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We introduce *Backström pairs* and *Backström rings*, study their derived categories and construct for them a sort of *categorical resolutions*. For the latter we define the global dimension, construct a sort of semiorthogonal decomposition of the derived category and deduce that the derived dimension of a Backström ring is at most 2. Using this semiorthogonal decomposition, we define a description of the derived category as the category of elements of a special bimodule. We also construct a partial tilting for a Backström pair to a ring of triangular matrices and define the global dimension of the latter.

Introduction

Backström orders were introduced in [Ringel and Roggenkamp 1979], where it was shown that their representations are in correspondence with those of quivers or species. A special class of Backström orders are *nodal orders*, which appeared (without this name) in [Drozd 1990] as such pure noetherian algebras that the classification of their finitely generated modules is tame. In [Burban and Drozd 2004] tameness was also proved for the derived categories of nodal orders. Global analogues of nodal algebras, called *nodal curves*, were considered in [Burban and Drozd 2011; Drozd and Voloshyn 2012; Voloshyn and Drozd 2013]. Namely, in [Burban and Drozd 2011] a sort of tilting theory for such curves was developed, which related them to some quasihereditary finite dimensional algebras. In [Drozd and Voloshyn 2012] a criterion was found for a nodal curve to be tame with respect to the classification of vector bundles, and in [Voloshyn and Drozd 2013] it was proved that the same class of curves has tame derived categories. It was clear that the tilting theory of [Burban and Drozd 2011] can be extended to a general situation, namely, to *Backström curves*, i.e., noncommutative curves having Backström orders as their localizations. Nodal orders and related gentle algebras appear in studying mirror symmetry, see for instance, [Lekili and Polishchuk 2018]. Finite dimensional

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analogues of nodal orders, called *nodal algebras*, were introduced in [Drozd and Zembyk 2013; Zembyk 2014]. In the latter paper their structure was completely described. In [Zembyk 2015] it was shown that certain important classes of algebras, such as gentle and skewed-gentle algebras, are nodal. In [Burban and Drozd 2017] a tilting theory was developed for nodal algebras, which was applied to the study of derived categories of gentle and skewed-gentle algebras.

This paper is devoted to a tilting theory for *Backström rings*, which are a straightforward generalization of Backström orders and algebras.

In Section 1, we propose a variant of partial tilting, which generalizes the technique of minors from [Burban et al. 2017].

In Section 2, we introduce *Backström pairs*, which are pairs of semiperfect rings $\boldsymbol{H} \supseteq \boldsymbol{A}$ with a common radical; (piecewise) *Backström rings* are likewise introduced as those rings $\boldsymbol{A}$ that occur in (piecewise) Backström pairs with (piecewise) hereditary $\boldsymbol{H}$. We construct the *Auslander envelope* $\tilde{\boldsymbol{A}}$ of a Backström pair and calculate its global dimension. It turns out that this global dimension only depends on the global dimension of $\boldsymbol{H}$. In particular, Auslander envelopes for Backström rings are of global dimension at most 2.

In Section 3, we apply the tilting technique to show that the derived category of the algebra $\boldsymbol{A}$ is connected by a recollement with the derived category of its Auslander envelope. This implies that the derived dimension of $\boldsymbol{A}$ in the sense of [Rouquier 2008] is not greater than that of the Auslander envelope.

In Section 4, we consider a recollement between the derived categories of the algebra $\boldsymbol{H}$ and of the Auslander envelope. It is used to calculate the derived dimension of the Auslander envelope, thus obtaining an upper bound for the derived dimension of the algebra $\boldsymbol{A}$. In particular, we prove that the derived dimension of a Backström or piecewise Backström algebra is at most 2. Moreover, if $\boldsymbol{A}$ is a Backström or piecewise Backström algebra of Dynkin type, then either it is piecewise hereditary of Dynkin type, so $\operatorname{der.dim} \boldsymbol{A} = 0$, or else $\operatorname{der.dim} \boldsymbol{A} = 1$.

In Section 5, we establish an equivalence between the category $\mathscr{D}(\tilde{\boldsymbol{A}})$ and a bimodule category. This gives a useful instrument for calculations in this derived category. (See, for instance, [Bekkert et al. 2003; Bekkert and Merklen 2003; Burban and Drozd 2004; 2006; 2017; Voloshyn and Drozd 2013].)

In Section 6, we consider another partial tilting for the Auslander envelope $\tilde{\boldsymbol{A}}$ of a Backström pair, relating its derived category by a recollement to the derived category of an algebra $\boldsymbol{B}$ of triangular matrices which looks simpler than the Auslander algebra. In this case, we calculate explicitly the global dimension of $\boldsymbol{B}$ and the kernel of the partial tilting functor

$$\mathsf{F} : \mathscr{D}(\boldsymbol{B}) \to \mathscr{D}(\boldsymbol{A}).$$

1. Partial tilting

Let $\mathscr{T}$ be a triangulated category, $\mathfrak{R} \subseteq \operatorname{Ob} \mathscr{T}$. We denote by $\operatorname{Tri}(\mathfrak{R})$ the smallest strictly full triangulated subcategory containing $\mathfrak{R}$ that is closed under coproducts (this means that if a coproduct of objects from $\operatorname{Tri}(\mathfrak{R})$ exists in $\mathscr{T}$, it belongs to $\operatorname{Tri}(\mathfrak{R})$). For a DG-category $\mathscr{R}$ we denote by $\mathscr{D}(\mathscr{R})$ its derived category [Keller 1994]. The following result is a generalization of [Lunts 2010, Proposition 2.6]:

Theorem 1.1. *Let $\mathfrak{R}$ be a subset of the set of compact objects of $\operatorname{Ob}\mathscr{D}(\mathscr{A})$, where $\mathscr{A}$ is a Grothendieck category. We consider the DG-category $\mathscr{R}$ with the set of objects $\mathfrak{R}$ and the sets of morphisms $\mathscr{R}(T, R) = \mathbb{R}\mathrm{Hom}(T, R)$. Define the functor $\mathsf{F} : \mathscr{D}(\mathscr{A}) \to \mathscr{D}(\mathscr{R}^{\mathrm{op}})$ by mapping a complex C to the DG-module $\mathsf{F}C = \mathbb{R}\mathrm{Hom}_{\mathscr{D}(\mathscr{A})}(-, C)$ restricted onto $\mathfrak{R}$.*

(1) *The restriction of F onto $\operatorname{Tri}(\mathfrak{R})$ is an equivalence $\operatorname{Tri}(\mathfrak{R}) \to \mathscr{D}(\mathscr{R}^{\mathrm{op}})$.*

(2) *There is a recollement diagram in the sense of* [Beĭlinson et al. 1982, 1.4.3]

$$\operatorname{Ker}\mathsf{F} \;\underset{\overset{\longleftarrow}{\mathsf{I}^!}}{\overset{\overset{\mathsf{I}^*}{\longleftarrow}}{\xrightarrow{\ \mathsf{I}\ }}}\; \mathscr{D}(\mathscr{A}) \;\underset{\overset{\longleftarrow}{\mathsf{F}^!}}{\overset{\overset{\mathsf{F}^*}{\longleftarrow}}{\xrightarrow{\ \mathsf{F}\ }}}\; \mathscr{D}(\mathscr{R}^{\mathrm{op}}), \tag{1-1}$$

where I is the embedding.[1]

Recall that this means that the following conditions hold:

(a) *F and I are exact.*

(b) $\mathsf{F}\mathsf{I} = 0$.

(c) *F^* and $F^!$ are left and right adjoint functors to F, respectively.*

(d) *Both adjunction morphisms $\eta : \mathrm{Id}_{\mathscr{D}(\mathscr{R}^{\mathrm{op}})} \to \mathsf{F}\mathsf{F}^*$ and $\zeta : \mathsf{F}\mathsf{F}^! \to \mathrm{Id}_{\mathscr{D}(\mathscr{R}^{\mathrm{op}})}$ are isomorphisms.*

(e) *The same holds for the triple $(\mathsf{I}, \mathsf{I}^*, \mathsf{I}^!)$.*

(Note that Condition 1.4.3.4 from [Beĭlinson et al. 1982] is a consequence of the other ones; see [Neeman 2001, 9.2].)

If $\mathfrak{R}$ generates $\mathscr{D}(\mathscr{A})$, we obtain an equivalence $\mathscr{D}(\mathscr{A}) \simeq \mathscr{D}(\mathscr{R}^{\mathrm{op}})$, as in [Lunts 2010]. *If $\mathfrak{R}$ consists of one object R, we obtain an equivalence $\operatorname{Tri}(R) \simeq \mathscr{D}(\boldsymbol{R}^{\mathrm{op}})$, where $\boldsymbol{R} = \mathbb{R}\mathrm{Hom}(R, R)$.*

Proof. (1) We identify $\mathscr{D}(\mathscr{A})$ with the homotopy category $\mathscr{I}(\mathscr{A})$ of *K-injective complexes*, i.e., complexes I such that $\mathrm{Hom}(C, I)$ is acyclic for every acyclic complex C, and suppose that $\mathfrak{R} \subseteq \mathscr{I}(\mathscr{A})$. Then, $\mathbb{R}\mathrm{Hom}$ coincides with Hom within the category $\mathscr{I}(\mathscr{A})$; so, for $C \in \mathscr{I}(\mathscr{A})$, $\mathsf{F}C = \mathrm{Hom}_{\mathscr{I}(\mathscr{A})}(-, C))$ restricted onto $\mathfrak{R}$. The full subcategory of $\mathscr{I}(\mathscr{A})$ consisting of complexes C such that the natural map $\mathrm{Hom}_{\mathscr{I}(\mathscr{A})}(R, C) \to \mathrm{Hom}_{\mathscr{D}(\mathscr{R}^{\mathrm{op}})}(\mathsf{F}R, \mathsf{F}C)$ is bijective for all $R \in \mathfrak{R}$ contains $\mathfrak{R}$, is strictly full, triangulated and closed under coproducts, since all objects from $\mathscr{R}$ are

[1] Note that $\mathfrak{R}$ is not necessarily *recollement-defining* in the sense of [Nicolás and Saorín 2009].

compact. Therefore, it contains $\mathrm{Tri}(\mathfrak{R})$. Quite analogously, the full subcategory of complexes C such that the natural map $\mathrm{Hom}_{\mathscr{I}(\mathscr{A})}(C, C') \to \mathrm{Hom}\,\mathscr{I}(\mathscr{A})(\mathsf{F}C, \mathsf{F}C')$ is bijective for every $C' \in \mathrm{Tri}(\mathfrak{R})$ also contains $\mathrm{Tri}(\mathfrak{R})$. Hence, the restriction of F onto $\mathrm{Tri}(\mathfrak{R})$ is fully faithful. Moreover, as the functors $\mathrm{Hom}_{\mathscr{R}}(-, R)$, where R runs through $\mathfrak{R}$, generate $\mathscr{D}(\mathscr{R}^{\mathrm{op}})$, the functor F is essentially surjective. Therefore, restricted to $\mathrm{Tri}(\mathfrak{R})$, it gives an equivalence $\mathrm{Tri}(\mathfrak{R}) \to \mathscr{D}(\mathscr{R})$.

(2) Note that $\mathscr{D}(\mathscr{R}^{\mathrm{op}})$ is cocomplete and compactly generated, hence satisfies the Brown representability theorem [Neeman 2001, Theorem 8.3.3]. Therefore, it is true for $\mathrm{Tri}(\mathfrak{R})$ too. Then, [Neeman 2001, Proposition 9.1.19] implies that a Bowsfield localization functor exists for $\mathrm{Tri}(\mathfrak{R}) \subseteq \mathscr{D}(\mathscr{A})$ and [Neeman 2001, Proposition 9.1.18] implies that the embedding $\mathsf{E} : \mathrm{Tri}(\mathfrak{R}) \to \mathscr{D}(\mathscr{A})$ has a right adjoint $\Theta : \mathscr{D}(\mathscr{A}) \to \mathrm{Tri}(\mathfrak{R})$. Let $\mathsf{F}' : \mathscr{D}(\mathscr{R}^{\mathrm{op}}) \to \mathrm{Tri}(\mathfrak{R})$ be a quasi-inverse to the restriction of F onto $\mathrm{Tri}(\mathfrak{R})$. In particular, F' is a left adjoint to this restriction and the adjunction $\mathsf{F}\mathsf{F}' \to \mathrm{Id}_{\mathscr{D}(\mathscr{R}^{\mathrm{op}})}$ is an isomorphism. Then,

$$\mathsf{F}C = \mathrm{Hom}_{\mathscr{I}(\mathscr{A})}(-, C)|_{\mathfrak{R}} \simeq \mathrm{Hom}_{\mathscr{I}(\mathscr{A})}(-, \Theta C)|_{\mathfrak{R}} = \mathsf{F}\Theta C.$$

Set $\mathsf{F}^* = \mathsf{E}\mathsf{F}'$. Since $\mathsf{F}'M \in \mathrm{Tri}(\mathfrak{R})$ for every $M \in \mathscr{D}(\mathscr{R}^{\mathrm{op}})$,

$$\begin{aligned}\mathrm{Hom}_{\mathscr{I}(\mathscr{A})}(\mathsf{F}^*M, C) &\simeq \mathrm{Hom}_{\mathrm{Tri}(\mathfrak{R})}(\mathsf{F}'M, \Theta C)\\ &\simeq \mathrm{Hom}_{\mathscr{D}(\mathscr{R}^{\mathrm{op}})}(M, \mathsf{F}\Theta C) \simeq \mathrm{Hom}_{\mathscr{D}(\mathscr{R}^{\mathrm{op}})}(M, \mathsf{F}C),\end{aligned}$$

for any $M \in \mathscr{D}(\mathscr{R}^{\mathrm{op}})$ and $C \in \mathsf{I}(\mathscr{A})$. Hence, F^* is a left adjoint to F. If, moreover, $C \in \mathrm{Tri}(\mathfrak{R})$, we obtain

$$\mathrm{Hom}_{\mathscr{D}(\mathscr{R}^{\mathrm{op}})}(\mathsf{F}\mathsf{F}^*M, \mathsf{F}C) \simeq \mathrm{Hom}_{\mathscr{I}(\mathscr{A})}(\mathsf{F}^*M, C) \simeq \mathrm{Hom}_{\mathscr{D}(\mathscr{R}^{\mathrm{op}})}(M, \mathsf{F}C).$$

As F is essentially surjective, this implies that $\eta : \mathsf{F}\mathsf{F}^* \to \mathrm{Id}_{\mathscr{D}(\mathscr{R}^{\mathrm{op}})}$ is an isomorphism. As all objects from $\mathfrak{R}$ are compact, F respects coproducts, hence has a right adjoint $\mathsf{F}^!$ [Neeman 2001, Theorem 8.4.4]. Now it follows from [Burban et al. 2017, Corollary 2.3] that $\zeta : \mathsf{F}\mathsf{F}^! \to \mathrm{Id}_{\mathscr{D}(\mathscr{R}^{\mathrm{op}})}$ is an isomorphism and there is a recollement diagram (1-1). □

Note that $\mathrm{Im}\,\mathsf{F}^* = \mathrm{Tri}(\mathfrak{R})$ by construction, but usually $\mathrm{Im}\,\mathsf{F}^! \neq \mathrm{Tri}(\mathfrak{R})$, though it is equivalent to $\mathrm{Tri}(\mathfrak{R})$.

Corollary 1.2. *Under the conditions and notations of the preceding theorem, suppose that* $\mathrm{Hom}_{\mathscr{D}(\mathscr{A})}(R, T[m]) = 0$ *for* $R, T \in \mathfrak{R}$ *and* $m \neq 0$. *Then, the functor* F *induces an equivalence* $\mathrm{Tri}(R) \xrightarrow{\sim} \mathscr{D}(\mathscr{R}^{\mathrm{op}})$, *where* $\mathscr{R}$ *is the category with the set of objects* $\mathfrak{R}$ *and the sets of morphisms* $\mathscr{R}(A, B) = \mathrm{Hom}_{\mathscr{D}(\mathscr{A})}(A, B)$.

In this situation, we call the functor F *a partial tilting functor.*

2. Backström pairs

Recall from [Bass 1960; Lambek 1976] that a *semiperfect ring* is a ring $\boldsymbol{A}$ such that $\boldsymbol{A}/\operatorname{rad}\boldsymbol{A}$ is a semisimple artinian ring and idempotents can be lifted modulo $\operatorname{rad}\boldsymbol{A}$. Equivalently, as a left (or as a right) $\boldsymbol{A}$-module, $\boldsymbol{A}$ decomposes into a direct sum of modules with local endomorphism rings.

Definition 2.1. (1) A *Backström pair* is a pair of semiperfect rings $\boldsymbol{H} \supseteq \boldsymbol{A}$ such that $\operatorname{rad}\boldsymbol{A} = \operatorname{rad}\boldsymbol{H}$. We denote by $\boldsymbol{C}(\boldsymbol{H}, \boldsymbol{A})$ the *conductor* of $\boldsymbol{H}$ in $\boldsymbol{A}$:

$$\boldsymbol{C}(\boldsymbol{H}, \boldsymbol{A}) = \{\alpha \in \boldsymbol{A} \mid \boldsymbol{H}\alpha \subseteq \boldsymbol{A}\} = \operatorname{ann}(\boldsymbol{H}/\boldsymbol{A})_{\boldsymbol{A}}$$

(the right subscript $_{\boldsymbol{A}}$ means that we consider $\boldsymbol{H}/\boldsymbol{A}$ as a right $\boldsymbol{A}$-module). Obviously, $\boldsymbol{C}(\boldsymbol{H}, \boldsymbol{A}) \supseteq \operatorname{rad}\boldsymbol{A}$, so both $\boldsymbol{A}/\boldsymbol{C}$ and $\boldsymbol{H}/\boldsymbol{C}$ are semisimple rings.

(2) We call a ring $\boldsymbol{A}$ a (left) *Backström ring* (resp. *piecewise Backström ring*) if there is a Backström pair $\boldsymbol{H} \supseteq \boldsymbol{A}$, where the ring $\boldsymbol{H}$ is left hereditary (resp. *left piecewise hereditary* [Happel 1988], i.e., derived equivalent to a left hereditary ring). If, moreover, both $\boldsymbol{A}$ and $\boldsymbol{H}$ are finite dimensional algebras over a field $\Bbbk$, we call $\boldsymbol{A}$ a *Backström algebra* (resp. *piecewise Backström algebra*).

Remark 2.2. If e is an idempotent in $\boldsymbol{A}$, then $\operatorname{rad}(e\boldsymbol{A}e) = e(\operatorname{rad}\boldsymbol{A})e$, hence, if $\boldsymbol{H} \supseteq \boldsymbol{A}$ is a Backström pair, so is $e\boldsymbol{H}e \supseteq e\boldsymbol{A}e$. This implies that if P is a finitely generated projective $\boldsymbol{A}$-module, $\boldsymbol{A}' = \operatorname{End}_{\boldsymbol{A}} P$ and $\boldsymbol{H}' = \operatorname{End}_{\boldsymbol{H}}(\boldsymbol{H} \otimes_{\boldsymbol{A}} P)$, then $\boldsymbol{H}' \supseteq \boldsymbol{A}'$ is also a Backström pair. Note that if $\boldsymbol{H}$ is left hereditary (or piecewise hereditary), so is $\boldsymbol{H}'$, hence $\boldsymbol{A}'$ is a Backström ring (piecewise Backström ring) whenever $\boldsymbol{A}$ is. In particular, the notion of Backström (or piecewise Backström) ring is Morita invariant. Note also that if $\boldsymbol{H}$ is left hereditary and noetherian, it is also right hereditary, so $\boldsymbol{A}^{\mathrm{op}}$ is also a Backström ring (piecewise Backström ring).

Examples 2.3. (1) An important example of Backström algebras are *nodal algebras* introduced in [Drozd and Zembyk 2013; Zembyk 2014]. By definition, they are finite dimensional algebras such that there is a Backström pair $\boldsymbol{H} \supseteq \boldsymbol{A}$, where $\boldsymbol{H}$ is a hereditary algebra and $\operatorname{length}_{\boldsymbol{A}}(\boldsymbol{H} \otimes_{\boldsymbol{A}} U) \le 2$ for every simple $\boldsymbol{A}$-module U. Their structure was completely described in [Zembyk 2014].

(2) Recall that a $\Bbbk$-algebra $\boldsymbol{A}$ is called *gentle* [Assem and Skowroński 1987] if $\boldsymbol{A} \simeq \Bbbk\Gamma/J$, where Γ is a finite quiver (oriented graph) and J is an ideal in the path algebra $\Bbbk\Gamma$ such that $(J_+)^2 \supseteq J \supseteq (J_+)^k$ for some k, where J_+ is the ideal generated by all arrows, and the following conditions hold:

(a) For every vertex $i \in \operatorname{Ver}\Gamma$ there are at most two arrows starting at i and at most two arrows ending at i.

(b) If an arrow a starts at i (resp. ends at i) and arrows b_1, b_2 end at i (resp. start at i), then either $ab_1 = 0$ or $ab_2 = 0$ (resp. either $b_1a = 0$ or $b_2a = 0$), but not both.

(c) The ideal J is generated by products of arrows of the sort ab.

It is proved in [Zembyk 2015] that such algebras are nodal, hence Backström algebras. The same is true for skewed-gentle algebras [Geißand de la Peña 1999] obtained from gentle algebras by blowing up some vertices.

(3) *Backström orders* are orders $\boldsymbol{A}$ over a discrete valuation ring such that there is a Backström pair $\boldsymbol{H} \supseteq \boldsymbol{A}$, where $\boldsymbol{H}$ is a hereditary order. They were considered in [Ringel and Roggenkamp 1979].

(4) Let $\boldsymbol{H} = T(n, \Bbbk)$ be the ring of upper triangular $n \times n$ matrices over a field $\Bbbk$ and $\boldsymbol{A} = \mathrm{UT}(n, \Bbbk)$ be its subring of unitriangular matrices M, i.e., such that all diagonal elements of M are equal. Then, $\boldsymbol{H}$ is hereditary and $\operatorname{rad} \boldsymbol{H} = \operatorname{rad} \boldsymbol{A}$, hence $\boldsymbol{A}$ is a Backström algebra. In this case, $\boldsymbol{C}(\boldsymbol{H}, \boldsymbol{A}) = \operatorname{rad} \boldsymbol{A}$.

(5) $\Lambda_n = \Bbbk[x_1, x_2, \dots, x_n]/(x_1, x_2, \dots, x_n)^2$ embeds into $\boldsymbol{H} = \prod_{i=1}^n \Bbbk\Gamma_i$, where $\Gamma_i = \bullet \xrightarrow{a_i} \bullet$ (x_i maps to a_i). Obviously, under this embedding $\operatorname{rad} \Lambda_n = \operatorname{rad} \boldsymbol{H}$, so Λ_n is a Backström algebra.

We consider a fixed Backström pair $\boldsymbol{H} \supseteq \boldsymbol{A}$, set $\mathfrak{r} = \operatorname{rad} \boldsymbol{A} = \operatorname{rad} \boldsymbol{H}$ and denote by $\boldsymbol{C}$ the conductor $\boldsymbol{C}(\boldsymbol{H}, \boldsymbol{A})$. Obviously, $\boldsymbol{C}$ is a two-sided $\boldsymbol{A}$-ideal and the biggest left $\boldsymbol{H}$-ideal contained in $\boldsymbol{A}$. Actually, it even turns out to be a two-sided $\boldsymbol{H}$-ideal and its definition is left-right symmetric.

Lemma 2.4. *Let $R \subseteq S$ be semisimple rings, $I = \{\alpha \in R \mid S\alpha \subseteq R\}$. Then, I is a two-sided S-ideal.*

Proof. Obviously, I is a left S-ideal and a two-sided R-ideal. As R is semisimple, $I = Re$ for some central idempotent $e \in R$. Then, $Se \subseteq Re$, so $Se = Re = eR$ and $(1-e)Se = 0$. Hence, $eS(1-e)$ is a left ideal in S and $(eS(1-e))^2 = 0$, so $eS(1-e) = 0$ and $I = Se = eS$ is also a right S-ideal. □

Proposition 2.5. *$\boldsymbol{C}$ is a two-sided $\boldsymbol{H}$-ideal. It is the biggest $\boldsymbol{H}$-ideal contained in $\boldsymbol{A}$. Therefore, it coincides with the set $\{\alpha \in \boldsymbol{A} \mid \alpha\boldsymbol{H} \subseteq \boldsymbol{A}\}$ or with $\operatorname{ann}_{\boldsymbol{A}}(\boldsymbol{H}/\boldsymbol{A})$ considered as a left $\boldsymbol{A}$-module.*

Proof. It follows from the preceding lemma applied to the rings $\boldsymbol{A}/\operatorname{rad} \boldsymbol{A}$ and $\boldsymbol{H}/\operatorname{rad} \boldsymbol{H}$. □

In what follows we assume that $\boldsymbol{A} \neq \boldsymbol{H}$, so $\boldsymbol{C} \neq \boldsymbol{A}$. To calculate $\boldsymbol{C}$, we consider a decomposition $\boldsymbol{A} = \bigoplus_{i=1}^m A_i$, where A_i are indecomposable projective left $\boldsymbol{A}$-modules. Arrange them so that $\boldsymbol{H}A_i \neq A_i$ for $1 \le i \le r$ and $\boldsymbol{H}A_i = A_i$ for $r < i \le m$, and set $A^0 = \bigoplus_{i=1}^r A_i$, $H^0 = \boldsymbol{H}A^0$ and $A^1 = \bigoplus_{i=r+1}^m A_i = \boldsymbol{H}A^1$. Then, $\boldsymbol{A} = A^0 \oplus A^1$ and $\boldsymbol{H} = H^0 \oplus A^1$ (possibly, $r = m$, so $A^0 = \boldsymbol{A}$ and $H^0 = \boldsymbol{H}$). Let $A^0 = \boldsymbol{A}e_0$ and $A^1 = \boldsymbol{A}e_1$, where e_0 and e_1 are orthogonal idempotents and $e_0 + e_1 = 1$. Set $A_b^a = e_b\boldsymbol{A}e_a$ and $H_b^a = e_b\boldsymbol{H}e_a$, where $a, b \in \{0, 1\}$. Note that $A_b^1 = H_b^1$ and $A_1^0 = H_1^0$. As A^0 and A^1 have no isomorphic direct summands,

$A^a_b \subseteq \operatorname{rad} \boldsymbol{A}$ if $a \neq b$. Hence, if we set $\mathfrak{r}^a = \operatorname{rad} A^a (a = 0, 1)$ and consider the Pierce decomposition of the ring $\boldsymbol{A}$

$$\boldsymbol{A} = \begin{pmatrix} A^0_0 & A^1_0 \\ A^0_1 & A^1_1 \end{pmatrix},$$

the Pierce decomposition of the ideal $\mathfrak{r}$ becomes

$$\mathfrak{r} = \begin{pmatrix} \mathfrak{r}^0_0 & A^1_0 \\ A^0_1 & \mathfrak{r}^1_1 \end{pmatrix},$$

where $\mathfrak{r}^a_a = \operatorname{rad} A^a_a,\ a = 0, 1.$ This implies that H^0 and H^1 have no isomorphic direct summands, the Pierce decomposition of $\boldsymbol{H}$ is

$$\boldsymbol{H} = \begin{pmatrix} H^0_0 & A^1_0 \\ A^0_1 & A^1_1 \end{pmatrix}$$

and $\mathfrak{r}^0_0 = \operatorname{rad} H^0_0$. Now, one easily sees that an element $a = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$ belongs to $\boldsymbol{C}$ if and only if $H^0\alpha \subseteq A^0$. We claim that in that case $H^0\alpha \subseteq \operatorname{rad} A^0$. Otherwise $H^0\alpha$ contains an idempotent, hence a direct summand of A^0, which is isomorphic to some A_i with $1 \leq i \leq r$. This is impossible, since $\boldsymbol{H}A_i \neq A_i$. Therefore, $\alpha \in \mathfrak{r}^0_0$ and we obtain the following result:

Proposition 2.6. *The Pierce decomposition of the ideal* $\boldsymbol{C}$ *is*

$$\boldsymbol{C} = \begin{pmatrix} \mathfrak{r}^0_0 & A^1_0 \\ A^0_1 & A^1_1 \end{pmatrix}.$$

Definition 2.7. Analogously to [Burban and Drozd 2011], we define the *Auslander envelope* of the Backström pair $\boldsymbol{H} \supseteq \boldsymbol{A}$ as the ring $\tilde{\boldsymbol{A}}$ of 2×2 matrices of the form

$$\tilde{\boldsymbol{A}} = \begin{pmatrix} \boldsymbol{A} & \boldsymbol{H} \\ \boldsymbol{C} & \boldsymbol{H} \end{pmatrix}$$

with the usual matrix multiplication.

Using Pierce decompositions of $\boldsymbol{A}$, $\boldsymbol{H}$ and $\boldsymbol{C}$, we also present $\tilde{\boldsymbol{A}}$ as the ring of 4×4 matrices

$$\tilde{\boldsymbol{A}} = \begin{pmatrix} A^0_0 & A^1_0 & H^0_0 & A^1_0 \\ A^0_1 & A^1_1 & A^0_1 & A^1_1 \\ \mathfrak{r}^0_0 & A^1_0 & H^0_0 & A^1_0 \\ A^0_1 & A^1_1 & A^0_1 & A^1_1 \end{pmatrix}. \tag{2-1}$$

We also define $\widetilde{\boldsymbol{H}}$ as the ring of 4×4 matrices of the form

$$\widetilde{\boldsymbol{H}} = \begin{pmatrix} \boldsymbol{H} & \boldsymbol{H} \\ \boldsymbol{C} & \boldsymbol{H} \end{pmatrix} \quad \text{or} \quad \widetilde{\boldsymbol{H}} = \begin{pmatrix} H_0^0 & A_0^1 & H_0^0 & A_0^1 \\ A_1^0 & A_1^1 & A_1^0 & A_1^1 \\ \mathfrak{r}_0^0 & A_0^1 & H_0^0 & A_0^1 \\ A_1^0 & A_1^1 & A_1^0 & A_1^1 \end{pmatrix}.$$

Obviously, $\operatorname{rad}\widetilde{\boldsymbol{H}} = \operatorname{rad}\tilde{\boldsymbol{A}}$, so $\widetilde{\boldsymbol{H}} \supseteq \tilde{\boldsymbol{A}}$ is also a Backström pair. $\tilde{\boldsymbol{A}}$ is left noetherian if and only if $\boldsymbol{A}$ is left noetherian and $\boldsymbol{H}$ is finitely generated as a left $\boldsymbol{A}$-module.

In the noetherian case one can calculate the global dimensions of $\tilde{\boldsymbol{A}}$ and $\widetilde{\boldsymbol{H}}$. It turns out that it only depends on $\boldsymbol{H}$.

Theorem 2.8. *Suppose that either $\boldsymbol{A}$* (*hence also $\boldsymbol{H}$*) *is left perfect or $\boldsymbol{A}$ is left noetherian and $\boldsymbol{H}$ is finitely generated as a left $\boldsymbol{A}$-module* (*hence also left noetherian*). *Then*

$$\begin{aligned} \operatorname{l.gl.dim}\tilde{\boldsymbol{A}} &= 1+\max(1+\operatorname{pr.dim}_{\boldsymbol{H}}\mathfrak{r}^0, \operatorname{pr.dim}_{\boldsymbol{H}}\mathfrak{r}^1) \\ &= \begin{cases} 1+\operatorname{l.gl.dim}\boldsymbol{H} & \text{if } \operatorname{pr.dim}_{\boldsymbol{H}}\mathfrak{r}^0 \geq \operatorname{pr.dim}_{\boldsymbol{H}}\mathfrak{r}^1, \\ \operatorname{l.gl.dim}\boldsymbol{H} & \text{if } \operatorname{pr.dim}_{\boldsymbol{H}}\mathfrak{r}^0 < \operatorname{pr.dim}_{\boldsymbol{H}}\mathfrak{r}^1 \end{cases} \end{aligned}$$

and

$$\operatorname{l.gl.dim}\widetilde{\boldsymbol{H}} = \operatorname{l.gl.dim}\boldsymbol{H},$$

where we set $\operatorname{pr.dim} 0 = -1$. *In particular*, *if $\boldsymbol{A}$ is a Backström ring*, *so is $\tilde{\boldsymbol{A}}$*, *and if $\boldsymbol{A}$ is not left hereditary*, *then* $\operatorname{l.gl.dim}\tilde{\boldsymbol{A}} = 2$.[2]

For instance, this is the case for nodal (in particular, gentle or skewed-gentle) algebras (Examples 2.3).

Proof. Under these conditions $\tilde{\boldsymbol{A}}$ and $\widetilde{\boldsymbol{H}}$ are either left perfect or left noetherian. We recall that if a ring Λ is left perfect or left noetherian and semiperfect, then $\operatorname{l.gl.dim}\Lambda = \operatorname{pr.dim}_\Lambda(\Lambda/\operatorname{rad}\Lambda) = 1+\operatorname{pr.dim}_\Lambda \operatorname{rad}\Lambda$. The 4×4 matrix presentation (2-1) of $\tilde{\boldsymbol{A}}$ implies that the corresponding presentation of $\operatorname{rad}\tilde{\boldsymbol{A}}$ is

$$\operatorname{rad}\tilde{\boldsymbol{A}} = \begin{pmatrix} \mathfrak{r}_0^0 & A_0^1 & H_0^0 & A_0^1 \\ A_1^0 & \mathfrak{r}_1^1 & A_1^0 & \mathfrak{r}_1^1 \\ \mathfrak{r}_0^0 & A_0^1 & \mathfrak{r}_0^0 & A_0^1 \\ A_1^0 & \mathfrak{r}_1^1 & A_1^0 & \mathfrak{r}_1^1 \end{pmatrix}. \tag{2-2}$$

An $\tilde{\boldsymbol{A}}$-module M is given by a quadruple (M', M'', ϕ, ψ), where M' is an $\boldsymbol{A}$-module, M'' is an $\boldsymbol{H}$-module, $\psi : M'' \to M'$ is a homomorphism of $\boldsymbol{A}$-modules and $\phi : \boldsymbol{C}\otimes_{\boldsymbol{A}} M' \to M''$ is a homomorphism of $\boldsymbol{H}$-modules. Namely, $M' = e'M$, $M'' = e''M$, where $e' = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $e'' = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $\psi(m'') = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)m''$ and $\phi(c\otimes m') = \left(\begin{smallmatrix} 0 & 0 \\ c & 0 \end{smallmatrix}\right)m'$.

[2]Note that if $\tilde{\boldsymbol{A}}$ is left hereditary, so is $\boldsymbol{A} = e'\tilde{\boldsymbol{A}}e'$ [Sandomierski 1969].

We frequently write $M = \binom{M'}{M''}$, not mentioning ϕ and ψ. For an $\boldsymbol{H}$-module N we define the $\tilde{\boldsymbol{A}}$-module $N^+ = \binom{N}{N}$. Then, $N \mapsto N^+$ is an exact functor mapping projective modules to projective ones, since $\boldsymbol{H}^+ = \binom{\boldsymbol{H}}{\boldsymbol{H}}$ is a projective $\tilde{\boldsymbol{A}}$-module.

We denote by L^i and by R^i the i-th column of the presentations (2-1) and (2-2), respectively. Then, $R^1 = (\mathfrak{r}^0)^+$ and $R^2 = R^4 = (\mathfrak{r}^1)^+$, where $\mathfrak{r}^a = \mathfrak{r}e_a$. If

$$\cdots \to F_k \to \cdots \to F_1 \to F_0 \to N \to 0$$

is a minimal projective resolution of an $\boldsymbol{H}$-module N,

$$\cdots \to F_k^+ \to \cdots \to F_1^+ \to F_0^+ \to N^+ \to 0$$

is a minimal projective resolution of N^+, so $\operatorname{pr.dim}_{\tilde{\boldsymbol{A}}} N^+ = \operatorname{pr.dim}_{\boldsymbol{H}} N$. In particular, $\operatorname{pr.dim}_{\tilde{\boldsymbol{A}}} R^1 = \operatorname{pr.dim}_{\boldsymbol{H}} \mathfrak{r}^0$ and $\operatorname{pr.dim}_{\tilde{\boldsymbol{A}}} R^2 = \operatorname{pr.dim}_{\boldsymbol{H}} \mathfrak{r}^1$. For the module R^3 we have an exact sequence

$$0 \to (\mathfrak{r}^0)^+ \to R^3 \to \begin{pmatrix} H^0/\mathfrak{r}^0 \\ 0 \end{pmatrix} \to 0. \tag{2-3}$$

Note that $H^0/\mathfrak{r}^0$ is a semisimple $\boldsymbol{A}$-module and $e_1(H^0/\mathfrak{r}^0) = 0$, hence it contains the same simple direct summands as $A^0/\mathfrak{r}^0$. The same is true for

$$\begin{pmatrix} H^0/\mathfrak{r}^0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} A^0/\mathfrak{r}^0 \\ 0 \end{pmatrix} = L^1/R^1.$$

Hence,

$$\operatorname{pr.dim}_{\tilde{\boldsymbol{A}}} \begin{pmatrix} H^0/\mathfrak{r}^0 \\ 0 \end{pmatrix} = 1 + \operatorname{pr.dim}_{\tilde{\boldsymbol{A}}} R^1 = 1 + \operatorname{pr.dim}_{\boldsymbol{H}} \mathfrak{r}^0.$$

Therefore, the exact sequence (2-3) shows that $\operatorname{pr.dim}_{\tilde{\boldsymbol{A}}} R^3 = 1 + \operatorname{pr.dim}_{\boldsymbol{H}} \mathfrak{r}^0$ and

$$\operatorname{pr.dim}_{\tilde{\boldsymbol{A}}} \operatorname{rad} \tilde{\boldsymbol{A}} = \max(1 + \operatorname{pr.dim}_{\boldsymbol{H}} \mathfrak{r}^0, \operatorname{pr.dim}_{\boldsymbol{H}} \mathfrak{r}^1),$$

which gives the necessary result for $\tilde{\boldsymbol{A}}$. On the other hand, R^3 is a projective $\widetilde{\boldsymbol{H}}$-module, whence $\operatorname{l.gl.dim} \widetilde{\boldsymbol{H}} = \operatorname{l.gl.dim} \boldsymbol{H}$. □

3. The structure of derived categories

In what follows we denote by $\mathscr{D}(\boldsymbol{A})$ the derived category $\mathscr{D}(\boldsymbol{A}\text{-Mod})$. We denote by $\mathscr{D}_f(\boldsymbol{A})$ the full subcategory of $\mathscr{D}(\boldsymbol{A})$ consisting of complexes quasi-isomorphic to complexes of finitely generated projective modules. If $\boldsymbol{A}$ is left noetherian, it coincides with the derived category of the category $\boldsymbol{A}$-mod of finitely generated $\boldsymbol{A}$-modules. We also use the usual superscripts $+, -, b$. By $\operatorname{Perf}(\boldsymbol{A})$ we denote the full subcategory of perfect complexes from $\mathscr{D}(\boldsymbol{A})$, i.e., complexes quasi-isomorphic to finite complexes of finitely generated projective modules. It coincides with the full subcategory of compact objects in $\mathscr{D}(\boldsymbol{A})$ [Rouquier 2008]. If $\boldsymbol{A}$ is left noetherian, an $\boldsymbol{A}$-module M belongs to $\operatorname{Perf}(A)$ if and only if it is finitely generated and of finite projective dimension.

There are close relations between the categories $\mathscr{D}(\boldsymbol{A})$, $\mathscr{D}(\boldsymbol{H})$ and $\mathscr{D}(\tilde{\boldsymbol{A}})$ based on the following construction [Burban et al. 2017]:

Let $\boldsymbol{P} = \binom{\boldsymbol{A}}{\boldsymbol{C}}$. It is a projective $\tilde{\boldsymbol{A}}$-module and $\operatorname{End} \boldsymbol{P} \simeq \boldsymbol{A}^{\mathrm{op}}$, so it can be considered as a right $\boldsymbol{A}$-module. Consider the functors

$$\begin{aligned} \mathsf{F} &= \operatorname{Hom}_{\tilde{\boldsymbol{A}}}(\boldsymbol{P}, -) \simeq \boldsymbol{P}^\vee \otimes_{\tilde{\boldsymbol{A}}} - : \tilde{\boldsymbol{A}}\text{-Mod} \to \boldsymbol{A}\text{-Mod}, \\ \mathsf{F}^* &= \boldsymbol{P} \otimes_{\boldsymbol{A}} - : \boldsymbol{A}\text{-Mod} \to \tilde{\boldsymbol{A}}\text{-Mod}, \\ \mathsf{F}^! &= \operatorname{Hom}_{\boldsymbol{A}}(\boldsymbol{P}^\vee, -) : \boldsymbol{A}\text{-Mod} \to \tilde{\boldsymbol{A}}\text{-Mod}, \end{aligned}$$

where $\boldsymbol{P}^\vee = \operatorname{Hom}_{\tilde{\boldsymbol{A}}}(P, \tilde{\boldsymbol{A}}) \simeq (\boldsymbol{A}\ \ \boldsymbol{H})$ is the dual right projective $\tilde{\boldsymbol{A}}$-module, the functor F is exact, F^* is its left adjoint and $\mathsf{F}^!$ is its right adjoint. Moreover, the adjunction morphisms $\mathsf{F}\mathsf{F}^* \to \operatorname{Id}_{\boldsymbol{A}\text{-Mod}}$ and $\operatorname{Id}_{\boldsymbol{A}\text{-Mod}} \to \mathsf{F}\mathsf{F}^!$ are isomorphisms [Burban et al. 2017, Theorem 4.3]. The functors F^* and $\mathsf{F}^!$ are fully faithful and F is essentially surjective, i.e., every $\boldsymbol{A}$-module is isomorphic to $\mathsf{F}M$ for some $\tilde{\boldsymbol{A}}$-module M. $\operatorname{Ker}\mathsf{F}$ is a Serre subcategory of $\tilde{\boldsymbol{A}}$-Mod equivalent to $\overline{\boldsymbol{H}}$-Mod, where $\overline{\boldsymbol{H}} = \boldsymbol{H}/\boldsymbol{C} \simeq \tilde{\boldsymbol{A}}/\begin{pmatrix} \boldsymbol{A} & \boldsymbol{H} \\ \boldsymbol{C} & \boldsymbol{C} \end{pmatrix}$. The embedding functor $\mathsf{I} : \operatorname{Ker}\mathsf{F} \to \tilde{\boldsymbol{A}}$-Mod has a left adjoint I^* and a right adjoint $\mathsf{I}^!$ and we obtain a recollement diagram

$$\operatorname{Ker}\mathsf{F} \;\underset{\overset{\longleftarrow\,\mathsf{I}^!\,}{}}{\overset{\overset{\longleftarrow\,\mathsf{I}^*\,}{}}{\xrightarrow{\;\mathsf{I}\;}}}\; \tilde{\boldsymbol{A}}\text{-Mod} \;\underset{\overset{\longleftarrow\,\mathsf{F}^!\,}{}}{\overset{\overset{\longleftarrow\,\mathsf{F}^*\,}{}}{\xrightarrow{\;\mathsf{F}\;}}}\; \boldsymbol{A}\text{-Mod}.$$

As the functor F is exact, it extends to the functor between the derived categories $\mathsf{DF} : \mathscr{D}(\tilde{\boldsymbol{A}}) \to \mathscr{D}(\boldsymbol{A})$ acting on complexes componentwise. The derived functors LF^* and $\mathsf{RF}^!$ are its left and right adjoints, respectively, the adjunction morphisms $\operatorname{Id}_{\mathscr{D}(\boldsymbol{A})} \to \mathsf{DF} \cdot \mathsf{LF}^*$ and $\mathsf{DF} \cdot \mathsf{LF}^* \to \operatorname{Id}_{\mathscr{D}(\boldsymbol{A})}$ are again isomorphisms and we have a recollement diagram

$$\operatorname{Ker}\mathsf{DF} \;\underset{\overset{\longleftarrow\,\mathsf{RI}^!\,}{}}{\overset{\overset{\longleftarrow\,\mathsf{LI}^*\,}{}}{\xrightarrow{\;\mathsf{DI}\;}}}\; \mathscr{D}(\tilde{\boldsymbol{A}}) \;\underset{\overset{\longleftarrow\,\mathsf{RF}^!\,}{}}{\overset{\overset{\longleftarrow\,\mathsf{LF}^*\,}{}}{\xrightarrow{\;\mathsf{DF}\;}}}\; \mathscr{D}(\boldsymbol{A}).$$

(It also follows from Corollary 1.2.) Here $\operatorname{Ker}\mathsf{DF} = \mathscr{D}_{\overline{\boldsymbol{H}}}(\tilde{\boldsymbol{A}})$, the full subcategory of complexes whose cohomologies are $\overline{\boldsymbol{H}}$-modules, i.e., are annihilated by the ideal $\begin{pmatrix} \boldsymbol{A} & \boldsymbol{H} \\ \boldsymbol{C} & \boldsymbol{C} \end{pmatrix}$. Note that, as a rule, it is not equivalent to $\mathscr{D}(\overline{\boldsymbol{H}})$. From the definition of F it follows that

$$\operatorname{Ker}\mathsf{DF} = \boldsymbol{P}^\perp = \{C \in \mathscr{D}(\tilde{\boldsymbol{A}}) \mid \operatorname{Hom}_{\mathscr{D}(\tilde{\boldsymbol{A}})}(\boldsymbol{P}, C[k]) = 0 \text{ for all } k\}.$$

Obviously, DF maps $\mathscr{D}^\sigma(\tilde{\boldsymbol{A}})$ to $D^\sigma(\boldsymbol{A})$ for $\sigma \in \{+, -, b\}$, LF^* maps $\mathscr{D}^-(\boldsymbol{A})$ to $\mathscr{D}^-(\tilde{\boldsymbol{A}})$ and $\mathsf{RF}^!$ maps $\mathscr{D}^+(\boldsymbol{A})$ to $\mathscr{D}^+(\tilde{\boldsymbol{A}})$. If $\tilde{\boldsymbol{A}}$ is left noetherian, DF maps $\mathscr{D}_f(\tilde{\boldsymbol{A}})$ to $\mathscr{D}_f(\boldsymbol{A})$ and LF^* maps $\mathscr{D}_f(\boldsymbol{A})$ to $\mathscr{D}_f(\tilde{\boldsymbol{A}})$. Finally, both DF and LF^* have right adjoints, hence map compact objects (i.e., perfect complexes) to compact ones.

On the contrary, usually LF^* does not map $\mathscr{D}^b(\boldsymbol{A})$ to $\mathscr{D}^b(\tilde{\boldsymbol{A}})$. For instance, it is definitely so if l.gl.dim $\tilde{\boldsymbol{A}} < \infty$ while l.gl.dim $\boldsymbol{A} = \infty$ as in Examples 2.3 (4, 5). If l.gl.dim $\boldsymbol{H}$ is finite, so is l.gl.dim $\tilde{\boldsymbol{A}}$, thus this recollement can be considered as a sort of categorical resolution of the category $\mathscr{D}(\boldsymbol{A})$. In any case, it is useful for studying the categories $\boldsymbol{A}$-Mod and $\mathscr{D}(\boldsymbol{A})$ if we know the structure of the categories $\tilde{\boldsymbol{A}}$-Mod and $\mathscr{D}(\tilde{\boldsymbol{A}})$. For instance, it is so if we are interested in the *derived dimension*, i.e., the dimension of the category $\mathscr{D}^b_f(\boldsymbol{A})$ in the sense of [Rouquier 2008].

Definition 3.1. Let $\mathscr{T}$ be a triangular category and $\mathfrak{M}$ be a set of objects from $\mathscr{T}$.

(1) We denote by $\langle\mathfrak{M}\rangle$ the smallest full subcategory of $\mathscr{T}$ containing $\mathfrak{M}$ and closed under direct sums, direct summands and shifts (not closed under cones, so not a triangulated subcategory).

(2) If $\mathfrak{N}$ is another subset of $\mathscr{T}$, we denote by $\mathfrak{M} \dagger \mathfrak{N}$ the set of objects C from $\mathscr{T}$ such that there is an exact triangle $A \to B \to C \xrightarrow{+}$, where $A \in \mathfrak{M}$, $B \in \mathfrak{N}$.

(3) We define $\langle\mathfrak{M}\rangle_k$ recursively, setting $\langle\mathfrak{M}\rangle_1 = \langle\mathfrak{M}\rangle$ and $\langle\mathfrak{M}\rangle_{k+1} = \langle\langle\mathfrak{M}\rangle \dagger \langle\mathfrak{M}\rangle_k\rangle$.

(4) The *dimension* $\dim \mathscr{T}$ of $\mathscr{T}$ is the smallest k such that there is a finite set of objects $\mathfrak{M}$ such that $\langle\mathfrak{M}\rangle_{k+1} = \mathscr{T}$ (if it exists). We call the dimension $\dim \mathscr{D}^b_f(\boldsymbol{A})$ the *derived dimension* of the ring $\boldsymbol{A}$ and denote it by der.dim $\boldsymbol{A}$.

As the functor F is exact and essentially surjective, the next result is evident:

Proposition 3.2. *We have* der.dim $\boldsymbol{A} \le$ der.dim $\tilde{\boldsymbol{A}}$. *Namely, if* $\mathscr{D}^b_f(\tilde{\boldsymbol{A}}) = \langle\mathfrak{M}\rangle_{k+1}$, *then* $\mathscr{D}^b_f(\boldsymbol{A}) = \langle \mathsf{DF}(\mathfrak{M})\rangle_{k+1}$.

4. Semiorthogonal decomposition

There is another recollement diagram for $\mathscr{D}(\tilde{\boldsymbol{A}})$ related to the projective module $\boldsymbol{Q} = \binom{\boldsymbol{H}}{\boldsymbol{H}}$ with End $\boldsymbol{Q} \simeq \boldsymbol{H}^{\mathrm{op}}$. Namely, we set

$$\begin{aligned}
\mathsf{G} &= \mathrm{Hom}_{\tilde{\boldsymbol{A}}}(\boldsymbol{Q}, -) \simeq \boldsymbol{Q}^\vee \otimes_{\tilde{\boldsymbol{A}}} - : \tilde{\boldsymbol{A}}\text{-Mod} \to \boldsymbol{H}\text{-Mod},\\
\mathsf{G}^* &= \boldsymbol{Q} \otimes_{\boldsymbol{H}} - : \boldsymbol{H}\text{-Mod} \to \tilde{\boldsymbol{A}}\text{-Mod},\\
\mathsf{G}^! &= \mathrm{Hom}_{\boldsymbol{H}}(\boldsymbol{Q}^\vee, -) : \boldsymbol{H}\text{-Mod} \to \tilde{\boldsymbol{A}}\text{-Mod},
\end{aligned}$$

where $\boldsymbol{Q}^\vee = \mathrm{Hom}_{\tilde{\boldsymbol{A}}}(\boldsymbol{Q}, \tilde{\boldsymbol{A}}) \simeq (\boldsymbol{C}\ \boldsymbol{H})$,

$$\begin{aligned}
\mathsf{DG} &: \mathscr{D}(\tilde{\boldsymbol{A}}) \to \mathscr{D}(\boldsymbol{H}) \text{ is } \mathsf{G} \text{ applied componentwise},\\
\mathsf{LG}^* &: \mathscr{D}(A) \to \mathscr{D}(\tilde{\boldsymbol{A}}) \text{ is the left adjoint of } \mathsf{DG},\\
\mathsf{RG}^! &: \mathscr{D}(A) \to \mathscr{D}(\tilde{\boldsymbol{A}}) \text{ is the right adjoint of } \mathsf{DG}.
\end{aligned}$$

We also set $\bar{\boldsymbol{A}} = \boldsymbol{A}/\boldsymbol{C} \simeq \tilde{\boldsymbol{A}}/\begin{pmatrix} \boldsymbol{C} & \boldsymbol{H} \\ \boldsymbol{C} & \boldsymbol{H} \end{pmatrix}$. Then, we have recollement diagrams

$$\operatorname{Ker}\mathsf{G} \;\underset{\mathsf{J}^!}{\overset{\mathsf{J}^*}{\leftleftarrows}}\!\!\xrightarrow{\;\mathsf{J}\;}\; \tilde{\boldsymbol{A}}\text{-Mod} \;\underset{\mathsf{G}^!}{\overset{\mathsf{G}^*}{\leftleftarrows}}\!\!\xrightarrow{\;\mathsf{G}\;}\; \boldsymbol{H}\text{-Mod}$$

and

$$\operatorname{Ker}\mathsf{DG} \;\underset{\mathsf{RJ}^!}{\overset{\mathsf{LJ}^*}{\leftleftarrows}}\!\!\xrightarrow{\;\mathsf{DJ}\;}\; \mathscr{D}(\tilde{\boldsymbol{A}}) \;\underset{\mathsf{RG}^!}{\overset{\mathsf{LG}^*}{\leftleftarrows}}\!\!\xrightarrow{\;\mathsf{DG}\;}\; \mathscr{D}(\boldsymbol{H}),$$

where $\operatorname{Ker}\mathsf{G} \simeq \bar{\boldsymbol{A}}$-Mod. Since the $\tilde{\boldsymbol{A}}$-ideal $(\boldsymbol{C}\ \ \boldsymbol{H})$ is projective as a right $\tilde{\boldsymbol{A}}$-module, [Burban et al. 2017, Theorem 4.6] implies that $\operatorname{Ker}\mathsf{DG} \simeq \mathscr{D}(\bar{\boldsymbol{A}})$.

As usual, this recollement diagram gives semiorthogonal decompositions [Burban et al. 2017, Corollary 2.6]

$$\mathscr{D}(\tilde{\boldsymbol{A}}) = (\operatorname{Ker}\mathsf{DG}, \operatorname{Im}\mathsf{LG}^*) = (\operatorname{Im}\mathsf{RG}^!, \operatorname{Ker}\mathsf{DG}) \tag{4-1}$$

with $\operatorname{Ker}\mathsf{DG} \simeq \mathscr{D}(\bar{\boldsymbol{A}})$ and $\operatorname{Im}\mathsf{LG}^* \simeq \operatorname{Im}\mathsf{RG}^! \simeq \mathscr{D}(\boldsymbol{H})$ (though usually $\operatorname{Im}\mathsf{LG}^* \neq \operatorname{Im}\mathsf{RG}^!$).

Recall from [Kuznetsov and Lunts 2015] that a *semiorthogonal decomposition* $\mathscr{T} = (\mathscr{T}_1, \mathscr{T}_2)$, where $\mathscr{T}_1$, $\mathscr{T}_2$ are full triangulated subcategories of $\mathscr{T}$, means that

$$\operatorname{Hom}_{\mathscr{T}}(T_2, T_1) = 0 \quad \text{for all } T_1 \in \mathscr{T}_1 \text{ and } T_2 \in \mathscr{T}_2,$$

and for every object $T \in \mathscr{T}$ there is an exact triangle $T_1 \to T_2 \to T \xrightarrow{+}$, where $T_i \in \mathscr{T}_i$.

Lemma 4.1.[3] *If $\mathscr{T} = (\mathscr{T}_1, \mathscr{T}_2)$ is a semiorthogonal decomposition of a triangulated category $\mathscr{T}$, then*

$$\dim \mathscr{T} \le \dim \mathscr{T}_1 + \dim \mathscr{T}_2 + 1.$$

Proof. First we show that for any subsets $\mathfrak{M}$, $\mathfrak{N}$ of objects of the category $\mathscr{T}$

$$\langle\mathfrak{M}\rangle_{k+1} \dagger \mathfrak{N} \subseteq \langle\mathfrak{M}\rangle \dagger \langle\langle\mathfrak{M}\rangle_k \dagger N\rangle \subseteq \underbrace{\langle\mathfrak{M}\rangle \dagger \langle\langle\mathfrak{M}\rangle \dagger \langle\langle\mathfrak{M}\rangle \dagger \cdots \langle\langle\mathfrak{M}\rangle}_{k+1} \dagger \mathfrak{N}\rangle \cdots\rangle\rangle. \tag{4-2}$$

Indeed, let $C \in \langle\mathfrak{M}\rangle_{k+1} \dagger \mathfrak{N}$, i.e., there is an exact triangle $A \to B \to C \xrightarrow{+}$, where $A \in \langle\mathfrak{M}\rangle_{k+1}$, $B \in \mathfrak{N}$. There is also an exact triangle $A_1 \to A \to A_2 \xrightarrow{+}$, where $A_1 \in \langle\mathfrak{M}\rangle_k$, $A_2 \in \langle\mathfrak{M}\rangle$. The octahedron axiom implies that there are exact triangles $A_1 \to B \to B' \xrightarrow{+}$ and $A_2 \to B' \to C \xrightarrow{+}$. Therefore, $B' \in \langle\mathfrak{M}\rangle_k \dagger \mathfrak{N}$ and $C \in \langle\mathfrak{M}\rangle \dagger \langle\langle\mathfrak{M}\rangle_k \dagger \mathfrak{N}\rangle$.

Now, let $\langle\mathfrak{M}\rangle_{k+1} = \mathscr{T}_1$ and $\langle\mathfrak{N}\rangle_{l+1} = \mathscr{T}_2$. Then, for every $T \in \mathscr{T}$ there is an exact triangle $T_1 \to T_2 \to T \xrightarrow{+}$, where $T_1 \in \langle\mathfrak{M}\rangle_{k+1}$, $T_2 \in \langle\mathfrak{N}\rangle_{l+1}$. But, according to (4-2), $\langle\mathfrak{M}\rangle_{k+1} \dagger \langle\mathfrak{N}\rangle_{l+1} \subseteq \langle\mathfrak{M}\cup\mathfrak{N}\rangle_{k+l+2}$, so $\mathscr{T} = \langle\mathfrak{M}\cup\mathfrak{N}\rangle_{k+l+2}$ and $\dim \mathscr{T} \le k+l+1$. □

[3] In [Psaroudakis 2014, Theorem 7.4] this result is proved in the case when this decomposition arises from a recollement.

Since $\bar{\boldsymbol{A}}$ is semisimple, any indecomposable object from $\mathscr{D}(\bar{\boldsymbol{A}})$ is just a shifted simple module, so $\mathscr{D}_f^b(\bar{\boldsymbol{A}}) = \langle \bar{\boldsymbol{A}} \rangle$ and $\operatorname{der.dim} \bar{\boldsymbol{A}} = 0$. If $\boldsymbol{H}$ is hereditary, every indecomposable object from $\mathscr{D}_f^b(\boldsymbol{H})$ is a shift of a module. For every module M there is an exact sequence $0 \to P' \to P \to M \to 0$ with projective modules P, P' and, since $\boldsymbol{H}$ is semiperfect, every indecomposable projective $\boldsymbol{H}$-module is a direct summand of $\boldsymbol{H}$. Hence, $\mathscr{D}_f^b(\boldsymbol{H}) = \langle \boldsymbol{H} \rangle_2$ and $\operatorname{der.dim} \boldsymbol{H} \leq 1$.

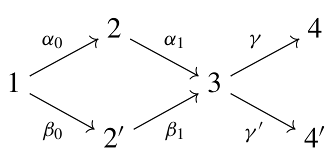
Corollary 4.2. *We have* $\operatorname{der.dim} \boldsymbol{A} \leq \operatorname{der.dim} \boldsymbol{H} + 1$. *In particular, if $\boldsymbol{A}$ is a Backström* (*or piecewise Backström*) *ring*, $\operatorname{der.dim} \boldsymbol{A} \leq 2$.

A finite dimensional hereditary algebra is said to be of *Dynkin type* if it has finitely many isomorphism classes of indecomposable modules. Such algebras, up to Morita equivalence, correspond to Dynkin diagrams [Dlab and Ringel 1976; Gabriel 1972]. If the derived category of an algebra $\boldsymbol{H}$ is equivalent to the derived category of a hereditary algebra of Dynkin type, we say that $\boldsymbol{H}$ is *piecewise hereditary of Dynkin type*.[4] We say that a Backström (or piecewise Backström) algebra $\boldsymbol{A}$ is of *Dynkin type* if there is a Backström pair $\boldsymbol{H} \supseteq \boldsymbol{A}$, where $\boldsymbol{H}$ is a hereditary (piecewise hereditary) algebra of Dynkin type. For instance, it is so if $\boldsymbol{A}$ is a gentle or skewed-gentle algebra [Zembyk 2015], or the algebra $\mathrm{UT}(n\Bbbk)$ of unitriangular matrices over a field (Examples 2.3 (4)), or the algebra Λ_n from Examples 2.3 (5). In this case, $\mathscr{D}_f^b(\boldsymbol{H}) = \langle M_1, M_2, \ldots, M_m \rangle_1$, where $M_1, M_2, \ldots, M_m$ are all pairwise nonisomorphic indecomposable $\boldsymbol{H}$-modules, so $\operatorname{der.dim} \boldsymbol{H} = 0$.

In [Chen et al. 2008] it was proved that $\operatorname{der.dim} \boldsymbol{A} = 0$ for a finite dimensional algebra $\boldsymbol{A}$ if and only if $\boldsymbol{A}$ is a piecewise hereditary algebra of Dynkin type.

Corollary 4.3. *If $\boldsymbol{A}$ is a Backström* (*or piecewise Backström*) *algebra of Dynkin type* (*for instance, gentle or skewed-gentle*), *but is not piecewise hereditary of Dynkin type, then* $\operatorname{der.dim} \boldsymbol{A} = 1$.

Example 4.4. The path algebra of the commutative quiver



$$\alpha_1\alpha_0 = \beta_1\beta_0$$

is a tilted (hence piecewise hereditary) algebra of type $\widetilde{D}_5$. At the same time it is a Backström algebra of type A_4. Namely, it is a skewed-gentle algebra obtained from the path algebra of the quiver $1 \to 2 \to 3 \to 4$ by blowing up vertices 2 and 4.[5]

[4] It is proved in [Happel 1988] that piecewise hereditary algebras of Dynkin type are just iterated tilted algebras of Dynkin type.

[5] See [Zembyk 2014] for the construction of blowing up and its relation to nodal algebras.

5. Relation to bimodule categories

In this section, we explain how a semiorthogonal decomposition allows us to apply to calculations in a triangulated category the technique of matrix problems, namely, of bimodule categories, as in [Drozd 2010].

Let $\mathscr{A}$ and $\mathscr{B}$ be additive categories, $\mathcal{U}$ be an $\mathscr{A}$-$\mathscr{B}$-bimodule, i.e., a biadditive functor $\mathscr{A}^{\mathrm{op}} \times \mathscr{B} \to \mathbf{Ab}$. Recall from [Drozd 2010] that the *bimodule category* or the *category of elements* of the bimodule $\mathcal{U}$ is the category $\mathsf{El}(\mathcal{U})$ whose set of objects is $\bigcup_{A\in\mathscr{A},B\in\mathscr{B}} \mathcal{U}(A, B)$ and whose morphisms from $u \in \mathcal{U}(A, B)$ to $v \in \mathcal{U}(A', B')$ are the pairs (α, β) such that $u\alpha = \beta v$, where $\alpha : A' \to A$, $\beta : B \to B'$. Here, as usual, we wrote $u\alpha$ and βv instead of $\mathcal{U}(\alpha, 1_B)u$ and $\mathcal{U}(1_{A'}, \beta)v$. Bimodule categories appear when there is a semiorthogonal decomposition of a triangulated category.

Theorem 5.1. *Let $(\mathscr{A}, \mathscr{B})$ be a semiorthogonal decomposition of a triangulated category $\mathscr{C}$. Consider the $\mathscr{A}$-$\mathscr{B}$-bimodule $\mathcal{U}$ such that $\mathcal{U}(A, B) = \mathrm{Hom}_{\mathscr{C}}(A, B)$, $A \in \mathscr{A}$, $B \in \mathscr{B}$. For every $f : A \to B$ fix a cone Cf, that is, an exact triangle $A \xrightarrow{f} B \xrightarrow{f_1} Cf \xrightarrow{f_2} A[1]$. The map $f \mapsto Cf$ induces an equivalence of categories $\mathsf{C} : \mathsf{El}(\mathcal{U}) \xrightarrow{\sim} \mathscr{C}/\mathscr{J}$, where $\mathscr{J}$ is the ideal of $\mathscr{C}$ consisting of morphisms η such that there are factorizations $\eta = \eta'\xi = \zeta\eta''$, where the source of η' is in $\mathscr{A}$ and the target of η'' is in $\mathscr{B}$. Moreover, $\mathscr{J}^2 = 0$, so C induces a bijection between isomorphism classes of objects from $\mathsf{El}(\mathcal{U})$ and from $\mathscr{C}$.*[6]

Proof. As $(\mathscr{A}, \mathscr{B})$ is a semiorthogonal decomposition of $\mathscr{C}$, every object from $\mathscr{C}$ occurs in an exact triangle $A \xrightarrow{f} B \to C \xrightarrow{+}$, where $A \in \mathscr{A}$, $B \in \mathscr{B}$, so f is an object from $\mathsf{El}(\mathcal{U})$ and $C \simeq Cf$. Let $f' : A' \to B'$ be another object of $\mathsf{El}(\mathcal{U})$ and $(\alpha, \beta) : f \to f'$ be a morphism from $\mathsf{El}(\mathcal{U})$. Fix a commutative diagram

$$
\begin{array}{ccccccc}
A & \xrightarrow{f} & B & \xrightarrow{f_1} & Cf & \xrightarrow{f_2} & A[1] \\
\downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\alpha[1]} \\
A' & \xrightarrow{f'} & B' & \xrightarrow{f_1'} & Cf' & \xrightarrow{f_2'} & A'[1]
\end{array}
\tag{5-1}
$$

It exists, though is not unique. Let γ' be another morphism making the diagram (5-1) commutative and set $\eta = \gamma - \gamma'$. Then, $\eta f_1 = 0$, hence η factors through f_2, and $f_2'\eta = 0$, hence η factors through f_1'. Thus, $\eta \in \mathscr{J}$. On the other hand, if $\eta : Cf \to Cf'$ is in $\mathscr{J}$, the decomposition $\eta = \eta'\xi$ implies that $\eta f_1 = \eta'\xi f_1 = 0$ and the decomposition $\eta = \zeta\eta''$ implies that $f_2'\eta = f_2'\zeta\eta'' = 0$, hence the morphism $\gamma' = \gamma + \eta$ makes the diagram (5-1) commutative. Therefore, the class $\mathsf{C}(\alpha, \beta)$ of γ modulo $\mathscr{J}$ is uniquely defined, so the maps $f \mapsto Cf$ and $(\alpha, \beta) \mapsto \mathsf{C}(\alpha, \beta)$ define a functor $\mathsf{C} : \mathsf{El}(\mathcal{U}) \to \mathscr{C}/\mathscr{J}$.

[6] This theorem is a partial case of [Drozd 2010, Theorem 1.1].

Let now $\gamma : Cf \to Cf'$ be any morphism. Then, $f_2'\gamma f_1 = 0$, so $\gamma f_1 = f_1'\beta$ for some $\beta : B \to B'$. Hence, there is a morphism $\alpha : A \to A'$ making the diagram (5-1) commutative, i.e., defining a morphism $(\alpha, \beta) : f \to f'$ such that $\gamma \equiv \mathsf{C}(\alpha, \beta) \pmod{\mathscr{J}}$. If (α', β') is another such morphism, $f_1'(\beta - \beta') = 0$, so $\beta - \beta' = f'\xi$ for some $\xi : B \to A$. But $\xi = 0$, so $\beta = \beta'$. In the same way $\alpha = \alpha'$. Hence, the functor C is fully faithful. As we have already noticed, it is essentially surjective, and therefore defines an equivalence $\mathsf{El}(\mathcal{U}) \xrightarrow{\sim} \mathscr{C}/\mathscr{J}$. The equality $\mathscr{J}^2 = 0$ follows immediately from the definition and the conditions of the theorem. □

We apply Theorem 5.1 to Backström pairs $\boldsymbol{H} \subseteq \boldsymbol{A}$ such that $\boldsymbol{A}$ is left noetherian and $\boldsymbol{H}$ is left hereditary and finitely generated as a left $\boldsymbol{A}$-module. For instance, it is so in the case of Backström algebras or Backström orders. Then, the ring $\tilde{\boldsymbol{A}}$ is also noetherian and $\boldsymbol{C}$ is projective as a left $\boldsymbol{H}$-module. According to (4-1), $(\operatorname{Ker}\mathsf{DG}, \operatorname{Im}\mathsf{LG}^*)$ is a semiorthogonal decomposition of $\mathscr{D}(\tilde{\boldsymbol{A}})$. Moreover, both G and G^* map finitely generated modules to finitely generated modules, so the same is valid if we consider their restrictions onto $\mathscr{D}_f(\tilde{\boldsymbol{A}})$ and $\mathscr{D}_f(\boldsymbol{H})$. Note also that G^* is exact, so G^* can be applied to complexes componentwise. The $\tilde{\boldsymbol{A}}$-module G^*M can be identified with the module of columns $M^+ = \binom{M}{M}$ with the action of $\tilde{\boldsymbol{A}}$ given by matrix multiplication. It gives an equivalence of $\mathscr{D}(\boldsymbol{H})$ with $\operatorname{Im}\mathsf{LG}^*$. As $\boldsymbol{H}$ is left hereditary, every complex from $\mathscr{D}(\boldsymbol{H})$ is equivalent to a direct sum of shifted modules (see [Keller 2007, Section 2.5]). On the other hand, $\operatorname{Ker}\mathsf{DG} \simeq \mathscr{D}(\bar{\boldsymbol{A}})$ and $\bar{\boldsymbol{A}}$ is semisimple, since $\boldsymbol{C} \supseteq \boldsymbol{r}$. Hence, every complex from $\mathscr{D}(\bar{\boldsymbol{A}})$ is isomorphic to a direct sum of shifted simple $\bar{\boldsymbol{A}}$-modules, which are direct summands of $\bar{\boldsymbol{A}}$. So, to calculate the bimodule $\mathcal{U}$, we only have to calculate $\operatorname{Ext}^i_{\tilde{\boldsymbol{A}}}(\bar{\boldsymbol{A}}, M^+)$, where M is an $\boldsymbol{H}$-module. Note also that $\boldsymbol{C}^+$ is a projective $\tilde{\boldsymbol{A}}$-module, since $\boldsymbol{C}$ is a projective $\boldsymbol{H}$-module. Therefore, a projective resolution of $\bar{\boldsymbol{A}}$ is $0 \to \boldsymbol{C}^+ \xrightarrow{\varepsilon} \boldsymbol{P} \to \bar{\boldsymbol{A}} \to 0$ and $\operatorname{pr.dim}_{\tilde{\boldsymbol{A}}} \bar{\boldsymbol{A}} = 1$. Hence, we only have to calculate $\operatorname{Hom}_{\tilde{\boldsymbol{A}}}(\bar{\boldsymbol{A}}, M^+)$ and $\operatorname{Ext}^1_{\tilde{\boldsymbol{A}}}(\bar{\boldsymbol{A}}, M^+)$.

Theorem 5.2. (1) $\operatorname{Hom}_{\tilde{\boldsymbol{A}}}(\bar{\boldsymbol{A}}, M^+) \simeq \operatorname{ann}_M \boldsymbol{C} = \{u \in M \mid \boldsymbol{C}u = 0\}$.

(2) $\operatorname{Ext}^1_{\tilde{\boldsymbol{A}}}(\bar{\boldsymbol{A}}, M^+) \simeq \operatorname{Hom}_{\boldsymbol{H}}(\boldsymbol{C}, M)/(M/\operatorname{ann}_M \boldsymbol{C})$, *where the quotient* $M/\operatorname{ann}_M \boldsymbol{C}$ *embeds into* $\operatorname{Hom}_{\boldsymbol{H}}(\boldsymbol{C}, M)$ *if we map an element* $u \in M$ *to the homomorphism* $\mu_u : c \mapsto cu$.

Proof. (1) $\operatorname{Hom}_{\tilde{\boldsymbol{A}}}(\bar{\boldsymbol{A}}, M^+)$ is identified with the set of homomorphisms $\phi : \boldsymbol{P} \to M^+$ such that $\phi\varepsilon = 0$. A homomorphism $\phi : \boldsymbol{P} \to M^+$ is uniquely defined by an element $u \in M$ such that $\phi\binom{1}{0} = \binom{u}{0}$. Namely, $\phi\binom{a}{c} = \binom{au}{cu}$. Obviously, $\phi\varepsilon = 0$ if and only if $\boldsymbol{C}u = 0$, i.e., $u \in \operatorname{ann}_M \boldsymbol{C}$.

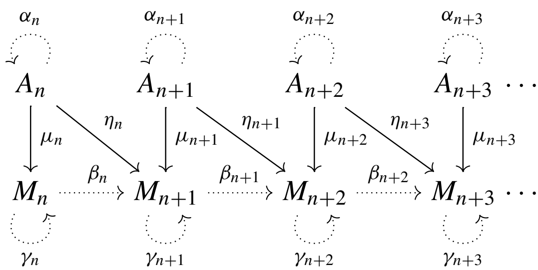
(2) $\operatorname{Ext}^1_{\tilde{\boldsymbol{A}}}(\bar{\boldsymbol{A}}, M^+) \simeq \operatorname{Hom}_{\tilde{\boldsymbol{A}}}(C^+, M^+)/\operatorname{Hom}_{\tilde{\boldsymbol{A}}}(\boldsymbol{P}, M^+)\varepsilon$. As the functor G^* is fully faithful, $\operatorname{Hom}_{\tilde{\boldsymbol{A}}}(C^+, M^+) \simeq \operatorname{Hom}_{\boldsymbol{H}}(\boldsymbol{C}, M)$. Namely, $\psi : \boldsymbol{C} \to M$ induces

$\psi^+ : \boldsymbol{C}^+ \to M^+$ mapping $\binom{a}{b}$ to $\binom{\psi(a)}{\psi(b)}$. Let $\phi : \boldsymbol{P} \to M^+$ correspond, as above, to an element $u \in M$. Then,

$$\phi\varepsilon \begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} au \\ cu \end{pmatrix},$$

so it equals μ_u, and $\mathrm{Hom}_{\tilde{\boldsymbol{A}}}(P, M^+)\varepsilon$ is identified with $M/\operatorname{ann}_M \boldsymbol{C}$ embedded into $\mathrm{Hom}_{\boldsymbol{H}}(\boldsymbol{C}, M)$ as above. □

Actually, in our case an object E from the category $\mathsf{El}(\mathcal{U})$ (therefore, also an object from $\mathscr{D}^b(\tilde{\boldsymbol{A}})$) is given by the vertices and solid arrows of a diagram



(of arbitrary length), where A_i are $\bar{\boldsymbol{A}}$-modules, M_i are $\boldsymbol{H}$-modules, μ_i belongs to $\mathrm{Hom}_{\tilde{\boldsymbol{A}}}(A_i, M_i^+)$ and η_i belongs to $\mathrm{Ext}^1_{\tilde{\boldsymbol{A}}}(A_i, M_{i-1}^+)$. A morphism between E and E' is given by the dotted arrows, where

$$\begin{aligned} \alpha_i &\in \mathrm{Hom}_{\bar{\boldsymbol{A}}}(A_i, A_i') \simeq \mathrm{Hom}_{\tilde{\boldsymbol{A}}}(A_i, A_i'), \\ \gamma_i &\in \mathrm{Hom}_{\boldsymbol{H}}(M_i, M_i') \simeq \mathrm{Hom}_{\tilde{\boldsymbol{A}}}(M_i^+, (M_i')^+), \\ \beta_i &\in \mathrm{Ext}^1_{\boldsymbol{H}}(M_i, M_{i+1}') \simeq \mathrm{Ext}^1_{\tilde{\boldsymbol{A}}}(M_i^+, (M_{i+1}')^+). \end{aligned}$$

These morphisms must satisfy the relations

$$\mu_i'\alpha_i = \gamma_i\mu_i, \quad \eta_i'\alpha_i = \gamma_{i+1}\eta_i + \beta_i\mu_i.$$

6. Partial tilting for Backström pairs

Let $\boldsymbol{H} \subseteq \boldsymbol{A}$ be a Backström pair. Consider the ring $\boldsymbol{B}$ of triangular matrices of the form

$$\boldsymbol{B} = \begin{pmatrix} \bar{\boldsymbol{A}} & \bar{\boldsymbol{H}} \\ 0 & \boldsymbol{H} \end{pmatrix}.$$

Let $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, and let $B_1 = \boldsymbol{B}e_1$ and $B_2 = \boldsymbol{B}e_2$ be projective $\boldsymbol{B}$-modules given by the first and the second column of $\boldsymbol{B}$, i.e.,

$$B_1 = \begin{pmatrix} \bar{\boldsymbol{A}} \\ 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} \bar{\boldsymbol{H}} \\ \boldsymbol{H} \end{pmatrix}.$$

A $\boldsymbol{B}$-module M is defined by a triple $\begin{pmatrix} M_1 \\ M_2 \end{pmatrix} \chi_M)$, where $M_1 = e_1 M$ is an $\bar{\boldsymbol{A}}$-module, $M_2 = e_2 M$ is an $\boldsymbol{H}$-module and $\chi_M : M_2 \to M_1$ is an $\boldsymbol{A}$-homomorphism such that $\operatorname{Ker} \chi_M \supseteq \boldsymbol{C} M_2$ (it is necessary since $\boldsymbol{C} M_1 = 0$). Namely, χ_M is multiplication by $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. We write an element $u \in M$ as a column $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, where $u_1 = e_1 u$, $u_2 = e_2 u$. Then,

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} a u_1 + \chi_M(b u_2) \\ c u_2 \end{pmatrix}.$$

A homomorphism $\alpha : M \to N$ is defined by two homomorphisms $\alpha_1 : M_1 \to N_1$ and $\alpha_2 : M_2 \to N_2$ such that $\alpha_1 \chi_M = \chi_N \alpha_2$. We write $\alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$.

Proposition 6.1. *We have* $\operatorname{l.gl.dim} \boldsymbol{B} = \max(\operatorname{l.gl.dim} \boldsymbol{H}, \operatorname{w.dim} \overline{\boldsymbol{H}}_{\boldsymbol{H}} + 1)$.

In particular, if $\boldsymbol{H}$ is left hereditary and $\overline{\boldsymbol{H}}$ is not flat as a right $\boldsymbol{H}$-module, then $\operatorname{l.gl.dim} \boldsymbol{B} = 2$.

Proof. [Palmér and Roos 1973, Theorem 5] shows that $\operatorname{l.gl.dim} \boldsymbol{B} \le n$ if and only if

$$\operatorname{l.gl.dim} \boldsymbol{H} \le n \text{ and } \mathbb{R}^n \operatorname{Hom}_{\bar{\boldsymbol{A}}}(\overline{\boldsymbol{H}} \otimes_{\boldsymbol{H}} - , -) = 0.$$

As the ring $\bar{\boldsymbol{A}}$ is semisimple,

$$\mathbb{R}^n \operatorname{Hom}_{\bar{\boldsymbol{A}}}(\overline{\boldsymbol{H}} \otimes_{\boldsymbol{H}} - , -) = \operatorname{Hom}_{\bar{\boldsymbol{A}}}(\operatorname{Tor}_n^{\boldsymbol{H}}(\overline{\boldsymbol{H}}, -), -).$$

This implies the first assertion. The second is obvious, since $\operatorname{Tor}_1^{\boldsymbol{H}}(\overline{\boldsymbol{H}}, -) = 0$ if and only if $\overline{\boldsymbol{H}}$ is flat as a right $\boldsymbol{H}$-module. □

We denote by R the $\boldsymbol{B}$-module given by the triple $\begin{pmatrix} \boldsymbol{H}/\boldsymbol{A} \\ \boldsymbol{H} \end{pmatrix} \pi)$, where $\pi : \boldsymbol{H} \to \boldsymbol{H}/\boldsymbol{A}$ is the natural surjection.

Proposition 6.2. (1) $\operatorname{End}_{\boldsymbol{B}} R \simeq \boldsymbol{A}^{\mathrm{op}}$.

(2) $\operatorname{pr.dim}_{\boldsymbol{B}} R = 1$.

(3) $\operatorname{Ext}^1_{\boldsymbol{B}}(R, R) = 0$.

Recall that conditions (2) *and* (3) *mean that R is a partial tilting $\boldsymbol{B}$-module.*

Proof. The minimal projective resolution of R is

$$0 \to B_1 \xrightarrow{\varepsilon} B_2 \to R \to 0,$$

where ε is the embedding, which gives (2). Any endomorphism γ of R induces a commutative diagram:

$$\begin{array}{ccc} B_1 & \xrightarrow{\varepsilon} & B_2 \\ {\scriptstyle\gamma_1}\downarrow & & \downarrow{\scriptstyle\gamma_2} \\ B_1 & \xrightarrow{\varepsilon} & B_2 \end{array}$$

As $\operatorname{End}_{\boldsymbol{B}} B_2 \simeq \boldsymbol{H}^{\mathrm{op}}$, γ_2 is given by multiplication with an element $h \in \boldsymbol{H}$ on the right. If there is a commutative diagram as above, necessarily $h \in \boldsymbol{A}$, which proves (1).

Finally, a homomorphism $\alpha : B_1 \to R$ maps the generator $\binom{1}{0}$ of B_1 to an element $\binom{\bar{h}}{0} \in R$. If h is a preimage of $\bar{h}$ in $\boldsymbol{H}$, then α extends to the homomorphism $B_2 \to R$ that maps the generator $\binom{0}{1}$ of B_2 to $\binom{0}{h} \in R$. This implies (3). □

Now Theorem 1.1 applied to the module R gives the following result:

Theorem 6.3. (1) *The functor* $\mathsf{F} = \mathbb{R}\mathrm{Hom}(R, -)$ *induces an equivalence*

$$\mathrm{Tri}(R) \xrightarrow{\sim} \mathscr{D}(\boldsymbol{A}).$$

(2) $\operatorname{Ker}\mathsf{F}$ *consists of complexes* C *such that the map* $\chi_{H^k(C)}$ *is bijective for all* k.

(3) *There is a recollement diagram*

$$\operatorname{Ker}\mathsf{F} \;\underset{\mathsf{I}^!}{\overset{\mathsf{I}^*}{\leftleftarrows}}\!\xrightarrow{\ \mathsf{I}\ } \mathscr{D}(\boldsymbol{B}) \;\underset{\mathsf{F}^!}{\overset{\mathsf{F}^*}{\leftleftarrows}}\!\xrightarrow{\ \mathsf{F}\ } \mathscr{D}(\boldsymbol{A}).$$

Actually, claim (2) *means that a complex* C *is in* $\operatorname{Ker}\mathsf{F}$ *if and only if its cohomologies are direct sums of* $\boldsymbol{B}$*-modules of the form* $\left(\begin{smallmatrix} U \\ U \end{smallmatrix}\ 1_U\right)$, *where* U *is a simple* $\overline{\boldsymbol{H}}$*-module.*

F is a partial tilting functor in the sense of Corollary 1.2.

Proof. (1) and (3) follow from Proposition 6.2 and Theorem 1.1, since the complex $P : 0 \to B_1 \xrightarrow{\varepsilon} B_2 \to 0$ is perfect, hence compact, and isomorphic to R in $\mathscr{D}(\mathscr{B})$. To find $\operatorname{Ker}\mathsf{F}$, consider a complex

$$C : \cdot \to C^{k-1} \xrightarrow{d^{k-1}} C^k \xrightarrow{d^k} C^{k+1} \to \cdots,$$

where C^k is defined by a triple $\left(\begin{smallmatrix} C_1^k \\ C_2^k \end{smallmatrix}\ \chi_k\right)$ and $d^k = \left(\begin{smallmatrix} d_1^k \\ d_2^k \end{smallmatrix}\right)$, where $d_1^k \chi_k = \chi_{k+1} d_2^k$ for all k. Note that $C_i = (C_i^k, d_i^k)$ $(i = 1, 2)$ are complexes, (χ_k) is a homomorphism of complexes and $H^k(C) = \left(\begin{smallmatrix} H^k(C_1) \\ H^k(C^2) \end{smallmatrix}\ \bar{\chi}_k\right)$, where $\bar{\chi}_k = \chi_{H^k(C)}$ is induced by χ_k. A homomorphism $P \to C[k]$ is a pair of homomorphisms $\alpha : B_2 \to C^k,\ \beta : B_1 \to C^{k-1}$ such that $\alpha_1 \pi = \chi_k \alpha_2,\ \beta_2 = 0,\ d_i^k \alpha_i = 0\ (i = 1, 2)$ and $d^{k-1}\beta_1 = \alpha_1|_{\bar{\boldsymbol{A}}}$. Let $\alpha_2(1) = x \in C_2^k$ and $\beta_1(1) = y \in C_1^{k-1}$. These values completely define α and β. The conditions for α and β mean that $d_2^k x = 0$ and $d^{k-1} y = \chi_k x$.

This morphism is homotopic to zero if and only if there are maps $\sigma : B_2 \to C^{k-1}$ and $\tau : B_1 \to C^{k-2}$ such that $\alpha = d^{k-1}\sigma$ and $\beta = \sigma\varepsilon + d^{k-2}\tau$. Again, σ is defined by the element $z = \sigma_2(1) \in C_2^{k-1}$ and τ is defined by the element $t = \tau_1(1) \in C_1^{k-2}$. Then, the conditions for α and β mean that $x = d_2^{k-1} z$ and $y = \chi_{k-1} z + d_1^{k-2} t$.

Suppose that any homomorphism $P \to C[k]$ is homotopic to zero. Let $\bar{x}$ in $H^k(C^2)$ be such that $\bar{\chi}_k(\bar{x}) = 0$ and $x \in \operatorname{Ker} d_2^k$ be a representative of $\bar{x}$. Then, $\chi_k(x) = d_1^{k-1} y$ for some $y \in C^{k-1}$, so the pair (x, y) defines a homomorphism $P \to C[k]$. Therefore, there must be $z \in C_2^{k-1}$ such that $x = d^{k-1} z$; thus $\bar{x} = 0$ and $\bar{\chi}_k$ is injective. Let now $\bar{y} \in H^{k-1}(C_2)$ and $y \in C_2^{k-1}$ be its representative. Then, the pair $(0, y)$ defines a homomorphism $P \to C[k]$, so there must be elements $z \in C_2^{k-1}$

and $t \in C_1^{k-2}$ such that $d_1^{k-1}z = 0$ and $y = \chi_{k-1}z + d_1^{k-2}t$. Hence, $\bar{y} = \bar{\chi}_{k-1}(\bar{z})$, so $\bar{\chi}_{k-1}$ is surjective. As this holds for all k, we have that all maps $\bar{\chi}_k$ are bijective.

On the contrary, suppose that all $\bar{\chi}_k$ are bijective. If a pair (x, y) defines a homomorphism $P \to C[k]$, then $\chi_k(x) = d_1^{k-1}y$, so $\bar{\chi}_k(x) = 0$. Therefore, $\bar{x} = 0$, i.e., $x = d_2^{k-1}z$ for some $z \in C_2^{k-1}$ and $\chi_k x = d_1^{k-1}\chi_{k-1}z$. Then, $d_1^{k-1}(y - \chi_{k-1}z) = 0$, hence there is an element $z' \in C_2^{k-1}$ such that $d_2^{k-1}z' = 0$ and the cohomology class of $y - \chi_{k-1}z$ equals $\bar{\chi}_{k-1}\bar{z}'$, i.e., $y - \chi_{k-1}z = \chi_{k-1}z' + d_1^{k-2}t$ for some t. Then, $x = d_2^{k-1}(z + z')$ and $y = \chi_{k-1}(z + z') + d_1^{k-2}t$, so this homomorphism is homotopic to zero. □

As usual, we identify the category $\boldsymbol{A}$-Mod with the full subcategory of $\mathscr{D}(\boldsymbol{A})$ consisting of the complexes C concentrated in degree 0. The following result shows how the partial titling functor F behaves with respect to modules:

Corollary 6.4. *Let a $\boldsymbol{B}$-module M be given by the triple $\left(\begin{smallmatrix} M_1 \\ M_2 \end{smallmatrix}\ \chi_M\right)$.*

(1) $\mathsf{F}M \in \boldsymbol{A}$-Mod *if and only if χ_M is surjective. Namely, then* $\mathsf{F}M \simeq \operatorname{Ker} \chi_M$.

(2) $\mathsf{F}M \in \boldsymbol{A}$-Mod[1] *if and only if χ_M is injective. Namely, then* $\mathsf{F}M \simeq \operatorname{Cok} \chi_M[1]$.

Proof. Note that $\operatorname{Hom}_{\boldsymbol{B}}(B_1, M) \simeq M_1$, $\operatorname{Hom}_{\boldsymbol{B}}(B_2, M) \simeq M_2$ and if $\phi : B_2 \to M$ maps $\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$ to $\left(\begin{smallmatrix} 0 \\ x \end{smallmatrix}\right)$, then $\phi\varepsilon$ maps $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ to $\left(\begin{smallmatrix} \chi_M(x) \\ 0 \end{smallmatrix}\right)$. Therefore, $\mathbb{R}\operatorname{Hom}_{\boldsymbol{B}}(R, M)$ is the complex

$$0 \to M_2 \xrightarrow{\chi_M} M_1 \to 0,$$

which proves the claim. □

Remark 6.5. There are several derived equivalences related to $\tilde{\boldsymbol{A}}$.

(1) If $\boldsymbol{A}$ is a Backström order, it is known (see [Burban et al. 2017]) that the complex $T = B_1[1] \oplus \boldsymbol{H}^+$, where $B_1 = \left(\begin{smallmatrix} \bar{\boldsymbol{A}} \\ 0 \end{smallmatrix}\right)$, is a tilting complex for $\tilde{\boldsymbol{A}}$ and $(\operatorname{End}_{\mathscr{D}(\tilde{\boldsymbol{A}})})^{\mathrm{op}} T \simeq \boldsymbol{B}$, hence $\tilde{\boldsymbol{A}}$ is derived equivalent to $\boldsymbol{B}$. Nevertheless, in the general situation of Backström rings (even of Backström algebras) this is not so. First of all, $\operatorname{Hom}_{\tilde{\boldsymbol{A}}}(B_1, \boldsymbol{H}^+) \simeq \operatorname{ann}_{\boldsymbol{H}} \boldsymbol{C}$, so it can happen that $\operatorname{Hom}_{\mathscr{D}(\tilde{\boldsymbol{A}})}(T, T[1]) \neq 0$. This is so, for instance, for the pair $(T(n, \Bbbk), \mathrm{UT}(n, \Bbbk))$ from Equation (2-3) (4), since in this case the matrix unit e_{nn} belongs to $\operatorname{ann}_{\boldsymbol{H}} \boldsymbol{C}$. This is also so for Equation (2-3) (5). Moreover, even if $\operatorname{ann}_{\boldsymbol{H}} \boldsymbol{C} = 0$, one can see that $\overline{\boldsymbol{H}}' = \operatorname{Ext}^1_{\tilde{\boldsymbol{A}}}(B_1, \boldsymbol{H}^+) \simeq \boldsymbol{C}^{-1}/{}_{\boldsymbol{C}}\boldsymbol{H}$, where $\boldsymbol{C}^{-1} = \operatorname{Hom}_{\boldsymbol{H}}(\boldsymbol{C}, \boldsymbol{H})$ and ${}_{\boldsymbol{C}}\boldsymbol{H} = \boldsymbol{H} / \operatorname{ann}_{\boldsymbol{H}} \boldsymbol{C}$ is naturally embedded into $\boldsymbol{C}^{-1}$. Therefore, in this case,

$$(\operatorname{End}_{\mathscr{D}(\tilde{\boldsymbol{A}})} T)^{\mathrm{op}} \simeq \boldsymbol{B}' = \begin{pmatrix} \bar{\boldsymbol{A}} & \overline{\boldsymbol{H}}' \\ 0 & \boldsymbol{H} \end{pmatrix},$$

which need not coincide with $\boldsymbol{B}$ (see Example 6.6 below). If $\boldsymbol{H}$ is a hereditary order, then $\operatorname{ann}_{\boldsymbol{H}} \boldsymbol{C} = 0$ and $\overline{\boldsymbol{H}}' \simeq \overline{\boldsymbol{H}}$, hence $\boldsymbol{B}' \simeq \boldsymbol{B}$, in accordance with [Burban et al. 2017].

(2) On the other hand, set $T' = \begin{pmatrix} A & H/A \\ C & \overline{H} \end{pmatrix}$ considered as a left $\tilde{A}$-module. One can check it is a tilting module for $\tilde{A}$ and

$$(\mathrm{End}_{\mathscr{D}(\tilde{A})} T')^{\mathrm{op}} \simeq \widetilde{B} = \begin{pmatrix} A & H/A \\ 0 & \overline{A} \end{pmatrix},$$

hence $\tilde{A}$ is derived equivalent to $\widetilde{B}$. Unfortunately, this ring can be not so good from the homological point of view. At least, it is not better than A itself. Namely, as one can easily check,

$$\mathrm{l.gl.dim}\, \widetilde{B} = \max(\mathrm{l.gl.dim}\, A,\, 1 + \mathrm{pr.dim}_A(H/A)),$$

which is either $\mathrm{l.gl.dim}\, A$ or (more often) $\mathrm{l.gl.dim}\, A + 1$.

(3) One more observation: Consider the right $\tilde{A}$-modules $(\overline{A}\ 0)$ and $(C\ H)$. One can check that $T'' = (\overline{A}\ 0)[1] \oplus (C\ H)$ is a tilting complex for $\mathscr{D}(\tilde{A}^{\mathrm{op}})$ and

$$\mathrm{End}_{\mathscr{D}(\tilde{A}^{\mathrm{op}})} T'' \simeq B'' = \begin{pmatrix} \overline{A} & 0 \\ \overline{H} & H \end{pmatrix},$$

hence $\tilde{A}^{\mathrm{op}}$ is derived equivalent to $(B'')^{\mathrm{op}}$.

Note that the functor $P \mapsto \mathrm{Hom}_R(P, R)$ induces an exact duality

$$\mathrm{Perf}(R) \to \mathrm{Perf}(R^{\mathrm{op}})$$

for any ring R. Hence, $\mathrm{Perf}(\tilde{A}) \simeq \mathrm{Perf}(B'')$.

Example 6.6. Let $H = \mathsf{T}(3, \Bbbk)$ and $A = \{(a_{ij}) \in H \mid a_{11} = a_{22}\}$. Set $H_i = He_{ii}$ and $U_i = H_i/\mathrm{rad}\, H_i$. Then, $C = \{(a_{ij}) \in H \mid a_{11} = a_{22} = 0\}$, hence $\overline{H} = U_1 \oplus U_2$. On the other hand, $C = \mathrm{rad}\, H_2 \oplus H_3 \simeq H_1 \oplus H_3$, so $C^{-1} = \mathrm{Hom}_H(C, H)$ can be identified with the set of 3×2 matrices (b_{ij}) such that $b_{12} = b_{22} = 0$. One can check that ${}_C H$ is identified with the subset $\{(b_{ij}) \mid b_{11} = 0\} \subset C^{-1}$ and $\overline{H}' \simeq U_2 \not\simeq \overline{H}$ (even $\dim_\Bbbk \overline{H}' \neq \dim_\Bbbk \overline{H}$).

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References

[Assem and Skowroński 1987] I. Assem and A. Skowroński, "Iterated tilted algebras of type $\tilde{\mathbf{A}}_n$", *Math. Z.* **195**:2 (1987), 269–290. MR Zbl

[Bass 1960] H. Bass, "Finitistic dimension and a homological generalization of semi-primary rings", *Trans. Amer. Math. Soc.* **95** (1960), 466–488. MR Zbl

[Bekkert and Merklen 2003] V. Bekkert and H. A. Merklen, "Indecomposables in derived categories of gentle algebras", *Algebr. Represent. Theory* **6**:3 (2003), 285–302. MR Zbl

[Bekkert et al. 2003] V. Bekkert, E. N. Marcos, and H. A. Merklen, "Indecomposables in derived categories of skewed-gentle algebras", *Comm. Algebra* **31**:6 (2003), 2615–2654. MR Zbl

[Beĭlinson et al. 1982] A. A. Beĭlinson, J. Bernstein, and P. Deligne, "Faisceaux pervers", pp. 5–171 in *Analysis and topology on singular spaces*, *I* (Luminy, 1981), Astérisque **100**, Soc. Math. France, Paris, 1982. MR Zbl

[Burban and Drozd 2004] I. Burban and Y. Drozd, "Derived categories of nodal algebras", *J. Algebra* **272**:1 (2004), 46–94. MR Zbl

[Burban and Drozd 2006] I. Burban and Y. Drozd, "Derived categories for nodal rings and projective configurations", pp. 23–46 in *Noncommutative algebra and geometry*, edited by C. De Concini et al., Lect. Notes Pure Appl. Math. **243**, Chapman & Hall/CRC, Boca Raton, FL, 2006. MR Zbl

[Burban and Drozd 2011] I. Burban and Y. Drozd, "Tilting on non-commutative rational projective curves", *Math. Ann.* **351**:3 (2011), 665–709. MR Zbl

[Burban and Drozd 2017] I. Burban and Y. Drozd, "On the derived categories of gentle and skew-gentle algebras: homological algebra and matrix problems", preprint, 2017. arXiv 1706.08358

[Burban et al. 2017] I. Burban, Y. Drozd, and V. Gavran, "Minors and resolutions of non-commutative schemes", *Eur. J. Math.* **3**:2 (2017), 311–341. MR Zbl

[Chen et al. 2008] X.-W. Chen, Y. Ye, and P. Zhang, "Algebras of derived dimension zero", *Comm. Algebra* **36**:1 (2008), 1–10. MR Zbl

[Dlab and Ringel 1976] V. Dlab and C. M. Ringel, *Indecomposable representations of graphs and algebras*, vol. 6, Mem. Amer. Math. Soc. **173**, American Mathematical Society, Providence, RI, 1976. MR Zbl

[Drozd 1990] Y. A. Drozd, "Finite modules over purely Noetherian algebras", pp. 86–96, 225 in *Galois theory*, *rings*, *algebraic groups and their applications*, vol. 183, 1990. In Russian; translated in *Proc. Steklov Inst. Math.* **4** (1991), 97–108. MR Zbl

[Drozd 2010] Y. A. Drozd, "Matrix problems, triangulated categories and stable homotopy types", *São Paulo J. Math. Sci.* **4**:2 (2010), 209–249. MR Zbl

[Drozd and Voloshyn 2012] Y. A. Drozd and D. E. Voloshyn, "Vector bundles over noncommutative nodal curves", *Ukrainian Math. J.* **64**:2 (2012), 208–224. MR Zbl

[Drozd and Zembyk 2013] Y. A. Drozd and V. V. Zembyk, "Representations of nodal algebras of type A", *Algebra Discrete Math.* **15**:2 (2013), 179–200. MR Zbl

[Gabriel 1972] P. Gabriel, "Unzerlegbare Darstellungen, I", *Manuscripta Math.* **6** (1972), 71–103; correction, ibid. 6, 309. MR Zbl

[Geißand de la Peña 1999] C. Geißand J. A. de la Peña, "Auslander–Reiten components for clans", *Bol. Soc. Mat. Mexicana* (3) **5**:2 (1999), 307–326. MR Zbl

[Happel 1988] D. Happel, *Triangulated categories in the representation theory of finite-dimensional algebras*, London Mathematical Society Lecture Note Series **119**, Cambridge University Press, 1988. MR Zbl

[Keller 1994] B. Keller, "Deriving DG categories", *Ann. Sci. École Norm. Sup.* (4) **27**:1 (1994), 63–102. MR Zbl

[Keller 2007] B. Keller, "Derived categories and tilting", pp. 49–104 in *Handbook of tilting theory*, edited by L. Angeleri Hügel et al., London Math. Soc. Lecture Note Ser. **332**, Cambridge Univ. Press, 2007. MR

[Kuznetsov and Lunts 2015] A. Kuznetsov and V. A. Lunts, "Categorical resolutions of irrational singularities", *Int. Math. Res. Not.* **2015**:13 (2015), 4536–4625. MR Zbl

[Lambek 1976] J. Lambek, *Lectures on rings and modules*, 2nd ed., Chelsea Publishing Co., New York, 1976. MR Zbl

[Lekili and Polishchuk 2018] Y. Lekili and A. Polishchuk, "Auslander orders over nodal stacky curves and partially wrapped Fukaya categories", *J. Topol.* **11**:3 (2018), 615–644. MR Zbl

[Lunts 2010] V. A. Lunts, "Categorical resolution of singularities", *J. Algebra* **323**:10 (2010), 2977–3003. MR Zbl

[Neeman 2001] A. Neeman, *Triangulated categories*, Annals of Mathematics Studies **148**, Princeton University Press, 2001. MR Zbl

[Nicolás and Saorín 2009] P. Nicolás and M. Saorín, "Parametrizing recollement data for triangulated categories", *J. Algebra* **322**:4 (2009), 1220–1250. MR Zbl

[Palmér and Roos 1973] I. Palmér and J.-E. Roos, "Explicit formulae for the global homological dimensions of trivial extensions of rings", *J. Algebra* **27** (1973), 380–413. MR Zbl

[Psaroudakis 2014] C. Psaroudakis, "Homological theory of recollements of abelian categories", *J. Algebra* **398** (2014), 63–110. MR Zbl

[Ringel and Roggenkamp 1979] C. M. Ringel and K. W. Roggenkamp, "Diagrammatic methods in the representation theory of orders", *J. Algebra* **60**:1 (1979), 11–42. MR Zbl

[Rouquier 2008] R. Rouquier, "Dimensions of triangulated categories", *J. K-Theory* **1**:2 (2008), 193–256. MR Zbl

[Sandomierski 1969] F. L. Sandomierski, "A note on the global dimension of subrings", *Proc. Amer. Math. Soc.* **23** (1969), 478–480. MR

[Voloshyn and Drozd 2013] D. E. Voloshyn and Y. A. Drozd, "Derived categories of nodal curves", *Ukrainian Math. J.* **64**:8 (2013), 1177–1184. MR Zbl

[Zembyk 2014] V. V. Zembyk, "Structure of finite-dimensional nodal algebras", *Ukrainian Math. J.* **66**:3 (2014), 467–472. MR Zbl

[Zembyk 2015] V. V. Zembyk, "Skewed-gentle algebras are nodal", *Ukrainian Math. J.* **67**:4 (2015), 648–651. MR Zbl

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