2-dimensional part of a (2+1)-TQFT. In this context the braid group is envisioned as the group of motions of point-like quasiparticles in a disk, with the trajectories of these *anyons* forming the braids in 3-dimensions. Here the braid group generators $\sigma_i$ correspond to exchanging the positions of the $i$ and $(i+1)$-st anyons. The density of such braid group representations in the group of (special) unitary matrices is intimately related to the universality of quantum computational models built on these topological phases of matter [Freedman et al. 2002a; 2002b], as well as the (classical) computational complexity of the associated link invariants [Rowell 2009]. Indeed, there is a circle of conjectures relating *finite* braid group images [Naidu and Rowell 2011; Rowell et al. 2009], *classical* link invariants, *nonuniversal* topological quantum computers and *localisable* unitary braid group representations [Rowell and Wang 2012; Galindo et al. 2013]. The other side of this conjectured coin relates the holy grail of universal topological quantum computation with powerful 3-manifold invariants through surgery on links in the three sphere.

What is a nontrivial generalisation of the braid group to 3-dimensions? Natural candidates are groups of motions: heuristically, the elements are classes of trajectories of a compact submanifold $N$ inside an ambient manifold $M$ for which the initial and final positions of $N$ are set-wise the same. The group of motions of points in a 3-manifold in effect simply permutes the points, but the motion of circles or more general links in a 3-manifold is highly nontrivial. This motivates the study of these 3-dimensional motion groups, as defined in the mid-20th century by Dahm [1962] and expounded upon by Goldsmith [1981; 1982].

More formally, a motion of $N$ inside $M$ is an ambient isotopy $f_t(x)$ of $N$ in $M$ so that

$$f_0 = \mathrm{id}_M \quad \text{and} \quad f_1(N) = N.$$

Such a motion is *stationary* if $f_t(N) = N$ for all $t$; and given any motion $f$, we have the usual notion of the reverse $\bar{f}$. We say two motions $f, g$ are equivalent if the composition of $f$ with $\bar{g}$ (via concatenation) gives a motion endpoint-fixed homotopic to a stationary motion as isotopies

$$M \times [0, 1] \to M.$$

The *motion group* $\mathcal{M}o(M, N)$ is the group of motions modulo this equivalence. When $M$ and $N$ are both oriented we will consider only motions $f$ so that $f_1(N) = N$ as an oriented submanifold, although one may consider the larger groups allowing for orientation reversing motions.

The motion groups of links inside $\mathbb{R}^3$, $S^3$ or $D^3$ and their representations are very rich, and only recently explored in the literature [Bellingeri and Bodin 2016; Damiani and Kamada 2019; Kádár et al. 2017; Bullivant et al. 2020; Baez et al. 2007; Bullivant et al. 2019]. Further enticement is provided by the prospect of applications to 3-dimensional topological phases of matter with loop-like excitations (i.e., vortices) [Wang and Levin 2014]. The fruitful symbiosis between braid group representations and 2-dimensional condensed matter systems give us hope that 3-dimensional systems could enjoy a similar relationship with motion group representations, (3+1)-TQFTs, and invariants of surfaces embedded in 4-manifolds; see e.g., [Kamada 2007; Carter et al. 2004].

There are a few hints in the literature that the (3+1)-dimensional story has some key differences from the (2+1)-dimensional situation. Reutter [2020] has shown that semisimple (3+1)-TQFTs cannot detect smooth structures on 4-manifolds. Wang and Qiu [2021] provided evidence that the mapping class group and motion group representations associated with (3+1)-dimensional Dijkgraaf–Witten TQFTs are determined via dimension reduction by the corresponding (2+1)-dimensional DW theory. As the representation theory of motion groups has been largely neglected until very recently, it is hard to speculate on precise statements analogous to the 2-dimensional conjectures and theorems.

In this article we take hints from the classical works [Burau 1935; Hoefsmit 1974], from the braid group revolution [Jones 1987], and more directly from statistical mechanics [Deguchi and Akutsu 1990; Kauffman and Saleur 1991; Martin and Rittenberg 1992; Deguchi and Martin 1992], to study representations of the motion group of free unlinked circles in 3-dimensional space, the *loop braid group* $\mathrm{LB}_n$. Presentations of $\mathrm{LB}_n$ are known; see [Fenn et al. 1997; Damiani 2017]. As $\mathrm{LB}_n$ contains the braid group $B_n$ as an abstract subgroup, a natural approach to finding linear representations is to extend known $B_n$ representations to $\mathrm{LB}_n$. This has been considered by various authors; see e.g., [Bruillard et al. 2015; Bardakov 2005; Kádár et al. 2017]. Another idea is to look for finite dimensional quotients of the group algebra, mimicking the techniques of [Jones 1987; Birman and Wenzl 1989]. As nontrivial finite-dimensional quotients of the braid group are not so easy to find, we take a hybrid approach: we combine the extension of the Burau representation to $\mathrm{LB}_n$ [Burau 1935; Bardakov 2005] with the Hecke algebras $\mathcal{H}_n$ obtained from $\mathbb{Q}(t)[B_n]$ as the quotient by the ideal generated by

$$(\sigma_i + 1)(\sigma_i - t).$$

While the naive quotient of $\mathbb{Q}(t)[\mathrm{LB}_n]$ by this ideal does not provide a finite dimensional algebra, certain additional quadratic relations (satisfied by the extended Burau representation) are sufficient for finite dimensionality, with quotient denoted $\mathrm{LH}_n$. We find a local representation of $\mathrm{LH}_n$ that aids in the analysis of its structure — the loop Burau–Rittenberg representation. One important feature of the algebras $\mathrm{LH}_n$ is that they are not semisimple; in fact, the image of the loop Burau–Rittenberg representation has a 1-dimensional center, but is far from simple. Its semisimple quotient by the Jacobson radical gives an interesting tower of algebras with Bratteli diagram exactly Pascal's triangle.

Our results suggest new lines of investigation into motion group representations. What other finite dimensional quotients of motion group algebras can we find (see e.g., [Banjo 2013])? What is the role of (non)semisimplicity in such quotients? Can useful topological invariants be derived from these quotients? What do these results say about (3+1)-dimensional TQFTs?

*Outline of the paper.* In Section 2 we recall the Burau representation and corresponding knot invariants. In Section 3 we introduce loop Hecke algebras and prove they are finite dimensional. In Section 4 we develop arithmetic tools (calculus) that we will need. In Section 5 we construct our local representations and hence prove our main structure Theorems. In Section 6 we apply the results from Section 5 to $\mathrm{LH}_n$, and make several conjectures on the open cases with $t^2 = 1$. We conclude with a discussion of new directions opened up by this work.

## 2. Burau representation, Hecke algebra and invariants of knots

Let $\underline{n} := \{1, 2, \ldots, n\}$. Then the braid group $B_n$ may be identified with the motion group $\mathcal{M}o(\mathbb{R}^2, \underline{n} \times \{0\})$. Artin showed that, for $n \geq 1$, $B_n$ admits the presentation

$$\left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{cl} \sigma_i\sigma_j = \sigma_j\sigma_i & \text{for } |i-j| > 1 \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} & \text{for } i = 1, \ldots, n-2 \end{array} \right\rangle \tag{2-1}$$

We will write $\mathfrak{A}_n(\sigma)$ for the set of relations here.

We will also need the symmetric group $S_n$. In a "motion group spirit" this can be identified with $\mathcal{M}o(\mathbb{R}^3, \underline{n} \times \{0\} \times \{0\})$. It can be presented as a quotient of $B_n$ by the relation ${\sigma_1}^2 = 1$ (however since we will often want to have both groups together we will soon rename the $S_n$ generators).

**2A. *Burau representation.*** We define Burau representation $\varrho\colon B_n \to \mathrm{GL}_n(\mathbb{Z}[t, t^{-1}])$ as follows:

$$\sigma_i \mapsto I_{i-1} \oplus \begin{pmatrix} 1-t & t \\ 1 & 0 \end{pmatrix} \oplus I_{n-i-1}. \tag{2-2}$$

The Burau representation has Jordan–Holder decomposition into a 1-dimensional representation (the vector $(1, \ldots, 1)^T$ remains fixed) and an $(n-1)$-dimensional irreducible representation known as reduced Burau representation $\bar{\varrho}\colon B_n \to \mathrm{GL}_{n-1}(\mathbb{Z}[t, t^{-1}])$. The decomposition is not split over $\mathbb{Z}[t, t^{-1}]$ — an inverse of $t+1$ is needed (see later).

**Remark 2.1.** One can also use the transpose matrix of (2-2) (depending on orientation choices while building the "carpark cover" of the punctured disc in the homological definition of Burau). The transpose fixes $(1, \ldots, 1, t, t^2, 1, \ldots, 1)^T$.

**2B. *Facts about the Burau representation.***

(1) Burau is unfaithful for $n \geq 5$ (Moody [1991] proved unfaithfulness for $n \geq 9$, Long and Paton [1993] for $n \geq 6$, Bigelow [1999] for $n = 5$).

(2) The case $n = 4$ is open, Beridze and Traczyk [2018] recently published some advances toward closing the problem.

(3) It is faithful for $n = 2, 3$ [Magnus and Peluso 1969].

(4) If we consider the braid group in its mapping class group formulation, it has a homological meaning (attached *a posteriori* to it, since Burau [1935] used only combinatorial aspects of matrices). The Burau representation describes the action of braids on the first homology group of the (covering of) the punctured disk. On the other hand the Alexander polynomial is extracted from the presentation matrix of the first homology group of the knot complement (the Alexander matrix). When we close up a braid, each element of homology of the punctured disk on the bottom

becomes identified with its image in the punctured disk at the top. At this point the Alexander matrix of the closed braid is (roughly) the Burau matrix of the braid with the modification of identifying the endpoints.

More specifically, let $K$ be a knot, and $b$ a braid such that $\hat{b}$ is equivalent to $K$. Then the Alexander polynomial $\Delta_K(t)$ can be obtained by computing:

$$\Delta_K(t) = \frac{\det(\bar{\varrho}(b) - I_{n-1})}{1 + t + \cdots + t^{n-1}}.$$

So one can think of the Alexander polynomial of $K \sim \hat{b}$ as a rescaling of the characteristic polynomial of the image of $b$ in the reduced representation.

Representations of $B_n$ are partially characterised by the eigenvalue spectrum of the image of $\sigma_i$. Observe that

$$\varrho({\sigma_i}^2) = (1-t)\varrho(\sigma_i) + tI_n, \tag{2-3}$$

i.e., the eigenvalue spectrum is $\mathrm{Spec}(\varrho(\sigma_i)) = \{1, -t\}$. Recall also that Kronecker products obey $\mathrm{Spec}(A \otimes B) = \mathrm{Spec}(A).\,\mathrm{Spec}(B)$, so $\mathrm{Spec}(\varrho(\sigma_i) \otimes \varrho(\sigma_i) = \varrho \otimes \varrho(\sigma_i)) = \{1, -t, t^2\}$. From this we see that the spectrum is fixed under tensor product only if $t = \pm 1$; see for example [Kauffman and Saleur 1991].

**2C. *Hecke algebras.*** Let $R$ be an integral domain and $q_1, q_2$ elements of $R$ with $q_2$ invertible. We define the Hecke algebra $H_n^R(q_1, q_2)$ to be the algebra with generators $\{1, T_1, \ldots, T_{n-1}\}$ and the following defining relations:

$$T_i T_j = T_j T_i \qquad \text{for } |i-j| > 1, \tag{2-4}$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \qquad \text{for } i = 1, \ldots, n-2, \tag{2-5}$$

$$T_i^2 = (q_1 + q_2)T_i - q_1 q_2 \qquad \text{for } i = 1, \ldots, n-1. \tag{2-6}$$

**Remark 2.2.** (1) Relation (2-6) coincides with the characteristic equation of the images of the generators under the Burau representation 2A when $(q_1, q_2) = (1, -t)$. We denote the resulting 1-parameter Iwahori–Hecke algebra by $H_n^R(t)$.

(2) If $t = 1$ then $H_n^R(t)$ is the group algebra $R[S_n]$ (the free $R$-module $RS_n$ made an $R$-algebra in the usual way).

(3) There is a map from $B_n$ to $H_n^R(t)$ sending $\sigma_i$ to $T_i$. Thus representations of $H_n^R(t)$ are equivalent to representations of $B_n$ for which the generators satisfy relation (2-6). This is described in [Bigelow 2006, Section 3; Jones 1987, Section 4; Martin 1991, Section 5.7] and many other places.

(4) Fixing $R = \mathbb{C}$, point (3) allows us to think of $H_n^R(t)$ as being isomorphic to the quotient $H_n(t) := \mathbb{C}[B_n]\sigma_i^2 = (1-t)\sigma_i + t$.

(5) Using the map in (3) we can represent any element of $H_n(t)$ as a linear combination of braid diagrams. The quadratic relation can be seen as a *skein relation* on elementary crossings. Knowing a basis for $H_n(t)$ makes this fact usable.

**Question 2.3.** Why these parameters and this quadratic relation?

As noted, Hecke algebras can be defined with two units of $R$ as parameters. We chose to fix these parameters to $(1, -t)$ because from this quotient one should recover the Alexander polynomial. Choosing $(-1, t)$ one should get the quotient on which Ocneanu traces are defined; see [Kassel and Turaev 2008, Chapter 4.2]. With the Ocneanu trace being a 1-parameter family over a 1-parameter algebra, we end up with polynomials in two variables. These polynomials are attached to the braid diagrams that we can see representing elements of $H_n(t)$. Moreover they are defined in such a way to respect Markov moves, so they are invariants for the closures of said braids. Hence, they are knot invariants. The quadratic relation from Remark 2.2(3) translates the trace in a *skein relation*. Through the Ocneanu trace (normalised) the invariant that is obtained is the HOMFLY-PT polynomial, which specialises in both Alexander and Jones. Each specialisation corresponds to factoring through a further quotient of the Hecke algebra (in the case of Jones, this is a quotient of the Temperley–Lieb algebra). Below we "reverse engineer" this process.

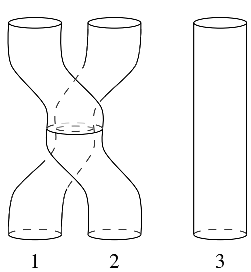
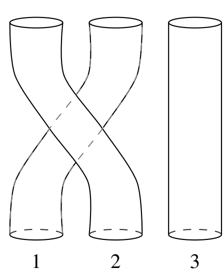
## 3. Generalising Burau and Hecke to loop braid groups

**3A.** ***The loop braid group.*** Here $S^1$ denotes the unit circle. We now consider the loop braid group

$$\mathrm{LB}_n = \mathcal{M}o(\mathbb{R}^3, \underline{n} \times S^1);$$

see e.g., [Goldsmith 1981; Savushkina 1996; Fenn et al. 1997; Brendle and Hatcher 2013; Damiani 2017; Kádár et al. 2017; Bruillard et al. 2015].

Consider the set $\Xi_n = \{\sigma_i, \rho_i, i = 1, 2, \ldots, n-1\}$ and group $\langle \Xi_n \mid \mathfrak{Q}_n \rangle$ presented by generators $\sigma_i$ and $\rho_i$, and relations $\mathfrak{Q}_n$ as follows. The generators may be visualised as the "leapfrog" and loop exchange, such as the following depictions of $\sigma_1$ and $\rho_1$ as generators of $\mathrm{LB}_3$ (motions read bottom-to-top):

$\sigma_1 =$  $\rho_1 =$ 

The $\sigma_i$ obey the braid relations as in (2-1); the $\rho_i$ obey the braid relations and also

$$\rho_i \rho_i = 1 \tag{3-1}$$

and then there are mixed braid relations

$$\rho_i \rho_{i+1} \sigma_i = \sigma_{i+1} \rho_i \rho_{i+1}, \tag{3-2}$$

$$\rho_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \rho_{i+1}, \tag{3-3}$$

$$\sigma_i \rho_{i\pm j} = \rho_{i\pm j} \sigma_i \qquad (j > 1) \text{ (all distant commutators).} \tag{3-4}$$

**Remark 3.1.** The first mixed relation (3-2) implies its reversed order counterpart:

$$\sigma_i \rho_{i+1} \rho_i = \rho_{i+1} \rho_i \sigma_{i+1} \tag{3-5}$$

whereas the reversed order second mixed relation does not hold. The relations also imply

$$\rho_2 \sigma_1 \rho_2 = \rho_1 \sigma_2 \rho_1. \tag{3-6}$$

We have (see e.g., [Fenn et al. 1997]) that

$$\mathrm{LB}_n \cong \langle \Xi_n \mid \mathfrak{Q}_n \rangle. \tag{3-7}$$

It will be convenient to give an *algebra presentation* for the group algebra. Recall that in an algebra presentation inverses are not present automatically by freeness, so we may put them in by hand as formal symbols and then impose the inverse relations. Thus as a presented algebra we have

$$k\langle \Xi_n \mid \mathfrak{Q}_n \rangle = \langle \Xi_n \cup \Xi_n^- \mid \mathfrak{Q}_n, \mathfrak{I}_n \rangle_k;$$

here $kG$ means the group $k$-algebra of group $G$, $\langle - \mid - \rangle_k$ means a $k$-algebra presentation and $\mathfrak{I}_n$ is the set of inverse relations $\sigma_i \sigma_i^- = 1$.

**3B. *The loop–Hecke algebra* $\mathbf{LH}_n$.** With Section 2 in mind, there *is* a suitable generalisation of the Burau representation to $\mathrm{LB}_n$.

**Proposition 3.2** [Vershinin 2001]. *The map on generators of* $\mathrm{LB}_n$ *given by*

$$\sigma_i \mapsto I_{i-1} \oplus \begin{pmatrix} 1-t & t \\ 1 & 0 \end{pmatrix} \oplus I_{n-i-1}. \tag{3-8}$$

$$\rho_i \mapsto I_{i-1} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus I_{n-i-1}. \tag{3-9}$$

*extends to a representation* $\varrho_{GB} : \mathrm{LB}_n \to \mathrm{GL}_n(\mathbb{Z}[t, t^{-1}])$.

*Proof.* Direct calculation. □

This group representation is not faithful for $n \geq 3$ [Bardakov 2005], and corresponds to an Alexander polynomial for welded knots.

We consider a quotient algebra of the group algebra (over a suitable commutative ring) of the group $\langle \Xi_n \mid \mathfrak{Q}_n \rangle$. The quotient algebra is

$$\mathrm{LH}_n^{\mathbb{Z}} := \mathbb{Z}[t, t^{-1}]\langle \Xi_n \mid \mathfrak{Q}_n \rangle \mathfrak{R}_n = \langle \Xi_n \cup \Xi_n^- \mid \mathfrak{Q}_n, \mathfrak{I}_n, \mathfrak{R}_n \rangle_{\mathbb{Z}[t,t^{-1}]} \tag{3-10}$$

where $\mathfrak{R}_n$ is the set of (algebra) relations:

$$\sigma_i^2 = (1-t)\sigma_i + t \qquad \text{(i.e., } (\sigma_i - 1)(\sigma_i + t) = 0\text{)}, \tag{3-11}$$

$$\rho_i \sigma_i = -t\rho_i + \sigma_i + t \qquad \text{(i.e., } (\rho_i - 1)(\sigma_i + t) = 0\text{)}, \tag{3-12}$$

$$\sigma_i \rho_i = -\sigma_i + \rho_i + 1 \qquad \text{(i.e., } (\sigma_i - 1)(\rho_i + 1) = 0\text{)}. \tag{3-13}$$

(NB we already have $(\rho_i - 1)(\rho_i + 1 = 0.$)

Observe that (3-11) yields an inverse for $\sigma_i$ (the inverse to $t$ is specifically needed), so we have

$$\mathrm{LH}_n^{\mathbb{Z}} = \langle \Xi_n \mid \mathfrak{Q}_n, \mathfrak{R}_n \rangle_{\mathbb{Z}[t,t^{-1}]}. \tag{3-14}$$

Observe then that the relations as such do not require an inverse to $t$, so we could consider the variant algebra over $\mathbb{Z}[t]$.

For any field $K$ that is a $\mathbb{Z}[t, t^{-1}]$ algebra we then define the base change $\mathrm{LH}_n^K = K \otimes_{\mathbb{Z}[t,t^{-1}]} \mathrm{LH}_n^{\mathbb{Z}}$ and, for given $t_c \in \mathbb{C}$,

$$\mathrm{LH}_n(t_c) = \mathrm{LH}_n = \mathrm{LH}_n^{\mathbb{C}}$$

where $\mathbb{C}$ is a $\mathbb{Z}[t]$-algebra by evaluating $t$ at $t_c$ (the choice of which we notationally suppress). Note that there is no reason to suppose that this gives a flat deformation (i.e., the same dimension) in all cases. (It will turn out that it does, at least in low rank, if we can localise at $t^2 - 1$. In particular, perhaps surprisingly, in the variant $t = 0$ is isomorphic to the generic case.)

**Remark.** The relations (3-11) et seq. are suggested by (2-3) and the following calculations (on $\sigma_1$ and $\rho_1$ in $\mathrm{LB}_3$, noting that blocks work the same way for all generators):

$$\varrho_{GB}(\sigma_1\rho_1) = \begin{pmatrix} t & 1-t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = -\begin{pmatrix} 1-t & t & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + I_3,$$

$$\varrho_{GB}(\rho_1\sigma_1) = \begin{pmatrix} 1 & 0 & 0 \\ 1-t & t & 0 \\ 0 & 0 & 1 \end{pmatrix} = -t\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1-t & t & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + tI_3.$$

**3C. *Notable direct consequences of the relations: Finiteness.*** Given a word in the generators, of form $\sigma_3\sigma_4\rho_2$ say, by a *translate* of it we mean the word obtained by shifting the indices thus: $\sigma_{3+i}\sigma_{4+i}\rho_{2+i}$.

With the $\mathfrak{Q}$ and $\mathfrak{R}$ relations we can derive the following ones, together with the natural translates thereof (here $\overset{*}{=}$ uses (3-1); $\overset{\rho\rho\sigma}{=}$ uses (3-2); $\overset{\sigma\rho}{=}$ uses (3-13), and so on):

$$
\begin{aligned}
\text{(M1)} \quad \boldsymbol{\sigma_2\rho_1\sigma_2} &\overset{*}{=} \sigma_2\rho_2\rho_2\rho_1\sigma_2 \\
&\overset{\rho\rho\sigma}{=} \sigma_2\rho_2\sigma_1\rho_2\rho_1 \\
&\overset{\sigma\rho}{=} -\sigma_2\sigma_1\rho_2\rho_1 + \rho_2\sigma_1\rho_2\rho_1 + \sigma_1\rho_2\rho_1 \\
&\overset{\rho\sigma\sigma,\rho\rho\sigma}{=} -\rho_1\sigma_2\sigma_1\rho_1 + \rho_2\rho_2\rho_1\sigma_2 + \sigma_1\rho_2\rho_1 \\
&\overset{\sigma\rho}{=} \boldsymbol{\sigma_1\rho_2\rho_1 + \rho_1\sigma_2\sigma_1 - \rho_1\sigma_2\rho_1},
\end{aligned}
$$

$$
\begin{aligned}
\text{(M2)} \quad \boldsymbol{\rho_2\sigma_1\sigma_2} &\overset{*}{=} \rho_2\sigma_1\rho_2\rho_2\sigma_2 \\
&\overset{\rho\sigma}{=} -t\rho_2\sigma_1\rho_2\rho_2 + \rho_2\sigma_1\rho_2\sigma_2 + t\rho_2\sigma_1\rho_2 \\
&\overset{*,\rho\sigma\rho}{=} -t\rho_2\sigma_1 + \rho_1\sigma_2\rho_1\sigma_2 + t\rho_1\sigma_2\rho_1 \\
&\overset{M1}{=} -t\rho_2\sigma_1 + \rho_1(\rho_1\sigma_2\sigma_1 - \rho_1\sigma_2\rho_1 + \sigma_1\rho_2\rho_1) + t\rho_1\sigma_2\rho_1 \\
&\overset{*,\rho\sigma}{=} -t\rho_2\sigma_1 + \sigma_2\sigma_1 - \sigma_2\rho_1 + (-t\rho_1 + \sigma_1 + t)\rho_2\rho_1 + t\rho_1\sigma_2\rho_1 \\
&= \sigma_1\rho_2\rho_1 + t\rho_1\sigma_2\rho_1 - t\rho_1\rho_2\rho_1 + \sigma_2\sigma_1 - \sigma_2\rho_1 - t\rho_2\sigma_1 + t\rho_2\rho_1.
\end{aligned}
$$

**Definition 3.3.** For given $n$ and $m \leq n$ let $\mathrm{LH}_m^{\langle}$ denote the subalgebra of $\mathrm{LH}_{n+1}$ generated by $\Xi_m$ (it is a quotient of $\mathrm{LH}_m$, as per the $\Psi$ map formalism in Section 4B).

**Lemma 3.4.** *For any $n$ let $X_i$ be the vector subspace of $LH_n$ spanned by* $\{1, \sigma_i, \rho_i\}$. *Then* $\mathrm{LH}_{n+1} = \mathrm{LH}_n^{\langle}\, X_n\, \mathrm{LH}_n^{\langle}$.

*Proof.* It is enough to show that $X_n\, \mathrm{LH}_n^{\langle}\, X_n$ lies in $\mathrm{LH}_n^{\langle}\, X_n\, \mathrm{LH}_n^{\langle}$. We work by induction on $n$. The case $n = 1$ is clear, since $LH_1 = \mathbb{C}$. Assume true in case $n-1$ and consider case $n$. We have

$$X_n\, \mathrm{LH}_n^{\langle}\, X_n = X_n\, \mathrm{LH}_{n-1}^{\langle}\, X_{n-1}\, \mathrm{LH}_{n-1}^{\langle}\, X_n$$

by assumption. But $\mathrm{LH}_{n-1}^{\langle}$ and $X_n$ commute so we have $\mathrm{LH}_{n-1}^{\langle}\, X_n X_{n-1} X_n\, \mathrm{LH}_{n-1}^{\langle}$. The inductive step follows from the relations $\mathfrak{Q}$ and $\mathfrak{R}$ and the relations (M1) and (M2) above. $\square$

**Corollary 3.5.** $\mathrm{LH}_n$ *is finite dimensional.* $\square$

**Remark 3.6.** We may also treat certain other quotients of $\mathbb{C}\,\mathrm{LB}_n$. For example, eliminating either relations (3-12) or (3-13) we still obtain finite dimensional quotients. In particular, if we only include (3-13) and not (3-12) then the analogous proof with $X_n$ replaced by $\{1, \rho_n, \sigma_n, \rho_n\sigma_n\}$ proves finite dimensionality.

**3D.** ***Refining the spanning set.*** Can we express elements of $\mathrm{LH}_3$ as sums of length-2 words (and hence eventually solve word problem)? We have, for example,

(3-15)
$$\rho_1\rho_2\rho_1 = -1+\rho_2+\frac{(-t-1)}{(t-1)}(-\rho_1+\rho_2\rho_1-\rho_1\rho_2)+\frac{2}{(t-1)}(-\sigma_1+\sigma_2\rho_1-\rho_1\sigma_2)$$

But in general this is not easy. And another problem is that we do not have immediately manifest relationships between different ranks (such as inclusion) that would be useful. With this (and several related points) in mind it would be useful to have a tensor space representation. In what follows we address the construction of such a representation.

## 4. Basic arithmetic with $\mathrm{LH}_n$

Here we briefly report some basic arithmetic in $\mathrm{LH}_n$ that gives the clues we need for our local representation constructions below.

**4A.** ***Fundamental tools, locality.*** In what follows, $\mathsf{B}$ denotes the *braid category*: a strict monoidal category with object monoid $(\mathbb{N}_0, +)$ generated by 1, and $\mathsf{B}(n, n) = B_n$, $\mathsf{B}(n, m) = \varnothing$ otherwise, and monoidal composition is via side-by-side concatenation of suitable braid representatives; see e.g., [Mac Lane 1998, XI.4]. Similarly $\mathsf{S}$ is the permutation category (of symmetric groups). Let $\mathsf{H}$ denote the ordinary Hecke category — again monoidal, but less obviously so [Humphreys 1990]. (We have not yet shown that $\mathsf{LH}$, the loop-Hecke category, is monoidal.)

Let $\mathsf{LB}$ denote the loop-braid category — this is the strict monoidal category analogous to the braid category where the object monoid is $(\mathbb{N}, +)$, $\mathsf{LB}(n, n) = \mathrm{LB}_n$, $\mathsf{LB}(n, m) = \varnothing$ otherwise, and monoidal composition $\otimes$ is side-by-side concatenation of loop-braids.

Suppose $\mathsf{C}$ is a strict monoidal category with object monoid $(\mathbb{N}_0, +)$ generated by 1 (for example, $\mathsf{LB}$). Write $1_1$ for the unique element of $\mathsf{C}(1, 1)$ and for $x \in \mathsf{C}(n, n)$ define the *translate*

(4-1)
$$x^{(t)} = 1_1^{\otimes t} \otimes x \in \mathsf{C}(n+t, n+t)$$

For $k$ a commutative ring, define translates of elements of $k\,\mathrm{LB}_n$ (i.e., $k\mathsf{LB}(n, n)$), and $kS_n$ and so on, by linear extension.

*Caveat.* Note that it is a property of the geometric topological construction of loop braids that the composition $\otimes$ in $\mathsf{LB}$ makes manifest sense. It requires that side-by-side concatenation of rank $n$ with rank $m$ passes to $n+m$. This is clear by construction. But in groups/algebras defined by generators and relations it would not be intrinsically clear. For example, how do we know that the subalgebra of $\mathrm{LH}_n$ generated *in* $\mathrm{LH}_n$ by the elements $p_i, s_i$, $i = 1, 2, \ldots, n-2$ is isomorphic to $\mathrm{LH}_{n-1}$? (Some of our notation requires care at this point since it may lead us to take isomorphism for granted!)

**4B.** ***The $\Psi$ maps.*** Let $A = \langle X \mid R \rangle_k$ be an algebra presented with generators $X$ and relations $R$. Then there is a homomorphism from the free algebra generated by any subset $X_1$ of $X$ to $A$, taking $s \in X_1$ to its image in $A$. This factors through the quotient by any relations, $R_1$ say, expressed only in $X_1$. We may consider it as a homomorphism from this quotient. But of course the kernel may be bigger — relations induced indirectly by the relations in $R$. A $\Psi$ map is such a homomorphism:

$$\langle X_1 \mid R_1 \rangle_k \xrightarrow{\Psi} \langle X_1 \mid R \rangle_k \hookrightarrow \langle X \mid R \rangle_k$$

Note that arithmetic properties such as idempotency, orthogonality and vanishing are preserved under $\Psi$ maps. Thus for example a decomposition of 1 into orthogonal idempotents in $kS_n$ passes to such a decomposition in $\mathrm{LH}_n$ (see (4-3)). However conditions such as primitivity, inequality and even nonzeroness are not preserved in general.

Note that there is a natural (not generally isomorphic) image of

$$kS_n \cong k\langle p_1, \dots, p_i, \dots, p_{n-1} \mid \mathfrak{A}_n(p),\, p_i p_i = 1 \rangle \tag{4-2}$$

in $\mathrm{LH}_n$ obtained by the map of generators $p_i \mapsto \rho_i$. Let us call it $\mathrm{LH}_n^\rho$. Thus

$$kS_n \xrightarrow{\Psi} \mathrm{LH}_n^\rho \hookrightarrow \mathrm{LH}_n \tag{4-3}$$

Similarly $H_n = \langle T_1, \dots, T_i, \dots, T_{n-1} \mid \mathfrak{A}_n(T), \dots \rangle_k$ has image $\mathrm{LH}_n^\sigma$ under $T_i \mapsto \sigma_i$:

$$H_n \twoheadrightarrow \mathrm{LH}_n^\sigma \hookrightarrow \mathrm{LH}_n \tag{4-4}$$

Let us consider the image of a primitive idempotent decomposition in $kS_n$

$$1 = \sum_{\lambda \in \Lambda_n} \sum_{i=1}^{d_\lambda} \boldsymbol{e}_\lambda^i$$

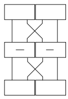
under $\Psi : kS_n \to \mathrm{LH}_n$. Here $\Lambda_n$ denotes the set of integer partitions of $n$, and $d_\lambda$ is the dimension of the $S_n$ irrep. See the Appendix for explicit constructions. We will also write $(\Lambda, \subseteq)$ for the poset of all integer partitions ordered by the usual inclusion as a Young diagram.

**Proposition 4.1.** *Let $k$ be the field of fractions of $\mathbb{Z}[t, t^{-1}]$:*

(I) *The image $\Psi(\boldsymbol{e}_\lambda^i)$ in $\mathrm{LH}_n^k$ of every idempotent with $(2, 2) \subseteq \lambda \in \Lambda_n$ is zero.*

(II) *On the other hand all other $\lambda \in \Lambda_n$, i.e., all hook shapes, give nonzero image.*

*Proof.* (I) Note that $\boldsymbol{e}^1_\mu$ with $\mu \in \Lambda_m$ is defined in $kS_n$ for $n \geq m$ by $S_m \hookrightarrow S_n$. It is shown for example in [Martin and Rittenberg 1992] that if the relation $\boldsymbol{e}^1_\mu = 0$ is imposed in a quotient of $kS_n$ then $\boldsymbol{e}^i_\nu = 0$ holds for $\mu \subseteq \nu \in \Lambda_n$ (a proof uses $S_{n-1} \hookrightarrow S_n$ restriction rules, from which we see that $\boldsymbol{e}^1_\mu$ is expressible as a sum of orthogonal such idempotents). Consider $\boldsymbol{e}^1_{(2,2)}$ (i.e., with $(2,2) \in \Lambda_4$) which may be expressed as

$\boldsymbol{e}^1_{(2,2)} =$  $\propto (p_1+1)(p_3+1)p_2(p_1-1)(p_3-1)p_2(p_1+1)(p_3+1)$

(using notation and a choice from (A-5)). By a direct calculation in $\mathrm{LH}_4$

$$\Psi((p_1+1)(p_3+1)p_2(p_1-1)(p_3-1)) = 0 \tag{4-5}$$

(NB we know no elegant way to do this calculation; the result holds also for generic $t$, but *not* for $t = 1$).

(II) This can be verified by evaluation as nonzero in a suitable representation. (For simplicity it is sufficient to work in the "SP quotient" that we give in Theorem 5.2 below, working with Kronecker products. We will omit the explicit calculation.) □

With identity (4-5) in mind, recall that in [Martin and Rittenberg 1992] local representations of ordinary Hecke (and hence $S_n$) with this property were constructed from spin chains. In Section 5 we will combine this with Burau and thus find the representations of loop-Hecke that we need here.

By Proposition 4.1 we have a decomposition of 1 in $\mathrm{LH}_n$ according to hook partitions

$$1 = \sum_{i=0}^{n-1} \sum_{j=1}^{d_{(n-i,1^i)}} \Psi(\boldsymbol{e}^j_{(n-i,1^i)}). \tag{4-6}$$

(NB $j$ varies over idempotents that are equivalent in the sense that they induce isomorphic modules — it will be sufficient to focus on $j = 1$.)

(Left) multiplying by $A = \mathrm{LH}_n$ we thus have a decomposition of the algebra

$$A \cong \oplus_{i=0}^{n-1} \oplus_j A\Psi(\boldsymbol{e}^j_{(n-i,1^i)})$$

as a left-module for itself, into projective summands.

We have not yet shown that these summands are indecomposable. But consider for a moment the action of $\mathrm{LH}_n$ on the image under $\Psi$ of

$$Y^n_\pm = \sum_{g \in S_n} (\pm 1)^{\mathrm{len}(g)} g$$

in $\mathrm{LH}_n$ (we write $Y^n_+$ for unnormalised $\boldsymbol{e}^1_{(n)}$ and $Y^n_-$ for $\boldsymbol{e}^1_{(1^n)}$; again, see the Appendix for a review). By abuse of notation we will write $Y^n_\pm$ also for the image. By (3-13) and the classical identities $Y_{\pm}^{a(1)} Y_{\pm}^{n} = a! Y_{\pm}^{n}$ (recall $Y_{\pm}^{a(1)}$ means $Y_{\pm}^{a}$ with indices shifted by +1, see (A-2) et seq.) we have

$$\sigma_i Y_+^n = Y_+^n, \qquad Y_-^n \sigma_i = -t Y_-^n. \tag{4-7}$$

It follows that $Y_+^n$ spans a 1-d left ideal in $\mathrm{LH}_n$. If we work over a field containing the rationals then it is normalisable as an idempotent, and so we have an indecomposable projective left module

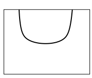
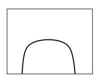
$$P_{(n)} = \mathrm{LH}_n\, Y_+^n = \mathrm{LH}_n\, \boldsymbol{e}_{(n)}^1 = k\boldsymbol{e}_{(n)}^1.$$

## 5. On local representations

Here $\mathsf{Mat}$ is the monoidal category of matrices over a given commutative ring (and $\mathsf{Mat}_k$ the case over commutative ring $k$), with object monoid $(\mathbb{N}, \times)$ and tensor product on morphisms given by a Kronecker product (NB there is a convention choice in defining the Kronecker product). We often focus on the monoidal subcategory $\mathsf{Mat}^m$ generated by a single object $m \in \mathbb{N}$ — usually $m = 2$. Then the object monoid $(2^{\mathbb{N}}, \times)$ becomes $(\mathbb{N}, +)$ in the natural way.

In the study of ordinary Hecke algebras (and particularly quantum-group-controlled quotients like Temperley–Lieb) a very useful tool is the beautiful set of local tensor space representations generalising those arising from XXZ spin chains and Schur–Weyl duality. For example we have the following.

Consider the TL diagram category $\mathsf{T}$ with object monoid $(\mathbb{N}, +)$ $k$-linear-monoidally generated by the morphisms represented by diagrams

$\mathsf{u} =$  $\in \mathsf{T}(2, 0)$ and $\mathsf{u}^* =$ .

This has a TQFT $\mathsf{F}_2$ given by $\mathsf{u} \mapsto (0, \tau, \tau^{-1}, 0)$ (the target category is $\mathsf{Mat}$) and taking $*$ to transpose. Of course for $1_1 \in \mathsf{T}(1, 1)$ we have $\mathsf{F}_2(1_1) = I_2$.

To pass to our present topic we note that $1_1 \otimes 1_1 = 1_2$ and that the Yang–Baxter construction $\sigma_1 \mapsto 1_2 - \tau^2 \mathsf{u}^* \mathsf{u}$ gives

$$\sigma_1 \mapsto \mathsf{F}_2(1_2) - \tau^2 \begin{pmatrix} 0 & & & \\ & \tau^2 & 1 & \\ & 1 & \tau^{-2} & \\ & & & 0 \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & 1-\tau^4 & -\tau^2 & \\ & -\tau^2 & 0 & \\ & & & 1 \end{pmatrix} \tag{5-1}$$

thus a representation of the braid category $\mathsf{B}$ (note that eigenvalues are 1 and $-\tau^4$ so $\tau^4$ here passes to $t$ in our parametrisation for loop-Hecke). But also note that $\mathsf{u}, \mathsf{u}^*$ can be used for a Markov trace. And also for idempotent localisation functors: let $\mathsf{U} = \mathsf{u}^* \mathsf{u}$, $\mathsf{U}_1 = \mathsf{U} \otimes 1_{n-2}$, and $T_n = \mathsf{T}(n, n)$ regarded as a $k$-algebra; then we have the algebra isomorphism $\mathsf{U}_1 T_n \mathsf{U}_1 \cong T_{n-2}$. This naturally gives a category embedding

$\mathsf{G}_\mathsf{U}$ of $T_{n-2}$-mod in $T_n$-mod. Recall that irreps are naturally indexed by partitions of $n$ into at most two parts: $\lambda = (n-m, m)$, or equivalently (for given $n$) by "charge" $\lambda_1 - \lambda_2 = n - 2m$, thus by $\Upsilon_n = \{n, n-2, n-4, \dots, 0/1\}$ (depending on $n$ is odd or even). This latter labeling scheme is stable under the embedding. That is, indecomposable projective modules are mapped by $\mathsf{G}_\mathsf{U}$ according to $\Upsilon_{n-2} \hookrightarrow \Upsilon_n$.

**5A.** ***Charge conservation.*** Another useful property of $\mathsf{F}_2$ is "charge conservation". We may label the row/column index for object 2 in $\mathsf{Mat}$ by $\{\varepsilon_1, \varepsilon_2\}$ or $\{+, -\}$. Then $2 \otimes 2$ has index set $\{\varepsilon_1 \otimes \varepsilon_1, \varepsilon_2 \otimes \varepsilon_1, \varepsilon_1 \otimes \varepsilon_2, \varepsilon_2 \otimes \varepsilon_2\}$ (which we may abbreviate to $\{11, 21, 12, 22\}$) and so on. The "charge" $ch$ of an index is $ch = \#1 - \#2$. Note from (5-1) that $\mathsf{F}_2$ does not mix between different charges (hence charge conservation).

For a functor with the charge conservation property the representation of $B_n$ (say) obtained has a direct sum decomposition according to charge, with "Young blocks" $\beta_i$ of charge $i = n, n-2, \dots, -n$. The dimensions of the blocks are given by Pascal's triangle. It will be convenient to express this with the semiinfinite Toeplitz matrices $\mathcal{U}$ and $\mathcal{T}$:

$$\mathcal{U} = \begin{pmatrix} 1 & 1 & & & \\ & 1 & 1 & & \\ & & 1 & 1 & \\ & & & 1 & 1 \\ & & & & \ddots \end{pmatrix}, \quad \mathcal{U}^2 = \begin{pmatrix} 1 & 2 & 1 & & & \\ & 1 & 2 & 1 & & \\ & & 1 & 2 & 1 & \\ & & & 1 & 2 & 1 \\ & & & & \ddots & \end{pmatrix}, \quad \mathcal{T} = \begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & 1 & & \\ & 1 & 0 & 1 & \\ & & 1 & 0 & 1 \\ & & & & \ddots \end{pmatrix}$$

and semiinfinite vectors $v_1 = (1, 0, 0, 0, \dots)$, $v_2 = (0, 1, 0, 0, \dots)$, .... Thus $v_1\mathcal{U}^n$ (respectively $v_{n+1}\mathcal{T}^n$) gives the numbers in the $n+1$-th row of Pascal (followed by a tail of zeros). (The two different formulations correspond to two different thermodynamic limits — $\mathcal{T}$ corresponds to the $\Upsilon_{n-2} \hookrightarrow \Upsilon_n$ limit — see later.) Then

$$\dim(\beta_i) = (v_1\mathcal{U}^n)_{(n-i+2)/2} = (v_{n+1}\mathcal{T}^n)_{n-i+1}. \tag{5-2}$$

In the case of $\mathsf{F}_2$ these blocks are not linearly irreducible in general (the generic irreducible dimensions are given by $v_1\mathcal{T}^n$). But they still provide a useful framework. We return to this later.

With this construction and Proposition 3.2 in mind, it is natural to ask if we can make a local version of generalised Burau. (Folklore is that this cannot work, and directly speaking it does not. But we now have some more clues at our disposal.)

**5B.** ***Representations of*** $\mathsf{B}$**.** Now we have in mind Proposition 4.1; and brute force calculations in low rank showing (see Section 6) that $\mathrm{LH}_n$ is nonsemisimple but has irreducible representations with dimensions given by Pascal's triangle. This is reminiscent of Rittenberg's analysis of the quantum spin chains over Lie superalgebras found in [Deguchi 1989; Deguchi et al. 1989; Deguchi and Akutsu 1990; Kauffman and Saleur 1991; Martin and Rittenberg 1992; Deguchi and Martin 1992]. It is also reminiscent of work of Saleur on "type-B" braids [Martin and Saleur 1994]; emdash but for this see e.g., [Bullivant et al. 2020]. Inspired by this and the Burau representation (and see [Damiani and Florens 2018]) we proceed as follows. Define

$$M_t(\sigma) = \begin{pmatrix} 1 & & & \\ & 1-t & t & \\ & 1 & 0 & \\ & & & 1 \end{pmatrix}, \quad M'_t(\sigma) = \begin{pmatrix} 1 & & & \\ & 1-t & t & \\ & 1 & 0 & \\ & & & -t \end{pmatrix} \tag{5-3}$$

as in [Deguchi 1989; Kauffman and Saleur 1991]. Fix a commutative ring $k$, $\tau \in k^\times$, and $t = \tau^4$. Observe that there is a monoidal functor $\mathsf{F}_M$ from the Braid category $\mathsf{B}$ to $\mathsf{Vect}$ (or at least $\mathsf{Mat}$) given by object 1 mapping to $V = \mathbb{C}\{e_1, e_2\}$ (i.e., to 2 in $\mathsf{Mat}_{\mathbb{Z}[t]}$) and the positive braid $\sigma$ in $\mathsf{B}(2,2)$ mapping to $M_t(\sigma)$. The conjugation of this matrix to $\mathsf{F}_2(\sigma)$ lifts to a natural isomorphism of functors. Another natural isomorphism class of charge conserving functors has representative functor $\mathsf{F}_{M'}$ given by $M'_t(\sigma)$. (According to the scheme of Deguchi et al., this is the (1,1)-super class; see for example [Deguchi 1989; Kauffman and Saleur 1991]. But note that in extending to $\mathsf{LB}$ below, isomorphism will not be preserved, so we are focusing on the specific representative.) In fact some elementary analysis shows that these two classes are all of this form that factor through Hecke (apart from the trivial class).

Let us formulate this in language that will be useful later. First note that (like any invertible matrix) $M_t(\sigma)$ and $M'_t(\sigma)$ extend to monoidal functors from the free monoidal category generated by $\sigma$ to $\mathsf{Mat}$. Thus, in particular,

$$M'_t(\sigma \otimes 1_1) = M'_t(\sigma) \otimes \mathrm{Id}_2 \in \mathsf{Mat}(2^3, 2^3).$$

Given the form of the construction, proof of the above factoring through $\mathsf{B}$ follows from a direct verification of the braid relation in each case. More interestingly we have, again by direct calculation, the stronger result

$$M'_t(\sigma \otimes 1_1) M'_t(1_1 \otimes \sigma) M'_s(\sigma \otimes 1_1) = M'_s(1_1 \otimes \sigma) M'_t(\sigma \otimes 1_1) M'_t(1_1 \otimes \sigma) \tag{5-4}$$

while the $tss$ version of this identity does *not* hold (unless we force $s = 1$, or $s = t$) (NB care must be taken with conventions here.)

To pass back from the basic-algebra/homology to the full algebra we need the dimensions of the irreducibles. For an algebra $A$ with Cartan matrix $C_L(A)$ and a vector $v_L(A)$ giving the dimensions of the irreducible heads of the projectives we have

$$\dim(A) = v_L(A) C_L(A) v_L(A)^T. \tag{5-5}$$

**Definition 5.1.** Let the $n \times n$ matrix $\overline{\mathcal{M}}_n$ be:

$$\overline{\mathcal{M}}_n = \begin{pmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ & 1 & 1 & & & \\ & & 1 & 1 & & \\ & & & \ddots & \ddots & \\ & & & & 1 & 1 \end{pmatrix}$$

We label columns left to right (and rows top to bottom) by the ordered set $h_n$ of hook integer partitions of $n$:

$$h_n = ((n), (n-1, 1), (n-2, 1^2), \dots, (1^n)).$$

We will see in Theorem 5.8 that $\overline{\mathcal{M}}_n$ is the left Cartan decomposition matrix of $\mathrm{SP}_n$ (it follows that the Ext-matrix is the same except without the main diagonal entries).

**5C. *Extending to* LB.** Recall we introduced the loop-braid category $\mathsf{LB}$. We write $\sigma \in \mathsf{LB}(2, 2)$ for the positive braid exchange and $\rho \in \mathsf{LB}(2, 2)$ for the symmetric exchange.

Formally extending with elementary transpositions (cf. $\varrho_{GB}$), the $\mathsf{F}_M$ construction fails to satisfy the mixed braid relation (3-3). However the functor $\mathsf{F}_{M'}$ fairs better.

**Theorem 5.2.** (i) *The $\sigma \mapsto M'_t(\sigma)$ construction extended using the super transposition $\rho \mapsto M'_1(\sigma)$ gives a monoidal functor $F^e_{M'}$ from the loop Braid category* $\mathsf{LB}$ *to* $\mathsf{Mat}$.

(ii) *$F^e_{M'}$ factors through* $\mathsf{LH}$.

*Proof.* The proof is a linear algebra calculation similar to the $\mathsf{B}$ cases above, using Kronecker product identities; but also using the appropriate special case of (5-4) for (3-3). □

**Definition 5.3.** Fix a field $k$ and $t \in k$. Then the $k$-algebra $\mathrm{SP}_n = k\,\mathrm{LB}_n \,/\, \mathrm{Ann}\, F^e_{M'}$.

We conjecture that the extended super representation, which we call Burau–Rittenberg, or “SP” rep for short, is faithful on LH unless $t^2 = 1$ (see later).

**Remark 5.4.** As the Hecke algebra is related to the quantum groups $U_q\mathfrak{sl}(k \mid m)$ via Schur–Weyl duality [Jimbo 1986; Deguchi and Akutsu 1990] one naturally wonders if local representations of LH can be obtained from the $R$-matrices coming from quantum groups, by extension. The results of [Kádár et al. 2017] suggest that $R$-matrices that extend to local representations of $\mathrm{LB}_n$ are in general somewhat rare. The SP representation is of this form: $M'_t$ comes from the super-quantum group $U_q\mathfrak{sl}(1 \mid 1)$. We are not aware of other $R$-matrices coming from quantum groups that extend to $\mathrm{LB}_n$, but this approach is nevertheless intriguing.

**Proposition 5.5.** *Fix a field $k$ and $t \in k$, $t \neq 1$. Let $\chi_i = (\sigma_i - \rho_i)/(1-t)$. Then*:

(a) *$\chi_i$ and $\rho_i$ $(i = 1, 2, \dots, n-1)$ are alternative generators of* $\mathrm{SP}_n$.

(b) *The $k$-algebra isomorphism class of* $\mathrm{SP}_n$ *is independent of $t$.*

*Proof.* (a) Elementary. (b) The images of the alternative generators in the defining representation are independent of $t$. □

**5D.** ***Towards linear structure of SP.*** Let us work out the linear structure of SP. (i.e., its Artin–Wedderburn linear representation theory over $\mathbb{C}$: simple modules, projective modules and so on. See Section 5E for a review.)

**Proposition 5.6.** *Suppose $t \neq 1 \in k$. Let $\chi = (\sigma - \rho)/(1-t)$ and $\chi_1 = (\sigma_1 - \rho_1)/(1-t) \in \mathrm{SP}_n$.*

(I) *Then*

$$\chi_1\, \mathrm{SP}_n\, \chi_1 \cong \mathrm{SP}_{n-1} \tag{5-6}$$

*and*

$$\mathrm{SP}_n \,/\, \mathrm{SP}_n\, \chi_1\, \mathrm{SP}_n \cong k. \tag{5-7}$$

(II) *In particular the map $f_\chi : \mathrm{SP}_{n-1} \to \chi_1\, \mathrm{SP}_n\, \chi_1$ given by $w \mapsto \chi_1 w^{(1)} \chi_1$ (recall the translation notation from (4-1)) is an algebra isomorphism.*

*Proof.* (I) Let us write simply $F = F_n$ for the defining representation $F^e_{M'}$ of $\mathrm{SP}_n$. We write $\{1, 2\}^n$ for the basis (i.e., we write simply symbols $1, 2$ for $e_1, e_2$ and the word $112$ for $e_1 \otimes e_1 \otimes e_2$ and so on). Our convention for ordering the basis is given by 11,21,12,22. First observe that the image in $F$ is (here with $n = 3$):

$$(\chi \otimes 1_2) \mapsto \begin{pmatrix} 0 & & & \\ & \boxed{1} & -1 & \\ & & 0 & \\ & & & \boxed{1} \end{pmatrix} \otimes 1_2 = \begin{pmatrix} 0 & & & & & & & \\ & \boxed{1} & -1 & & & & & \\ & & 0 & & & & & \\ & & & \boxed{1} & & & & \\ & & & & 0 & & & \\ & & & & & \boxed{1} & -1 & \\ & & & & & & 0 & \\ & & & & & & & \boxed{1} \end{pmatrix} \tag{5-8}$$

Note that the basis change conjugating by

$$\begin{pmatrix} 1 & & & \\ & 1 & 1 & \\ & & 1 & \\ & & & 1 \end{pmatrix} \otimes 1_2 \tag{5-9}$$

(again this is the example with $n = 3$) brings this into diagonal form, projecting onto the $2\{1, 2\}^{n-1}$ subspace (the subspace of $V^n = \mathbb{C}\{1, 2\}^n$ spanned by basis elements of form $2w$ with $w \in \{1, 2\}^{n-1}$, i.e., of form $e_2 \otimes \ldots$). That is: $\chi \mapsto \binom{0}{1} \otimes 1_2$. Furthermore:

$$(\chi \otimes 1_2)(1_2 \otimes \sigma)(\chi \otimes 1_2) \mapsto \begin{pmatrix} 0 &&&&&&& \\ & \boxed{1} & -1 &&&&& \\ && 0 &&&&& \\ &&& \boxed{1-t} & 0 & \boxed{t} & -t & \\ &&&& 0 &&& \\ &&& \boxed{1} & 0 & \boxed{0} && \\ &&&&&& 0 & \\ &&&&&&& \boxed{-t} \end{pmatrix} \tag{5-10}$$

Note that after the (5-9) basis change this decomposes as a sum of several copies of the 0 module together with the submodule $N$ with basis $2\{1, 2\}^{n-1}$. Then the map from $N$ to $\{1, 2\}^{n-1}$ given by $2w \mapsto w$ gives

$$\chi_1\sigma_1\chi_1 = -t.\chi_1 \mapsto F_{n-1}(-t.1)$$

and $\chi_1\sigma_2\chi_1 \mapsto F_{n-1}(\sigma_1)$ and $\chi_1\rho_2\chi_1 \mapsto F_{n-1}(\rho_1)$. Also note that $\chi_1$ commutes with $\sigma_i$ for $i > 2$ so we have

$$\chi_1\sigma_i\chi_1 = \chi_1\sigma_i \mapsto F_{n-1}(\sigma_{i-1}), \quad \chi_1\rho_i\chi_1 \mapsto F_{n-1}(\rho_{i-1}) \quad i > 2. \tag{5-11}$$

Thus the images of the generators under $w \mapsto \chi_1 w \chi_1$ are the generators of $\mathrm{SP}_{n-1}$, establishing (5-6) on generators. To show that the images of the generators span we proceed as follows. From Lemma 3.4 and the (sufficient) symmetry of the relations under $i \mapsto n - i$ on indices, writing $\mathrm{LH}_n = L_n$ for short, we have

$$\begin{aligned} L_{n+1} &= L_n X_n L_n \\ &= L_n^{(1)} X_1 L_n^{(1)} \\ &= L_{n-1}^{(2)} X_2 L_{n-1}^{(2)} X_1 L_{n-1}^{(2)} X_2 L_{n-1}^{(2)} \\ &= L_{n-1}^{(2)} X_2 L_{n-2}^{(3)} X_3 L_{n-2}^{(3)} X_1 X_2 L_{n-1}^{(2)} \\ &= L_{n-1}^{(2)} X_2 X_3 X_1 X_2 L_{n-1}^{(2)}. \end{aligned}$$

Thus

$$\chi_1 L_{n+1} \chi_1 = \chi_1 L_{n-1}^{(2)} X_2 X_3 X_1 X_2 L_{n-1}^{(2)} \chi_1 = L_{n-1}^{(2)} \chi_1 X_2 X_3 X_1 X_2 \chi_1 L_{n-1}^{(2)}.$$

We can show by direct calculations that $\chi_1 X_2 X_3 X_1 X_2 \chi_1$ lies in the algebra generated by the images of the generators. (We can do this even in $\mathrm{LH}_4$. The result then holds in $\mathrm{SP}_4$ since it is a quotient; and then in $\mathrm{SP}_n$ by construction. Note however that we have not shown that it holds in $\mathrm{LH}_n$.) Also $L^{(2)}_{n-1}\chi_1$ evidently lies in the algebra generated by the images of the generators, by commutation, so we are done.

Finally (5-7) follows on noting that the quotient corresponds to imposing $\chi_1 = 0$, i.e., $\sigma_1 = \rho_1$. Noting that $t \neq 1$, this gives $\sigma_i = 1$.

(II) Note that $f_\chi$ inverses the map from (5-11) above. □

**5E.** ***Aside on linear/Artinian representation theory.*** Since this paper bridges between topology and linear representation theory it is perhaps appropriate to say a few words on the bridge. While topology focuses on topological invariants, linear rep theory is concerned with invariants such as the spectrum of linear operators (and the generalised "spectrum" of algebras of linear operators). The former is thus of interest for topological quantum field theories, and the latter for usual quantum field theories (where notions such as mass are defined). In this section we recall a few key points of linear/Artinian rep theory that are useful for us. (So of course it can be skipped if you are not interested in this aspect, or are already familiar.)

Recall that every finite dimensional algebra over an algebraically closed field is Morita equivalent to a basic algebra; see e.g., [Nesbitt and Scott 1943; Jacobson 1974; Benson 1991]. This allows us to track separately the combinatorial and homological data of an algebra.

Let $A$ be a finite dimensional algebra over an algebraically closed field $k$; see, e.g., [Benson 1991]. Let $J(A)$ denote the radical. Let $L = \{L_1, \dots, L_r\}$ be an ordered set of the isomorphism classes of simple $A$-modules, with projective covers $P_i = Ae_i$ (i.e., the $e_i$s are a set of primitive idempotents). Given an $A$-module $M$ let $\mathrm{Rad}(M)$ denote the intersection of the maximal proper submodules. Now suppose $A$ is basic. Recall that $\mathrm{Ext}^1_A(L_i, L_j)$ codifies the nonsplit extensions between these modules; i.e., the "atomic" components of nonsemisimplicity. The corresponding "Ext-matrix" $E_L(A)$ is given by

$$(E_L(A))_{ij} = \dim_k \mathrm{Ext}^1_A(L_i, L_j)$$

or equivalently

$$\begin{aligned}\dim_k \mathrm{Ext}^1_A(L_i, L_j) &= \dim_k(\mathrm{Hom}_A(P_j, \mathrm{Rad}(P_i))/\mathrm{Hom}_A(P_j, \mathrm{Rad}^2(P_i))) \\ &= \dim_k(e_j J(A)e_i / e_j J^2(A)e_i).\end{aligned}$$

This perhaps looks technical, but note that $e_j J(A)e_i = e_j Ae_i$ when $i \neq j$ and so then is essentially what we study in Section 4B et seq. (and in our case the quotient factor is even conjecturally zero, so in fact we are already studying the Ext-matrix!). Note that the Ext-matrix defines a quiver and hence a path algebra $kE_L(A)$. For any finite dimensional algebra $A$, basic or otherwise, the Cartan decomposition matrix $C_L(A)$ is given by

$$(C_L(A))_{ij} = \dim_k \mathrm{Hom}_A(P_j, P_i) \tag{5-12}$$

that is, the $i$-th row gives the number of times each simple module occurs in $P_i$.

**5F. *Linear structure of* $\mathbf{SP}_n$.** A corollary of Proposition 5.6 is that we have an embedding of module categories $G_\chi : \mathrm{SP}_{n-1} - \mathrm{mod} \to \mathrm{SP}_n - \mathrm{mod}$. In fact we can use this (together with our earlier calculations) to determine the structure of these algebras. Before giving the structure theorem let us recall the relevant general theory.

**Lemma 5.7** (see, e.g., [Green 1980, Section 6.2]). *Let $A$ be an algebra and $e \in A$ an idempotent. Then*:

(i) *The functor $Ae \otimes_{eAe} - : eAe - \mathrm{mod} \to A - \mathrm{mod}$ takes a complete set of inequivalent indecomposable projective left $eAe$-modules to a set of inequivalent indecomposable projective $A$-modules that is complete except for the projective covers of simple modules $L$ in which $eL = 0$.* (*There is a corresponding right-module version.*)

(ii) *This functor and the functor $\overline{G}_e : A - \mathrm{mod} \to eAe - \mathrm{mod}$ given by $M \mapsto eM$ form a left-right adjoint pair.*

(iii) *The Cartan decomposition matrix of $eAe$ embeds in that of $A$ according to the labeling of modules in* (*i*). □

**Theorem 5.8.** (i) *Isomorphism classes of irreps of* $\mathrm{SP}_n$ *are naturally indexed by $h_n$.* (*Indeed* $\mathrm{SP}_n/rad \cong \mathbb{Q}S_n/\boldsymbol{e}^1_{2,2}$ *so the dimensions are given by the $n$-th row of Pascal's triangle*; *see Figure 1.*)

(ii) *The left Cartan decomposition matrix is $\overline{\mathcal{M}}_n$. Note that this determines the structure of* $\mathrm{SP}_n$. *It gives the dimension as*

$$\dim = \binom{2(n-1)}{n-1} + \binom{2(n-1)}{n} = \tfrac{1}{2}\binom{2n}{n}.$$

(iii) *The image of the decomposition (4-6) is complete in* $\mathrm{SP}_n$.

*Proof.* (i) Consider Lemma 5.7(i). In our case, putting

$$A = \mathrm{SP}_n \quad \text{and} \quad e = \chi_1,$$

then by (5-7) there is exactly one module $L$ such that $eL = 0$ (at each $n$) — the trivial module. Thus by Proposition 5.6 $\mathrm{SP}_n$ has one more class of projectives and hence irreps than $\mathrm{SP}_{n-1}$.

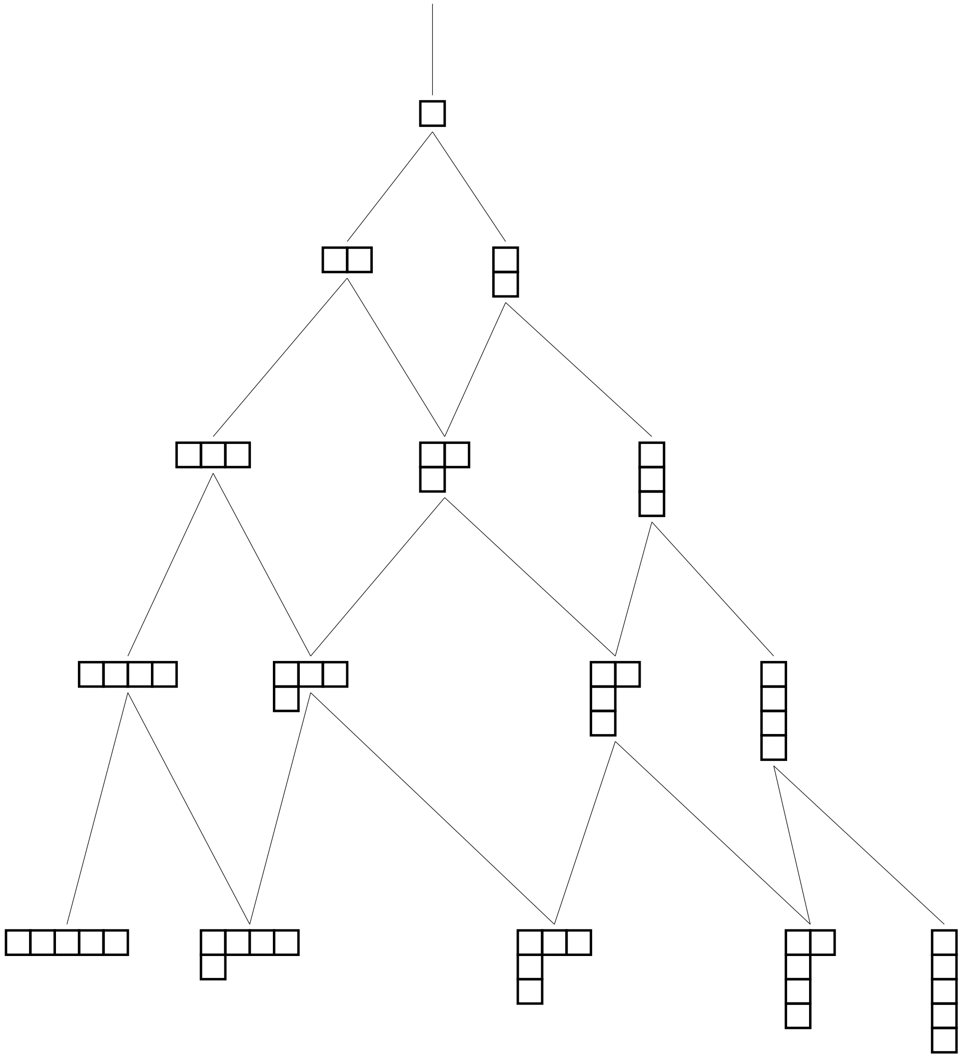


**Figure 1.** Young graph up to rank 5 with 22-diagrams removed.

In particular write

$$G_\chi : \mathrm{SP}_{n-1} - \mathrm{mod} \to \mathrm{SP}_n - \mathrm{mod}$$

for the functor in our case obtained using (5-6) from Proposition 5.6, that is

$$G_\chi(M) = \mathrm{SP}_n\, \chi \otimes_{\chi\, \mathrm{SP}_n\, \chi} f_\chi M,$$

suppressing the index $n$, where $f_\chi$ is as described above. Then a complete set of indecomposable projectives is

$$\begin{aligned}
P^n_n &= \mathrm{SP}_n\, e^1_{(n)},\\
P^n_{n-1} &= G_\chi(P^{n-1}_{n-1}) = G_\chi(\mathrm{SP}_{n-1}\, e^1_{(n-1)}),\\
P^n_{n-2} &= G_\chi(G_\chi(\mathrm{SP}_{n-2}\, e^1_{(n-2)}),\\
&\ \ \vdots\\
P^n_{n-j} &= G^{\circ j}_\chi(\mathrm{SP}_{n-j}\, e^1_{(n-j)}),\\
&\ \ \vdots\\
P^n_1 &= G^{\circ n-1}_\chi(k).
\end{aligned}$$

It follows that the Cartan decomposition matrix $C(n)$ contains $C(n-1)$ as a submatrix, with one new row and column with the label $n$. The new row gives the simple content of $P^n_n$. But by (4-7) (noting Theorem 5.2(ii)) this projective is simple. Iterating, we deduce that $C(n)$ is lower-unitriangular.

Working by induction, suppose $C(n)$ is of the claimed form in (ii) at level $n-1$. Then at level $n$ we have

$$C(n) = \left(\begin{array}{c|cccccc}
1 & & & & & & \\
\hline
* & 1 & & & & & \\
* & 1 & 1 & & & & \\
* & & 1 & 1 & & & \\
* & & & 1 & 1 & & \\
\vdots & & & & \ddots & \ddots & \\
* & & & & & 1 & 1
\end{array}\right) \tag{5-13}$$

(omitted entries 0). To complete the inductive step we need to compute the $e^1_{(n)} P_{n-j}$ for each $j$. Write $G^m_\chi$ for $G_\chi$ and $f^m_\chi$ for $f_\chi$ at level $m < n$, and note that

$$\begin{aligned}
G_\chi(\mathrm{SP}_{n-1}\, e^1_\lambda) &= \mathrm{SP}_n\, \chi_1 \otimes_{\chi\, \mathrm{SP}_n\, \chi} f_\chi(\mathrm{SP}_{n-1}\, e^1_\lambda)\\
&= \mathrm{SP}_n\, \chi_1 \otimes_{\chi\, \mathrm{SP}_n\, \chi} \chi_1\, \mathrm{SP}^{(1)}_{n-1}\, e^{(1)}_\lambda \chi_1\\
&= \mathrm{SP}_n\, \chi_1\, \mathrm{SP}^{(1)}_{n-1}\, e^{(1)}_\lambda \chi_1 \otimes_{\chi\, \mathrm{SP}_n\, \chi} \chi_1\\
&\cong \mathrm{SP}_n\, \chi_1\, \mathrm{SP}^{(1)}_{n-1}\, e^{(1)}_\lambda \chi_1
\end{aligned}$$

where we have used that these modules are idempotently generated ideals to apply the tensor product up to isomorphism (and where again we use the notation from (4-1), so $\mathrm{SP}^{(1)}_{n-1}$ is the 1-step translated copy of $\mathrm{SP}_{n-1}$ in $\mathrm{SP}_n$). So in particular

$$e^1_{(n)}\, \mathrm{SP}_n\, G_\chi(\mathrm{SP}_{n-1}\, e^1_{(n-1)}) \cong e^1_{(n)}\, \mathrm{SP}_n\, \chi_1\, \mathrm{SP}^{(1)}_{n-1}\, e^{(1)}_{(n-1)} \chi_1 \subseteq e^1_{(n)}\, \mathrm{SP}_n\, \chi_1.$$

It follows from the form of the image of $e^1_{(n)}$ in the SP representation (see [Hamermesh 1962; Martin 1992, Appendix B; Martin and Rittenberg 1992]) that the dimension of $e^1_{(n)}\,\mathrm{SP}_n\,\chi_1$ is 1, so the first $*$ is 1. Specifically we have for example

$$e_2=\tfrac{1}{2}(1+p_1)\mapsto\left(\begin{array}{c|cc|c} 1 & & & \\ \hline & 1/2 & 1/2 & \\ & 1/2 & 1/2 & \\ \hline & & & 0 \end{array}\right),\qquad \chi\mapsto\left(\begin{array}{c|cc|c} 0 & & & \\ \hline & 1 & -1 & \\ & 0 & 0 & \\ \hline & & & 1 \end{array}\right)$$

and

$$e_3\mapsto\frac{1}{3}\left(\begin{array}{c|ccc|ccc|c} 3 & & & & & & & \\ \hline & 1 & 1 & 1 & & & & \\ & 1 & 1 & 1 & & & & \\ & 1 & 1 & 1 & & & & \\ \hline & & & & 0 & 0 & 0 & \\ & & & & 0 & 0 & 0 & \\ & & & & 0 & 0 & 0 & \\ \hline & & & & & & & 0 \end{array}\right),\qquad \chi_1\mapsto\left(\begin{array}{c|ccc|ccc|c} 0 & & & & & & & \\ \hline & 0 & & & & & & \\ & & 1 & -1 & & & & \\ & & 0 & 0 & & & & \\ \hline & & & & 1 & -1 & 0 & \\ & & & & 0 & 0 & 0 & \\ & & & & 0 & 0 & 1 & \\ \hline & & & & & & & 1 \end{array}\right)$$

where we have reordered the basis into fixed charge sectors, i.e., as 111, 112, 121, 211, 122, 212, 221, 222 (the charge of a basis element is $\#(1)-\#(2)$, where $\#(1)$ is the number of 1's [Baxter 1982; Martin 1992]). Note from the construction that charge is conserved in SP, so each charge sector is a submodule. We see that in each charge sector except $(n-1,\,1)$ we have that either the image of $e^1_{(n)}$ is zero or the image of $\chi_1$ is zero. Finally in the $(n-1,\,1)$ sector both of these have rank 1. We deduce that $e^1_n A\chi_1$ is 1-dimensional as required.

Similarly we have to consider

$$\begin{aligned} G_\chi G_\chi^{n-1}(\mathrm{SP}_{n-2}\,e^1_{(n-2)}) &\cong \mathrm{SP}_n\,\chi_1 f_\chi f_\chi^{n-1}(\mathrm{SP}_{n-2}\,e^1_{(n-2)})\\ &\cong \mathrm{SP}_n\,\chi_1 f_\chi(\chi_1\,\mathrm{SP}^{(1)}_{n-2}\,e^{(1)}_{(n-2)}\chi_1)\\ &= \mathrm{SP}_n\,\chi_1\chi_1\chi_1^{(1)}\,\mathrm{SP}^{(2)}_{n-2}\,e^{(2)}_{(n-2)}\chi_1^{(1)}\chi_1, \end{aligned}$$

(NB $\chi_1^{(1)}=\chi_2$) giving

$$e^1_{(n)}\,\mathrm{SP}_n\,G_\chi G_\chi(\mathrm{SP}_{n-2}\,e^1_{(n-2)})\cong e^1_{(n)}\,\mathrm{SP}_n\,f_\chi f_\chi(\mathrm{SP}_{n-2}\,e^1_{(n-2)})=e^1_{(n)}\,\mathrm{SP}_n\,\chi_1\chi_2\ldots.$$

We have, in the charge block basis,

$$\chi_2 \mapsto \left(\begin{array}{c|ccc|ccc|c} 0 & & & & & & & \\ \hline & 1 & -1 & & & & & \\ & & 0 & 0 & & & & \\ & & 0 & 0 & & & & \\ \hline & & & & 1 & 0 & 0 & \\ & & & & 0 & 1 & -1 & \\ & & & & 0 & 0 & 0 & \\ \hline & & & & & & & 1 \end{array}\right), \quad \chi_1\chi_2 \mapsto \left(\begin{array}{c|ccc|ccc|c} 0 & & & & & & & \\ \hline & 0 & 0 & & & & & \\ & & 0 & 0 & & & & \\ & & 0 & 0 & & & & \\ \hline & & & & 1 & -1 & 1 & \\ & & & & 0 & 0 & 0 & \\ & & & & 0 & 0 & 0 & \\ \hline & & & & & & & 1 \end{array}\right)$$

(in general for a nonzero entry in $\chi_1\chi_2$ we need basis elements with 2 in the first and second position) so

$$e^1_{(n)}\,\mathrm{SP}_n\,\chi_1\chi_2 = 0.$$

**Remark.** Indeed we can verify that $e^1_{(n)}\chi_2\chi_1 = 0$ holds in $\mathrm{LH}_n$) so the second $*$ and indeed the other $*$s in (5-13) are all zero. We have verified the inductive step for (ii).

Statement (iii) may be deduced from (i,ii) as follows. Note that we have $n$ isomorphism classes in the decomposition, and their multiplicities are the dimensions of the hook irreps of $S_n$ in the natural order. On the other hand the $n+1$ charge blocks of the SP representation are each either an irrep or contains two irreps, since each contains one or two irreps upon restricting to $S_n$. The first is an irrep (since dimension 1). By the proof of (ii) the second contains the first irrep, so two irreps in total, and the other again has the same dimension as the corresponding $S_n$ hook representation. Furthermore no other block contains the first irrep so this block must be indecomposable (else the SP representation could not be faithful, which it is by definition). Proceeding through the blocks then by (ii) the first $n$ of them are a complete set of projective modules, so each one except the first and last contains two simple modules ("adjacent" in the hook order). But then by the construction of the Pascal triangle and (ii) these simple modules have the same dimension as the corresponding $S_n$ irreps, and (iii) follows. □

## 6. On representation theory of $\mathrm{LH}_n$

Combining (5-2) with (5-5) and Theorem 5.8 we have

$$\dim(\mathrm{SP}_n(t \neq 1)) = v_1\mathcal{U}^{n-1} \begin{pmatrix} 1 & & & & \\ 1 & 1 & & & \\ & 1 & 1 & & \\ & & 1 & 1 & \\ & & & \ddots & \ddots \end{pmatrix} (v_1\mathcal{U}^{n-1})^T = \binom{2n-1}{n-1} = \tfrac{1}{2}\binom{2n}{n}.$$

| | $t=1$ | $t=-1$ | $t^2\neq 1$ | $t\neq 1$ | irreps/dimensions | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $n$ | dim | dim | dim | $ss$ dim | –6 | –5 | –4 | –3 | –2 | –1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 1 | 1 | 1 | 1 | 1 | | | | | | | 1 | | | | | | |
| 2 | 3 | 3 | 3 | 2 | | | | | | 1 | | 1 | | | | | |
| 3 | 15 | 11 | 10 | 6 | | | | | 1 | | 2 | | 1 | | | | |
| 4 | 114 | 42 | 35 | 20 | | | | 1 | | 3 | | 3 | | 1 | | | |
| 5 | 1170 | 163 | 126 | 70 | | | 1 | | 4 | | 6 | | 4 | | 1 | | |
| 6 | 15570 | 638 | 462 | 252 | | 1 | | 5 | | 10 | | 10 | | 5 | | 1 | |
| 7 | | 2510 | 1716 | 924 | 1 | | 6 | | 15 | | 20 | | 15 | | 6 | | 1 |

**Table 1.** A summary of what we learn for the algebra dimensions, and irreducible reps, of $\mathrm{LH}_n$. The irrep labels here are given by $(n-i, 1^i) \mapsto n-2i-1$.

NB we have used the obvious "global" limit of all the Cartan matrices (it is a coincidence that this and the $\mathcal{U}$ matrix are similar).

Given a vector $v$ we write $\mathrm{Diag}(v)$ for the diagonal matrix with $v$ down the diagonal. Let $p^n$ be the vector with the $n$-th row of Pascal's triangle as the entries, thus for example $p^4 = (1, 3, 3, 1)$. We have

$$\mathcal{M}_n^p := \mathrm{Diag}(p^n)\overline{\mathcal{M}}_n \,\mathrm{Diag}(p^n)$$

(examples are given in (6-2) below) and the dimension is the sum of all the entries. The closed form follows readily from this. Also from Theorem 5.8 we have:

**Corollary 6.1.** *For $t \neq 1$ the Morita class of $\mathrm{SP}_n$ is of the path algebra with $A_n$ quiver (directed $1 \to 2 \to \cdots \to n$) and relations given by vanishing of all proper paths of length 2. In particular the radical-squared vanishes.*

**6A.** ***Properties determined from Theorem 5.8 and direct calculation in low rank.*** Our results for $\mathrm{LH}_n$ may be neatly given as follows. Firstly,

**Proposition 6.2.** *For $t^2 \neq 1$ and $n < 8$,*

$$\mathrm{LH}_n \cong \mathrm{SP}_n\,.$$

*Proof.* Here we can compute dimensions directly, which saturates the bound on the kernel. □

**Conjecture 6.3.** *For $t^2 \neq 1$,*

$$\mathrm{LH}_n \cong \mathrm{SP}_n\,.$$

Combining (5-2) with (5-5), Theorems 5.8 and 6.3 we have the conjecture

$$\dim(\mathrm{LH}_n(t^2 \neq 1)) = v_1\mathcal{U}^{n-1} \begin{pmatrix} 1 &&&& \\ 1 & 1 &&& \\ & 1 & 1 && \\ && 1 & 1 & \\ &&& \ddots & \ddots \end{pmatrix} (v_1\mathcal{U}^{n-1})^T = \binom{2n-1}{n-1} = \tfrac{1}{2}\binom{2n}{n}.$$

For $t = -1$ we note that $\mathrm{SP}_n$ is generally a proper quotient of $\mathrm{LH}_n$, and that $\mathrm{LH}_n$ has larger radical (the square does not vanish). We define the semiinfinite matrix

$$C(\mathrm{LH}(t = -1)) = \begin{pmatrix} 1 &&&& \\ 1 & 1 &&& \\ 1 & 1 & 1 && \\ 1 & 1 & 1 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and conjecture that the Cartan matrix $C(\mathrm{LH}_n(t = -1))$ is this truncated at $n \times n$ (i.e., the quiver is the same as the generic case, but without quotient relations); and thus we conjecture

$$\dim(\mathrm{LH}_n(t = -1)) = v_1\mathcal{U}^{n-1} \begin{pmatrix} 1 &&&& \\ 1 & 1 &&& \\ 1 & 1 & 1 && \\ 1 & 1 & 1 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} (v_1\mathcal{U}^{n-1})^T = \frac{n^2 + \binom{2n-2}{n-1}}{2}; \tag{6-1}$$

see OEIS A032443. Note that our calculations verify this for $n \leq 7$.

For $t = 1$ we see that $\mathrm{LH}_n(t = 1)$ has semisimple quotient at least as big as $\mathbb{C}S_n$, which is of dimension $n!$. Indeed, in this case the quotient by the relation $\sigma_i = \rho_i$ is precisely $\mathbb{C}S_n$, since in this case $\sigma_i^2 = 1$. For $n \leq 4$ we have computationally verified that the semisimple subalgebra of $\mathrm{LH}_n(t = 1)$ is precisely $\mathbb{C}S_n$, and we conjecture that this is the case for all $n$. The Jacobson radical grows quite quickly however, and we do not have a conjecture on the general structure.

Observe that the numbers in Table 1 follow the conjectured patterns. Since the vector $v_1$ has finite support the nominally infinite sums above are all finite. To inspect the supported part, in the generic case consider matrices $\mathcal{M}_n^p$ ($n = 2, 3, 4, 5$)

$$\begin{pmatrix} 1 & \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 && \\ 2 & 2^2 & \\ & 2 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 &&& \\ 3 & 3^3 && \\ & 9 & 3^3 & \\ && 3 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 &&&& \\ 4 & 4^2 &&& \\ & 24 & 6^2 && \\ && 24 & 4^2 & \\ &&& 4 & 1 \end{pmatrix}. \tag{6-2}$$

Here the semisimple dimension is given by the sum down the diagonal and the radical dimension is given by the sum in the off-diagonal.

For $t=-1$

$$\begin{pmatrix}1&\\1&1\end{pmatrix},\quad \begin{pmatrix}1&&\\2&2^2&\\1&2&1\end{pmatrix},\quad \begin{pmatrix}1&&&\\3&3^3&&\\3&9&3^3&\\1&3&3&1\end{pmatrix},\quad \begin{pmatrix}1&&&&\\4&4^2&&&\\6&24&6^2&&\\4&16&24&4^2&\\1&4&6&4&1\end{pmatrix}.$$

**6B. *On $\chi$ elements.*** Let us define

$$\chi^{(m+1)}=(\sigma_1-\rho_1)(\sigma_2-\rho_2)\dots(\sigma_m-\rho_m), \tag{6-3}$$

understood as an element in $\mathrm{LH}_n$ with $n>m$. Thus in particular $\chi^{(2)}=\chi_1$. Similarly for sequence $X=(x_1,x_2,\dots,x_k)$ define

$$\chi^{(X)}=(\sigma_{x_1}-\rho_{x_1})(\sigma_{x_2}-\rho_{x_2})\cdots(\sigma_{x_k}-\rho_{x_k}), \tag{6-4}$$

and

$$\chi_-^{(m+1)}=(\sigma_m-\rho_m)(\sigma_{m-1}-\rho_{m-1})\cdots(\sigma_2-\rho_2)(\sigma_1-\rho_1). \tag{6-5}$$

It is easy to verify that if $X$ is nonincreasing then $\chi^{(X)}\chi^{(X)}=(1-t)^k\chi^{(X)}$. Thus (for $t\neq 1$) the nonincreasing cases can all be normalised as idempotents. However it is also easy to check that no increasing case can. (A nice illustration of the “chirality” present in the defining relations.)

Observe that imposing the relation $\sigma_1=\rho_1$ in $\mathrm{LH}_n$ forces $\sigma_1=1$, unless $t=1$. Thus the quotient algebra

$$\mathrm{LH}_n/\chi^{(2)}\cong k,\qquad t\neq 1 \tag{6-6}$$

i.e., only the trivial, or label $\lambda=+n$, irrep survives. And the same holds for $\mathrm{SP}_n$. The following has been checked up to rank 5.

**Conjecture 6.4.** *The structure of the quotient* $\mathrm{LH}_n/\chi^{(j+1)}$ *is given by the* $j\times j$ *truncation of* $\mathcal{M}_n^p$.

## 7. Discussion and avenues for future work

Above we give answers to the main structural questions for $\mathrm{SP}_n$ and $\mathrm{LH}_n$. But exploration of generalisations is also well-motivated, since these algebras (even taken together with the constructions discussed in [Kádár et al. 2017]) cover a relatively small quotient inside $\mathrm{Rep}(\mathrm{LB}_n)$. With this in mind, there are a number of other questions worth addressing around $\mathrm{SP}_n$ and $\mathrm{LH}_n$, offering clues on generalisation, and hence towards understanding more of the structure of the group algebra. Remark 3.6 suggests that for most values of $t$ we obtain larger finite dimensional quotients by eliminating one of the local relations (3-12) or (3-13). Computational experiments suggest that for $t = 0$ eliminating (3-13) yields infinite dimensional algebras. This parameter-dependence should be further explored.

In light of the results of [Reutter 2020] the nonsemisimplicity of $\mathrm{LH}_n$ is an important feature, rather than a shortcoming. Extracting topological information from the nonsemisimple part requires some further work, as Markov traces typically "see" the semisimple part. Another aspect of our work is the (conjectural) localisation of the regular representation of $\mathrm{LH}_n$. It is worth pointing out that localisations of *unitary* sequences of $B_n$ representations are relatively rare, conjecturally corresponding to representations with finite braid group image [Rowell and Wang 2012; Galindo et al. 2013]. Since $\mathrm{LH}_n$ is nonsemisimple and hence nonunitary this does not contradict this conjectural relationship, but gives us some hope that localisations are possible for other parameter choices and other quotients.

The quotient of $\mathrm{LB}_n$ by the relation ${\sigma_i}^2 = 1$ is a potentially interesting infinite group, which we call the mixed double symmetric group $\mathrm{MDS}_n$. The reason for this nomenclature is that $\mathrm{MDS}_n$ is a quotient of the free product of two copies of the symmetric group. In particular, $\mathrm{MDS}_n$ surjects onto $S_n$ by $\sigma_i \to \rho_i$. It is of special interest here as $\mathrm{LH}_n(1)$ is a quotient of $\mathbb{Z}[\mathrm{MDS}_n]$. We expect it could be of quite general interest.

In [Kádár et al. 2017] constructions are developed based on BMW algebras, but still starting from "classical" precepts. It would be very interesting to meld the super-Burau–Rittenberg construction to the KMRW construction. For example, one might try to use cubic local (eigenvalue) relations among the generators $\rho_i, \sigma_i$ to obtain finite dimensional quotients, possibly inspired by the relations satisfied by a subsequence of $\mathrm{LB}_n$ lifts of BMW algebra representations.

## Appendix: Preparatory arithmetic and notation for left ideals

**AA.** ***Symmetric group and Hecke algebra arithmetic.*** Recall Young's (anti) symmetrisers in $kS_n$. Unnormalised in $\mathbb{Z}S_n$ they are

$$Y^n_\pm = \sum_{g \in S_n} (\pm 1)^{\mathrm{len}(g)} g \tag{A-1}$$

where $\mathrm{len}(g)$ is the usual Coxeter length function. If $k$ has characteristic 0 then $kS_n$ is semisimple and these elements are simply the (unnormalised) idempotents corresponding to the trivial and alternating representations respectively. Note that exactly the same classical construction works for the Hecke algebra over any field where it is semisimple. (The corresponding idempotents are sometimes called Jones–Wenzl projectors.) Specifically (see e.g., [Curtis and Reiner 1981, Section 9B])

$$X^n_\pm = \sum_{g \in S_n} (-\lambda_\mp)^{-\mathrm{len}(g)} T_g, \quad \text{i.e., } X^2_- = 1 - \sigma_1,\ X^2_+ = 1 + t^{-1}\sigma_1, \ldots$$

where for us $\lambda_- = -t$ and $\lambda_+ = 1$ (the apparent flip of labels is just because we use non-Lusztig scaling), and $T_g$ is the product of generators obtained by writing $g$ in reduced form then applying $\rho_i \mapsto T_i$.

Working in $kS_{n+m}$ we understand $Y^n_\pm$ and translates such as $Y^{n(1)}_+$ in the obvious way. Note then that we have many identities like

$$Y^2_+ Y^n_+ = 2Y^n_+, \quad Y^{a(1)} Y^n_+ = a! Y^n_+ \quad (a < n). \tag{A-2}$$

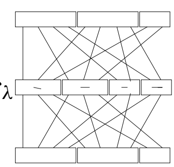
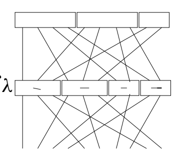
Recall $\Lambda_n$ denotes the set of integer partitions of $n$. Over the rational field we have a decomposition of $1 \in kS_n$ into primitive central idempotents

$$1 = \sum_{\lambda \in \Lambda_n} \epsilon_\lambda \tag{A-3}$$

where each $\epsilon_\lambda$ is a known unique element; see e.g., [Cohn 1977, Section 7.6] or [Curtis and Reiner 1981] for gentle expositions. There is a further (not generally unique) decomposition of each $\epsilon_\lambda$ into primitive orthogonal idempotents

$$\epsilon_\lambda = \sum_{i=1}^{\dim_\lambda} \boldsymbol{e}^i_\lambda \tag{A-4}$$

where $\dim_\lambda$ is the number of walks from the root to $\lambda$ on the directed Young graph. The elements $\boldsymbol{e}^i_\lambda$ are conjugate to each other. The elements $\boldsymbol{e}^i_\lambda$ are not uniquely defined in general. Two possible constructions of one for each $\lambda$ are exemplified pictorially by (case $\lambda = 442$)

(A-5) $e^1_\lambda = c_\lambda$ , $\quad \hat{e}^1_\lambda = c_\lambda$ 

where an undecorated box is a symmetriser and a "$-$" decorated box an antisymmetriser, and the factor $c_\lambda$ is just a scalar. (NB For the moment we write $e^1_\lambda$ instead of $\boldsymbol{e}^1_\lambda$ for this specific choice.) In particular though, $\boldsymbol{e}^1_{(n)}$ *is* unique: $\boldsymbol{e}^1_{(n)} = \frac{1}{n!} Y^n_+$. (The whole story lifts to the Hecke case; see e.g., [Martin 1991] for a gentle exposition.)

An idempotent decomposition of 1 in a subalgebra $B$ of an algebra $A$ is of course a decomposition in $A$. Thus in particular we can take an idempotent in $kS_n$ and consider it as an idempotent in $kS_{n+1}$ by the inclusion that is natural from the presentation ($p_i \mapsto p_i$). Understanding $\boldsymbol{e}^1_\lambda$ with $\lambda \vdash n$ in $kS_{n+1}$ in this way, a useful property in our $k = \mathbb{C}$ case will be

$$\boldsymbol{e}^j_\lambda = \sum_{\mu \in \lambda +} \boldsymbol{e}'_\mu \tag{A-6}$$

where $\lambda+$ denotes the set of partitions obtained from $\lambda$ by adding a box, and the prime indicates that we identify this idempotent only up to equivalence. (Various proofs exist. For example note that the existence of such a decomposition follows from the induction rules for $S_n \hookrightarrow S_{n+1}$.) For example

$$\boldsymbol{e}^1_{(2,2)} = \boldsymbol{e}'_{(3,2)} + \boldsymbol{e}'_{(2,2,1)}.$$

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