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GROUPS WITH 2-GENERATED SYLOW SUBGROUPS AND THEIR CHARACTER TABLES

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Let G be a finite group with a Sylow p-subgroup P. We show that the character table of G determines whether P has maximal nilpotency class and whether P is a minimal nonabelian group. The latter result is obtained from a precise classification of the corresponding groups G in terms of their composition factors. For p-constrained groups G we prove further that the character table determines whether P can be generated by two elements.

1. Introduction

Recently, Navarro and Sambale [2023] have investigated finite groups G with a Sylow p-subgroup P such that $|P : P'| = p^2$ or $|P : \mathrm{Z}(P)| = p^2$ where $P' = [P, P]$ denotes the commutator subgroup and $\mathrm{Z}(P)$ is the center of P. It was proved that both properties can be read off from the character table $X(G)$ of G. This was another contribution to Richard Brauer's Problem 12 [1963], which asks what properties of a Sylow p-subgroup P are determined by $X(G)$. We refer the reader to the introduction of [Navarro and Sambale 2023] and [Sambale 2020] for a collection of the known results on this problem. We just mention that one important property is that $X(G)$ knows whether P is abelian. While there is an elementary proof of the case $p = 2$ by Camina and Herzog [1980], the full solution has required the classification of finite simple groups (see [Kimmerle and Sandling 1995; Navarro et al. 2015; Malle and Navarro 2021]).

After dealing with P' and $\mathrm{Z}(P)$, it is natural to turn our attention to the Frattini subgroup $\Phi(P)$ of P. Recall that $|P : \Phi(P)| \le p$ holds if and only if P is cyclic. It is easy to show that this property can be read off from $X(G)$ (see [Navarro 2018, Corollary 3.12]). In the first part of the present paper we consider groups G with $|P : \Phi(P)| = p^2$, i.e., P is generated by two elements, but not by one. For $p = 2$ this property is detectable by $X(G)$ as was shown in [Navarro et al. 2021]. We obtain the corresponding result for odd primes p provided that G is p-constrained in Corollary 5. In the general case we offer a partial solution depending on the socle of G (see Proposition 6 and the subsequent remark).

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Our next objective are groups with Sylow p-subgroups P of maximal nilpotency class. For $p=2$, this property is equivalent to $|P:P'|=4$. This case was previously handled in an elementary fashion by Navarro, Sambale, and Tiep [Navarro et al. 2018]. The general result is our first main theorem.

Theorem A. *The character table of a finite group G determines whether G has Sylow p-subgroups of maximal nilpotency class.*

It is known that $X(G)$ determines the isomorphism types of abelian Sylow subgroups. Of course we cannot expect this for maximal class Sylow subgroups as $X(D_8)=X(Q_8)$. Perhaps surprisingly, $X(G)$ does not even determine $X(P)$. Counterexamples for $p=3$ arise as semidirect products of nonequivalent faithful actions of $\mathrm{SL}(2,3)$ on $C_9 \times C_9$ (the groups are $\texttt{SmallGroup}(2^3 3^5, a)$ with $a \in \{2289, 2290\}$ in GAP [2020]). Here P indeed has maximal class. This is related to [Navarro et al. 2022, Question E].

We obtain Theorem A as a consequence of the following structure description, which might be of independent interest:

Theorem B. *Let G be a finite group with a Sylow p-subgroup P of maximal class. Suppose that $\mathrm{O}_{p'}(G)=1$ and $\mathrm{O}^{p'}(G)=G$. Then one of the following holds*:

(i) *There exists $x \in P$ such that $|\mathrm{C}_G(x)|_p = p^2$.*

(ii) *G is quasisimple and $|\mathrm{Z}(G)| \leq p$.*

The proof uses recent work by Grazian and Parker [2022] on fusion systems and is given in Section 3.

In the final part of the paper we study groups with minimal nonabelian Sylow p-subgroups P, i.e., P is nonabelian, but every proper subgroup of P is abelian. It is easy to see that this happens if and only if $|P:\mathrm{Z}(P)| = |P:\Phi(P)| = p^2$ (see Lemma 9 below). Refining [Navarro and Sambale 2023, Theorem 7.5], we obtain in Section 4 the following description:

Theorem C. *Let G be a finite group with a minimal nonabelian Sylow p-subgroup P and $\mathrm{O}_{p'}(G)=1$. Then one of the following holds*:

(i) *$p=2$, $P \in \{D_8, Q_8\}$ and $\mathrm{O}^{2'}(G) \in \{\mathrm{SL}(2,q), \mathrm{PSL}(2,q'), A_7\}$ where $q \equiv \pm 3 \pmod 8$ and $q' \equiv \pm 7 \pmod{16}$.*

(ii) *$|P| = p^3$ and $\exp(P) = p > 2$.*

(iii) *$G = P \rtimes Q$ where $Q \leq \mathrm{GL}(2,p)$.*

(iv) *$p>2$, $\mathrm{O}^{p'}(G) = S \rtimes C_{p^a}$ where S is a simple group of Lie type with cyclic Sylow p-subgroups. The image of C_{p^a} in $\mathrm{Out}(S)$ has order p.*

(v) *$p=2$ and $G = \mathrm{PSL}(2,q^f) \rtimes C_{2^a d}$ where q is a prime, $q^f \equiv \pm 3 \pmod 8$ and $d \mid f$. Moreover, C_{2^a} acts as a diagonal automorphism of order 2 on $\mathrm{PSL}(2,q^f)$ and C_d induces a field automorphism of order d.*

(vi) $p=3$ *and* $\mathrm{O}^{3'}(G)=\mathrm{PSL}^{\epsilon}(3,q^f)\rtimes C_{3^a}$ *where* $\epsilon=\pm 1$, q *is prime*, $(q^f-\epsilon)_3=3$ *and* $G/\mathrm{O}^{3'}(G)\le C_f\times C_2$.

Here, PSL^{ϵ} stands for PSL if $\epsilon=1$ and PSU otherwise. Again the proof is based on the classification of the corresponding fusion systems. To show that Case (iv) in Theorem C occurs for all odd primes p, we will exhibit appropriate examples after the proof.

Corollary D. *The character table of a finite group G determines whether G has minimal nonabelian Sylow p-subgroups.*

2. 2-generated Sylow subgroups

In the following G will always denote a finite group. The exponent of G is denoted by $\exp(G)$. The core of a subgroup $H\le G$ is defined by $\mathrm{core}_G(H):=\bigcap_{g\in G} gHg^{-1}\trianglelefteq G$. For $x,y\in G$ let $[x,y]:=xyx^{-1}y^{-1}$. The Fitting subgroup and the generalized Fitting subgroup of G are denoted by $\mathrm{F}(G)$ and $\mathrm{F}^*(G)=\mathrm{F}(G)\mathrm{E}(G)$ respectively. We write $\mathrm{Irr}(G)$ to denote the set of ordinary complex irreducible characters of G. For $g\in G$ and $\chi\in\mathrm{Irr}(G)$ let

$$\mathbb{Q}(g):=\mathbb{Q}(\chi(g):\chi\in\mathrm{Irr}(G)),$$
$$\mathbb{Q}(\chi):=\mathbb{Q}(\chi(g):g\in G).$$

It is well-known that $\mathbb{Q}(\chi)$ lies in the cyclotomic field $\mathbb{Q}_n$ where $n=|G|$. Let f_χ be the smallest positive integer such that $\mathbb{Q}(\chi)\subseteq\mathbb{Q}_{f_\chi}$ (f_χ is called the *Feit number* in [Navarro 2018]). Let $\mathrm{Irr}_{p'}(G):=\{\chi\in\mathrm{Irr}(G):p\nmid\chi(1)\}$ as usual. The p-part and the p'-part of an integer n are denoted by n_p and $n_{p'}$ respectively.

Our first lemma is applied multiple times throughout the paper.

Lemma 1. *Let A be an abelian normal subgroup of G such that $G=\langle x\rangle A$ for some $x\in G$. Then the map $A\to G'$, $a\mapsto[x,a]$ is an epimorphism with kernel $\mathrm{C}_A(x)$. In particular, $|G'|=|A/\mathrm{C}_A(x)|$.*

Proof. See [Isaacs 2008, Lemma 4.6]. □

To get from P' to $\Phi(P)$ we need the following variant:

Lemma 2. *Let P be a p-group with a proper normal subgroup Q and $x\in P$ such that $P=\langle x\rangle Q$ and $\langle x\rangle\cap Q\le P'$. Then $|P:\Phi(P)|=p^2$ if and only if $|\mathrm{C}_{Q/\Phi(Q)}(x)|=p$.*

Proof. Since $\langle x\rangle\cap Q\le P'\le\Phi(P)$ and $Q<P$, we have

$$P/\Phi(P)=Q\Phi(P)/\Phi(P)\times\langle x\rangle\Phi(P)/\Phi(P)\cong Q/(Q\cap\Phi(P))\times C_p.$$

Moreover,

$$\Phi(P)\cap Q=P'\Phi(Q)\langle x^p\rangle\cap Q=P'\Phi(Q)(\langle x^p\rangle\cap Q)=P'\Phi(Q).$$

Now $|P:\Phi(P)|=p^2$ if and only if

$$|Q/\Phi(Q):(P/\Phi(Q))'|=|Q:P'\Phi(Q)|=p.$$

By Lemma 1 applied to $Q/\Phi(Q)\trianglelefteq P/\Phi(Q)$, this is equivalent to

$$|\mathrm{C}_{Q/\Phi(Q)}(x)|=p. \qquad \square$$

The next result is a variation of [Navarro and Sambale 2023, Theorem 6.1].

Lemma 3. *Let G be a finite group with a Sylow p-subgroup P and $\mathrm{O}_{p'}(G)=1$. Then*

$$K:=\bigcap_{\substack{\chi\in\mathrm{Irr}_{p'}(G)\\ p^2\nmid f_\chi}}\mathrm{Ker}(\chi)=\mathrm{core}_G(\Phi(P)).$$

Proof. Let $n:=|G|$. If $n_p=1$, then the claim holds since $\bigcap_{\chi\in\mathrm{Irr}(G)}\mathrm{Ker}(\chi)=1=P$. Thus, let $n_p\ne 1$. Then $\mathcal{G}:=\mathrm{Gal}(\mathbb{Q}_n|\mathbb{Q}_{pn_{p'}})$ is a p-group. Let $N:=\mathrm{core}_G(\Phi(P))$ and $\chi\in\mathrm{Irr}_{p'}(G)$ with $p^2\nmid f_\chi$. Since $\mathbb{Q}(\chi_P)\subseteq\mathbb{Q}(\chi)\subseteq\mathbb{Q}_{pn_{p'}}$, $\mathcal{G}$ permutes the irreducible constituents of χ_P. Since the sizes of the $\mathcal{G}$-orbits are p-powers and $p\nmid\chi(1)$, there must be a linear constituent $\lambda\in\mathrm{Irr}(P|\chi)$ fixed by $\mathcal{G}$, i.e., $\mathbb{Q}(\lambda)\subseteq\mathbb{Q}_p$. It follows that $N\subseteq\Phi(P)\subseteq\mathrm{Ker}(\lambda)$. By Clifford theory, χ_N is a sum of conjugates of λ_N. Hence, $N\subseteq\mathrm{Ker}(\chi)$. This shows that $N\le K$.

Now let $\lambda\in\mathrm{Irr}(P/\Phi(P))$. This time, $\mathcal{G}$ acts on the irreducible constituents of λ^G. Since $p\nmid|G:P|=\lambda^G(1)$, there must be a constituent $\chi\in\mathrm{Irr}_{p'}(G|\lambda)$ fixed by $\mathcal{G}$, i.e., $p^2\nmid f_\chi$. This implies $\chi_{P\cap K}=\chi(1)1_{P\cap K}$. On the other hand, $\lambda_{P\cap K}$ is a constituent of $\chi_{P\cap K}$. Therefore, $P\cap K\subseteq\mathrm{Ker}(\lambda)$. Since $\lambda\in\mathrm{Irr}(P/\Phi(P))$ was arbitrary, we obtain $P\cap K\le\Phi(P)$. Now Tate's theorem (see [Huppert 1967, Satz IV.4.7]) yields that K is p-nilpotent. By hypothesis, $\mathrm{O}_{p'}(K)\le\mathrm{O}_{p'}(G)=1$ and K is a p-group. Finally, $K\le\mathrm{O}_p(G)\cap K\le P\cap K\le\Phi(P)$ and $K\le N$. $\square$

We mention that the characters χ with $p^2\nmid f_\chi$ are precisely the *almost p-rational* characters introduced in [Hung et al. 2022]. Lemma 3 allows to read off $K:=\mathrm{core}_G(\Phi(P))$ from the character table. Since $|P/K:\Phi(P/K)|=|P:\Phi(P)|$, it is therefore no loss to assume that $K=1$. The next theorem comes close to [Navarro and Sambale 2023, Theorem 3.1].

Theorem 4. *Let G be a finite group with a nonabelian Sylow p-subgroup P such that $|P:\Phi(P)|=p^2$ and $\mathrm{O}_{p'}(G)=1=\mathrm{core}_G(\Phi(P))$. Then $\mathrm{F}^*(G)$ is the unique minimal normal subgroup of G and $P\mathrm{F}^*(G)/\mathrm{F}^*(G)$ is cyclic. If $\mathrm{F}^*(G)$ is nonabelian, then P permutes the simple components of $\mathrm{F}^*(G)$ transitively. In particular, their number is a p-power in this case.*

Proof. Let N be a minimal normal subgroup of G. Then

$$\begin{aligned}|PN/N:\Phi(PN/N)| &= |P/P\cap N:\Phi(P/P\cap N)|\\ &= |P/P\cap N:\Phi(P)(P\cap N)/P\cap N|\\ &= |P:\Phi(P)(P\cap N)|\\ &\le |P:\Phi(P)|\\ &= p^2,\end{aligned}$$

where the second equality follows from [Isaacs 2008, Lemma 4.5], for instance. Suppose first that $P\cap N\le\Phi(P)$. Then by Tate's theorem (see [Huppert 1967, Satz IV.4.7]), N is a p-group and $N\le\Phi(P)$. This contradicts $\mathrm{core}_G(\Phi(P))=1$. Consequently, $|PN/N:\Phi(PN/N)|\le p$ and PN/N is cyclic. Let $M\ne N$ be another minimal normal subgroup of G. Then G/N and similarly G/M have cyclic Sylow p-subgroups. Since G is isomorphic to a subgroup of $G/M\times G/N$, G has abelian Sylow p-subgroups, which we have excluded explicitly. This shows that N is the unique minimal normal subgroup.

Assume now that N is nonabelian. Then $\mathrm{F}(G)\cap N=1$ implies $\mathrm{F}(G)=1=\mathrm{Z}(G)$ and $\mathrm{F}^*(G)=\mathrm{E}(G)=N$. Write $N=T_1\times\cdots\times T_n$ with nonabelian simple groups $T_1\cong\cdots\cong T_n$. If $P\le N$, using that P is 2-generated and nonabelian, we conclude that $n=1$ and P certainly acts transitively on $\{T_1,\dots,T_n\}$. Hence, we may assume that $P\nsubseteq N$ and $n\ge 2$. Let $Q_i:=P\cap T_i$ for $i=1,\dots,n$. Let $x\in P$ such that $PN/N=\langle xN\rangle$. Since $P\cap N\nsubseteq\Phi(P)$, there exists some $1\le i\le n$ with $Q_i\nsubseteq\Phi(P)$. Without loss of generality, let $i=1$. Choose $y\in Q_1\setminus\Phi(P)$. For all $j\in\mathbb{Z}$ we note that $xy^j\notin N\supseteq\Phi(P)$. Since $|P:\Phi(P)|=p^2$, it follows that $P=\langle x,y\rangle$. Without loss of generality, let $T_1,\dots,T_k$ be the orbit of T_1 under P. Suppose by way of contradiction that $k<n$. Then $Q_1\cdots Q_k\trianglelefteq P$ and $Q_{k+1}\times\cdots\times Q_n\le P/Q_1\cdots Q_k=\langle xQ_1\cdots Q_k\rangle$ is cyclic. This is only possible if $n=k+1$ and Q_n is cyclic. Moreover, $Q_n=\langle x^{p^a}z\rangle$ for some $a\ge 1$ and $z\in Q_1\cdots Q_k$. Since a nonabelian simple group cannot have a cyclic Sylow 2-subgroup, $p>2$. It follows from [Gross 1982, theorem A] that x induces an inner automorphism on T_n. This is impossible since x^{p^a} induces an inner automorphism of order $|T_n|_p$. This contradiction shows that P permutes the T_i transitively.

Finally, assume that N is elementary abelian. Since $\mathrm{O}_{p'}(G)=1$, we have $F:=\mathrm{F}(G)=\mathrm{O}_p(G)$. Suppose that $N<F$. Then $\Phi(F)\le\Phi(P)$ yields $\Phi(F)\le\mathrm{core}_G(\Phi(P))=1$, i.e., F is elementary abelian. Now the existence of an element of order p in $P\setminus N$ implies the existence of a (cyclic) complement of N in P. By a theorem of Gaschütz (see [Huppert 1967, Hauptsatz I.17.4]), N has a complement K in G. Since F centralizes N, we obtain $1\ne K\cap F\trianglelefteq NK=G$. This contradicts the fact that N is the unique minimal normal subgroup of G. Hence, $F=N$. Suppose that $\mathrm{E}(G)\ne 1$ and choose a central product $M\trianglelefteq G$ of quasisimple components. Then

$N \le \mathrm{Z}(M)$, because $1 \ne N \cap M \trianglelefteq G$. Since M/N has cyclic Sylow p-subgroups, the order of the Schur multiplier of M/N is not divisible by p. This contradicts $N \le \mathrm{Z}(M)$. We have therefore shown that $N = \mathrm{F}^*(G)$. □

In order to decide whether $|P : \Phi(P)| = p^2$, we may assume that the hypotheses of Theorem 4 are fulfilled. The situation now splits into two cases. When $\mathrm{F}^*(G)$ is abelian, the group G is p-constrained (recall that in general a group G is called p-constrained if $\mathrm{C}_{\overline{G}}(\mathrm{O}_p(\overline{G})) \le \mathrm{O}_p(\overline{G})$ where $\overline{G} := G/\mathrm{O}_{p'}(G)$). In this case we solve the problem completely. To do so, we will use a result of Higman (see [Navarro 2018, Corollary 7.18]) that allows to locate the p-elements in $X(G)$.

Corollary 5. *The character table of a p-constrained group G determines whether a Sylow p-subgroup P is generated by two elements.*

Proof. Let P be a Sylow p-subgroup of G. Since the character table $X(G)$ determines $X(G/\mathrm{O}_{p'}(G))$, we may assume that $\mathrm{O}_{p'}(G) = 1$. Since G is p-constrained, $\mathrm{O}_p(G) > 1$. By Lemma 3, we may assume that $\mathrm{core}_G(\Phi(P)) = 1$. Moreover, the orders and embeddings of the normal subgroups of G can be read off from $X(G)$. Hence by Theorem 4, we may assume that $N = \mathrm{O}_p(G) = \mathrm{F}(G)$ is the only minimal normal subgroup of G. If $P = N$, then $|P : \Phi(P)| = |P|$ and we are done. Hence, let $N < P$. By [Navarro 2018, Corollary 3.12], $X(G/N)$ detects whether P/N is cyclic. By Theorem 4, we can assume that this is the case. Choose $x \in P$ with $P/N = \langle xN\rangle$ (note that x can be spotted in $X(G)$ using [Navarro 2018, Corollary 3.12]). Since $P = N\langle x\rangle = \mathrm{O}_p(G)\langle x\rangle$ is the only Sylow p-subgroup of G containing x, $\mathrm{C}_P(x) = \mathrm{C}_N(x)\langle x\rangle$ is a Sylow p-subgroup of $\mathrm{C}_G(x)$. In particular, $|\mathrm{C}_N(x)| = |\mathrm{C}_G(x)|_p/|P/N|$ is determined by $X(G)$. By Lemma 1, we have

$$P' = [x, N] = \{[x, y] : y \in N\} \tag{2-1}$$

and $|P'| = |N/\mathrm{C}_N(x)|$ can be computed from $X(G)$. Let $|P/N| = p^a$ and $|N/P'| = p^n$. If $x^{p^a} \in P'$, then $P/P' \cong C_{p^a} \times C_p^n$ and otherwise $P/P' \cong C_{p^{a+1}} \times C_p^{n-1}$. Since $\mathbb{Q}(x)$ can be read off from $X(G)$, it suffices to show that

$$p|\mathbb{Q}(x) : \mathbb{Q}|_p = \exp(P/P').$$

Taking only $X(G/N)$ into account, we obtain $\mathbb{Q}(xN) = \mathbb{Q}_{p^a}$ or equivalently $|\mathbb{Q}(xN) : \mathbb{Q}|_p = p^{a-1}$ by [Navarro 2018, Theorem 3.11]. Thus $|\mathbb{Q}(x) : \mathbb{Q}|_p \ge p^{a-1}$. If $x^{p^a} = 1$, then $p|\mathbb{Q}(x) : \mathbb{Q}|_p = p^a = \exp(P/P')$ as desired. Now let $|\langle x\rangle| = p^{a+1}$. If $x^{p^a} \in P'$, then there exists $y \in N$ with $x^{p^a} = [x, y] = xyx^{-1}y^{-1}$ by (2-1). It follows that $yxy^{-1} = x^{1-p^a}$ and $|\mathrm{N}_G(\langle x\rangle) : \mathrm{C}_G(x)|_p = p$. Again by [Navarro 2018, Theorem 3.11], we have $p|\mathbb{Q}(x) : \mathbb{Q}|_p = p^a = \exp(P/P')$. Assume conversely that $|\mathbb{Q}(x) : \mathbb{Q}|_p = p^{a-1}$. Then there exists $y \in G$ with $yxy^{-1} = x^{1+kp^a}$ for some $0 < k < p$. We observe that $y \in \mathrm{N}_G(\langle x\rangle N) = \mathrm{N}_G(P)$. Replacing y by its p-part, we get $y \in P$. Now $x^{-kp^a} = [x, y] \in P'$ and $\exp(P/P') = p^a$ as desired. □

If G is p-solvable in the situation of Corollary 5 (recall that every p-solvable group is p-constrained), then $\mathrm{O}_p(G)$ has a complement K in $\mathrm{O}_{pp'}(G)$ by the Schur–Zassenhaus theorem. Using the Frattini argument, it is easy to show that $\mathrm{N}_G(K)$ is a complement of N in G. In this situation, G is a primitive permutation group on N of affine type.

On the other hand, every nonabelian simple group S gives rise to a nonsplit extension $G = N.S$ where $N = \Phi(G)$ is elementary abelian without complement (see [Doerk and Hawkes 1992, Theorem B.11.8]). Garrison [1976] has exhibited examples to show that $X(G)$ does not determine whether G splits over N. For instance,

$$\texttt{PerfectGroup}(7500, 1) \cong C_5^3 \rtimes A_5 \quad \text{and} \quad \texttt{PerfectGroup}(7500, 2) \cong C_5^3.A_5$$

in GAP [2020] have the same character table and the Sylow 5-subgroup is 2-generated in both cases.

Now assume that $N = \mathrm{F}^*(G)$ in the situation of Theorem 4 is nonabelian. If $N \cap P$ is abelian, then N has a complement in PN by [Huppert 1967, Satz IV.3.8]. In this case PN is a twisted wreath product. The nonsplit extension $M_{10} = A_6.C_2$ with $P = SD_{16}$, a semidihedral group, shows that this is not always the case. Even when N is not simple, $P \cap N$ is not always abelian (as in [Navarro and Sambale 2023, Theorem 3.1]). One example is

$$G = \mathrm{PSL}(2, 7)^2 \rtimes \langle x \rangle \cong \mathrm{PSL}(2, 7)^2 \rtimes C_4 \leq \mathrm{PGL}(2, 7) \wr C_2,$$

where x^2 acts as a diagonal automorphism on both factors $\mathrm{PSL}(2, 7)$ simultaneously. Here $P = D_8^2 \rtimes C_4$ is 2-generated. Nevertheless, we provide the following reduction theorem:

Proposition 6. *Let G be a finite group with Sylow p-subgroup P such that $\mathrm{O}_{p'}(G) = 1$ and $N = \mathrm{F}^*(G)$ is the unique minimal normal subgroup of G. Suppose that N is nonabelian and PN/N is cyclic. Let S be a simple component of N. Assume that $|G : \mathrm{N}_G(S)|$ is a p-power. Then the following hold*:

(i) *$G = \mathrm{N}_G(S)P$.*

(ii) *$\widetilde{P} := \mathrm{N}_P(S)\mathrm{C}_G(S)/\mathrm{C}_G(S)$ is a Sylow p-subgroup of the almost simple group $\mathrm{N}_G(S)/\mathrm{C}_G(S)$ with socle $\widetilde{S} := S\mathrm{C}_G(S)/\mathrm{C}_G(S) \cong S$. Moreover, $\widetilde{P}\widetilde{S}/\widetilde{S}$ is cyclic.*

(iii) *$|P : \Phi(P)| \leq p^2$ if and only if $|\widetilde{P} : \Phi(\widetilde{P})| \leq p^2$.*

(iv) *S and $|\widetilde{P}|$ are determined by $X(G)$.*

Proof. (i) Since $|G : \mathrm{N}_G(S)|$ is a p-power, $|\mathrm{N}_G(S)P| = |\mathrm{N}_G(S) : \mathrm{N}_P(S)||P| = |G|$ and $G = \mathrm{N}_G(S)P$.

(ii) By (i), $\mathrm{N}_P(S)$ is a Sylow p-subgroup of $\mathrm{N}_G(S)$. Hence, $\widetilde{P}$ is a Sylow p-subgroup of $\mathrm{N}_G(S)/\mathrm{C}_G(S)$. Let $Q := N \cap P \trianglelefteq P$. Then $P/Q \cong PN/N$ is cyclic by hypothesis. Let $x \in P$ such that $P = \langle x \rangle Q$. Then $\widetilde{P}\widetilde{S}/\widetilde{S} \cong \mathrm{N}_P(S)S\mathrm{C}_G(S)/S\mathrm{C}_G(S) \leq \langle x \rangle S\mathrm{C}_G(S)/S\mathrm{C}_G(S)$ is cyclic.

(iii) If $P \leq N \leq \mathrm{N}_G(S)$, then $S \trianglelefteq G$ and $N = S$. Here, $P \cong \widetilde{P}$, so we are done. Now assume $PN/N \neq 1$. As in (ii), let $Q := N \cap P \trianglelefteq P$. Since $\mathrm{O}^p(PN) = N$, there exists $x \in P$ such that $P = \langle x \rangle Q$ and $\langle x \rangle \cap Q \leq P'$ (see [Brandis 1978, Satz 3.3]). Lemma 2 yields $|P : \Phi(P)| = p^2$ if and only if $|\mathrm{C}_{Q/\Phi(Q)}(x)| = p$.

By (i), we may write $N = T_1 \times \cdots \times T_{p^a}$ such that $T_i = x^{i-1}Sx^{1-i}$ for $i = 1, \ldots, p^a$. Let $Q_i := T_i \cap P \leq Q$. Then $\widetilde{Q} := Q_1\mathrm{C}_G(S)/\mathrm{C}_G(S) \cong Q_1$ is a normal subgroup of $\widetilde{P}$. Since $\mathrm{N}_P(S) = \langle x^{p^a} \rangle Q$, we have $\widetilde{P} = \langle \tilde{x} \rangle \widetilde{Q}$ where $\tilde{x} := x^{p^a}\mathrm{C}_G(S)$. It is easy to see that the map

$$\mathrm{C}_{Q_1/\Phi(Q_1)}(x^{p^a}) \to \mathrm{C}_{Q/\Phi(Q)}(x), \quad y\Phi(Q_1) \mapsto \prod_{i=0}^{p^a-1} x^i y x^{-i} \Phi(Q)$$

is an isomorphism. In particular, $|\mathrm{C}_{Q/\Phi(Q)}(x)| = |\mathrm{C}_{Q_1/\Phi(Q_1)}(x^{p^a})|$. Assume for the moment that $x^{p^a} \in Q$. Then

$$\widetilde{P} = \widetilde{Q} \leq \widetilde{S} \quad \text{and} \quad |\mathrm{C}_{Q_1/\Phi(Q_1)}(x^{p^a})| = |Q_1/\Phi(Q_1)| = |\widetilde{P}/\Phi(\widetilde{P})|.$$

In this case, $|P : \Phi(P)| = p^2$ if and only if $\widetilde{P}$ is cyclic, i.e., $|\widetilde{P} : \Phi(\widetilde{P})| = p$. Now let $x^{p^a} \notin Q$. By way of contradiction, suppose that $x^{p^a} \in Q_1\mathrm{C}_G(S)$. Then there exists $y \in Q_1$ such that $x^{p^a}y \in \mathrm{C}_G(S)$. Now also

$$z := x^{p^a} \prod_{i=0}^{p^a-1} x^i y x^{-i} \in \mathrm{C}_G(S).$$

Since z is centralized by x, it follows that $z \in x^i\mathrm{C}_G(S)x^{-i} = \mathrm{C}_G(T_i)$ for $i = 1, \ldots, p^a$. Hence, $z \in \mathrm{C}_G(N) = 1$ and $x^{p^a} \in Q$, a contradiction. Thus, $\widetilde{Q} < \widetilde{P}$ and

$$\widetilde{Q} \cap \langle \tilde{x} \rangle = (Q \cap \langle x^{p^a} \rangle)\mathrm{C}_G(S)/\mathrm{C}_G(S) \leq P'\mathrm{C}_G(S)/\mathrm{C}_G(S) = \widetilde{P}'.$$

Lemma 2 shows that $|\widetilde{P} : \Phi(\widetilde{P})| = p^2$ if and only if

$$|\mathrm{C}_{Q_1/\Phi(Q_1)}(x^{p^a})| = |\mathrm{C}_{\widetilde{Q}/\Phi(\widetilde{Q})}(\tilde{x})| = p.$$

Now the claim follows.

(iv) The isomorphism types of N and S are determined by $X(G)$ according to [Navarro and Sambale 2023, Theorem 4.1]. We obtain $|\mathrm{N}_P(S)|$ from $|N| = |S|^{|P:\mathrm{N}_P(S)|}$. Arguing as in (iii), shows that $\mathrm{C}_P(S) = \mathrm{C}_Q(S) = Q_2 \cdots Q_{p^a}$. Hence, $|\mathrm{C}_P(S)| = |S|_p^{p^a-1}$ is computable from $X(G)$. The claim follows from $\widetilde{P} \cong \mathrm{N}_P(S)/\mathrm{C}_P(S)$. □

To decide whether $|P : \Phi(P)| = p^2$ holds, it suffices to obtain the structure of $\widetilde{P}$ with the notation from Proposition 6. If $p \geq 5$ and S is neither a linear nor a unitary group, then $\mathrm{Out}(S)$ has a cyclic Sylow p-subgroup by [Conway et al. 1985, Table 5]. In this case the isomorphism type of $\widetilde{P}$ is uniquely determined by $X(G)$ and the problem is solved. On the other hand, the proof of [Navarro and Sambale 2023, Lemma 5.1] shows that for linear and unitary groups S the condition $|P:\Phi(P)| = p^2$ is not determined by $|\widetilde{P}|$ alone. It remains a challenge to settle these cases (and $p = 3$ with $S = D_4(q)$, $E_6(q)$ and ${}^2E_6(q)$).

3. p-groups of maximal class

We start by introducing some terminology of (saturated, nonexotic) fusion systems. Let P be a Sylow p-subgroup of G as before. The *fusion system* $\mathcal{F} = \mathcal{F}_P(G)$ of G on P is a category whose objects are the subgroups of P and the morphisms of $\mathcal{F}$ have the form $f : S \to T$, $x \mapsto gxg^{-1}$ where $S, T \le P$ and $g \in G$. Then $\mathrm{Aut}_{\mathcal{F}}(S) \cong \mathrm{N}_G(S)/\mathrm{C}_G(S)$ and $\mathrm{Out}_{\mathcal{F}}(S) \cong \mathrm{N}_G(S)/S\mathrm{C}_G(S)$. Elements $x, y \in P$ (or subsets $S, T \subseteq P$) are called *$\mathcal{F}$-conjugate* if there exists a morphism f such that $f(x) = y$ (or $f(S) = T$). A subgroup $S \le P$ is called

- *fully normalized*, if $|\mathrm{N}_P(T)| \le |\mathrm{N}_P(S)|$ for all $\mathcal{F}$-conjugates T of S,
- *centric*, if $\mathrm{C}_P(T) = \mathrm{Z}(T)$ for all $\mathcal{F}$-conjugates T of S,
- *radical*, if $\mathrm{O}_p(\mathrm{Aut}_{\mathcal{F}}(S)) = \mathrm{Inn}(S)$ (equivalently, $\mathrm{O}_p(\mathrm{Out}_{\mathcal{F}}(S)) = 1$),
- *essential*, if S is fully normalized, centric and $\mathrm{Out}_{\mathcal{F}}(S)$ contains a strongly p-embedded subgroup (see [Aschbacher et al. 2011, Definition A.6]). For our purpose, it is enough to know that S is radical in this case.

By Alperin's fusion theorem, every morphism in $\mathcal{F}$ is a composition of restrictions of morphisms $f \in \mathrm{Aut}_{\mathcal{F}}(S)$ where $S = P$ or S is essential (see [Aschbacher et al. 2011, Theorem I.3.5]). Note that $\mathrm{Aut}_{\mathcal{F}}(P)$ permutes the essential subgroups by conjugation. Hence, if $Q \le P$ does not lie in any essential subgroup, then Q is fully normalized. In this case, $\mathrm{N}_P(Q)$ is a Sylow p-subgroup of $\mathrm{N}_G(Q)$ (see [Aschbacher et al. 2011, Lemma I.1.2]). Consequently, $\mathrm{C}_P(Q) = \mathrm{N}_P(Q) \cap \mathrm{C}_G(P)$ is a Sylow p-subgroup of $\mathrm{C}_G(P)$.

We call $\mathcal{F}$ *controlled* if $\mathrm{N}_G(P)$ controls the fusion in P with respect to G, i.e., every morphism $S \to T$ has the form $x \mapsto gxg^{-1}$ for some $g \in \mathrm{N}_G(P)$. Abstractly, this means that there are no essential subgroups and $\mathcal{F} = \mathcal{F}_P(P \rtimes A)$ for some Schur–Zassenhaus complement A of $\mathrm{Inn}(P)$ in $\mathrm{Aut}_{\mathcal{F}}(P)$. More generally, $\mathcal{F}$ is called *constrained* if there exists $Q \trianglelefteq P$ such that $\mathrm{C}_P(Q) = \mathrm{Z}(Q)$ and $\mathrm{N}_G(Q)$ controls the fusion in P. By the model theorem (see [Aschbacher et al. 2011, Theorem I.4.9]), a constrained fusion system is realized by a unique group G such that $\mathrm{C}_G(\mathrm{O}_p(G)) \le \mathrm{O}_p(G)$ (note that G is p-constrained). The largest subgroup $Q \trianglelefteq P$ such that $\mathrm{N}_G(Q)$ controls the fusion in P is denoted by $\mathrm{O}_p(\mathcal{F})$. Note that $\mathrm{O}_p(G) \le \mathrm{O}_p(\mathcal{F})$.

It is well-known that a p'-automorphism of $Q \le P$ acts nontrivially on $Q/\Phi(Q)$. If Q is radical, it follows that $\mathrm{Out}_{\mathcal{F}}(Q)$ acts faithfully on $Q/\Phi(Q)$. Now assume that there exists a series of characteristic subgroups $\Phi(Q) = Q_0 < \cdots < Q_n = Q$ of Q. Then $\mathrm{Out}_{\mathcal{F}}(Q)$ acts faithfully on $Q_n/Q_{n-1} \times \cdots \times Q_1/Q_0$ by [Gorenstein 1980, 5.3.2]. This argument will often be applied in the following to exclude same candidates of essential subgroups.

We say that a p-group P of order p^n has *maximal class* if the nilpotency class is $n-1$. This may include the case $|P| = p^2$. The 2-groups of maximal class are the dihedral groups (including C_2^2), the semidihedral groups, the (generalized) quaternion groups and C_4 (see [Huppert 1967, Satz III.11.9]). Now assume that $n \ge 4$ and $p > 2$ to avoid some degenerate cases. Let $\mathrm{K}_2(P) = P'$ and $\mathrm{K}_{i+1}(P) = [P, \mathrm{K}_i(P)]$ for $i \ge 2$. Let $\mathbb{Z}_0(P) := 1$ and $\mathbb{Z}_{i+1}(P/\mathbb{Z}_i(P)) := \mathrm{Z}(P/\mathbb{Z}_i(P))$ for $i \ge 0$. Then $\mathrm{K}_i(P) = \mathbb{Z}_{n-i}(P)$ is the only normal subgroup of P of index p^i by [Huppert 1967, Hilfssatz III.14.2]. It is easy to see that the characteristic subgroups $P_1 := \mathrm{C}_P(\mathrm{K}_2(P)/\mathrm{K}_4(P))$ and $P_2 := \mathrm{C}_P(\mathbb{Z}_2(P))$ are maximal in P.

Lemma 7. *Let P be a p-group with a nonabelian subgroup $Q \le P$ of order p^3 and exponent p. If $\mathrm{C}_P(Q) = \mathrm{Z}(Q)$, then $\mathbb{Z}_2(P) \le Q$.*

Proof. Since $\mathrm{Z}(P) \le \mathrm{C}_P(Q)$, we have $Z := \mathrm{Z}(P) = \mathrm{Z}(Q) \cong C_p$. Let $xZ \in \mathrm{C}_{P/Z}(Q/Z)$. Then $x \in \mathrm{N}_P(Q)$. By [Winter 1972], $\mathrm{N}_P(Q)/Q \le \mathrm{Out}(Q) \cong \mathrm{GL}(2, p)$. As mentioned above, the kernel of the action of $\mathrm{Aut}(Q)$ on Q/Z is a p-group. Since $\mathrm{O}_p(\mathrm{GL}(2, p)) = 1$, we obtain $x \in Q$. Hence, $\mathbb{Z}_2(P)/Z = \mathrm{Z}(P/Z) \le \mathrm{C}_{P/Z}(Q/Z) = Q/Z$ and $\mathbb{Z}_2(P) \le Q$. □

Lemma 8. *Let G be a finite group with Sylow p-subgroup P of maximal class. Let $N \trianglelefteq G$ such that $p^2 \le |N|_p < |P|$. Then there exists $x \in P$ such that $|\mathrm{C}_G(x)|_p = p^2$.*

Proof. By hypothesis, $|P| \ge p|N|_p \ge p^3$. In particular, $\mathrm{Z}(P)$ is the unique normal subgroup of order p of P. Since $M := P \cap N \trianglelefteq P$, we have $\mathrm{Z}(P) \le N$. If $|P| = p^3$, every element $x \in P \setminus N$ cannot be conjugate to an element of $\mathrm{Z}(P) \le N$. Hence, $|\mathrm{C}_G(x)|_p = p^2$. Now assume that $|P| \ge p^4$. If $p = 2$, P is a dihedral, semidihedral or quaternion group and we choose $x \in P$ outside the cyclic maximal subgroup of P. For $p > 2$, let $x \in P \setminus (P_1 \cup P_2)$. By [Huppert 1967, Hilfssatz III.14.13], we have $|\mathrm{C}_P(x)| = p^2$. Since $|P| \ge p^4$, $\mathbb{Z}_2(P)$ is the unique normal subgroup of order p^2 in P. In particular, $\mathbb{Z}_2(P) \le M$ since $|M| \ge p^2$. If $p = 2$, we may assume that $x \notin M$. For $p > 2$, we have $P_1 \cup P_2 \cup M \subsetneq P$. Again we may choose $x \notin M$.

Let $\mathcal{F}$ be the fusion system of G on P. If x is not contained in any essential subgroup, then $\langle x \rangle$ is fully normalized as explained above. It follows that $|\mathrm{C}_G(x)|_p = |\mathrm{C}_P(x)| = p^2$ and we are done. Now let $Q < P$ be essential containing x. By [Grazian and Parker 2022, Theorem D], Q is a so-called pearl, i.e., Q is elementary abelian of order p^2 or nonabelian of order p^3 and exponent p (or $Q = Q_8$ if $p = 2$, see [Grazian and Parker 2022, Lemma 6.1]). As an essential subgroup, Q is centric and $\mathrm{C}_P(Q) = \mathrm{Z}(Q)$. Assume first that $|Q| = p^2$. Then

$$Z := \mathrm{Z}(P) = M \cap Q = N \cap Q \trianglelefteq \mathrm{N}_G(Q).$$

Since Q is radical, $\mathrm{Out}_{\mathcal{F}}(Q) \cong \mathrm{N}_G(Q)/Q$ acts faithfully on $Z \times Q/Z \cong C_p^2$. But then $\mathrm{Out}_{\mathcal{F}}(Q)$ would be a p'-group in contradiction to $Q < \mathrm{N}_P(Q)$. Next let $|Q| = p^3$. Here, Lemma 7 shows that $\mathbb{Z}_2(P) = M \cap Q = N \cap Q \trianglelefteq \mathrm{N}_G(Q)$. Then $\mathrm{Out}_{\mathcal{F}}(Q)$ acts faithfully on $\mathbb{Z}_2(P)/Z \times Q/\mathbb{Z}_2(P) \cong C_p^2$ and we derive another contradiction. □

Theorem B. *Let G be a finite group with a Sylow p-subgroup P of maximal class. Suppose that $\mathrm{O}_{p'}(G) = 1$ and $\mathrm{O}^{p'}(G) = G$. Then one of the following holds*:

(i) *There exists $x \in P$ such that $|\mathrm{C}_G(x)|_p = p^2$.*

(ii) *G is quasisimple and $|\mathrm{Z}(G)| \leq p$.*

Proof. We may assume that G is not simple and $|P| \geq p^3$. Let $N < G$ be a maximal normal subgroup. Then $1 < |N|_p < |P|$ as $\mathrm{O}_{p'}(G) = 1$ and $\mathrm{O}^{p'}(G) = G$. If $|N|_p \geq p^2$, then the claim follows from Lemma 8. Hence, let $|N|_p = p$. Then $P \cap N \trianglelefteq P$ has index $p^s \geq p^2$ and therefore $P \cap N = \mathrm{K}_s(P) \leq P'$. By Tate's theorem (see [Huppert 1967, Satz IV.4.7]), N has a normal p-complement. Since $\mathrm{O}_{p'}(G) = 1$, this forces $|N| = p$. Since $|G : \mathrm{C}_G(N)|$ divides $p - 1$, we further have $N \leq \mathrm{Z}(G)$. Since G/N is simple, G is quasisimple with $|\mathrm{Z}(G)| \leq p$. □

If Case (ii) in Theorem B applies with $|\mathrm{Z}(G)| = p$ and (i) fails, then Robinson's ordinary weight conjecture predicts the existence of an irreducible character χ in the principal p-block such that $p^2\chi(1)_p = |G|_p$ (see [Robinson 2008, Lemma 4.7]). Conversely, such a character can only appear when P has maximal class. Examples are $\mathrm{SL}(2, 9)$ for $p = 2$, $\mathrm{SL}(3, 19)$ for $p = 3$ and $\mathrm{SL}(p, q)$ for $p \geq 5$ where $q - 1$ is divisible by p just once. Our proof of Theorem A does however not rely on any conjecture.

Theorem A. *The character table of a finite group G determines whether G has Sylow p-subgroups of maximal class.*

Proof. Let P be a Sylow p-subgroup of G. We may assume that $\mathrm{O}_{p'}(G) = 1$ and $|P| \geq p^3$. Let $K := \mathrm{O}^{p'}(G)$. The character table detects elements $x \in P$ such that $|\mathrm{C}_G(x)|_p = |\mathrm{C}_K(x)|_p = p^2$. In this case $|\mathrm{C}_P(x)| = p^2$ and P has maximal class by [Huppert 1967, Satz III.14.23]. Hence, by Theorem B we may assume that K is quasisimple with $|\mathrm{Z}(K)| \leq p$. Note that the character table of G determines the isomorphism type of the simple chief factor $K/\mathrm{Z}(K)$ (see [Navarro and Sambale 2023, Theorem 4.1]). In this way we confirm that the Sylow p-subgroup $P/\mathrm{Z}(K)$ of $K/\mathrm{Z}(K)$ has maximal class. If $\mathrm{Z}(K) = 1$, then we are done. Otherwise, P

has maximal class if and only if $\mathrm{Z}(K) = \mathrm{Z}(P)$. This happens if and only if $|\mathrm{C}_G(x)|_p < |P|$ for all $x \in P \setminus \mathrm{Z}(K)$. □

4. Minimal nonabelian Sylow subgroups

The following elementary lemma underlines the importance of minimal nonabelian groups. For elements x, y, z of a group we use the commutator convention $[x, y, z] := [x, [y, z]]$.

Lemma 9. *For a p-group P the following assertions are equivalent*:

(1) *P is minimal nonabelian.*

(2) $|P : \Phi(P)| = |P : \mathrm{Z}(P)| = p^2$.

(3) $|P : \Phi(P)| = p^2$ *and* $|P'| = p$.

Proof. (1) ⇒ (2): Since P is nonabelian, there exist noncommuting elements $x, y \in P$. Since $\langle x, y\rangle$ is nonabelian, we have $P = \langle x, y\rangle$. By Burnside's basis theorem, $|P : \Phi(P)| = p^2$. Choose distinct maximal subgroups $S, T < P$. Since S and T are abelian and $P = ST$, it follows that $\Phi(P) = S \cap T \subseteq \mathrm{Z}(P)$. It is well-known that $P/\mathrm{Z}(P)$ cannot be a nontrivial cyclic group. In particular, $|P : \mathrm{Z}(P)| \geq p^2$ and $\Phi(P) = \mathrm{Z}(P)$.

(2) ⇒ (3): Let $\mathrm{Z}(P) < S < P$. Since $S/\mathrm{Z}(P)$ is cyclic and $\mathrm{Z}(P) \leq \mathrm{Z}(S)$, we obtain that S is abelian. Pick $x \in P \setminus S$. Then Lemma 1 yields that $|P'| = |S : \mathrm{Z}(P)| = p$.

(3) ⇒ (1): Obviously, P is nonabelian since $P' \neq 1$. For $g, x \in P$ we have $gxg^{-1} = [g, x]x \in P'x$. Thus, every conjugacy class lies in a coset of P'. The hypothesis $|P'| = p$ implies $|P : \mathrm{C}_P(x)| \leq p$ for every $x \in P$. Since $\Phi(P)$ is the intersection of the maximal subgroups of P, we deduce $\Phi(P) \leq \bigcap_{x \in P} \mathrm{C}_P(x) = \mathrm{Z}(P)$. Now for every maximal subgroup $S < P$, we see that $S/\mathrm{Z}(S)$ is cyclic and S must be abelian. We conclude that P is minimal nonabelian. □

The nonnilpotent, minimal nonabelian groups were classified by Miller and Moreno [1903]. The nilpotent ones are p-groups and have been determined by Rédei [1947]. For the convenience of the reader we give a proof.

Lemma 10 (Rédei). *Every minimal nonabelian p-group belongs to one of the following classes*:

(i) $\Gamma(a, b) := \langle x, y \mid x^{p^a} = y^{p^b} = 1,\ yxy^{-1} = x^{1+p^{a-1}}\rangle$ *a metacyclic group where* $a \geq 2$ *and* $b \geq 1$,

(ii) $\Delta(a, b) := \langle x, y \mid x^{p^a} = y^{p^b} = [x, y]^p = [x, x, y] = [y, x, y] = 1\rangle$ *where* $a \geq b \geq 1$,

(iii) Q_8.

Proof. Let P be minimal nonabelian. By Lemma 9, there exist $x, y \in P$ such that $P/P' = \langle xP' \rangle \times \langle yP' \rangle \cong C_{p^a} \times C_{p^b}$. Since $|P'| = p$, we have $P' = \langle z \rangle$ where $z := [x, y]$. Note that $P' \leq \Phi(P) = \mathrm{Z}(P)$ and $[x, z] = [y, z] = 1$. We distinguish three cases:

Case 1: $x^{p^a} = y^{p^b} = 1$. Here P fulfills the same relations as $\Delta(a, b)$, so it must be a quotient of the latter group. Moreover, every element of P can be written uniquely in the form $x^i y^j z^k$ with $1 \leq i \leq p^a$, $1 \leq j \leq p^b$ and $1 \leq k \leq p$. Consequently, $|P| = p^{a+b+1}$. For the same reason we have $|\Delta(a, b)| \leq p^{a+b+1}$. Therefore, $P \cong \Delta(a, b)$.

Case 2: Either $x^{p^a} = 1$ or $y^{p^b} = 1$. Without loss of generality, let $x^{p^a} \neq 1$ and $y^{p^b} = 1$. Then $P' \leq \langle x \rangle \trianglelefteq P$ and $yxy^{-1} = x^k$ for some $k \in \mathbb{Z}$. Since $\langle x^p, y \rangle < P$ is abelian, $x^p = yx^p y^{-1} = x^{kp}$ and $p \equiv kp \pmod{p^{a+1}}$ as $|\langle x \rangle| = p^{a+1}$. Hence, we may assume that $k = 1 + p^a l$ for some $0 < l < p$. Let $0 < l' < p$ such that $ll' \equiv 1 \pmod p$. Then $y^{l'} x y^{-l'} = x^{(1+p^a l)^{l'}} = x^{1+p^a}$. Thus, after replacing y by $y^{l'}$, we obtain $yxy^{-1} = x^{1+p^a}$. Now P satisfies the relations of $\Gamma(a+1, b)$. It is clear that these groups have the same order, so $P \cong \Gamma(a+1, b)$.

Case 3: $x^{p^a} \neq 1 \neq y^{p^b}$. Without loss of generality, let $a \geq b$. Let $x^{p^a} = z^i$ and $y^{p^b} = z^j$ where $0 < i, j < p$. Then $(x^j)^{p^a} = z^{ij}$, $(y^i)^{p^b} = z^{ij}$ and $[x^j, y^i] = z^{ij}$ by [Huppert 1967, Hilfssatz III.1.3] (using $z \in \mathrm{Z}(P)$). Hence, replacing x by x^j and y by y^i, we may assume that $x^{p^a} = z = y^{p^b}$. Again by [Huppert 1967, Hilfssatz III.1.3],

$$(x^{p^{a-b}} y^{-1})^{p^b} = x^{p^a} y^{-p^b} [y^{-1}, x^{p^{a-b}}]^{\binom{p^b}{2}} = z^{p^{a-b}\binom{p^b}{2}} = 1$$

unless $p^b = p^a = 2$. In this exceptional case, $P \cong Q_8$. Otherwise, we replace y by $x^{p^{a-b}} y^{-1}$. Afterwards we still have $P/P' = \langle xP' \rangle \times \langle yP' \rangle$, but now $y^{p^b} = 1$. Thus, we are in Case (2). □

The metacyclic groups $\Gamma(a, b)$ can of course be constructed as semidirect products, while the groups $\Delta(a, b)$ can be constructed as subgroups of $\Gamma(a, b) \times C_{p^a}$. For $p = 2$, note that $\Gamma(2, 1) \cong D_8 \cong \Delta(1, 1)$. Apart from that, the groups in Lemma 10 are pairwise nonisomorphic (for different parameters a, b).

We digress slightly to present a counterexample to a related question. Since for p-groups P in general we have $\Phi(P) = P'\mho(P)$ where $\mho(P) = \langle x^p : x \in P \rangle$, one might wonder if $X(G)$ determines the property $|P : \mho(P)| = p^2$. For $p = 2$, it is well-known that $\mho(P) = \Phi(P)$, so the answer is yes in this case. For $p > 2$, $|P : \mho(P)| = p^2$ holds if and only if P is metacyclic (see [Huppert 1967, Satz III.11.4]). The following example shows that this is not even determined by $X(P)$.

Proposition 11. *For $a \geq 2$ and all primes p the groups $\Gamma(2, a)$ and $\Delta(a, 1)$ have the same character table.*

Proof. We denote the generators of $P := \Gamma(2, a)$ by x, y and those of $\widetilde{P} := \Delta(a, 1)$ by $\tilde{x}$, $\tilde{y}$ as in Lemma 10. Additionally, let $\tilde{z} := [\tilde{x}, \tilde{y}]$. We consider the maximal subgroups $Q := \langle x^p, y\rangle \le P$ and $\widetilde{Q} := \langle \tilde{x}, \tilde{z}\rangle \le \widetilde{P}$. Since $xyx^{-1} = x^{-p}y$ and $\tilde{y}\tilde{x}\tilde{y}^{-1} = \tilde{z}^{-1}\tilde{y}$, the map

$$Q \to \widetilde{Q}, \quad x^p \mapsto z,\ y \mapsto \tilde{x}$$

is an isomorphism compatible with the action of P and $\widetilde{P}$. The irreducible characters of P of degree p are induced from linear characters of Q, which are not P-invariant. Since these characters vanish outside Q, they correspond naturally to irreducible characters of $\widetilde{P}$. On the other hand, the linear characters of P are extensions of characters of Q with x^p in their kernel. For $\lambda \in \mathrm{Irr}(Q/P')$ the extensions $\hat{\lambda}$ are determined by $\hat{\lambda}(x) = \zeta$ where ζ is a p-th root of unity. Similarly, for $\lambda \in \mathrm{Irr}(\widetilde{Q}/\widetilde{P}')$ the extensions are determined by $\hat{\lambda}(\tilde{y}) = \zeta$. Therefore, the bijection $P \to \widetilde{P}$, $x^{i+jp}y^k \mapsto \tilde{x}^k\tilde{y}^i\tilde{z}^j$ where $0 \le i, j < p$ and $0 \le k < p^a$ induces the equality of the matrices $X(P)$ and $X(\widetilde{P})$. □

The second author has investigated fusion systems on minimal nonabelian 2-groups in order to classify blocks with such defect groups (see e.g., [Sambale 2016]). We now determine the fusion systems for odd primes too (partial results were obtained in [Yang and Gao 2011]). It turns out that they all come from finite groups unless $|P| = 7^3$. We make use of the Frobenius group $M_9 \cong \mathrm{PSU}(3, 2) \cong C_3^2 \rtimes Q_8$ with $\mathrm{Out}(M_9) \cong S_3$.

Theorem 12. *Let $\mathcal{F}$ be a saturated fusion system on a minimal nonabelian p-group P. Then one of the following holds*:

(i) $P \in \{D_8, Q_8\}$ *and* $\mathcal{F} = \mathcal{F}_P(G)$ *where* $G \in \{P, S_4, \mathrm{GL}(3, 2), \mathrm{SL}(2, 3)\}$.

(ii) $|P| = p^3$, $\exp(P) = p > 2$ *and the possibilities for $\mathcal{F}$ are given in* [Ruiz and Viruel 2004].

(iii) $P \cong \Gamma(a, b)$, $a \ge 2, b \ge 1$ *and* $\mathcal{F} = \mathcal{F}_P(C_{p^a} \rtimes C_{p^b d})$ *for some* $d \mid p - 1$.

(iv) $P \cong \Delta(a, b)$, $a > b$ *and* $\mathcal{F} = \mathcal{F}_P(P \rtimes Q)$ *where* $Q \le C_{p-1}^2$.

(v) $P \cong \Delta(a, a)$, $a \ge 2$ *and* $\mathcal{F} = \mathcal{F}_P(P \rtimes Q)$ *for some p'-group* $Q \le \mathrm{GL}(2, p)$.

(vi) $p = 2$, $P \cong \Delta(a, 1)$, $a \ge 2$ *and* $\mathcal{F} = \mathcal{F}_P(A_4 \rtimes C_{2^a})$ *where C_{2^a} acts as a transposition in* $\mathrm{Aut}(A_4) = S_4$.

(vii) $p = 3$, $P \cong \Delta(a, 1)$, $a \ge 2$ *and* $\mathcal{F} = \mathcal{F}_P(G)$ *where* $G \in \{M_9 \rtimes C_{3^a}, M_9 \rtimes D_{2\cdot 3^a}\}$. *Here the image of C_{3^a} and $D_{2\cdot 3^a}$ in* $\mathrm{Out}(M_9)$ *is C_3 and S_3 respectively.*

Proof. The case $P \in \{D_8, Q_8\}$ is well-known and can be found in [Craven and Glesser 2012, Theorem 5.3], for instance. If $p = 2$ and $P = \Gamma(a, b)$ with $|P| \ge 16$, then $\mathcal{F}$ is trivial, i.e., $\mathcal{F} = \mathcal{F}_P(P)$ by [Craven and Glesser 2012, Theorem 3.7]. Then (iii) holds. Now suppose that $p > 2$ and $P = \Gamma(a, b)$. Then $\mathcal{F}$ is controlled, i.e., $\mathcal{F} = \mathcal{F}_P(P \rtimes Q)$ for some p'-group $Q \le \mathrm{Aut}(P)$ by [Stancu 2006] (see also [Craven and Glesser 2012, Theorem 3.10]). By [Sasaki 1997, Lemma 2.4], $\mathrm{Aut}(P) = A \rtimes \langle\sigma\rangle$ where A is a p-group, $|\langle\sigma\rangle| = p-1$, $\sigma(x) \in \langle x\rangle$ and $\sigma(y) = y$. Hence, Q is conjugate to a subgroup of $\langle\sigma\rangle$. After renaming the generators of P, we may assume that $Q \le \langle\sigma\rangle$. Now (iii) holds.

Next assume that $P \cong \Delta(a, b)$ for some $a \ge b \ge 1$. If $a = 1$ and $p > 2$, then $|P| = p^3$ and $\exp(P) = p$, so (ii) holds. Hence, let $a \ge 2$. Set $z := [x, y] \in P$. Since the p'-group $\mathrm{Out}_{\mathcal{F}}(P)$ acts faithfully on $P/\Phi(P) \cong C_p^2$, we have $\mathrm{Out}_{\mathcal{F}}(P) \le \mathrm{GL}(2, p)$. If $a > b$, then $\mathrm{Out}_{\mathcal{F}}(P)$ acts on $P/\Omega_{a-1}(P) \times \Omega_{a-1}(P)/\Phi(P)$ where $\Omega_{a-1}(P) = \langle g \in P : g^{p^{a-1}} = 1\rangle = \langle x^p, y, z\rangle$. In this case $\mathrm{Out}_{\mathcal{F}}(P) \le C_{p-1}^2$. If $\mathcal{F}$ is controlled, then we are in Case (iv) or (v). Hence, we may assume that $\mathcal{F}$ is not controlled. Then there exists an essential subgroup $Q \le P$. Since Q is centric and $\Phi(P) = \mathrm{Z}(P) \le \mathrm{C}_P(Q) \le Q$, Q is a maximal subgroup. Those are given by

$$\langle xy^i, y^p, z\rangle \cong C_{p^a} \times C_{p^{b-1}} \times C_p, \quad i = 0, \dots, p-1,$$
$$\langle x^p, y, z\rangle \cong C_{p^{a-1}} \times C_{p^b} \times C_p.$$

By [Gorenstein 1980, Theorem 5.2.4], $A := \mathrm{Aut}_{\mathcal{F}}(Q)$ acts faithfully on $\Omega(Q) = \{g \in Q : g^p = 1\}$. Since $P/Q \le A$, this implies $\Omega(Q) \nsubseteq \mathrm{Z}(P)$ and $Q = \langle x^p, y, z\rangle$ with $b = 1$. Now Q is the only maximal subgroup of P isomorphic to $C_{p^{a-1}} \times C_p^2$. In particular, Q is characteristic in P. By Alperin's fusion theorem, $\mathcal{F}$ is constrained with $\mathrm{O}_p(\mathcal{F}) = Q$. By the model theorem, there exists a unique p-constrained group H with $P \in \mathrm{Syl}_p(H)$, $\mathrm{O}_{p'}(H) = 1$ and $\mathcal{F} = \mathcal{F}_P(H)$. We will construct H in the following.

By [Oliver 2014, Lemma 1.11], there exists an A-invariant decomposition $Q = Q_1 \times Q_2$ with $Q_1 \cong C_p^2$ and $Q_2 \cong C_{p^{a-1}}$. Moreover, $\mathrm{O}^{p'}(A) \cong \mathrm{SL}(2, p)$ acts faithfully on Q_1 and trivially on Q_2. Since $P/Q \le \mathrm{O}^{p'}(A)$, it follows that $Q_2 \le \mathrm{Z}(P) = \langle x^p, z\rangle$. Moreover, $xyx^{-1} = yz$ implies $z \in Q_1$. Let $\alpha \in A$ be a p'-automorphism acting trivially on Q_1. Then α commutes with the action of P/Q. Since Q is receptive (see [Aschbacher et al. 2011, Definition I.2.2]), α extends to an automorphism of P. Suppose that $\alpha \ne 1$. Since $Q_2 \le \mathrm{Z}(P) = \Phi(P)$, α must act nontrivially on P/Q_2. Note that P/Q_2 is nonabelian of order p^3 as $z \in Q_1$. An analysis of $\mathrm{Aut}(P/Q_2)$ reveals that α cannot act trivially on $Q/Q_2 \cong Q_1$. Hence, $\alpha = 1$ and A acts faithfully on Q_1. In particular, $A \le \mathrm{GL}(2, p)$. If $p = 2$, then

$$A \cong \mathrm{SL}(2, 2) = \mathrm{GL}(2, 2) \cong S_3.$$

It is easy to see that (vi) holds here. If $p = 3$, then $\mathrm{SL}(2, 3) \cong Q_8 \rtimes C_3$, $\mathrm{GL}(2, 3) \cong Q_8 \rtimes S_3$ and (vii) is satisfied. Thus, let $p \ge 5$. Then the Sylow normalizer in $\mathrm{SL}(2, p)$ acts nontrivially on a Sylow p-subgroup of $\mathrm{SL}(2, p)$. Hence, there exists $\alpha \in \mathrm{O}^{p'}(A)$

acting nontrivially P/Q. But then α acts nontrivially on $\langle x^p\rangle Q_1/Q_1 = Q/Q_1 \cong Q_2$. This contradicts [Oliver 2014, Lemma 1.11]. ☐

The groups $A_4 \rtimes C_4$, $M_9 \rtimes C_9$ and $M_9 \rtimes D_{18}$ can be constructed in GAP [2020] as `SmallGroup`(n,k) where $(n,k) \in \{(48,39), (648,534), (6^4, 2892)\}$ respectively.

Corollary 13. *Let $\mathcal{F}$ be a fusion system on a minimal nonabelian p-group P with $|P| \geq p^4$. Then $\mathcal{F}$ is constrained. If $p \geq 5$, then $\mathcal{F}$ is controlled.*

We now gather some information on simple groups in order to prove Theorem C. As customary, if q is a prime power, let

$$\mathrm{PSL}^{\epsilon}(n,q) := \begin{cases} \mathrm{PSL}(n,q) & \text{if } \epsilon = 1, \\ \mathrm{PSU}(n,q) & \text{if } \epsilon = -1. \end{cases}$$

The following is certainly known, but included for convenience.

Lemma 14. *Let q be a prime power. Let $S = \mathrm{PSL}^{\epsilon}(n,q)$ with a cyclic Sylow p-subgroup and $n \geq 3$. Then there exists a unique integer $2 \leq d \leq n$ such that p divides $q^d - \epsilon^d$.*

Proof. Since a nonabelian simple group cannot have cyclic Sylow 2-subgroups, we have $p > 2$. If $p \mid q$, then a Sylow p-subgroup of S is given by the set of unitriangular matrices. This subgroup is nonabelian since $n \geq 3$. Now let $p \nmid q$. If $q \equiv \epsilon \pmod{p}$, then S contains a subgroup of diagonal matrices isomorphic to C_p^2. Hence, let $q \not\equiv \epsilon \pmod{p}$. In the following we write $q^* := q$ if $\epsilon = 1$ and $q^* := q^2$ if $\epsilon = -1$. Let $x \in S$ be a generator of a Sylow p-subgroup of S. We identify x with a preimage in $\mathrm{GL}(n,q^*)$. We may assume that x has order p^k. Let e be the order of q^* modulo p^k. Then x has an eigenvalue $\zeta \in \mathbb{F}_{(q^*)^e}^{\times}$ of order p^k. Since $\mathrm{tr}(x) \in \mathbb{F}_{q^*}$, the elements $\zeta^{(q^*)^i}$ for $i = 0, \dots, e-1$ are distinct eigenvalues of x. In particular, $e \leq n$. If $\epsilon = 1$, then $e \geq 2$ we can choose $d := e$ in the statement. If $2e \leq n$, we obtain $q^d \equiv 1 \equiv \epsilon^d \pmod{p}$ for $d := 2e$.

Now suppose that $\epsilon = -1$ and $2e > n$. Since x is a unitary matrix, we have $\bar{x}x^{\mathrm{t}} = 1$ where $\bar{x} = (x_{ij}^q)_{i,j}$ and x^{t} is the transpose of x. It follows that ζ^{-q} is an eigenvalue of x. Since $n < 2e$, there must be some i with $\zeta^{q^{2i}} = \zeta^{-q}$. This shows that $q^{2i-1} \equiv -1 \equiv \epsilon^{2i-1} \pmod{p^k}$. Since $q^{2(2i-1)} \equiv 1 \pmod{p^k}$, we have

$$e \mid 2i-1 \leq 2(e-1)-1 < 2e \quad \text{and} \quad e = 2i-1.$$

Hence, we can set $d := e$.

For the uniqueness of d, we note that

$$|S| = \frac{q^{n(n-1)/2}}{\gcd(n, q-\epsilon)} \prod_{i=2}^{n} (q^i - \epsilon^i),$$

is not divisible by p^{k+1}, since $p^k = |\langle x\rangle| = |S|_p$. ☐

Lemma 15. *Let S be a finite simple group with Sylow 3-subgroup C_3^2 and outer automorphism of order 3. Then $S \cong \mathrm{PSL}^{\epsilon}(3, p^f)$ where $\epsilon = \pm 1$, p is a prime and $(p^f - \epsilon)_3 = 3$. Moreover, $\mathrm{Out}(S) \cong C_3 \rtimes (C_f \times C_2)$.*

Proof. The simple groups with Sylow 3-subgroup C_3^2 were classified in [Koshitani and Miyachi 2001, Proposition 1.2]. The alternating groups and sporadic groups do not have outer automorphisms of order 3. Now let S be a classical group of dimension d over $\mathbb{F}_{p^f}$. Then $p^f \not\equiv \pm 1 \pmod 9$. This implies $3 \nmid f$ and S does not have field automorphisms of order 3. According to [Conway et al. 1985, Table 5], there must be a diagonal automorphism of order 3. This forces $d = 3$ and $S = \mathrm{PSL}^{\epsilon}(3, p^f)$ such that $(p^f - \epsilon)_3 = 3$. If $\epsilon = 1$, then $\mathrm{Out}(S) = C_3 \rtimes (C_f \times C_2)$ as desired. If $\epsilon = -1$, then there is no graph automorphism and instead we have a field automorphism of order $2f$. However, since $p^f \equiv 2, 5 \pmod 9$, f must be odd and $C_{2f} \cong C_f \times C_2$. $\square$

Theorem C. *Let G be a finite group with a minimal nonabelian Sylow p-subgroup P and $\mathrm{O}_{p'}(G) = 1$. Then one of the following holds*:

(i) $p = 2$, $P \in \{D_8, Q_8\}$ *and* $\mathrm{O}^{2'}(G) \in \{\mathrm{SL}(2, q), \mathrm{PSL}(2, q'), A_7\}$ *where* $q \equiv \pm 3 \pmod 8$ *and* $q' \equiv \pm 7 \pmod{16}$.

(ii) $|P| = p^3$ *and* $\exp(P) = p > 2$.

(iii) $G = P \rtimes Q$ *where* $Q \leq \mathrm{GL}(2, p)$.

(iv) $p > 2$, $\mathrm{O}^{p'}(G) = S \rtimes C_{p^a}$ *where S is a simple group of Lie type with cyclic Sylow p-subgroups. The image of* C_{p^a} *in* $\mathrm{Out}(S)$ *has order* p.

(v) $p = 2$ *and* $G = \mathrm{PSL}(2, q^f) \rtimes C_{2^a d}$ *where q is a prime,* $q^f \equiv \pm 3 \pmod 8$ *and* $d \mid f$. *Moreover,* C_{2^a} *acts as a diagonal automorphism of order 2 on* $\mathrm{PSL}(2, q^f)$ *and* C_d *induces a field automorphism of order* d.

(vi) $p = 3$ *and* $\mathrm{O}^{3'}(G) = \mathrm{PSL}^{\epsilon}(3, q^f) \rtimes C_{3^a}$ *where* $\epsilon = \pm 1$, *q is prime,* $(q^f - \epsilon)_3 = 3$ *and* $G/\mathrm{O}^{3'}(G) \leq C_f \times C_2$.

Proof. By Lemma 9, $|P : \mathrm{Z}(P)| = p^2$ and G is described in [Navarro and Sambale 2023, Theorem 7.5]. We go through the various cases in the notation used there:

In Case (A), using that P is 2-generated and $\mathrm{O}_p(G)$ is not cyclic, we deduce that $S = 1$. Here $P = \mathrm{F}^*(G) \trianglelefteq G$ and $\mathrm{C}_G(P) \leq P$. Since G/P acts faithfully on $P/\Phi(P) \cong C_p^2$, we have $G/P \leq \mathrm{GL}(2, p)$ and (iii) holds. Assume now that $P < G$. In Case (B), the quasisimple group C has a nonabelian Sylow p-subgroup of order p^3 which must coincide with P. If $P = D_8$, then (i) or (v) holds by the Gorenstein–Walter theorem (there are no field automorphisms of order 2) [Gorenstein 1980, p. 462]. If $P = Q_8$, the claim follows from the Brauer–Suzuki theorem [Gorenstein 1980, Theorem 12.1.1] and Walter's theorem [Gorenstein 1980, p. 485]. If $p > 2$, then we must have $\exp(P) = p$, since otherwise the focal subgroup theorem [Isaacs

2008, Theorem 5.21] and Theorem 12 lead to the contradiction $|P| = |P : P \cap G'| \geq p$. Thus, (ii) holds. Case (D) is impossible, since then P has a nonabelian maximal subgroup.

Now consider Case (C), i.e., $\mathrm{F}^*(G) = \mathrm{O}_p(G) \times S$ has abelian Sylow p-subgroups, S is a direct product of simple groups and $|G : \mathrm{F}^*(G)|_p = p$. Let $x \in P \setminus \mathrm{F}^*(G)$.

Case 1: $S = 1$. Since $S = 1$, $\mathrm{F}^*(G) = \mathrm{O}_p(G)$ and so

$$\mathrm{C}_G(\mathrm{F}^*(G)) = \mathrm{C}_G(\mathrm{O}_p(G)) \leq \mathrm{O}_p(G).$$

Therefore, $\mathrm{C}_P(\mathrm{O}_p(G)) = \mathrm{O}_p(G)$ and we have $\mathrm{C}_G(\mathrm{O}_p(G)) = \mathrm{O}_p(G) \times K$ where $K \leq \mathrm{O}_{p'}(G) = 1$. Hence, G is p-constrained and $\mathcal{F}_P(G)$ is given by (vi) or (vii) of Theorem 12. By the model theorem, the isomorphism type of G is uniquely determined by $\mathcal{F}_P(G)$. Since $\mathrm{PSL}(2,3) \cong A_4$ and $\mathrm{PSU}(3,2) \cong M_9$, we obtain (v) or (vi).

Case 2: $S \neq 1$ is not simple. By Lemma 10, the maximal subgroups of P are generated by at most three elements. Hence, S is a direct product of two or three simple groups, say $S = T_1 \times T_2$ or $T_1 \times T_2 \times T_3$. Since a Sylow 2-subgroup of a simple group cannot be generated by less than 2 elements, we deduce that $p > 2$ and the T_i have cyclic Sylow p-subgroups. If x does not normalize some T_i, then $p = 3$ and x permutes $T_1 \cong T_2 \cong T_3$. However, $C_{3^n} \wr C_3$ is not minimal nonabelian. Hence, x acts on each T_i. If x acts nontrivially on $\mathrm{O}_p(G)$, then $\mathrm{O}_p(G)\langle x\rangle$ is nonabelian and $P = \mathrm{O}_p(G)\langle x\rangle$. But then S would be simple. Similarly, if x acts nontrivially on $Q_1 := P \cap T_1$, then $P = Q_1\langle x\rangle$. Write $Q_2 := P \cap T_2 = \langle y\rangle$ such that $x^p \in yQ_1$. Then x centralizes y. By [Gross 1982, Theorem B], this implies that x induces an inner automorphism on T_2. However, x^p induces the inner automorphism by y. Hence, x cannot have order greater than $|T_2|_p$. Another contradiction.

Case 3: S is simple. Let $Q := P \cap S \trianglelefteq P$ be a Sylow p-subgroup of S. Arguing as in Case 2, we see that x acts nontrivially on Q and therefore $P = Q\langle x\rangle$. First let Q be cyclic. Then $p > 2$ and P is metacyclic. Since $\mathrm{Out}(S)$ needs to have an element of order p, S must be of Lie type. To obtain (iv), it remains to show that PS is normal in G. Assume the contrary. By the structure of $\mathrm{Out}(S)$ (see [Conway et al. 1985, Table 5]), P induces a field or graph automorphism of order p on S which acts nontrivially on the subgroup of outer diagonal automorphisms of S. In particular, the diagonal automorphism group must have order at least $p+1$, in fact $2p+1 \geq 7$ since $p > 2$. This excludes all families of simple groups except $S = \mathrm{PSL}^\epsilon(d, q^f)$ where $p \mid f$ and $d \geq 2p+1$. Since Q is cyclic and $f > 1$, we have $q \neq p$. By Fermat's little theorem,

$$q^{(p-1)f} \equiv q^{2(p-1)f} \equiv 1 \pmod{p}.$$

This contradicts Lemma 14 (note that $p-1$ is even). Hence, $PS \trianglelefteq G$ and (iv) holds.

Let Q be noncyclic. Recall that in general Q is homocyclic and $\mathrm{N}_S(Q)$ acts irreducibly on $\Omega(Q)$ (see [Flores and Foote 2009, Proposition 2.5]). This implies that P cannot be metacyclic, as otherwise the fusion in P is controlled by $\mathrm{N}_G(P)$ and $\mathrm{N}_G(Q)=\mathrm{N}_G(P)\mathrm{C}_G(Q)$ acts reducibly on Q according to Theorem 12. Hence, let $P\cong\Delta(a,b)$. Then P' is a direct factor of Q and we obtain $Q=\Omega(Q)$. If Q has rank 3, then $P\cong\Delta(2,1)$. However, by Theorem 12, $\mathrm{N}_G(Q)/\mathrm{C}_G(Q)\le\mathrm{GL}(2,p)$ does not act irreducibly on Q. Hence, we may assume that Q has rank 2. Now $P\cong\Delta(a,1)$ with $a\ge 2$. If $\mathrm{N}_G(P)$ controls the fusion in P, then $\mathrm{N}_G(Q)$ would fix P'. Hence, we are in Case (vi) or (vii) of Theorem 12. Consider $p=2$ first. By Walter's theorem (see [Gorenstein 1980, p. 485]), $S\cong\mathrm{PSL}(2,q^f)$ with $q^f\equiv\pm 3 \pmod 8$. It follows that f is odd and $G/PS\le\mathrm{Out}(S)\le C_{2f}$ by [Conway et al. 1985, Table 5]. Here C_2 induces a diagonal automorphism and C_f is caused by a field automorphism. So (v) holds. Finally, let $p=3$. Here the claim follows easily from Lemma 15. □

Examples for Theorem C (iv) can be constructed as follows: Let $p>2$ and $a\ge 2$. By Dirichlet's theorem, there exists a prime $q\equiv 1+p^{a-1} \pmod{p^{a+1}}$. Then $q^p\equiv 1+p^a \pmod{p^{a+1}}$ and $S:=\mathrm{PSL}(2,q^p)$ has a cyclic Sylow p-subgroup Q of order p^a. Let $R\cong C_{p^b}$ and construct $G:=S\rtimes R$ where R acts as the field automorphism $\mathbb{F}_{q^p}\to\mathbb{F}_{q^p}$, $\lambda\mapsto\lambda^q$ on S. By [Gross 1982], R acts nontrivially on Q and $P:=Q\rtimes R\cong\Gamma(a,b)$. A different example is $G=\mathrm{Sz}(2^5)\rtimes C_5$ for $p=5$.

Corollary 16. *Let G be a finite group with a minimal nonabelian Sylow p-subgroup and $\mathrm{O}_{p'}(G)=1$. Then G has at most one nonabelian composition factor.*

Proof. We may assume that G is nonsolvable. If $|G|_p=p^3$, then $\mathrm{F}^*(G)$ is quasisimple and $G/\mathrm{F}^*(G)\le\mathrm{Aut}(\mathrm{F}^*(G))\le\mathrm{Aut}(\mathrm{F}^*(G)/\mathrm{Z}(F^*(G)))$ is solvable by Schreier's conjecture. Otherwise we have $\mathrm{F}^*(G)=S\times C_{p^b}$ for a simple group S and $b\ge 0$ by the proof of Theorem C. Since $\mathrm{Aut}(C_{p^b})$ is abelian, the claim follows again from Schreier's conjecture. □

Corollary D. *The character table of a finite group G determines whether G has minimal nonabelian Sylow p-subgroups.*

Proof. Let P be a Sylow p-subgroup of G. We may assume that $\mathrm{O}_{p'}(G)=1$. By [Navarro and Sambale 2023, Theorem B], the character table determines whether $|P:\mathrm{Z}(P)|=p^2$. Suppose that this is the case. By Lemma 9, it remains to detect whether $|P:\Phi(P)|=p^2$. This is true for $|P|=p^3$, so let $|P|\ge p^4$. By Theorem 4 and Corollary 5, we may assume that $\mathrm{O}_p(G)=1$. Now by Theorem C we expect that $\mathrm{O}^{p'}(G)=S\rtimes C_p$ for a simple group S with a cyclic Sylow p-subgroup Q. As usual, $X(G)$ determines the isomorphism type of S. If Q is indeed cyclic, then clearly P is 2-generated and we are done. □

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