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SPIKE SOLUTIONS FOR A FRACTIONAL ELLIPTIC EQUATION IN A COMPACT RIEMANNIAN MANIFOLD

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Given an n-dimensional compact Riemannian manifold (M, g) without boundary, we consider the nonlocal equation

$$\varepsilon^{2s} P_g^s u + u = u^p \quad \text{in } (M, g),$$

where P_g^s stands for the fractional Paneitz operator with principal symbol $(-\Delta_g)^s$, $s \in (0, 1)$, $p \in (1, 2_s^* - 1)$ with $2_s^* := \frac{2n}{n-2s}$, $n > 2s$, represents the critical Sobolev exponent and $\varepsilon > 0$ is a small real parameter. We construct a family of positive solutions u_ε that concentrate, as $\varepsilon \to 0$ goes to zero, near critical points of the mean curvature H for $0 < s < \frac{1}{2}$ and near critical points of a reduced function involving the scalar curvature of the manifold M for $\frac{1}{2} \le s < 1$.

1. Introduction and preliminary results

Let $s \in (0, 1)$ and let (M, g) be an n-dimensional smooth compact Riemannian manifold without boundary with $n > 2s$. We consider the nonlocal problem

$$\varepsilon^{2s} P_g^s u + u = u^p, \quad u > 0 \text{ in } (M, g), \tag{1-1}$$

where P_g^s is the fractional Paneitz operator whose principal symbol is exactly $(-\Delta_g)^s$, $p \in (1, 2_s^* - 1)$ with $2_s^* := \frac{2n}{n-2s}$ is the critical Sobolev exponent and $\varepsilon > 0$ is a small real parameter. In this paper we study concentration phenomena of solutions to problem (1-1) as the parameter ε goes to zero. We prove that such solutions exist

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and concentration occur near critical points of the mean curvature H for $0 < s < \frac{1}{2}$ and near critical points of a reduced function involving the scalar curvature of the manifold M for $\frac{1}{2} \le s < 1$.

In the local setting (i.e., $s = 1$), an analogue-type result has been obtained by Micheletti and Pistoia [32]. They considered the following problem:

$$-\varepsilon^2 \Delta_g u + u = u^p, \quad u > 0 \text{ in } (M, g), \tag{1-2}$$

where (M, g) is a smooth compact Riemannian manifold of dimension $n \ge 2$, Δ_g is the Laplace–Beltrami operator on M, $p > 1$ for $n = 2$ and

$$1 < p < 2^* - 1 = \frac{n+2}{n-2} \quad \text{for } n \ge 3.$$

They constructed a family of positive solutions which concentrate, for sufficiently small values of ε, near stable critical points of the scalar curvature S_g of the metric g. Precisely, if J_ε is the energy functional defined by

$$J_\varepsilon(u) = \frac{1}{\varepsilon^n} \int_M \left[|\nabla_g u|^2 + \tfrac{1}{2} u^2 - \frac{1}{p+1} u^{p+1} \right] d\mu_g,$$

they proved that the following asymptotic expansion holds:

$$J_\varepsilon(u_\varepsilon) = c_0 - c_1 \varepsilon^2 S_g(\xi) + o(\varepsilon^2), \tag{1-3}$$

where c_0 and c_1 are explicit constants. Since any critical point of J_ε is a solution to problem (1-2), it turns out that is the scalar curvature function which is relevant for point concentration in M for problem (1-2). On the other hand, consider the following local singular perturbed Neumann problem:

$$-\varepsilon^2 \Delta u + u = u^p, \quad u > 0 \text{ in } \Omega, \quad \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega, \tag{1-4}$$

on a smooth bounded domain Ω in $\mathbb{R}^n$, where ε is a small parameter, ν denotes the outward normal to $\partial\Omega$, and the exponent $p > 1$. Lin, Ni and Takagi [30; 33; 34] proved that equation (1-4) possesses a least-energy solution u_ε which concentrate near maximum points of the mean curvature H of $\partial\Omega$ for ε sufficiently small. As above, the proof is based on an asymptotic expansion of the associated energy functional. They showed that

$$J_\varepsilon(u_\varepsilon) = \tfrac{1}{2} I(\omega) - c\varepsilon H(\xi) + o(\varepsilon), \tag{1-5}$$

where $c > 0$ is an explicit constant, w is the unique ground state solution of

$$\begin{cases} \Delta w - w + w^p = 0, & w > 0 \text{ in } \mathbb{R}^n, \\ w(0) = \max_{y \in \mathbb{R}^n} w(y), \\ \lim_{|y| \to +\infty} w(y) = 0, \end{cases}$$

and $I[w]$ is the ground-state energy

$$I[w] = \tfrac{1}{2}\int_{\mathbb{R}^n} |\nabla w|^2 \, dy + \tfrac{1}{2}\int_{\mathbb{R}^n} w^2 \, dy - \frac{1}{p+1}\int_{\mathbb{R}^n} w^{p+1} dy.$$

This time it turns out that the mean curvature of the boundary of Ω is relevant for point concentration of problem (1-4).

The main objective of this paper is to extend the previous results to the nonlocal setting. Before stating our main results we introduce some preliminary notations and definitions, we refer to [5; 6; 8; 20; 26] for more precise details.

Given an n-dimensional smooth compact Riemannian manifold $M = M^n$ without boundary, with $n \geq 2$ and let $X = X^{n+1}$ be a smooth $(n+1)$-dimensional manifold whose boundary is M^n. A function ρ is said to be a defining function of the boundary M^n in X^{n+1} if

$$\rho > 0 \text{ in } X^{n+1}, \quad \rho = 0 \text{ on } M^n \quad \text{and} \quad d\rho \neq 0 \text{ on } M^n. \tag{1-6}$$

We say that g^+ is conformally compact if, there exists a defining function ρ, such that the setting $\bar{g} = \rho^2 g^+$, the closure $(\overline{X}^{n+1}, \bar{g})$ is compact. This induces a conformal class of metrics $g = \bar{g}_{|TM^n}$ on M^n as defining functions vary. The conformal manifold $(M^n, [g])$ is called the conformal infinity of (X^{n+1}, g^+).

A metric g^+ is said to be asymptotically hyperbolic if it is conformally compact and the sectional curvature approaches -1 at infinity.

Given a conformally compact, asymptotically hyperbolic manifold (X^{n+1}, g^+) and a representative g in $[g]$ on the conformal infinity M, there is a uniquely defining function ρ such that, on $M \times (0, \delta)$ in (X, g^+), has the normal form

$$g^+ = \rho^{-2}(d\rho^2 + g_\rho),$$

where g_ρ is a one-parameter family of metrics on M satisfying $g_{\rho|M} = g$. Moreover, g_ρ has an asymptotic expansion which contains only even powers of ρ, at least up to degree n. It is well known (see Mazzeo and Melrose [31], Graham and Zworski [27]) that, given $f \in C^\infty(M)$ and $z \in \mathbb{C}$, the eigenvalue problem

$$-\Delta_{g^+} v - z(n-z)\, v = 0 \quad \text{in } X \tag{1-7}$$

has a solution of the form

$$v = F\rho^{n-z} + G\rho^z, \quad F, G \in C^\infty(X) \quad \text{and} \quad F_{|\rho=0} = f \tag{1-8}$$

for all $z \in \mathbb{C}$ unless $z(n-z)$ belongs to the pure point spectrum of $-\Delta_{g^+}$. Now, the scattering operator on M is defined by

$$S(z) f := G_{|M}, \tag{1-9}$$

which is a meromorphic family of pseudodifferential operator in $\left\{z \in \mathbb{C};\ \mathrm{Re}(z) > \frac{n}{2}\right\}$.

We define the conformally covariant fractional powers of the Laplacian by

$$P_g^s = P^s[g^+, g] := \begin{cases} \frac{-2^{2s} s\Gamma(s)}{\Gamma(1-s)} S\big(\frac{n}{2}+s\big) & \text{if } s \notin \mathbb{N}, \\ (-1)^s 2^{2s} s!(s-1)\operatorname{Res}_{z=\frac{n}{2}+s} S(z) & \text{for } s \in \mathbb{N}, \end{cases} \tag{1-10}$$

whose principal symbol is exactly $(-\Delta_g)^s$. Here $\operatorname{Res}_{z=s_0} S(z)$ is the residue at s_0 of S.

Notice that if (X, g^+) is Poincaré–Einstein, we have for $s = 1$

$$P_g^1 u = -\Delta_g u + \frac{n-2}{4(n-1)} R_g(u),$$

which is nothing but the usual conformal Laplacian, and for $s = 2$ we have

$$P_g^2 u = (-\Delta_g)^2 u - \operatorname{div}_g\big((c_1 R_g - c_2 \operatorname{Ric}_g)\, du\big) + \frac{n-4}{2} Q_g u,$$

which is nothing but the Paneitz operator.

The operator $P_g^s = P^s[g^+, g]$ satisfy an important conformal covariance property (see [8] and [27]). Indeed, for a conformal change of metric

$$g_v := v^{4/(n-2s)} g, \quad v > 0,$$

we have that

$$P^s[g^+, g_v]\phi = v^{-(n+2s)/(n-2s)} P^s[g^+, g](v\phi)$$

for all smooth functions ϕ defined on M.

Finally, we define the fractional scalar curvature Q_g^s associated to the conformal fractional Laplacian P_g^s by

$$Q_g^s := P_g^s(1). \tag{1-11}$$

According to [8], it is natural to consider the following degenerate equation with the weighted Neumann boundary condition:

$$\begin{cases} -\operatorname{div}(\rho^{1-2s}\nabla U) + E(\rho) U = 0 & \text{in } (X, \bar{g}), \\ \partial_\nu^s U = 0 & \text{on } (M, g), \end{cases} \tag{1-12}$$

where $\bar{g} := \rho^2 g^+$ is a compact metric on the closure $\overline{X}$ of X, g its restriction onto M ($g = \bar{g}_{|M}$) and

$$E(\rho) := \rho^{-1-z}(-\Delta_{g^+} - z(n-z))\,\rho^{n-z},$$

with $2z := n + 2s$ and

$$\partial_\nu^s U := -\kappa_s \lim_{\rho\to 0} \rho^{1-2s} \frac{\partial U}{\partial \rho}, \tag{1-13}$$

where ν is the outward normal vector to $M = \partial X$ and

$$\kappa_s := \frac{\Gamma(s)}{2^{1-2s}\Gamma(1-s)}. \tag{1-14}$$

Let $\mathrm{Ext}^s(u)$ be the s-harmonic extension of u and denote it by U. Chang and González [8] proved that the generalized eigenvalue problem (1-7) on a noncompact manifold (X^{n+1}, g^+) is equivalent to a linear degenerate elliptic problem on the compact manifold $(\overline{X}^{n+1}, \bar{g})$ for $\bar{g} = \rho^2 g^+$. Moreover, they identify the fractional Laplacian defined above with the normalized scattering operators and the one given in the spirit of the Dirichlet-to-Neumann operator by Caffarelli and Silvestre in [6]. Precisely, they proved the following result, which will play a crucial role in this paper and provides an alternative way to study problem (1-1).

Proposition 1.1 [8, Theorems 4.3 and 5.1]. *Let (X^{n+1}, g^+) be a asymptotically hyperbolic manifold with the conformal infinity $(M^n, [g])$ and ρ the geodesic defining function of g. Assume also that the trace H of the second fundamental form $\pi_{ij} = -\langle \nabla_{\partial_\rho}\partial_i, \partial_j\rangle_g$ on $M = \partial X$ vanishes if $s \in \left(\frac{1}{2}, 1\right)$. For a smooth function u on M, if v is a solution of* (1-7) *and satisfies* (1-8), *then the function $U := \rho^{z-n} v$ solves*

$$-\mathrm{div}(\rho^{1-2s}\nabla U) + E(\rho)U = 0 \quad \textit{in } (X, \bar{g}) \qquad \textit{and} \qquad U = u \quad \textit{on } (M, g), \tag{1-15}$$

where $\bar{g} := \rho^2 g^+$, $E(\rho) := \rho^{-1-z}(-\Delta_{g^+} - z(n-z))\rho^{n-z}$, and $2z := n + 2s$. Moreover,

$$P_g^s(u) = \begin{cases} \partial_\nu^s U & \textit{for } s \in (0,1)\backslash\left\{\frac{1}{2}\right\}, \\ \partial_\nu^s U + \frac{n-1}{2n} H u & \textit{for } s = \frac{1}{2}. \end{cases} \tag{1-16}$$

Here the operator $\partial_\nu^s U$ denotes the weighted normal derivative defined in (1-13).

For $r_0 > 0$ sufficiently small, it also holds that

$$E(\rho) = \frac{n-2s}{4n}\left[R_{\bar{g}}\rho^{1-2s} - (R_{g^+} + n(n+1))\rho^{-1-2s}\right] \quad \text{on } M \times (0, r_0). \tag{1-17}$$

Notice that the transformation law of the scalar curvature (see (1.1) in [20] and (2.3) in [28]) implies that

$$R_{g^+} = -n(n+1) + n\rho\partial_\rho \log(\det g(\rho)) + \rho^2 R_{\bar{g}} \quad \text{on } M \times (0, r_0), \tag{1-18}$$

then, using the fact that

$$\partial_\rho \log(\det g(\rho))_{|\rho=0} = \mathrm{Tr}\big(g(\rho)^{-1}\partial_\rho g(\rho)\big)_{\rho=0} = -2H, \tag{1-19}$$

the term $E(\rho)$ in (1-17) becomes

$$
\begin{aligned}
E(\rho)(z) &= -\Big(\frac{n-2s}{4}\Big)\,\partial_\rho \log(\det g(\rho))(\sigma)\,\rho^{-2s} \\
&= -\Big(\frac{n-2s}{4}\Big)\,\partial_\rho \log(\det g(\rho))_{|\rho=0}(\sigma)\,\rho^{-2s} + \mathcal{O}(\rho^{1-2s}) \\
&= \Big(\frac{n-2s}{2}\Big) H(\sigma)\,\rho^{-2s} + \mathcal{O}(\rho^{1-2s})
\end{aligned} \tag{1-20}
$$

for all $z = (\sigma, \rho) \in M \times (0, r_0)$.

Observe that (1-18) yields

$$R_{g^+} + n(n+1) = o(1)$$

near M for all asymptotically hyperbolic manifolds, where $o(1)$ is a quantity which goes to 0 uniformly as $\rho \to 0$. We assume that for $\frac{1}{2} \le s < 1$, the scalar curvature R_{g^+} in X satisfies the following decay assumption

$$R_{g^+} + n(n+1) = o(\rho^2) \quad \text{as } \rho \to 0 \text{ uniformly on } M. \tag{1-21}$$

Assumption (1-21) naturally appears to control extrinsic quantities such as the mean curvature H or the second fundamental form π on M, on the other hand, it is an intrinsic curvature condition of an asymptotically hyperbolic manifold, which is independent of the choice of a representative of the class $[g]$. Consequently, we have immediately from (1-21) (see, for instance, [11, Lemma 3.2]) that

$$H = 0 \quad \text{and} \quad R_{\rho\rho}[\bar{g}] = \frac{1-2n}{2(n-1)}\|\pi\|_g^2 + \frac{1}{2(n-1)} R[g]. \tag{1-22}$$

Before stating our main result, we define on M

$$\Xi(\xi) := \tfrac{1}{6}(\tilde{d} + \tilde{d}_1 \tilde{C}_{n,s}^2) R_g(\xi) + \tfrac{1}{6}\tilde{d}_1 \tilde{C}_{n,s}^3 \|\pi\|^2(\xi), \tag{1-23}$$

where the constants $\tilde{d}$, $\tilde{d}_1$, $\tilde{C}_{n,s}^2$ and $\tilde{C}_{n,s}^3$ will be defined later in (4-7), (4-8) and (4-19) respectively. Our main theorem reads as:

Theorem 1.2. *Let (X^{n+1}, g^+) be an asymptotically hyperbolic manifold with the conformal infinity $(M^n, [g])$ such that $M = \partial X$. Assume that $n > 2s + 2$ and let H be the trace of the second fundamental form of (M, g). Then there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, problem* (1-1) *has a solution u_ε which concentrates at a point $\xi \in M$ as ε goes to zero, where ξ is a critical point of H for $0 < s < \frac{1}{2}$, and is a critical point of the function Ξ defined in* (1-23), *for $\frac{1}{2} \le s < 1$ provided that* (1-21) *holds.*

Observe that, solving our main equation (1-1) is equivalent to finding a positive solution U to the problem

$$\begin{cases} -\mathrm{div}(\rho^{1-2s}\nabla U)+E(\rho)U=0 & \text{in } (X,\bar{g}), \\ \varepsilon^{2s}\partial_\nu^s U=u^p-u & \text{on } (M,g), \\ U|_M=u. \end{cases} \tag{1-24}$$

Up to a scaling in the second equation in the above problem (1-24), we are led to study the following nonlocal equation

$$(-\Delta_{g_\varepsilon})^s v+v=v^p, \quad v>0 \text{ in } (M_\varepsilon, g_\varepsilon), \tag{1-25}$$

where $M_\varepsilon=\frac{1}{\varepsilon}M$ endowed with the scaled metric $g_\varepsilon=\frac{1}{\varepsilon^2}g$. For $\varepsilon>0$ sufficiently small, we will construct an approximate solution to our problem whose leading term is a solution of the limit equation

$$(-\Delta)^s u+u=u^p, \quad u>0 \text{ in } H^s(\mathbb{R}^n). \tag{1-26}$$

Precisely, we will look for a solution u_ε to problem (1-1) that concentrate at interior points ξ of the manifold M which, at main order, looks like

$$u_\varepsilon(x)\approx\omega\Big(\frac{x-\xi}{\varepsilon}\Big), \tag{1-27}$$

where ω is the solution of the limit problem (1-26).

We recall that, for $s\in(0,1)$, the fractional Laplacian operator $(-\Delta)^s$ is defined at any point $x\in\mathbb{R}^n$ by

$$(-\Delta)^s u(x)=c_{n,s}\int_{\mathbb{R}^n}\frac{2u(x)-u(x+y)-u(x-y)}{|y|^{n+2s}}\,dy, \tag{1-28}$$

where $c_{n,s}$ is an explicit positive normalizing constant and $H^s(\mathbb{R}^n)$ is the fractional Sobolev space of order s on $\mathbb{R}^n$, defined by

$$H^s(\mathbb{R}^n):=\bigg\{u\in L^2(\mathbb{R}^n):\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy<\infty\bigg\}, \tag{1-29}$$

endowed with the norm

$$\|u\|_{H^s(\mathbb{R}^n)}:=\bigg(\int_{\mathbb{R}^n}|u(x)|^2\,dx+\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\bigg)^{1/2}. \tag{1-30}$$

We refer to [18; 29; 40] for an introduction to the fractional Laplacian operator.

Concentration phenomenon for related nonlocal PDEs in the euclidean space have attracted lot of attention. For example, if we consider the fractional Schrödinger equation

$$(-\Delta)^s u+V(x)u=f(x,u) \quad \text{in } \mathbb{R}^n \tag{1-31}$$

under suitable conditions on the potential V and the nonlinearity f, existence and multiplicity results of spike layer solutions have been obtained (see, for instance, Alves and Miyagaki [1], Alves, de Lima and Nóbrega [2], Autuori and Pucci [3], Felmer, Quaas and Tan [22], Cheng [12], Secchi [35], Dávila, del Pino and Wei [16], Dipierro, Palatucci and Valdinoci [19], Fall, Mahmoudi and Valdinoci [21], Bisci and Rădulescu [4], Servadei and Valdinoci [36; 37], Shang and Zhang [38; 39], Caponi and Pucci [7], Fiscella, Pucci and Saldi [23]). See also [9; 10; 36].

Some results have also been obtained for the fractional nonlinear Schrödinger (NLS) equation in bounded domains under Dirichlet and Neumann boundary conditions. We mention the result of Dávila et al. [17] who built a family of solutions that concentrate at an interior point of the domain for a fractional NLS with zero Dirichlet datum. The Neumann fractional NLS have been considered in [41]. See also [13] where concentration phenomena for a perturbed fractional Yamabe problem has been considered.

The rest of the paper is organized as follows. In Section 2, we first give some properties of the limit profile and the linearized operator around it. Then, we give the asymptotic expansion of the metric and we prove some preliminary results. Finally, we construct the first ansatz of the approximate solution and its decay properties. Section 3 is devoted to the finite dimensional reduction procedure. In Section 4, we prove our main result using the asymptotic expansions of the finite dimensional problem obtained in Sections 4A and 4B. Finally, in Appendix, we prove Lemma 2.7.

2. Setting-up of the problem

2A. ***Uniqueness and nondegeneracy for the limit equation.*** In this subsection, we recall some known results for the limit equation (1-26). Frank, Lenzmann and Silvestre [25] proved uniqueness and nondegeneracy of ground state solutions for (1-26) in arbitrary dimension $n \geq 1$ and any admissible exponent $1 < p < \frac{n+2s}{n-2s}$. We summarize the results of [24] and [25] in the following lemmas.

Lemma 2.1. *Let $n \geq 1$, $s \in (0, 1)$ and $p \in \left(1, \frac{n+2s}{n-2s}\right)$. Then there exists a unique solution* (*up to translation*) *$\omega \in H^s(\mathbb{R}^n)$ of* (1-26). *Moreover, ω is radial, positive, strictly decreasing in $|x|$ and satisfies*

$$\frac{C_1}{1+|x|^{n+2s}} \leq \omega(x) \leq \frac{C_2}{1+|x|^{n+2s}} \quad \text{for } x \in \mathbb{R}^n, \tag{2-1}$$

with some constants $C_2 \geq C_1 > 0$.

Lemma 2.2. *Let $n \geq 1$, $s \in (0, 1)$ and $p \in \left(1, \frac{n+2s}{n-2s}\right)$. Suppose that ω is the solution of the limit problem* (1-26). *Then the linearized operator*

$$L_0(\phi) := (-\Delta)^s \phi + \phi - p\omega^{p-1}\phi$$

is nondegenerate. That is, its kernel is given by

$$\ker L_0 = \mathrm{Span}\{\partial_{x_1}\omega, \dots, \partial_{x_n}\omega\}. \tag{2-2}$$

The nondegeneracy implies that 0 is an isolated spectral point of L_0. More precisely, for all $\phi \in (\ker L_0)^{\perp}$, one has

$$\|L_0(\phi)\|_{L^2(\mathbb{R}^n)} \geq c\|\phi\|_{H^{2s}(\mathbb{R}^n)} \tag{2-3}$$

for some positive constant c. By Lemma C.2 of [25], it holds that, for $j = 1, \dots, n$, $\partial_{x_j}\omega$ has the decay estimate

$$|\partial_{x_j}\omega| \leq \frac{C}{1+|x|^{n+2s}}. \tag{2-4}$$

It is well known that when $s = 1$, the ground state solution of (1-26) decays exponentially at infinity. However, when $s \in (0, 1)$, the corresponding ground bound state solution decays polynomially like $\frac{1}{|x|^{n+2s}}$ when $|x| \to \infty$.

Let W denote the s-harmonic extension of ω to $\mathbb{R}^{n+1}_+$, that is, W satisfies

$$\begin{cases} \mathrm{div}(t^{1-2s}\nabla W) = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ \partial^s_\nu W = \omega^p - \omega & \text{on } \mathbb{R}^n, \\ W = \omega & \text{on } \mathbb{R}^n. \end{cases} \tag{2-5}$$

Next, we define for all $i = 1, \dots, n$

$$z_i(x) := \partial_{x_i}\omega(x), \quad x \in \mathbb{R}^n \tag{2-6}$$

and we set $Z_i(x, t) = \mathrm{Ext}^s(z_i(x))$, the s-harmonic extension of z_i. It has been proven in [15] that any bounded solution on $\mathbb{R}^n \times \{0\}$ of the linearized equation

$$\begin{cases} \mathrm{div}(t^{1-2s}\nabla \Phi) = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ \partial^s_\nu \Phi = p\omega^{p-1}\Phi - \Phi & \text{on } \mathbb{R}^n \end{cases} \tag{2-7}$$

is a linear combination of Z_i.

2B. *Preliminary results.* We first give the asymptotic expansion of the metric of an asymptotically hyperbolic manifold X near its boundary M. Next, we introduce the functional setting and we give the first ansatz of the approximate solution and its decay properties.

Asymptotic expansion of the metric $\bar{g}$ near the boundary. Let (X, g^+) be an asymptotically hyperbolic manifold with boundary (M, g) and let ρ be the geodesic defining function, so that $(\overline{X}, \bar{g})$ is a compact manifold where $\bar{g} = \rho^2 g^+$. Assume $0 \in M$ and let $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ be normal coordinates on M at 0 and

$$(x, x_{n+1}) \in \mathbb{R}^n \times (0, +\infty)$$

be the Fermi coordinates on X at 0. We set $N = n+1$ and

$$\bar{g} = dx_N^2 + g_{ij}(x, x_N)\,dx_i\,dx_j, \tag{2-8}$$

so that $\bar{g}_{|M} = g$. Here the indices i, j run from 1 to n and summations over repeated indices is understood. We have the following asymptotic expansion of the metric $\bar{g}$ near 0, see Lemmas 3.1 and 3.2 in [20] and Lemma 2.2 in [28]. Precisely, we have:

Lemma 2.3. *For $(x_1, \dots, x_n) \in \mathbb{R}^n$ and $x_N = x_{n+1} > 0$ small, it holds that*

$$g^{ij} = \bar{g}^{ij} = \delta_{ij} + 2\pi_{ij}x_N + \tfrac{1}{3}R_{ikjl}x_k x_l + 2\pi_{ij,k}x_k x_N + (3\pi_{ih}\pi_{hj} R_{iNjN})x_N^2 + \mathcal{O}(|(x, x_N)|^3) \tag{2-9}$$

and

$$\det \bar{g} = \det g = 1 - 2Hx_N + (H^2 - \|\pi\|^2 - R_{NN})x_N^2 - 2H_{,k}\, x_k x_N - \tfrac{1}{3}R_{kl}x_k x_l + \mathcal{O}(|(x, x_N)|^3), \tag{2-10}$$

$$\sqrt{\det \bar{g}} = \sqrt{\det g} = 1 - Hx_N + \tfrac{1}{2}(H^2 - \|\pi\|^2 - R_{NN})x_N^2 - H_{,k}\, x_k x_N - \tfrac{1}{6}R_{kl}x_k x_l + \mathcal{O}(|(x, x_N)|^3). \tag{2-11}$$

Here π stands for the second fundamental form of $M = \partial X$, H is its trace, R_{ij} are the components of the Ricci tensor, R_{ijkl} are the components of the Riemannian tensor and $R_{NN} = \bar{g}^{ij}R_{iNjN}$. The indices i, j, k, and l run from 1 to n, summations over repeated indices is understood and every tensors are computed at 0.

The functional setting. We define the space $H(X, \rho^{1-2s})$ to be the weighted Sobolev space endowed with the inner product

$$\langle U, V\rangle_\varepsilon := \frac{1}{\varepsilon^{n-2s}} \int_X \rho^{1-2s}[(\nabla U, \nabla V)_{\bar{g}} + UV]\,d\mathrm{vol}_{\bar{g}} \tag{2-12}$$

and the corresponding norm

$$\|U\|_\varepsilon = \left(\frac{1}{\varepsilon^{n-2s}} \int_X \rho^{1-2s}[|\nabla U|_{\bar{g}}^2 + U^2]\,d\mathrm{vol}_{\bar{g}}\right)^{1/2}. \tag{2-13}$$

Let L_ε^q be the Banach space $L_g^q(M)$ equipped the norm

$$|U|_{q,\varepsilon} := \left(\frac{1}{\varepsilon^n} \int_M |U|^q\,d\mathrm{vol}_g\right)^{1/q}. \tag{2-14}$$

It is clear that for any $1 \le q < \frac{2n}{n-2s}$, the embedding of $H^1(X, \rho^{1-2s})$ in $L^q(M)$ is continuous and compact. Particularly, there exists a constant $c = c(s, n, X)$ such that

$$|U|_{q,\varepsilon} \le c\|U\|_\varepsilon. \tag{2-15}$$

The next lemmas provide equivalent norms to the $\|\cdot\|_\varepsilon$-norm.

Lemma 2.4. *The norm*

$$\|U\|_{\varepsilon,*} := \left(\frac{1}{\varepsilon^{n-2s}} \int_X \rho^{1-2s} |\nabla U|^2_{\bar{g}} \, d\mathrm{vol}_{\bar{g}} + \frac{1}{\varepsilon^n} \int_M U^2 \, d\mathrm{vol}_g \right)^{1/2} \tag{2-16}$$

is equivalent to the norm $\|\cdot\|_\varepsilon$ *defined in* (2-13).

Lemma 2.5. *Assume that the mean curvature* H *on* $M = \partial X$ *vanishes for* $s \in \left[\frac{1}{2}, 1\right)$ (*which is the case when* (1-21) *holds*) *and there exists a constant* $C > 0$ *such that the coercivity assumption*

$$\frac{1}{\varepsilon^{n-2s}} \left(\int_X \rho^{1-2s} |\nabla U|^2_{\bar{g}} + E(\rho) U^2 \right) d\mathrm{vol}_{\bar{g}} + \frac{1}{\varepsilon^n} \int_M U^2 \, d\mathrm{vol}_g \geq \frac{C}{\varepsilon^{n-2s}} \int_X \rho^{1-2s} U^2 \, d\mathrm{vol}_{\bar{g}} \tag{2-17}$$

holds for arbitrary function $U \in H^1(X, \rho^{1-2s})$. *Then the norm*

$$\|U\|_{\varepsilon,**} := \left(\frac{1}{\varepsilon^{n-2s}} \kappa_s \int_X \left(\rho^{1-2s} |\nabla U|^2_{\bar{g}} + E(\rho) U^2 \right) d\mathrm{vol}_{\bar{g}} + \frac{1}{\varepsilon^n} \int_M U^2 \, d\mathrm{vol}_g \right)^{1/2}$$

is an equivalent norm to $\|\cdot\|_\varepsilon$.

Proof. For the proof of the previous lemmas, we refer the reader to Lemmas 3.1 and 3.2 in [13]. □

We next define the trace operator

$$i : H^1(X, \rho^{1-2s}) \to L^p(M) \tag{2-18}$$

by $i(U) = U_{|M} := u$. The operator i is well defined, continuous and, compact for $1 \leq p < \frac{2n}{n-2s}$. The adjoint operator $i^* : L^{p'} \to H^1(X, \rho^{1-2s})$, where $\frac{1}{p'} = \frac{1}{p} + \frac{2s}{n}$, is a continuous map defined by the equation

$$\begin{cases} -\mathrm{div}(t^{1-2s} \nabla U) + E(\rho) U = 0 & \text{in } (X, \bar{g}), \\ \varepsilon^{2s} \partial_\nu^s U = v - u & \text{on } (M, g), \\ U = u & \text{on } (M, g), \end{cases} \tag{2-19}$$

where $U = i^*(v)$ is bounded thanks to Lemma 2.5. The above properties are proved in [13]. We summarize them in the next lemma.

Lemma 2.6 [13, Lemma 3.3 and Corollary 3.4]. *Assume* $n > 2s$ *and* $p \in \left(1, \frac{n+2s}{n-2s}\right)$. *Then the embedding* $i : H^1(X, \rho^{1-2s}) \hookrightarrow L^p(M)$ *is compact continuous map. The adjoint operator* $i^* : L^{p'} \to H^1(X, \rho^{1-2s})$, *where* p' *satisfying* $\frac{1}{p} = \frac{1}{p'} - \frac{2s}{n}$, *is a continuous map. In other words, if* $v \in L^{p'}(M)$ *such that* $U = i^*(v)$ *and* $u = i(U)$, *then there exists* $C = C(p) > 0$ *such that*

$$\|u\|_{L^p(M)} \leq C \|v\|_{L^{p'}(M)}. \tag{2-20}$$

Furthermore, for $n > 2s$ and for any fixed $q \in \left(1, \frac{n+2s}{n-2s}\right)$, the adjoint map i^* : $L^q(M) \to H^1(X, \rho^{1-2s})$ is compact.

By Lemma 2.6, we can rewrite problem (1-24) in the equivalent way

$$U = i^*(f(u)) \text{ and } U = u > 0 \quad \text{on } M \tag{2-21}$$

for $U \in H^1(X, \rho^{1-2s})$ and $f(u) := u^p$.

Decay properties of approximate solutions. Recall that we want to find a solution U to the problem

$$\begin{cases} -\operatorname{div}(\rho^{1-2s}\nabla U) + E(\rho)U = 0 & \text{in } (X, \bar{g}), \\ \varepsilon^{2s}\partial_\nu^s U = u^p - u & \text{on } (M, g). \end{cases} \tag{2-22}$$

Let r_0 be a small positive real number be as in (1-17), we choose $r < r_0$ a positive number less than quarter of the injectivity radius of (M, g). We define χ_r to be a smooth cut-off function such that $\chi_r = 1$ in $(0, r)$ and 0 in $(2r, \infty)$. Observe that, any point $z \in X$ near the boundary M can be described as $z = (\hat{\xi}, \rho)$ for some $\hat{\xi} \in M$ and $\rho \in (0, \infty)$.

Let $W(\,\cdot\,,\,\cdot\,)$ be the s-harmonic extension of ω, solution of the limit problem (1-26) and define the scaled function W_ε $(\varepsilon > 0)$ by

$$W_\varepsilon(x, x_N) := W\left(\frac{x}{\varepsilon}, \frac{x_N}{\varepsilon}\right), \quad x \in \mathbb{R}^n,\ x_N > 0. \tag{2-23}$$

Fix a point $\xi \in M$, we define the functions $\mathcal{W}_{\varepsilon,\xi}$ on X by

$$\mathcal{W}_{\varepsilon,\xi}(z) = \mathcal{W}_{\varepsilon,\xi}(\hat{\xi}, \rho) = \begin{cases} \chi_r(d(z, \xi))\, W_\varepsilon(\exp_\xi^{-1}(\hat{\xi}), \rho) & \text{if } d(z, \xi) < 2r, \\ 0, & \text{otherwise,} \end{cases} \tag{2-24}$$

where exp is the exponential map on (M, g) and $d(\,\cdot\,, \xi)$ is the function defined near the boundary of $(X, \bar{g})$ by

$$d(z, \xi)^2 = d((\hat{\xi}, \rho), \xi)^2 = d_M(\hat{\xi}, \xi)^2 + \rho^2,$$

where $d_M(\,\cdot\,, \xi)$ is the geodesic distance from ξ on (M, g).

We look for a solution of problem (1-24) of the form

$$U = \mathcal{W}_{\varepsilon,\xi} + \Phi, \tag{2-25}$$

where Φ is a function defined on X whose $H^1(X, \rho^{1-2s})$-norm is sufficiently small and $\mathcal{W}_{\varepsilon,\xi}$ is the global approximation given in (2-24). Now, for $\xi \in M$, $\varepsilon > 0$ and $i = 1, \dots, n$, we introduce the functions

$$\mathcal{Z}^i_{\varepsilon,\xi}(z) = \mathcal{Z}^i_{\varepsilon,\xi}(\hat{\xi}, \rho) = \begin{cases} \chi_r(d(z, \xi))\, Z^i_\varepsilon(\exp_\xi^{-1}(\hat{\xi}), \rho) & \text{if } d(z, \xi) < 2r, \\ 0, & \text{otherwise,} \end{cases} \tag{2-26}$$

where $Z_\varepsilon^i,\ i = 1, \dots, n$, are defined by

$$Z_\varepsilon^i(x, x_N) := Z_i\Big(\frac{x}{\varepsilon}, \frac{x_N}{\varepsilon}\Big), \tag{2-27}$$

with $Z_i = \mathrm{Ext}^s(z_i)$, the s-harmonic extension of the functions z_i defined in (2-6).

Next, we introduce the subspace

$$K_{\varepsilon,\xi} := \mathrm{Span}\{\mathcal{Z}^1_{\varepsilon,\xi}, \dots, \mathcal{Z}^n_{\varepsilon,\xi}\} \tag{2-28}$$

and we let $K^\perp_{\varepsilon,\xi}$ be its orthogonal complement with respect to the inner product $\langle\,\cdot\,,\cdot\,\rangle_{\varepsilon,**}$, that is,

$$K^\perp_{\varepsilon,\xi} := \big\{U \in H^1(X, \rho^{1-2s}) : \langle \mathcal{Z}^i_{\varepsilon,\xi}, U\rangle_{\varepsilon,**} = 0 \text{ for all } i = 1, \dots, n\big\}. \tag{2-29}$$

Furthermore, denote by

$$\Pi_{\varepsilon,\xi} : H^1(X, \rho^{1-2s}) \to K_{\varepsilon,\xi} \quad \text{and} \quad \Pi^\perp_{\varepsilon,\xi} : H^1(X, \rho^{1-2s}) \to K^\perp_{\varepsilon,\xi} \tag{2-30}$$

the orthogonal projections onto $K_{\varepsilon,\xi}$ and $K^\perp_{\varepsilon,\xi}$ respectively.

The function $U = \mathcal{W}_{\varepsilon,\xi} + \Phi$ is a solution of (1-24) if and only if Φ solves

$$\Pi^\perp_{\varepsilon,\xi}\big\{\mathcal{W}_{\varepsilon,\xi} + \Phi - i^*\big(i\big(f(\mathcal{W}_{\varepsilon,\xi} + \Phi)\big)\big)\big\} = 0, \tag{2-31}$$

$$\Pi_{\varepsilon,\xi}\big\{\mathcal{W}_{\varepsilon,\xi} + \Phi - i^*\big(i\big(f(\mathcal{W}_{\varepsilon,\xi} + \Phi)\big)\big)\big\} = 0. \tag{2-32}$$

We end this section by the following result which concerns the decay property of W_ε and the functions Z_ε^i defined in (2-26). We postpone its proof to Appendix.

Lemma 2.7. *Assume that $n \geq 2$, fix any $0 < R_1 < R_2$ and set $A^+_{(R_1,R_2)} := B^+_{R_2} \backslash B^+_{R_1}$. Then as $\varepsilon \to 0$ the following estimates hold true*:

$$\int_{\mathbb{R}^{n+1}_+ \backslash B^+_{R_1}} x_N^{1-2s} |\nabla W_\varepsilon|^2 \, dx \, dx_N = \mathcal{O}(\varepsilon^{2n-4s}), \tag{2-33}$$

$$\int_{B^+_{R_1}} x_N^{2-2s} |\nabla W_\varepsilon|^2 \, dx \, dx_N = \begin{cases} \mathcal{O}(\varepsilon^{n+1-2s}) & \textit{for } n > 2s+1, \\ \mathcal{O}(\varepsilon^2 |\ln \varepsilon|) & \textit{for } n = 2s+1, \\ \mathcal{O}(\varepsilon^{2n-4s}) & \textit{for } n < 2s+1, \end{cases} \tag{2-34}$$

$$\int_{A^+_{(R_1,R_2)}} x_N^{1-2s} W_\varepsilon^2 \, dx \, dx_N = \begin{cases} \mathcal{O}(\varepsilon^{2n-4s}) & \textit{for } n \neq 2s+2, \\ \mathcal{O}(\varepsilon^4 |\ln \varepsilon|) & \textit{for } n = 2s+2. \end{cases} \tag{2-35}$$

Moreover, *we have*

$$\int_{\mathbb{R}^{n+1}_+ \backslash B^+_{R_1}} x_N^{1-2s} |\nabla Z^i_\varepsilon|^2 \, dx \, dx_N = \mathcal{O}(\varepsilon^{2n-4s}) \quad \textit{for } i = 1, \dots, n. \tag{2-36}$$

$$\int_{A^+_{(R_1,R_2)}} x_N^{1-2s} (Z^i_\varepsilon)^2 \, dx \, dx_N = \mathcal{O}(\varepsilon^{2n-4s}) \quad \textit{for } i = 1, \dots, n, \tag{2-37}$$

and

$$\text{(2-38)}\quad \int_{B^+_{R_1}} x_N^{1-2s}\mathcal{O}(|(x,x_N)|^2)\,|\nabla W_\varepsilon|^2\,dx\,dx_N = \begin{cases} \mathcal{O}(\varepsilon^{n+2-2s}) & \text{for } n > 2s+2, \\ \mathcal{O}(\varepsilon^4|\ln\varepsilon|) & \text{for } n = 2s+2, \\ \mathcal{O}(\varepsilon^{2n-4s}) & \text{for } n < 2s+2. \end{cases}$$

3. The finite-dimensional reduction

In this section we will solve (2-31). Let us introduce the linear operator

$$L_{\varepsilon,\xi} : (K_{\varepsilon,\xi})^\perp \to (K_{\varepsilon,\xi})^\perp$$

defined by

$$\text{(3-1)}\qquad L_{\varepsilon,\xi}(\Phi) := \Pi^\perp_{\varepsilon,\xi}\big(\Phi - i^*\big(i(f'(\mathcal{W}_{\varepsilon,\xi})\Phi)\big)\big), \quad \Phi \in (K_{\varepsilon,\xi})^\perp.$$

Clearly, equation (2-31) is equivalent to

$$\text{(3-2)}\qquad L_{\varepsilon,\xi}(\Phi) = N_{\varepsilon,\xi}(\Phi) + R_{\varepsilon,\xi},$$

where

$$\text{(3-3)}\qquad R_{\varepsilon,\xi} := \Pi^\perp_{\varepsilon,\xi}\big[i^*\big(i(f(\mathcal{W}_{\varepsilon,\xi}))\big) - \mathcal{W}_{\varepsilon,\xi}\big],$$

$$\text{(3-4)}\qquad N_{\varepsilon,\xi}(\Phi) := \Pi^\perp_{\varepsilon,\xi}\big[i^*\big(i\big(f(\mathcal{W}_{\varepsilon,\xi}+\Phi) - f(\mathcal{W}_{\varepsilon,\xi}) - f'(\mathcal{W}_{\varepsilon,\xi})\Phi\big)\big)\big].$$

Our first task is to study the invertibility of $L_{\varepsilon,\xi}$. This is given by the next lemma.

Lemma 3.1. *Suppose that $n > 2s$. Then, there exists $\varepsilon_0 > 0$ and $c > 0$ such that for any $\xi \in M$ and for any $\varepsilon \in (0, \varepsilon_0)$*

$$\text{(3-5)}\qquad \|L_{\varepsilon,\xi}(\Phi)\|_{\varepsilon,**} \geq c\|\Phi\|_{\varepsilon,**}$$

for all $\Phi \in (K_{\varepsilon,\xi})^\perp$.

Proof. The proof is based on classical blow up argument. We argue by contradiction, assuming that there exist sequences $\varepsilon_m \to 0$, $\xi_m \in M$ such that $\xi_m \to \xi$, $\Phi_m \in K^\perp_{\varepsilon_m,\xi_m}$, with $\|\Phi_m\|_{\varepsilon_m,**} = 1$ such that

$$\text{(3-6)}\qquad L_{\varepsilon_m,\xi_m}(\Phi_m) = \psi_m \quad \text{and} \quad \|\psi_m\|_{\varepsilon_m,**} \to 0.$$

We can write, by the above decomposition, that

$$\text{(3-7)}\qquad \Phi_m - i^*\big(i(f'(\mathcal{W}_{\varepsilon,\xi})\Phi_m)\big) = \psi_m + \zeta_m,$$

with $\zeta_m = \sum_{k=1}^n (c_k)_m\, \mathcal{Z}^k_{\varepsilon_m,\xi_m} \in K_{\varepsilon,\xi}$. We claim that

$$\text{(3-8)}\qquad \|\zeta_m\|_{\varepsilon_m,**} \to 0.$$

Indeed, multiplying (3-7) by $\mathcal{Z}^l_{\varepsilon_m,\xi_m}$ for $l = 1, \dots, n$ and integrating, taking into account that $\Phi_m, \psi_m \in K^{\perp}_{\varepsilon_m,\xi_m}$, we get

$$\sum_{k=1}^{n} (c_k)_m \langle \mathcal{Z}^k_{\varepsilon_m,\xi_m}, \mathcal{Z}^l_{\varepsilon_m,\xi_m} \rangle_{\varepsilon_m,**} = -\frac{1}{\varepsilon_m^n} \int_M f'(\mathcal{W}_{\varepsilon_m,\xi_m}) \Phi_m \mathcal{Z}^l_{\varepsilon_m,\xi_m} \, d\mathrm{vol}_g. \tag{3-9}$$

A straightforward computations yield

$$\begin{aligned}
&\langle \mathcal{Z}^k_{\varepsilon_m,\xi_m}, \mathcal{Z}^l_{\varepsilon_m,\xi_m} \rangle_{\varepsilon_m,**} \\
&= \frac{1}{\varepsilon_m^{n-2s}} \kappa_s \left[\int_X \rho^{1-2s} \nabla_{\bar g} \mathcal{Z}^k_{\varepsilon_m,\xi_m} \nabla_{\bar g} \mathcal{Z}^l_{\varepsilon_m,\xi_m} + E(\rho) \mathcal{Z}^k_{\varepsilon_m,\xi_m} \mathcal{Z}^l_{\varepsilon_m,\xi_m} \right] d\mathrm{vol}_{\bar g} \\
&\qquad + \frac{1}{\varepsilon_m^n} \int_M \mathcal{Z}^k_{\varepsilon_m,\xi_m} \mathcal{Z}^l_{\varepsilon_m,\xi_m} \, d\mathrm{vol}_g \\
&= \kappa_s \int_{B^+_{r/\varepsilon_m}} x_N^{1-2s} |\bar g(\varepsilon_m x, \varepsilon_m x_N)|^{\frac12} \\
&\quad \big[g^{ij}(\varepsilon_m x, \varepsilon_m x_N) \, \partial_i \big(Z_k(x, x_N) \chi(\varepsilon_m x, \varepsilon_m x_N) \big) \\
&\qquad \cdot \partial_j \big(Z_l(x, x_N) \chi(\varepsilon_m x, \varepsilon_m x_N) \big) \\
&\qquad + \partial_N \big(Z_k(x, x_N) \chi(\varepsilon_m x, \varepsilon_m x_N) \big) \, \partial_N \big(Z_l(x, x_N) \chi(\varepsilon_m x, \varepsilon_m x_N) \big) \big] \, dx \, dx_N \\
&\quad + \varepsilon_m^{1+2s} \kappa_s \int_{B^+_{r/\varepsilon_m}} E(t) |\bar g(\varepsilon_m x, \varepsilon_m x_N)|^{\frac12} \\
&\qquad \big(Z_k(x, x_N) \chi(\varepsilon_m x, \varepsilon_m x_N) \big) \big(Z_l(x, x_N) \chi(\varepsilon_m x, \varepsilon_m x_N) \big) \, dx \, dx_N \\
&\quad + \int_{B^+_{r/\varepsilon_m}} |\bar g(\varepsilon x, \varepsilon x_N)|^{\frac12} \\
&\qquad \big(Z_k(x, x_N) \chi(\varepsilon_m x, \varepsilon_m x_N) \big) \big(Z_l(x, x_N) \chi(\varepsilon_m x, \varepsilon_m x_N) \big) \\
&= c\delta_{kl} + o(1),
\end{aligned} \tag{3-10}$$

where c is a positive constant. Then, setting

$$\tilde\Phi_m(y) = \begin{cases} \chi_r(\varepsilon_m y) \, \Phi_m(\exp_{\xi_m}(\varepsilon_m y)) & \text{if } y \in B(0, r/\varepsilon_m), \\ 0, & \text{otherwise,} \end{cases}$$

it is easy to check that

$$\|\tilde\Phi_m\|_{H^1(\mathbb{R}^{n+1}_+, x_N^{1-2s})} \le C$$

for some positive constant C. Hence,

$$\tilde\Phi_m \rightharpoonup \tilde\Phi \quad \text{in } H^1(\mathbb{R}^{n+1}_+, x_N^{1-2s}),$$

and by the compactness of the trace operator we deduce that

$$\tilde\Phi_m \to \tilde\Phi \quad \text{in } L^q_{\mathrm{loc}}(\mathbb{R}^n) \text{ for any } 1 \le q < \frac{2n}{n-2s}.$$

Using this, together with the fact that $\Phi_m \in K^{\perp}_{\varepsilon_m,\xi_m}$, we get

$$
\begin{aligned}
(3\text{-}11)\quad & -\frac{1}{\varepsilon_m^n}\int_M f'(\mathcal{W}_{\varepsilon_m,\xi_m})\Phi_m\,\mathcal{Z}^l_{\varepsilon_m,\xi_m}\,d\mathrm{vol}_g \\
&= \frac{1}{\varepsilon_m^{n-2s}}\kappa_s\int_X\big(\rho^{1-2s}\nabla_{\bar g}\mathcal{Z}^l_{\varepsilon_m,\xi_m}\nabla_{\bar g}\Phi_m + E(\rho)\mathcal{Z}^l_{\varepsilon_m,\xi_m}\Phi_m\big)\,d\mathrm{vol}_{\bar g} \\
&\qquad\qquad + \frac{1}{\varepsilon_m^n}\int_M (1-f'(\mathcal{W}_{\varepsilon_m,\xi_m}))\Phi_m\,\mathcal{Z}^l_{\varepsilon_m,\xi_m}\,d\mathrm{vol}_g \\
&= \kappa_s\int_{\mathbb{R}^{n+1}_+} x_N^{1-2s}\nabla Z_i\nabla\tilde\Phi\,dx\,dx_N + \int_M (1-f'(W))\tilde\Phi Z_i\,dx + o(1) = o(1).
\end{aligned}
$$

Combining (3-9)–(3-11), we deduce that $(c_k)_m \to 0$ for any $k = 1,\dots,n$, and the claim (3-8) is proved.

Now, we consider the functions φ_m defined by

$$
\varphi_m(y) = \begin{cases} \chi_r(\varepsilon_m y)\,\varphi(\exp_{\xi_m}(\varepsilon_m y)) & \text{if } y \in B(0, r/\varepsilon_m), \\ 0, & \text{otherwise} \end{cases}
$$

for any function $\varphi \in C_0^\infty(\mathbb{R}^{n+1})$. We multiply (3-7) by φ_m, we get

$$
\langle \Phi_m, \varphi_m\rangle_{\varepsilon_m,**} = \big\langle i^*\big(i(f'(\mathcal{W}_{\varepsilon_m,\xi_m})\Phi_m), \varphi_m\big)\big\rangle_{\varepsilon_m,**} + \Big\langle \Psi_m + \sum_{k=1}^n c_k\mathcal{Z}^k_{\varepsilon_m,\xi_m}, \varphi_m\Big\rangle_{\varepsilon_m,**}.
$$

Since

$$
\Big\langle \Psi_m + \sum_{k=1}^n c_k\mathcal{Z}^k_{\varepsilon_m,\xi_m}, \varphi_m\Big\rangle_{\varepsilon_m,**} = o(1),
$$

then, taking $\varepsilon_m \to 0$, we obtain

$$
\int_{\mathbb{R}^{n+1}_+} x_N^{1-2s}\nabla\tilde\Phi\nabla\varphi\,dx\,dx_N = p\int_{\mathbb{R}^n}\omega^{p-1}\tilde\Phi\varphi\,dx - \int_{\mathbb{R}^n}\tilde\Phi\varphi\,dx
$$

for any function $\varphi \in C_0^\infty(\mathbb{R}^{n+1})$. This clearly implies that $\tilde\Phi$ is a weak solution of (2-7).

Moreover,

$$
\|\tilde\Phi\|_{H^1(\mathbb{R}^{n+1}_+, x_N^{1-2s})} < \infty,
$$

so the Moser iteration argument works and it reveals that $\tilde\Phi$ is $L^\infty(\mathbb{R}^n)$-bounded (see the proof of Lemma 5.1 in [14]). This with (3-6), implies $\tilde\Phi = 0$ in $\mathbb{R}^n$.

On the other hand, using the fact that

$$
-\frac{1}{\varepsilon_m^n}\int_M f'(\mathcal{W}_{\varepsilon_m,\xi_m})\Phi_m^2\,d\mathrm{vol}_g = -p\int_{B^+_{r/\varepsilon_m}} \mathcal{W}^{p-1}_{\varepsilon_m,\xi_m}\Phi_m^2|\bar g(\varepsilon x, \varepsilon x_N)|^{\frac12}\,dx = o(1),
$$

together with (3-7), one deduces that

$$\|\Phi_m\|^2_{\varepsilon_m,**} = \frac{1}{\varepsilon_m^n}\int_M f'(\mathcal{W}_{\varepsilon_m,\xi_m})\Phi_m^2\,d\mathrm{vol}_g + \left\langle \Psi_m + \sum_{k=1}^n c_k \mathcal{Z}^k_{\varepsilon_m,\xi_m},\, \Phi_m \right\rangle_{\varepsilon_m,**} = o(1),$$

which gives a contradiction with the fact that $\|\Phi_m\|_{\varepsilon_m,**} = 1$. This concludes the proof of the desired result. □

We next prove the following estimate on $R_{\varepsilon,\xi}$.

Lemma 3.2. *Assume that* $n > 2s+2$, *there exists* $\varepsilon_0 > 0$ *and* $c > 0$ *such that for any* $\xi \in M$ *and for any* $\varepsilon \in (0, \varepsilon_0)$, *we have*

$$\|R_{\varepsilon,\xi}\|_{\varepsilon,**} \le c\varepsilon^\gamma, \tag{3-12}$$

where γ *is given by*

$$\gamma = \begin{cases} \frac{1}{2}+\zeta & \text{if } 0 < s < \frac{1}{2}, \\ 1+\zeta & \text{if } \frac{1}{2} \le s < 1, \end{cases} \tag{3-13}$$

where ζ *can be taken sufficiently small.*

Proof. We first introduce some notations. Given $R > 0$, we denote by $B^+_{\bar g}(\xi, R)$ and $B_g(\xi, R)$ the balls defined respectively by

$$B^+_{\bar g}(\xi, R) := \{z \in X : d(z, \xi) < R\} \quad\text{and}\quad B_g(\xi, R) := \{\hat\xi \in M : d_M(\hat\xi, \xi) < R\}. \tag{3-14}$$

Next, we define by duality, the norm

$$\|U\| = \sup\{\langle U, \Phi\rangle : \|\Phi\|_{\varepsilon,**} \le 1\}$$

for $U \in H^1(X; \rho^{1-2s})$. Given $\Phi \in H^1(X; \rho^{1-2s})$ with $\|\Phi\|_{\varepsilon,**} \le 1$ and set $\phi = i(\Phi)$, we clearly have

$$\begin{aligned}
\langle \mathcal{W}_{\varepsilon,\xi}, \Phi\rangle_{\varepsilon,**} &- \langle i(\mathcal{W}^p_{\varepsilon,\xi}), \phi\rangle \\
&= \frac{1}{\varepsilon^{n-2s}}\kappa_s \int_{B^+_{\bar g}(\xi,2r_0)} \rho^{1-2s}(\nabla \mathcal{W}_{\varepsilon,\xi}, \nabla\Phi)_{\bar g}\,d\mathrm{vol}_{\bar g} \\
&\qquad + \frac{1}{\varepsilon^n}\int_{B_g(\xi,2r_0)} (\mathcal{W}_{\varepsilon,\xi} - \mathcal{W}^p_{\varepsilon,\xi})\,\phi\,d\mathrm{vol}_g \\
&\qquad + \frac{1}{\varepsilon^{n-2s}}\kappa_s \int_{B^+_{\bar g}(\xi,2r_0)} E(\rho)\mathcal{W}_{\varepsilon,\xi}\Phi\,d\mathrm{vol}_{\bar g} \\
&= -\frac{1}{\varepsilon^{n-2s}}\kappa_s \int_{B^+_{\bar g}(\xi,2r_0)} \mathrm{div}_{\bar g}(\rho^{1-2s}\nabla \mathcal{W}_{\varepsilon,\xi})\,\Phi\,d\mathrm{vol}_{\bar g} \\
&\qquad + \frac{1}{\varepsilon^{n-2s}}\kappa_s \int_{B^+_{\bar g}(\xi,2r_0)} E(\rho)\mathcal{W}_{\varepsilon,\xi}\Phi\,d\mathrm{vol}_{\bar g}.
\end{aligned} \tag{3-15}$$

We set

$$\mathcal{I}_1 := -\frac{1}{\varepsilon^{n-2s}}\kappa_s \int_{B^+_{\bar g}(\xi,2r_0)} \operatorname{div}_{\bar g}(\rho^{1-2s}\nabla \mathcal{W}_{\varepsilon,\xi})\, \Phi \, d\mathrm{vol}_{\bar g}, \tag{3-16}$$

$$\mathcal{I}_2 := \frac{1}{\varepsilon^{n-2s}}\kappa_s \int_{B^+_{\bar g}(\xi,2r_0)} E(\rho)\mathcal{W}_{\varepsilon,\xi}\, \Phi \, d\mathrm{vol}_{\bar g}. \tag{3-17}$$

We first estimate $\mathcal{I}_1$. Recalling the definition of $\mathcal{W}_{\varepsilon,\xi}$ given in (2-24) we can write

$$\begin{aligned}\mathcal{I}_1 = -\frac{1}{\varepsilon^{n-2s}}\kappa_s \int_{B^+_{\bar g}(\xi,2r_0)} \chi_r \operatorname{div}_{\bar g}(\rho^{1-2s}\nabla W_\varepsilon)\, \Phi \, d\mathrm{vol}_{\bar g} \\ - \frac{1}{\varepsilon^{n-2s}}\kappa_s \int_{B^+_{\bar g}(\xi,2r_0)} \operatorname{div}_{\bar g}(\rho^{1-2s}\nabla \chi_r)\, W_\varepsilon \, \Phi \, d\mathrm{vol}_{\bar g} \\ - \frac{2}{\varepsilon^{n-2s}}\kappa_s \int_{B^+_{\bar g}(\xi,2r_0)} \rho^{1-2s}\nabla \chi_r \cdot \nabla W_\varepsilon \, \Phi \, d\mathrm{vol}_{\bar g}.\end{aligned}$$

Using the Taylor expansions of the metric given in Lemma 2.3, we get

$$\begin{aligned}\mathcal{I}_1 = -\frac{1}{\varepsilon^{n-2s}}\kappa_s \int_{\mathbb{R}^{n+1}_+} \operatorname{div}(t^{1-2s}\nabla W_\varepsilon)\, \Phi \, dx\, dt \\ + \frac{1}{\varepsilon^{n-2s}} \int t^{1-2s}\mathcal{O}(t + |(x,t)|^2)\, |\nabla W_\varepsilon|\, |\nabla \Phi| \, dx\, dt \\ + \frac{1}{\varepsilon^{n-2s}}\mathcal{O}\biggl(\varepsilon^2 \int_{B^+_{\bar g}(\xi,2r_0)} \rho^{1-2s}|W_\varepsilon|\, |\Phi| \, d\mathrm{vol}_{\bar g}\biggr) \\ + \frac{1}{\varepsilon^{n-2s}}\mathcal{O}\biggl(\varepsilon \int_{B^+_{\bar g}(\xi,2r_0)} \rho^{-2s}|W_\varepsilon|\, |\Phi| \, d\mathrm{vol}_{\bar g}\biggr) \\ + \frac{1}{\varepsilon^{n-2s}}\mathcal{O}\biggl(\varepsilon^2 \int_{B^+_{\bar g}(\xi,2r_0)} \rho^{1-2s}|\nabla W_\varepsilon|\, |\Phi| \, d\mathrm{vol}_{\bar g}\biggr).\end{aligned}$$

Using the fact that W_ε solves (2-5), the estimates of Lemma 2.7 and Cauchy–Schwarz inequality, we can easily deduce that

$$|\mathcal{I}_1| = \mathcal{O}(\varepsilon^{1+\zeta})$$

for some $\zeta > 0$ which can be chosen small enough.

Now, to estimate the second term $\mathcal{I}_2$ we argue as in the proof of Lemma 4.1 in [13].

- For $0 < s < \frac{1}{2}$, we can choose $\zeta_1 > 0$ small enough so that $s + \zeta_1 < \frac{1}{2}$. We obtain

$$\begin{aligned}|\mathcal{I}_2| &= \biggl|\kappa_s \frac{1}{\varepsilon^{n-2s}} \int_{B^+_{\bar g}} E(\rho)\mathcal{W}_{\varepsilon,\xi}\, \Phi \, dv_{\bar g}\biggr| \\ &\le \frac{C}{\varepsilon^{n-2s}} \int_{B^+_{\bar g}} \rho^{-2s} |\mathcal{W}_{\varepsilon,\xi}|\, |\Phi| \, d\mathrm{vol}_{\bar g}\end{aligned}$$

$$\le C\bigg(\frac{1}{\varepsilon^{n-2s}}\int_{B_{\bar g}^+}\rho^{1-2s-2(s+\zeta_1)}\mathcal{W}_{\varepsilon,\xi}^2\,d\mathrm{vol}_{\bar g}\bigg)^{1/2}\bigg(\frac{1}{\varepsilon^{n-2s}}\int_{B_{\bar g}^+}\rho^{-1+2\zeta_1}\Phi^2\,d\mathrm{vol}_{\bar g}\bigg)^{1/2}$$

$$\le C\bigg(\frac{1}{\varepsilon^{n-2s}}\int_{B_{2r_0}^+}t^{1-2s-2(s+\zeta_1)}W_\varepsilon^2(z)\,dz\bigg)^{1/2}=\mathcal{O}(\varepsilon^{1-(s+\zeta_1)})=\mathcal{O}(\varepsilon^{\frac12+\zeta_3}),$$

with $\xi_3=\frac12-(s+\xi_1)>0$.

- For $\frac12<s<1$ and $H=0$, we can choose $\zeta_2>0$ small enough so that $s+\zeta_2<1$. Arguing as above, we get by (1-20) that

$$\text{(3-18)}\quad \bigg|\kappa_s\frac{1}{\varepsilon^{n-2s}}\int_{B_{\bar g}^+}E(\rho)\mathcal{W}_{\varepsilon,\xi}\Phi\,dv_{\bar g}\bigg|$$

$$\le\frac{C}{\varepsilon^{n-2s}}\int_{B_{\bar g}^+}\rho^{1-2s}|\mathcal{W}_{\varepsilon,\xi}||\Phi|\,dv_{\bar g}$$

$$\le C\bigg(\frac{1}{\varepsilon^{n-2s}}\int_{B_{\bar g}^+}\rho^{1-2s+2\zeta_2}\mathcal{W}_{\varepsilon,\xi}^2\,dv_{\bar g}\bigg)^{1/2}\bigg(\frac{1}{\varepsilon^{n-2s}}\int_{B_{\bar g}^+}\rho^{1-2s-2\zeta_2}\Phi^2\,dv_{\bar g}\bigg)^{1/2}$$

$$\le C\bigg(\frac{1}{\varepsilon^{n-2s}}\int_{B_{2r_0}^+}t^{1-2s+2\zeta_2}W_\varepsilon^2(z)\,dz\bigg)^{1/2}=\mathcal{O}(\varepsilon^{1+\zeta_2}).$$

Combining the above estimates, the desired result follows at once. □

Finally and in order to solve (2-31), it is important to study the linear operator $L_{\varepsilon,\xi}$ defined in (3-1). To this aim, we let $\Psi=L_{\varepsilon,\xi}(\Phi)$, we have that

$$\text{(3-19)}\quad \begin{cases}\Phi-i^*\big(i(f'(\mathcal{W}_{\varepsilon,\xi})\Phi)\big)=\Psi+\sum_{k=1}^n c_\varepsilon^k\mathcal{Z}_{\varepsilon,\xi}^k & \text{in } X\\ \langle\Phi,\mathcal{Z}_{\varepsilon,\xi}^k\rangle=0 & \text{for all } k=1,\dots,n\end{cases}$$

for some constants $c_\varepsilon^1,\dots,c_\varepsilon^n\in\mathbb{R}$. In the next proposition, we prove that for a fixed $\Psi\in(K_{\varepsilon,\xi})^\perp$ there are a unique function $\Phi\in(K_{\varepsilon,\xi})^\perp$ and an (n)-*tuple* $(c_\varepsilon^1,\dots,c_\varepsilon^n)\in\mathbb{R}^n$ satisfying the linear problem (3-19). Precisely, we prove the following result.

Proposition 3.3. *Given $n>2s$, $\xi\in M$ and $\varepsilon>0$ a small parameter. Then, for any $\Psi\in(K_{\varepsilon,\xi})^\perp$, there exists a unique solution $(\Phi,(c_\varepsilon^1,\dots,c_\varepsilon^n))$ to the equation* (3-19) *such that the estimate* (3-5) *holds.*

Proof. The existence of a unique solution follows directly from Lemma 2.6 and the Fredholm alternative for compact operator. □

A consequence of the above proposition is the following result.

Proposition 3.4. *Under the assumption of Proposition 3.3, equation* (2-31) *possesses a unique solution $\Phi=\Phi_{\varepsilon,\xi}\in(K_{\varepsilon,\xi})^\perp$ such that*

$$\text{(3-20)}\quad \|\Phi_{\varepsilon,\xi}\|_{\varepsilon,**}\le c\varepsilon^\gamma,$$

with a positive constant c and where γ is defined in (3-13).

Proof. The proof is based on a contraction mapping theorem. Indeed, let us define the operator $T_{\varepsilon,\xi} : (K_{\varepsilon,\xi})^{\perp} \to (K_{\varepsilon,\xi})^{\perp}$ by

$$T_{\varepsilon,\xi}(\Phi) := (L_{\varepsilon,\xi})^{-1}(N_{\varepsilon,\xi}(\Phi) + R_{\varepsilon,\xi}). \tag{3-21}$$

Using Lemma 3.2, a straightforward computations show that $T_{\varepsilon,\xi}$ is a contraction map from the ball

$$\mathcal{B} := \{\Phi \in (K_{\varepsilon,\xi})^{\perp} : \|\Phi\|_{\varepsilon,**} \leq C\varepsilon^{\gamma}\} \tag{3-22}$$

into itself, for some large constant $C > 0$. Hence, $T_{\varepsilon,\xi}$ possesses a unique fixed point $\Phi_{\varepsilon,\xi} \in \mathcal{B}$, which is a solution to (3-2), or equivalently to (2-31). □

4. Asymptotic expansion of the finite-dimensional functional

The goal is to solve (2-32). Let $J_\varepsilon : H^1(X, \rho^{1-2s}) \to \mathbb{R}$ be defined by

$$J_\varepsilon(U) := \frac{1}{2\varepsilon^{n-2s}}\kappa_s \int_X \big(\rho^{1-2s}|\nabla U|^2_{\bar{g}} + E(\rho)U^2\big)\, d\mathrm{vol}_{\bar{g}} + \frac{1}{\varepsilon^n}\int_M \tfrac{1}{2}U^2 - F(U)\, d\mathrm{vol}_g,$$

where $u_+ = \max(u, 0)$ and $F(u) := \frac{1}{p+1}u_+^{p+1}$ so that $F'(u) = f(u)$. It is well known that any critical point of J_ε is solution to problem (1-1).

Next, we introduce the reduced functional $\tilde{J}_\varepsilon : M \to \mathbb{R}$ defined by

$$\tilde{J}_\varepsilon(\xi) := J_\varepsilon(\mathcal{W}_{\varepsilon,\xi} + \Phi_{\varepsilon,\xi}), \quad \xi \in M, \tag{4-1}$$

where $\mathcal{W}_{\varepsilon,\xi}$ is the global approximate solution given in (2-24) and $\Phi_{\varepsilon,\xi}$ is a small perturbation defined in (2-25). Applying a finite dimensional reduction procedure, we prove the following result

Lemma 4.1. *The reduced energy functional $\tilde{J}_\varepsilon$ is continuously differentiable. Moreover, if ξ_0 is a critical point of $\tilde{J}_\varepsilon$, then $\mathcal{W}_{\varepsilon,\xi_0} + \Phi_{\varepsilon,\xi_0}$ is a positive solution to problem* (1-1) *or equivalently to problem* (2-32).

Proof. Given $\xi \in M$, we define the linear operator, $\mathcal{H}(\xi, \cdot) : H^1(X, \rho^{1-2s}) \to \mathbb{R}$ by

$$\mathcal{H}(\xi, U) := U + \Pi^{\perp}_{\varepsilon,\xi}\big[\mathcal{W}_{\varepsilon,\xi} - i^*_\varepsilon\big(i\big(f(\mathcal{W}_{\varepsilon,\xi} + U)\big)\big)\big] \tag{4-2}$$

for $U \in H^1(X, \rho^{1-2s})$. We clearly have

$$\mathcal{H}(\xi, \Phi_{\varepsilon,\xi}) = 0 \quad \text{and} \quad \frac{\partial \mathcal{H}}{\partial U}(\xi, \Phi_{\varepsilon,\xi})U = U - \Pi^{\perp}_{\varepsilon,\xi}\big[i^*_\varepsilon\big(i\big(f'(\mathcal{W}_{\varepsilon,\xi} + \Phi_{\varepsilon,\xi})U\big)\big)\big].$$

On the other hand, using Lemma 2.6, we deduce that

$$i\big(f'(\mathcal{W}_{\varepsilon,\xi} + \Phi_{\varepsilon,\xi})U\big) \in L^q(M)$$

for some $q \in \left(1, \frac{n+2s}{n-2s}\right)$ with

$$U \in H^1(X, \rho^{1-2s}) \quad \text{and} \quad \frac{\partial \mathcal{H}}{\partial U}(\xi, \Phi_{\varepsilon,\xi}) : H^1(X, \rho^{1-2s}) \to H^1(X, \rho^{1-2s})$$

is a Fredholm operator of index 0. Moreover, using (3-12) one can easily check that it is also injective. Therefore $\frac{\partial \mathcal{H}}{\partial U}(\xi, \Phi_{\varepsilon,\xi})$ is invertible and by the implicit function theorem, we deduce that the mapping

$$\xi \in M \mapsto \Phi_{\varepsilon,\xi} \in H^1(X, \rho^{1-2s})$$

is C^1. This proves that $\tilde{J}_\varepsilon$ is of class C^1. It then remains to prove that $\tilde{J}'_\varepsilon(\xi) = 0$ implies that

$$J'_\varepsilon(\mathcal{W}_{\varepsilon,\xi} + \Phi_{\varepsilon,\xi}) = 0.$$

Let $\xi_0 \in M$ and define

$$\xi = \xi(y) = \exp_{\xi_0}(y), \qquad y \in B(0, r) \subset T_{\xi_0} M$$

with $r > 0$. A straightforward computations yield

$$\begin{aligned}
&\frac{\partial}{\partial y_k} \tilde{J}_\varepsilon(\exp_{\xi_0}(y)) \\
&= J'_\varepsilon\big(\mathcal{W}_{\varepsilon,\xi(y)} + \Phi_{\varepsilon,\xi(y)}\big)\left[\frac{\partial}{\partial y_k}\mathcal{W}_{\varepsilon,\xi(y)} + \frac{\partial}{\partial y_k}\Phi_{\varepsilon,\xi(y)}\right] \\
&= \left\langle \mathcal{W}_{\varepsilon,\xi(y)} + \Phi_{\varepsilon,\xi(y)} - i^*\big(i(f(\mathcal{W}_{\varepsilon,\xi(y)} + \Phi_{\varepsilon,\xi(y)}))\big), \left(\frac{\partial}{\partial y_k}\mathcal{W}_{\varepsilon,\xi(y)} + \frac{\partial}{\partial y_k}\Phi_{\varepsilon,\xi(y)}\right)\right\rangle_{\varepsilon,**}.
\end{aligned}$$

On the other hand, by (3-19), there exist some constants c^l_ε, $1 \le l \le n$, such that

$$\mathcal{W}_{\varepsilon,\xi(y)} + \Phi_{\varepsilon,\xi(y)} - i^*\big(i\big(f(\mathcal{W}_{\varepsilon,\xi(y)} + \Phi_{\varepsilon,\xi(y)})\big)\big) = \sum_{l=1}^{n} c^l_\varepsilon \mathcal{Z}^l_{\varepsilon,\xi(y)}.$$

Therefore

$$\frac{\partial}{\partial y_k} \tilde{J}_\varepsilon(\exp_{\xi_0}(y)) = \sum_{l=1}^{n} c^l_\varepsilon \left\langle \mathcal{Z}^l_{\varepsilon,\xi(y)}, \frac{\partial}{\partial y_k}\mathcal{W}_{\varepsilon,\xi(y)} + \frac{\partial}{\partial y_k}\Phi_{\varepsilon,\xi(y)}\right\rangle_{\varepsilon,**}. \tag{4-3}$$

Assuming now that ξ_0 is a critical point of $\tilde{J}_\varepsilon$. That is,

$$\left(\frac{\partial}{\partial y_k} \tilde{J}_\varepsilon(\exp_{\xi_0}(y))\right)\Big|_{y=0} = 0 \quad \text{for all } k = 1, \dots, n. \tag{4-4}$$

Evaluating (4-3) at $y = 0$ and assuming ε sufficiently small, we immediately get from Lemma 4.8 that $c^l_\varepsilon = 0$ for all $l = 1, \dots, n$.

To prove that $\mathcal{W}_{\varepsilon,\xi(y)} + \Phi_{\varepsilon,\xi(y)}$ is positive, we argue as in [13, Proof of Proposition 5.1]. In fact, given any $\Psi \in H^1(X, \rho^{1-2s})$ we have

$$\frac{1}{\varepsilon^{n-2s}} \int_X \Big(\rho^{1-2s} \big(\nabla(\mathcal{W}_{\varepsilon,\xi(y)} + \Phi_{\varepsilon,\xi(y)}), \nabla\Psi\big)_{\bar{g}} + E(\rho)(\mathcal{W}_{\varepsilon,\xi(y)} + \Phi_{\varepsilon,\xi(y)})\,\Psi\Big)\, d\mathrm{vol}_{\bar{g}}$$
$$+ \frac{1}{\varepsilon^n} \int_M (\mathcal{W}_{\varepsilon,\xi(y)} + \Phi_{\varepsilon,\xi(y)})\,\Psi\, d\mathrm{vol}_g$$
$$= \frac{p}{\varepsilon^n} \int_M (\mathcal{W}_{\varepsilon,\xi(y)} + \Phi_{\varepsilon,\xi(y)})_+^{p-1}\,\Psi\, d\mathrm{vol}_g.$$

Then, choosing $\Psi = (\mathcal{W}_{\varepsilon,\xi(y)} + \Phi_{\varepsilon,\xi(y)})_-$ into the above identity and using (2-17) we immediately get that $\mathcal{W}_{\varepsilon,\xi(y)} + \Phi_{\varepsilon,\xi(y)}$ is nonnegative in $\overline{X}$. The fact that it is positive follows from the inequality

$$\|\mathcal{W}_{\varepsilon,\xi(y)} + \Phi_{\varepsilon,\xi(y)}\|_{\varepsilon,**} \geq \|\mathcal{W}_{\varepsilon,\xi(y)}\|_{\varepsilon,**} - \|\Phi_{\varepsilon,\xi(y)}\|_{\varepsilon,**} \geq C + O(\varepsilon^\gamma) > 0$$

and that (2-22) is a uniformly elliptic equation in divergence form away from the boundary M. □

4A. *C^0-estimates of the energy.* This section is devoted to the expansion of the energy functional $\tilde{J}_\varepsilon$ in powers of ε. The first important result is the following one.

Lemma 4.2. *Assume that $n > 2s + 2$, for $\varepsilon > 0$ sufficiently small, we suppose that $H = 0$ if $s \in \big[\frac{1}{2}, 1\big)$ (which is the case when* (1-21) *holds). We have the validity of the following expansion for the function J_ε*

$$J_\varepsilon(\mathcal{W}_{\varepsilon,\xi}) - \tilde{C} = \begin{cases} -\varepsilon \tilde{d}_2 H(\xi) + o(\varepsilon) & \text{if } 0 < s < \frac{1}{2}, \\ -\frac{1}{6}\varepsilon^2 \big[\big(\tilde{d} + \tilde{d}_1 \tilde{C}_{n,s}^2\big) R_g(\xi) + \tilde{d}_1 \tilde{C}_{n,s}^3 \|\pi\|^2(\xi)\big] + o(\varepsilon^2) & \text{if } \frac{1}{2} \leq s < 1, \end{cases} \tag{4-5}$$

uniformly with respect to ξ as ε goes to zero. Here R_g is the scalar curvature of (M, g), π is the second fundamental form on M, the constants $\tilde{C}$, $\tilde{d}$, $\tilde{d}_1$ and $\tilde{d}_2$ are defined respectively by

$$\tilde{C} := \Big(\tfrac{1}{2}\kappa_s \int_{\mathbb{R}_+^{n+1}} x_N^{1-2s} |\nabla W|^2\, dx\, dx_N + \tfrac{1}{2} \int_{\mathbb{R}^n} \omega^2\, dx - \tfrac{1}{p+1} \int_{\mathbb{R}^n} \omega^{p+1}\, dx\Big), \tag{4-6}$$

$$\tilde{d} := \Big(\tfrac{1}{2}\kappa_s \int_{\mathbb{R}_+^{n+1}} x_N^{1-2s} x_1^2 |\nabla W|^2\, dx\, dx_N + \tfrac{1}{2} \int_{\mathbb{R}^n} x_1^2 \omega^2\, dx - \tfrac{1}{p+1} \int_{\mathbb{R}^n} x_1^2 \omega^{p+1}\, dx\Big), \tag{4-7}$$

$$\tilde{d}_1 := \kappa_s \int_{\mathbb{R}_+^{n+1}} x_N^{1-2s} W^2\, dx\, dx_N, \tag{4-8}$$

$$\tilde{d}_2 := C_{n,s}^2 \kappa_s \int_{\mathbb{R}_+^{n+1}} x_N^{2-2s} |\nabla W|^2\, dx\, dx_N. \tag{4-9}$$

Here $W = \mathrm{Ext}^s(w)$ is the s-harmonic extension of ω to $\mathbb{R}^{n+1}_+$ s harmonic of w defined in Section 2.

Proof. We recall that

$$J_\varepsilon(\mathcal{W}_{\varepsilon,\xi}) := \frac{1}{2\varepsilon^{n-2s}}\kappa_s \int_X \big(\rho^{1-2s}|\nabla \mathcal{W}_{\varepsilon,\xi}|^2_{\bar g} + E(\rho)\mathcal{W}^2_{\varepsilon,\xi}\big)\, d\mathrm{vol}_{\bar g} + \frac{1}{\varepsilon^n}\int_M \tfrac{1}{2}\mathcal{W}^2_{\varepsilon,\xi} - F(\mathcal{W}_{\varepsilon,\xi})\, d\mathrm{vol}_g.$$

According to Lemma 2.3 and 4.4 and for $0 < s < \frac{1}{2}$, we obtain that

$$\begin{aligned} \frac{1}{2\varepsilon^{n-2s}}\kappa_s \int_X \rho^{1-2s}|\nabla \mathcal{W}_{\varepsilon,\xi}|^2_{\bar g}\, d\mathrm{vol}_{\bar g} &= \tfrac{1}{2}\kappa_s \int_{B^+_{2r_0/\varepsilon}} x_N^{1-2s}[\bar g^{ij}(\varepsilon x, \varepsilon x_N)\,\partial_i W \partial_j W + (\partial_N W)^2]|\bar g(\varepsilon x, \varepsilon x_N)|^{\frac{1}{2}}\, dx\, dx_N \\ &= \tfrac{1}{2}\kappa_s \int_{\mathbb{R}^{n+1}_+} x_N^{1-2s}|\nabla W|^2\, dx\, dx_N \\ &\quad - \varepsilon\kappa_s\bigg[\tfrac{1}{2}H\int_{\mathbb{R}^{n+1}_+} x_N^{2-2s}|\nabla W|^2\, dx\, dx_N - \pi_{ij}\int_{\mathbb{R}^{n+1}_+} x_N^{2-2s}\partial_i W \partial_j W\, dx\, dx_N\bigg] \\ &= \tfrac{1}{2}\kappa_s \int_{\mathbb{R}^{n+1}_+} x_N^{1-2s}|\nabla W|^2\, dx\, dx_N - C^0_{n,s}\varepsilon H\kappa_s \int_{\mathbb{R}^{n+1}_+} x_N^{2-2s}|\nabla W|^2\, dx\, dx_N \\ &\quad + o(\varepsilon). \end{aligned} \tag{4-10}$$

Also, in view of (1-20) and for $0 < s < \frac{1}{2}$, we get

$$E(x_N) = \Big(\frac{n-2s}{2}\Big)H\rho^{-2s} \tag{4-11}$$

for $x_N \ge 0$ small. So

$$\begin{aligned} \frac{1}{2\varepsilon^{n-2s}}\kappa_s \int_X (E(\rho)\mathcal{W}^2_{\varepsilon,\xi})\, d\mathrm{vol}_{\bar g} &= \frac{\varepsilon^{1+2s}}{2}\kappa_s \int_{B^+_{2r_0/\varepsilon}} (E(\varepsilon x_N)\, W^2)|\bar g(\varepsilon x, \varepsilon x_N)|^{\frac{1}{2}}\, dx\, dx_N \\ &= \varepsilon H\kappa_s\Big(\frac{n-2s}{4}\Big)\int_{\mathbb{R}^{n+1}_+} x_N^{-2s} W^2\, dx\, dx_N + o(\varepsilon^2) \\ &= C^1_{n,s}\varepsilon H\kappa_s \int_{\mathbb{R}^{n+1}_+} x_N^{2-2s}|\nabla W|^2\, dx\, dx_N + o(\varepsilon). \end{aligned} \tag{4-12}$$

Using the fact that $x_N = 0$ on M, we get

$$\frac{1}{2\varepsilon^n}\int_M \mathcal{W}^2_{\varepsilon,\xi}\, d\mathrm{vol}_g = \tfrac{1}{2}\int_{B_{2r_0/\varepsilon}} \omega^2|\bar g(\varepsilon x, \varepsilon x_N)|^{\frac{1}{2}}\, dx = \tfrac{1}{2}\int_{\mathbb{R}^n} \omega^2\, dx + o(\varepsilon), \tag{4-13}$$

and

$$-\frac{1}{(p+1)\varepsilon^n}\int_M \mathcal{W}_{\varepsilon,\xi}^{p+1}\,d\mathrm{vol}_g = -\frac{1}{p+1}\int_{B_{2r_0/\varepsilon}} W^{p+1}|\bar g(\varepsilon x,\varepsilon x_N)|^{\frac12}\,dx = -\frac{1}{p+1}\int_{\mathbb{R}^n} W^{p+1}\,dx + o(\varepsilon). \tag{4-14}$$

According to (4-10), (4-12), (4-13) and (4-14), then for $0 < s < \frac12$, we get

$$J_\varepsilon(\mathcal{W}_{\varepsilon,\xi}) - \left(\tfrac12\kappa_s\int_{\mathbb{R}^{n+1}_+} x_N^{1-2s}|\nabla W|^2\,dx\,dx_N + \tfrac12\int_{\mathbb{R}^n}\omega^2\,dx - \frac{1}{p+1}\int_{\mathbb{R}^n}\omega^{p+1}\,dx\right) = -C^2_{n,s}\varepsilon H\kappa_s\int_{\mathbb{R}^{n+1}_+} x_N^{2-2s}|\nabla W|^2\,dx\,dx_N + o(\varepsilon).$$

In the above estimates, the constants $C^i_{n,s},\ i = 0, 1, 2$, are defined by

$$C^0_{n,s} := \frac{2(n-s)-1}{4n},\qquad C^1_{n,s} := \frac{n-2s}{2-4s}\quad\text{and}\quad C^2_{n,s} := C^0_{n,s} - C^1_{n,s}.$$

Now, we deal with the case where $\frac12 \le s < 1$. By Lemmas 2.3 and 2.5, using the result of Lemma 7.2 in [26], we get

$$\begin{aligned}
&\frac{1}{2\varepsilon^{n-2s}}\kappa_s\int_X \rho^{1-2s}|\nabla\mathcal{W}_{\varepsilon,\xi}|^2_{\bar g}\,d\mathrm{vol}_{\bar g} \\
&= \tfrac12\kappa_s\int_{B^+_{2r_0/\varepsilon}} x_N^{1-2s}[\bar g^{ij}(\varepsilon x,\varepsilon x_N)\,\partial_i W\partial_j W + (\partial_N W)^2]|\bar g(\varepsilon x,\varepsilon x_N)|^{\frac12}\,dx\,dx_N \\
&= \tfrac12\kappa_s\int_{\mathbb{R}^{n+1}_+} x_N^{1-2s}|\nabla W|^2\,dx\,dx_N + \varepsilon\kappa_s\pi_{ij}\int_{\mathbb{R}^{n+1}_+} x_N^{2-2s}\partial_i W\partial_j W\,dx\,dx_N \\
&\qquad + \varepsilon^2\kappa_s\pi_{ij,k}\int_{\mathbb{R}^{n+1}_+} x_N^{2-2s}x_k\partial_i W\partial_j W\,dx\,dx_N \\
&\qquad - \tfrac16\varepsilon^2\kappa_s\Big[\tfrac12 R_{kl}\int_{\mathbb{R}^{n+1}_+} x_N^{1-2s}x_kx_l|\nabla W|^2\,dx\,dx_N \\
&\qquad\qquad - R_{ikjl}\int_{\mathbb{R}^{n+1}_+} x_N^{1-2s}x_kx_l\partial_i W\partial_j W\,dx\,dx_N\Big] \\
&\qquad + \tfrac12\varepsilon^2\kappa_s(3\pi_{ih}\pi_{hj} + R_{iNjN})\int_{\mathbb{R}^{n+1}_+} x_N^{3-2s}\partial_i W\partial_j W\,dx\,dx_N \\
&\qquad - \tfrac14\varepsilon^2\kappa_s(\|\pi\|^2 + R_{NN})\int_{\mathbb{R}^{n+1}_+} x_N^{3-2s}|\nabla W|^2\,dx\,dx_N \\
&\qquad + o(\varepsilon^2) \\
&= \tfrac12\kappa_s\int_{\mathbb{R}^{n+1}_+} x_N^{1-2s}|\nabla W|^2\,dx\,dx_N - \tfrac1{12}\varepsilon^2\kappa_s R_{kk}\int_{\mathbb{R}^{n+1}_+} x_N^{1-2s}x_1^2|\nabla W|^2\,dx\,dx_N \\
&\qquad - \tfrac16\varepsilon^2\kappa_s[\tilde C^0_{n,s}(\|\pi\|^2 + R_{NN}) + \tilde C^1_{n,s}\|\pi\|^2]\int_{\mathbb{R}^{n+1}_+} x_N^{1-2s}W^2\,dx\,dx_N \\
&\qquad + o(\varepsilon^2).
\end{aligned}\tag{4-15}$$

Using the fact that $x_N = 0$ on M, we get

$$\frac{1}{2\varepsilon^n}\int_M \mathcal{W}^2_{\varepsilon,\xi}\, d\mathrm{vol}_g = \tfrac{1}{2}\int_{B_{2r_0/\varepsilon}} \omega^2 |\bar g(\varepsilon x, \varepsilon x_N)|^{\frac12}\, dx = \tfrac12 \int_{\mathbb{R}^n}\omega^2\, dx - \tfrac{1}{12}\varepsilon^2 R_{kk}\int_{\mathbb{R}^n} x_1^2\omega^2\, dx + o(\varepsilon^2), \tag{4-16}$$

and

$$-\frac{1}{(p+1)\varepsilon^n}\int_M \mathcal{W}^{p+1}_{\varepsilon,\xi}\, d\mathrm{vol}_g = -\frac{1}{p+1}\int_{B_{2r_0/\varepsilon}} \omega^{p+1}|\bar g(\varepsilon x, \varepsilon x_N)|^{\frac12}\, dx = -\frac{1}{p+1}\int_{\mathbb{R}^n}\omega^{p+1}\, dx + \frac{1}{p+1}\varepsilon^2 R_{kk}\int_{\mathbb{R}^n} x_1^2 \omega^{p+1}\, dx + o(\varepsilon^2). \tag{4-17}$$

Here $\tilde C^0_{n,s}$ and $\tilde C^1_{n,s}$ are the constants defined by

$$\tilde C^0_{n,s} := \Big(\frac{3n-2(1+s)}{n}\Big)(1-s), \quad \tilde C^1_{n,s} := \frac{4}{n}(1-s^2). \tag{4-18}$$

Then according to (4-15), (4-16) and (4-17), we get

$$\begin{aligned} J_\varepsilon(\mathcal{W}_{\varepsilon,\xi}) &- \Big(\tfrac12\kappa_s\int_{\mathbb{R}^{n+1}_+} x_N^{1-2s}|\nabla W|^2\, dx\, dx_N + \tfrac12\int_{\mathbb{R}^n}\omega^2\, dx - \frac{1}{p+1}\int_{\mathbb{R}^n}\omega^{p+1}\, dx\Big)\\ &= -\tfrac16\varepsilon^2 R_{kk}(\xi)\Big[\tfrac12\kappa_s\int_{\mathbb{R}^{n+1}_+} x_N^{1-2s}x_1^2|\nabla W|^2\, dx\, dx_N + \tfrac12\int_{\mathbb{R}^n} x_1^2\omega^2\, dx - \frac{1}{p+1}\int_{\mathbb{R}^n} x_1^2\omega^{p+1}\, dx\Big]\\ &\quad - \tfrac16\varepsilon^2\kappa_s[\tilde C^0_{n,s}(\|\pi\|^2 + R_{NN}) + \tilde C^1_{n,s}\|\pi\|^2](\xi)\int_{\mathbb{R}^{n+1}_+} x_N^{1-2s} W^2\, dx\, dx_N\\ &\quad + o(\varepsilon^2). \end{aligned}$$

Substituting the second identity in (1-22) into the above, we obtain

$$\begin{aligned} J_\varepsilon(\mathcal{W}_{\varepsilon,\xi}) &- \Big(\tfrac12\kappa_s\int_{\mathbb{R}^{n+1}_+} x_N^{1-2s}|\nabla W|^2\, dx\, dx_N + \tfrac12\int_{\mathbb{R}^n}\omega^2\, dx - \frac{1}{p+1}\int_{\mathbb{R}^n}\omega^{p+1}\, dx\Big)\\ &= -\tfrac16\varepsilon^2 R_{kk}(\xi)\Big[\tfrac12\kappa_s\int_{\mathbb{R}^{n+1}_+} x_N^{1-2s}x_1^2|\nabla W|^2\, dx\, dx_N + \tfrac12\int_{\mathbb{R}^n} x_1^2\omega^2\, dx - \frac{1}{p+1}\int_{\mathbb{R}^n} x_1^2\omega^{p+1}\, dx\Big]\\ &\quad - \tfrac16\varepsilon^2\kappa_s[\tilde C^3_{n,s}\|\pi\|^2 + \tilde C^2_{n,s} R_{kk}](\xi)\int_{\mathbb{R}^{n+1}_+} x_N^{1-2s} W^2\, dx\, dx_N\\ &\quad + o(\varepsilon^2), \end{aligned}$$

where

$$\tilde{C}^2_{n,s} := \frac{\tilde{C}^0_{n,s}}{2(n-1)} \quad \text{and} \quad \tilde{C}^3_{n,s} := \tilde{C}^1_{n,s} - \tilde{C}^2_{n,s}. \tag{4-19}$$

Recalling the definitions of the constants $\tilde{C}$, $\tilde{d}$, $\tilde{d}_1$ and $\tilde{d}_2$ given respectively in (4-6), (4-7), (4-8) and (4-9), the proof follows at once. □

Lemma 4.3. *Given $\varepsilon > 0$ sufficiently small, we have*

$$\tilde{J}_\varepsilon(\xi) := J_\varepsilon(\mathcal{W}_{\varepsilon,\xi} + \Phi_{\varepsilon,\xi}) = J_\varepsilon(\mathcal{W}_{\varepsilon,\xi}) + \mathcal{O}(\varepsilon^{2\gamma}), \tag{4-20}$$

uniformly with respect to $\xi \in M$ as ε goes to zero, where $J_\varepsilon(\mathcal{W}_{\varepsilon,\xi})$ is defined in (4-5) *and γ is defined in* (3-13).

Proof. The proof is based on a Taylor expansion in the neighborhood of $\mathcal{W}_{\varepsilon,\xi}$ and the fact that $\Phi_{\varepsilon,\xi}$ is orthogonal to the space $K_{\varepsilon,\xi}$. Then a straightforward computations yield

$$\begin{aligned}
\tilde{J}_\varepsilon(\xi) - J_\varepsilon(\mathcal{W}_{\varepsilon,\xi}) &= \langle \mathcal{W}_{\varepsilon,\xi} + \Phi_{\varepsilon,\xi}, \Phi_{\varepsilon,\xi}\rangle_{\varepsilon,**} - \frac{1}{\varepsilon^n}\int_M \big(F(\mathcal{W}_{\varepsilon,\xi} + \Phi_{\varepsilon,\xi}) - F(\mathcal{W}_{\varepsilon,\xi})\big) \\
&= \frac{1}{\varepsilon^n}\int_M \big(f(\mathcal{W}_{\varepsilon,\xi} + \Phi_{\varepsilon,\xi}) - f(\mathcal{W}_{\varepsilon,\xi})\big)\Phi_{\varepsilon,\xi} \\
&\qquad - \frac{1}{\varepsilon^n}\int_M \big(F(\mathcal{W}_{\varepsilon,\xi} + \Phi_{\varepsilon,\xi}) - F(\mathcal{W}_{\varepsilon,\xi}) - f(\mathcal{W}_{\varepsilon,\xi})\,\Phi_{\varepsilon,\xi}\big) \\
&= \mathcal{O}(\|\Phi_{\varepsilon,\xi}\|^2_{\varepsilon,**}),
\end{aligned}$$

where

$$F(u) := \frac{1}{p+1}(u^+)^{p+1}.$$

Then, using Proposition 3.4 we immediately get the desired result. □

Lemma 4.4. *Suppose that $s \in \big(0, \frac{1}{2}\big)$, $n > 2s + 1$ and W the s-harmonic extension defined in* (2-5). *Then,*

$$\int_{\mathbb{R}^{n+1}_+} x_N^{2-2s} |\nabla W|^2 \, dx \, dx_N = \frac{4}{1+2s} \int_{\mathbb{R}^{n+1}_+} x_N^{2-2s} |\nabla_x W|^2 \, dx \, dx_N, \tag{4-21}$$

$$= \frac{1-2s}{2} \int_{\mathbb{R}^{n+1}_+} x_N^{-2s} W^2 \, dx \, dx_N < \infty. \tag{4-22}$$

Proof. We refer to Lemma 6.3 in [11] and Lemma 7.2 in [27] for the proof. □

As a consequence of the above lemmas, we have the validity of the following C^0-estimate.

Proposition 4.5. *We suppose that $H = 0$ if $s \in \left[\frac{1}{2}, 1\right)$ for $\varepsilon > 0$ sufficiently small* (*which is the case when* (1-21) *holds*). *We have the validity of the following expansion for $\tilde{J}_\varepsilon$*:

$$\tag{4-23} \tilde{J}_\varepsilon(\xi) = \begin{cases} \tilde{C} - \varepsilon \tilde{d}_2 H(\xi) + o(\varepsilon) & \text{if } 0 < s < \frac{1}{2}, \\ \tilde{C} - \frac{1}{6}\varepsilon^2[(\tilde{d} + \tilde{d}_1 \tilde{C}_{n,s}^2) R_g(\xi) + \tilde{d}_1 \tilde{C}_{n,s}^3 \|\pi\|^2(\xi)] + o(\varepsilon^2) & \text{if } \frac{1}{2} \le s < 1, \end{cases}$$

uniformly for $\xi \in M$ as ε goes to zero.

Proof. It follows directly from Lemmas 4.2, 4.3 and 4.4. □

4B. C^1-*estimates of the energy.* The aim is to improve Proposition 4.5 by showing that the $o(1)$-terms go to 0 in C^1-sense.

Proposition 4.6. *Estimate* (4-23) *is valid C^1-uniformly for $\xi_0 \in M$. Precisely, the following holds for each fixed point $\xi_0 \in M$. Suppose that $y \in \mathbb{R}^n$ is a point near the origin. Under the assumption in Proposition 4.5, we have*

$$\tag{4-24} \begin{aligned} \frac{\partial}{\partial y_k} \tilde{J}_\varepsilon(\xi)(\exp_{\xi_0}(y))|_{y=0} &= \frac{\partial}{\partial y_k}(J_\varepsilon(\mathcal{W}_{\varepsilon,\xi} + \Phi_{\varepsilon,\xi}))|_{y=0} \\ &= \frac{\partial}{\partial y_k}(J_\varepsilon(\mathcal{W}_{\varepsilon,\xi}))|_{y=0} + o(\varepsilon^\gamma) \end{aligned}$$

for each $1 \le k \le n$.

For the proof of Proposition 4.6, we first need to establish several preliminary lemmas. We fix $\xi_0 \in M$ and set

$$\xi(y) = \exp_{\xi_0}(y) \quad \text{for } y \in B^n(0, 4r_0)$$

(recall that $4r_0 > 0$ is chosen to be smaller than the injectivity radius of M). Recall the definition of the cutoff function χ_r in (2-24) and observe that any point $z \in X$ located sufficiently close to $\xi_0 \in M$ can be written as $z = (\xi(x), x_N)$ for some $x \in B^n(0, 2r_0)$ and $x_N \in (0, r_0)$, The first key result in the proof of Proposition 4.6 is:

Lemma 4.7. *For any $1 \le k \le n$, we have*

$$\tag{4-25} \begin{aligned} &\frac{\partial}{\partial y_k} \mathcal{W}_{\varepsilon,\xi(y)}\Big|_{y=0}(\exp_{\xi_0}(x), x_N) \\ &= \frac{1}{\varepsilon}\chi_r(\varepsilon x, \varepsilon x_N) \sum_{j=1}^{N} \left[\partial_j W(x, x_N) \frac{\partial \mathcal{K}_j}{\partial y_k}(0, \varepsilon x)\right] \\ &\qquad + \mathcal{O}\left(\varepsilon^{n-2s} \frac{|\nabla \chi_r|(|(x-y, x_N)|)}{|(x-y, x_N)|^{N-2s}}\right). \end{aligned}$$

Moreover, for any z near the point ξ_0 and $1 \le i \le n$, it holds

$$\frac{\partial}{\partial y_k}\mathcal{Z}_{\varepsilon,\xi(y)}\Big|_{y=0}(\exp_{\xi_0}(x), x_N) = \frac{1}{\varepsilon}\chi_r(\varepsilon x, \varepsilon x_N)\sum_{j=1}^{N}\Big[\partial_j Z_i(x, x_N)\frac{\partial \mathcal{K}_j}{\partial y_k}(0, \varepsilon x)\Big] + \mathcal{O}\Big(\varepsilon^{n-2s}\frac{|\nabla\chi_r|(|(x-y, x_N)|)}{|(x-y, x_N)|^{N-2s}}\Big), \tag{4-26}$$

uniformly with respect to ξ as ε goes to zero.

Proof. Let $\xi_0 \in M$ be fixed, define

$$\xi = \xi(y) = \exp_{\xi_0}(y), \quad y \in B^n(0, 4r_0),$$

and set

$$\mathcal{K}(y, x) = \exp^{-1}_{\xi(y)}(\xi(x)) = (\mathcal{K}_1(y, x), \dots, \mathcal{K}_n(y, x)) \in \mathbb{R}^n.$$

Using the chain rule and Lemma A.2, a straightforward computations yield

$$\begin{aligned}\frac{\partial}{\partial y_k}\mathcal{W}_{\varepsilon,\xi(y)}(z) &= \chi_r\big(d((\exp_{\xi(y)}(x), x_N), \xi(y))\big)\\ &\quad \sum_{j=1}^{N}\Big[\partial_j W_\varepsilon(\mathcal{K}(y, x), x_N)\frac{\partial \mathcal{K}_j}{\partial y_k}(y, \xi^{-1}(\exp_{\xi(y)})(x))\Big]\\ &\quad + \mathcal{O}\Big(\varepsilon^{n-2s}\frac{|\nabla\chi_r|(|(x, x_N)|)}{|(x, x_N)|^{N-2s}}\Big)\\ &= \frac{1}{\varepsilon}\chi_r\big(d((\exp_{\xi(y)}(\varepsilon x), \varepsilon x_N), \xi(y))\big)\\ &\quad \sum_{j=1}^{N}\Big[\partial_j W(\mathcal{K}(y, x), x_N)\frac{\partial \mathcal{K}_j}{\partial y_k}(y, \xi^{-1}(\exp_{\xi(y)})(\varepsilon x))\Big]\\ &\quad + \mathcal{O}\Big(\varepsilon^{n-2s}\frac{|\nabla\chi_r|(|(x, x_N)|)}{|(x, x_N)|^{N-2s}}\Big).\end{aligned} \tag{4-27}$$

Taking $y = 0$ on the both sides of (4-27), we get

$$\begin{aligned}&\frac{\partial}{\partial y_k}\mathcal{W}_{\varepsilon,\xi(y)}\Big|_{y=0}(\exp_{\xi_0}(x), x_N)\\ &\quad = \frac{1}{\varepsilon}\chi_r(\varepsilon x, \varepsilon x_N)\sum_{j=1}^{N}\Big[\partial_j W(x, x_N)\frac{\partial \mathcal{K}_j}{\partial y_k}(0, \varepsilon x)\Big] + \mathcal{O}\Big(\varepsilon^{n-2s}\frac{|\nabla\chi_r|(|(x, x_N)|)}{|(x, x_N)|^{N-2s}}\Big).\end{aligned}$$

This proves the first identity (4-25). Reasoning similarly, we prove the second identity (4-26). □

Let $\Phi_{\varepsilon,\xi}$ be the solution of (2-31) given by Proposition 3.4. Then for some constants $c_\varepsilon^l \in \mathbb{R},\ 1 \le l \le n$, we have

$$\Phi_{\varepsilon,\xi} = -\mathcal{W}_{\varepsilon,\xi} + i^*\big(i(f(\mathcal{W}_{\varepsilon,\xi} + \Phi_{\varepsilon,\xi}))\big) + \sum_{l=1}^{n} c_\varepsilon^l \mathcal{Z}_{\varepsilon,\xi}^l. \tag{4-28}$$

We next have the following result which will be crucial in the proofs of Lemma 4.1 and Proposition 4.6.

Lemma 4.8. *The constants* $c_\varepsilon^l,\ 1 \le l \le n$, *defined in* (4-28) *satisfy*

$$c_\varepsilon^l = \mathcal{O}(\varepsilon^\gamma) \quad \textit{for all } 1 \le l \le n. \tag{4-29}$$

Proof. Let $\Phi_{\varepsilon,\xi(y)} \in K_{\varepsilon,\xi(y)}^\perp$ be given by (4-28) and let $l \in \{1, \dots, n\}$, we clearly have that

$$\begin{aligned} J_\varepsilon'(\mathcal{W}_{\varepsilon,\xi(y)}+\Phi_{\varepsilon,\xi(y)})\mathcal{Z}_{\varepsilon,\xi(y)}^l &= \langle \mathcal{W}_{\varepsilon,\xi}, \mathcal{Z}_{\varepsilon,\xi}^l\rangle_{\varepsilon,**} - \frac{1}{\varepsilon^n}\int_M f(\mathcal{W}_{\varepsilon,\xi}+\Phi_{\varepsilon,\xi})\mathcal{Z}_{\varepsilon,\xi}^l \\ &= \left(\langle \mathcal{Z}_{\varepsilon,\xi}^l, \mathcal{W}_{\varepsilon,\xi}\rangle_{\varepsilon,**} - \frac{1}{\varepsilon^n}\int_M f(\mathcal{W}_{\varepsilon,\xi})\mathcal{Z}_{\varepsilon,\xi}^l\right) \\ &\quad + \left(\frac{1}{\varepsilon^n}\int_M \big(f(\mathcal{W}_{\varepsilon,\xi}) - f(\mathcal{W}_{\varepsilon,\xi}+\Phi_{\varepsilon,\xi})\big)\mathcal{Z}_{\varepsilon,\xi}^l\right) \\ &= I_1 + I_2. \end{aligned} \tag{4-30}$$

We first estimate I_1 in (4-30). Replacing Φ by $\mathcal{Z}_{\varepsilon,\xi}^l$ in the proof of Lemma 3.2 and using the fact that $\|\mathcal{Z}_{\varepsilon,\xi}^l\|_{\varepsilon,**} = \mathcal{O}(1)$, we get

$$\langle \mathcal{Z}_{\varepsilon,\xi}^l, \mathcal{W}_{\varepsilon,\xi}\rangle_{\varepsilon,**} - \frac{1}{\varepsilon^n}\int_M \big(f(\mathcal{W}_{\varepsilon,\xi})\big)\mathcal{Z}_{\varepsilon,\xi}^l = \mathcal{O}(\varepsilon^\gamma).$$

Now, to estimate I_2 we use the mean value theorem. We get, for some $\tau \in [0, 1]$, that

$$\begin{aligned} &\left|\frac{1}{\varepsilon^n}\int_M \big(f(\mathcal{W}_{\varepsilon,\xi}) - f(\mathcal{W}_{\varepsilon,\xi} + \Phi_{\varepsilon,\xi})\big)\mathcal{Z}_{\varepsilon,\xi}^l\right| \\ &\qquad = \left|\frac{1}{\varepsilon^n}\int_M f'(\mathcal{W}_{\varepsilon,\xi} + \tau\Phi_{\varepsilon,\xi})\Phi_{\varepsilon,\xi}\mathcal{Z}_{\varepsilon,\xi}^l\right| \\ &\qquad \le c\frac{1}{\varepsilon^n}\int_M |\mathcal{W}_{\varepsilon,\xi}|^{p-1}|\Phi_{\varepsilon,\xi}|\,|\mathcal{Z}_{\varepsilon,\xi}^l| + c\frac{1}{\varepsilon^n}\int_M |\Phi_{\varepsilon,\xi}|^p|\mathcal{Z}_{\varepsilon,\xi}^l| \\ &\qquad \le c\|\Phi_{\varepsilon,\xi}\|_{\varepsilon,**}\|\mathcal{Z}_{\varepsilon,\xi}^l\|_{\varepsilon,**} + \|\Phi_{\varepsilon,\xi}\|_{\varepsilon,**}^p\|\mathcal{Z}_{\varepsilon,\xi}^l\|_{\varepsilon,**} = \mathcal{O}(\varepsilon^\gamma). \end{aligned}$$

Combining the two above estimates, it follows that

$$J_\varepsilon'(\mathcal{W}_{\varepsilon,\xi(y)} + \Phi_{\varepsilon,\xi(y)})\mathcal{Z}_{\varepsilon,\xi(y)}^k = \mathcal{O}(\varepsilon^\gamma). \tag{4-31}$$

On the other hand, using (3-10) and (2-31), we conclude that

$$J'_\varepsilon(\mathcal{W}_{\varepsilon,\xi(y)}+\Phi_{\varepsilon,\xi(y)})\mathcal{Z}^k_{\varepsilon,\xi(y)} = \sum_{l=1}^{n} c^l_\varepsilon \langle \mathcal{Z}^l_{\varepsilon_m,\xi_m}, \mathcal{Z}^k_{\varepsilon_m,\xi_m}\rangle_\varepsilon = \sum_{l=1}^{n} c^l_\varepsilon(\delta_{kl}+o(1)) = c^l_\varepsilon \langle \mathcal{Z}^l_{\varepsilon,\xi}, \mathcal{Z}^l_{\varepsilon,\xi}\rangle_{\varepsilon,**} + \sum_{l\neq k} c^k_\varepsilon \langle \mathcal{Z}^k_{\varepsilon,\xi}, \mathcal{Z}^l_{\varepsilon,\xi}\rangle_{\varepsilon,**} = \mathcal{O}(\varepsilon^\gamma). \tag{4-32}$$

Using (4-31) and (4-32), the result follows at once. □

Lemma 4.9. *There exist $\varepsilon_0 > 0$ and $c > 0$ such that for any $\xi \in M$ and for any $\varepsilon \in (0, \varepsilon_0)$, we have*

$$\|\mathcal{Z}^l_{\varepsilon,\xi} - i^*\big(i(f'(\mathcal{W}_{\varepsilon,\xi(y)})\mathcal{Z}^l_{\varepsilon,\xi(y)})\big)\|_{\varepsilon,**} \le c\varepsilon^\gamma, \quad \Big\|\frac{1}{\varepsilon}\mathcal{Z}^l_{\varepsilon,\xi} + \Big(\frac{\partial}{\partial y_l}\mathcal{W}_{\varepsilon,\xi(y)}\Big)\Big|_{y=0}\Big\|_{\varepsilon,**} \le c\varepsilon.$$

Proof. The proof of the first estimate follows the same arguments as the proof of Lemma 3.2. To prove the second estimate, it is convenient to write

$$\begin{aligned}
\Big\|\frac{1}{\varepsilon}\mathcal{Z}^l_{\varepsilon,\xi} + \Big(\frac{\partial}{\partial y_l}\mathcal{W}_{\varepsilon,\xi(y)}\Big)\Big|_{y=0}\Big\|^2_{\varepsilon,**}
&= \frac{1}{\varepsilon^{n-2s}}\kappa_s \int_X \Bigg(\rho^{1-2s}\Big|\nabla\Big(\frac{1}{\varepsilon}\mathcal{Z}^l_{\varepsilon,\xi} + \Big(\frac{\partial}{\partial y_l}\mathcal{W}_{\varepsilon,\xi(y)}\Big)\Big|_{y=0}\Big)\Big|^2_{\bar g} \\
&\qquad + E(\rho)\Big(\frac{1}{\varepsilon}\mathcal{Z}^l_{\varepsilon,\xi} + \Big(\frac{\partial}{\partial y_l}\mathcal{W}_{\varepsilon,\xi(y)}\Big)\Big|_{y=0}\Big)^2\Bigg)\, d\mathrm{vol}_{\bar g} \\
&\qquad + \frac{1}{\varepsilon^n}\int_M \Big(\frac{1}{\varepsilon}\mathcal{Z}^l_{\varepsilon,\xi} + \Big(\frac{\partial}{\partial y_l}\mathcal{W}_{\varepsilon,\xi(y)}\Big)\Big|_{y=0}\Big)^2 d\mathrm{vol}_g \\
&= \Theta_1+\Theta_2+\Theta_3 \quad \text{for any } l=1,\dots,n,
\end{aligned}$$

where we have denoted by Θ_1, Θ_2 and Θ_3 respectively the first, second and third term in the right hand side of the above equality. To estimate Θ_1, we write

$$\begin{aligned}
\Theta_1 = \kappa_s \int_{B(0,r/\varepsilon)} & x_N^{1-2s}|\bar g_{\xi(y)}(\varepsilon x,\varepsilon x_N)|^{\frac12} \\
&\Bigg[\bar g^{ij}_{\xi(y)}(\varepsilon x,\varepsilon x_N)\,\partial_i\Big(\frac{1}{\varepsilon}\mathcal{Z}^l_{\varepsilon,\xi} + \Big(\frac{\partial}{\partial y_l}\mathcal{W}_{\varepsilon,\xi(y)}\Big)\Big|_{y=0}\Big) \\
&\quad \cdot \partial_j\Big(\frac{1}{\varepsilon}\mathcal{Z}^l_{\varepsilon,\xi} + \Big(\frac{\partial}{\partial y_l}\mathcal{W}_{\varepsilon,\xi(y)}\Big)\Big|_{y=0}\Big) \\
&\quad + \Big(\partial_N\Big(\frac{1}{\varepsilon}\mathcal{Z}^l_{\varepsilon,\xi} + \Big(\frac{\partial}{\partial y_l}\mathcal{W}_{\varepsilon,\xi(y)}\Big)\Big|_{y=0}\Big)\Big)^2\Bigg]\, dx\, dx_N,
\end{aligned} \tag{4-33}$$

where summation over repeated indices is understood. Using Lemma 4.7 we obtain

$$
\begin{aligned}
&\frac{1}{\varepsilon}\mathcal{Z}^l_{\varepsilon,\xi}+\frac{1}{\varepsilon}\left(\frac{\partial}{\partial y_l}\mathcal{W}_{\varepsilon,\xi(y)}\right)\Big|_{y=0}\\
&=\frac{1}{\varepsilon}\chi_r(\varepsilon x,\varepsilon x_N)\sum_{k=1}^{n}\left[\partial_k W(x,x_N)\frac{\partial\mathcal{K}_k}{\partial y_l}(0,\varepsilon x)\right]\\
&\qquad\qquad+\frac{1}{\varepsilon}\chi_r(\varepsilon x,\varepsilon x_N)Z_l(x,x_N)+\mathcal{O}\left(\varepsilon^{n-2s}\frac{|\nabla\chi_r|(|(x-y,x_N)|)}{|(x-y,x_N)|^{N-2s}}\right)\\
&=\frac{1}{\varepsilon}\chi_r(\varepsilon x,\varepsilon x_N)Z_l(x,x_N)\left[\frac{\partial\mathcal{K}_l}{\partial y_l}(0,\varepsilon x)+1\right]\\
&\quad+\frac{1}{\varepsilon}\sum_{k=1,k\neq l}^{n}\chi_r(\varepsilon x,\varepsilon x_N)Z_k(x,x_N)\frac{\partial\mathcal{K}_k}{\partial y_l}(0,\varepsilon x)+\mathcal{O}\left(\varepsilon^{n-2s}\frac{|\nabla\chi_r|(|(x-y,x_N)|)}{|(x-y,x_N)|^{N-2s}}\right).
\end{aligned}
$$

This implies that

$$
\begin{aligned}
&\partial_i\left(\frac{1}{\varepsilon}\mathcal{Z}^l_{\varepsilon,\xi}+\left(\frac{\partial}{\partial y_l}\mathcal{W}_{\varepsilon,\xi(y)}\right)\Big|_{y=0}\right)\\
&\quad=\frac{1}{\varepsilon}\chi_r(\varepsilon x,\varepsilon x_N)\,\partial_i Z_l(x,x_N)\left[\frac{\partial\mathcal{K}_l}{\partial y_l}(0,\varepsilon x)+1\right]\\
&\qquad\qquad+\partial_i\chi_r(\varepsilon x,\varepsilon x_N)Z_l(x,x_N)\left[\frac{\partial\mathcal{K}_l}{\partial y_l}(0,\varepsilon x)+1\right]\\
&\qquad\qquad+\frac{1}{\varepsilon}\chi_r(\varepsilon x,\varepsilon x_N)Z_l(x,x_N)\,\partial_i\left(\frac{\partial\mathcal{K}_l}{\partial y_l}(0,\varepsilon x)\right)\\
&\qquad\qquad+\frac{1}{\varepsilon}\sum_{k=1,k\neq l}^{n}\chi_r(\varepsilon x,\varepsilon x_N)\,\partial_i Z_k(x,x_N)\frac{\partial\mathcal{K}_k}{\partial y_l}(0,\varepsilon x)\\
&\qquad\qquad+\sum_{k=1,k\neq l}^{n}\partial_i\chi_r(\varepsilon x,\varepsilon x_N)Z_k(x,x_N)\frac{\partial\mathcal{K}_k}{\partial y_l}(0,\varepsilon x)\\
&\qquad\qquad+\frac{1}{\varepsilon}\sum_{k=1,k\neq l}^{n}\chi_r(\varepsilon x,\varepsilon x_N)Z_k(x,x_N)\,\partial_i\left(\frac{\partial\mathcal{K}_k}{\partial y_l}(0,\varepsilon x)\right)\\
&\qquad\qquad+\mathcal{O}\left(\varepsilon^{n-2s}\,\partial_i\frac{|\nabla\chi_r|(|(x-y,x_N)|)}{|(x-y,x_N)|^{N-2s}}\right).
\end{aligned}
$$

Recalling the definition of the cutoff function χ_r defined above, we obtain

$$
\Theta_1\le c\sum_{h=1}^{7}\Theta_{1h}, \tag{4-34}
$$

where the quantities Θ_{1h}'s are given by

$$\Theta_{11} = \frac{1}{\varepsilon^2}\kappa_s \int_{B^+_{2r_0/\varepsilon}} x_N^{1-2s}\big\{|\partial_i Z_l(x,x_N)|^2 + |\partial_N Z_l(x,x_N)|^2\big\}\Big|\frac{\partial \mathcal{K}_l}{\partial y_l}(0,\varepsilon x)+1\Big|^2,$$

$$\Theta_{12} = \kappa_s \int_{B^+_{2r_0/\varepsilon}} x_N^{1-2s}\big\{|\partial_i \chi_r(\varepsilon x,\varepsilon x_N) Z_l(x,x_N)|^2 + |\partial_N \chi_r(\varepsilon x,\varepsilon x_N) Z_l(x,x_N)|^2\big\} \cdot \Big|\frac{\partial \mathcal{K}_l}{\partial y_l}(0,\varepsilon x)+1\Big|^2,$$

$$\Theta_{13} = \frac{1}{\varepsilon^2}\kappa_s \int_{B^+_{2r_0/\varepsilon}} x_N^{1-2s}\left\{\Big|\partial_i\Big(\frac{\partial \mathcal{K}_l}{\partial y_l}(0,\varepsilon x)\Big)\Big|^2 + \Big|\partial_N\Big(\frac{\partial \mathcal{K}_l}{\partial y_l}(0,\varepsilon x)\Big)\Big|^2\right\} \cdot |\chi_r(\varepsilon x,\varepsilon x_N) Z_l(x,x_N)|^2,$$

$$\Theta_{14} = \frac{1}{\varepsilon^2}\sum_{k=1,k\neq l}^{n} \kappa_s \int_{B^+_{2r_0/\varepsilon}} x_N^{1-2s}\big\{|\partial_i Z_k(x,x_N)|^2 + |\partial_N Z_k(x,x_N)|^2\big\}\Big|\frac{\partial \mathcal{K}_k}{\partial y_l}(0,\varepsilon x)\Big|^2,$$

$$\Theta_{15} = \sum_{k=1,k\neq l}^{n} \kappa_s \int_{B^+_{2r_0/\varepsilon}} x_N^{1-2s}\big\{|\partial_i \chi_r(\varepsilon x,\varepsilon x_N) Z_k(x,x_N)|^2 + |\partial_N \chi_r(\varepsilon x,\varepsilon x_N) Z_k(x,x_N)|^2\big\}\Big|\frac{\partial \mathcal{K}_k}{\partial y_l}(0,\varepsilon x)\Big|^2,$$

$$\Theta_{16} = \frac{1}{\varepsilon^2}\sum_{k=1,k\neq l}^{n} \kappa_s \int_{B^+_{2r_0/\varepsilon}} x_N^{1-2s}\left\{\Big|\partial_i\Big(\frac{\partial \mathcal{K}_k}{\partial y_l}(0,\varepsilon x)\Big)\Big|^2 + \Big|\partial_N\Big(\frac{\partial \mathcal{K}_k}{\partial y_l}(0,\varepsilon x)\Big)\Big|^2\right\} \cdot |\chi_r(\varepsilon x,\varepsilon x_N) Z_k(x,x_N)|^2,$$

$$\Theta_{17} = \kappa_s \int_{B^+_{2r_0/\varepsilon}} x_N^{1-2s}\left\{\left(\varepsilon^{n-2s}\partial_i \frac{|\nabla\chi_r|(|(x-y,x_N)|)}{|(x-y,x_N)|^{N-2s}}\right)^2 + \left(\varepsilon^{n-2s}\partial_N \frac{|\nabla\chi_r|(|(x-y,x_N)|)}{|(x-y,x_N)|^{N-2s}}\right)^2\right\}.$$

On the other hand, using (6.12) of [32] (see also [13]), we have that

$$\frac{\partial \mathcal{K}_k}{\partial y_l}\big(y,\xi^{-1}(\exp_{\xi(y)})(\varepsilon x)\big)\Big|_{y=0} = \frac{\partial \mathcal{K}_k}{\partial y_l}(0,\varepsilon x) = -\delta_{kl} + \mathcal{O}(\varepsilon^2|x|^2). \tag{4-35}$$

Then by (4-34) we get

$$\Theta_1 \le c\varepsilon^2.$$

Arguing similarly, we easily obtain

$$
\begin{aligned}
\Theta_2 = \varepsilon^{1+2s}\kappa_s \int_{B^+_{2r_0/\varepsilon}} E(\varepsilon x_N)\bigg\{ &\frac{1}{\varepsilon}\chi_r(\varepsilon x, \varepsilon x_N) Z_l(x, x_N)\bigg[\frac{\partial \mathcal{K}_l}{\partial y_l}(0, \varepsilon x) + 1\bigg] \\
&+ \sum_{k=1, k\neq l}^{n} \chi_r(\varepsilon x, \varepsilon x_N) Z_k(x, x_N)\frac{\partial \mathcal{K}_k}{\partial y_l}(0, \varepsilon x) \\
&+ \mathcal{O}\bigg(\varepsilon^{n-2s}\frac{|\nabla \chi_r|(|(x-y, x_N)|)}{|(x-y, x_N)|^{N-2s}}\bigg)\bigg\}^2 \\
&\cdot |\bar{g}_{\xi(y)}(\varepsilon x, \varepsilon x_N)|^{\frac{1}{2}}\, dx\, dx_N \\
\leq c\varepsilon^2 &
\end{aligned}
\tag{4-36}
$$

and

$$
\begin{aligned}
\Theta_3 &= \int_{B_{2r_0/\varepsilon}} \bigg(\frac{1}{\varepsilon}\mathcal{Z}^l_{\varepsilon,\xi} + \bigg(\frac{\partial}{\partial y_l}\mathcal{W}_{\varepsilon,\xi(y)}\bigg)\bigg|_{y=0}\bigg)^2 \Big|g_{\xi(y)}(\varepsilon x, \varepsilon x_N)\Big|^{\frac{1}{2}}\, dx \\
&= \int_{B_{2r_0/\varepsilon}} \bigg[\frac{1}{\varepsilon}\chi_r(\varepsilon x, \varepsilon x_N)\, Z_l(x, x_N)\bigg[\frac{\partial \mathcal{K}_l}{\partial y_l}(0, \varepsilon x) + 1\bigg] \\
&\qquad + \sum_{k=1, k\neq l}^{n} \chi_r(\varepsilon x, \varepsilon x_N) Z_k(x, x_N)\frac{\partial \mathcal{K}_k}{\partial y_l}(0, \varepsilon x) \\
&\qquad + \mathcal{O}\bigg(\varepsilon^{n-2s}\frac{|\nabla \chi_r|(|(x-y, x_N)|)}{|(x-y, x_N)|^{N-2s}}\bigg)\bigg]^2 |g_{\xi(y)}(\varepsilon x, \varepsilon x_N)|^{\frac{1}{2}}\, dx \\
&\leq c\varepsilon^2.
\end{aligned}
\tag{4-37}
$$

This prove the desired estimate. □

We go back now to the proof of Proposition 4.6. For simplicity, we will use the notation

$$
(\chi_r \partial_k W_\varepsilon)(z) = \chi_r(|(x, x_N)|)\, W_\varepsilon(x, x_N)
$$

for $z = (\xi(x), x_N) \in X$ near $\xi_0 \in M$. We may assume that the domain of these functions is the Euclidean space $\mathbb{R}^{n+1}_+$.

By the previous lemma, we have

$$
\begin{aligned}
&\frac{\partial}{\partial_{y_k}} J_\varepsilon(\mathcal{W}_{\varepsilon,\xi(y)} + \Phi_{\varepsilon,\xi(y)}) - \frac{\partial}{\partial y_k} J_\varepsilon(\mathcal{W}_{\varepsilon,\xi(y)}) \\
&= J'_\varepsilon(\mathcal{W}_{\varepsilon,\xi(y)} + \Phi_{\varepsilon,\xi(y)})(\partial_{y_k}\mathcal{W}_{\varepsilon,\xi(y)} + \partial_{y_k}\Phi_{\varepsilon,\xi(y)}) - J'_\varepsilon(\mathcal{W}_{\varepsilon,\xi(y)})(\partial_k \mathcal{W}_{\varepsilon,\xi(y)}) \\
&= J'_\varepsilon(\mathcal{W}_{\varepsilon,\xi(y)} + \Phi_{\varepsilon\xi(y)})[\partial_{y_k}\Phi_{\varepsilon,\xi(y)}] + [J'_\varepsilon(\mathcal{W}_{\varepsilon,\xi(y)} + \Phi_{\varepsilon,\xi(y)}) - J'_\varepsilon(\mathcal{W}_{\varepsilon,\xi(y)})][\partial_k \mathcal{W}_{\varepsilon,\xi(y)}] \\
&= J_1 + J_2,
\end{aligned}
$$

where we have set

$$
\begin{aligned}
J_1 &:= J'_\varepsilon(\mathcal{W}_{\varepsilon,\xi(y)} + \Phi_{\varepsilon\xi(y)})[\partial_{y_k}\Phi_{\varepsilon,\xi(y)}], \\
J_2 &:= [J'_\varepsilon(\mathcal{W}_{\varepsilon,\xi(y)} + \Phi_{\varepsilon,\xi(y)}) - J'_\varepsilon(\mathcal{W}_{\varepsilon,\xi(y)})][\partial_k \mathcal{W}_{\varepsilon,\xi(y)}].
\end{aligned}
$$

We first estimate the term J_1. Using Lemma 4.8, Proposition 3.4 and the fact that $\|\partial_{y_k}\mathcal{Z}^l_{\varepsilon,\xi}\|_{\varepsilon,**} = \mathcal{O}(1)$, we get

$$
\begin{aligned}
J_1 &= J'_\varepsilon(\mathcal{W}_{\varepsilon,\xi(y)} + \Phi_{\varepsilon\xi(y)})[\partial_{y_k}\Phi_{\varepsilon,\xi(y)}] \\
&= \sum_{l=1}^{n} c^l_\varepsilon \langle \mathcal{Z}^l_{\varepsilon,\xi}, \partial_{y_k}\Phi_{\varepsilon,\xi(y)}\rangle_{\varepsilon,*} \\
&= -\sum_{l=1}^{n} c^l_\varepsilon \langle \partial_{y_k}\mathcal{Z}^l_{\varepsilon,\xi}, \Phi_{\varepsilon,\xi(y)}\rangle_{\varepsilon,*} \\
&\le \sum_{l=1}^{n} |c^l_\varepsilon|\,|\langle \partial_{y_k}\mathcal{Z}^l_{\varepsilon,\xi}, \Phi_{\varepsilon,\xi(y)}\rangle_{\varepsilon,*}| \\
&\le c\sum_{l=1}^{n} |c^l_\varepsilon|\,\|\partial_{y_k}\mathcal{Z}^l_{\varepsilon,\xi}\|_{\varepsilon,*}\|\Phi_{\varepsilon,\xi(y)}\|_{\varepsilon,*} = \mathcal{O}(\varepsilon^{2\gamma}).
\end{aligned}
\tag{4-38}
$$

Concerning the term J_2, we write

$$
\begin{aligned}
J_2 &= [J'_\varepsilon(\mathcal{W}_{\varepsilon,\xi(y)} + \Phi_{\varepsilon,\xi(y)}) - J'_\varepsilon(\mathcal{W}_{\varepsilon,\xi(y)})][\partial_k\mathcal{W}_{\varepsilon,\xi(y)}] \\
&= \langle \Phi_{\varepsilon,\xi(y)}, \partial_k\mathcal{W}_{\varepsilon,\xi(y)}\rangle_{\varepsilon,*} - \frac{1}{\varepsilon^n}\int_M [f(\mathcal{W}_{\varepsilon,\xi(y)} + \Phi_{\varepsilon,\xi(y)}) - f(\mathcal{W}_{\varepsilon,\xi(y)})]\partial_k\mathcal{W}_{\varepsilon,\xi(y)} \\
&= \Big\langle \Phi_{\varepsilon,\xi(y)} - i^*\big(i(f'(\mathcal{W}_{\varepsilon,\xi(y)})\Phi_{\varepsilon,\xi(y)})\big), \partial_k\mathcal{W}_{\varepsilon,\xi(y)} + \frac{1}{\varepsilon}\mathcal{Z}^l_{\varepsilon,\xi(y)}\Big\rangle_{\varepsilon,*} \\
&\quad - \frac{1}{\varepsilon^n}\int_M [f(\mathcal{W}_{\varepsilon,\xi(y)} + \Phi_{\varepsilon,\xi(y)}) - f(\mathcal{W}_{\varepsilon,\xi(y)}) - f'(\mathcal{W}_{\varepsilon,\xi(y)})\Phi_{\varepsilon,\xi(y)}]\partial_k\mathcal{W}_{\varepsilon,\xi(y)} \\
&\quad - \frac{1}{\varepsilon^n}\big\langle \Phi_{\varepsilon,\xi(y)}, \mathcal{Z}^l_{\varepsilon,\xi(y)} - i^*\big(i(f'(\mathcal{W}_{\varepsilon,\xi(y)})\mathcal{Z}^l_{\varepsilon,\xi(y)})\big)\big\rangle_{\varepsilon,*} \\
&= J_{21} + J_{22} + J_{23}.
\end{aligned}
$$

To estimate J_{21}, we use (3-20), (2-15) and Lemma 4.9. We get

$$
\begin{aligned}
|J_{21}| &\le \|\Phi_{\varepsilon,\xi(y)} - i^*\big(i(f'(\mathcal{W}_{\varepsilon,\xi(y)})\Phi_{\varepsilon,\xi(y)})\big)\|_{\varepsilon,*}\,\|\partial_k\mathcal{W}_{\varepsilon,\xi(y)} + \frac{1}{\varepsilon}\mathcal{Z}^l_{\varepsilon,\xi(y)}\|_{\varepsilon,*} \\
&\le c\varepsilon\|\Phi_{\varepsilon,\xi(y)}\|_{\varepsilon,*} = \mathcal{O}(\varepsilon^{\gamma+1}).
\end{aligned}
\tag{4-39}
$$

Next, we compute the second term J_{22}, by (3-20), we obtain

$$
|J_{22}| \le c(\|\Phi_{\varepsilon,\xi(y)}\|^2_{\varepsilon,**} + \|\Phi_{\varepsilon,\xi(y)}\|^{p+1}_{\varepsilon,*})\,\|\partial_k\mathcal{W}_{\varepsilon,\xi(y)}\|_{\varepsilon,*} = \mathcal{O}(\varepsilon^{2\gamma})
\tag{4-40}
$$

We now estimate the second term J_{22}. For $p \ge 2$, we have that

$$
|J_{22}| \le c\|\Phi_{\varepsilon,\xi(y)}\|^2_{\varepsilon,**}\|\partial_k\mathcal{W}_{\varepsilon,\xi(y)}\|_{\varepsilon,*} + \|\Phi_{\varepsilon,\xi(y)}\|^p_{\varepsilon,*}\|\partial_k\mathcal{W}_{\varepsilon,\xi(y)}\|_{\varepsilon,*}.
\tag{4-41}
$$

While, for $1 < p < 2$, we have that

$$
\begin{aligned}
|J_{22}| &\le c\frac{1}{\varepsilon^n}\int_M \mathcal{W}^{p-2}_{\varepsilon,\xi(y)}\Phi^2_{\varepsilon,\xi(y)}|\partial_k\mathcal{W}_{\varepsilon,\xi(y)}| \\
&\le c\|\Phi_{\varepsilon,\xi(y)}\|^2_{\varepsilon,**}\|\partial_k\mathcal{W}_{\varepsilon,\xi(y)}\|_{\varepsilon,*}\|\mathcal{W}_{\varepsilon,\xi(y)}\|^{p-2}_{\varepsilon,*}.
\end{aligned}
\tag{4-42}
$$

Then, using (4-28), we conclude, for every $p \in (1, 2^*_s - 1)$

$$|J_{22}| = \mathcal{O}(\varepsilon^{2\gamma}).$$

Finally, using (3-20) and Lemma 4.9, the last term J_{23} can be estimated as

$$|J_{23}| \le \|\Phi_{\varepsilon,\xi(y)}\|_{\varepsilon,*}\|\mathcal{Z}^l_{\varepsilon,\xi(y)} - i^*\big(i(f'(\mathcal{W}_{\varepsilon,\xi(y)})\mathcal{Z}^l_{\varepsilon,\xi(y)})\big)\|_{\varepsilon,*} = \mathcal{O}(\varepsilon^{2\gamma}).$$

Collecting the previous estimates. we deduce that

$$J_2 = \mathcal{O}(\varepsilon^{2\gamma}) + \mathcal{O}(\varepsilon^{\gamma+1}). \tag{4-43}$$

Combining (4-38) and (4-43), the result follows at once.

Proposition 4.10. *Define $\xi(y) = \exp_\xi(y)$, $y \in B^n(0, 4r_0)$. It holds the following.*

- *For* $0 < s < \frac{1}{2}$

$$
\begin{aligned}
&\left(\frac{\partial}{\partial y_h}J_\varepsilon(\mathcal{W}_{\varepsilon,\xi})\right)_{|y=0} \\
&\qquad = -\varepsilon d_1\left(\frac{\partial}{\partial y_k}H(\xi(y))\right)_{|y=0} + \varepsilon d_2\left(\frac{\partial}{\partial y_k}\pi_{ij}(\xi(y))\right)_{|y=0} + o(\varepsilon).
\end{aligned}
\tag{4-44}
$$

- *For* $\frac{1}{2} \le s < 1$

$$
\begin{aligned}
&\left(\frac{\partial}{\partial y_h}J_\varepsilon(\mathcal{W}_{\varepsilon,\xi})\right)_{|y=0} \\
&= \frac{\varepsilon^2}{12}b_1\left(\frac{\partial}{\partial y_m}R_{kl}(\xi(y))\right)_{|y=0} - \frac{\varepsilon^2}{6}b_2\left(\frac{\partial}{\partial y_m}R_{ikjl}(\xi(y))\right)_{|y=0} \\
&\quad - \frac{\varepsilon^2}{2}b_3\left(-\left(\frac{\partial}{\partial y_k}R_{NN}(\xi(y))\right) + \left(\frac{\partial}{\partial y_k}\pi_{is}(\xi(y))\right)\pi_{si}\right)_{|y=0} \\
&\quad + \varepsilon^2 b_4\left(\left(\frac{\partial}{\partial y_k}R_{iNjN}(\xi(y))\right) + 7\left(\frac{\partial}{\partial y_k}\pi_{jh}(\xi(y))\right)\pi_{hi}(\xi(y))\right)_{|y=0} + o(\varepsilon^2)
\end{aligned}
\tag{4-45}
$$

uniformly in ξ as ε goes to zero. Here the constants b_1, b_2, b_3 b_4, d_1 and d_2 are explicit constants given below.

Proof. We have

$$
\begin{aligned}
&\frac{\partial}{\partial y_h} J_\varepsilon(\mathcal{W}_{\varepsilon,\xi}) \\
&= J'_\varepsilon(\mathcal{W}_{\varepsilon,\xi})\left[\frac{\partial}{\partial y_h}\mathcal{W}_{\varepsilon,\xi}\right] \\
&= \frac{1}{\varepsilon^{n-2s}}\kappa_s \int_X \left(\rho^{1-2s}\left(\nabla \mathcal{W}_{\varepsilon,\xi(y)}, \nabla \frac{\partial}{\partial y_h}\mathcal{W}_{\varepsilon,\xi(y)}\right)_{\bar g} + E(\rho)\mathcal{W}_{\varepsilon,\xi(y)}\frac{\partial}{\partial y_h}\mathcal{W}_{\varepsilon,\xi(y)}\right) d\mathrm{vol}_{\bar g} \\
&\qquad + \frac{1}{\varepsilon^n}\int_M \mathcal{W}_{\varepsilon,\xi(y)}\frac{\partial}{\partial y_h}\mathcal{W}_{\varepsilon,\xi(y)} - f(\mathcal{W}_{\varepsilon,\xi(y)})\frac{\partial}{\partial y_h}\mathcal{W}_{\varepsilon,\xi(y)}\, d\mathrm{vol}_g \\
&= \kappa_s \int_{B^+_{2r_0/\varepsilon}} x_N^{1-2s}\left\{\bar g^{ij}_{\xi(y)}(\varepsilon x, \varepsilon x_N)\,\partial_i \mathcal{W}_{\varepsilon,\xi(y)}\partial_j\left(\frac{\partial}{\partial y_h}\mathcal{W}_{\varepsilon,\xi(y)}\right)\right. \\
&\qquad \left. + \partial_N \mathcal{W}_{\varepsilon,\xi(y)}\partial_N\left(\frac{\partial}{\partial y_h}\mathcal{W}_{\varepsilon,\xi(y)}\right)\right\}|\bar g_{\xi(y)}(\varepsilon x, \varepsilon x_N)|^{\frac12}\, dx\, dx_N \\
&\qquad + \varepsilon^{1+2s}\kappa_s \int_{B^+_{2r_0/\varepsilon}} E(\varepsilon\rho)\mathcal{W}_{\varepsilon,\xi(y)}\frac{\partial}{\partial y_h}\mathcal{W}_{\varepsilon,\xi(y)}|\bar g_{\xi(y)}(\varepsilon x, \varepsilon x_N)|^{\frac12}\, dx\, dx_N \\
&\qquad + \int_{B_{2r_0/\varepsilon}} \mathcal{W}_{\varepsilon,\xi(y)}\frac{\partial}{\partial y_h}\mathcal{W}_{\varepsilon,\xi(y)}|g_{\xi(y)}(\varepsilon x)|^{\frac12}\, dx\, dx_N \\
&\qquad - \int_{B_{2r_0/\varepsilon}} f(\mathcal{W}_{\varepsilon,\xi(y)})\frac{\partial}{\partial y_h}\mathcal{W}_{\varepsilon,\xi(y)}|g_{\xi(y)}(\varepsilon x)|^{\frac12}\, dx\, dx_N \\
&= \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3 - \mathcal{J}_4.
\end{aligned}
$$

Using Taylor's expansions, we get

$$
\begin{aligned}
\sqrt{\det \bar g} &= \sqrt{\det g} \\
&= 1 - H x_N + \tfrac12 (H^2 - \|\pi\|^2 - R_{NN}) x_N^2 - H_{,k} x_k x_N - \tfrac16 R_{kl} x_k x_l - \tfrac{1}{12} R_{kl,m} x_k x_l x_m \\
&\quad + \tfrac12\big(-H_{,kl} + \tfrac13 R_{iksl}\pi_{si}\big) x_k x_l x_N + \tfrac12(-R_{NN,k} + \pi_{is,k}\pi_{si}) x_k x_N^2 \\
&\quad + \tfrac16\big(-R_{NN,N} + 2(\pi_{is}R_{sNiN}) - 4H^3 + 12H(\pi_{is})^2 - 8\pi_{is}\pi_{sr}\pi_{ri}\big) x_N^3 \\
&\quad + \mathcal{O}(|(x, x_N)|^4),
\end{aligned}
$$

and

$$
\begin{aligned}
g^{ij}_\xi &= \bar g^{ij}_\xi \\
&= \delta_{ij} + 2\pi_{ij} x_N + \tfrac13 R_{ikjl} x_k x_l + 2\pi_{ij,k} x_k x_N + (3\pi_{ih}\pi_{hj} + R_{iNjN}) x_N^2 \\
&\quad + \tfrac16 R_{ikjl,m} x_k x_l x_m + (\pi_{ij,kl} + R_{jkhl}\pi_{hi}) x_k x_l x_N + (R_{iNjN,k} + 7\pi_{jh,k}\pi_{hi}) x_k x_N^2 \\
&\quad + \tfrac13\big(R_{iNjN,N} + 10(\pi_{ih}R_{hNjN}) + 12\pi_{ih}\pi_{hr}\pi_{rj}\big) x_N^3 + \mathcal{O}(|(x, x_N)|^4).
\end{aligned}
$$

• For $0 < s < \frac{1}{2}$, we first compute the term $\mathcal{J}_1$. Using Lemma 4.7, we have

$$\begin{aligned}
\mathcal{J}_1 = {} & \kappa_s \int_{B^+_{2r_0/\varepsilon}} x_N^{1-2s} |\bar{g}_{\xi(y)}(\varepsilon x, \varepsilon x_N)|^{\frac{1}{2}} \\
& \Big\{ \bar{g}^{ij}_{\xi(y)}(\varepsilon x, \varepsilon x_N)\, \partial_i \mathcal{W}_{\varepsilon,\xi(y)} \partial_j \Big(\frac{\partial}{\partial y_h} \mathcal{W}_{\varepsilon,\xi(y)} \Big) \\
& \qquad + \partial_N \mathcal{W}_{\varepsilon,\xi(y)} \partial_N \Big(\frac{\partial}{\partial y_h} \mathcal{W}_{\varepsilon,\xi(y)} \Big) \Big\} \, dx\, dx_N \\
= {} & -\varepsilon H_{,k} \kappa_s \sum_{e=1}^{n} \frac{\partial \mathcal{K}_e}{\partial y_h}(y, \xi(y)) \\
& \int_{\mathbb{R}^{n+1}_+} x_N^{2-2s} x_k (\partial_i W \partial^2_{ie} W + \partial_N W \partial^2_{Ne} W) \, dx\, dx_N \\
& + 2\varepsilon \pi_{ij,k} \kappa_s \sum_{e=1}^{n} \frac{\partial \mathcal{K}_e}{\partial y_h}(y, \xi(y)) \int_{\mathbb{R}^{n+1}_+} x_N^{2-2s} x_k \partial_i W \partial^2_{je} W \, dx\, dx_N \\
& + o(\varepsilon).
\end{aligned} \tag{4-46}$$

Similarly, we can estimate the second term $\mathcal{J}_2$ as

$$\begin{aligned}
\mathcal{J}_2 = {} & \varepsilon^{1+2s} \kappa_s \int_{B^+_{2r_0/\varepsilon}} E(\varepsilon x_N) \mathcal{W}_{\varepsilon,\xi(y)} \frac{\partial}{\partial y_h} \mathcal{W}_{\varepsilon,\xi(y)} |\bar{g}_{\xi(y)}(\varepsilon x, \varepsilon x_N)|^{\frac{1}{2}} \, dx\, dx_N \\
= {} & \varepsilon^{1+2s} \kappa_s \int_{B^+_{2r_0/\varepsilon}} E(\varepsilon x_N) W \chi(\varepsilon x, \varepsilon x_N) \\
& \cdot \Big(\frac{1}{\varepsilon} \chi_r(\varepsilon x, \varepsilon x_N) \sum_{e=1}^{n} \Big[\partial_e W(x, x_N) \frac{\partial \mathcal{K}_e}{\partial y_h} \big(y, \xi^{-1}(\exp_{\xi(y)})(\varepsilon x) \big) \Big] \\
& \qquad + \mathcal{O}\Big(\varepsilon^{n-2s} \frac{|\nabla \chi_r|(|(x-y, x_N)|)}{|(x-y, x_N)|^{N-2s}} \Big) \Big) \\
& \cdot |\bar{g}_{\xi(y)}(\varepsilon x, \varepsilon x_N)|^{\frac{1}{2}} \, dx\, dx_N \\
= {} & o(\varepsilon).
\end{aligned} \tag{4-47}$$

On the other hand, similar arguments yield

$$\begin{aligned}
\mathcal{J}_3 = {} & \int_{B_{2r_0/\varepsilon}} \mathcal{W}_{\varepsilon,\xi(y)} \frac{\partial}{\partial y_h} \mathcal{W}_{\varepsilon,\xi(y)} |g_{\xi(y)}(\varepsilon x, \varepsilon x_N)|^{\frac{1}{2}} \, dx \\
= {} & \int_{B_{2r_0/\varepsilon}} \omega \chi_{r/\varepsilon} \Big(\frac{1}{\varepsilon} \chi_{r/\varepsilon} \sum_{e=1}^{n} \Big[\partial_e \omega \frac{\partial \mathcal{K}_e}{\partial y_h} \big(y, \xi^{-1}(\exp_{\xi(y)})(\varepsilon x) \big) \Big] \\
& \qquad + \mathcal{O}\Big(\varepsilon^{n-2s} \frac{|\nabla \chi_r|(|(x-y)|)}{|(x-y)|^{N-2s}} \Big) \Big) |g_{\xi(y)}(\varepsilon x)|^{\frac{1}{2}} \, dx \\
= {} & o(\varepsilon).
\end{aligned} \tag{4-48}$$

Finally

$$
\begin{aligned}
(4\text{-}49)\quad \mathcal{J}_4 &= \int_{B_{2r_0/\varepsilon}} f(\mathcal{W}_{\varepsilon,\xi(y)})\frac{\partial}{\partial y_h}\mathcal{W}_{\varepsilon,\xi(y)}|g_{\xi(y)}(\varepsilon x)|^{\frac{1}{2}}\,dx \\
&= \int_{B_{2r_0/\varepsilon}} f(\omega\chi_{r/\varepsilon})\biggl(\frac{1}{\varepsilon}\chi_{r/\varepsilon}\sum_{e=1}^{n}\Bigl[\partial_e\omega\frac{\partial\mathcal{K}_e}{\partial y_h}\bigl(y,\xi^{-1}(\exp_{\xi(y)})(\varepsilon x)\bigr)\Bigr] \\
&\qquad\qquad + \mathcal{O}\biggl(\varepsilon^{n-2s}\frac{|\nabla\chi_r|(|(x-y)|)}{|(x-y)|^{N-2s}}\biggr)\biggr)|g_{\xi(y)}(\varepsilon x)|^{\frac{1}{2}}\,dx \\
&= o(\varepsilon).
\end{aligned}
$$

Using (4-46)–(4-49), we deduce that

$$
\begin{aligned}
&\frac{\partial}{\partial y_h}J_\varepsilon(\mathcal{W}_{\varepsilon,\xi}) \\
&\quad = -\varepsilon H_{,k}\kappa_s\sum_{e=1}^{n}\frac{\partial\mathcal{K}_e}{\partial y_h}(y,\xi(y))\int_{\mathbb{R}^{n+1}_+} x_N^{2-2s}x_k(\partial_i W\partial^2_{ie}W+\partial_N W\partial^2_{Ne}W)\,dx\,dx_N \\
&\qquad\qquad + 2\varepsilon\pi_{ij,k}\kappa_s\sum_{e=1}^{n}\frac{\partial\mathcal{K}_e}{\partial y_h}(y,\xi(y))\int_{\mathbb{R}^{n+1}_+} x_N^{2-2s}x_k\partial_i W\partial^2_{je}W\,dx\,dx_N \\
&\qquad\qquad + o(\varepsilon).
\end{aligned}
$$

Then

$$
\begin{aligned}
(4\text{-}50)\quad &\biggl(\frac{\partial}{\partial y_h}J_\varepsilon(\mathcal{W}_{\varepsilon,\xi})\biggr)_{|y=0} \\
&\qquad = -\varepsilon d_1\biggl(\frac{\partial}{\partial y_k}H(\xi(y))\biggr)_{|y=0} + \varepsilon d_2\biggl(\frac{\partial}{\partial y_k}\pi_{ij}(\xi(y))\biggr)_{|y=0} + o(\varepsilon),
\end{aligned}
$$

where

$$
d_1 := \kappa_s\int_{\mathbb{R}^{n+1}_+} x_N^{2-2s}x_k\nabla W\nabla\partial_h W\,dx\,dx_N
$$

and

$$
d_2 := 2\kappa_s\int_{\mathbb{R}^{n+1}_+} x_N^{2-2s}x_k\partial_i W\partial^2_{je}W\,dx\,dx_N.
$$

• For $\frac{1}{2}\le s<1$, using again Lemma 4.7, we get

$$
\begin{aligned}
(4\text{-}51)\quad \mathcal{J}_1 = \kappa_s\int_{B^+_{2r_0/\varepsilon}} & x_N^{1-2s}|\bar{g}_{\xi(y)}(\varepsilon x,\varepsilon x_N)|^{\frac{1}{2}} \\
&\biggl\{\bar{g}^{ij}_{\xi(y)}(\varepsilon x,\varepsilon x_N)\,\partial_i\mathcal{W}_{\varepsilon,\xi(y)}\partial_j\frac{\partial\mathcal{W}_{\varepsilon,\xi(y)}}{\partial y_h} \\
&\qquad + \partial_N\mathcal{W}_{\varepsilon,\xi(y)}\partial_N\frac{\partial\mathcal{W}_{\varepsilon,\xi(y)}}{\partial y_h}\biggr\}\,dx\,dx_N
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{\varepsilon^2}{12} R_{kl,m}\kappa_s \sum_{e=1}^{n} \frac{\partial \mathcal{K}_e}{\partial y_h}(y,\xi(y)) \int_{\mathbb{R}^{n+1}_+} x_N^{1-2s} x_k x_l x_m \nabla W \nabla \partial_e W \, dx\, dx_N \\
&\quad + \frac{\varepsilon^2}{6} R_{ikjl,m}\kappa_s \sum_{e=1}^{n} \frac{\partial \mathcal{K}_e}{\partial y_h}(y,\xi(y)) \\
&\qquad\qquad \cdot \int_{\mathbb{R}^{n+1}_+} x_N^{1-2s} x_k x_l x_m \partial_i W \partial^2_{je} W(x,x_N)\, dx\, dx_N \\
&\quad - \frac{\varepsilon^2}{2}(-R_{NN,k} + \pi_{is,k}\pi_{si})\kappa_s \sum_{e=1}^{n} \frac{\partial \mathcal{K}_e}{\partial y_h}(y,\xi(y)) \\
&\qquad\qquad \cdot \int_{\mathbb{R}^{n+1}_+} x_N^{3-2s} x_k \nabla W \nabla \partial_e W \, dx\, dx_N \\
&\quad + (R_{iNjN,k} + 7\pi_{jh,k}\pi_{hi})\kappa_s \sum_{e=1}^{n} \frac{\partial \mathcal{K}_e}{\partial y_h}(y,\xi(y)) \\
&\qquad\qquad \cdot \int_{\mathbb{R}^{n+1}_+} x_N^{3-2s} x_k \partial_i W \partial^2_{je} W \, dx\, dx_N \\
&\quad + o(\varepsilon^2).
\end{aligned}
$$

The second term $\mathcal{J}_2$ can be estimated as

$$
\begin{aligned}
\text{(4-52)}\quad \mathcal{J}_2 &= \varepsilon^{1+2s}\kappa_s \int_{B^+_{2r_0/\varepsilon}} E(\varepsilon x_N) \mathcal{W}_{\varepsilon,\xi(y)} \frac{\partial}{\partial y_h} \mathcal{W}_{\varepsilon,\xi(y)} |\bar{g}_{\xi(y)}(\varepsilon x, \varepsilon x_N)|^{\frac12}\, dx\, dx_N \\
&= \varepsilon^{1+2s}\kappa_s \int_{B^+_{2r_0/\varepsilon}} E(\varepsilon x_N) W \chi(\varepsilon x, \varepsilon x_N) \\
&\quad \cdot \Biggl(\frac{1}{\varepsilon}\chi_r(\varepsilon x, \varepsilon x_N) \sum_{e=1}^{n} \Bigl[\partial_e W(x,x_N) \frac{\partial \mathcal{K}_e}{\partial y_h}\bigl(y, \xi^{-1}(\exp_{\xi(y)})(\varepsilon x)\bigr) \Bigr] \\
&\qquad + \mathcal{O}\Bigl(\varepsilon^{n-2s} \frac{|\nabla \chi_r|(|(x-y,x_N)|)}{|(x-y,x_N)|^{N-2s}} \Bigr) \Biggr) |\bar{g}_{\xi(y)}(\varepsilon x, \varepsilon x_N)|^{\frac12}\, dx\, dx_N \\
&= o(\varepsilon^2).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\text{(4-53)}\quad \mathcal{J}_3 &= \int_{B_{2r_0/\varepsilon}} \mathcal{W}_{\varepsilon,\xi(y)} \frac{\partial}{\partial y_h} \mathcal{W}_{\varepsilon,\xi(y)} |g_{\xi(y)}(\varepsilon x, \varepsilon x_N)|^{\frac12}\, dx \\
&= \int_{B_{2r_0/\varepsilon}} \omega \chi_{r/\varepsilon} \Biggl(\frac{1}{\varepsilon}\chi_{r/\varepsilon} \sum_{e=1}^{n} \Bigl[\partial_e \omega \frac{\partial \mathcal{K}_e}{\partial y_h}\bigl(y, \xi^{-1}(\exp_{\xi(y)})(\varepsilon x)\bigr) \Bigr] \\
&\qquad + \mathcal{O}\Bigl(\varepsilon^{n-2s} \frac{|\nabla \chi_r|(|(x-y)|)}{|(x-y)|^{N-2s}} \Bigr) \Biggr) |g_{\xi(y)}(\varepsilon x)|^{\frac12}\, dx \\
&= -\frac{\varepsilon^2}{12} R_{kl,m} \sum_{e=1}^{n} \frac{\partial \mathcal{K}_e}{\partial y_h}(y,\xi(y)) \int_{\mathbb{R}^n} x_k x_l x_m\, \omega \partial_e \omega \, dx + o(\varepsilon^2)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{J}_4 &= \int_{B_{2r_0/\varepsilon}} f(\mathcal{W}_{\varepsilon,\xi(y)}) \frac{\partial}{\partial y_h} \mathcal{W}_{\varepsilon,\xi(y)} |g_{\xi(y)}(\varepsilon x)|^{\frac{1}{2}}\, dx \\
&= \int_{B_{2r_0/\varepsilon}} f(\omega\chi_{r/\varepsilon}) \biggl(\frac{1}{\varepsilon} \chi_{r/\varepsilon} \sum_{e=1}^{n} \Bigl[\partial_e \omega \frac{\partial \mathcal{K}_e}{\partial y_h} \bigl(y, \xi^{-1}(\exp_{\xi(y)})(\varepsilon x)\bigr) \Bigr] \\
&\qquad\qquad + \mathcal{O}\biggl(\varepsilon^{n-2s} \frac{|\nabla \chi_r|(|(x-y)|)}{|(x-y)|^{N-2s}} \biggr) \biggr) |g_{\xi(y)}(\varepsilon x)|^{\frac{1}{2}}\, dx \\
&= -\frac{\varepsilon^2}{12} R_{kl,m} \sum_{e=1}^{n} \frac{\partial \mathcal{K}_e}{\partial y_h}(y, \xi(y)) \int_{\mathbb{R}^n} x_k x_l x_m f(\omega)\, \partial_e \omega\, dx + o(\varepsilon^2).
\end{aligned} \tag{4-54}
$$

Arguing as in the first case and using (4-51)–(4-54), we deduce that

$$
\begin{aligned}
&\frac{\partial}{\partial y_h} J_\varepsilon(\mathcal{W}_{\varepsilon,\xi}) \\
&= -\frac{\varepsilon^2}{12} R_{kl,m} \sum_{e=1}^{n} \frac{\partial \mathcal{K}_e}{\partial y_h}(y, \xi(y)) \\
&\qquad \cdot \biggl[\kappa_s \int_{\mathbb{R}^{n+1}_+} x_N^{1-2s} x_k x_l x_m \nabla W \nabla \partial_e W\, dx\, dx_N + \int_{\mathbb{R}^n} x_k x_l x_m (\omega - f(\omega))\, \partial_e \omega\, dx \biggr] \\
&\quad + \frac{\varepsilon^2}{6} R_{ikjl,m} \kappa_s \sum_{e=1}^{n} \frac{\partial \mathcal{K}_e}{\partial y_h}(y, \xi(y)) \int_{\mathbb{R}^{n+1}_+} x_N^{1-2s} x_k x_l x_m \partial_i W \partial^2_{je} W(x, x_N)\, dx\, dx_N \\
&\quad - \frac{\varepsilon^2}{2} (-R_{NN,k} + \pi_{is,k}\pi_{si}) \kappa_s \sum_{e=1}^{n} \frac{\partial \mathcal{K}_e}{\partial y_h}(y, \xi(y)) \int_{\mathbb{R}^{n+1}_+} x_N^{3-2s} x_k \nabla W \nabla \partial_e W\, dx\, dx_N \\
&\quad + (R_{iNjN,k} + 7\pi_{jh,k}\pi_{hi}) \kappa_s \sum_{e=1}^{n} \frac{\partial \mathcal{K}_e}{\partial y_h}(y, \xi(y)) \int_{\mathbb{R}^{n+1}_+} x_N^{3-2s} x_k \partial_i W \partial^2_{je} W\, dx\, dx_N \\
&\quad + o(\varepsilon^2).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\biggl(\frac{\partial}{\partial y_h} J_\varepsilon(\mathcal{W}_{\varepsilon,\xi}) \biggr)_{|y=0} \\
&\quad = \frac{\varepsilon^2}{12} b_1 \biggl(\frac{\partial}{\partial y_m} R_{kl}(\xi(y)) \biggr)_{|y=0} - \frac{\varepsilon^2}{6} b_2 \biggl(\frac{\partial}{\partial y_m} R_{ikjl}(\xi(y)) \biggr)_{|y=0} \\
&\qquad\qquad - \frac{\varepsilon^2}{2} b_3 \biggl(-\biggl(\frac{\partial}{\partial y_k} R_{NN}(\xi(y)) \biggr) + \biggl(\frac{\partial}{\partial y_k} \pi_{is}(\xi(y)) \biggr) \pi_{si}(\xi(y)) \biggr)_{|y=0} \\
&\qquad\qquad + \varepsilon^2 b_4 \biggl(\biggl(\frac{\partial}{\partial y_k} R_{iNjN}(\xi(y)) \biggr) + 7 \biggl(\frac{\partial}{\partial y_k} \pi_{jh}(\xi(y)) \biggr) \pi_{hi}(\xi(y)) \biggr)_{|y=0} \\
&\qquad\qquad + o(\varepsilon^2),
\end{aligned}
$$

where we have set

$$b_1 := \kappa_s \int_{\mathbb{R}^{n+1}_+} x_N^{1-2s} x_k x_l x_m \nabla W \nabla \partial_h W \, dx\, dx_N + \int_{\mathbb{R}^n} x_k x_l x_m (\omega - f(\omega)) \partial_h \omega \, dx,$$

$$b_2 := \kappa_s \int_{\mathbb{R}^{n+1}_+} x_N^{1-2s} x_k x_l x_m \partial_i W \partial^2_{jh} W \, dx\, dx_N,$$

$$b_3 := \kappa_s \int_{\mathbb{R}^{n+1}_+} x_N^{3-2s} x_k \nabla W \nabla \partial_e W \, dx\, dx_N,$$

$$b_4 := \kappa_s \int_{\mathbb{R}^{n+1}_+} x_N^{3-2s} x_k \partial_i W \partial^2_{je} W \, dx\, dx_N.$$

This prove the desired result. □

Appendix: Proof of Lemma 2.7

The proof of Lemma 2.7 is based on the following preliminary results.

Lemma A.1. *Let* $0 < s < 1$, $a \in R$ *and* $0 < R_1 < R_2$. *We denote*

$$A^+_{\varepsilon^{-1}} = B^+_{R_2\varepsilon^{-1}} \backslash B^+_{R_1\varepsilon^{-1}}.$$

Then, as $\varepsilon \to 0$, *we have the*

$$\int_{A^+_{\varepsilon^{-1}}} \frac{x_N^{1-2s}}{|(x, x_N)|^{n-2s+2+a}} \, dx\, dx_N = \begin{cases} \mathcal{O}(\varepsilon^a) & \text{for } a \neq 0, \\ \mathcal{O}(|\log \varepsilon|) & \text{for } a = 0, \end{cases} \tag{A-1}$$

$$\int_{A^+_{\varepsilon^{-1}}} \frac{x_N^{2s-1}}{|(x, x_N)|^{n+2s+a}} \, dx\, dx_N = \begin{cases} \mathcal{O}(\varepsilon^a) & \text{for } a \neq 0, \\ \mathcal{O}(|\log \varepsilon|) & \text{for } a = 0. \end{cases} \tag{A-2}$$

Proof. To prove the first inequality, we decompose the domain of integration

$$A^+_{\varepsilon^{-1}} = (A^+_{\varepsilon^{-1}} \cup \{|x_N| \geq |x|\}) \cup (A^+_{\varepsilon^{-1}} \cup \{|x_N| \leq |x|\})$$

and estimate each part separately. If $|x_N| \geq |x|$, then it holds that

$$|x_N| \leq |(x, x_N)| \leq \sqrt{2}|x_N|.$$

Hence we get

$$\begin{aligned} &\int_{A^+_{\varepsilon^{-1}} \cup \{|x_N| \geq |x|\}} \frac{x_N^{1-2s}}{|(x, x_N)|^{n-2s+2+a}} \, dx\, dx_N \\ &\qquad \leq \max\{1, \sqrt{2}^{2s-1}\} \int_{A^+_{\varepsilon^{-1}} \cup \{|x_N| \geq |x|\}} \frac{1}{|(x, x_N)|^{n+a+1}} \, dx\, dx_N \\ &\qquad \leq C \int_{A^+_{\varepsilon^{-1}}} \frac{1}{|(x, x_N)|^{n+a+1}} dx\, dx_N = \begin{cases} \mathcal{O}(\varepsilon^a) & \text{for } a \neq 0, \\ \mathcal{O}(|\log \varepsilon|) & \text{for } a = 0. \end{cases} \end{aligned} \tag{A-3}$$

Now, if $|x_N| \le |x|$, we have that

$$\frac{1}{\sqrt{2}\varepsilon} \le \frac{1}{\sqrt{2}}|(x, x_N)| \le |x| \le |(x, x_N)| \le \frac{2}{\varepsilon}$$

for $(x, x_N) \in A^+_{\varepsilon^{-1}}$. Therefore,

$$\begin{aligned}
&\int_{A^+_{\varepsilon^{-1}} \cup \{|x_N| \le |x|\}} \frac{x_N^{1-2s}}{|(x, x_N)|^{n-2s+2+a}}\, dx\, dx_N \\
&\qquad \le \int_{\{\frac{1}{\sqrt{2}\varepsilon} \le |x| \le \frac{2}{\varepsilon}\}} \int_{\{|x_N| \le |x|\}} \frac{x_N^{1-2s}}{|x|^{n-2s+2+a}}\, dx\, dx_N \\
&\qquad = \frac{1}{1-s} \int_{\{\frac{1}{\sqrt{2}\varepsilon} \le |x| \le \frac{2}{\varepsilon}\}} \frac{|x|^{2-2s}}{|x|^{n-2s+2+a}}\, dx \\
&\qquad = \frac{1}{1-s} \int_{\{\frac{1}{\sqrt{2}\varepsilon} \le |x| \le \frac{2}{\varepsilon}\}} \frac{1}{|x|^{n+a}}\, dx = \begin{cases} \mathcal{O}(\varepsilon^a) & \text{for } a \ne 0, \\ \mathcal{O}(|\log \varepsilon|) & \text{for } a = 0. \end{cases}
\end{aligned} \tag{A-4}$$

Combining the above two estimates, we achieve the proof of the lemma. □

The second preliminary result is:

Lemma A.2. *Assume that* $|(x, x_N)| \ge R_0$ *for some fixed* $R_0 > 0$ *sufficiently large. Then*:

(i) $|W(x, x_N)| \le \frac{C}{|(x,x_N)|^{n-2s}}$.

(ii) $|\nabla_x W(x, x_N)| \le \frac{C}{|(x,x_N)|^{n-2s+1}}$ *and* $|\partial_{x_N} W(x, x_N)| \le \left(\frac{C}{|(x,x_N)|^{n-2s+1}} + \frac{C x_N^{2s-1}}{|(x,x_N)|^{n+2s}}\right)$.

(iii) *For* $i = 1, \dots, n$

$$|\nabla \partial_i W(x, x_N)| \le \left(\frac{C}{|(x, x_N)|^{n-2s+2}} + \frac{C x_N^{2s-1}}{|(x, x_N)|^{n+2s+1}}\right)$$

for some positive constant $C = C(s, n, R_0)$.

Proof. Using Green's representation formula for (2-5) we have that

$$W(x, x_N) = a_{n,s} \int_{\mathbb{R}^n} \frac{\omega^p - \omega}{|(x - y, x_N)|^{n-2s}}\, dy, \tag{A-5}$$

where $1 < p < \frac{n+2s}{n-2s}$ and $a_{n,s}$ is a positive constant depending only on n and s (see [13; 14]).

To estimate $W(x, x_N)$ we discuss two cases. In the range $|x| \le |x_N|$, we have $|(x, x_N)| \le \sqrt{2}|x_N|$. Then, by using the fact that the function $\omega = W(x, 0)$ satisfies (2-1), we obtain

$$\begin{aligned}
\text{(A-6)}\quad |W(x, x_N)| &= a_{n,s}\Big|\int_{\mathbb{R}^n} \frac{\omega^p}{|(x-y, x_N)|^{n-2s}} - \frac{\omega}{|(x-y, x_N)|^{n-2s}}\,dy\Big| \\
&\le C\int_{\mathbb{R}^n}\Big(\frac{1}{1+|y|^{n+2s}}\Big)\frac{1}{|(x-y, x_N)|^{n-2s}}\,dy \\
&\le C\int_{\mathbb{R}^n}\frac{1}{1+|y|^{n+2s}}\,\frac{1}{|x_N|^{n-2s}}\,dy \le \frac{C}{|x_N|^{n-2s}} \le \frac{C}{|(x, x_N)|^{n-2s}}
\end{aligned}$$

for $|(x, x_N)| \ge R_0$ large and $|x_N| \ge |x|$ where here and below C is a positive constant, depending only on n and s, which is allowed to vary from one formula to another. Now, in the range $|x| \ge |x_N|$ we have that $|(x, x_N)| \le \sqrt{2}|x|$. Then arguing as before we get

$$\begin{aligned}
\text{(A-7)}\quad |W(x, x_N)| &\le C\int_{\mathbb{R}^n}\Big(\frac{1}{1+|y|^{n+2s}}\Big)\frac{1}{|(x-y, x_N)|^{n-2s}}\,dy \\
&\le C\int_{\mathbb{R}^n}\frac{1}{1+|y|^{n+2s}}\,\frac{1}{|x-y|^{n-2s}}\,dy \\
&= C\Big\{\int_{|y-x|>\frac{1}{2}|x|}\frac{1}{1+|y|^{n+2s}}\,\frac{1}{|x-y|^{n-2s}}\,dy \\
&\qquad\qquad + \int_{|y-x|<\frac{1}{2}|x|}\frac{1}{1+|y|^{n+2s}}\,\frac{1}{|x-y|^{n-2s}}\,dy\Big\} \\
&\le \frac{C}{|x|^{n-2s}} + \frac{C}{|x|^n} \le \frac{C}{|(x, x_N)|^{n-2s}}
\end{aligned}$$

for $|(x, x_N)| \ge R_0$ large and $|x| \ge |x_N|$. Combining the above two estimates, we get the first estimate (i).

To estimate $|\nabla W|$ we can argue similarly. First, for $|x| \le |x_N|$, we have $|(x, x_N)| \le \sqrt{2}|x_N|$ and from (2-1), one deduces that

$$\begin{aligned}
\text{(A-8)}\quad |\nabla_{(x,x_N)} W(x, x_N)| &\le \Big|a_{n,s}\int_{\mathbb{R}^n}\nabla_{(x,x_N)}\frac{\omega^p-\omega}{|(x-y, x_N)|^{n-2s}}\,dy\Big| \\
&\le C\int_{\mathbb{R}^n}\frac{1}{1+|y|^{n+2s}}\Big|\nabla_{(x,x_N)}\frac{1}{|(x-y, x_N)|^{n-2s}}\Big|\,dy \\
&\le C\int_{\mathbb{R}^n}\frac{1}{1+|y|^{n+2s}}\,\frac{1}{|(x-y, x_N)|^{n-2s+1}}\,dy \\
&\le C\int_{\mathbb{R}^n}\frac{1}{1+|y|^{n+2s}}\,\frac{1}{|x_N|^{n-2s+1}}\,dy \\
&= \frac{C}{|x_N|^{n-2s+1}} \le \frac{C}{|(x, x_N)|^{n-2s+1}}.
\end{aligned}$$

Now, for $|x| \geq |x_N|$, we have that $|(x, x_N)| \leq \sqrt{2}|x|$. Then, integrating by parts one gets

$$\begin{aligned}
\nabla_x W(x, x_N) &= -a_{n,s} \int_{\mathbb{R}^n} (\omega^p - \omega) \nabla_y \left(\frac{1}{|(x-y, x_N)|^{n-2s}} \right) dy \\
&= - \int_{|y-x| \geq \frac{1}{2}|x|} (\omega^p - \omega) \nabla_y \left(\frac{1}{|(x-y, x_N)|^{n-2s}} \right) dy \\
&\qquad + \int_{|y-x| \leq \frac{1}{2}|x|} \nabla_y (\omega^p - \omega) \left(\frac{dy}{|(x-y, x_N)|^{n-2s}} \right) \\
&\qquad - \int_{|y-x| = \frac{1}{2}|x|} (\omega^p - \omega) \left(\frac{\sigma_y \, dS_y}{|(x-y, x_N)|^{n-2s}} \right),
\end{aligned}$$

where σ_y and dS_y are respectively the outward unit normal vector and the surface measure on the sphere $|y - x| = \frac{1}{2}|x|$ respectively. Notice that if $|y - x| \leq \frac{1}{2}|x|$ then $|y| \geq \frac{1}{2}|x|$ and we derive from the above that

$$\begin{aligned}
&|\nabla_x W(x, x_N)| \\
&\quad \leq \frac{C}{|x|^{n-2s+1}} \int_{|y-x| \geq \frac{1}{2}|x|} \frac{dy}{1 + |y|^{n+2s}} \\
&\qquad + \frac{C}{|x|^{n+2s+1}} \int_{|y-x| \leq \frac{1}{2}|x|} \frac{dy}{|(x-y, x_N)|^{n-2s}} + \mathcal{O}\left(\frac{|x|^{n-1}}{|x|^{(n+2s)+(n-2s)}} \right) \\
&\quad = \mathcal{O}\left(\frac{1}{|x|^{n-2s+1}} \right) + \mathcal{O}\left(\frac{1}{|x|^{n+2s+1}} . |x|^{2s} \right) + \mathcal{O}\left(\frac{1}{|x|^{n+1}} \right) \\
&\quad \leq \frac{C}{|x|^{n-2s+1}} \\
&\quad \leq \frac{C}{|(x, x_N)|^{n-2s+1}}.
\end{aligned} \tag{A-9}$$

This together with (A-8) implies the first inequality of (ii).

Now, in the range $|x| \geq |x_N|$ and $|y - x| \geq \frac{1}{2}|x|$, we have that

$$\begin{aligned}
&\int_{|y-x| \geq \frac{1}{2}|x|} \frac{1}{1 + |x-y|^{n+2s}} \frac{x_N}{|(y, x_N)|^{n-2s+2}} \, dy \\
&\quad \leq \frac{1}{|x|^{n+2s}} \int_{\mathbb{R}^n} \frac{x_N}{|(y, x_N)|^{n-2s+2}} \, dy = \frac{1}{|x|^{n+2s}} \int_{\mathbb{R}^n} \frac{x_N . x_N^n}{x_N^{n-2s+2} |(y, 1)|^{n-2s+2}} \, dy \\
&\quad = \frac{C x_N^{2s-1}}{|x|^{n+2s}} \\
&\quad \leq \frac{C x_N^{2s-1}}{|(x, x_N)|^{n+2s}}.
\end{aligned} \tag{A-10}$$

On the other hand, for $|x| \geq |x_N|$ and $|y-x| \leq \frac{1}{2}|x|$, we have that $|y| \geq \frac{1}{2}|x|$. Hence

$$\text{(A-11)} \quad \begin{aligned} \int_{|y-x|\geq\frac{1}{2}|x|} \frac{1}{1+|x-y|^{n+2s}} \frac{x_N}{|(y,x_N)|^{n-2s+2}}\,dy &\leq \frac{x_N}{|x|^{n-2s+2}} \int_{\mathbb{R}^n} \frac{1}{1+|x-y|^{n+2s}}\,dy \\ &= \frac{Cx_N}{|x|^{n-2s+2}} \leq \frac{Cx_N}{|(x,x_N)|^{n-2s+2}} \leq \frac{C}{|(x,x_N)|^{n-2s+1}}. \end{aligned}$$

Combining the above two estimates (A-10) and (A-11), we get that

$$\text{(A-12)} \quad \begin{aligned} |\partial_{x_N} W(x,x_N)| &\leq C \int_{\mathbb{R}^n} \frac{1}{1+|x-y|^{n+2s}} \frac{x_N}{|(y,x_N)|^{n-2s+2}}\,dy \\ &\leq C\left(\frac{x_N^{2s-1}}{|(x,x_N)|^{n+2s}} + \frac{1}{|(x,x_N)|^{n-2s+1}} \right). \end{aligned}$$

Now thanks to (A-8), (A-9) and (A-12), we get the second estimate of (i).

The last estimate (iii) for $|\nabla \partial_i W|$ can be obtained adapting the same procedure with obvious modifications. This concludes the proof of Lemma A.2. □

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