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This paper is devoted to studying geometric and analytic properties of g-starlike mappings of complex order λ. By using Loewner chains, we obtain the growth theorems for g-starlike mappings of complex order λ on the unit ball in reflexive complex Banach spaces, which generalize some results of Graham, Hamada and Kohr. As applications, several different kinds of distortion theorems for g-starlike mappings of complex order λ are obtained. Finally, we prove that the Roper–Suffridge extension operators preserve the property of g-starlike mappings of complex order λ in complex Banach spaces, which generalizes many classical results.

1. Introduction

Let $f = z + \sum_{n=2}^{\infty} a_n z^n$ be a normalized univalent function on the unit disk $\mathbb{D}$ in $\mathbb{C}$. The growth theorem shows that the modulus of a normalized univalent function $|f|$ has a finite upper and positive lower bound depending only on the modulus of the variable $|z|$, and the image of f contains a disk centered at origin with radius $\frac{1}{4}$. The distortion theorem gives explicit upper and lower bounds on $|f'(z)|$ in terms of $|z|$. The term distortion arises from the geometric interpretation of $|f'(z)|$ as the

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infinitesimal magnification factor of arc length and the interpretation of the square of $|f'(z)|$ as the infinitesimal magnification factor of area.

However, in the case of several complex variables, H. Cartan pointed out that the growth theorem and distortion theorem do not hold for normalized biholomorphic mappings. In addition, he suggested that one should investigate the important geometrically defined subfamilies of convex and starlike mappings. As a matter of fact, there was little work in the geometric directions suggested by Cartan, until the 1970s, when a number of results dealing with the convex and starlike biholomorphic mappings appeared. As a direct generalization of the growth theorem for univalent function on the unit disk $\mathbb{D}$, the growth theorem for normalized biholomorphic starlike mappings on the unit ball $\mathbb{B}_n$ was obtained by Barnard, Fitzgerald and Gong [Barnard et al. 1991] using the analytical characterization of starlikeness, and by Kubicka and Poreda [1988] using the method of Loewner chains.

Theorem A [Barnard et al. 1991; Kubicka and Poreda 1988]. *Let f be a starlike mapping on the unit ball $\mathbb{B}_n$. Then, for any point $z \in \mathbb{B}_n$, we have*

$$\frac{\|z\|}{(1+\|z\|)^2} \le \|f(z)\| \le \frac{\|z\|}{(1-\|z\|)^2}.$$

Furthermore, the above estimates are sharp.

If the convexity restriction is attached to the family of normalized locally biholomorphic mapping f, the following growth theorem for convex mappings is due to Suffridge [1977], Thomas [1991], Liu [1989] or Liu and Ren [1998].

Theorem B. *Let f be a convex mapping on the unit ball $\mathbb{B}_n$. Then, for any point $z \in \mathbb{B}_n$, we have*

$$\frac{\|z\|}{1+\|z\|} \le \|f(z)\| \le \frac{\|z\|}{1-\|z\|}.$$

Moreover, the above estimates are sharp.

In several complex variables, Barnard, Fitzgerald and Gong [Barnard et al. 1994] were the first to show that the version of the distortion theorem for the determinant of the Jacobian of normalized biholomorphic convex mappings holds on the unit ball $\mathbb{B}_2$ in $\mathbb{C}^2$, but there does not exist a direct generalization of the distortion theorem in the case of the family of starlike mappings. The monograph of Graham and Kohr [2003, Chapter 7] and Gong [1998, Chapter 3, Chapter 4] contain a nice development of the growth theorem and distortion theorem for starlike mappings and convex mappings. And for a more classical results concerning starlike mappings and convex mappings in n-dimensional Euclidean space or complex Banach space; see [Gurganus 1975; Kikuchi 1973; Pfaltzgraff 1974; Poreda 1989; Roper and Suffridge 1995; Suffridge 1970; 1973; 1977].

Hamada and Honda [2008] introduced a subfamily of starlike mappings on the unit ball in complex Banach spaces, which is called g-starlike mappings. They also obtained a sharp growth theorem for this mappings by using the method of parametric representation. Recently, the distortion theorem for g-starlike mappings on the unit ball $\mathbb{B}_n$ was obtained by Graham, Hamada and Kohr [Graham et al. 2020a] using the Schwarz lemma at the boundary. As a generalization of spirallike mappings, Bălăeţi and Nechita [2010] defined almost starlike mappings of complex order λ on the unit ball $\mathbb{B}_n$ and gave an equivalent characterization in terms of Loewner chains. It is interesting to note that the family of g-starlike mappings gives a unified representation of some well-known subfamilies of starlike mappings, and the family of almost starlike mappings of complex order λ gives a unified expression of some well-known subfamilies of spirallike mappings of type β. There is a lot of results concerning g-starlike mappings and almost starlike mappings of complex order λ; see [Chirilă 2014; 2015; Graham et al. 2002a; 2020b; Hamada and Kohr 2004; Hamada et al. 2006; 2021; Li and Zhang 2019; Zhang et al. 2018].

In view of the above results, the motivation for this paper can be summarized in terms of the following question:

Question. Can we unify g-starlike mappings and complex order λ on the unit ball of complex Banach spaces and characterize their geometric and analytic properties?

We manage to answer the above questions affirmatively in the case of the unit ball of some complex Banach space. In Section 2, the definition of g-starlike mappings of complex order λ is given by combining the definition of g-starlike mappings with the definition of almost starlike mappings of complex order λ on the unit ball in complex Banach spaces. As mentioned in Remark 2.4, it gives a unified expression of a variety of biholomorphic mappings, which includes g-starlike mappings, almost starlike mappings of complex order λ as the special case. In Section 3, by using Loewner chains idea, we establish a growth theorem of g-starlike mappings of complex order λ in reflexive complex Banach spaces, which is a generalization of [Hamada and Honda 2008, Theorem 3.1]. Because the family of g-starlike mappings of complex order λ contains most of the biholomorphic mappings that have geometry meaning in higher dimensions, this result essentially corresponds to giving a unified form of the growth theorems for some subfamilies of starlike mappings and spirallike mappings. As applications, in Section 4, we obtain distortion theorems for g-starlike mappings of complex order λ on the unit polydisk $\mathbb{D}^n$ and the unit ball $\mathbb{B}_n$ respectively, which is a generalization of [Graham et al. 2020a, Theorem 5.6, Theorem 5.11; Liu et al. 2015, Theorem 4.2; 2011, Theorem 3.1, Theorem 3.2]. In Section 5, we will prove that the Roper–Suffridge type extension operator and the Muir type extension operator preserve g-starlike mappings of complex order λ on domain Ω_r respectively, where g is a univalent convex function on $\mathbb{D}$. In particular, if $\lambda = 0$, then the results obtained in this paper are generalizations of results in [Graham et al. 2020b; Muir 2005].

2. Preliminaries

2A. ***Notations and definitions.*** Let $\mathbb{D}_r = \{\zeta \in \mathbb{C} : |\zeta| < r\}$ be the disk of radius r in the complex plane $\mathbb{C}$, and let $\mathbb{D}_1 = \mathbb{D}$. Let $\mathbb{C}^n$ denote the space of n complex variables $u = (u_1, \dots, u_n)'$ equipped with inner product $\langle u, v\rangle = \sum_{k=1}^{n} u_k \overline{v_k}$, and the Euclidean norm $\|u\| = \sqrt{\sum_{k=1}^{n} |u_k|^2}$, the symbol $'$ means the transpose of vectors and matrices. The open ball centered at zero and radius r in $\mathbb{C}^n$ is denoted by $\mathbb{B}_n(0, r) = \{u \in \mathbb{C}^n : \|u\| < r\}$, the closed ball is denoted by $\overline{\mathbb{B}_n(0, r)}$, the unit ball is denoted by $\mathbb{B}_n$. Let $\mathbb{D}^n(0, r) = \{u = (u_1, \dots, u_n)' \in \mathbb{C}^n : |u_k| < r, k = 1, \dots, n\}$ be the polydisk of radius r. The unit polydisk is denoted by $\mathbb{D}^n$. The boundary of $\mathbb{B}_n$ is denoted by $\partial\mathbb{B}_n = \left\{u \in \mathbb{C}^n : \sum_{k=1}^{n} |u_k|^2 = 1\right\}$, the distinguished boundary of the polydisk $\mathbb{D}^n$ is denoted by $(\partial\mathbb{D})^n = \{u \in \mathbb{C}^n : |u_k| = 1, k = 1, \dots, n\}$. Let X be a complex Banach space with respect to the norm $\|\cdot\|_X$. Let $\mathcal{B}_r = \{x \in X : \|x\|_X < r\}$ be the open ball centered at zero and of radius r, and let $\mathcal{B}$ be the open unit ball in X. Let $\overline{\mathcal{B}_r}$ be the closed ball centered at zero and of radius r. Let $\Omega \subseteq X$ be a domain which contains the origin, we denote by $H(\Omega)$ the set of holomorphic mappings from Ω to X. If $f \in H(\Omega)$, and $f(0) = 0$, $Df(0) = I$, then we say that f is normalized, where $Df(0)$ is the Fréchet derivative of f at 0, I is the identity operator on X. A holomorphic mapping $f \in H(\Omega)$ is said to be biholomorphic if the inverse f^{-1} exists and it is holomorphic on the open set $f(\Omega)$. A mapping $f \in H(\Omega)$ is said to be locally biholomorphic if each $x \in \Omega$ has a neighborhood V such that $f|_V$ is biholomorphic. If $X = \mathbb{C}^n$, then $Df(z) = J_f(z)$ is the Jacobian matrix of f.

Let $T : X \to \mathbb{C}$ be a continuous linear functional. Then

$$\|T\| = \sup\{|Tx| : x \in \partial\mathcal{B}\}.$$

For each $x \in X\backslash\{0\}$, we define $T(x) = \{T_x \in X^* : \|T_x\| = 1, \|T_x(x)\| = \|x\|\}$. According to the Hahn–Banach theorem, $T(x)$ is nonempty. For any fixed $x \in X$, $\zeta \in \mathbb{C}\backslash\{0\}$, we have $T_{\zeta x} = (|\zeta|/\zeta)T_x$. In particular, $T_{rx} = T_x$ when $r > 0$.

The following elementary definitions are used:

- If for any $x \in \Omega$, $t \in [0, 1]$, $(1-t)x \in \Omega$ holds, then Ω is said to be starlike (with respect to the origin).

- A domain $\Omega \subseteq X$ is said to be convex if given $x_1, x_2 \in \Omega$, $tx_1 + (1-t)x_2 \in \Omega$, for all $t \in [0, 1]$.

- A domain $\Omega \subseteq X$ is said to be ε-starlike if there exists a positive number $\varepsilon \in [0, 1]$, such that for any $z, w \in \Omega$, one has $(1-t)z + \varepsilon t w \in \Omega$ for all $t \in [0, 1]$.

In particular, if $\varepsilon = 0$ or $\varepsilon = 1$, then the ε-starlike domain reduces to starlike domain with respect to the origin or convex domain, respectively.

• Let $f \in H(\Omega)$ be biholomorphic mapping with $0 \in f(\Omega)$. If $f(\Omega)$ is starlike (with respect to the origin), then f is said to be starlike. If $f(\Omega)$ is convex, then f is said to be convex. If $f(\Omega)$ is ε-starlike, then f is said to be ε-starlike, where $\varepsilon \in [0, 1]$.

• Let $g : \mathbb{D} \to \mathbb{C}$ be a holomorphic univalent function, $g(0) = 1$ and $\Re g(\zeta) > 0$. Furthermore, let g be symmetric along the real axis, i.e., $g(\bar{\zeta}) = \overline{g(\zeta)}$, and satisfy the condition

$$\begin{cases} \min_{|\zeta|=r} \Re g(\zeta) = \min\{g(r), g(-r)\}; \\ \max_{|\zeta|=r} \Re g(\zeta) = \max\{g(r), g(-r)\}. \end{cases}$$

Let $G(\mathbb{D})$ denote the family of holomorphic functions g defined as above.

2B. *Loewner chains.* We next recall the notions of subordination and Loewner chains on the unit ball $\mathcal{B}$ in X. Some results may be found in [Graham et al. 2013; 2020b].

A mapping $v \in H(\mathcal{B})$ is called a Schwarz mapping if $v(0) = 0$ and $\|v(x)\|_X < 1$, $x \in \mathcal{B}$.

If $f, g \in H(\mathcal{B})$, and there exists a Schwarz mapping v such that $f = g \circ v$, then we say that f is subordinate to g, denoted by $f \prec g$.

If g is biholomorphic on $\mathcal{B}$, then $f \prec g$ is equivalent to requiring that $f(0) = g(0)$ and $f(\mathcal{B}) \subseteq g(\mathcal{B})$.

Definition 2.1. Let $\mathcal{B}$ be the unit ball of a complex Banach space X. A mapping $f : \mathcal{B} \times [0, \infty) \to X$ is called a univalent subordination chain if $f(\cdot, t)$ is univalent on $\mathcal{B}$, $f(0, t) = 0$ for $t \geq 0$, and $f(\cdot, s) \prec f(\cdot, t)$ when $0 \leq s \leq t < \infty$. A univalent subordination chain $f : \mathcal{B} \times [0, \infty) \to X$ is called a Loewner chain if $f(\cdot, t)$ is biholomorphic on $\mathcal{B}$ and $Df(0, t) = e^t I$, for all $t \geq 0$.

The subordination condition of Loewner chain is equivalent to the existence of a unique biholomorphic Schwarz mapping $v = v(\cdot, s, t)$, called the transition mapping associated with $f(x, t)$, such that $f(x, s) = f(v(x, s, t), t)$ for $x \in \mathcal{B}$ and $0 \leq s \leq t$.

Let $g \in G(\mathbb{D})$ be defined as above. The family $\mathcal{M}_g(\mathcal{B})$ of holomorphic mappings $h : \mathcal{B} \to X$ that is analogous to the analytic functions on the unit disk in the complex plane, with positive real part, is defined as follows.

$$\mathcal{M}_g(\mathcal{B}) = \left\{ h \in H(\mathcal{B}) : h(0) = 0, Dh(0) = I, \frac{1}{\|x\|_X} T_x\{h(x)\} \in g(\mathbb{D}), T_x \in T(x), x \in \mathcal{B}\setminus\{0\} \right\}.$$

If $g(\zeta) = (1 + \zeta)/(1 - \zeta)$, $\zeta \in \mathbb{D}$, then $\mathcal{M}_g(\mathcal{B})$ reduces to the Carathéodory family $\mathcal{M}(\mathcal{B})$ on the unit ball $\mathcal{B}$ in a complex Banach space.

We know that both $\mathcal{M}(\mathcal{B})$ and $\mathcal{M}_g(\mathcal{B})$ consist of so-called holomorphically accretive mappings, which were intensively studied in Euclidean space $\mathbb{C}^n$ or complex Banach spaces during the last decades. Some related results may be found in [Duren et al. 2010; Elin et al. 2019; Graham et al. 2002a; 2013; Hamada and Kohr 2004; Pfaltzgraff 1974; Reich and Shoikhet 1996; 2005; Suffridge 1973].

Definition 2.2 [Bracci et al. 2009; Duren et al. 2010; Graham et al. 2002a]. A Herglotz vector field associated with the family $\mathcal{M}(\mathcal{B})$ on $\mathcal{B}$ is a mapping $h = h(x, t) : \mathcal{B} \times [0, \infty) \to X$ satisfying the following conditions:

(i) $h(\cdot, t) \in \mathcal{M}(\mathcal{B})$, for a.e. $t \geq 0$.

(ii) $h(x, \cdot)$ is strongly measurable on $[0, \infty)$, for all $x \in \mathcal{B}$.

Hamada and Kohr [2004] proved that if X is a reflexive complex Banach space, and $h(x, t) : \mathcal{B} \times [0, \infty) \to X$ is a Herglotz vector field, then for each $s \geq 0$ and $x \in \mathcal{B}$, the initial value problem

$$\begin{cases} \frac{\partial v}{\partial t} = -h(v, t), & a.e.\ s \leq t, \\ v(x, s, s) = x, & t = s \end{cases}$$

has a unique solution $v = v(x, s, t)$ such that $v(\cdot, s, t)$ is a univalent Schwarz mapping, $v(x, s, \cdot)$ is Lipschitz continuous on $[s, \infty)$ uniformly with respect to $x \in \overline{\mathcal{B}_r}$, $r \in (0, 1)$, $Dv(0, s, t) = e^{s-t} I$ for $0 \leq s \leq t$. Furthermore, the following limit

$$\lim_{t \to \infty} e^t v(x, s, t) = f(x, s)$$

exists uniformly on each closed ball $\overline{\mathcal{B}_r}$ for $r \in (0, 1)$, $s \in [0, \infty)$. And $f(x, t)$ is a univalent subordination chain.

2C. *g-starlike mappings of complex order λ.*

Definition 2.3. Let $g \in G(\mathbb{D})$, $\lambda \in \mathbb{C}$ with $\Re\lambda \leq 0$. And let $f : \mathcal{B} \to X$ be a normalized locally biholomorphic mapping. If

$$\{(1-\lambda)(Df(x))^{-1} f(x) + \lambda x\} \in \mathcal{M}_g(\mathcal{B}),$$

then f is called a g-starlike mapping of complex order λ.

We denote by $\mathcal{S}^*_{g,\lambda}(\mathcal{B})$ the family of g-starlike mapping of complex order λ on $\mathcal{B}$.

Obviously, for the case of $X = \mathbb{C}$, $\mathcal{B} = \mathbb{D}$, the above definition shows that $f \in \mathcal{S}^*_{g,\lambda}(\mathbb{D})$ if and only if $(1-\lambda) f(z)/(zf'(z)) + \lambda \prec g$.

Remark 2.4. (i) Let $\lambda = 0$. Then $f \in \mathcal{S}^*_{g,\lambda}(\mathcal{B})$ is a g-starlike mapping on the unit ball $\mathcal{B}$, some results of g-starlike mappings may be found in [Chirilă 2014; 2015; Graham et al. 2002a; Hamada et al. 2021].

(ii) Let $\alpha \in [0, 1)$, $\beta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $\lambda = (\alpha - i \tan\beta)/(\alpha - 1)$. Then $\mathcal{S}^*_{g,\lambda}(\mathcal{B}) = \widehat{\mathcal{S}}^{\alpha,\beta}_g(\mathcal{B})$, the definition on the unit ball $\mathbb{B}_n$ in Euclidean space can be found in [Tu and Xiong 2019].

(iii) Let $g(\zeta) = (1+\zeta)/(1-\zeta)$, $\zeta \in \mathbb{D}$. Then $f \in \mathcal{S}^*_{g,\lambda}(\mathcal{B})$ means that

$$\frac{1}{\|x\|} T_x\{(1-\lambda)(Df(x))^{-1} f(x) + \lambda x\}$$

maps the unit ball $\mathcal{B}\backslash\{0\}$ into the right half plane, i.e.,

$$\Re T_x\{(1-\lambda)(Df(x))^{-1} f(x)\} \geq -\|x\| \Re\lambda,\ x \in \mathcal{B}\backslash\{0\}.$$

This is the definition of almost starlike mappings of complex order λ; see [Bălăeţi and Nechita 2010; Zhang et al. 2018].

(iv) Let $g(\zeta) = (1 + A\zeta)/(1 + B\zeta)$, $-1 \leq B < A \leq 1$, $\zeta \in \mathbb{D}$. Then $\mathcal{S}^*_{g,\lambda}(\mathcal{B}) = \mathcal{S}^*_{\mathcal{B}}[A, B, \lambda]$ is the Janowski-starlike mappings of complex order λ on the unit ball $\mathcal{B}$; see [Li and Zhang 2019].

Let $\alpha \in (0, 1)$, $\beta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. If $A = 1$, $B = 2\alpha - 1$, $\lambda = i \tan\beta$, then $f \in \mathcal{S}^*_{g,\lambda}(\mathcal{B})$ means that $\frac{1}{\|x\|} T_x\{(1-\lambda)(Df(x))^{-1} f(x) + \lambda x\}$ maps the unit ball $\mathcal{B}\backslash\{0\}$ into the domain $\Sigma_1 = \left\{\zeta \in \mathbb{C} : \left|\zeta - \frac{1}{2\alpha}\right| < \frac{1}{2\alpha}\right\}$, i.e.,

$$\left| e^{-i\beta} \frac{1}{\|x\|} T_x\{(Df(x))^{-1} f(x)\} - \left(\frac{\cos\beta}{2\alpha} - i \sin\beta\right)\right| < \frac{\cos\beta}{2\alpha}, \quad x \in \mathcal{B}\backslash\{0\}.$$

This is the definition of spirallike mappings of type β and order α; see [Feng et al. 2007].

(v) Let $\rho \in [0, 1)$, $\beta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\lambda = i \tan\beta$. If

$$g(\zeta) = 1 + 4(1-\rho)/\pi^2 (\log(1 + \sqrt{\zeta})/(1 - \sqrt{\zeta}))^2, \quad \zeta \in \mathbb{D},$$

then $f \in \mathcal{S}^*_{g,\lambda}(\mathcal{B})$ means that $\frac{1}{\|x\|} T_x\{(1-\lambda)(Df(x))^{-1} f(x) + \lambda x\}$ maps the unit ball $\mathcal{B}\backslash\{0\}$ into the domain $\Sigma_2 = \{\zeta \in \mathbb{C} : |\zeta - 1| < (1 - 2\rho) + \Re\{\zeta\}\}$, i.e.,

$$\left| \frac{1}{\|x\|} T_x\{(Df(x))^{-1} f(x) - 1\} \right| < (1 - 2\rho)\cos\beta + \Re\left\{ e^{-i\beta} \frac{1}{\|x\|} T_x\{(Df(x))^{-1} f(x)\}\right\}, \quad x \in \mathcal{B}\backslash\{0\},$$

where the branch of the logarithm function is chosen such that $\log 1 = 0$, which reduces to the definition of parabolic spirallike mappings of type β and order ρ; see [Zhang and Yan 2016].

Next, we give two examples in higher dimensions.

Example 2.5. Assume $\lambda \in \mathbb{C}$, $\Re\lambda \le 0$ and $g \in G(\mathbb{D})$ is a convex function. Suppose that $f : \mathbb{B}_n \to \mathbb{C}^n$ is holomorphic with $f(z) = (f_1(z_1), f_2(z_2), \dots, f_n(z_n))'$, where $f_j(z_j)$, $j = 1, 2, \dots, n$, are normalized biholomorphic functions on $\mathbb{D}$. If

$$(1-\lambda)\frac{f_j(z_j)}{z_j f_j'(z_j)} + \lambda \prec g(z_j), \quad z_j \in \mathbb{D},\ j = 1, 2, \dots, n,$$

then $f \in S^*_{g,\lambda}(\mathbb{B}_n)$.

Proof. Since

$$\frac{1}{\sum_{j=1}^n |z_j|^2}\langle (1-\lambda)(Df(z))^{-1} f(z) + \lambda z, z\rangle$$
$$= \frac{1}{\sum_{j=1}^n |z_j|^n}\sum_{j=1}^n |z_j|^2\left((1-\lambda)\frac{f_j(z_j)}{z_j f_j'(z_j)} + \lambda\right) \in g(\mathbb{D}),$$

we have $f \in S^*_{g,\lambda}(\mathbb{B}_n)$. □

Example 2.6. Let $a, \lambda \in \mathbb{C}$, $\Re\lambda \le 0$, $g \in G(\mathbb{D})$. Assume that $f : \mathbb{B}_n \to \mathbb{C}^n$ is a holomorphic mapping with $f(z) = (z_1 + a z_2^2, z_2, \dots, z_n)'$. If

$$|a| \le \frac{3\sqrt{3}}{2}\frac{1}{|1-\lambda|}\operatorname{dist}(1, \partial g(\mathbb{D})),$$

then $f \in S^*_{g,\lambda}(\mathbb{B}_n)$.

Proof. By some elementary calculations, we get

$$(Df(z))^{-1} = \begin{pmatrix} 1 & -2az_2 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Then

$$\frac{1}{\|z\|^2}\langle (1-\lambda)(Df(z))^{-1} f(z) + \lambda z, z\rangle = 1 - \frac{a(1-\lambda)\bar{z}_1 z_2^2}{|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2}.$$

Since $|a| \le \frac{3\sqrt{3}}{2}\frac{1}{|1-\lambda|}\operatorname{dist}(1, \partial g(\mathbb{D}))$, it yields that

$$\left|\frac{a(1-\lambda)\bar{z}_1 z_2^2}{|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2}\right| \le \left|\frac{a(1-\lambda)\bar{z}_1 z_2^2}{|z_1|^2 + |z_2|^2}\right| < \frac{2}{3\sqrt{3}}|a||1-\lambda| \le \operatorname{dist}(1, \partial g(\mathbb{D})).$$

This implies

$$1 - \frac{a(1-\lambda)\bar{z}_1 z_2^2}{|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2} \in g(\mathbb{D}).$$

Thus $f \in S^*_{g,\lambda}(\mathbb{B}_n)$. □

3. Growth theorems for g-starlike mappings of complex order λ

In the next subsection, we utilize the method of Loewner chains to deal with the growth theorem of g-starlike mappings of complex order λ on the unit ball in a reflexive complex Banach space X. The family of g-starlike mappings of complex order λ unifies the family of almost starlike mappings of complex order λ and the family of g-starlike mappings, and the result in the forthcoming subsection will lead to a number of well known statements.

3A. ***Several lemmas.*** We begin this subsection with the following equivalent characterization for almost starlike mappings of complex order λ in terms of Loewner chains on the unit ball $\mathcal{B}$.

Lemma 3.1 [Zhang et al. 2018]. *Let f be a normalized locally biholomorphic mapping on $\mathcal{B}$, and let $\lambda \in \mathbb{C}$ with $\Re\lambda \le 0$. Then f is an almost starlike mapping of complex order λ on $\mathcal{B}$ if and only if*

$$F(x,t) = e^{(1-\lambda)t} f(e^{\lambda t}x), \quad \forall x \in \mathcal{B}, t \in [0,+\infty)$$

is a Loewner chain.

The following lemma is due to Kato.

Lemma 3.2 [Kato 1967]. *Let $x : [0,+\infty) \to X$ be differentiable at the point $s \in (0,+\infty)$, and let $\|x(t)\|$ be also differentiable at the point s with respect to t. Then*

$$\Re\left\{T_{x(s)}\left[\frac{dx}{dt}(s)\right]\right\} = \frac{d\|x(s)\|}{dt}, \quad s \in [0,+\infty).$$

In fact, the following lemma shows that Loewner chain is generated by its transition mapping. It is due to Graham et al. [2013].

Lemma 3.3. *Suppose that X is a reflexive complex Banach space. Let $f(x,t) : \mathcal{B} \times [0,\infty) \to X$ be a Loewner chain. And let $v(x,s,t)$ be the transition mapping associated with $f(x,t)$. If for each $r \in (0,1)$, there exists $M = M(r) > 0$ such that*

$$\|e^{-t} f(x,t)\|_X \le M(r), \quad x \in \mathcal{B}_r, t \in [0,\infty),$$

then

$$f(x,s) = \lim_{t\to\infty} e^t v(x,s,t)$$

uniformly on $\overline{\mathcal{B}_r}$ for $r \in (0,1)$.

In fact, the following lemma plays an important role in the proof of growth theorem.

Lemma 3.4. *Let* $\lambda \in \mathbb{C}$ *with* $\Re\lambda \leq 0$, $g \in G(\mathbb{D})$, *and let* $f : \mathcal{B} \to X$ *be a* g*-starlike mapping of complex order* λ *on* $\mathcal{B}$. *Then*

$$\begin{aligned}(3.1)\quad -\|x\|\Re\lambda + \|x\| \min\{g(\|x\|), g(-\|x\|)\} \\ &\leq \Re\{(1-\lambda)T_x[(Df(x))^{-1} f(x)]\} \\ &\leq -\|x\|\Re\lambda + \|x\| \max\{g(\|x\|), g(-\|x\|)\}.\end{aligned}$$

Moreover, if $g \in G(\mathbb{D})$ *also satisfies* $\max_{|\zeta|=r}|g(\zeta)| = \max\{g(r), g(-r)\}$, $r \in (0, 1)$, *then*

$$\begin{aligned}-\|x\|\Re\lambda + \|x\| \min\{g(\|x\|), g(-\|x\|)\} &\leq |(1-\lambda)T_x[(Df(x))^{-1} f(x)]| \\ &\leq \|x\||\lambda| + \|x\| \max\{g(\|x\|), g(-\|x\|)\}.\end{aligned}$$

Proof. Fixing $x \in \mathcal{B}\backslash\{0\}$, let $x_0 = \frac{x}{\|x\|}$. Then the holomorphic function

$$q(\zeta) = \begin{cases}(1-\lambda)\frac{1}{\zeta}T_{x_0}[(Df(\zeta x_0))^{-1} f(\zeta x_0)] + \lambda, & \zeta \in \mathbb{D}\backslash\{0\}, \\ 1, & \zeta = 0,\end{cases}$$

is well defined on the unit disk $\mathbb{D}$. Since

$$q(\zeta) = (1-\lambda)\frac{1}{|\zeta|}T_{\zeta x_0}[(Df(\zeta x_0))^{-1} f(\zeta x_0)] + \lambda, \zeta \neq 0,$$

from Definition 2.3, it yields that $q(0) = g(0) = 1$, $q(\mathbb{D}) \subseteq g(\mathbb{D})$, i.e., $q \prec g$.

By the subordination principle, it follows that $q(r\mathbb{D}) \subseteq g(r\mathbb{D})$, $r \in (0, 1)$. Hence

$$\min\{g(r), g(-r)\} \leq \Re q(\zeta) \leq \max\{g(r), g(-r)\}.$$

Let $\zeta = \|x\|$. Then

$$\begin{aligned}-\|x\|\Re\lambda+\|x\| \min\{g(\|x\|), g(-\|x\|)\} &\leq \Re\{(1-\lambda)T_x[(Df(x))^{-1} f(x)]\} \\ &\leq -\|x\|\Re\lambda+\|x\| \max\{g(\|x\|), g(-\|x\|)\}.\end{aligned}$$

If we impose the condition $\max_{|\zeta|=r}|g(\zeta)| = \max\{g(r), g(-r)\}$, $r \in (0, 1)$, then

$$\begin{aligned}-\|x\|\Re\lambda+\|x\| \min\{g(\|x\|), g(-\|x\|)\} &\leq |(1-\lambda)T_x[(Df(x))^{-1} f(x)]| \\ &\leq \|x\||\lambda|+\|x\| \max\{g(\|x\|), g(-\|x\|)\}. \quad \square\end{aligned}$$

Remark 3.5. If $\mathcal{B} = \mathbb{B}_n \subseteq \mathbb{C}^n$, then the inequality (3.1) is equivalent to the following form:

$$\begin{aligned}-\|z\|^2\Re\lambda+\|z\|^2 \min\{g(\|z\|), g(-\|z\|)\} &\leq \Re\{(1-\lambda)\bar{z}'[(Df(z))^{-1} f(z)]\} \\ &\leq -\|z\|^2\Re\lambda+\|z\|^2\max\{g(\|z\|), g(-\|z\|)\}.\end{aligned}$$

3B. ***Growth theorems of the classes*** $\mathcal{S}^*_{g,\lambda}(\mathcal{B})$**.** The method to approach the following theorem is analogous to that of [Zhang et al. 2018], although we are now considering normalized biholomorphic mappings on the unit ball $\mathcal{B}$ in an infinite dimensional complex Banach space.

Theorem 3.6. *Let $\lambda \in \mathbb{C}$ with $\Re\lambda \le 0$, $g \in G(\mathbb{D})$, and let $f : \mathcal{B} \to X$ be a g-starlike mapping of complex order λ on $\mathcal{B}$ in reflexive complex Banach space. Then*

$$\begin{aligned}\|x\| \exp\biggl(\int_0^{\|x\|}\biggl[\frac{1-\Re\lambda}{\max\{g(y), g(-y)\}-\Re\lambda}-1\biggr]\frac{dy}{y}\biggr) &\\ \le \|f(x)\| &\\ \le \|x\| \exp\biggl(\int_0^{\|x\|}\biggl[\frac{1-\Re\lambda}{\min\{g(y), g(-y)\}-\Re\lambda}-1\biggr]\frac{dy}{y}\biggr).&\end{aligned}$$

Proof. Since $\Re g(\zeta) > 0$, $\zeta \in \mathbb{D}$, we have $f \in \mathcal{S}^*_{g,\lambda}(\mathcal{B})$ is also an almost starlike mapping of complex order λ. By Lemma 3.1 we know that

$$F(x,t) = e^{(1-\lambda)t} f(e^{\lambda t}x)$$

is a Loewner chain, hence we have

$$F(x,s) \prec F(x,t), \quad \forall 0 \le s \le t,$$

i.e., there is a Schwarz mapping $v(x,s,t)$ such that $F(x,s) = F(v(x,s,t),t)$. By some calculation, we obtain

$$\begin{aligned}&\frac{\partial F}{\partial t}(x,t) = (1-\lambda)e^t e^{-\lambda t} f(e^{\lambda t}x) + \lambda e^t Df(e^{\lambda t}x)x,\\ &DF(x,t) = e^t Df(e^{\lambda t}x).\end{aligned}$$

Let $\frac{\partial F}{\partial t}(x,t) = DF(x,t)h(x,t)$. Then

$$h(x,t) = (1-\lambda)e^{-\lambda t}(Df(e^{\lambda t}x))^{-1} f(e^{\lambda t}x) + \lambda x.$$

For fixed $x \in \mathcal{B}\backslash\{0\}$, $s \ge 0$, let $v(t) = v(x,s,t)$. Then

$$\frac{\partial v}{\partial t}(t) = -(DF(v(t),t))^{-1}\frac{\partial F}{\partial t}(v(t),t) = -h(v(t),t).$$

Since for all $x \in \mathcal{B}$, we have

$$\|e^{-t}F(x,t)\|_X \le \begin{cases}\frac{\|x\|_X}{(1-\|x\|_X)^{2/(1+\Re\lambda)}}, & \Re\lambda \ne -1,\\ \|x\|_X \exp(\|x\|_X), & \Re\lambda = -1,\end{cases}$$

here we use the fact that f is also an almost starlike mapping of complex order λ and the upper bound of $\|f(x)\|_X$; see [Zhang et al. 2018, Theorem 3.1]. By

Lemma 3.3, we obtain

$$\lim_{t\to\infty} e^t v(x,s,t) = F(x,s).$$

Furthermore, by Lemmas 3.2 and 3.4 we see that

$$\begin{aligned}
(3.2)\quad & \frac{d\|v(t)\|}{dt} \\
&= \Re T_{v(t)}\left[\frac{dv(t)}{dt}\right] \\
&= -\Re T_{v(t)}[(1-\lambda)e^{-\lambda t}(Df(e^{\lambda t}v(t)))^{-1}f(e^{\lambda t}v(t)) + \lambda v(t)] \\
&= -\Re \frac{e^{\lambda t}}{|e^{\lambda t}|} T_{e^{\lambda t}v(t)}[(1-\lambda)e^{-\lambda t}(Df(e^{\lambda t}v(t)))^{-1}f(e^{\lambda t}v(t))] - \|v(t)\|\Re\lambda \\
&= -\frac{1}{|e^{\lambda t}|}\Re\{(1-\lambda)T_{e^{\lambda t}v(t)}[(Df(e^{\lambda t}v(t)))^{-1}f(e^{\lambda t}v(t))]\} - \|v(t)\|\Re\lambda \\
&\leq -\|v(t)\|\min\{g(\|e^{\lambda t}v(t)\|), g(-\|e^{\lambda t}v(t)\|)\}.
\end{aligned}$$

By Lemma 3.4 and equality (3.2), we have

$$\begin{aligned}
-\|e^{\lambda t}v(t)\| & \max\{g(\|e^{\lambda t}v(t)\|), g(-\|e^{\lambda t}v(t)\|)\} \\
&\leq -\Re\{(1-\lambda)T_{e^{\lambda t}v(t)}[(Df(e^{\lambda t}v(t)))^{-1}f(e^{\lambda t}v(t))]\} - \|e^{\lambda t}v(t)\|\Re\lambda \\
&= |e^{\lambda t}|\frac{d\|v(t)\|}{dt} \\
&\leq -\|e^{\lambda t}v(t)\|\min\{g(\|e^{\lambda t}v(t)\|), g(-\|e^{\lambda t}v(t)\|)\}.
\end{aligned}$$

Since

$$\frac{d\|e^{\lambda t}v(t)\|}{dt} = |e^{\lambda t}|\frac{d\|v(t)\|}{dt} + \|e^{\lambda t}v(t)\|\Re\lambda,$$

we have

$$\begin{aligned}
(3.3)\quad & \|e^{\lambda t}v(t)\|\Re\lambda - \|e^{\lambda t}v(t)\|\max\{g(\|e^{\lambda t}v(t)\|), g(-\|e^{\lambda t}v(t)\|)\} \\
&\leq \frac{d\|e^{\lambda t}v(t)\|}{dt} \\
(3.4)\quad &\leq \|e^{\lambda t}v(t)\|\Re\lambda - \|e^{\lambda t}v(t)\|\min\{g(\|e^{\lambda t}v(t)\|), g(-\|e^{\lambda t}v(t)\|)\} \\
&< 0,
\end{aligned}$$

which implies that $\|e^{\lambda t}v(t)\|$ is decreasing on $[s,\infty)$.

Integrating on both sides of the inequality (3.4) with respect to $\tau \in [s,t]$, we infer that

$$\begin{aligned}(1-\Re\lambda)(t-s) &\le \int_s^t \frac{1-\Re\lambda}{\|e^{\lambda\tau}v(\tau)\|\Re\lambda - \|e^{\lambda\tau}v(\tau)\|\min\{g(\|e^{\lambda\tau}v(\tau)\|),\, g(-\|e^{\lambda\tau}v(\tau)\|)\}} \\ &\qquad\qquad \times \frac{d\|e^{\lambda\tau}v(\tau)\|}{d\tau}d\tau \\ &= \int_{\|e^{\lambda s}x\|}^{\|e^{\lambda t}v(t)\|} \frac{1-\Re\lambda}{y\Re\lambda - y\min\{g(y),\, g(-y)\}}dy \\ &= \int_{\|e^{\lambda s}x\|}^{\|e^{\lambda t}v(t)\|}\left[\frac{1-\Re\lambda}{\Re\lambda - \min\{g(y),\, g(-y)\}}+1\right]\frac{dy}{y} - \int_{\|e^{\lambda s}x\|}^{\|e^{\lambda t}v(t)\|}\frac{1}{y}dy,\end{aligned}$$

hence

$$(1-\Re\lambda)(t-s) \le \int_{\|e^{\lambda s}x\|}^{\|e^{\lambda t}v(t)\|}\left[\frac{1-\Re\lambda}{\Re\lambda - \min\{g(y),\, g(-y)\}}+1\right]\frac{dy}{y} + \log\frac{\|e^{\lambda s}x\|}{\|e^{\lambda t}v(t)\|},$$

i.e.,

$$e^{(t-s)} \le \frac{\|x\|}{\|v(t)\|}\exp\left(\int_{\|e^{\lambda s}x\|}^{\|e^{\lambda t}v(t)\|}\left[\frac{1-\Re\lambda}{\Re\lambda - \min\{g(y),\, g(-y)\}}+1\right]\frac{dy}{y}\right). \tag{3.5}$$

By using Lemma 3.3 we have $\|e^t v(t)\| \to \|e^{(1-\lambda)s} f(e^{\lambda s}x)\|$ as $t \to +\infty$. Because $\lim_{t\to+\infty}\|e^{\lambda t}v(t)\| = 0$, then taking $t \to +\infty$ on the both sides of the inequality (3.5), and taking $s = 0$, we see that

$$\|f(x)\| \le \|x\|\exp\left(\int_0^{\|x\|}\left[\frac{1-\Re\lambda}{\min\{g(y),\, g(-y)\} - \Re\lambda} - 1\right]\frac{dy}{y}\right).$$

By using the same method for obtaining inequality (3.3), we get

$$\|f(x)\| \ge \|x\|\exp\left(\int_0^{\|x\|}\left[\frac{1-\Re\lambda}{\max\{g(y),\, g(-y)\} - \Re\lambda} - 1\right]\frac{dy}{y}\right). \qquad \square$$

Remark 3.7. In particular, if $g \in G(\mathbb{D})$ and $\lambda \in \mathbb{C}$ with $\Re\lambda \le 0$ are some special functions and special complex number, such as in Remark 2.4, we can get the growth theorems of starlike mappings, spirallike mappings of type β, etc. This is one of the reasons for the interest in this normalized biholomorphic mappings.

4. Distortion theorems for g-starlike mappings of complex order λ

4A. ***Distortion theorems along a unit direction.*** In this subsection, we obtain the distortion theorems for g-starlike mappings of complex order λ along a unit direction on the unit polydisk $\mathbb{D}^n$ and the unit ball $\mathcal{B}$, respectively.

Theorem 4.1. *Let $\lambda \in \mathbb{C}$ with $\Re\lambda \le 0$, $g \in G(\mathbb{D})$ with*

$$\max_{|\zeta|=r}|g(\zeta)| = \max\{g(r), g(-r)\},$$

$r \in (0, 1)$, and let $f : \mathbb{D}^n \to \mathbb{C}^n$ be a g-starlike mapping of complex order λ. Then, for all $z \in \mathbb{D}^n\setminus\{0\}$, there exists a unit vector $\zeta(z)$ such that

$$\begin{aligned}\frac{|1-\lambda|}{|\lambda|+\max\{g(\|z\|), g(-\|z\|)\}}&\exp\left(\int_0^{\|z\|}\left[\frac{1-\Re\lambda}{\max\{g(y), g(-y)\}-\Re\lambda}-1\right]\frac{dy}{y}\right)\\ &\le \|Df(z)\zeta(z)\|\\ &\le \frac{|1-\lambda|}{\min\{g(\|z\|), g(-\|z\|)\}-\Re\lambda}\exp\left(\int_0^{\|z\|}\left[\frac{1-\Re\lambda}{\min\{g(y), g(-y)\}-\Re\lambda}-1\right]\frac{dy}{y}\right).\end{aligned}$$

Proof. The proof is divided into the following two steps:

Step 1. Let $\xi = (\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{D}^n$ with $|\xi_1| = |\xi_2| = \cdots = |\xi_n| = \|\xi\|$. Then $T_\xi = \left(0, \dots, 0, \frac{\|\xi\|}{\xi_j}, 0, \dots, 0\right) \in T(\xi)$.

Taking $w(z) = (w_1(z), \dots, w_n(z))' = (Df(z))^{-1}f(z)$, then there exists $1 \le j \le n$ such that

$$\begin{aligned}\|w(z)\| &= |w_j(z)|\\ &\le \max_{\xi\in(\partial\mathbb{D}(0,\|z\|))^n}|w_j(\xi)|\\ &= \max_{\xi\in(\partial\mathbb{D}(0,\|z\|))^n}\left|\frac{\|\xi\|}{\xi_j}w_j(\xi)\right|\\ &= \max_{\xi\in(\partial\mathbb{D}(0,\|z\|))^n}|T_\xi[w(\xi)]|\\ &= \max_{\xi\in(\partial\mathbb{D}(0,\|z\|))^n}|T_\xi[(Df(\xi))^{-1}f(\xi)]|.\end{aligned}$$

By using Lemma 3.4, we have

$$\begin{aligned}|1-\lambda||T_\xi[(Df(\xi))^{-1}f(\xi)]| &\le \|\xi\||\lambda| + \|\xi\|\max\{g(\|\xi\|), g(-\|\xi\|)\}\\ &\le \|z\||\lambda| + \|z\|\max\{g(\|z\|), g(-\|z\|)\}.\end{aligned}$$

Hence

$$\|(Df(z))^{-1}f(z)\| \le \frac{\|z\|}{|1-\lambda|}(|\lambda| + \max\{g(\|z\|), g(-\|z\|)\}). \tag{4.1}$$

Since $\|T_z\| \le 1$, by Lemma 3.4 we get

$$\begin{aligned}\|(Df(z))^{-1}f(z)\| &\ge |T_z[(Df(z))^{-1}f(z)]|\\ &\ge \frac{\|z\|}{|1-\lambda|}(-\Re\lambda + \min\{g(\|z\|), g(-\|z\|)\}).\end{aligned} \tag{4.2}$$

Step 2. Let $\zeta(z) = (Df(z))^{-1} f(z)/\|(Df(z))^{-1} f(z)\|$, $z \in \mathbb{D}^n \backslash \{0\}$. Then

$$f(z) = Df(z)(Df(z))^{-1} f(z) = \|(Df(z))^{-1} f(z)\| Df(z)\zeta(z).$$

Hence, by Theorem 3.6, (4.1) and (4.2), we have

$$\begin{aligned}
&\frac{|1-\lambda|}{|\lambda| + \max\{g(\|z\|), g(-\|z\|)\}} \exp\biggl(\int_0^{\|z\|} \biggl[\frac{1-\Re\lambda}{\max\{g(y), g(-y)\} - \Re\lambda} - 1\biggr]\frac{dy}{y}\biggr) \\
&\quad \le \|Df(z)\zeta(z)\| = \frac{\|f(z)\|}{\|(Df(z))^{-1} f(z)\|} \\
&\quad \le \frac{|1-\lambda|}{\min\{g(\|z\|), g(-\|z\|)\} - \Re\lambda} \exp\biggl(\int_0^{\|z\|} \biggl[\frac{1-\Re\lambda}{\min\{g(y), g(-y)\} - \Re\lambda} - 1\biggr]\frac{dy}{y}\biggr),
\end{aligned}$$

which completes the proof. □

Theorem 4.2. *Let $\lambda \in \mathbb{C}$ with $\Re\lambda \le 0$, $g \in G(\mathbb{D})$ with*

$$\max_{|\zeta|=r} |g(\zeta)| = \max\{g(r), g(-r)\},$$

$r \in (0, 1)$, *and let $f : \mathcal{B} \to X$ be a g-starlike mapping of complex order λ in reflexive complex Banach spaces. Then, for all $x \in \mathcal{B}\backslash\{0\}$, there exists a unit vector $\zeta(x)$ such that*

$$\begin{aligned}
&\|Df(x)\zeta(x)\| \\
&\quad \le \frac{|1-\lambda|}{\min\{g(\|x\|), g(-\|x\|)\} - \Re\lambda} \exp\biggl(\int_0^{\|x\|} \biggl[\frac{1-\Re\lambda}{\min\{g(y), g(-y)\} - \Re\lambda} - 1\biggr]\frac{dy}{y}\biggr).
\end{aligned}$$

Proof. Let $\zeta(x) = \frac{(Df(x))^{-1} f(x)}{\|(Df(x))^{-1} f(x)\|} \in \partial\mathcal{B}$. Then

$$f(x) = Df(x)(Df(x))^{-1} f(x) = \|(Df(x))^{-1} f(x)\| Df(x)\zeta(x).$$

By using Lemma 3.4, we get

$$\begin{aligned}
-\|x\|\Re\lambda + \|x\| \min\{g(\|x\|), g(-\|x\|)\} &\le \bigl|(1-\lambda)T_x[(Df(x))^{-1} f(x)]\bigr| \\
&\le |1-\lambda| \|(Df(x))^{-1} f(x)\|.
\end{aligned}$$

Hence, by Theorem 3.6, we have

$$\begin{aligned}
\|Df(x)\zeta(x)\| &= \frac{\|f(x)\|}{\|(Df(x))^{-1} f(x)\|} \\
&\le \frac{|1-\lambda|}{\min\{g(\|x\|), g(-\|x\|)\} - \Re\lambda} \\
&\qquad \times \exp\biggl(\int_0^{\|x\|} \biggl[\frac{1-\Re\lambda}{\min\{g(y), g(-y)\} - \Re\lambda} - 1\biggr]\frac{dy}{y}\biggr). \quad \square
\end{aligned}$$

Remark 4.3. If $\lambda = 0$ and g is some biholomorphic function in Definition 2.3, we can get the results in [Liu et al. 2011; 2012] from Theorems 4.1 and 4.2.

4B. ***Distortion theorems on the unit ball $\mathbb{B}_n$.*** In this subsection, the distortion theorems for g-starlike mappings of complex order λ at extreme points are established on the unit ball $\mathbb{B}_n$ in $\mathbb{C}^n$. Denote by $T_{z_0}^{(1,0)}(\partial\mathbb{B}_n) = \{w \in \mathbb{C}^n : \overline{z_0}' w = 0\}$ the complex tangent space at $z_0 \in \partial\mathbb{B}_n$. The following boundary Schwarz lemma is due to Liu et al. [2015] and Graham et al. [2020a], which plays an important role in the proof of the following theorem.

Lemma 4.4 [Graham et al. 2020a; Liu et al. 2015]. *Let $f : \mathbb{B}_n \to \mathbb{B}_n$ be a holomorphic mapping. If f is holomorphic at $z_0 \in \partial\mathbb{B}_n$, $f(z_0) = w_0 \in \partial\mathbb{B}_n$, then $Df(z_0)$ has the following properties*:

(i) *There is a $\mu \in \mathbb{R}$ such that $\overline{Df(z_0)}' w_0 = \mu z_0$ and*

$$\mu = \overline{w_0}' Df(z_0) z_0 \geq \frac{|1 - \bar{c}' w_0|^2}{1 - \|c\|^2} > 0,$$

where $c = f(0)$.

(ii) $\|Df(z_0)\beta\| \leq \sqrt{\mu}$, *for all $\beta \in T_{z_0}^{(1,0)}(\partial\mathbb{B}_n)$ with $\|\beta\| = 1$.*

(iii) $|\det Df(z_0)| \leq \mu^{(n+1)/2}$.

Theorem 4.5. *Let $\lambda \in \mathbb{C}$ with $\Re\lambda \leq 0$, $g \in G(\mathbb{D})$, and let $f : \mathbb{B}_n \to \mathbb{C}^n$ be a g-starlike mapping of complex order λ*:

(1) *If $z \in \mathbb{B}_n$ satisfies $\max_{\|\zeta\|=\|z\|} \|f(\zeta)\| = \|f(z)\|$, then*

$$|\det Df(z)| \leq \left(\frac{\Re(1-\lambda)}{-\Re\lambda + \min\{g(\|z\|), g(-\|z\|)\}} \right)^{(n+1)/2} \times \exp\left(n \int_0^{\|z\|} \left[\frac{1 - \Re\lambda}{\min\{g(y), g(-y)\} - \Re\lambda} - 1 \right] \frac{dy}{y} \right).$$

(2) *If $z \in \mathbb{B}_n$ satisfies $\min_{\|\zeta\|=\|z\|} \|f(\zeta)\| = \|f(z)\|$, then*

$$|\det Df(z)| \geq \left(\frac{\Re(1-\lambda)}{-\Re\lambda + \max\{g(\|z\|), g(-\|z\|)\}} \right)^{(n+1)/2} \times \exp\left(n \int_0^{\|z\|} \left[\frac{1 - \Re\lambda}{\max\{g(y), g(-y)\} - \Re\lambda} - 1 \right] \frac{dy}{y} \right).$$

Proof. Without loss of generality, let $\|z\| = r \in (0, 1)$, $M = \max_{\|\zeta\|=r} \|f(\zeta)\|$ and $m = \min_{\|\zeta\|=r} \|f(\zeta)\|$:

(1) Let $\eta(w) = f(rw)/M$, $w \in \mathbb{B}_n$, then $\eta : \mathbb{B}_n \to \mathbb{B}_n$, $\eta(0) = 0$ and η is biholomorphic in a neighborhood of $\overline{\mathbb{B}_n}$. Take $z_0 = z/r$ and $w_0 = \eta(z_0) = f(z)/M$, then $z_0 \in \partial\mathbb{B}_n$, $w_0 \in \partial\mathbb{B}_n$. By Lemma 4.4, there is a $\mu \in \mathbb{R}$ such that $\overline{D\eta(z_0)}' w_0 = \mu z_0$ and $1 \le \mu = \overline{w_0}' D\eta(z_0) z_0 = \overline{f(z)}' Df(z) z / M^2$. Because $\overline{w_0}' = \mu \overline{z_0}' (D\eta(z_0))^{-1}$, we know that $\overline{f(z)}'$ and $\overline{z}'(Df(z))^{-1}$ have the same direction.

Furthermore, since

$$
\begin{aligned}
\mu &= \frac{\overline{f(z)}' Df(z) z}{M^2} \\
&= \frac{\|f(z)\| \overline{z}'(Df(z))^{-1} Df(z) z}{\|f(z)\|^2 \|\overline{z}'(Df(z))^{-1}\|} \\
&= \frac{\|z\|^2}{\|f(z)\| \|\overline{z}'(Df(z))^{-1}\|} \\
&= \frac{\|z\|^2}{\overline{z}'(Df(z))^{-1} f(z)} \\
&= \frac{\Re\{(1-\lambda)\|z\|^2\}}{\Re\{(1-\lambda)\overline{z}'(Df(z))^{-1} f(z)\}} \\
&\le \frac{\Re(1-\lambda)}{-\Re\lambda + \min\{g(\|z\|), g(-\|z\|)\}},
\end{aligned}
$$

by Lemma 4.4 we have

$$
|\det D\eta(z_0)| \le \mu^{(n+1)/2} \le \left(\frac{\Re(1-\lambda)}{-\Re\lambda + \min\{g(\|z\|), g(-\|z\|)\}} \right)^{(n+1)/2}.
$$

Because $D\eta(z_0) = \frac{r}{M} Df(rz_0) = \frac{r}{M} Df(z)$, by Theorem 3.6, we obtain

$$
\begin{aligned}
|\det Df(z)| &= \left(\frac{M}{r}\right)^n |\det D\eta(z_0)| \\
&\le \left(\frac{\|f(z)\|}{\|z\|}\right)^n \left(\frac{\Re(1-\lambda)}{-\Re\lambda + \min\{g(\|z\|), g(-\|z\|)\}} \right)^{(n+1)/2} \\
&\le \left(\frac{\Re(1-\lambda)}{-\Re\lambda + \min\{g(\|z\|), g(-\|z\|)\}} \right)^{(n+1)/2} \\
&\qquad \times \exp\left(n \int_0^{\|z\|} \left[\frac{1-\Re\lambda}{\min\{g(y), g(-y)\} - \Re\lambda} - 1 \right] \frac{dy}{y} \right).
\end{aligned}
$$

(2) Let $h(w) = f(rw)/m$, $w \in \mathbb{B}_n$, then $h(0) = 0$ and h is biholomorphic in a neighborhood of $\overline{\mathbb{B}_n}$ with $h(\mathbb{B}_n) \supset \mathbb{B}_n$. Take $z_0 = z/r$ and $w_0 = h(z_0) = f(z)/m$, then $z_0 \in \partial\mathbb{B}_n$ and $w_0 \in \partial\mathbb{B}_n$. Furthermore, $h^{-1} : \mathbb{B}_n \to \mathbb{B}_n$, $h^{-1}(0) = 0$ and h^{-1} is holomorphic in a neighborhood of $\overline{\mathbb{B}_n}$ with $h^{-1}(w_0) = z_0$. For the same reason as in the proof of (1) we conclude that $\overline{f(z)}'$ and $\overline{z}'(Df(z))^{-1}$ have the same direction.

By Lemmas 4.4 and 3.4, there exists a $\mu \in \mathbb{R}$ such that

$$\begin{aligned}1 \le \mu &= \overline{z_0}' Dh^{-1}(w_0) w_0 \\ &= \overline{z_0}'(Dh(z_0))^{-1} w_0 \\ &= \frac{\bar{z}'\left(\frac{r}{m} Df(z)\right)^{-1}}{r} \frac{f(z)}{m} \\ &= \frac{\bar{z}'(Df(z))^{-1} f(z)}{\|z\|^2} \\ &= \frac{\Re\{(1-\lambda)\bar{z}'(Df(z))^{-1} f(z)\}}{\Re\{(1-\lambda)\|z\|^2\}} \\ &\le \frac{1}{\Re(1-\lambda)}(-\Re\lambda + \max\{g(\|z\|), g(-\|z\|)\}).\end{aligned}$$

By Lemma 4.4 we have

$$\begin{aligned}|\det Dh^{-1}(w_0)| &= \frac{1}{|\det Dh(z_0)|} \\ &\le \mu^{(n+1)/2} \\ &\le \left(\frac{-\Re\lambda + \max\{g(\|z\|), g(-\|z\|)\}}{\Re(1-\lambda)}\right)^{(n+1)/2}.\end{aligned}$$

Since $Dh(z_0) = \frac{r}{m} Df(z)$, we obtain

$$\begin{aligned}\frac{1}{|\det Df(z)|} &= \left(\frac{r}{m}\right)^n \frac{1}{|\det Dh(z_0)|} \\ &\le \left(\frac{\|z\|}{\|f(z)\|}\right)^n \left(\frac{-\Re\lambda + \max\{g(\|z\|), g(-\|z\|)\}}{\Re(1-\lambda)}\right)^{(n+1)/2} \\ &\le \left(\frac{-\Re\lambda + \max\{g(\|z\|), g(-\|z\|)\}}{\Re(1-\lambda)}\right)^{(n+1)/2} \\ &\qquad \times \exp\left(-n \int_0^{\|z\|} \left[\frac{1-\Re\lambda}{\max\{g(y), g(-y)\} - \Re\lambda} - 1\right] \frac{dy}{y}\right),\end{aligned}$$

where we have used Theorem 3.6, i.e.,

$$\begin{aligned}|\det Df(z)| \ge &\left(\frac{\Re(1-\lambda)}{-\Re\lambda + \max\{g(\|z\|), g(-\|z\|)\}}\right)^{(n+1)/2} \\ &\times \exp\left(n \int_0^{\|z\|} \left[\frac{1-\Re\lambda}{\max\{g(y), g(-y)\} - \Re\lambda} - 1\right] \frac{dy}{y}\right). \quad \square\end{aligned}$$

Remark 4.6. Note that if $\lambda = 0$, Theorem 4.5 reduces to [Graham et al. 2020a, Theorem 5.6].

Theorem 4.7. *Let $\lambda \in \mathbb{C}$ with $\Re\lambda \le 0$, $g \in G(\mathbb{D})$, and let $f : \mathbb{B}_n \to \mathbb{C}^n$ be a g-starlike mapping of complex order λ:*

(1) *If $z \in \mathbb{B}_n$ satisfies*

$$\max_{\|\zeta\|=\|z\|} \|f(\zeta)\| = \|f(z)\|,$$

then for all $\beta \in T_z^{(1,0)}(\partial\mathbb{B}_n(0, \|z\|))$ there holds

$$\|Df(z)\beta\| \le \left(\frac{\Re(1-\lambda)}{-\Re\lambda + \min\{g(\|z\|), g(-\|z\|)\}}\right)^{1/2} \times \exp\left(\int_0^{\|z\|}\left[\frac{1-\Re\lambda}{\min\{g(y), g(-y)\} - \Re\lambda} - 1\right]\frac{dy}{y}\right)\|\beta\|.$$

(2) *If $z \in \mathbb{B}_n$ satisfies*

$$\min_{\|\zeta\|=\|z\|} \|f(\zeta)\| = \|f(z)\|,$$

then for all $\beta \in T_z^{(1,0)}(\partial\mathbb{B}_n(0, \|z\|))$ there holds

$$\|Df(z)\beta\| \ge \left(\frac{\Re(1-\lambda)}{-\Re\lambda + \max\{g(\|z\|), g(-\|z\|)\}}\right)^{1/2} \times \exp\left(\int_0^{\|z\|}\left[\frac{1-\Re\lambda}{\max\{g(y), g(-y)\} - \Re\lambda} - 1\right]\frac{dy}{y}\right)\|\beta\|.$$

Proof. (1) From the proof of Theorem 4.5, we know that if $z \in \mathbb{B}_n$ is the maximum module point of f in the ball $\overline{\mathbb{B}_n(0, \|z\|)}$, there exists a real number $\mu > 0$ such that

$$\mu \le \frac{\Re(1-\lambda)}{-\Re\lambda + \min\{g(\|z\|), g(-\|z\|)\}}.$$

Using Lemma 4.4, we obtain

$$\|D\eta(z_0)\beta\| \le \sqrt{\mu} \le \left(\frac{\Re(1-\lambda)}{-\Re\lambda + \min\{g(\|z\|), g(-\|z\|)\}}\right)^{1/2}$$

for all $\beta \in T_{z_0}^{(1,0)}(\partial\mathbb{B}_n)$ with $\|\beta\| = 1$, i.e.,

$$\|D\eta(z_0)\beta\| \le \left(\frac{\Re(1-\lambda)}{-\Re\lambda + \min\{g(\|z\|), g(-\|z\|)\}}\right)^{1/2}\|\beta\|, \quad \forall\beta \in T_{z_0}^{(1,0)}(\partial\mathbb{B}_n).$$

Since

$$T_{z_0}^{(1,0)}(\partial\mathbb{B}_n) = T_z^{(1,0)}(\partial\mathbb{B}_n(0, \|z\|))$$

and

$$D\eta(z_0) = \frac{r}{M}Df(rz_0) = \frac{\|z\|}{\|f(z)\|}Df(z),$$

we get

$$\begin{aligned}\|Df(z)\beta\| &= \frac{\|f(z)\|}{\|z\|}\|D\eta(z_0)\beta\| \\ &\leq \frac{\|f(z)\|}{\|z\|}\left(\frac{\Re(1-\lambda)}{-\Re\lambda+\min\{g(\|z\|), g(-\|z\|)\}}\right)^{1/2}\|\beta\|,\end{aligned}$$

for all $\beta \in T_z^{(1,0)}(\partial\mathbb{B}_n(0, \|z\|))$. By Theorem 3.6 we can obtain

$$\|Df(z)\beta\| \leq \left(\frac{\Re(1-\lambda)}{-\Re\lambda+\min\{g(\|z\|), g(-\|z\|)\}}\right)^{1/2} \times \exp\left(\int_0^{\|z\|}\left[\frac{1-\Re\lambda}{\min\{g(y), g(-y)\}-\Re\lambda}-1\right]\frac{dy}{y}\right)\|\beta\|$$

for all $\beta \in T_z^{(1,0)}(\partial\mathbb{B}_n(0, \|z\|))$.

(2) From the proof of Theorem 4.5, we know that if $z \in \mathbb{B}_n$ is the minimum module point of f in the ball $\overline{\mathbb{B}_n(0, \|z\|)}$, there is a real number $\mu > 0$ such that

$$\mu \leq \frac{1}{\Re(1-\lambda)}(-\Re\lambda+\max\{g(\|z\|), g(-\|z\|)\}).$$

By Lemma 4.4 we have

$$\|Dh^{-1}(w_0)\gamma\| \leq \sqrt{\mu} \leq \left(\frac{-\Re\lambda+\max\{g(\|z\|), g(-\|z\|)\}}{\Re(1-\lambda)}\right)^{1/2},$$

for all $\gamma \in T_{w_0}^{(1,0)}(\partial\mathbb{B}_n)$ with $\|\gamma\| = 1$, i.e.,

$$\|Dh^{-1}(w_0)\gamma\| \leq \left(\frac{-\Re\lambda+\max\{g(\|z\|), g(-\|z\|)\}}{\Re(1-\lambda)}\right)^{1/2}\|\gamma\|, \tag{4.3}$$

for all $\gamma \in T_{w_0}^{(1,0)}(\partial\mathbb{B}_n)$.

Noting that

$$Dh^{-1}(w_0) = (Dh(z_0))^{-1} = \left(\frac{r}{m}Df(z)\right)^{-1} = \frac{\|f(z)\|}{\|z\|}(Df(z))^{-1}$$

and

$$Dh^{-1}(w_0)T_{w_0}^{(1,0)}(\partial\mathbb{B}_n) = T_{z_0}^{(1,0)}(\partial\mathbb{B}_n) = T_z^{(1,0)}(\partial\mathbb{B}_n(0, \|z\|)),$$

we know that

$$Df(z)T_z^{(1,0)}(\partial\mathbb{B}_n(0, \|z\|)) = T_{w_0}^{(1,0)}(\partial\mathbb{B}_n).$$

Replacing $Df(z)\beta$ with γ in the inequality (4.3), where $\beta \in T_z^{(1,0)}(\partial\mathbb{B}_n(0,\|z\|))$, we obtain

$$\frac{\|f(z)\|}{\|z\|}\|\beta\| \le \left(\frac{-\Re\lambda+\max\{g(\|z\|),g(-\|z\|)\}}{\Re(1-\lambda)}\right)^{1/2}\|Df(z)\beta\|,$$

i.e.,

$$\|Df(z)\beta\| \ge \frac{\|f(z)\|}{\|z\|}\left(\frac{\Re(1-\lambda)}{-\Re\lambda+\max\{g(\|z\|),g(-\|z\|)\}}\right)^{1/2}\|\beta\|,$$

for all $\beta \in T_z^{(1,0)}(\partial\mathbb{B}_n(0,\|z\|))$. By Theorem 3.6 we see that

$$\|Df(z)\beta\| \ge \left(\frac{\Re(1-\lambda)}{-\Re\lambda+\max\{g(\|z\|),g(-\|z\|)\}}\right)^{1/2} \times \exp\left(\int_0^{\|z\|}\left[\frac{1-\Re\lambda}{\max\{g(y),g(-y)\}-\Re\lambda}-1\right]\frac{dy}{y}\right)\|\beta\|$$

for all $\beta \in T_z^{(1,0)}(\partial\mathbb{B}_n(0,\|z\|))$. □

Remark 4.8. Note that if $\lambda = 0$, Theorem 4.7 reduces to [Graham et al. 2020a, Theorem 5.11].

5. Roper–Suffridge extension operators and the families $\mathcal{S}^*_{g,\lambda}(\mathcal{B})$

5A. ***Roper–Suffridge extension operators.*** The challenge of constructing examples of starlike mappings and of convex mappings in higher dimensions was well-known, until the introduction of the Roper–Suffridge operator [Roper and Suffridge 1995]. This operator is used to construct starlike mappings and convex mappings in higher dimensions via starlike functions and convex functions in the unit disk, respectively. In the same paper, Roper and Suffridge proved that if f is a normalized locally biholomorphic convex function on the unit disk $\mathbb{D}$, then

$$\Phi_n(f)(u) = (f(u_1), \sqrt{f'(u_1)}\tilde{u}),\ u = (u_1, \tilde{u}) \in \mathbb{B}_n,$$

is a normalized locally biholomorphic convex mapping on the Euclidean unit ball $\mathbb{B}_n$, where $\tilde{u} \in \mathbb{C}^{n-1}$, $\sqrt{f'(0)} = 1$. Graham and Kohr [2000] used the analytic definition of starlike mappings to prove that if f is a starlike function on $\mathbb{D}$, then $\Phi_n(f)$ is a starlike mapping on the unit ball $\mathbb{B}_n$. Furthermore, Graham and Hamada et al. [2002b] proved that if f is a normalized locally biholomorphic starlike function on $\mathbb{D}$, then $\Phi_{n,\alpha,\beta}(f)$ is a normalized locally biholomorphic starlike mapping on $\mathbb{B}_n$ for $\alpha \in [0,1]$, $\beta \in \left[0,\frac{1}{2}\right]$, $\alpha+\beta \le 1$; if f is a normalized locally biholomorphic convex function on $\mathbb{D}$, then $\Phi_{n,\alpha,\beta}(f)$ is a normalized locally biholomorphic convex mapping on $\mathbb{B}_n$ if and only if $(\alpha,\beta) = \left(0,\frac{1}{2}\right)$, where

$$\Phi_{n,\alpha,\beta}(f)(u) = \left(f(u_1), \left(\frac{f(u_1)}{u_1}\right)^{\alpha}(f'(u_1))^{\beta}\tilde{u}\right), \quad u = (u_1,\tilde{u}) \in \mathbb{B}_n,$$

$\alpha \in [0, 1]$, $\beta \in \left[0, \frac{1}{2}\right]$, $\alpha+\beta \le 1$, and the branches of the power functions are chosen such that $(f(u_1)/u_1)^\alpha \mid_{u_1=0}= 1$, $(f'(u_1))^\beta \mid_{u_1=0}= 1$.

In the above, the Roper–Suffridge operator is only defined on the unit ball $\mathbb{B}_n$. Graham and Kohr [2000], raised the following question:

Question. Consider the egg domain $\Omega_{2,p} = \{(u_1, u_2) \in \mathbb{C}^2 : |u_1|^2 + |u_2|^p < 1\}$, where $p > 1$. Does the operator

$$\Phi_{n,1/p}(f)(u) = (f(u_1), (f'(u_1))^{1/p}u_2), u = (u_1, u_2) \in \Omega_{2,p},$$

extend convex functions on $\mathbb{D}$ to convex mappings on the egg domain $\Omega_{2,p}$?

Gong and Liu [2002] gave an affirmative answer to the above question. They used the contractive property of Carathéodory metric under holomorphic mappings to show that if f is a normalized locally biholomorphic ε starlike function on $\mathbb{D}$, then

$$\Phi_{n,1/p}(f)(u) = (f(u_1), (f'(u_1))^{1/p}\tilde{u}), \quad u = (u_1, \tilde{u}) \in \Omega_p,$$

is a normalized locally biholomorphic ε starlike mapping on Ω_p, where $\Omega_p = \left\{(u_1, \dots, u_n) \in \mathbb{C}^n : |u_1|^2 + \sum_{j=2}^n |u_j|^p < 1\right\}$.

Muir [2005] introduced an extension operator from a new perspective as follows:

$$\Phi_{n,P}(f)(u) = (f(u_1) + P(\tilde{u})f'(u_1), \sqrt{f'(u_1)}\tilde{u}), \quad u = (u_1, \tilde{u}) \in \mathbb{B}_n,$$

where f is a normalized locally biholomorphic function on $\mathbb{D}$ and $P : \mathbb{C}^{n-1} \to \mathbb{C}$ is a homogeneous polynomial mapping of degree 2, and $\sqrt{f'(0)} = 1$. Furthermore, he showed that if f is a normalized locally biholomorphic starlike function on $\mathbb{D}$, then $\Phi_{n,P}(f)$ is a normalized locally biholomorphic starlike mapping on $\mathbb{B}_n$ if and only if $\|P\| \le \frac{1}{4}$; if f is a normalized locally biholomorphic convex function on $\mathbb{D}$, then $\Phi_{n,P}(f)$ is a normalized locally biholomorphic convex mapping on $\mathbb{B}_n$ if and only if $\|P\| \le \frac{1}{2}$.

Recently, Graham and Hamada et al. [Graham et al. 2020b] consider the extension operator $\Phi_{\alpha,\beta}$ and Φ_{P_r} on some unit ball in the complex Banach space $Z = \mathbb{C} \times Y$, where

$$\Phi_{\alpha,\beta}(f)(z) = \left(f(z_1), \left(\frac{f(z_1)}{z_1}\right)^\alpha (f'(z_1))^\beta w\right), \qquad z= (z_1, w) \in \Omega_r,$$

$$\Phi_{P_r}(f)(z) = (f(z_1) + P_r(w)f'(z_1), (f'(u_1))^{1/r}w), \quad z= (z_1, w) \in \Omega_r,$$

$\alpha \in [0, 1]$, $\beta \in [, , 1/r]$, $\alpha+\beta \le 1$, f is a normalized locally biholomorphic function on $\mathbb{D}$, the branches of the power functions are chosen such that $(f(z_1)/z_1)^\alpha \mid_{z_1=0}= 1$, $(f'(z_1))^\beta \mid_{z_1=0}= 1$, $P_r : Y \to \mathbb{C}$ is a homogeneous polynomial mapping of degree r, $2 \le r$ and

$$\Omega_r = \{z = (z_1, w) \in Z = \mathbb{C} \times Y : |z_1|^2 + \|w\|_Y^r < 1\},$$

where Y is a complex Banach space. The Minkowski function of Ω_r is a complete norm $\|\cdot\|_Z$ on Z, Ω_r is the unit ball of Z with respect to this norm. They proved that these two extension operators had the following properties: Let $g \in G(\mathbb{D})$ be a convex function, and the normalized locally biholomorphic function f can be embedded as the first element of a g-Loewner chain on $\mathbb{D}$. Then $\Phi_{\alpha,\beta}(f)$ can be embedded as the first element of a g-Loewner chain on Ω_r. If $\|P_r\| \le \frac{1}{4}\operatorname{dist}(1, \partial g(\mathbb{D}))$, then $\Phi_{P_r}(f)$ can be embedded as the first element of a g-Loewner chain on Ω_r. The extension operators for normalized locally biholomorphic functions on the unit disk $\mathbb{D}$ to higher dimensional spaces have been extensively studied in the literature, see, e.g., [Elin 2011; Elin and Levenshtein 2014; Feng and Liu 2008; Gong and Liu 2003; Graham et al. 2012; Liu et al. 2019; Liu and Xu 2006; Wang 2013; Wang and Liu 2010; 2018].

In the next subsection, we study the extension operators $\Phi_{\alpha,\beta}$ and Φ_{P_r} associated with the g-starlike mappings of complex order λ on Ω_r by using two different methods.

5B. *Some lemmas.* In order to prove the main theorems in this subsection, we need the following lemmas.

Lemma 5.1 [Graham et al. 2020b]. *Let Y be a complex Banach space and let $\Omega_r = \{z = (z_1, w) \in \mathbb{C} \times Y : |z_1|^2 + \|w\|_Y^r < 1\}$ be the unit ball of $Z = \mathbb{C} \times Y$, where $r \ge 1$. Let $z = (z_1, w) \ne 0$. Then*

$$T_z((z_1, 0)) = \frac{2|z_1|^2 \|z\|_Z}{2|z_1|^2 + r(\|z\|_Z^2 - |z_1|^2)}$$

and

$$T_z((0, w)) = \frac{r(\|z\|_Z^2 - |z_1|^2)\|z\|_Z}{2|z_1|^2 + r(\|z\|_Z^2 - |z_1|^2)}$$

for any $T_z \in T(z)$.

Lemma 5.2 [Muir 2008]. *Let $f : \mathbb{D} \to \mathbb{C}$ be a normalized biholomorphic function, $k \ge 2$. Then*

$$\left|(1 - |\zeta|^2)\frac{f''(\zeta)}{f'(\zeta)} - k\bar{\zeta}\right| \le k + 2, \quad \forall \zeta \in \mathbb{D}.$$

Lemma 5.3 [Pommerenke 1975]. *Let $g : \mathbb{D} \to \mathbb{C}$ be a convex function. Then for any $a \in \mathbb{D}$, $g(\mathbb{D})$ contains the disk of radius $\frac{1}{2}|g'(a)|(1 - |a|^2)$ centered at $g(a)$.*

Lemma 5.4 [Graham et al. 2020b]. *Let $g \in G(\mathbb{D})$. We say that a mapping $f = f(x, t) : \mathcal{B} \times [0, \infty) \to X$ is a g-Loewner chain if the following conditions hold:*

(i) *$f(x, t)$ is a Loewner chain such that $\{e^{-t} f(\cdot, t)\}_{t \ge 0}$ is uniformly bounded on each ball $\mathcal{B}_r$ $(0 < r < 1)$;*

(ii) $\frac{\partial f}{\partial t}(x,t)$ *exists for a.e.* $t \geq 0$ *and for all* $x \in \mathcal{B}$, *and there exists a Herglotz vector field* $h = h(x,t) : \mathcal{B} \times [0,\infty) \to X$ *with* $h(\cdot,t) \in \mathcal{M}_g(\mathcal{B})$ *for a.e.* $t \geq 0$ *such that*

$$\frac{\partial f}{\partial t}(x,t) = Df(x,t)h(x,t), \quad a.e.\ t \geq 0, x \in \mathcal{B}.$$

Remark 5.5. Let $g \in G(\mathbb{D})$. It is not difficult to deduce that $f : \mathcal{B} \to X$ is a g-starlike mapping of complex order λ if and only if $F(x,t) = e^{(1-\lambda)t} f(e^{\lambda t}x)$, $\forall x \in \mathcal{B}$, $t \in [0,\infty)$, is a g-Loewner chain.

5C. *Examples of $\mathcal{S}^*_{g,\lambda}(\Omega_r)$.* By using Roper–Suffridge extension operators, we can construct many examples of $\mathcal{S}^*_{g,\lambda}(\Omega_r)$ via holomorphic functions of $\mathcal{S}^*_{g,\lambda}(\mathbb{D})$.

Theorem 5.6. *Let* $g \in G(\mathbb{D})$ *be a convex function , and let* Y *be a complex Banach space. Denote* $\Omega_r = \{z = (z_1, w) \in Z : |z_1|^2 + \|w\|_Y^r < 1\}$ *by the unit ball of* $Z = \mathbb{C} \times Y$, *where* $r \geq 1$. *Suppose that* $\alpha \in [0,1]$, $\beta \in [0, 1/r]$, $\alpha + \beta \leq 1$. *If* f *is a* g*-starlike function of complex order* λ *on* $\mathbb{D}$, *then*

$$F(z) = \Phi_{\alpha,\beta}(f)(z) = \left(f(z_1), \left(\frac{f(z_1)}{z_1}\right)^{\alpha} (f'(z_1))^{\beta} w \right) \in \mathcal{S}^*_{g,\lambda}(\Omega_r),$$

where $z = (z_1, w) \in \Omega_r$ *and the branches of the power functions are chosen such that* $(f(z_1)/z_1)^{\alpha} |_{z_1=0} = 1$ *and* $(f'(z_1))^{\beta} |_{z_1=0} = 1$.

Proof. Note that

$$f(z_1, t) = e^{(1-\lambda)t} f(e^{\lambda t} z_1), \forall z_1 \in \mathbb{D}, t \in [0,\infty)$$

is a g-Loewner chain, since f is a g-starlike function of complex order λ on the unit disk $\mathbb{D}$. According to a result of Graham et al. [2020b, Theorem 3.1], we know that

$$\begin{aligned} F(z,t) &= e^t \Phi_{\alpha,\beta}(e^{-t} f(\cdot,t))(z,t) \\ &= \left(f(z_1,t), e^{(1-\alpha-\beta)t} \left(\frac{f(z_1,t)}{z_1}\right)^{\alpha} (f'(z_1,t))^{\beta} w \right) \end{aligned}$$

is a g-Loewner chain. Furthermore,

$$\begin{aligned} F(z,t) &= \left(e^{(1-\lambda)t} f(e^{\lambda t} z_1), e^{(1-\lambda)t} \left(\frac{f(e^{\lambda t} z_1)}{e^{\lambda t} z_1}\right)^{\alpha} (f'(e^{\lambda t} z_1))^{\beta} (e^{\lambda t} w) \right) \\ &= e^{(1-\lambda)t} \Phi_{\alpha,\beta}(f)(e^{\lambda t} z). \end{aligned}$$

It yields that $F = \Phi_{\alpha,\beta}(f) \in \mathcal{S}^*_{g,\lambda}(\Omega_r)$. $\square$

Theorem 5.7. *Let Y be a complex Banach space and let $k \geq 2$ be an integer. Denote*

$$\Omega_r = \{z = (z_1, w) \in \mathbb{C} \times Y : |z_1|^2 + \|w\|_Y^r < 1\},$$

$r \geq 1$, and let $P_k : Y \to \mathbb{C}$ be a homogeneous polynomial mapping of degree k, $r \leq k$. Assume that f is a g-starlike function of complex order λ on $\mathbb{D}$, where $g \in G(\mathbb{D})$ is a convex function. If

$$\|P_k\| \leq r/(2(k+r)|1-\lambda|)\,\mathrm{dist}(1, \partial g(\mathbb{D})),$$

then

$$F(z) = \Phi_{P_k}(f)(z_1, w) = (f(z_1) + P_k(w) f'(z_1), (f'(z_1))^{1/k} w) \in \mathcal{S}^*_{g,\lambda}(\Omega_r),$$

where $(z_1, w) \in \Omega_r$ and the branch of the power function is chosen such that $(f'(z_1))^{1/k} |_{z_1=0} = 1$.

Proof. For any holomorphic mapping $\eta(z) = (\eta_1(z), \eta_0(z)) : \Omega_r \to \mathbb{C} \times Y$, we have

$$\begin{aligned} DF(z)\eta(z) = \big(&\eta_1(z)(f'(z_1) + P_k(w) f''(z_1)) + \nabla P_k(w) f'(z_1)\eta_0(z), \\ &\tfrac{1}{k}(f'(z_1))^{(1/k)-1} f''(z_1)\eta_1(z) w + (f'(z_1))^{1/k}\eta_0(z)\big). \end{aligned}$$

Let $DF(z)\eta(z) = F(z)$. Then

$$\begin{aligned} \eta(z) &= (DF(z))^{-1} F(z) \\ &= \left(\frac{f(z_1)}{f'(z_1)} - (k-1) P_k(w), \left(1 - \frac{1}{k}\frac{f(z_1) f''(z_1)}{(f'(z_1))^2} + \left(1 - \frac{1}{k}\right) P_k(w) \frac{f''(z_1)}{f'(z_1)}\right) w\right). \end{aligned}$$

Hence,

$$\begin{aligned} (5.1)\quad &(1-\lambda)(DF(z))^{-1} F(z) + \lambda z \\ &= \left((1-\lambda)\left(\frac{f(z_1)}{f'(z_1)} - (k-1) P_k(w)\right) + \lambda z_1,\right. \\ &\qquad \left.(1-\lambda)\left(1 - \frac{1}{k}\frac{f(z_1) f''(z_1)}{(f'(z_1))^2} + \left(1 - \frac{1}{k}\right) P_k(w) \frac{f''(z_1)}{f'(z_1)}\right) w + \lambda w\right). \end{aligned}$$

Next, we will show that

$$\frac{1}{\|z\|_Z} T_z\{(1-\lambda)(DF(z))^{-1} F(z) + \lambda z\} \in g(\mathbb{D}),\ z \in \Omega_r \backslash \{0\}.$$

It is equivalent to prove

$$\frac{1}{\rho} T_z\{(1-\lambda)(DF(\rho z))^{-1} F(\rho z) + \lambda \rho z\} \in g(\mathbb{D}),\ z \in \partial\Omega_r,\ \rho \in (0, 1).$$

Indeed, if $z = (z_1, 0) \in \partial\Omega_r$, then

$$\frac{1}{\rho} T_z\{(1-\lambda)(DF(\rho z))^{-1} F(\rho z) + \lambda \rho z\} = (1-\lambda)\frac{f(\rho z_1)}{\rho z_1 f'(\rho z_1)} + \lambda \in g(\mathbb{D}).$$

If $z=(z_1,w)\in\partial\Omega_r$ with $w\neq 0$, then by using Lemma 5.1 and (5.1), we have

$$\begin{aligned}
&\frac{(2|z_1|^2+r(1-|z_1|^2))}{\rho}T_z\{(1-\lambda)(DF(\rho z))^{-1}F(\rho z)+\lambda\rho z\}\\
&\quad=2|z_1|^2\left((1-\lambda)\frac{f(\rho z_1)}{\rho z_1 f'(\rho z_1)}+\lambda\right)-2(1-\lambda)(k-1)\rho^{k-1}P_k(w)\overline{z_1}\\
&\qquad\quad+r(1-|z_1|^2)(1-\lambda)\left(1-\frac{1}{k}\frac{f(\rho z_1)f''(\rho z_1)}{(f'(\rho z_1))^2}+\left(1-\frac{1}{k}\right)\rho^k P_k(w)\frac{f''(\rho z_1)}{f'(\rho z_1)}\right)\\
&\qquad\quad+r\lambda(1-|z_1|^2)\\
&\quad=2|z_1|^2\left((1-\lambda)\frac{f(\rho z_1)}{\rho z_1 f'(\rho z_1)}+\lambda\right)+\frac{r}{k}(1-|z_1|^2)\left(1-(1-\lambda)\frac{f(\rho z_1)f''(\rho z_1)}{(f'(\rho z_1))^2}\right)\\
&\qquad+\frac{r(k-1)}{k}(1-|z_1|^2)\\
&\qquad\quad\times\left[1+(1-\lambda)\rho^{k-2}\frac{P_k(w)}{1-|z_1|^2}\left(\rho^2(1-|z_1|^2)\frac{f''(\rho z_1)}{f'(\rho z_1)}-\frac{2k}{r}\overline{\rho z_1}\right)\right].
\end{aligned}$$

Let

$$\psi(\zeta)=(1-\lambda)\frac{f(\zeta)}{\zeta f'(\zeta)}+\lambda,\quad \forall\zeta\in\mathbb{D}.$$

Since f is a g-starlike function of complex order λ on $\mathbb{D}$, we have

$$\psi(\zeta)\in g(\mathbb{D}), \tag{5.2}$$

and $\psi(0)=g(0)=1$, i.e., $\psi\prec g$. Hence, there is a Schwarz mapping $v:\mathbb{D}\to\mathbb{D}$ such that $v(0)=0$ and $\psi(\zeta)=g(v(\zeta))$. Furthermore, it is easy to see that, for all $\zeta\in\mathbb{D}$,

$$|v'(\zeta)|\le\frac{1-|v(\zeta)|^2}{1-|\zeta|^2}\quad\text{and}\quad \psi(\zeta)+\zeta\psi'(\zeta)=1-(1-\lambda)\frac{f(\zeta)f''(\zeta)}{(f'(\zeta))^2},$$

Therefore, we have

$$\begin{aligned}
&\frac{(2|z_1|^2+r(1-|z_1|^2))}{\rho}T_z\{(1-\lambda)(DF(\rho z))^{-1}F(\rho z)+\lambda\rho z\}\\
&\quad=2|z_1|^2\psi(\rho z_1)+\frac{r}{k}(1-|z_1|^2)(\psi(\rho z_1)+\rho z_1\psi'(\rho z_1))+\frac{r(k-1)}{k}(1-|z_1|^2)\\
&\qquad\quad\times\left[1+(1-\lambda)\rho^{k-2}\frac{P_k(w)}{1-|z_1|^2}\left(\rho^2(1-|z_1|^2)\frac{f''(\rho z_1)}{f'(\rho z_1)}-\frac{2k}{r}\overline{\rho z_1}\right)\right]\\
&\quad=\frac{2(k-r)}{k}|z_1|^2\psi(\rho z_1)+\frac{r}{k}(1+|z_1|^2)\\
&\qquad\times\left(\psi(\rho z_1)+\frac{\rho z_1\psi'(\rho z_1)(1-|z_1|^2)}{1+|z_1|^2}\right)+\frac{r(k-1)}{k}(1-|z_1|^2)\\
&\qquad\quad\times\left[1+(1-\lambda)\rho^{k-2}\frac{P_k(w)}{1-|z_1|^2}\left(\rho^2(1-|z_1|^2)\frac{f''(\rho z_1)}{f'(\rho z_1)}-\frac{2k}{r}\overline{\rho z_1}\right)\right].
\end{aligned}$$

Since $g(\mathbb{D})$ contains a disk with $g(a)$ as center and $\frac{1}{2}|g'(a)|(1-|a|^2)$ as radius, where $a = v(\rho z_1)$, and

$$\left|\frac{\rho z_1 \psi'(\rho z_1)(1-|z_1|^2)}{1+|z_1|^2}\right| = \frac{\rho|z_1|}{1+|z_1|^2}|g'(a)||v'(\rho z_1)|(1-|z_1|^2) < \tfrac{1}{2}|g'(a)|(1-|a|^2),$$

we have

$$\psi(\rho z_1) + \frac{\rho z_1 \psi'(\rho z_1)(1-|z_1|^2)}{1+|z_1|^2} \in g(\mathbb{D}). \tag{5.3}$$

On the other hand, since

$$|P_k(w)| \le \|P_k\| \|w\|_Y^k \quad \text{and} \quad \|P_k\| \le \frac{r}{2(k+r)|1-\lambda|} \operatorname{dist}(1, \partial g(\mathbb{D})),$$

by using Lemma 5.2, we have

$$\begin{aligned}\rho^{k-2}\left|(1-\lambda)\frac{P_k(w)}{1-|z_1|^2}\left(\rho^2(1-|z_1|^2)\frac{f''(\rho z_1)}{f'(\rho z_1)} - \frac{2k}{r}\overline{\rho z_1}\right)\right| & \\ &\le \rho^{k-1}|1-\lambda|\|P_k\|\left(\frac{2k}{r}+2\right) \\ &\le \operatorname{dist}(1, \partial g(\mathbb{D})).\end{aligned}$$

It yields that

$$1 + (1-\lambda)\rho^{k-2}\frac{P_k(w)}{1-|z_1|^2}\left(\rho^2(1-|z_1|^2)\frac{f''(\rho z_1)}{f'(\rho z_1)} - \frac{2k}{r}\overline{\rho z_1}\right) \in g(\mathbb{D}). \tag{5.4}$$

Putting the equation (5.2), (5.3) and (5.4) together, we get

$$\frac{1}{\rho}T_z\{(1-\lambda)(DF(\rho z))^{-1}F(\rho z) + \lambda\rho z\} \in g(\mathbb{D}), z \in \partial\Omega_r, \quad \forall \rho \in (0,1). \qquad \square$$

Remark 5.8. If $r = k$ and $\lambda = 0$, then Theorem 5.7 reduces to [Graham et al. 2020b, Theorem 4.1].

In particular, when $Y = \mathbb{C}^{n-1}$, we do have the following corollary, which is a generalization of [Muir 2005, Theorem 4.1].

Corollary 5.9. *Let $k \ge 2$ be an integer. And let $P_k : \mathbb{C}^{n-1} \to \mathbb{C}$ be a homogeneous polynomial mapping of degree k. Assume that f is a g-starlike function of complex order λ on $\mathbb{D}$, where $g \in G(\mathbb{D})$ is a convex function. If $\|P_k\| \le 1/((k+2)|1-\lambda|)$* $\operatorname{dist}(1, \partial g(\mathbb{D}))$, *then*

$$F(z_1, w) = \Phi_{P_k}(f)(z_1, w) = (f(z_1) + P_k(w)f'(z_1), \quad (f'(z_1))^{1/k}w) \in \mathcal{S}^*_{g,\lambda}(\mathbb{B}_n),$$

where $z = (z_1, w) \in \mathbb{B}_n$ and the branch of the power function is chosen such that $(f'(z_1))^{1/k}|_{z_1=0} = 1$.

Remark 5.10. Since the functions g in Remark 2.4 are all convex functions, the extension operators $\Phi_{\alpha,\beta}$ and Φ_{P_k} preserve the geometric properties of the normalized locally biholomorphic mappings, which we have displayed in Remark 2.4, respectively.

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References

[Barnard et al. 1991] R. W. Barnard, C. H. FitzGerald, and S. Gong, "The growth and $\frac{1}{4}$-theorems for starlike mappings in $\mathbb{C}^n$", *Pacific J. Math.* **150**:1 (1991), 13–22. MR

[Barnard et al. 1994] R. W. Barnard, C. H. FitzGerald, and S. Gong, "A distortion theorem for biholomorphic mappings in $\mathbb{C}^2$", *Trans. Amer. Math. Soc.* **344**:2 (1994), 907–924. MR

[Bracci et al. 2009] F. Bracci, M. D. Contreras, and S. Díaz-Madrigal, "Evolution families and the Loewner equation, II: Complex hyperbolic manifolds", *Math. Ann.* **344**:4 (2009), 947–962. MR Zbl

[Bălăeţi and Nechita 2010] C. M. Bălăeţi and V. O. Nechita, "Loewner chains and almost starlike mappings of complex order λ", *Carpathian J. Math.* **26**:2 (2010), 146–157. MR Zbl

[Chirilă 2014] T. Chirilă, "Subclasses of biholomorphic mappings associated with g-Loewner chains on the unit ball in $\mathbb{C}^n$", *Complex Var. Elliptic Equ.* **59**:10 (2014), 1456–1474. MR Zbl

[Chirilă 2015] T. Chirilă, "Extreme points, support points and g-Loewner chains associated with Roper–Suffridge and Pfaltzgraff–Suffridge extension operators", *Complex Anal. Oper. Theory* **9**:8 (2015), 1781–1799. MR Zbl

[Duren et al. 2010] P. Duren, I. Graham, H. Hamada, and G. Kohr, "Solutions for the generalized Loewner differential equation in several complex variables", *Math. Ann.* **347**:2 (2010), 411–435. MR Zbl

[Elin 2011] M. Elin, "Extension operators via semigroups", *J. Math. Anal. Appl.* **377**:1 (2011), 239–250. MR Zbl

[Elin and Levenshtein 2014] M. Elin and M. Levenshtein, "Covering results and perturbed Roper–Suffridge operators", *Complex Anal. Oper. Theory* **8**:1 (2014), 25–36. MR Zbl

[Elin et al. 2019] M. Elin, S. Reich, and D. Shoikhet, *Numerical range of holomorphic mappings and applications*, Springer, 2019. MR Zbl

[Feng and Liu 2008] S. Feng and T. Liu, "The generalized Roper–Suffridge extension operator", *Acta Math. Sci. Ser. B* (*Engl. Ed.*) **28**:1 (2008), 63–80. MR Zbl

[Feng et al. 2007] S. X. Feng, T. S. Liu, and G. B. Ren, "The growth and covering theorems for several mappings on the unit ball in complex Banach spaces", *Chinese Ann. Math. Ser. A* **28**:2 (2007), 215–230. MR

[Gong 1998] S. Gong, *Convex and starlike mappings in several complex variables*, Mathematics and its Applications (China Series) **435**, Kluwer Academic Publishers, Dordrecht, 1998. MR Zbl

[Gong and Liu 2002] S. Gong and T. Liu, "On the Roper–Suffridge extension operator", *J. Anal. Math.* **88** (2002), 397–404. MR Zbl

[Gong and Liu 2003] S. Gong and T. Liu, "The generalized Roper–Suffridge extension operator", *J. Math. Anal. Appl.* **284**:2 (2003), 425–434. MR Zbl

[Graham and Kohr 2000] I. Graham and G. Kohr, "Univalent mappings associated with the Roper–Suffridge extension operator", *J. Anal. Math.* **81** (2000), 331–342. MR Zbl

[Graham and Kohr 2003] I. Graham and G. Kohr, *Geometric function theory in one and higher dimensions*, Monographs and Textbooks in Pure and Applied Mathematics **255**, Marcel Dekker, New York, 2003. MR Zbl

[Graham et al. 2002a] I. Graham, H. Hamada, and G. Kohr, "Parametric representation of univalent mappings in several complex variables", *Canad. J. Math.* **54**:2 (2002), 324–351. MR Zbl

[Graham et al. 2002b] I. Graham, H. Hamada, G. Kohr, and T. J. Suffridge, "Extension operators for locally univalent mappings", *Michigan Math. J.* **50**:1 (2002), 37–55. MR Zbl

[Graham et al. 2012] I. Graham, H. Hamada, and G. Kohr, "Extension operators and subordination chains", *J. Math. Anal. Appl.* **386**:1 (2012), 278–289. MR Zbl

[Graham et al. 2013] I. Graham, H. Hamada, G. Kohr, and M. Kohr, "Univalent subordination chains in reflexive complex Banach spaces", pp. 83–111 in *Complex analysis and dynamical systems V*, edited by M. L. Agranovsky et al., Contemp. Math. **591**, Amer. Math. Soc., Providence, RI, 2013. MR Zbl

[Graham et al. 2020a] I. Graham, H. Hamada, and G. Kohr, "A Schwarz lemma at the boundary on complex Hilbert balls and applications to starlike mappings", *J. Anal. Math.* **140**:1 (2020), 31–53. MR Zbl

[Graham et al. 2020b] I. Graham, H. Hamada, G. Kohr, and M. Kohr, "g-Loewner chains, Bloch functions and extension operators in complex Banach spaces", *Anal. Math. Phys.* **10**:1 (2020), Paper No. 5, 28. MR Zbl

[Gurganus 1975] K. R. Gurganus, "Φ-like holomorphic functions in $\mathbb{C}^n$ and Banach spaces", *Trans. Amer. Math. Soc.* **205** (1975), 389–406. MR Zbl

[Hamada and Honda 2008] H. Hamada and T. Honda, "Sharp growth theorems and coefficient bounds for starlike mappings in several complex variables", *Chinese Ann. Math. Ser. B* **29**:4 (2008), 353–368. MR Zbl

[Hamada and Kohr 2004] H. Hamada and G. Kohr, "Loewner chains and the Loewner differential equation in reflexive complex Banach spaces", *Rev. Roumaine Math. Pures Appl.* **49**:3 (2004), 247–264. MR Zbl

[Hamada et al. 2006] H. Hamada, T. Honda, and G. Kohr, "Growth theorems and coefficients bounds for univalent holomorphic mappings which have parametric representation", *J. Math. Anal. Appl.* **317**:1 (2006), 302–319. MR Zbl

[Hamada et al. 2021] H. Hamada, G. Kohr, and M. Kohr, "The Fekete–Szegö problem for starlike mappings and nonlinear resolvents of the Carathéodory family on the unit balls of complex Banach spaces", *Anal. Math. Phys.* **11**:3 (2021), Paper No. 115, 22. MR Zbl

[Kato 1967] T. Kato, "Nonlinear semigroups and evolution equations", *J. Math. Soc. Japan* **19** (1967), 508–520. MR Zbl

[Kikuchi 1973] K. Kikuchi, "Starlike and convex mappings in several complex variables", *Pacific J. Math.* **44** (1973), 569–580. MR Zbl

[Kubicka and Poreda 1988] E. Kubicka and T. Poreda, "On the parametric representation of starlike maps of the unit ball in $\mathbb{C}^n$ into $\mathbb{C}^n$", *Demonstratio Math.* **21**:2 (1988), 345–355. MR Zbl

[Li and Zhang 2019] X. Li and X. Zhang, "Janowski-starlike mappings of complex order λ on the unit ball B^n", *Results Math.* **74**:4 (2019), Paper No. 141, 13. MR Zbl

[Liu 1989] T. Liu, *The growth theorems, covering theorems and distortion theorems for biholomorphic mappings on classical domains*, Ph.D. thesis, Hefei: University of Science and Technology of China, 1989.

[Liu and Ren 1998] T. Liu and G. Ren, "Growth theorem of convex mappings on bounded convex circular domains", *Sci. China Ser. A* **41**:2 (1998), 123–130. MR Zbl

[Liu and Xu 2006] T.-S. Liu and Q.-H. Xu, "Loewner chains associated with the generalized Roper–Suffridge extension operator", *J. Math. Anal. Appl.* **322**:1 (2006), 107–120. MR Zbl

[Liu et al. 2011] T. Liu, J. Wang, and J. Lu, "Distortion theorems of starlike mappings in several complex variables", *Taiwanese J. Math.* **15**:6 (2011), 2601–2608. MR Zbl

[Liu et al. 2015] T. Liu, J. Wang, and X. Tang, "Schwarz lemma at the boundary of the unit ball in $\mathbb{C}^n$ and its applications", *J. Geom. Anal.* **25**:3 (2015), 1890–1914. MR Zbl

[Liu et al. 2019] T. Liu, X. Tang, and J. Wang, "Rigidity for convex mappings of Reinhardt domains and its applications", *Sci. China Math.* **62**:5 (2019), 901–920. MR Zbl

[Lu et al. 2012] J. Lu, T. Liu, and J. Wang, "Distortion theorems for subclasses of starlike mappings along a unit direction in $\mathbb{C}^n$", *Acta Math. Sci. Ser. B* (*Engl. Ed.*) **32**:4 (2012), 1675–1680. MR Zbl

[Muir 2005] J. R. Muir, Jr., "A modification of the Roper–Suffridge extension operator", *Comput. Methods Funct. Theory* **5**:1 (2005), 237–251. MR Zbl

[Muir 2008] J. R. Muir, Jr., "A class of Loewner chain preserving extension operators", *J. Math. Anal. Appl.* **337**:2 (2008), 862–879. MR Zbl

[Pfaltzgraff 1974] J. A. Pfaltzgraff, "Subordination chains and univalence of holomorphic mappings in C^n", *Math. Ann.* **210** (1974), 55–68. MR Zbl

[Pommerenke 1975] C. Pommerenke, *Univalent functions*, Studia Mathematica/Mathematische Lehrbücher **25**, Vandenhoeck & Ruprecht, Göttingen, 1975. MR Zbl

[Poreda 1989] T. Poreda, "On the univalent subordination chains of holomorphic mappings in Banach spaces", *Comment. Math. Prace Mat.* **28**:2 (1989), 295–304. MR Zbl

[Reich and Shoikhet 1996] S. Reich and D. Shoikhet, "Generation theory for semigroups of holomorphic mappings in Banach spaces", *Abstr. Appl. Anal.* **1**:1 (1996), 1–44. MR Zbl

[Reich and Shoikhet 2005] S. Reich and D. Shoikhet, *Nonlinear semigroups, fixed points, and geometry of domains in Banach spaces*, Imperial College Press, London, 2005. MR Zbl

[Roper and Suffridge 1995] K. A. Roper and T. J. Suffridge, "Convex mappings on the unit ball of $\mathbb{C}^n$", *J. Anal. Math.* **65** (1995), 333–347. MR Zbl

[Suffridge 1970] T. J. Suffridge, "The principle of subordination applied to functions of several variables", *Pacific J. Math.* **33** (1970), 241–248. MR Zbl

[Suffridge 1973] T. J. Suffridge, "Starlike and convex maps in Banach spaces", *Pacific J. Math.* **46** (1973), 575–589. MR Zbl

[Suffridge 1977] T. J. Suffridge, "Starlikeness, convexity and other geometric properties of holomorphic maps in higher dimensions", pp. 146–159 in *Complex analysis*, edited by J. D. Buckholtz and T. J. Suffridge, Lecture Notes in Math. **599**, Springer, 1977. MR Zbl

[Thomas 1991] C. R. Thomas, *Extensions of classical results in one complex variable to several complex variables*, Ph.D. thesis, University of California, San Diego, 1991, available at https://www.proquest.com/docview/303950067.

[Tu and Xiong 2019] Z. Tu and L. Xiong, "Growth and distortion results for a class of biholomorphic mapping and extremal problem with parametric representation in $\mathbb{C}^n$", *Complex Anal. Oper. Theory* **13**:6 (2019), 2747–2769. MR Zbl

[Wang 2013] J. Wang, "Modified Roper–Suffridge operator for some subclasses of starlike mappings on Reinhardt domains", *Acta Math. Sci. Ser. B* (*Engl. Ed.*) **33**:6 (2013), 1627–1638. MR Zbl

[Wang and Liu 2010] J. F. Wang and T. S. Liu, "A modified Roper–Suffridge extension operator for some holomorphic mappings", *Chinese Ann. Math. Ser. A* **31**:4 (2010), 487–496. MR

[Wang and Liu 2018] J. Wang and T. Liu, "The Roper–Suffridge extension operator and its applications to convex mappings in $\mathbb{C}^2$", *Trans. Amer. Math. Soc.* **370**:11 (2018), 7743–7759. MR Zbl

[Zhang and Yan 2016] X. Zhang and C. Yan, "Growth theorem and distortion theorem for a subclass of parabolic starlike mappings", *Chin. Quart. J. of Math.* **31**:2 (2016), 211–220. Zbl

[Zhang et al. 2018] X. Zhang, J. Lu, and X. Li, "Growth and distortion theorems for almost starlike mappings of complex order λ", *Acta Math. Sci. Ser. B* (*Engl. Ed.*) **38**:3 (2018), 769–777. MR Zbl

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