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By using Gromov's μ-bubble technique, we show that the 3-dimensional spherical caps are rigid under perturbations that do not reduce the metric, the scalar curvature, and the mean curvature along its boundary. Several generalizations of this result will be discussed.

1. Introduction

In recent decades, a lot of progress has been made toward understanding the scalar curvature of a Riemannian manifold; see [Gromov 2023]. A particular medium for gaining such understanding is to answer whether one can perturb the metric of a "model space" in certain ways without reducing its scalar curvature. This viewpoint was famously represented by the positive mass theorem and its various generalizations and analogues. One analogue, which motivated the current work, is the following conjecture proposed by Min-Oo around 1995; see [Min-Oo 1998, Theorem 4].

Conjecture 1.1 (Min-Oo). *Suppose that g is a smooth Riemannian metric on the* (*topological*) *hemisphere $\boldsymbol{S}^n_+$* ($n \geq 3$) *with the properties*:

(1) *The scalar curvature R_g satisfies $R_g \geq n(n-1)$ on $\boldsymbol{S}^n_+$.*

(2) *The boundary $\partial \boldsymbol{S}^n_+$ is totally geodesic with respect to g.*

(3) *The induced metric on $\partial \boldsymbol{S}^n_+$ agrees with the standard metric on $\boldsymbol{S}^{n-1}$.*

Then g is isometric to the standard metric on $\boldsymbol{S}^n_+$.

Unlike its counterparts modeled on $\mathbb{R}^n$ and $\boldsymbol{H}^n$,[1] Min-Oo's conjecture turned out to admit counterexamples; see [Brendle et al. 2011]. Yet, its statement remains interesting, especially when it is compared with the following theorem of Llarull [1998, Theorem A].

Theorem 1.2 (Llarull). *Let $(\boldsymbol{S}^n, \hat{g})$ be the standard n-sphere ($n \geq 3$). Suppose that g is another Riemannian metric on $\boldsymbol{S}^n$ satisfying $g \geq \hat{g}$ and $R_g \geq R_{\hat{g}}$. Then $g = \hat{g}$.*

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[1] See [Schoen and Yau 1979, Corollary 2; Gromov and Lawson 1983, Theorem A; Min-Oo 1989; Andersson et al. 2008].

A side-by-side view of Min-Oo's conjecture and Llarull's theorem suggests the following.

Conjecture 1.3. *Let* $(\boldsymbol{S}^n_+, \hat{g})$ *be the standard n-dimensional hemisphere. Then Conjecture 1.1 holds under the additional assumption*: $g \geq \hat{g}$.

Our first result in this article is that Conjecture 1.3 holds when $n = 3$; here is a more precise statement; also see Corollary 3.12 below.

Theorem 1.4. *Let* $(\boldsymbol{S}^3_+, \hat{g})$ *be the standard* 3*-dimensional hemisphere. Suppose that* g *is another Riemannian metric on* $\boldsymbol{S}^3_+$ *with the properties*:

(1) $g \geq \hat{g}$ *and* $R_g \geq R_{\hat{g}}$ *on* $\boldsymbol{S}^3_+$.

(2) *the mean curvature* H_g *on* $\partial \boldsymbol{S}^3_+$ *satisfies* $H_g \geq 0$.[2]

(3) *The induced metrics on* $\partial \boldsymbol{S}^3_+$ *satisfy* $g_{\partial \boldsymbol{S}^3_+} = \hat{g}_{\partial \boldsymbol{S}^3_+}$.

Then $g = \hat{g}$.

As we will see below, Theorem 1.4 admits a somewhat direct proof. With more technical work, we can generalize it in the following aspects: (i) the assumption (3) in Theorem 1.4 will be removed; and (ii) the model space will not need to be the standard hemisphere — it can be a "spherical cap" or, more generally, a geodesic ball inside a space form. To make these points explicit, we now state our main result; also see Theorem 5.3 below.

Theorem 1.5. *For any suitable constants* κ, μ, *let* $(\boldsymbol{B}_{\kappa,\mu}, \hat{g}_\kappa)$ *be a geodesic ball in the* 3*-dimensional space form with sectional curvature* κ *such that* $\partial \boldsymbol{B}_{\kappa,\mu}$ *has mean curvature* μ. *Suppose that* g *is another Riemannian metric on* $\boldsymbol{B}_{\kappa,\mu}$ *satisfying*

$$g \geq \hat{g}_\kappa, \quad R_g \geq 6\kappa \text{ on } \boldsymbol{B}_{\kappa,\mu} \quad \text{and} \quad H_g \geq \mu \text{ on } \partial \boldsymbol{B}_{\kappa,\mu}.$$

Then $g = \hat{g}_\kappa$.

In Gromov's first preprint of [2019a], a (more general) version of Theorem 1.5 was stated as a "nonexistence" result (see [Gromov 2019b, Theorem 1]); an outline of proof was sketched, which relied on a "generalized Llarull's theorem". Following Gromov's main idea, we present a detailed and purely variational proof of Theorem 1.5; this theorem also confirms, in the case of $n = 3$, a rigidity statement mentioned in [Gromov 2019b, Remark (d)] without proof.

A simple modification of the proof of Theorem 1.5 yields the following; also see Theorem 5.1.

[2] Given a domain Ω in a Riemannian manifold, unless we specify otherwise, we shall adopt the (sign) convention for the mean curvature of $\partial\Omega$ to be $H = \mathrm{tr}(\nabla \nu)$, where ν is the *outward* unit normal along $\partial\Omega$. Under this convention, the mean curvature of the boundary of the unit ball in $\mathbb{R}^n$ is $n - 1$.

Theorem 1.6. *Let $(S^3 \setminus \{O, O'\}, \hat{g})$ be the standard 3-sphere with a pair of antipodal points removed, and let $h \geq 1$ be a smooth function on $S^3 \setminus \{O, O'\}$. Suppose that g is another Riemannian metric on $S^3 \setminus \{O, O'\}$ satisfying*

$$g \geq h^4 \hat{g} \quad \text{and} \quad R_g \geq h^{-2} R_{\hat{g}}.$$

Then $h \equiv 1$, and $g = \hat{g}$.

When $h \equiv 1$, Theorem 1.6 is a special case of Gromov's theorem of "extremality of doubly punctured spheres" (see [Gromov 2023, Sections 5.5 and 5.7]), and it implies Theorem 1.2 in the case of $n = 3$. We also remark that Theorem 1.6 would fail without the assumption $h \geq 1$ (see Remark 5.2 below). We tend to believe that the conclusion of Theorem 1.6 still holds when the condition $g \geq h^4 \hat{g}$ is replaced by $g \geq h^2 \hat{g}$; a condition such as $\inf h > 0$ would still be needed, otherwise, the metric in Remark 5.2 would serve as a counterexample.

Before sketching our technical ingredients, let us remind the reader that since the early 1980s, two different approaches — variational and spinorial — have been developed for studying the scalar curvature. Yet, for more than two decades, extensions of Llarull's rigidity theorem, like Llarull's original proof, had been mainly carried out from the spinorial approach. See, for example, [Goette and Semmelmann 2002; Herzlich 2005; Listing 2009; Cecchini and Zeidler 2022, especially Theorem 1.15, Corollary 1.17; Lott 2021; Su et al. 2022; Zhang 2020]. It is relatively recent that variational methods have also become available for proving results of Llarull type.[3] A key in this new development, which is also a main tool for the current paper, is Gromov's *μ-bubble* technique [2023, Section 5].

Roughly speaking, given a function μ on a Riemannian manifold (M^n, g), a μ-bubble is a minimizer (and a critical point) of the functional

$$\Omega \mapsto \mathrm{vol}_{n-1}(\partial\Omega) - \int_{\Omega} \mu \tag{1-1}$$

defined for suitable subsets $\Omega \subset M$; given a μ-bubble, useful geometric information can be extracted from its first and second variation formulae. In order to guarantee that a nondegenerate μ-bubble exists, (M, g) is often assumed to be a *Riemannian band*,[4] and μ is often required to satisfy a *barrier condition* (see (2-2) below), which prevents minimizing sequences from collapsing either to a point or into ∂M.

In some cases, even without the assumption of either a Riemannian band or a barrier condition, a μ-bubble may still be found by direct observation of the functional (1-1). This is the case with our proof of Theorem 1.4. In fact, if we

[3] To our best knowledge, a purely variational proof of Llarull's original theorem remains to be found.

[4] See Section 2A below for definition, and see [Gromov 2018; Räde 2021] for related discussion.

modify (1-1) by considering the new functional

$$\Omega \mapsto \operatorname{vol}_{n-1}(\partial\Omega) + \int_{S^3_+\setminus\Omega} \mu, \tag{1-2}$$

the variational properties remain unchanged; in our situation, the new functionals associated to g and $\hat{g}$ admit an inequality, which becomes an equality when $\Omega = S^3_+$, and then direct comparison shows that S^3_+ is a μ-bubble (see the proof of Corollary 3.12). We note that this argument crucially relies on the assumption (3) in Theorem 1.4.

Now let us continue to take Theorem 1.4 as an example to explain how to obtain rigidity results from having an "initial" μ-bubble Ω. Although Ω need not be S^3_+, we do, for a technical reason, require that $\partial\Omega$ has a connected component Σ_0 whose projection onto S^2 has *nonzero degree* (see (3-6)) — for simplicity, let us call such a Σ_0 a "good component". By using the second variation and the Gauss–Bonnet formulae, we show that, under certain extra assumptions, Σ_0 must be a 2-sphere parallel (with respect to $\hat{g}$) to the equator ∂S^3_+; furthermore, along Σ_0 the ambient metric g must agree with $\hat{g}$ (Proposition 3.4). This obtained, a standard foliation lemma (Lemma 3.8) and minimality of Ω imply that g must agree with $\hat{g}$ in a *neighborhood* of Σ_0 (Lemma 3.10). Finally, with an "open-closed" argument and standard facts in geometric measure theory, we show that such a neighborhood can be extended to the whole manifold, thus completing the proof (Proposition 3.11).

In the more general setting of Theorem 1.5, the existence of an "initial" μ-bubble becomes less direct to prove. For simplicity, let us still assume that the model space is the standard hemisphere. Although (S^3_+, g) is *not* a Riemannian band, we may consider creating one from it by removing a small geodesic ball centered at the north pole $O \in (S^3_+, \hat{g})$, but an immediate problem arises: the natural choice $\mu = \hat{H}$ (see (3-3)), which corresponds to the mean curvature of the geodesic spheres centered at O with respect to $\hat{g}$, may not satisfy the barrier condition.

To address this problem, we construct a sequence of perturbations μ_ϵ (see (4-3); also see [Zhu 2021, Section 3]) of $\hat{H}$ that *do* satisfy the barrier conditions on a corresponding sequence of Riemannian bands $M_\epsilon \subset S^3_+$. In particular, in each M_ϵ there exists a μ_ϵ-bubble Ω_ϵ (Lemma 4.1). By construction, μ_ϵ tends to $\hat{H}$, and M_ϵ tends to S^3_+, as ϵ approaches 0. However, two new questions arise:

(a) *As ϵ tends to* 0, *do the Ω_ϵ subconverge to an $\hat{H}$-bubble Ω in* (S^3_+, g)?

(b) *If so, does $\partial\Omega$ possess a component whose projection to S^2 has nonzero degree*?

To put these in a slightly different way, regarding (a), we worry that Ω_ϵ may become degenerate in the limit; regarding (b), we worry that the "good components" of $\partial\Omega_\epsilon$ may either approach the north pole O and thus lose the "degree" property, or "meet and cancel" each other so that none of them is actually preserved in the limit.

In Sections 4C and 4D, we answer both questions (a) and (b) in the affirmative. A key step is to argue that each $\partial\Omega_\epsilon$ not only possesses a "good component" Σ_0^ϵ, but such a component must be disjoint from a fixed neighborhood of $O \in S^3_+$ provided that ϵ is small (Proposition 4.7), which is, again, enforced by the Gauss–Bonnet theorem. This step allows us to obtain a universal upper bound for the norm of the second fundamental form on Σ_0^ϵ, which is then used to prove the existence of a limiting hypersurface Σ_0 that is indeed a component of $\partial\Omega$ (Lemma 4.11).

Once having an "initial" μ-bubble, one may complete the proof of Theorem 1.5 by the foliation argument described above.

Regarding Theorem 1.6, we may consider Riemannian bands in $S^3 \setminus \{O, O'\}$ bounded by small geodesic spheres in $(S^3, \hat{g})$ centered at O and O', but because of the lack of mean curvature information with respect to g along those boundaries, perturbations of the form (4-3) are no longer adequate for meeting the barrier condition. To address this issue, we construct new functions μ_α by composing the function $\hat{H}$ with dilations of $S^3 \setminus \{O, O'\}$ in the "longitude" direction, and then μ_α will satisfy the desired barrier conditions; see Section 5 for more detail. The rest of the proof is similar to the other cases.

Remark 1.7. After our paper was submitted, an analogous result of Theorem 1.5 for higher dimensional spherical domains was proved in [Lee and Tam 2022]. Relying on harmonic maps flow and Ricci flow their argument works only for the case of compact domains in sphere for the time being.

2. Elements of Gromov's μ-bubble technique

In this section we recall some elements of Gromov's μ-bubble technique. Our discussion follows Section 5 of [Gromov 2023], Section 2 of [Zhu 2021] and Section 3 of [Zhou and Zhu 2020].

2A. *μ-bubbles in a Riemannian band.* Let (M^n, g) be a compact Riemannian manifold whose boundary ∂M is expressed as a disjoint union $\partial M = \partial_- \sqcup \partial_+$ where both ∂_- and ∂_+ are closed hypersurfaces. Such a quadruple $(M, g; \partial_-, \partial_+)$ is called a *Riemannian band*. Given a Riemannian band, let $\Omega_0 \subset M$ be a fixed smooth Caccioppoli set that contains a neighborhood of ∂_- and is disjoint from a neighborhood of ∂_+;[5] we call such an Ω_0 a *reference set*. Let $\mathcal{C}_{\Omega_0}$ denote the collection of Caccioppoli sets $\Omega \subset M$ satisfying $\Omega\Delta\Omega_0 \Subset \mathring{M}$ ("$\Subset$" reads "is compactly contained in"); here $\Omega\Delta\Omega_0$ denotes the symmetric difference between Ω and Ω_0, and $\mathring{M}$ stands for the interior of M.

[5] Also known as "sets of locally finite perimeter"; see [Giusti 1984] for details.

Let μ be either a smooth function on M, or a smooth function defined on $\mathring{M}$ satisfying $\mu \to \pm\infty$ on $\partial_{\mp}$. For $\Omega \in \mathcal{C}_{\Omega_0}$ consider the *brane action*

$$\mathcal{A}^{\mu}_{\Omega_0}(\Omega) := \mathcal{H}^{n-1}(\partial\Omega) - \mathcal{H}^{n-1}(\partial\Omega_0) - \int_M (\chi_\Omega - \chi_{\Omega_0})\mu \, d\mathcal{H}^n \tag{2-1}$$

where $\mathcal{H}^k$ is the k-dimensional Hausdorff measure induced by g and χ_Ω denotes the characteristic function associated to Ω. A minimizer Ω of (2-1) is called a *μ-bubble*.

Remark 2.1. (1) For $\Omega_1, \Omega_2 \in \mathcal{C}_{\Omega_0}$, we have $\mathcal{A}^{\mu}_{\Omega_0}(\Omega_2) - \mathcal{A}^{\mu}_{\Omega_1}(\Omega_2) = \mathcal{A}^{\mu}_{\Omega_0}(\Omega_1)$; thus, in a sense, minimizers are independent of the choice of a reference set. (2) The brane action (2-1) may be defined on manifolds that are not necessarily Riemannian bands; in those cases, one may replace $\mathcal{H}^{n-1}(\partial\Omega)$ by $\mathcal{H}^{n-1}(\partial(\Omega \cap K))$ and similarly for $\mathcal{H}^{n-1}(\partial\Omega_0)$, where K is a compact set such that $\Omega \Delta \Omega_0 \subset K$.

2B. *Existence and regularity.*

Definition 2.2. Given a Riemannian band $(M, g; \partial_-, \partial_+)$, a function μ is said to satisfy the *barrier condition* if either $\mu \in C^\infty(\mathring{M})$ with $\mu \to \pm\infty$ on $\partial_{\mp}$, or $\mu \in C^\infty(M)$ with

$$\mu|_{\partial_-} > H_{\partial_-}, \quad \mu|_{\partial_+} < H_{\partial_+} \tag{2-2}$$

where H_{∂_-} is the mean curvature of ∂_- with respect to the inward normal and H_{∂_+} is the mean curvature of ∂_+ with respect to the outward normal.

Lemma 2.3 [Zhu 2021, Proposition 2.1]. *Let $(M^n, g; \partial_-, \partial_+)$ be a Riemannian band with $n \le 7$, and let Ω_0 be a reference set. If μ satisfies the barrier condition, then there exists an $\Omega \in \mathcal{C}_{\Omega_0}$ with smooth boundary such that*

$$\mathcal{A}^{\mu}_{\Omega_0}(\Omega) = \inf_{\Omega' \in \mathcal{C}_{\Omega_0}} \mathcal{A}^{\mu}_{\Omega_0}(\Omega').$$

Remark 2.4. In Lemma 2.3 the smooth hypersurface $\Sigma := \partial\Omega \setminus \partial_-$ is homologous to ∂_+.

2C. *Variational properties.* Let Ω be a smooth μ-bubble in a Riemannian band $(M^n, g; \partial_-, \partial_+)$, and let $\Sigma = \partial\Omega \setminus \partial_-$. One may derive variation formulae for $\mathcal{A}^\mu$ at Ω; see (2.3) in [Zhu 2021] and the unnumbered equation above it. Specifically, the first variation implies that the mean curvature of Σ (with its outward normal ν) is equal to $\mu|_\Sigma$; the second variation implies that the Jacobi operator

$$J_\Sigma := -\Delta_\Sigma + \tfrac{1}{2}(R_\Sigma - R_g - \mu^2 - |\mathrm{II}|^2) - \nu(\mu) \tag{2-3}$$

is nonnegative, where Δ_Σ and R_Σ are respectively the g-induced Laplacian and scalar curvature of Σ; R_g is the scalar curvature of (M, g); and II is the second fundamental form of Σ.

Definition 2.5. Let μ be a smooth function on a Riemannian manifold (M^n, g). A smooth two-sided hypersurface $\mathcal{S} \subset M$ with unit normal ν is said to be a μ-*hypersurface* if its mean curvature taken with respect to ν is equal to $\mu|_{\mathcal{S}}$.

Clearly, (2-3) also makes sense when Σ is replaced by a μ-hypersurface; this motivates the following notion of stability.

Definition 2.6. A μ-hypersurface $\mathcal{S} \subset M$ with unit normal ν is said to be *stable* if $J_{\mathcal{S}}$ is nonnegative on $C_0^\infty(\mathcal{S})$.

Remark 2.7. If μ satisfies the barrier condition, then for any μ-bubble Ω each connected component of $\partial\Omega \backslash \partial_-$ with its outward unit normal is a stable μ-hypersurface.

Let $\mathcal{S}$ be a μ-hypersurface. Following [Gromov 2023, Section 5.1] we consider the operator

$$L_{\mathcal{S}} := -\Delta_{\mathcal{S}} + \tfrac{1}{2}(R_{\mathcal{S}} - R_+^\mu) \tag{2-4}$$

where

$$R_+^\mu := R_g + \frac{n}{n-1}\mu^2 - 2|\mathrm{d}\mu|_g. \tag{2-5}$$

In fact, $L_{\mathcal{S}}$ is obtained from applying the obvious inequalities

$$-\partial_\nu \mu \le |\mathrm{d}\mu|_g, \quad |\mathrm{II}|^2 \ge \frac{1}{n-1}\mu^2 \tag{2-6}$$

to $J_{\mathcal{S}}$. One can easily verify that the following holds when $\mathcal{S}$ is stable:

$$L_{\mathcal{S}} \ge J_{\mathcal{S}} \ge 0. \tag{2-7}$$

Example 2.8. Consider $\boldsymbol{S}^2 \times [t_1, t_2]$ $(0 < t_1 < t_2 < \pi)$ equipped with the metric $g = (\sin^2 t) g_{\boldsymbol{S}^2} + \mathrm{d}t^2$ where $g_{\boldsymbol{S}^2}$ is the standard metric on $\boldsymbol{S}^2$. This represents an annular region in the standard $\boldsymbol{S}^3$. Take $\mu(t) = 2\cot t$. It is easy to see that each t-level set S_t, with the unit normal $\nu = \partial_t$, is a μ-hypersurface. Moreover, on S_t we have

$$R_g = 6, \quad R_{S_t} = \frac{2}{\sin^2 t}, \quad |\mathrm{II}|^2 = 2\cot^2 t, \quad \nu(\mu) = \mu'(t) = -\frac{2}{\sin^2 t}.$$

In this case, both J_{S_t} and L_{S_t} reduce to $-\Delta_{S_t}$.

The following lemma is a direct consequence of Theorem 3.6 in [Zhou and Zhu 2020].

Lemma 2.9. *Let (M^n, g) be a closed Riemannian manifold with $2 \le n \le 6$, and let $\mu \in C^\infty(M)$. Let $\mathcal{S}$ be an immersed stable μ-hypersurface contained in an open subset $V \subset M$ and satisfying $\partial\mathcal{S} \cap V = \varnothing$. If* $\operatorname{area}(\mathcal{S}) \le C$ *for some constant C, then there exists a constant $C_1 = C_1(M, n, \|\mu\|_{C^3(M)}, C)$ such that*

$$|\mathrm{II}|^2(x) \le \frac{C_1}{\operatorname{dist}_g^2(x, \partial V)} \quad \text{for all } x \in \mathcal{S}. \tag{2-8}$$

2D. ***Comparison with a warped-product metric.*** Given a Riemannian manifold (N^{n-1}, g_N), an interval I (with coordinate t) and a function $\varphi : I \to \mathbb{R}_+$, consider the warped product metric defined on $\hat{N} := N \times I$

$$\hat{g} := \varphi(t)^2 g_N + \mathrm{d}t^2. \tag{2-9}$$

A standard calculation shows that the mean curvature on each slice $N \times \{t\}$ with respect to the ∂_t-direction is

$$\hat{H}(t) = (n-1)\frac{\varphi'(t)}{\varphi(t)}; \tag{2-10}$$

moreover, one may verify that the scalar curvature $R_{\hat{g}}$ of $\hat{g}$ satisfies

$$0 = -R_{\hat{g}} + \frac{1}{\varphi^2} R_N - \frac{n}{n-1}\hat{H}^2 - 2\frac{d\hat{H}}{dt}, \tag{2-11}$$

where R_N is the scalar curvature of (N, g_N).

Now suppose that $f : M \to \hat{N}$ is a smooth map from a Riemannian band $(M, g; \partial_-, \partial_+)$ to $\hat{N}$. By pulling back all functions in (2-11) via f and adding the resulting equation with (2-5), we obtain

$$R_+^\mu = \frac{1}{\varphi^2} R_N + (R_g - R_{\hat{g}}) + \frac{n}{n-1}(\mu^2 - \hat{H}^2) - 2(\partial_t \hat{H} + |\mathrm{d}\mu|_g), \tag{2-12}$$

where pull-back symbols are omitted for clarity. The expression (2-12) will be useful in our analysis of μ-bubbles.

3. Rigidity of 3D spherical caps

A spherical cap of radius $T \in (0, \pi)$ in the standard $\boldsymbol{S}^3$ may be represented by the closed ball $\boldsymbol{B}_T := \{\boldsymbol{x} \in \mathbb{R}^3 : |\boldsymbol{x}| \le T\}$ equipped with the metric

$$\hat{g} = \varphi(t)^2 g_{\boldsymbol{S}^2} + \mathrm{d}t^2 \quad \text{with } \varphi(t) = \sin t, \tag{3-1}$$

where $t \in [0, T]$ serves as the radial coordinate on $\boldsymbol{B}_T$ and $g_{\boldsymbol{S}^2}$ is the standard metric on $\boldsymbol{S}^2$. For $t \in (0, T]$, let $S_t := \partial \boldsymbol{B}_t$. For $0 < t_1 < t_2 \le T$, let $\boldsymbol{B}_{[t_1,t_2]} := \boldsymbol{B}_{t_2} \setminus \mathring{\boldsymbol{B}}_{t_1}$; similarly, let $\boldsymbol{B}_{(t_1,t_2]} := \boldsymbol{B}_{t_2} \setminus \boldsymbol{B}_{t_1}$. Given a domain $\Omega \subset \boldsymbol{B}_T$ with smooth boundary Σ, the outward normal along Σ with respect to the metric $\hat{g}$ will be denoted by $\hat{\nu}$.

The objective of this section and the next is to prove the following rigidity theorem.

Theorem 3.1. *Let $(\boldsymbol{B}_T, \hat{g})$ be the 3-dimensional spherical cap of radius $T \in (0, \pi)$. Suppose that g is another Riemannian metric on $\boldsymbol{B}_T$ satisfying*

$$g \geq \hat{g}, \quad R_g \geq R_{\hat{g}} \text{ on } \boldsymbol{B}_T, \quad \text{and} \quad H_g \geq H_{\hat{g}} = 2\cot T \text{ on } \partial \boldsymbol{B}_T. \tag{3-2}$$

Then $g = \hat{g}$.

Our proof begins by establishing a key ingredient: certain stable μ-hypersurfaces are necessarily t-level sets in $\boldsymbol{B}_T$ (Proposition 3.4), the justification of which hinges on an integral inequality (see (3-14)) involving an application of the Gauss–Bonnet formula. This result is followed by a classical foliation lemma (Lemma 3.8). Under a suitable "minimality" assumption (Assumption 3.9), each leaf in that foliation turns out to be stable, which implies local rigidity of the metric (Lemma 3.10). Section 3 culminates at Proposition 3.11, which justifies Theorem 3.1 assuming the existence of an "initial" minimizer (Assumption 3.9); this assumption will be verified in Section 4 via a perturbation argument (see Proposition 4.12).

3A. *Stable μ-hypersurfaces and t-level sets.* The metric (3-1) is of the form (2-9); thus, (2-10) applies to give

$$\hat{H}(t) = 2\cot t. \tag{3-3}$$

It will be useful to define, for $\mu = \mu(t)$, the function (see the last two terms in (2-12))

$$\begin{aligned} Z_\mu(t) &:= \tfrac{3}{2}(\mu(t)^2 - \hat{H}(t)^2) - 2(\hat{H}'(t) - \mu'(t)) \\ &= \tfrac{3}{2}\mu(t)^2 + 2\mu'(t) - 6\cot^2 t + \frac{4}{\sin^2 t}. \end{aligned} \tag{3-4}$$

Notice, in particular, that $Z_{\hat{H}}(t) \equiv 0$. As t is a coordinate on $\boldsymbol{B}_T$, we may regard μ and Z_μ as functions defined on $\boldsymbol{B}_T \setminus \{\boldsymbol{0}\}$.

Lemma 3.2. *Let $\mu(t)$ be a smooth, decreasing function defined on $(0, T]$, and let g be a Riemannian metric on $\boldsymbol{B}_T$ satisfying (3-2). At a point $q \in \boldsymbol{B}_T$, if $Z_\mu \geq 0$, then $R_+^\mu \geq 2/\varphi^2 > 0$.*

Proof. On the right-hand side of (2-12), the second term is nonnegative by assumption. Moreover, $g \geq \hat{g}$ implies

$$|\mathrm{d}\mu|_g \leq |\mathrm{d}\mu|_{\hat{g}} = |\partial_t \mu| = -\mu'(t). \tag{3-5}$$

Substituting this in the last term of (2-12) and noticing that $R_{S^2} = 2$, we obtain the desired inequality. $\square$

Now let Σ_0 be a hypersurface in $\boldsymbol{B}_{(0,T]}$, and let Φ denote the projection map from Σ_0 to $\boldsymbol{S}^2$, namely,

$$\Phi : \Sigma_0 \hookrightarrow \boldsymbol{B}_{(0,T]} \cong (0, T] \times \boldsymbol{S}^2 \to \boldsymbol{S}^2. \tag{3-6}$$

Lemma 3.3. *Let $d\sigma_{\hat{g}}$ be the area form on Σ_0 induced by $\hat{g}$. We have*

$$d\sigma_{\hat{g}} \geq \varphi^2 |\Phi^* \, d\sigma_{\boldsymbol{S}^2}| \tag{3-7}$$

where the absolute-value sign is put to eliminate the ambiguity of orientation.

Proof. Let (θ_α) $(\alpha = 1, 2)$ be local coordinates on $\boldsymbol{S}^2$, and write $g_{\boldsymbol{S}^2} = h_{\alpha\beta} \mathrm{d}\theta_\alpha \mathrm{d}\theta_\beta$. We get

$$\Phi^*(g_{\boldsymbol{S}^2}) = h_{\alpha\beta} \, \mathrm{d}\theta_\alpha \, \mathrm{d}\theta_\beta \leq \frac{1}{\varphi^2}(\mathrm{d}t^2 + \varphi^2 h_{\alpha\beta} \, \mathrm{d}\theta_\alpha \, \mathrm{d}\theta_\beta) = \frac{1}{\varphi^2} \hat{g}_{\Sigma_0}, \tag{3-8}$$

where the functions and forms are restricted to Σ_0. The conclusion follows. $\square$

Proposition 3.4. *Let $\mu(t)$ be a smooth, decreasing function defined on $(0, T]$. Suppose that $\Sigma_0 \hookrightarrow (\boldsymbol{B}_T \setminus \{\boldsymbol{0}\}, g)$ is a stable, closed μ-hypersurface with unit normal ν, where g satisfies $g \geq \hat{g}$ and $R_g \geq R_{\hat{g}}$. Moreover, suppose that $Z_\mu \geq 0$ on Σ_0 and that the projection Φ from Σ_0 to $\boldsymbol{S}^2$ has nonzero degree. Then:*

(a) *$\Sigma_0 = S_\tau$ for some $\tau \in (0, T]$.*

(b) *$J_{\Sigma_0} = L_{\Sigma_0} = -\Delta_{\Sigma_0}$; see (2-3), (2-4).*

(c) *$\Sigma_0 \subset (\boldsymbol{B}_T, g)$ is umbilic with constant mean curvature $\mu(\tau)$.*

(d) *$g(p) = \hat{g}(p)$ at all points $p \in \Sigma_0$; in particular, $g_{\Sigma_0} = \hat{g}_{\Sigma_0} = (\sin^2)\tau g_{\boldsymbol{S}^2}$.*

(e) *On Σ_0, $\nu = \partial_t$.*

(f) *On Σ_0, $R_+^\mu = 2/\varphi^2$ and $Z_\mu = 0$.*

We prepare our proof of this proposition with the following two lemmas.

Lemma 3.5. *Under the assumption of Proposition 3.4, Σ_0 is homeomorphic to $\boldsymbol{S}^2$.*

Proof. By stability, the operator L_{Σ_0} defined by (2-4) is nonnegative. Let $u \in C^\infty(\Sigma_0)$ be a principal eigenfunction of L_{Σ_0}, and let $\lambda_1 \geq 0$ be the corresponding eigenvalue. By the maximum principle, we can always choose u to be strictly positive. Thus,

$$-u^{-1} \Delta_{\Sigma_0} u + \tfrac{1}{2}(R_{\Sigma_0} - R_+^\mu) = \lambda_1 \geq 0. \tag{3-9}$$

Expanding

$$\mathrm{div}(u^{-1} \nabla_{\Sigma_0} u) = -u^{-2} |\nabla_{\Sigma_0} u|^2 + u^{-1} \Delta_{\Sigma_0} u, \tag{3-10}$$

applying it in the previous equation and integrating over Σ_0, we obtain

$$\frac{1}{2}\int_{\Sigma_0}(R_{\Sigma_0}-R_+^\mu)\,d\sigma_g=\int_{\Sigma_0}(\lambda_1+u^{-2}|\nabla_{\Sigma_0}u|^2)\,d\sigma_g\ge 0. \tag{3-11}$$

From (3-11), the Gauss–Bonnet formula, and Lemma 3.2, we deduce

$$4\pi\chi(\Sigma_0)=\int_{\Sigma_0}R_{\Sigma_0}\,d\sigma_g\ge\int_{\Sigma_0}R_+^\mu\,d\sigma_g>0; \tag{3-12}$$

since Σ_0 is a connected oriented surface, it is homeomorphic to $\boldsymbol{S}^2$. □

Remark 3.6. Lemma 3.5 remains true if we assume $R_+^\mu>0$ instead of $Z_\mu\ge 0$ on Σ_0.

Lemma 3.7. *Under the assumption of Proposition 3.4, if* $J_{\Sigma_0}=L_{\Sigma_0}=-\Delta_{\Sigma_0}$, *then*:

(i) $\Sigma_0\subset(\boldsymbol{B}_T,g)$ *is umbilic.*

(ii) $\Sigma_0=S_\tau$ *for some* $\tau\in(0,T]$.

(iii) $\mu|_{\Sigma_0}=\mu(\tau)$.

Proof. By assumption, (2-6) must be equalities. In particular, the traceless part of II_{Σ_0} must vanish, and thus $\Sigma_0\subset(\boldsymbol{B}_T,g)$ is umbilic, justifying (i). Moreover, $-\nu(\mu)=|\mathrm{d}\mu|_g$, and so ν must be parallel to $\nabla_g\mu$. Thus, for any tangent vector $X\in T\Sigma_0$, we have that $\mathrm{d}\mu(X)=g(\nabla_g\mu,X)$ is proportional to $g(\nu,X)=0$; this implies that μ is constant along Σ_0. Combining with the fact that $\Sigma_0\cong\boldsymbol{S}^2$ (Lemma 3.5), we conclude that Σ_0 is a level set S_τ, justifying (ii), and (iii) immediately follows. □

Proof of Proposition 3.4. The assumption $g\ge\hat g$ implies the relation between area forms on Σ_0:

$$d\sigma_g\ge d\sigma_{\hat g}. \tag{3-13}$$

We deduce

$$\begin{aligned}\int_{\Sigma_0}R_+^\mu\,d\sigma_g&\ge\int_{\Sigma_0}\frac{2}{\varphi^2}\,d\sigma_{\hat g}\\&\ge 2\int_{\Sigma_0}|\Phi^*\,d\sigma_{\boldsymbol{S}^2}|\\&\ge 2\left|\int_{\Sigma_0}\Phi^*\,d\sigma_{\boldsymbol{S}^2}\right|\\&=2k\int_{\boldsymbol{S}^2}d\sigma_{\boldsymbol{S}^2}\\&=8k\pi,\end{aligned} \tag{3-14}$$

where $k := |\deg(\Phi)| \geq 1$ by assumption. In (3-14), the first inequality is due to (3-13) and Lemma 3.2; the second inequality follows from Lemma 3.3; the remaining (in)equalities are obvious.

On combining (3-12) with (3-14), we obtain

$$8\pi = \int_{\Sigma_0} R_{\Sigma_0}\, d\sigma_g \geq \int_{\Sigma_0} R_+^{\mu}\, d\sigma_g \geq 8k\pi, \quad (k \geq 1). \tag{3-15}$$

This enforces the two inequalities in (3-15) to become equalities. Saturation of the first inequality, which we deduced from (3-11), implies that $\lambda_1 = 0$ and that u is a constant; hence, by (3-9), $R_{\Sigma_0} = R_+^{\mu}$; then, by (2-4), $L_{\Sigma_0} = -\Delta_{\Sigma_0}$. With this established, the relation (2-7) would enforce that $J_{\Sigma_0} = L_{\Sigma_0} = -\Delta_{\Sigma_0}$, justifying (b). By Lemma 3.7, (a) and (c) follow.

Next consider saturation of the second inequality in (3-15), or rather (3-14). Because we have already deduced that Σ_0 is a t-level set, the second and third inequalities in (3-14) automatically become equalities. Saturation of the first inequality in (3-14), on the other hand, has two implications:

$$d\sigma_g = d\sigma_{\hat{g}} \quad \text{and} \quad R_+^{\mu} = \frac{2}{\varphi(\tau)^2}.$$

The former, along with $g \geq \hat{g}$, implies that

$$g_{\Sigma_0} = \hat{g}_{\Sigma_0} = \varphi(\tau)^2 g_{S^2}; \tag{3-16}$$

the latter, along with the proof of Lemma 3.2, implies that $Z_\mu(\tau) = 0$ and $|\mathrm{d}\mu|_g = |\mathrm{d}\mu|_{\hat{g}}$, which is just $-\nu(\mu) = |\partial_t \mu|$ (see the proof of Lemma 3.7). Hence, $\nu = \partial_t + X$ for some vector field X on $\Sigma_0 = S_\tau$. Note that

$$1 = |\nu|_g \geq |\nu|_{\hat{g}} = \sqrt{|\partial_t|_{\hat{g}}^2 + |X|_{\hat{g}}^2} = \sqrt{1 + |X|_{\hat{g}}^2}; \tag{3-17}$$

we have $X = 0$ and $\nu = \partial_t$. Combining this with (3-16), we get $g(p) = \hat{g}(p)$ for all $p \in \Sigma_0$. This justifies (d), (e) and (f), completing the proof. $\square$

3B. *Foliation, minimality and rigidity.* The following “foliation” lemma is standard; see [Ye 1991; Andersson et al. 2008; Nunes 2013; Zhu 2021].

Lemma 3.8. *Suppose that $\Sigma_0 \subset (\boldsymbol{B}_T, g)$ is a μ-hypersurface (with unit normal ν) on which the stability operator J (see (2-3)) reduces to $-\Delta_{\Sigma_0}$. Then there exists an interval I and a map $\phi : \Sigma_0 \times I \to \boldsymbol{B}_T$ such that:*[6]

(1) *ϕ is a diffeomorphism onto a neighborhood of $\Sigma_0 \subset \boldsymbol{B}_T$.*

(2) *The family $\Sigma_s = \phi(\Sigma_0, s)$ is a normal variation of Σ_0 with $\partial_s \phi = \nu$ along Σ_0.*

(3) *On each Σ_s, the difference $H_{\Sigma_s} - \mu$ is a constant k_s.*

[6] If $0 < \tau < T$, I can be taken to be an open interval containing 0; if $\tau = T$, I is of the form $(a, 0]$; and if $\tau = \delta$, I is of the form $[0, b)$.

Proof. The proof is the same as that of Lemma 3.4 in [Zhu 2021], except for the extra step: once having obtained a foliation, we reexpress it as a normal variation by using a vector field normal to all its leaves; see [Andersson et al. 2008, page 6, second paragraph]. □

Before proceeding further, let us state a recurring assumption.

Assumption 3.9. Let g be a metric on $\boldsymbol{B}_T$ satisfying (3-2), and let $\Omega \subset (\boldsymbol{B}_T, g)$ be a Caccioppoli set such that $\partial\Omega \setminus \{\mathbf{0}\}$ is smooth and embedded. Define the class $\mathcal{C}_\Omega$ of Caccioppoli sets by

$$\mathcal{C}_\Omega := \{\Omega' \subset \boldsymbol{B}_T \text{ Caccioppoli set} : \Omega' \Delta \Omega \Subset \boldsymbol{B}_T \setminus \{\mathbf{0}\}\}. \tag{3-18}$$

Suppose that Ω is a minimizer in the sense that for any $\Omega' \in \mathcal{C}_\Omega$, we have $\mathcal{A}_\Omega^{\hat{H}}(\Omega') \geq 0$; and assume that there is a connected component $\Sigma_0 \subset \partial\Omega$ that is a stable $\hat{H}$-hypersurface,[7] disjoint from $\mathbf{0} \in \boldsymbol{B}_T$ and with nonzero-degree projection onto $\boldsymbol{S}^2$. Assume that $\operatorname{dist}_g(\Sigma_0, \partial\Omega \setminus \Sigma_0) > 0$.

Lemma 3.10 (compare to [Gromov 2023, Section 5.7]). *If Assumption 3.9 holds, then*:

1. *There exists a constant $\tau \in (0, T]$ such that $\Sigma_0 = S_\tau$ with outward normal ∂_t.*
2. *There exists an open neighborhood $\mathcal{U}$ of $\Sigma_0 = S_\tau$, disjoint from $\partial\Omega \setminus \Sigma_0$, on which $g = \hat{g}$.*

Proof. Since Σ_0 is assumed to be a stable, closed $\hat{H}$-hypersurface, and since $Z_{\hat{H}} \equiv 0$ (see (3-4)), Proposition 3.4 applies and yields (1).

To prove (2), first note that Proposition 3.4 and Lemma 3.8 together imply that a neighborhood $\mathcal{U}$ of Σ_0 is foliated by a normal variation $\{\Sigma_s\}$ ($s \in I$) of Σ_0; moreover, on each leaf Σ_s the difference $H_{\Sigma_s} - \hat{H}$ is a constant k_s. Since $\mathbf{0} \notin \Sigma_0$ and $\operatorname{dist}_g(\Sigma_0, \partial\Omega \setminus \Sigma_0) > 0$, $\mathcal{U}$ can be chosen to be disjoint from both $\partial\Omega \setminus \Sigma_0$ and $\mathbf{0}$.

For $s_1, s_2 \in I$ with $s_1 < s_2$ define $\Sigma_{[s_1,s_2]} \subset \boldsymbol{B}_T$ to be the (compact) subset with boundary $\Sigma_{s_1} \cup \Sigma_{s_2}$; then consider Ω_s defined by

$$\Omega_s := \begin{cases} \Omega \cup \Sigma_{[0,s]} & \text{if } s \geq 0, \\ \Omega \setminus \Sigma_{[-s,0]} & \text{if } s < 0. \end{cases} \tag{3-19}$$

Clearly, these Ω_s belong to the class $\mathcal{C}_\Omega$. Let us denote $\mathcal{A}_\Omega^{\hat{H}}(\Omega_s)$ by $\mathcal{A}(s)$ for brevity, and write $u_s = \langle \partial_s \phi, \nu_s \rangle > 0$ where ν_s is the (suitably oriented) unit normal along Σ_s. By Lemma 3.8 and the first variation formula,

$$\mathcal{A}'(s) = \int_{\Sigma_s} k_s u_s. \tag{3-20}$$

[7] We allow Σ_0 to overlap with $\partial\boldsymbol{B}_T$.

Since $\mathcal{A}(0)$ attains the minimum, it is necessary that:

(i) Either $\mathcal{A}(s) \equiv 0$ for all $s \geq 0$, or $\mathcal{A}'(s) > 0$ (equivalently, $k_s > 0$) for some $s > 0$.

(ii) Either $\mathcal{A}(s) \equiv 0$ for all $s \leq 0$, or $\mathcal{A}'(s) < 0$ (equivalently, $k_s < 0$) for some $s < 0$.

To complete the proof, it suffices to show that $\mathcal{A}(s) \equiv 0$ for all $s \in I$. If this does not hold, first suppose that $k_s > 0$ for some $s > 0$. Then on the Riemannian band $\Sigma_{[0,s]}$ with $\partial_- = \Sigma_0$ and $\partial_+ = \Sigma_s$ define the function

$$\tilde{\mu}(t) = \hat{H}(t) + \frac{\epsilon}{\sin^3 t}, \tag{3-21}$$

which is smooth and decreasing in t. By choosing sufficiently small ϵ, we can arrange that $\tilde{\mu} > H_{\Sigma_0}$ on Σ_0 and that $\tilde{\mu} < H_{\Sigma_s}$ on Σ_s. Thus, by Lemma 2.3, there exists a $\tilde{\mu}$-bubble $\tilde{\Omega}$ in $\Sigma_{[0,s]}$; in particular, $\tilde{\Sigma} = \partial\tilde{\Omega} \setminus \Sigma_0$ has a component $\tilde{\Sigma}_0$ whose projection to $\boldsymbol{S}^2$ has nonzero degree. However, by a direct calculation using (3-4), we get

$$Z_{\tilde{\mu}}(t) = \frac{3\epsilon^2}{2\sin^6 t} > 0, \tag{3-22}$$

contradicting Proposition 3.4(f).

The case when $k_s < 0$ for some $s < 0$ may be similarly and independently ruled out; it suffices to consider $\Sigma_{[s,0]}$ with $\partial_- = \Sigma_s$ and $\partial_+ = \Sigma_0$ and the following analogue of (3-21): $\tilde{\mu}(t) = \hat{H}(t) - \epsilon \sin^{-3} t$.

Finally, since we have proved that all Ω_s are $\mathcal{A}^{\hat{H}}$-minimizing in the class $\mathcal{C}_\Omega$, each Σ_s must be a t-level set. By Proposition 3.4(d), $g = \hat{g}$ on $\mathcal{U}$, and this completes the proof. □

Proposition 3.11. *If Assumption 3.9 holds, then $g = \hat{g}$ on $\boldsymbol{B}_T$.*

Proof. By Lemma 3.10, $\Sigma_0 = S_\tau$ for some $\tau \in (0, T]$, and its outward normal is ∂_t. Without loss of generality, we assume $\tau \in (0, T)$. Let $I = (t_1, t_2)$ be the maximum open interval containing τ such that $\boldsymbol{B}_{(t_1,t_2)}$ is disjoint from $\partial\Omega \setminus \Sigma_0$ and that $g = \hat{g}$ on $\boldsymbol{B}_{(t_1,t_2)}$. For $t \in I$, let Ω_t denote $\Omega \setminus \boldsymbol{B}_{(t,\tau]}$ if $t < \tau$ and $\Omega \cup \boldsymbol{B}_{[\tau,t]}$ if $t \geq \tau$. In particular, $\partial\Omega_t = (\partial\Omega \setminus \Sigma_0) \cup S_t$.

It suffices to show that $t_1 = 0$ and $t_2 = T$, and we argue by contradiction. First suppose that $t_1 > 0$. Then Ω_{t_1} is in the class $\mathcal{C}_\Omega$, and it satisfies $\mathcal{A}^{\hat{H}}_{\Omega}(\Omega_{t_1}) = 0$. If S_{t_1} were disjoint from $\partial\Omega \setminus \Sigma_0$, then, by Lemma 3.10, the interval I can be extended further, violating its maximality. On the other hand, if S_{t_1} were to touch a connected component $\Sigma' \subset \partial\Omega \setminus \Sigma_0$, then by smoothness and embeddedness $\partial\Omega_{t_1} \setminus \{\mathbf{0}\}$ (see [Zhou and Zhu 2020, Theorem 2.2]), Σ' must be equal to Σ_{t_1} but with the opposite outward normal, violating Proposition 3.4(e). Thus, we conclude that $t_1 = 0$. The proof of $t_2 = T$ is similar. □

With Proposition 3.11, it becomes clear that Theorem 3.1 would follow if one can verify Assumption 3.9. To illustrate this point, we now discuss a special case of Theorem 3.1 which admits a more direct proof. (The general situation is more subtle and will be addressed in the next section.)

Corollary 3.12. *Let $(\boldsymbol{B}_T, \hat{g})$ be the 3-dimensional spherical cap of radius $T \in (0, \pi/2]$. Suppose that g is another Riemannian metric on $\boldsymbol{B}_T$ satisfying $g \geq \hat{g}$ and $R_g \geq R_{\hat{g}}$ on $\boldsymbol{B}_T$; in addition, suppose that $H_g \geq H_{\hat{g}} = 2\cot T$ and $g_{\partial \boldsymbol{B}_T} = \hat{g}_{\partial \boldsymbol{B}_T}$ on $\partial \boldsymbol{B}_T$. Then $g = \hat{g}$.*

Proof. Take $\mu = \hat{H}$, which is in $L^1(\boldsymbol{B}_T)$. Since adding a constant to a functional does not affect its variational properties, we may consider, instead of (2-1),

$$\mathcal{B}^\mu(\Omega) := \mathcal{H}^2(\partial\Omega) + \int_{\boldsymbol{B}_T \setminus \Omega} \mu \, d\mathcal{H}^3, \tag{3-23}$$

for all smooth Caccioppoli sets $\Omega \subset \boldsymbol{B}_T$ with $\Omega \Delta \boldsymbol{B}_T \Subset \boldsymbol{B}_T \setminus \{\boldsymbol{0}\}$, and underlying metrics will be specified in subscripts. Since $\hat{H} = \operatorname{div}(\partial_t)$ on $\boldsymbol{B}_T \setminus \{\boldsymbol{0}\}$, we have

$$\mathcal{B}^\mu_{\hat{g}}(\Omega) = \mathcal{H}^2_{\hat{g}}(\partial\Omega) - \int_{\partial\Omega} \langle \partial_t, \hat{\nu} \rangle_{\hat{g}} \, d\mathcal{H}^2_{\hat{g}} + \mathcal{H}^2_{\hat{g}}(S_T) \geq \mathcal{H}^2_{\hat{g}}(S_T) = \mathcal{B}^\mu_{\hat{g}}(\boldsymbol{B}_T), \tag{3-24}$$

where the first equality is an application of the divergence formula, and the inequality is derived from the relation $\langle \partial_t, \hat{\nu} \rangle_{\hat{g}} \leq 1$. Now, since $g \geq \hat{g}$ and $\mu \geq 0$ on $\boldsymbol{B}_T$ $(T \leq \pi/2)$, we have $\mathcal{B}^\mu_g(\Omega) \geq \mathcal{B}^\mu_{\hat{g}}(\Omega)$; moreover, by $g_{\partial \boldsymbol{B}_T} = \hat{g}_{\partial \boldsymbol{B}_T}$, we have $\mathcal{B}^\mu_{\hat{g}}(\boldsymbol{B}_T) = \mathcal{B}^\mu_g(\boldsymbol{B}_T)$. Combining these with (3-24) gives $\mathcal{B}^\mu_g(\Omega) \geq \mathcal{B}^\mu_g(\boldsymbol{B}_T)$; and using $H_g \geq 2\cot T = \hat{H}|_{\partial \boldsymbol{B}_T}$, we deduce that $H_g = 2\cot T$ and hence, for any $\phi \in Lip(S_T)$ and $\phi \geq 0$, we have

$$D^2\mathcal{A}(\phi, \phi) := \int_{S_T} |\nabla \phi|^2 + \int_{S_T} (R_{S_T} - R_g - \hat{H}^2 - |\mathrm{II}|^2 - 2\nu(\hat{H}))\phi^2 \geq 0,$$

and then clearly, for all $\varphi \in C^\infty(S_T)$ we have

$$D^2\mathcal{A}(\varphi, \varphi) \geq D^2\mathcal{A}(|\varphi|, |\varphi|) \geq 0,$$

hence, S_T is a stable $\hat{H}$-hypersurface. Now it is easy to see that the pair $(\boldsymbol{B}_T, S_T)$ satisfies Assumption 3.9. The conclusion then follows from Proposition 3.11. □

4. Existence of an initial minimizer

Throughout this section, let g be a Riemannian metric on $\boldsymbol{B}_T$ satisfying (3-2). Our goal is to obtain an "initial" minimizer Ω and a connected component $\Sigma_0 \subset \partial\Omega$ which satisfy Assumption 3.9. To achieve this, we consider perturbations μ_ϵ of $\hat{H} = 2\cot t$ (see (4-3)). For each ϵ, we find a Riemannian band $M_\epsilon \subset \boldsymbol{B}_T$ on which μ_ϵ satisfies the barrier condition; thus, a μ_ϵ-bubble Ω_ϵ exists, and $\partial\Omega_\epsilon \cap \mathring{M}_\epsilon$ has a component Σ_0^ϵ which projects onto $\boldsymbol{S}^2$ with nonzero degree. One may wonder whether this "degree" property is preserved in the limit as $\epsilon \to 0$; this led us to find that each Σ_0^ϵ must be disjoint from a fixed open neighborhood of $\mathbf{0} \in \boldsymbol{B}_T$, provided ϵ is small (Proposition 4.7). Then we verify Assumption 3.9 by analyzing the limits of Ω_ϵ and Σ_0^ϵ (Proposition 4.12).

4A. ***A choice of μ_ϵ.*** Let $\epsilon > 0$ be a small constant, and define

$$t_c := \min\left\{\tfrac{\pi}{4}, \tfrac{T}{2}\right\}. \tag{4-1}$$

Moreover, we shall fix a function $\beta \in C^\infty((0, T])$ which is strictly decreasing and satisfies

$$\beta(t) = \cot t \text{ on } (0, t_c] \quad \text{and} \quad \beta(T) = -1; \tag{4-2}$$

such a β clearly exists. Now consider the function defined on $(0, T]$:

$$\mu_\epsilon(t) \equiv \hat{H}(t) + \epsilon\beta(t) = 2\cot t + \epsilon\beta(t). \tag{4-3}$$

Writing Z^ϵ for Z_{μ_ϵ}, we have (see (3-4))

$$Z^\epsilon(t) = \tfrac{3}{2}[\epsilon\beta(t)]^2 + 2\epsilon\beta'(t) + 6\epsilon(\cot t)\beta(t), \tag{4-4}$$

and, in particular,

$$Z^\epsilon(t) = \frac{\epsilon}{2\sin^2 t}[(3\epsilon + 12)\cos^2 t - 4] > 0 \quad \text{for } t \in (0, t_c]. \tag{4-5}$$

Moreover, by (4-4), it is clear that there exists a constant $b_0 > 0$, depending only on β, such that

$$Z^\epsilon(t) \geq -\epsilon\, b_0 \quad \text{for } t \in (0, T]. \tag{4-6}$$

4B. ***Existence of a μ_ϵ-bubble.*** Let $S(r, g)$ (resp., $B(r, g)$) denote the geodesic sphere (resp., open geodesic ball) of radius r, taken with respect to the metric g and centered at $\mathbf{0} \in \boldsymbol{B}_T$. An asymptotic expansion of the mean curvature function (see Lemma 3.4 of [Fan et al. 2009]) gives: for small $r > 0$ and all $q \in S(r, g)$,

$$H_{S(r,g)}(q) = \frac{2}{r} + O(r), \quad \hat{H}(q) = \frac{2}{t(q)} + O(t(q)). \tag{4-7}$$

Since $g \geq \hat{g}$, we have $r \geq t(q)$; then by (4-3) and (4-2), as long as $r < t_c$, we have

$$\mu_\epsilon(t(q)) = \frac{2+\epsilon}{t(q)} + O(t(q)) \geq \frac{2+\epsilon}{r} + O(t(q)), \quad q \in S(r, g). \tag{4-8}$$

It is now clear that there exists an $r_\epsilon < \epsilon$ such that $\mu_\epsilon > H_{S(r_\epsilon, g)}$ on $S(r_\epsilon, g)$. On the other hand, we have $H_g \geq 2\cot T > \mu_\epsilon(T)$ on S_T, where the first inequality is part of (3-2), and the second inequality is due to the choice of μ_ϵ and β. Therefore, μ_ϵ satisfies the barrier condition (see Definition 2.2) applied to the Riemannian band

(M_ϵ, g), where $M_\epsilon = \boldsymbol{B}_T \setminus B(r_\epsilon, g)$, with the distinguished boundaries: $\partial_- = S(r_\epsilon, g)$ and $\partial_+ = S_T$. The lemma below follows directly from Lemma 2.3.

Lemma 4.1. *In the Riemannian band $(M_\epsilon, g; S(r_\epsilon, g), S_T)$ there exists a minimal μ_ϵ-bubble Ω_ϵ; moreover, $\partial\Omega_\epsilon \setminus S(r_\epsilon, g)$ is disjoint from S_T, and it has a connected component Σ_0^ϵ whose projection onto $\boldsymbol{S}^2$ has nonzero degree.*

Lemma 4.2. $\Sigma_0^\epsilon \cap \boldsymbol{B}_{[t_c,T]}$ *is nonempty.*

Proof. Otherwise, $Z^\epsilon > 0$ on Σ_0^ϵ, which contradicts Proposition 3.4(f). □

4C. ***A "no-crossing" property of $\boldsymbol{\Sigma_0^\epsilon}$.*** From now on, let $t_* \in (0, t_c)$ be fixed. We will begin by assuming that $\Sigma_0^\epsilon \cap \boldsymbol{B}_{t_*}$ were nonempty; consequences of this hypothesis will be developed progressively with three lemmas (Lemmas 4.3, 4.5 and 4.6). Based on these lemmas, we prove that Σ_0^ϵ must be disjoint from $\boldsymbol{B}_{t_*}$ for small enough ϵ (Proposition 4.7).

In the following, let $\hat{\nu}$ denote the outward-pointing unit normal on Σ_0^ϵ with respect to $\hat{g}$, and let Φ denote the projection map from Σ_0^ϵ to $\boldsymbol{S}^2$; see (3-6).

Lemma 4.3. *If $\Sigma_0^\epsilon \cap \boldsymbol{B}_{t_*}$ were nonempty, then there would exist a point $q \in \Sigma_0^\epsilon \cap \boldsymbol{B}_{[t_*,T]}$ such that the angle $\angle_{\hat{g}}(\hat{\nu}, \partial_t) \in [\alpha, \pi - \alpha]$ at q, where*

$$\alpha = \min\left\{\arctan\left(\frac{t_c - t_*}{2\pi}\right), \frac{\pi}{4}\right\}. \tag{4-9}$$

Proof. We argue by contradiction, so let us assume that $\angle_{\hat{g}}(\hat{\nu}, \partial_t) \in [0, \alpha) \cup (\pi - \alpha, \pi]$ everywhere on $\Sigma_0^\epsilon \cap \boldsymbol{B}_{[t_*,T]}$. Because Σ_0^ϵ is connected and intersects both S_{t_*} (by assumption) and S_{t_c} (by Lemma 4.2), the image of $t|_{\Sigma_0^\epsilon}$ contains the interval $[t_*, t_c]$.

Let $t' \in (t_*, t_c)$ be a regular value of $t|_{\Sigma_0^\epsilon}$ that is sufficiently close to t_*. Because Σ_0^ϵ is connected, there exists a connected component $\mathcal{E} \subset \Sigma_0^\epsilon \cap \boldsymbol{B}_{(t',t_c]}$ whose closure $\bar{\mathcal{E}}$ intersects both $S_{t'}$ and S_{t_c}. On $\mathcal{E}$, the angle $\angle_{\hat{g}}(\hat{\nu}, \partial_t)$ can only take value in one of the intervals $[0, \alpha)$ and $(\pi - \alpha, \pi]$, but not both. Without loss of generality, let us assume that $\angle_{\hat{g}}(\hat{\nu}, \partial_t) \in [0, \alpha)$ on $\mathcal{E}$.

Since t' is a regular value of $t|_{\Sigma_0^\epsilon}$, $\bar{\mathcal{E}}$ meets $S_{t'}$ transversely. In particular, $\mathscr{C} := \bar{\mathcal{E}} \cap S_{t'}$ is a disjoint union of finitely many circles. It is easy to see that $S_{t'} \setminus \mathscr{C} = U_1 \cup U_2$ for some open subsets $U_i \subset S_{t'}$ with $\partial U_i = \mathscr{C}$ $(i = 1, 2)$.

Both U_i and $\mathcal{E}$ are oriented, and the orientations are associated to the respective normal directions, ∂_t and $\hat{\nu}$, by the right-hand rule. The orientation on $\mathscr{C}$ induced by $\mathcal{E}$ must completely agree with that induced by either U_1 or U_2; otherwise, gluing $\mathcal{E}$ with either U_1 or U_2 along $\mathscr{C}$ and smoothing would yield a nonorientable closed surface embedded in $\boldsymbol{B}_T$, which is impossible.

Thus, we can assume that U_1 and $\mathcal{E}$ induce *opposite* orientations on $\mathscr{C}$. Since $\angle_{\hat{g}}(\hat{\nu}, \partial_t) \in [0, \alpha)$ on $\mathcal{E}$, it is easy to see that the restriction of Φ to $\bar{\mathcal{E}} \cup U_1$ is a local homeomorphism to $\boldsymbol{S}^2$. Since $\bar{\mathcal{E}} \cup U_1$ is compact, $\Phi|_{\bar{\mathcal{E}} \cup U_1}$ is a covering map;

this map must be a homeomorphism, since $\boldsymbol{S}^2$ is simply connected and $\bar{\mathcal{E}} \cup U_1$ is connected.

Pick any $x \in \mathcal{E} \cap S_{t_c}$. Choose a shortest (regular) curve $\Gamma : [0, 1] \to \Phi(\bar{\mathcal{E}})$ connecting $\Gamma(0) = \Phi(x)$ and $\partial(\Phi(\mathcal{E}))$; in particular,

$$\text{length}_{g_{S^2}}(\Gamma) \leq \pi. \tag{4-10}$$

Now let $\gamma = (\Phi|_{\bar{\mathcal{E}}})^{-1} \circ \Gamma$, and write its tangent vectors γ' as the sum of γ'_N (parallel to ∂_t) and γ'_T (tangent to t-level sets). By $\hat{g} \leq g_{\boldsymbol{S}^2} + \mathrm{d}t^2$ and the hypothesis $\angle_{\hat{g}}(\hat{\nu}, \partial_t) \in [0, \alpha) \cup (\pi - \alpha, \pi]$, we obtain the estimate

$$|\gamma'_N|_{\hat{g}} \leq (\tan \alpha)|\gamma'_T|_{\hat{g}} \leq (\tan \alpha)|\mathrm{d}\Phi(\gamma')|_{g_{S^2}}. \tag{4-11}$$

Hence,

$$t_c - t' \leq \int_\gamma |\gamma'_N|_{\hat{g}} \leq (\tan \alpha) \cdot \text{length}_{g_{S^2}}(\Phi(\gamma)) \leq \pi \tan \alpha \leq \tfrac{1}{2}(t_c - t_*), \tag{4-12}$$

where the first inequality holds because $\gamma(0) \in S_{t_c}$ and $\gamma(1) \in S_{t'}$; the second and third inequalities are due to (4-11) and (4-10), respectively; the last inequality holds by the choice of α. Since t' is close to t_*, (4-12) is a contradiction. □

Corollary 4.4. *In Lemma 4.3 we can choose q such that: $\angle_{\hat{g}}(\hat{\nu}, \partial_t) = \alpha$ or $\pi - \alpha$ at q.*

Proof. In Σ_0^ϵ there exists a point at which t attains global maximum. At that point $\hat{\nu} = \pm\partial_t$. Thus, by continuity of angle, there exists a point $q \in \Sigma_0^\epsilon \cap \boldsymbol{B}_{[t_*,T]}$ at which the angle between $\hat{\nu}$ and ∂_t is equal to either α or $\pi - \alpha$. □

Lemma 4.5. *Let α be defined by (4-9). If $\Sigma_0^\epsilon \cap \boldsymbol{B}_{t_*}$ were nonempty, then there would exist a constant $S = S(g, \hat{g}, \beta, t_*) > 0$, independent of ϵ, and an open subset $U_\epsilon \subset \Sigma_0^\epsilon \cap \boldsymbol{B}_{[t_*/2,T]}$ such that:*

(1) *At each point $q \in U_\epsilon$, $\angle_{\hat{g}}(\hat{\nu}, \partial_t) \in (\alpha/2, 2\alpha) \cup (\pi - 2\alpha, \pi - \alpha/2)$.*

(2) *$\int_{U_\epsilon} |\Phi^* \, d\sigma_{S^2}| \geq S$.*

Proof. To begin with, let q be as in Corollary 4.4. For any unit tangent vector X (with respect to $\hat{g}$) of Σ_0^ϵ, we have

$$|X\langle\hat{\nu}, \partial_t\rangle_{\hat{g}}| = |\langle\hat{\nabla}_X \hat{\nu}, \partial_t\rangle_{\hat{g}} + \langle\hat{\nu}, \hat{\nabla}_X \partial_t\rangle_{\hat{g}}| \leq |\hat{\mathrm{II}}|_{\hat{g}} + |\hat{\nabla}\partial_t|_{\hat{g}}. \tag{4-13}$$

where $\hat{\nabla}$ is the connection of $\hat{g}$. It is clear that there exists a constant $C = C(\hat{g}, t_*)$ such that $|\hat{\nabla}\partial_t|_{\hat{g}} \leq C$ on $\boldsymbol{B}_{[t_*/2,T]}$. Moreover, by applying Lemma 2.9 (if necessary, extend g to a smooth metric on $\boldsymbol{B}_{T+\delta_0}$ for some fixed $\delta_0 > 0$, and let $V = \boldsymbol{B}_{(t_*/2,T+\delta_0)}$) and by comparing between $|\mathrm{II}|_g$ and $|\hat{\mathrm{II}}|_{\hat{g}}$, it is not difficult to see that there exists a constant $C' = C'(g, \hat{g}, \beta, t_*)$ such that $|\hat{\mathrm{II}}|_{\hat{g}} \leq C'$ on $\Sigma_0^\epsilon \cap \boldsymbol{B}_{[t_*/2,T]}$ for all sufficiently small ϵ. Thus, there exists a constant $\rho = \rho(g, \hat{g}, \beta, t_*) > 0$ such that on the geodesic ball

$$U_\epsilon := \{x \in \Sigma_0^\epsilon : \operatorname{dist}_{\hat{g}_{\Sigma_0^\epsilon}}(x, q) \leq \rho\}$$

we have

$$\angle_{\hat{g}}(\hat{\nu}, \partial_t) \in \left(\tfrac{\alpha}{2}, 2\alpha\right) \cup \left(\pi - 2\alpha, \pi - \tfrac{\alpha}{2}\right). \tag{4-14}$$

It is easy to see that $\Phi(U_\epsilon)$ contains a ball $\mathscr{B}$ of radius $\cos(2\alpha)\rho$ in $\boldsymbol{S}^2$. The proof is complete by taking $S := \operatorname{area}_{g_{S^2}}(\mathscr{B})$. ☐

Lemma 4.6. *If $\Sigma_0^\epsilon \cap \boldsymbol{B}_{t_*}$ were nonempty, then we would have*

$$\int_{\Sigma_0^\epsilon} \frac{2}{\varphi^2}\, d\sigma_{\hat{g}} - 2\int_{\Sigma_0^\epsilon} |\Phi^*\, d\sigma_{\boldsymbol{S}^2}| \geq A_0 \tag{4-15}$$

for some positive constant A_0 that is independent of ϵ.

Proof. Up to sign, the area form $d\sigma_{\hat{g}}$ induced by $\hat{g}$ on each tangent space of Σ_0^ϵ is equal to

$$\frac{1}{\cos(\angle_{\hat{g}}(\hat{\nu}, \partial_t))}\varphi^2 \Phi^* d\sigma_{\boldsymbol{S}^2}$$

provided that $\hat{\nu}$ is not orthogonal to ∂_t. Thus, by Lemma 4.5, we have

$$\begin{aligned}\int_{U_\epsilon} \frac{2}{\varphi^2}\, d\sigma_{\hat{g}} &\geq \int_{U_\epsilon} \frac{2}{\varphi^2}\frac{1}{\cos(\alpha/2)}\varphi^2 |\Phi^*\, d\sigma_{\boldsymbol{S}^2}| \\ &\geq 2S\left(\frac{1}{\cos(\alpha/2)} - 1\right) + 2\int_{U_\epsilon} |\Phi^*\, d\sigma_{\boldsymbol{S}^2}|.\end{aligned} \tag{4-16}$$

On the other hand, by Lemma 3.3,

$$\int_{\Sigma_0^\epsilon \setminus U_\epsilon} \frac{2}{\varphi^2}\, d\sigma_{\hat{g}} \geq 2\int_{\Sigma_0^\epsilon \setminus U_\epsilon} |\Phi^*\, d\sigma_{\boldsymbol{S}^2}|. \tag{4-17}$$

Adding (4-16) with (4-17) and rearranging terms, we get

$$\int_{\Sigma_0^\epsilon} \frac{2}{\varphi^2}\, d\sigma_{\hat{g}} - 2\int_{\Sigma_0^\epsilon} |\Phi^*\, d\sigma_{\boldsymbol{S}^2}| \geq 2S\left(\frac{1}{\cos(\alpha/2)} - 1\right). \tag{4-18}$$

The proof is complete by taking A_0 to be the right-hand side of (4-18). ☐

Proposition 4.7. *For sufficiently small ϵ, Σ_0^ϵ must be disjoint from the set $\boldsymbol{B}_{t_*} \subset \boldsymbol{B}_T$.*

Proof. By (4-6) and the proof of Lemma 3.2, we obtain

$$R_+^{\mu_\epsilon} \geq \frac{2}{\varphi^2} - 2b_0\epsilon \quad \text{on } \Sigma_0^\epsilon. \tag{4-19}$$

For small ϵ, Remark 3.6 and Lemma 3.5 imply that Σ_0^ϵ is homeomorphic to $\boldsymbol{S}^2$. Moreover, since Ω_ϵ is a μ_ϵ-bubble, the area of Σ_0^ϵ with respect to g has an upper bound $C_0 > 0$, which can be chosen to depend only on the metric g and not on ϵ.

Now suppose that $\Sigma_0^\epsilon \cap \boldsymbol{B}_{t_*} \neq \varnothing$. Then from (4-19), (3-13) and (4-15), we obtain

$$\int_{\Sigma_0^\epsilon} R_+^{\mu_\epsilon}\, d\sigma_g \geq \int_{\Sigma_0^\epsilon} \frac{2}{\varphi^2}\, d\sigma_{\hat{g}} - 2\epsilon b_0 C_0 \geq (A_0 - 2\epsilon b_0 C_0) + 2\int_{\Sigma_0^\epsilon} |\Phi^* \, d\sigma_{\boldsymbol{S}^2}|. \tag{4-20}$$

For small enough ϵ, $A_0 > 2\epsilon b_0 C_0$; by stability of Σ_0^ϵ, the analogue of (3-12) reads

$$4\pi\chi(\boldsymbol{S}^2) = \int_{\Sigma_0^\epsilon} R_{\Sigma_0^\epsilon}\, d\sigma_g \geq \int_{\Sigma_0^\epsilon} R_+^{\mu_\epsilon}\, d\sigma_g > 2\int_{\Sigma_0^\epsilon} |\Phi^*\, d\sigma_{\boldsymbol{S}^2}| \geq 8\pi; \tag{4-21}$$

a contradiction. □

Remark 4.8. There is another way to get (4-21), which does not rely on the assumption of an upper bound C_0 of $\operatorname{area}_g(\Sigma_0^\epsilon)$ but does rely on the fact that $\varphi \leq 1$. In fact, (4-19) implies that $R_+^{\mu_\epsilon} \geq 2\varphi^{-2}(1 - b_0\epsilon)$, and again by (3-13), (4-15) and the degree assumption we have

$$\int_{\Sigma_0^\epsilon} R_+^{\mu_\epsilon}\, d\sigma_g \geq (1 - b_0\epsilon)\Big(A_0 + 2\int_{\Sigma_0^\epsilon} |\Phi^*\, d\sigma_{\boldsymbol{S}^2}|\Big) \geq (1 - b_0\epsilon)(A_0 + 8\pi) > 8\pi$$

for small enough ϵ.

4D. ***Existence of a minimizer.*** Let M_ϵ, Ω_ϵ and Σ_0^ϵ be as in Lemma 4.1. We now study how Ω_ϵ and Σ_0^ϵ behave as $\epsilon \to 0$.

Recall from (4-1) the definition of t_c, and let $t_* \in (0, t_c)$ be fixed. By considering small enough ϵ, we can assume Σ_0^ϵ to be homeomorphic to $\boldsymbol{S}^2$ and disjoint from $\boldsymbol{B}_{t_*}$.

For a fixed ϵ, since Σ_0^ϵ is disjoint from S_T, the Jordan–Brouwer separation theorem applies. As a result, $\boldsymbol{B}_T \setminus \Sigma_0^\epsilon$ has exactly two connected components, say $\mathcal{U}_-^\epsilon$ and $\mathcal{U}_+^\epsilon$. Without loss of generality, let us assume that ν points away from $\mathcal{U}_-^\epsilon$ along Σ_0^ϵ. Given any constant $\delta > 0$, let us define

$$\begin{aligned} W_{-\delta}^\epsilon &:= \{x \in \mathcal{U}_-^\epsilon : \operatorname{dist}_g(x, \Sigma_0^\epsilon) \leq \delta\}, \\ W_{+\delta}^\epsilon &:= \{x \in \mathcal{U}_+^\epsilon : \operatorname{dist}_g(x, \Sigma_0^\epsilon) \leq \delta\}, \end{aligned} \tag{4-22}$$

where distance is taken in $(\boldsymbol{B}_T, g)$.

Lemma 4.9. *There exists a constant $\delta > 0$, independent of ϵ, such that for all small enough ϵ we have $W_{-\delta}^\epsilon \subset \mathring{\Omega}_\epsilon$ and $W_{+\delta}^\epsilon \cap \Omega_\epsilon = \varnothing$.*

Proof. Since in $\boldsymbol{B}_{[t_*/2,T]}$ all derivatives of μ_ϵ are uniformly bounded, it follows from Lemma 2.9 that the norm of the second fundamental form of $\partial\Omega_\epsilon \cap \boldsymbol{B}_{[t_*/2,T]}$ is also uniformly bounded. If some other component Σ' in $\partial\Omega_\epsilon$ were to get arbitrarily close to Σ_0^ϵ, then a suitable surgery (i.e., a connected sum of Σ_0^ϵ and Σ'

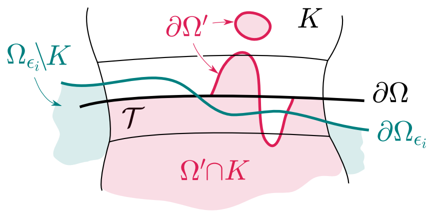

Figure 1. The shaded regions represent Ω'_{ϵ_i} (left figure) and $\Omega^*_{\epsilon_i}$ (right figure).

performed within M_ϵ) would yield a Caccioppoli set that has strictly less brane action, contradicting the minimality of Ω_ϵ. □

Now we fix a sequence $\{\epsilon_i\} \to 0$ and corresponding sequences of Ω_{ϵ_i} and $\Sigma_0^{\epsilon_i}$.

Lemma 4.10. *The sequence $\{\Omega_{\epsilon_i}\}$ subconverges to a Caccioppoli set $\Omega \subset \boldsymbol{B}_T$ where convergence is interpreted via the characteristic functions with respect to the L^1_{loc}-norm. Moreover*:

(1) $\partial\Omega \setminus \{\mathbf{0}\}$ *is smooth and embedded.*

(2) Ω *is a minimizer in the sense that* $\mathcal{A}^{\hat{H}}_{\Omega}(\Omega') \geq 0$ *for any Caccioppoli set* Ω' *with* $\Omega' \Delta \Omega \Subset \boldsymbol{B}_T \setminus \{\mathbf{0}\}$.

Proof. The existence of a convergent subsequence and that of Ω follow from standard theory of BV functions (see [Giusti 1984, Theorem 1.20]), and let us replace $\{\Omega_{\epsilon_i}\}$ by that subsequence.

Now let $K \subset \boldsymbol{B}_T \setminus \{\mathbf{0}\}$ be any compact domain. For sufficiently large i, the second fundamental form of $\partial\Omega_{\epsilon_i} \cap K$ has a uniform upper bound, and thus $\partial\Omega_{\epsilon_i} \cap K$ subconverges to a smooth hypersurface $\mathcal{S} \subset K$ in the graph sense. By using Lemma 4.9, it is easy to see that $\mathcal{S}$ is embedded and $\mathcal{S} = \partial\Omega \cap K$. Since K is arbitrary, we conclude (1).

To show that Ω is a minimizer, we argue by contradiction. Suppose that there exists a Caccioppoli set Ω' and a constant $c > 0$ such that $\Omega' \Delta \Omega \Subset \boldsymbol{B}_T \setminus \{\mathbf{0}\}$ and $\mathcal{A}^{\hat{H}}_{\Omega}(\Omega') \leq -c < 0$. Let us choose a compact domain $K \subset \boldsymbol{B}_T \setminus \{\mathbf{0}\}$ with smooth boundary such that $\Omega' \Delta \Omega \Subset \mathring{K}$. Consider a thin tubular neighborhood $\mathcal{T}$ of $\partial\Omega \cap K$ that is generated by the unit normal field along $\partial\Omega \cap K$; as $\mathcal{T}$ is diffeomorphic to $(\partial\Omega \cap K) \times I$ for some interval I, we may modify K such that the image of $(\partial\Omega \cap \partial K) \times I$ is equal to $\partial\mathcal{T} \cap \partial K$ (in particular, $\partial\Omega$ is transversal to ∂K). Note that for large i, $S(r_{\epsilon_i}, g)$ would be disjoint from K, and $\partial\Omega_{\epsilon_i} \cap K$ would be completely contained in $\mathcal{T}$.

Now consider the following Caccioppoli sets (see Figure 1):

$$\Omega'_{\epsilon_i} := (\Omega_{\epsilon_i} \setminus K) \cup (\Omega' \cap K), \quad \Omega^*_{\epsilon_i} := (\Omega_{\epsilon_i} \setminus K) \cup (\Omega \cap K). \tag{4-23}$$

We claim that, for sufficiently large i,

$$\mathcal{A}^{\mu_{\epsilon_i}}_{\Omega_{\epsilon_i}}(\Omega^*_{\epsilon_i}) \le \frac{c}{4}. \tag{4-24}$$

To see this, note that $\chi_{\Omega^*_{\epsilon_i}} - \chi_{\Omega_{\epsilon_i}}$ is just $\chi_{\Omega_{\epsilon_i}\cap K} - \chi_{\Omega\cap K}$; since $\mu_{\epsilon_i}|_K$ is uniformly bounded and $\chi_{\Omega_{\epsilon_i}} \to \chi_\Omega$ in L^1, we have

$$\int_{\boldsymbol{B}_T} (\chi_{\Omega^*_{\epsilon_i}} - \chi_{\Omega_{\epsilon_i}})\mu_{\epsilon_i} \to 0 \quad (i \to \infty). \tag{4-25}$$

Moreover, it is easy to see that

$$\mathcal{H}^2(\partial\Omega^*_{\epsilon_i}) - \mathcal{H}^2(\partial\Omega_{\epsilon_i}) \le [\mathcal{H}^2(\partial\Omega\cap K) - \mathcal{H}^2(\partial\Omega_{\epsilon_i}\cap K)] + \mathcal{H}^2(\partial\mathcal{T}\cap\partial K). \tag{4-26}$$

Thus, by graph convergence of $\partial\Omega_{\epsilon_i}\cap K$, we can choose $\mathcal{T}$ and i such that

$$\mathcal{H}^2(\partial\Omega^*_{\epsilon_i}) - \mathcal{H}^2(\partial\Omega_{\epsilon_i}) \le \frac{c}{8}. \tag{4-27}$$

On combining (4-25) and (4-27), we obtain (4-24) for large i.

Now, since $\mu_{\epsilon_i} \to \mu$ in $L^1(K)$ and $\Omega'_{\epsilon_i}\Delta\Omega^*_{\epsilon_i} = \Omega\Delta\Omega' \Subset \mathring{K}$, we have, for sufficiently large i,

$$\mathcal{A}^{\mu_{\epsilon_i}}_{\Omega^*_{\epsilon_i}}(\Omega'_{\epsilon_i}) \le -\frac{c}{2}. \tag{4-28}$$

On comparing (4-24) and (4-28), we get $\mathcal{A}^{\mu_{\epsilon_i}}_{\Omega_{\epsilon_i}}(\Omega'_{\epsilon_i}) \le -c/4 < 0$, contradicting the minimality of Ω_{ϵ_i}. This proves (2). ☐

Lemma 4.11. *Let Ω be as in Lemma 4.10. The sequence $\{\Sigma_0^{\epsilon_i}\}$ subconverges to a smooth, closed stable $\hat{H}$-hypersurface $\Sigma_0 \subset \boldsymbol{B}_{[t_*,T]}$, which is a t-level set in $\boldsymbol{B}_T$; moreover, $\Sigma_0 \subset \partial\Omega$ and $\partial\Omega\setminus\Sigma_0 \Subset \boldsymbol{B}_T\setminus\Sigma_0$.*

Proof. By our choice of $\{\epsilon_i\}$, all $\Sigma_0^{\epsilon_i}$ are contained in the compact set $\boldsymbol{B}_{[t_*,T]}$ and have a uniform upper bound on their second fundamental form. Thus, by standard minimal surface theory (see [Colding and Minicozzi 2011, Proposition 7.14]), $\{\Sigma_0^{\epsilon_i}\}$ subconverges to a smooth closed hypersurface Σ_0 whose projection onto $\boldsymbol{S}^2$ has nonzero degree. Now recall that each $\Sigma_0^{\epsilon_i}$ is a stable μ_{ϵ_i}-hypersurface. Since all derivatives of μ_{ϵ_i} respectively and uniformly converge to those $\hat{H}$, by passing stability to limit, Σ_0 is a stable $\hat{H}$-hypersurface; hence, Σ_0 is a t-level set, by Proposition 3.4.

To see that $\Sigma_0 \subset \partial\Omega$, first suppose that $\Sigma_0 \ne S_T$; in this case, it suffices to show that each open neighborhood of any $x \in \Sigma_0$ must intersect both $\mathring{\Omega}$ and $\boldsymbol{B}_T\setminus\Omega$, and this can be easily deduced from Lemma 4.9. The case of $\Sigma_0 = S_T$ is similar. Also by Lemma 4.9, Σ_0 has a tubular neighborhood that is disjoint from all other components of $\partial\Omega$, hence $\mathrm{dist}_g(\Sigma_0, \partial\Omega\setminus\Sigma_0) > 0$. ☐

On combining Lemmas 4.10 and 4.11, we immediately get the following.

Proposition 4.12. *Let g be a Riemannian metric on $\boldsymbol{B}_T$ satisfying (3-2). Then there exists a Caccioppoli set $\Omega \subset \boldsymbol{B}_T$ and a connected component $\Sigma_0 \subset \partial\Omega$ that satisfy Assumption 3.9.*

Theorem 3.1 follows directly from Propositions 3.11 and 4.12.

5. Generalizations

In this section we discuss a few variants of Theorem 3.1.

To begin with, we consider a version of Gromov's rigidity theorem for the doubly punctured sphere (see [Gromov 2023, Sections 5.5 and 5.7]), restricted to the 3-dimensional case.

Theorem 5.1. *Let $(\boldsymbol{S}^3 \setminus \{O, O'\}, \hat{g})$ be the standard 3-sphere with a pair of antipodal points removed, and let $h \geq 1$ be a smooth function on $\boldsymbol{S}^3 \setminus \{O, O'\}$. Suppose that g is another Riemannian metric on $\boldsymbol{S}^3 \setminus \{O, O'\}$ satisfying*

$$g \geq h^4 \hat{g} \quad \text{and} \quad R_g \geq h^{-2} R_{\hat{g}}. \tag{5-1}$$

Then $h \equiv 1$, and $g = \hat{g}$.

Proof. For convenience, let us use slightly different notations than those introduced at the beginning of Section 3 by representing $\boldsymbol{S}^3 \setminus \{O, O'\}$ as $\mathbb{B}_{(-\pi/2,\pi/2)} \cong \boldsymbol{S}^2 \times (-\pi/2, \pi/2)$ with t being the coordinate on $(-\pi/2, \pi/2)$. Under this representation we have $\varphi(t) = \cos t$ and

$$\hat{H}(t) = -2\tan t \tag{5-2}$$

instead of (3-3). Now for $\alpha \in (0, \pi/2)$ sufficiently close to $\pi/2$, consider the Riemannian band $\mathscr{B}_\alpha := (\mathbb{B}_{[-\alpha,\alpha]}, g; S_{-\alpha}, S_\alpha)$ and the functions

$$t_\alpha = \frac{t}{\alpha} \cdot \frac{\pi}{2} \quad \text{and} \quad \mu_\alpha = -2\tan t_\alpha \text{ on } \mathbb{B}_{(-\alpha,\alpha)}, \tag{5-3}$$

and consider the problem of finding μ_α-bubbles in $\mathscr{B}_\alpha$. Since $\mu_\alpha \to \pm\infty$ as $t \to \mp\alpha$, μ_α satisfies the barrier condition; thus, there exists a μ_α-bubble $\Omega_\alpha \subset \mathscr{B}_\alpha$, which satisfies analogous properties as described in Lemma 4.1. Let Σ_0^α be a connected component of $\partial\Omega_\alpha \setminus S_{-\alpha}$ whose projection to $\boldsymbol{S}^2$ has nonzero degree; Σ_0^α is a stable μ_α-hypersurface, on which

$$\begin{aligned} R_+^{\mu_\alpha} &= R_g + \tfrac{3}{2}(\mu_\alpha)^2 - 2|\mathrm{d}\mu_\alpha|_g \\ &\geq \frac{1}{h^2}\left(\frac{R_{\boldsymbol{S}^2}}{\varphi^2} - \frac{3}{2}\hat{H}^2 + 2|\mathrm{d}\hat{H}|_{\hat{g}}\right) + \frac{3}{2}(\mu_\alpha)^2 - \frac{2}{h^2}|\mathrm{d}\mu_\alpha|_{\hat{g}} \\ &\geq \frac{1}{h^2}\left(\frac{R_{\boldsymbol{S}^2}}{\varphi^2} + Z_{\mu_\alpha}\right) \end{aligned} \tag{5-4}$$

where the last step follows from the assumption $h \geq 1$ and the definition

$$Z_{\mu_\alpha} := \tfrac{3}{2}(\mu_\alpha^2 - \hat{H}^2) + 2(\partial_t \mu_\alpha - \partial_t \hat{H}).$$

By a careful estimate of Z_{μ_α} using the mean value theorem, it is not difficult to show that there exists a constant $t_c > 0$ such that

$$Z_{\mu_\alpha} > 0 \quad \text{for } t \in (-\alpha, -t_c) \cup (t_c, \alpha) \quad \text{and} \quad \varphi^2 Z_{\mu_\alpha} \geq C(\alpha) \quad \text{for } t \in (-\alpha, \alpha), \tag{5-5}$$

where $C(\alpha) < 0$ is a constant depending only on α and satisfies $C(\alpha) \to 0$ as $\alpha \to \pi/2$. Similar to the proof of Proposition 4.7, here (5-5) implies that Σ_0^α is contained in a fixed compact domain in $\mathbb{B}_{(-\pi/2,\pi/2)}$ that is independent of the choice of α. Thus, as $\alpha \to \pi/2$, such Σ_0^α subconverge to a stable $\hat{H}$-hypersurface, and an analogue of Proposition 4.12 can be obtained. An analogue of Proposition 3.4 and a foliation argument yield that $h \equiv 1$ and $g = \hat{g}$. □

Remark 5.2. The assumption $h \geq 1$ is important for Theorem 5.1 to hold. Without this assumption, one may let $g = \cos^2 t (\mathrm{d}t^2 + g_{\boldsymbol{S}^2}) \neq \hat{g}$ on $\boldsymbol{S}^3 \setminus \{O, O'\} \cong \boldsymbol{S}^2 \times (-\pi/2, \pi/2)$ and take $h = (\cos t)^{1/2}$, and it is easy to check that (5-1) is satisfied — in particular, $R_g = (2 + 4\cos^2 t)(\cos t)^{-4}$ and $h^{-2} R_{\hat{g}} = 6(\cos t)^{-1}$, so $R_g \geq h^{-2} R_{\hat{g}}$.

Theorem 3.1 has Euclidean and hyperbolic analogues. Putting together, let us take

$$\hat{g}_\kappa = \varphi_\kappa(t)^2 g_{\boldsymbol{S}^2} + \mathrm{d}t^2 \quad \text{on } \boldsymbol{B}_T \tag{5-6}$$

where

$$\varphi_\kappa(t) = \begin{cases} \sin \sqrt{\kappa} t, & \kappa > 0, \\ t, & \kappa = 0, \\ \sinh \sqrt{-\kappa} t, & \kappa < 0, \end{cases}$$

and $T \in (0, \pi/\sqrt{\kappa})$ if $\kappa > 0$; $T > 0$ if $\kappa \leq 0$. In particular, $\sec(\hat{g}_\kappa) = \kappa$, and $\hat{H}_\kappa(t) = 2\varphi_\kappa'(t)/\varphi_\kappa(t)$.

Theorem 5.3. *Let $\boldsymbol{B}_T$, $\hat{g}_\kappa$ be as above. Let g be a Riemannian metric on $\boldsymbol{B}_T$ satisfying*

$$g \geq h^4 \hat{g}_\kappa, \quad R_g \geq h^{-2} R_{\hat{g}_\kappa}, \quad H_{\partial \boldsymbol{B}_T} \geq \hat{H}_\kappa(T),$$

for some smooth function $h \geq 1$ defined on $\boldsymbol{B}_T$. Then $h \equiv 1$, and $g = \hat{g}_\kappa$.

As pointed out by Gromov [2023, Section 5.5], a key fact that allows the different cases (corresponding to different choices of κ) in Theorem 5.3 to be treated similarly is that the function $\varphi_\kappa(t)$ is "log-concave" — in other words, $\hat{H}_\kappa(t)$ is strictly decreasing in t; see Lemma 3.2 and Proposition 3.4. Having this in mind, the proof proceeds as that of either Theorem 3.1 or 5.1, and we leave the details to the interested reader.

Remark 5.4. When $\kappa \leq 0$ and $T = +\infty$, whether Theorem 5.3 holds remains unknown to us.

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