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We construct ribbon surfaces of Euler characteristic one for several infinite families of alternating 3-braid closures. We also use a twisted Alexander polynomial obstruction to conclude the classification of smoothly slice knots which are closures of alternating 3-braids with up to 20 crossings.

1. Introduction

By an *alternating braid* we mean a braid such that along any strand, over- and undercrossings alternate. Let σ_1 and σ_2 be the standard generators of the braid group on three strands B_3. If the closure of an alternating 3-braid has nonzero determinant, then it is isotopic to the closure of a braid

$$(\star) \qquad \sigma_1^{a_1}\sigma_2^{-b_1}\sigma_1^{a_2}\sigma_2^{-b_2}\ldots\sigma_1^{a_n}\sigma_2^{-b_n},$$

with $n \geqslant 1$ for some $a_i, b_i \geqslant 1$ for all i. Every 3-braid of the form $(\star)$ can be equivalently described by its *associated string* $\boldsymbol{a} = (2^{[a_1-1]}, b_1+2, \ldots, 2^{[a_n-1]}, b_n+2)$, where $2^{[a_i-1]}$ represents the substring consisting of the number 2 repeated $a_i - 1$ times. Cyclic rotations and reversals of $\boldsymbol{a}$ do not change the isotopy class of respective braid closures in S^3, so we consider associated strings up to those two operations. The *linear dual* of a string $\boldsymbol{b} = (b_1, \ldots, b_k)$ with all $b_i \geqslant 2$ is defined as follows: if $b_j \geqslant 3$ for some j, write $\boldsymbol{b}$ in the form $\boldsymbol{b} = (2^{[m_1]}, 3+n_1, 2^{[m_2]}, 3+n_2, \ldots, 2^{[m_l]}, 2+n_l)$ with $m_i, n_i \geqslant 0$ for all i. Then its linear dual is $\boldsymbol{c} = (2+m_1, 2^{[n_1]}, 3+m_2, 2^{[n_2]}, 3+m_3, \ldots, 3+m_l, 2^{[n_l]})$. If $\boldsymbol{b}$ is $(2^{[k]})$ or (1), define its linear dual as $(k+1)$ or the empty string, respectively.

Given a link $L \subset S^3$, by a *ribbon surface* we mean a surface F bounded by L that is properly smoothly embedded in D^4, has no closed components, and may be isotoped rel boundary so that the radial distance function $D^4 \to [0, 1]$ induces a handle decomposition on F with only 0- and 1-handles. By a *slice surface* we mean a surface S bounded by L that is properly smoothly embedded in D^4 and has no closed components; neither F nor S are required to be connected or orientable. Following [5], we say that L which bounds a ribbon (or slice) surface of Euler characteristic one is *χ-ribbon* (or *χ-slice*); these definitions coincide with the usual

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definitions of ribbon and slice in the case of knots. Clearly, if L is χ-ribbon, then it is also χ-slice.

Simone [20] has classified associated strings of all alternating 3-braid closures L with nonzero determinant such that $\Sigma_2(S^3, L)$, the double branched cover of S^3 over L, is unobstructed by Donaldson's theorem from bounding a rational ball, into five families:

$$\mathcal{S}_{2a} = \{(b_1+3, b_2, \dots, b_k, 2, c_l, \dots, c_1)\},$$
$$\mathcal{S}_{2b} = \{(3+x, b_1, \dots, b_{k-1}, b_k+1, 2^{[x]}, c_l+1, c_{l-1}, \dots, c_1) \mid x \geqslant 0 \text{ and } k+l \geqslant 2\},$$
$$\mathcal{S}_{2c} = \{(3+x_1, 2^{[x_2]}, 3+x_3, 2^{[x_4]}, \dots, 3+x_{2k+1}, 2^{[x_1]}, 3+x_2, 2^{[x_3]}, \dots, 3+x_{2k}, 2^{[x_{2k+1}]}) \mid k \geqslant 0 \text{ and } x_i \geqslant 0 \text{ for all } i\},$$
$$\mathcal{S}_{2d} = \{(2, 2+x, 2, 3, 2^{[x-1]}, 3, 4) \mid x \geqslant 1\} \cup \{(2, 2, 2, 4, 4)\},$$
$$\mathcal{S}_{2e} = \{(2, b_1+1, b_2, \dots, b_k, 2, c_l, \dots, c_2, c_1+1, 2) \mid k+l \geqslant 3\} \cup \{(2, 2, 2, 3)\}.$$

Here strings $(b_1, \dots, b_k)$ and $(c_1, \dots, c_l)$ are linear duals of each other. Since $\Sigma_2(S^3, L)$ of a χ-slice link L bounds a rational ball [5, Proposition 2.6], every χ-slice alternating 3-braid closure with nonzero determinant has its associated string in one of these families. Moreover, Simone has explicitly constructed rational balls for all such alternating 3-braid closures.

We show that alternating 3-braid closures whose associated strings lie in $\mathcal{S}_{2a} \cup \mathcal{S}_{2b} \cup \mathcal{S}_{2d} \cup \mathcal{S}_{2e}$ are χ-ribbon by exhibiting band moves, defined in Section 2, which make their link diagrams isotopic to the two- or three-component unlink. In Section 3, we consider the set $\mathcal{S}_{2c} \setminus (\mathcal{S}_{2a} \cup \mathcal{S}_{2b} \cup \mathcal{S}_{2d} \cup \mathcal{S}_{2e})$ that includes strings associated to known non-χ-slice alternating 3-braid closures, such as certain Turk's head knots, and list more examples of potentially non-χ-slice knots and links. In Section 4 we follow [11] and [1] in applying a twisted Alexander polynomial obstruction to show that among these examples, three knots are indeed not slice; this concludes the classification of smoothly slice knots which are closures of alternating 3-braids with up to 20 crossings.

2. Ribbon surfaces for $\mathcal{S}_{2a} \cup \mathcal{S}_{2b} \cup \mathcal{S}_{2d} \cup \mathcal{S}_{2e}$

One may exhibit a ribbon surface for a link L as follows. By a *band move* on L we mean choosing an embedding $\varphi : D^1 \times D^1 \hookrightarrow S^3$ of a *band* so that the image of φ is disjoint from L except for $\varphi(\partial D^1 \times D^1)$ coincident with two segments of L, removing those segments, joining corresponding ends along $\varphi(D^1 \times \partial D^1)$ and smoothing the corners. This operation amounts to removing a 1-handle in the putative ribbon surface F. If after n band moves, the resulting link is isotopic to the $(n+1)$-component unlink, one has indeed obtained a ribbon surface F of Euler characteristic one bounded by L, since each component of the unlink bounds a 0-handle

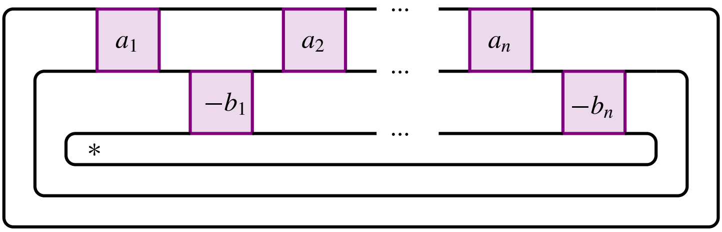

Figure 1. A generic alternating 3-braid closure. We denote sequences of positive (negative) crossings by blocks annotated by positive (negative) coefficients.

of F. Each band may be represented on a link diagram by an arc with endpoints on L that crosses the strands of L transversally, has no self-crossings, and is annotated by the number of half-twists in the band relative to the blackboard framing.

Given a 3-braid $\beta = \sigma_1^{a_1}\sigma_2^{-b_1}\dots\sigma_1^{a_n}\sigma_2^{-b_n}$, we draw it from left to right, as shown in Figure 1, and orient all strings in the closure $\hat{\beta}$ clockwise. Choose the chessboard colouring of the diagram for $\hat{\beta}$ where the unbounded region is white. Then there are $m = \left(\sum_{i=1}^{n} a_i\right) + 1$ black regions. We can index the black regions, excluding the one not adjacent to the unbounded region (marked by $*$ in Figure 1), by $\{1, \dots, m-1\}$ such that the number of crossings along the boundary of the region indexed by i is given by the i-th entry of the associated string $\boldsymbol{a} = (2^{[a_1-1]}, b_1+2, \dots, 2^{[a_n-1]}, b_n+2)$, and the region indexed by i shares one crossing with each of the regions indexed by $i-1$ and $i+1$ (mod $m-1$).

Proposition 2.1. *Let $\boldsymbol{a}$ be the associated string of an alternating 3-braid closure $\hat{\beta}$. If $\boldsymbol{a} \in \mathcal{S}_{2a} \cup \mathcal{S}_{2d} \cup \mathcal{S}_{2e}$, then $\hat{\beta}$ bounds a ribbon surface with a single 1-handle. If $\boldsymbol{a} \in \mathcal{S}_{2b}$, then $\hat{\beta}$ bounds a ribbon surface with at most two 1-handles.*

Our main observation, previously used by Lisca [14] and Lecuona [13], is that if $\boldsymbol{a}$ contains two disjoint linearly dual substrings (possibly perturbed on the ends), then the link diagram of $\hat{\beta}$ contains sub-braids which, if connected to each other by a half-twist $(\sigma_2\sigma_1\sigma_2)^{-1}$, may be cancelled out via successive isotopies. More precisely, suppose that $(b_1, \dots, b_k)$ and $(c_1, \dots, c_l)$ are linear duals. Let $\boldsymbol{b}' = (b_1 + x_l, b_2, \dots, b_k + x_r)$ and $\boldsymbol{c}' = (c_l + y_l, c_{l-1}, \dots, c_1 + y_r)$ with $x_i, y_i \geqslant 0$ for $i \in \{l, r\}$ and suppose that $\boldsymbol{a} = \boldsymbol{b}' \mid \boldsymbol{t} \mid \boldsymbol{c}' \mid \boldsymbol{s}$, where $\boldsymbol{t}$ and $\boldsymbol{s}$ are arbitrary strings, the length of $\boldsymbol{t}$ is $t \geqslant 0$, and $\mid$ denotes string concatenation. Consider the sub-braid B in the link diagram of $\hat{\beta}$ that exactly contains all crossings along the boundary of black regions $2, \dots, k-1$, all but x_l+1 leftmost crossings along the boundary of region 1, and all but x_r+1 rightmost crossings along the boundary of region k. Consider also the sub-braid C that exactly contains all crossings along the boundary of regions $k+t+2, \dots, k+t+l-1$, all but y_l+1 leftmost crossings along the boundary of region $k+t+1$, and all but y_r+1 rightmost crossings along the boundary of region $k+t+l$. Then $B(\sigma_2\sigma_1\sigma_2)^{-1}C = (\sigma_2\sigma_1\sigma_2)^{-1}$. Hence, if after applying a band move to $\hat{\beta}$ away from B and C, they are connected by a half-twist of the three strands, one may remove all crossings in B and C via isotopies illustrated in Figure 2. We call B and C *dual sub-braids* and enclose them in all following figures in blue and chartreuse rectangles, respectively.

Proof of Proposition 2.1. See Figures 4–7. □

In searching for the band moves in Figures 4–7, we have used the algorithm of Owens and Swenton implemented in the KLO program [16]. The band moves we exhibit for these four families of alternating 3-braid closures are *algorithmic* in the sense of [16].

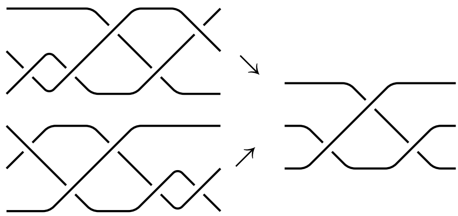

Figure 2. Undoing flyped tongues [22] to cancel dual sub-braids.

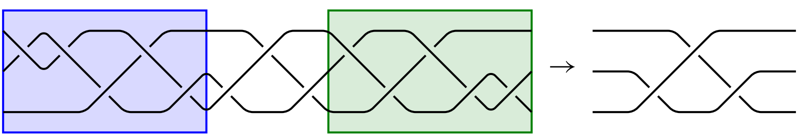

Figure 3. Cancellation of dual sub-braids for $(b_1, \ldots, b_k) = (2, 2, 3, 3)$ and $(c_l, \ldots, c_1) = (2, 3, 4)$ with $x_l = x_r = y_l = y_r = 0$. Fixing the ends on the braid shown, one may remove all crossings in B and C via moves illustrated in Figure 2.

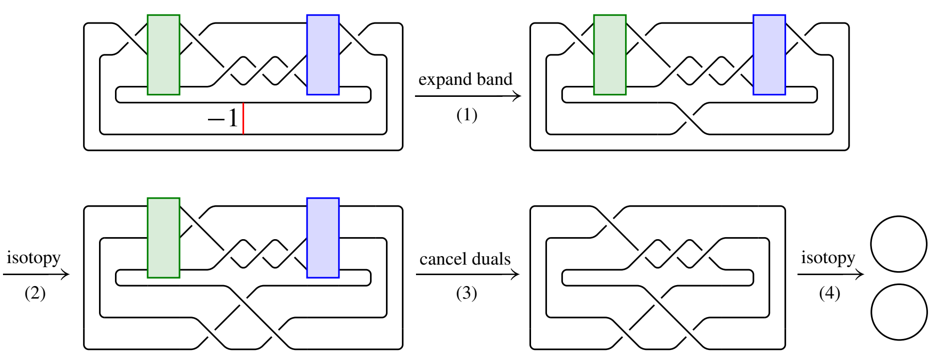

Figure 4. Band move for the $\mathcal{S}_{2a}$ case.

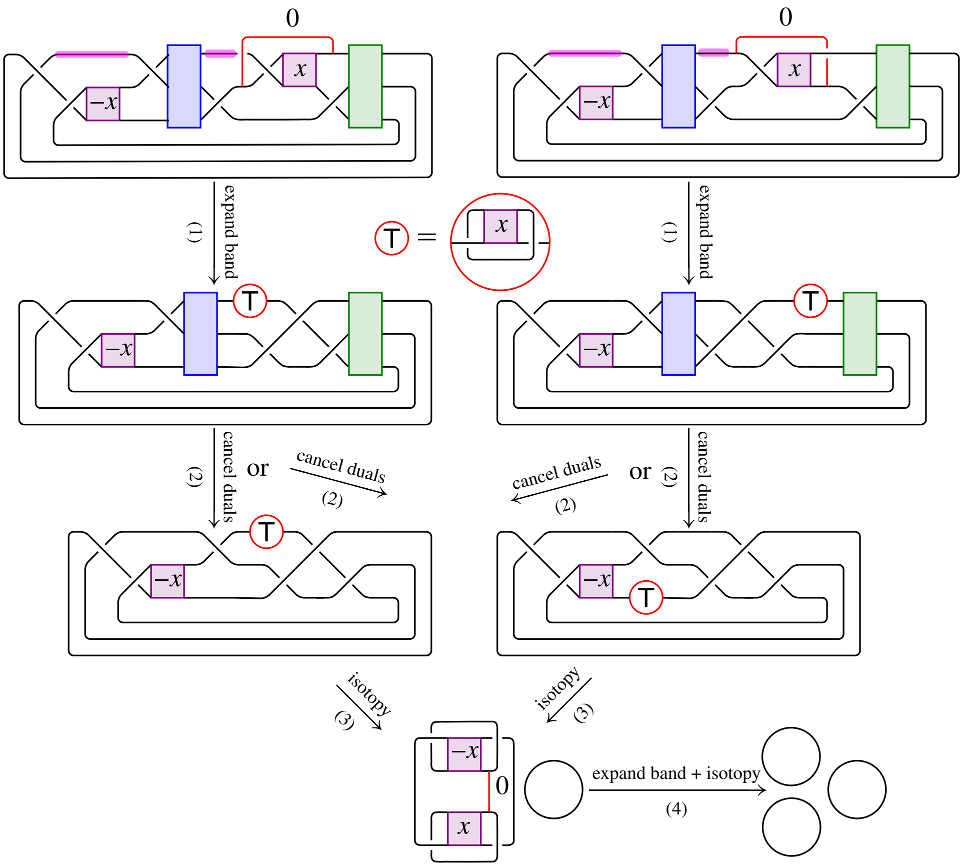

Figure 5. Band moves for the $\mathcal{S}_{2b}$ case. Start with the top left diagram if the two segments highlighted in purple do not lie on the same strand, otherwise start with the top right; this ensures that after step (2), the tangle T does not lie on the otherwise unknotted split component. The nontrivial component of the link obtained after step (3) is the connected sum $T(2, x+2) \# T(2, -(x+2))$ of two torus links.

3. The case of $\mathcal{S}_{2c} \setminus (\mathcal{S}_{2a} \cup \mathcal{S}_{2b} \cup \mathcal{S}_{2d} \cup \mathcal{S}_{2e})$

The remaining $\mathcal{S}_{2c}$ family is of special interest because it contains strings associated to known examples of nonslice, nonzero determinant alternating 3-braid closures, specifically Turk's head knots K_7 [19], K_{11}, K_{17} and K_{23} [1]; the associated string of K_i for $i \in \{7, 11, 17, 23\}$ is $(3^{[i]})$. Thus, we should not expect to find a set of band moves for all links with strings in $\mathcal{S}_{2c}$. We also note that knots of finite concordance order belonging to Family (3) in [15] have associated strings in $\mathcal{S}_{2c}$.

We have that $\mathcal{S}_{2c} \cap \mathcal{S}_{2d} = \mathcal{S}_{2c} \cap \mathcal{S}_{2e} = \varnothing$: this can be seen by computing the $I(\boldsymbol{a}) = \sum_{a \in \boldsymbol{a}} 3 - a$ invariant [14] which is 0 for strings in $\mathcal{S}_{2c}$, but 1 or 3 for strings

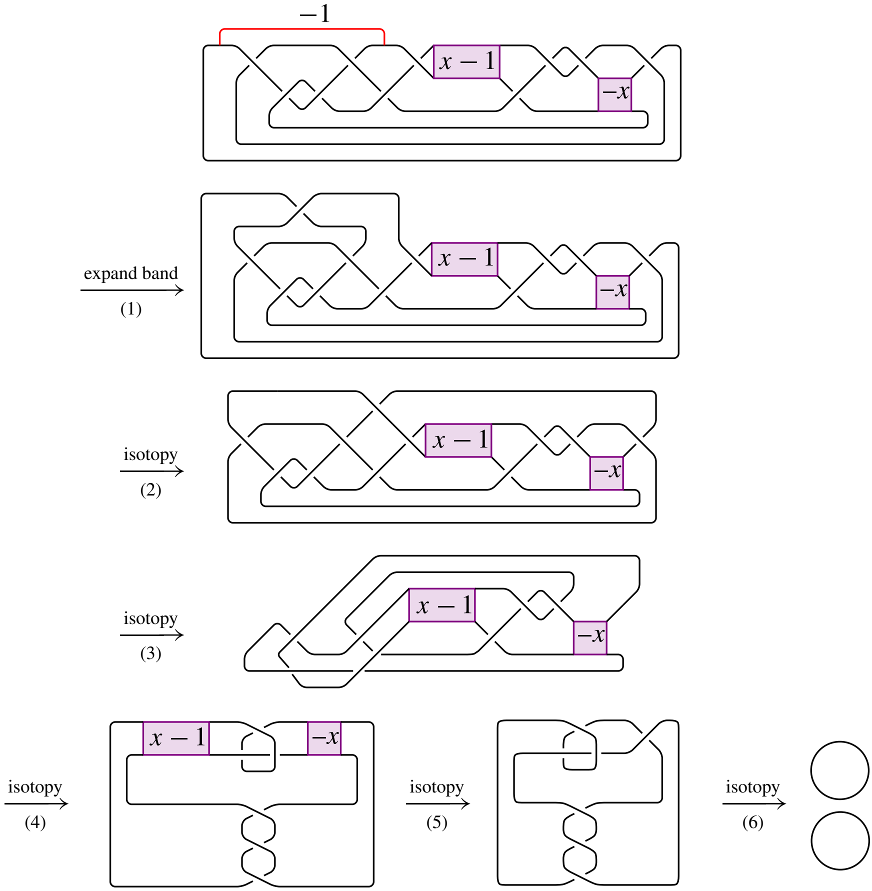

Figure 6. Band moves for the $\mathcal{S}_{2d}$ case with $x \geqslant 1$. In step (5), we undo $x-1$ crossings in both blocks by flyping the tangle on the bottom of the diagram and performing Reidemeister II moves. A similar band gives the two-component unlink for the alternating 3-braid closure with associated string (2, 2, 2, 4, 4).

in $\mathcal{S}_{2d}$ or $\mathcal{S}_{2e}$, respectively.[1] However, $\mathcal{S}_{2c}$ has nonzero intersection with $\mathcal{S}_{2a}$ and $\mathcal{S}_{2b}$: if one defines a *palindrome* to be a string $(a_1, \ldots, a_n)$ such that $a_i = a_{n-(i-1)}$ for all $1 \leqslant i \leqslant n$, then the following lemma holds.

Lemma 3.1 [20, Lemma 3.6]. *Let* $\boldsymbol{a} = (b_1+3, b_2, \ldots, b_k, 2, c_l, \ldots, c_1) \in \mathcal{S}_{2a}$ *and* $\boldsymbol{b} = (3+x, b_1, \ldots, b_{k-1}, b_k+1, 2^{[x]}, c_l+1, c_{l-1}, \ldots, c_1) \in \mathcal{S}_{2b}$. *Then* $\boldsymbol{a} \in \mathcal{S}_{2c}$ *if and only if* $(b_1+1, b_2, \ldots, b_k)$ *is a palindrome and* $\boldsymbol{b} \in \mathcal{S}_{2c}$ *if and only if* $(b_1 \ldots, b_k)$ *is a palindrome.*

[1] Observe that if $\boldsymbol{b} = (b_1, \ldots, b_k)$ and $\boldsymbol{c} = (c_1, \ldots, c_l)$ are linearly dual to each other and $k+l \geqslant 2$, then $I(\boldsymbol{b} \mid \boldsymbol{c}) = 2$.

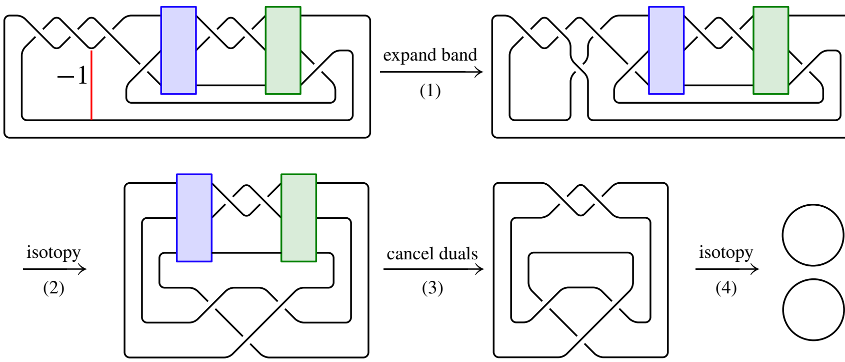

Figure 7. Band move for the $\mathcal{S}_{2e}$ case. A similar band move gives the two-component unlink for the alternating 3-braid closure with associated string (2, 2, 2, 3).

We seek to find an easier description of the complement $\mathcal{S}_{2c}^{\dagger} := \mathcal{S}_{2c} \setminus (\mathcal{S}_{2a} \cup \mathcal{S}_{2b} \cup \mathcal{S}_{2d} \cup \mathcal{S}_{2e})$. Let

$$(*)\qquad \boldsymbol{c} = (3+x_1, 2^{[x_2]}, 3+x_3, 2^{[x_4]}, \ldots, 3+x_{2k+1}, 2^{[x_1]}, \\ 3+x_2, 2^{[x_3]}, \ldots, 3+x_{2k}, 2^{[x_{2k+1}]}) \in \mathcal{S}_{2c},$$

where $k \geqslant 0$ and $x_i \geqslant 0$ for all i. One can more compactly describe $\boldsymbol{c}$ by its $\boldsymbol{x}$*-string* $\boldsymbol{x}(\boldsymbol{c}) = [x_1, \ldots, x_{2k+1}]$ (we use square brackets to denote $\boldsymbol{x}$-strings and, as with associated strings, consider them up to cyclic rotations and reversals). For example, the $\boldsymbol{x}$-string of $(3^{[i]})$ associated with K_i is $[0^{[i]}]$. Also, when writing $\boldsymbol{c}$ in the form $(*)$ with the first element being at least 3, call every maximal substring of the form $(2^{[x]})$ or $(3+x)$ for $x \geqslant 0$ an *entry*; the total number of entries $e(\boldsymbol{c})$ in $\boldsymbol{c}$ is congruent to 2 mod 4.

Lemma 3.2. *Let* $\boldsymbol{a} = (b_1+3, b_2, \ldots, b_k, 2, c_l, \ldots, c_1) \in \mathcal{S}_{2a} \cap \mathcal{S}_{2c}$ *and* $\boldsymbol{b} = (3+y, b_1, \ldots, b_{k-1}, b_k+1, 2^{[y]}, c_l+1, c_{l-1}, \ldots, c_1) \in \mathcal{S}_{2b} \cap \mathcal{S}_{2c}$. *Then*:

- $\boldsymbol{x}(\boldsymbol{a}) = [z_1]$ *with* $z_1 \geqslant 1$ *or* $\boldsymbol{x}(\boldsymbol{a}) = [z_1, \ldots, z_{\lfloor \frac{n}{2} \rfloor}, z_{\lfloor \frac{n}{2} \rfloor+1}, z_{\lfloor \frac{n}{2} \rfloor}, \ldots, z_2, z_1-2]$ *with* $z_1 \geqslant 2$ *and* $n \geqslant 3$ *odd.*
- $\boldsymbol{x}(\boldsymbol{b}) = [y, 0, z_2]$ *or* $\boldsymbol{x}(\boldsymbol{b}) = [y, 0, z_2, z_3, \ldots, z_{\frac{n}{2}}, z_{\frac{n}{2}+1}, z_{\frac{n}{2}}, \ldots, z_3, z_2+1]$ *with* $n \geqslant 4$ *even.*

Proof. Consider $\boldsymbol{a}$ and define $\boldsymbol{a_c} = (2, c_l, \ldots, c_1)$. Notice that $\boldsymbol{a_c}$ is the linear dual of the string

$$\boldsymbol{a_c^*} = (b_k+1, b_{k-1}, \ldots, b_1),$$

which by Lemma 3.1 must be a palindrome, and that $\boldsymbol{a} = (b_1+3, b_2, \ldots, b_k \mid \boldsymbol{a_c})$. If $(b_1, \ldots, b_k)$ is the empty string, then $\boldsymbol{a} = (2, 1) \notin \mathcal{S}_{2c}$. Otherwise, write

$$\boldsymbol{a_c} = (2^{[z_1]}, 3+z_2, \ldots, 2^{[z_n]})$$

for $n \geqslant 1$ odd and $z_1 \geqslant 1$. If $n = 1$, then $\boldsymbol{a_c} = (2^{[z_1]})$ and $\boldsymbol{a} = (3+z_1, 2^{[z_1]})$, so $\boldsymbol{x}(\boldsymbol{a}) = [z_1]$. If $n > 1$, then

$$\boldsymbol{a_c^*} = (2+z_1, 2^{[z_2]}, 3+z_3, \dots, 2^{[z_{n-1}]}, 2+z_n). \tag{$**$}$$

Thus,

$$\boldsymbol{a} = (3+(z_n+2), 2^{[z_{n-1}]}, \dots, 2^{[z_2]}, 1+z_1, 2^{[z_1]}, 3+z_2, \dots, 2^{[z_n]}).$$

If $z_1 = 1$, then

$$\boldsymbol{a} = (3+(z_n+2), 2^{[z_{n-1}]}, \dots, 3+z_3, 2^{[z_1+z_2+1]}, 3+z_2, \dots, 2^{[z_n]})$$

does not belong to $\mathcal{S}_{2c}$ because $e(\boldsymbol{a}) \equiv 0 \bmod 4$. If $z_1 > 1$, then

$$\boldsymbol{a} = (3+(z_n+2), 2^{[z_{n-1}]}, \dots, 2^{[z_2]}, 3+(z_1-2), 2^{[z_1]}, 3+z_2, \dots, 2^{[z_n]}).$$

Now, by considering $(**)$ we see that $\boldsymbol{a_c^*}$ is a palindrome if and only if

$$z_1 = z_n + 2, \quad z_2 = z_{n-1}, \quad \dots, \quad z_{\lfloor \frac{n}{2} \rfloor} = z_{\lfloor \frac{n}{2} \rfloor + 2},$$

so we conclude that $\boldsymbol{a} \in \mathcal{S}_{2a} \cap \mathcal{S}_{2c}$ if and only if $\boldsymbol{x}(\boldsymbol{a}) = [z_1]$ for $z_1 \geqslant 1$ or

$$\boldsymbol{x}(\boldsymbol{a}) = [z_1, z_2, \dots, z_{\lfloor \frac{n}{2} \rfloor}, z_{\lfloor \frac{n}{2} \rfloor + 1}, z_{\lfloor \frac{n}{2} \rfloor}, \dots, z_2, z_1 - 2] \quad \text{for } z_1 \geqslant 2 \text{ and } n \geqslant 3 \text{ odd}.$$

Similarly, if $(b_1, \dots, b_k)$ is empty, then $\boldsymbol{b} = (3+y, 2^{[y]}, 2) = (3+y, 2^{[y+1]}) \notin \mathcal{S}_{2c}$. If $k = 1$, then the linear dual of (b_1) with $b_1 \geqslant 2$ is $(2^{[b_1-1]})$, so

$$\begin{aligned}\boldsymbol{b} &= (3+y, 2^{[0]}, b_1+1, 2^{[y]}, 3+0, 2^{[b_1-2]}) \\ &= (3+y, 2^{[0]}, 3+(b_1-2), 2^{[y]}, 3+0, 2^{[b_1-2]})\end{aligned}$$

is indeed in $\mathcal{S}_{2c}$ and $\boldsymbol{x}(\boldsymbol{b}) = [y, 0, b_1-2]$. If $k > 1$, write

$$(b_1, \dots, b_k) = (2^{[z_1]}, 3+z_2, \dots, 2^{[z_{n-1}]}, 2+z_n)$$

for $n \geqslant 2$ even and $z_n \geqslant 1$; its linear dual is

$$(c_1, \dots, c_l) = (2+z_1, 2^{[z_2]}, 3+z_3, \dots, 2^{[z_{n-2}]}, 3+z_{n-1}, 2^{[z_n]}).$$

When $n = 2$, we recover the $k = 1$ case above, so suppose $n > 2$. Then we have

$$\begin{aligned}\boldsymbol{b} = (3+y, 2^{[z_1]}, \dots, 2^{[z_{n-1}]}, 3+z_n, 2^{[y]}, 3+0, 2^{[z_n-1]}, \\ 3+z_{n-1}, 2^{[z_{n-2}]}, \dots, 3+z_3, 2^{[z_2+1]}).\end{aligned}$$

By comparing this with $(*)$, we see that z_1 (which corresponds to x_2) must be zero, and

$$(b_1, \dots, b_k) = (3+z_2, 2^{[z_3]}, \dots, 2^{[z_{n-1}]}, 3+(z_n-1)).$$

The string $(b_1, \dots, b_k)$ is thus a palindrome precisely when

$$z_2 = z_n - 1, \quad z_3 = z_{n-1}, \quad \dots, \quad z_{\frac{n}{2}} = z_{\frac{n}{2}+2},$$

i.e., $\boldsymbol{x}(\boldsymbol{b}) = [y, 0, z_2, z_3, \dots, z_{\frac{n}{2}}, z_{\frac{n}{2}+1}, z_{\frac{n}{2}}, \dots, z_3, z_2+1]$. $\square$

In particular, we can draw the easy conclusion that if $\boldsymbol{x}(\boldsymbol{c})$ contains neither two adjacent elements differing by 2 nor a 0, then $\boldsymbol{c} \in \mathcal{S}_{2c}^{\dagger}$. We now show that infinitely many χ-ribbon links have their associated strings in $\mathcal{S}_{2c}^{\dagger}$.

Lemma 3.3. *Let $\hat{\beta}$ be the closure of $\beta = \sigma_1^{m+1}(\sigma_2^{-1}\sigma_1)^2\sigma_2^{-(m+1)}(\sigma_1\sigma_2^{-1})^2$ with the associated string $\boldsymbol{c} = (3+m, 3, 3, 2^{[m]}, 3, 3)$ and $m \geqslant 3$. Then $\boldsymbol{c} \in \mathcal{S}_{2c}^{\dagger}$ and $\hat{\beta}$ admits a ribbon surface with a single* 1*-handle.*

Proof. We have $\boldsymbol{x}(\boldsymbol{c}) = [m, 0, 0, 0, 0]$, so by Lemma 3.2, $\boldsymbol{c} \in \mathcal{S}_{2c}^{\dagger}$. For the band move, see Figure 8. □

Using KLO, we have found that 22 out of 33 closures of alternating 3-braids with up to 20 crossings whose associated strings belong to $\mathcal{S}_{2c}^{\dagger}$ are algorithmically ribbon, in each instance via at most two band moves. It is known that the Turk's head knot K_7 with the associated string in $\mathcal{S}_{2c}^{\dagger}$ and 14 crossings is not slice [19]. The remaining 10 examples for which we were unable to find band moves exhibiting a χ-ribbon surface are listed in Table 1. By a straightforward application of the Gordon–Litherland signature formula [10, Theorems 6 and 6″], the signature of the closure of a braid $\beta = \sigma_1^{a_1}\sigma_2^{-b_1}\dots\sigma_1^{a_n}\sigma_2^{-b_n}$ with $\sum_i a_i$ and $\sum_i b_i$ both greater than one is

$$\sigma(\hat{\beta}) = \sum_{i=1}^{n} b_i - a_i.$$

Thus, for all links with associated strings in $\mathcal{S}_{2a} \cup \mathcal{S}_{2b} \cup \mathcal{S}_{2c}$ satisfying this condition (in particular, for those in Table 1), the signature vanishes, which means that for knots, so do the Ozsváth and Szabó's τ and Rasmussen's s invariants [17; 18] without giving us any sliceness obstructions; Tristram–Levine signatures for knots in Table 1 are also zero. Moreover, by comparing their hyperbolic volumes, we have verified that none of the entries in Table 1 belong to the list of "escapee" χ-ribbon links described in [16]: this further advances them as candidates for more careful study. In Section 4 we will show that the three knots K_1, K_2 and K_3 in Table 1 are not slice, which lets us conclude that every knot which is a closure of an alternating 3-braid with up to 20 crossings and whose double branched cover bounds a rational ball, except K_1, K_2, K_3 and K_7, is slice.

Remark 3.4. We note that not all alternating knots can be represented as closures of alternating braids. This implies that our list of smoothly nonslice knots which are closures of alternating 3-braids with up to 20 crossings does not include, for example, the nonslice alternating knot 5_2, which has braid index 3, but cannot be represented as a closure of any alternating braid [3]. A full classification of braid presentations of alternating links with braid index 3 has been given by Stoimenow in [21].

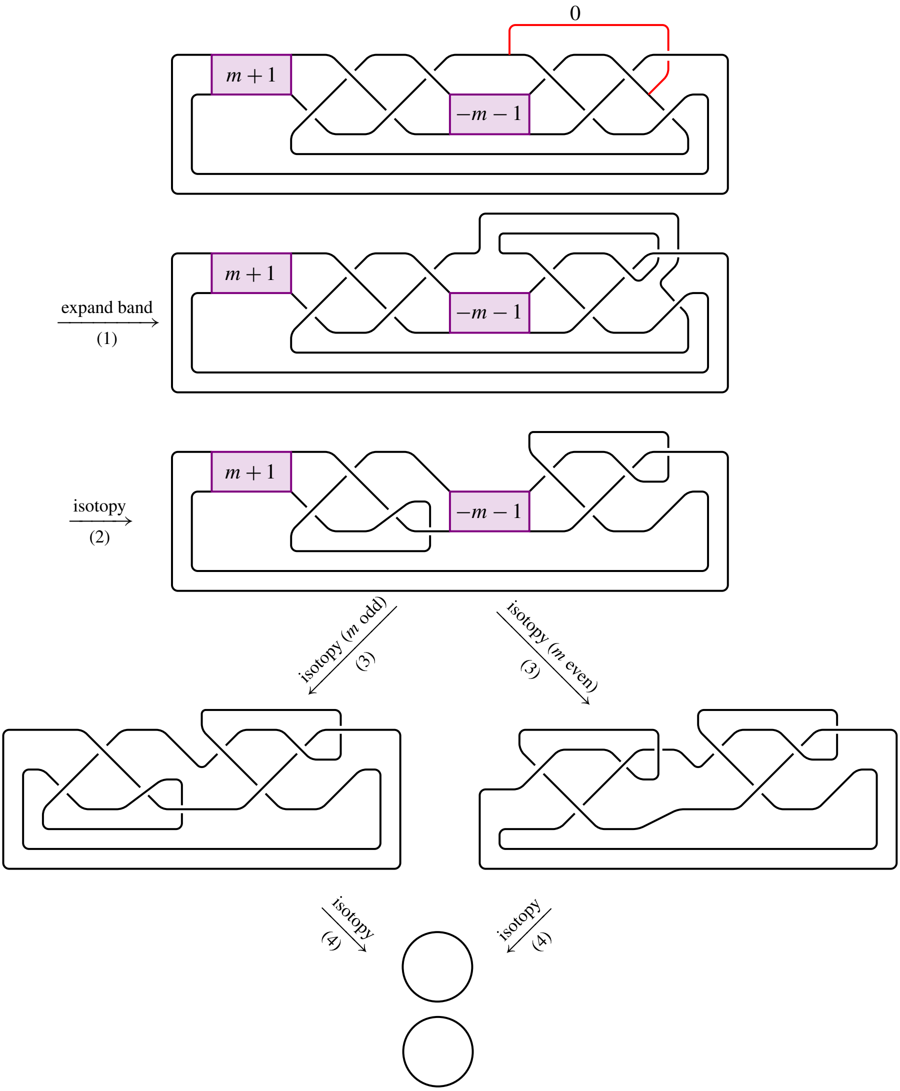

Figure 8. Band moves for an alternating 3-braid closure with $\boldsymbol{x}$-string $[m, 0, 0, 0, 0]$ for $m \geqslant 3$. In (3), we perform $m+1$ flypes of the tangle between two blocks with m crossings followed by Reidemeister II moves.

4. Three more nonslice knots in $\mathcal{S}_{2c}^{\dagger}$

In this section we restrict our attention to the three knots in Table 1. Let

$$\beta_1 = \sigma_1^2\sigma_2^{-2}\sigma_1^2\sigma_2^{-2}\sigma_1\sigma_2^{-2}\sigma_1^2\sigma_2^{-2}\sigma_1^2\sigma_2^{-1},$$
$$\beta_2 = \sigma_1^3\sigma_2^{-2}\sigma_1\sigma_2^{-1}\sigma_1^2\sigma_2^{-3}\sigma_1^2\sigma_2^{-1}\sigma_1\sigma_2^{-2},$$
$$\beta_3 = \sigma_1^2\sigma_2^{-1}\sigma_1\sigma_2^{-2}\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-2}\sigma_1\sigma_2^{-1}\sigma_1^2\sigma_2^{-1}\sigma_1\sigma_2^{-1},$$

and let $K_i = \hat{\beta}_i$ for $i = 1, 2, 3$. We will show that the knots K_i are not slice by adapting the approach of Aceto et al. [1], based in turn on work of Herald, Kirk and Livingston [11], and demonstrating that certain reduced twisted Alexander polynomials do not factor as norms; this is a generalisation of the Fox–Milnor condition on Alexander polynomials of K_i which is passed by these knots. Fix distinct primes p and q, and let ζ_q denote a primitive q-th root of unity. The general outline of the algorithm is the following.

(1) Construct the Seifert matrix S_i for K_i coming from the standard Seifert surface F_i associated to K_i viewed as a 3-braid closure.

(2) By considering the presentation matrix $P_i = tS_i - S_i^T \in \mathrm{Mat}(\mathbb{Z}[t^{\pm 1}])$ of the Alexander module $\mathcal{A}(K_i)$, determine the structure of $H_1(\Sigma_p(K_i))$, the first homology of the p-fold cover of S^3 branched over K_i, as well as a basis of $H_1(\Sigma_p(K_i))$ given by lifts of curves in $S^3 \setminus \nu(F)$.

(3) Calculate the Blanchfield pairings $\mathrm{Bl}_i : \mathcal{A}(K_i) \times \mathcal{A}(K_i) \to \mathbb{Q}(t)/\mathbb{Z}[t^{\pm 1}]$ and deduce the linking pairings $\lambda_i : H_1(\Sigma_p(K_i)) \times H_1(\Sigma_p(K_i)) \to \mathbb{Q}/\mathbb{Z}$.

(4) Enumerate all $\mathbb{Z}[t^{\pm 1}]$-submodules N of $H_1(\Sigma_p(K_i))$ with $|N|^2 = |H_1(\Sigma_p(K_i))|$ and thus find all metabolisers of $H_1(\Sigma_p(K_i))$, i.e., those N on which λ_i vanishes.

(5) Construct nontrivial characters $\chi : H_1(\Sigma_p(K_i)) \to \mathbb{Z}/q$ that vanish on the metabolisers.

(6) Using a Wirtinger presentation of $\pi_1(X_i)$, where X_i is the knot complement of K_i, construct a certain homomorphism $\pi_1(X_i) \to \mathbb{Z} \ltimes H_1(\Sigma_p(K_i))$ that induces a representation $\varphi_\chi : \pi_1(X_i) \to \mathrm{GL}(p, \mathbb{Q}(\zeta_q)[t^{\pm 1}])$ for each character in (5).

# of crossings	associated string	$\boldsymbol{x}$-string	# of components
18	$(3^{[9]})$	$[0^{[9]}]$	3
18	(2, 4, 2, 4, 4, 2, 4, 2, 3)	[1, 1, 1, 1, 0]	1
18	(2, 2, 4, 3, 2, 5, 2, 3, 4)	[2, 1, 0, 0, 1]	1
18	(2, 3, 4, 3, 4, 3, 2, 3, 3)	[1, 0, 0, 0, 1, 0, 0]	1
20	(2, 2, 2, 3, 3, 3, 6, 3, 3, 3)	$[3, 0^{[6]}]$	3
20	(2, 4, 2, 4, 2, 4, 2, 4, 2, 4)	$[1^{[5]}]$	3
20	(2, 4, 2, 3, 3, 4, 2, 4, 3, 3)	[1, 1, 1, 0, 0, 0, 0]	3
20	(2, 4, 3, 2, 3, 4, 2, 3, 4, 3)	[1, 1, 0, 0, 1, 0, 0]	3
20	(2, 3, 2, 3, 2, 3, 4, 4, 4, 3)	[1, 0, 1, 0, 1, 0, 0]	3
20	(2, 2, 2, 4, 3, 2, 6, 2, 3, 4)	[3, 1, 0, 0, 1]	3

Table 1. Links in $\mathcal{S}_{2c}^{\dagger}$ with up to 20 crossings which are potentially non-χ-slice. In the following we show that the three knots in this table are not slice.

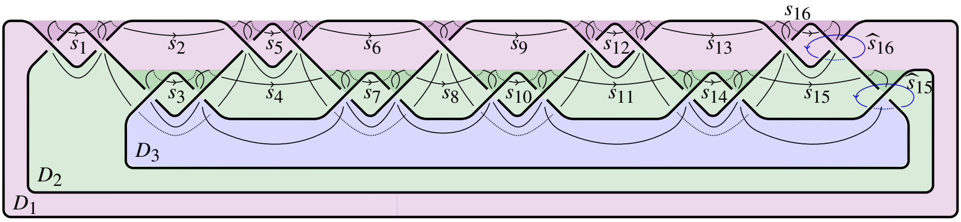

Figure 9. Our choice of a Seifert surface F_1 for K_1. Lifts of Alexander dual curves $\hat{s}_{15}$ and $\hat{s}_{16}$ to generate $H_1(\Sigma_3(K_1))$.

(7) Use the Fox matrix for a Wirtinger presentation of $\pi_1(X_i)$ to obtain a matrix Φ_χ for each χ in (5), whose determinant $\det \Phi_\chi$ is the reduced twisted Alexander polynomial $\tilde{\Delta}^\chi_{K_i}(t)$.

(8) Verify that none of the $\tilde{\Delta}^\chi_{K_i}(t)$ factor as norms, hence providing an obstruction to sliceness of all K_i.

For reference about various terms used in this outline, we direct the reader in the first instance to [11] and [1], as well as to the survey [9]. The computations were performed in `SageMath` notebooks available on the author's website.[2]

4A. ***The Seifert matrix.*** Let β be a 3-braid. A Seifert surface F for $\hat{\beta}$ can be constructed by joining three discs D_1, D_2 and D_3 by half-twisted bands, where each band between D_1 and D_2 comes from a σ_1 term in β, and each band between D_2 and D_3 from a σ_2 term; identify the bands with σ_i's. Let g be the genus of F. We can choose the generators of $H_1(F)$ to be the loops running once through consecutive σ_1's and σ_2's, except for the loop between the first and last σ_1 and the first and last σ_2. We order these generators $s_1, \dots, s_{2g}$ by when the first σ_i through which s_j runs appears in β. With this setup, the Seifert matrix S can be obtained using the algorithm of Collins [2]. Such F with $s_1, \dots, s_{2g}$ for K_1 is shown in Figure 9. Also, for $\nu(F)$ an open tubular neighbourhood of F, denote by $\hat{s}_i$ a choice of a simple closed curve in $S^3 \setminus \nu(F)$ that is *Alexander dual* to $\{s_1, \dots, s_{2g}\}$, i.e., which satisfies $\operatorname{lk}(s_i, \hat{s}_j) = \delta_{ij}$.

4B. ***Structure and bases of $H_1(\Sigma_3(K_i))$.*** We may perform column operations on the presentation matrices $P_i = tS_i - S_i^T$ of the Alexander modules $\mathcal{A}(K_i)$ to transform them into the forms

$$\left(\begin{array}{c|cc} I & \multicolumn{2}{c}{0} \\ \hline * & p_1(t) & 0 \\ & 0 & p_1(t) \end{array}\right), \quad \left(\begin{array}{c|ccc} I & \multicolumn{3}{c}{0} \\ \hline & p_2(t) & 0 & 0 \\ * & 0 & 1 & 0 \\ & 0 & * & p_2(t) \end{array}\right), \quad \left(\begin{array}{c|cc} I & \multicolumn{2}{c}{0} \\ \hline * & p_3(t) & 0 \\ & 0 & p_3(t) \end{array}\right)$$

[2] https://sites.google.com/view/vbrej

for $i = 1, 2, 3$, respectively, where each $p_i(t)$ is the square root of the untwisted Alexander polynomial $\Delta_{K_i}(t)$, I is the identity matrix and $*$ represents other entries. Specifically,

$$\begin{aligned} p_1(t) &= 1 - 3t + 7t^2 - 10t^3 + 11t^4 - 10t^5 + 7t^6 - 3t^7 + t^8, \\ p_2(t) &= 1 - 3t + 6t^2 - 9t^3 + 11t^4 - 9t^5 + 6t^6 - 3t^7 + t^8, \\ p_3(t) &= 1 - 4t + 8t^2 - 11t^3 + 13t^4 - 11t^5 + 8t^6 - 4t^7 + t^8. \end{aligned}$$

Recall that the Alexander module $\mathcal{A}(K)$ of a knot K is the $\mathbb{Z}[t^{\pm 1}]$-module $H_1(\widetilde{X_K^\infty})$, where $\widetilde{X_K^\infty}$ is the infinite cyclic cover of the knot complement X_K and t acts by deck transformations. Choose a preferred copy of $S^3 \setminus \nu(F_i)$ in $\widetilde{X_{K_i}^\infty}$ for all i. From [8, Theorems 1.3 and 1.4], summarised in the present context in [1, Theorem 3.6], it follows that

$$\mathcal{A}(K_i) \cong \mathbb{Z}[t^{\pm 1}]/\langle p_i(t)\rangle \oplus \mathbb{Z}[t^{\pm 1}]/\langle p_i(t)\rangle,$$

where $\mathcal{A}(K_i)$ for $i \in \{1, 3\}$ is generated by the lifts of $\hat{s}_{15}$ and $\hat{s}_{16}$ to the preferred copy of $S^3 \setminus \nu(F_i)$ in $\widetilde{X_i^\infty}$, while $\mathcal{A}(K_2)$ is generated by the lifts of $\hat{s}_{14}$ and $\hat{s}_{16}$; in each case, call these generators a and b, respectively. Choose $p = 3$. By, e.g., [6, Section 6.1], we have

$$\begin{aligned} H_1(\Sigma_3(K_i)) &\cong \mathcal{A}(K_i)/\langle t^2 + t + 1\rangle \\ &\cong \mathbb{Z}[t^{\pm 1}]/\langle p_i(t), t^2 + t + 1\rangle \oplus \mathbb{Z}[t^{\pm 1}]/\langle p_i(t), t^2 + t + 1\rangle \\ &\cong \mathbb{Z}[t^{\pm 1}]/\langle 7t, t^2 + t + 1\rangle \oplus \mathbb{Z}[t^{\pm 1}]/\langle 7t, t^2 + t + 1\rangle \\ &\cong (\mathbb{Z}/7)[t^{\pm 1}]/\langle t^2 + t + 1\rangle \oplus (\mathbb{Z}/7)[t^{\pm 1}]/\langle t^2 + t + 1\rangle \end{aligned}$$

in each of the three cases, since all of $p_i(t)$ are congruent to $7t$ modulo $t^2 + t + 1$. Hence, we fix $q = 7$. The generators of $\mathcal{A}(K_i)$ descend to $H_1(\Sigma_3(K_i))$, so by abuse of notation we also denote them by a and b. As a group, $H_1(\Sigma_3(K_i)) \cong (\mathbb{Z}/7)^4$, and we may treat it as a $(\mathbb{Z}/7)$-module generated by a, ta, b and tb.

4C. ***Blanchfield and linking forms.*** From [8, Theorems 1.3, 1.4; 1, Theorem 3.6] and a calculation in the accompanying notebooks, we obtain that the Blanchfield pairings on $\mathcal{A}(K_i)$ are given, with respect to the ordered basis $\{a, b\}$ and after reducing both the numerators and denominators modulo $t^3 - 1$, by

$$\tfrac{1}{7}\begin{pmatrix} 2t^2 + 2t - 4 & -2t^2 + 4t - 2 \\ 4t^2 - 2t - 2 & -4t^2 - 4t + 8 \end{pmatrix}, \quad \tfrac{1}{7}\begin{pmatrix} -3t^2 - 3t + 6 & 3t^2 - 3t \\ -3t^2 + 3t & 3t^2 + 3t - 6 \end{pmatrix},$$
$$\tfrac{1}{7}\begin{pmatrix} -4t^2 - 4t + 8 & 4t^2 - 2t - 2 \\ -2t^2 + 4t - 2 & 2t^2 + 2t - 4 \end{pmatrix}$$

for $i = 1, 2, 3$, respectively. Via [7, Chapter 2.6], applied similarly to [1, Proposition 3.7], we read off that the linking forms $\lambda_i : H_1(\Sigma_3(K_i)) \times H_1(\Sigma_3(K_i)) \to \mathbb{Q}/\mathbb{Z}$

with respect to the ordered basis $\{a, ta, b, tb\}$ are given by

$$\frac{1}{7}\begin{pmatrix} -4 & 2 & -2 & 4 \\ 2 & -4 & -2 & -2 \\ -2 & -2 & 1 & -4 \\ 4 & -2 & -4 & 1 \end{pmatrix}, \quad \frac{1}{7}\begin{pmatrix} 6 & -3 & 0 & -3 \\ -3 & 6 & 3 & 0 \\ 0 & 3 & -6 & 3 \\ -3 & 0 & 3 & -6 \end{pmatrix} \quad \text{and} \quad \frac{1}{7}\begin{pmatrix} 1 & -4 & -2 & -2 \\ -4 & 1 & 4 & -2 \\ -2 & 4 & -4 & 2 \\ -2 & -2 & 2 & -4 \end{pmatrix}.$$

4D. ***Metabolisers of $H_1(\Sigma_3(K_i))$.*** Write $M = (\mathbb{Z}/7)[t^{\pm 1}]/\langle t^2+t+1\rangle$ so that, as a $(\mathbb{Z}/7)[t^{\pm 1}]$-module, $H_1(\Sigma_3(K_i)) \cong M \oplus M$. Since the order $|H_1(\Sigma_3(K_i))| = 7^4$, we seek to describe all its $\mathbb{Z}[t^{\pm 1}]$-submodules of order $7^2 = 49$. Since t^2+t+1 has irreducible factors $(t-2), (t+3) \in (\mathbb{Z}/7)[t^{\pm 1}]$, the set $\{\langle 0\rangle, \langle 1\rangle, \langle t-2\rangle, \langle t+3\rangle\}$ contains precisely the $(\mathbb{Z}/7)[t^{\pm 1}]$-submodules of M; since the $\mathbb{Z}[t^{\pm 1}]$-action on M factors through $(\mathbb{Z}/7)[t^{\pm 1}]$, these are also precisely the $\mathbb{Z}[t^{\pm 1}]$-submodules of M. Observe that $|\langle 0\rangle| = 1$, $|\langle 1\rangle| = 49$ and $|\langle t-2\rangle| = |\langle t+3\rangle| = 7$. Now let N be a $\mathbb{Z}[t^{\pm 1}]$-submodule of $H_1(\Sigma_3(K_i))$, and consider the commutative diagram

$$\begin{array}{ccccc} M \oplus \{0\} & \longrightarrow & M \oplus M & \xrightarrow{\ \pi\ } & \{0\} \oplus M \\ \uparrow & & \uparrow & & \uparrow \\ \ker \pi|_N & \longrightarrow & N & \xrightarrow{\ \pi|_N\ } & \operatorname{im} \pi|_N \end{array}$$

where $\pi(x, y) = (0, y)$ for all $x, y \in M$, and unlabelled arrows are inclusions; $\ker \pi|_N$ and $\operatorname{im} \pi|_N$ are submodules of $M \oplus \{0\}$ and $\{0\} \oplus M$, respectively. Since $|N| = |\ker \pi|_N| \cdot |\operatorname{im} \pi|_N|$, we can deduce what N could be by order considerations.

- If $|\ker \pi|_N| = 49$, then $|\operatorname{im} \pi|_N| = 1$ and $N = \ker \pi|_N = \operatorname{span}_{(\mathbb{Z}/7)[t^{\pm 1}]}\{(1, 0)\}$.
- If $|\ker \pi|_N| = 1$, then $N \cong \operatorname{im} \pi|_N = \operatorname{span}_{(\mathbb{Z}/7)[t^{\pm 1}]}\{(k, 1)\}$ for some $k \in (\mathbb{Z}/7)[t^{\pm 1}]$.

Now, let $\{\langle t-2\rangle, \langle t+3\rangle\} = \{\langle \alpha\rangle, \langle \beta\rangle\}$; we have $\operatorname{Ann} \alpha = \langle \beta\rangle$ and $\operatorname{Ann} \beta = \langle \alpha\rangle$. There are two remaining cases to consider.

- Suppose $\ker \pi|_N \cong \operatorname{im} \pi|_N \cong \langle \alpha\rangle$. Then N contains $\{(\alpha, 0), (k, \alpha)\}$ for some $k \in (\mathbb{Z}/7)[t^{\pm 1}]$. Since $\beta(k, \alpha) = (\beta k, 0) \in \ker \pi|_N$, we must have $\beta k \in \langle \alpha\rangle$, so $k \in \langle \alpha\rangle$, i.e., $k = l\alpha$ for some $l \in (\mathbb{Z}/7)[t^{\pm 1}]$. Then $-l(\alpha, 0) + (k, \alpha) = (0, \alpha) \in N$, so N contains two linearly independent elements $(\alpha, 0)$ and $(0, \alpha)$ of order 7, and hence is generated by them for any choice of k. This yields two submodules $N = \operatorname{span}_{(\mathbb{Z}/7)[t^{\pm 1}]}\{(t-2, 0), (0, t-2)\}$ and $N = \operatorname{span}_{(\mathbb{Z}/7)[t^{\pm 1}]}\{(t+3, 0), (0, t+3)\}$.
- Suppose $\ker \pi|_N = \langle \alpha\rangle$ and $\operatorname{im} \pi|_N \cong \langle \beta\rangle$. We similarly observe that N contains $\{(\alpha, 0), (k, \beta)\}$ for some $k \in (\mathbb{Z}/7)[t^{\pm 1}]$. We have $\alpha(k, \beta) = (\alpha k, 0) \in \ker \pi|_N$, so we can take k modulo α, i.e., $k \in \mathbb{Z}/7$. Then $\{(\alpha, 0), (k, \beta)\}$ is a linearly independent set generating N for any choice of $k \in \mathbb{Z}/7$. Thus, $N = \operatorname{span}_{(\mathbb{Z}/7)[t^{\pm 1}]}\{(t-2, 0), (k, t+3)\}$ or $N = \operatorname{span}_{(\mathbb{Z}/7)[t^{\pm 1}]}\{(t+3, 0), (k, t-2)$ for $k \in \mathbb{Z}/7$.

K_1	K_2	K_3
$\chi_6^{\alpha\beta} : (1, 2, 1, -2)$	$\chi_1^{\alpha\beta} : (1, 2, 1, -4)$	$\chi_2^{\alpha\beta} = \chi_0^{\alpha} : (1, 2, 1, 2)$
$\chi_4^{\beta\alpha} : (1, -3, 1, -2)$	$\chi_1^{\beta\alpha} : (1, -3, 1, 1)$	$\chi_3^{\beta\alpha} : (1, -3, 1, 1)$

Table 2. Our choice of characters $\chi : H_1(\Sigma_3(K_i)) \to \mathbb{Z}/7$ vanishing on the metabolisers of K_1, K_2 and K_3; the characters χ_0^{α} and χ_0^{β} are given for all K_i by $(1, 2, 1, 2)$ and $(1, -3, 1, -3)$.

To summarise, writing elements of $H_1(\Sigma_3(K_i)) \cong M \oplus M$ additively with the first copy of M generated by a and the second by b, the desired submodules are

$$\begin{aligned}
N_0 &= \operatorname{span}_{(\mathbb{Z}/7)[t^{\pm 1}]}\{a\},\\
N_{k_0,k_1} &= \operatorname{span}_{(\mathbb{Z}/7)[t^{\pm 1}]}\{ka + b\} \text{ for } k \in (\mathbb{Z}/7)[t^{\pm 1}]\\
&= \operatorname{span}_{(\mathbb{Z}/7)[t^{\pm 1}]}\{(k_0 + k_1 t)a + b\} \text{ for } k_0, k_1 \in \mathbb{Z}/7,\\
N_0^{\alpha} &= \operatorname{span}_{(\mathbb{Z}/7)[t^{\pm 1}]}\{(t-2)a, (t-2)b\},\\
N_0^{\beta} &= \operatorname{span}_{(\mathbb{Z}/7)[t^{\pm 1}]}\{(t+3)a, (t+3)b\},\\
N_{k_0}^{\alpha\beta} &= \operatorname{span}_{(\mathbb{Z}/7)[t^{\pm 1}]}\{(t-2)a, k_0 a + (t+3)b\} \text{ for } k_0 \in \mathbb{Z}/7,\\
N_{k_0}^{\beta\alpha} &= \operatorname{span}_{(\mathbb{Z}/7)[t^{\pm 1}]}\{(t+3)a, k_0 a + (t-2)b\} \text{ for } k_0 \in \mathbb{Z}/7.
\end{aligned}$$

By a direct computation carried out in the accompanying notebooks, the submodules N_0^{α} and N_0^{β} are metabolisers for K_i for all i; in addition, K_1 has metabolisers $N_6^{\alpha\beta}$ and $N_4^{\beta\alpha}$, K_2 has metabolisers $N_1^{\alpha\beta}$ and $N_1^{\beta\alpha}$, and K_3 has metabolisers $N_2^{\alpha\beta}$ and $N_3^{\beta\alpha}$.

4E. ***Characters vanishing on the metabolisers.*** It is easy to define characters $\chi : H_1(\Sigma_3(K_i)) \to \mathbb{Z}/7$ that vanish on the metabolisers. Let subscripts and superscripts denote corresponding metabolisers and 4-tuples in parentheses represent the values a character takes on the ordered basis $\{a, ta, b, tb\}$. Then we can take χ_0^{α} and χ_0^{β} as defined by $(1, 2, 1, 2)$ and $(1, -3, 1, -3)$, respectively. The rest of the characters are presented in Table 2.

4F. ***Representations of the knot groups into*** $\mathbf{GL(3, \mathbb{Q}(\zeta_7)[t^{\pm 1}])}$**.** Let

$$K \in \{K_1, K_2, K_3\}.$$

We follow [1, Appendix A] and [11, Chapters 5–7] to construct representations

$$\varphi_\chi : \pi_1(X_K) \to \mathrm{GL}(3, \mathbb{Q}(\zeta_7)[t^{\pm 1}])$$

of the knot group of K that determine twisted Alexander polynomials for each character in Table 2. Fix a basepoint x_0 in $S^3 \setminus \nu(F)$ and let $\tilde{x}_0$ be its lift to the preferred copy of $S^3 \backslash \nu(F)$ in $\widetilde{X^3_K}$, the triple cyclic cover of the knot complement X_K. Also fix a based meridian μ_0 in $S^3 \setminus K$ and let $\varepsilon : \pi_1(X_K) \to \mathbb{Z}$ be the abelianisation homomorphism. Define a map $l : \ker \varepsilon \to H_1(\Sigma_3(K))$ that takes a simple closed curve $\gamma \subset S^3 \setminus K$ based at x_0 with $\mathrm{lk}(K, \gamma) = 0$ to the homology class of the well-defined lift $\tilde{\gamma}$ in $\widetilde{X^3_K} \subset \Sigma_3(K)$ based at $\tilde{x}_0$. In particular, l has the property that for any $\gamma \in \ker \varepsilon$, we have

$$l(\mu_0 \gamma \mu_0^{-1}) = t \cdot l(\gamma). \tag{‡}$$

Now consider the semidirect product $\mathbb{Z} \ltimes H_1(\Sigma_3(K))$, with $\mathbb{Z} = \langle t \rangle$, whose product structure is given by $(t^{m_1}, x_1) \cdot (t^{m_2}, x_2) = (t^{m_1+m_2}, t^{-m_2} \cdot x_1 + x_2)$ with t acting on elements of $H_1(\Sigma_3(K))$ by deck transformations. Fix a Wirtinger presentation of $\pi_1(X_K) \cong \langle g_1, \ldots, g_n \mid r_1, \ldots, r_n \rangle$ and define a homomorphism

$$\psi : \pi_1(X_K) \to \mathbb{Z} \ltimes H_1(\Sigma_3(K)), \quad g_i \mapsto (t, l(\mu_0^{-1} g_i)) =: (t, v_i)$$

on the generators of $\pi_1(X_K)$, since clearly $\mu_0^{-1} g_i \in \ker \varepsilon$. Observe that a relation $g_i g_j g_i^{-1} g_k^{-1} = 1$ imposes, via the group structure on $\mathbb{Z} \ltimes H_1(\Sigma_3(K))$, the condition

$$(1 - t) v_i + t v_j - v_k = 0. \tag{‡‡}$$

Finally, for a character $\chi : H_1(\Sigma_3(K)) \to \mathbb{Z}/7$, we obtain a representation

$$\varphi_\chi : \pi_1(X_K) \to \mathrm{GL}(3, \mathbb{Q}(\zeta_7)[t^{\pm 1}])$$

by setting $\varphi_\chi = \tau_\chi \circ \psi$, where

$$\tau_\chi : \mathbb{Z} \ltimes H_1(\Sigma_3(K)) \to \mathrm{GL}(3, \mathbb{Q}(\zeta_7)[t^{\pm 1}]),$$
$$(t^m, v) \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ t & 0 & 0 \end{pmatrix}^m \begin{pmatrix} \zeta_7^{\chi(v)} & 0 & 0 \\ 0 & \zeta_7^{\chi(t \cdot v)} & \\ 0 & 0 & \zeta_7^{\chi(t^2 \cdot v)} \end{pmatrix}.$$

We shall apply the equation (‡) to determine the form of the first few v_k for K in terms of the generators $\{a, b\}$ of $H_1(\Sigma_3(K))$ and then deduce the rest of v_k using (‡‡), giving us the desired φ_χ. We illustrate the process in more detail for K_1, with K_2 and K_3 cases being analogous.

Recall that we orient K_1 clockwise. Index the arcs in the diagram of K_1 as shown in Figure 10, starting with 1 at the top left and increasing the index at every undercrossing. This yields the following Wirtinger presentation of $\pi_1(X_1)$, with generators being the meridians g_i about each arc i based at x_0:

$$\pi_1(X_1) = \left\langle g_1, \ldots, g_{18} \;\middle|\; \begin{array}{l} g_1 g_{13} g_1^{-1} g_{12}^{-1},\ g_3 g_{17} g_3^{-1} g_{16}^{-1},\ g_7 g_1 g_7^{-1} g_{18}^{-1},\ g_{16} g_9 g_{16}^{-1} g_{10}^{-1}, \\ g_{13} g_2 g_{13}^{-1} g_1^{-1},\ g_{17} g_5 g_{17}^{-1} g_4^{-1},\ g_8 g_{13} g_8^{-1} g_{14}^{-1},\ g_{10} g_3 g_{10}^{-1} g_4^{-1},\ g_6 g_{11} g_6^{-1} g_{12}^{-1}, \\ g_2 g_{15} g_2^{-1} g_{14}^{-1},\ g_5 g_{18} g_5^{-1} g_{17}^{-1},\ g_{14} g_8 g_{14}^{-1} g_9^{-1},\ g_4 g_{10} g_4^{-1} g_{11}^{-1},\ g_{12} g_7 g_{12}^{-1} g_8^{-1} \\ g_{15} g_3 g_{15}^{-1} g_2^{-1},\ g_{18} g_7 g_{18}^{-1} g_6^{-1},\ g_9 g_{15} g_9^{-1} g_{16}^{-1},\ g_{11} g_5 g_{11}^{-1} g_6^{-1}, \end{array} \right\rangle$$

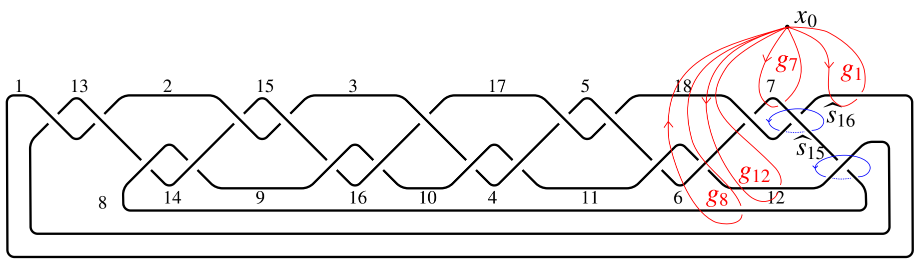

Figure 10. Choice of arc labels for K_1 giving the Wirtinger presentation of $\pi_1(X_1)$.

Observe that $\hat{s}_{15} = g_8 g_{12}^{-1}$ and $\hat{s}_{16} = g_1^{-1} g_7$. Fix $\mu_0 = g_1$. Then $v_1 = l(g_1^{-1} g_1) = 0$ and $v_7 = l(g_1^{-1} g_7) = b$. Also, using the property (‡), we have

$$\begin{aligned} a = l(g_8 g_{12}^{-1}) = l(g_8 g_1^{-1} g_1 g_{12}^{-1}) &= l(g_8 g_1^{-1}) + l(g_1 g_{12}^{-1}) \\ &= l(g_1 g_1^{-1} g_8 g_1^{-1}) - l(g_{12} g_1^{-1}) \\ &= l(g_1 g_1^{-1} g_8 g_1^{-1}) - l(g_1 g_1^{-1} g_{12} g_1^{-1}) = t v_8 - t v_{12}. \end{aligned}$$

Applying (‡‡) to the relation $g_{12} g_7 g_{12}^{-1} g_8^{-1} = 1$ and recalling we are working modulo $t^2 + t + 1$, we get

$$\begin{aligned} (1-t) v_{12} + t v_7 - v_8 = 0 \quad &\Longrightarrow \quad (1-t) v_{12} - v_8 = -tb \mid \cdot (-t) \\ &\Longrightarrow \quad (t v_8 - t v_{12}) + t^2 v_{12} = t^2 b \\ &\Longrightarrow \quad a + t^2 v_{12} = t^2 b \mid \cdot t \\ &\Longrightarrow \quad v_{12} = -ta + b. \end{aligned}$$

Now we can use (‡‡) repeatedly to obtain the values of all v_i. With the same conventions and the choice $\mu_0 = g_1$, for K_2 we have

$$l(\hat{s}_{14}) = l(g_1^{-1} g_6) = a \quad \text{and} \quad l(\hat{s}_{16}) = l(g_{14} g_7^{-1}) = b,$$

while for K_3,

$$l(\hat{s}_{15}) = l(g_1^{-1} g_7) = a \quad \text{and} \quad l(\hat{s}_{16}) = l(g_8 g_{13}^{-1}) = b;$$

this lets us calculate the values of v_i in Table 3 analogously. With that, constructing representations φ_χ for the characters in Section 4E is mechanical.

4G. ***Calculating twisted Alexander polynomials.*** Again, let $K \in \{K_1, K_2, K_3\}$ and fix the Wirtinger presentation of $\pi_1(X_K)$ as in Section 4F. Given a representation $\varphi_\chi : \pi_1(X_K) \to \mathrm{GL}(3, \mathbb{Q}(\zeta_7)[t^{\pm 1}])$, let $\Phi : \mathbb{Z}[\pi_1(X_K)] \to \mathrm{Mat}_3(\mathbb{Q}(\zeta_7)[t^{\pm 1}])$ be its natural extension to the group ring $\mathbb{Z}[\pi_1(X_K)]$ taking values in the set of 3×3

matrices with $\mathbb{Q}(\zeta_7)[t^{\pm 1}]$ coefficients. Let

$$\Psi = \left(\frac{\partial r_i}{\partial g_j}\right)_{i,j=1,\dots,18}$$

be the Fox matrix for the Wirtinger presentation of $\pi_1(X_K)$; the row of Ψ corresponding to the relation $g_i g_j g_i^{-1} g_k^{-1}$ has $1 - g_k$ in the i-th column, g_i in the j-th column, -1 in the k-th column and zeros elsewhere. Write $r(\Psi)$ for the reduced Fox matrix obtained by dropping the first row and column from Ψ and let Φ_χ be the 51×51 matrix obtained by applying Φ to $r(\Psi)$ entrywise. By [11, Section 9], the reduced twisted Alexander polynomial $\tilde{\Delta}_K^\chi(t)$ of (K, χ) (for nontrivial χ) is given by

$$\tilde{\Delta}_K^\chi(t) = \frac{\det \Phi_\chi}{(t-1)\det(\varphi_\chi(g_1) - I)}.$$

Thus we obtain the 11 reduced twisted Alexander polynomials listed in the Appendix associated with our characters of interest.

4H. ***Obstructing sliceness of K_i.*** To show that K_1, K_2 and K_3 are not slice, we use the following generalisation of the Fox–Milnor condition, due to Kirk and Livingston [12].

	$\pi_1(X_1)$	$\pi_1(X_2)$	$\pi_1(X_3)$
v_1	0	0	0
v_2	$(6t+5)a+(5t+6)b$	$(5t+6)a+(4t+4)b$	$(5t+6)a+(6t+5)b$
v_3	$5ta+5b$	$3a+(3t+1)b$	$(4t+3)a+(t+1)b$
v_4	$(2t+5)a+6b$	$(2t+6)a+2b$	$(6t+3)a+b$
v_5	$(6t+5)a+(5t+3)b$	$(4t+1)a+(6t+5)b$	$(6t+4)a+(4t+6)b$
v_6	$5tb$	a	$(4t+1)a+(t+6)b$
v_7	b	$a+(6t+1)b$	a
v_8	$(5t+6)a+b$	$(6t+6)a+(6t+5)b$	$a+(5t+6)b$
v_9	$(3t+2)a+(4t+1)b$	$5ta+(3t+5)b$	$(3t+6)a+(5t+3)b$
v_{10}	$(t+2)a+(5t+1)b$	$(2t+3)a+(3t+3)b$	$(4t+6)a+(3t+3)b$
v_{11}	$6a+(4t+1)b$	$(3t+6)a+5b$	$(3t+6)a+2tb$
v_{12}	$6ta+b$	$(6t+2)a+(6t+6)b$	$(6t+2)a+6b$
v_{13}	$6a+(6t+6)b$	$a+b$	$a+6tb$
v_{14}	$(3t+4)a+(6t+2)b$	$a+5tb$	$(6t+6)a+6b$
v_{15}	$3a+(2t+4)b$	$(5t+3)a+6b$	$(6t+2)a+(3t+4)b$
v_{16}	$5a+(2t+3)b$	$(5t+5)a+(3t+5)b$	$(t+1)a+(2t+6)b$
v_{17}	$4a+(2t+2)b$	$ta+(5t+3)b$	$ta+(2t+5)b$
v_{18}	$(6t+1)b$	$(6t+1)a$	$(6t+1)a$

Table 3. Values of $v_k = l(\mu_0^{-1} g_k) \in H_1(\Sigma_3(K_i))$.

Theorem 4.1 [12, Proposition 6.1]. *Let $K \subset S^3$ be a slice knot and fix distinct primes p and q. Then there exists a covering transformation invariant metaboliser N in $H_1(\Sigma_p(K))$ such that the following condition holds: for every character $\chi : H_1(\Sigma_p(K)) \to \mathbb{Z}/q$ that vanishes on N, the associated reduced twisted Alexander polynomial $\tilde{\Delta}^{\chi}_K(t) \in \mathbb{Q}(\zeta_q)[t^{\pm 1}]$ is a **norm**, i.e., $\tilde{\Delta}^{\chi}_K(t)$ can be written as*

$$\tilde{\Delta}^{\chi}_K(t) = \lambda t^k f(t)\overline{f(t)}$$

for some $\lambda \in \mathbb{Q}(\zeta_q)$, $k \in \mathbb{Z}$ and $\overline{f(t)}$ obtained from $f(t) \in \mathbb{Q}(\zeta_q)[t^{\pm 1}]$ by the involution $t \mapsto t^{-1}$, $\zeta_q \mapsto \zeta_q^{-1}$.

Using the routine implemented in SnapPy [4] for determining whether an element of $\mathbb{Q}(\zeta_q)[t^{\pm 1}]$ is a norm, which relies on the SageMath algorithm for factoring polynomials over cyclotomic fields, we conclude via a calculation in the accompanying notebooks that none of the 11 polynomials in the Appendix are norms. This implies that K_1, K_2 and K_3 are not slice.

Appendix: Reduced twisted Alexander polynomials for K_1, K_2 and K_3

The following table contains reduced twisted Alexander polynomials for knots K_1, K_2 and K_3 associated to characters vanishing on the metabolisers of respective knots; for brevity, we write $\zeta = \zeta_7$ and $\theta = \zeta_7 + \zeta_7^2 + \zeta_7^4$.

(K_i, χ)	$\tilde{\Delta}^{\chi}_{K_i}(t)$
(K_1, χ_0^{α})	$-t^{15} + (-2\theta - 1)t^{14} + (-8\theta - 3)t^{13} + 15t^{12} + (-3\theta + 48)t^{11} + (-8\theta + 33)t^{10} + (-48\theta + 34)t^9 + 199t^8 + (48\theta + 82)t^7 + (8\theta + 41)t^6 + (3\theta + 51)t^5 + 15t^4 + (8\theta + 5)t^3 + (2\theta + 1)t^2 - t$
(K_1, χ_0^{β})	$-t^{15} + (-4\theta + 5)t^{14} + (24\theta - 15)t^{13} + (-93\theta - 14)t^{12} + (98\theta + 11)t^{11} + (-2\theta + 71)t^{10} + (-11\theta - 154)t^9 + 360t^8 + (11\theta - 143)t^7 + (2\theta + 73)t^6 + (-98\theta - 87)t^5 + (93\theta + 79)t^4 + (-24\theta - 39)t^3 + (4\theta + 9)t^2 - t$
$(K_1, \chi_6^{\alpha\beta})$	$-t^{15} + (2\zeta^5 - \zeta^4 + 4\zeta^3 - \zeta^2 - 2\zeta + 5)t^{14} + (-3\zeta^5 + 7\zeta^4 - 24\zeta^3 - 3\zeta^2 + 2\zeta - 20)t^{13} + (7\zeta^5 - 67\zeta^4 + 41\zeta^3 - 8\zeta^2 - 35\zeta + 7)t^{12} + (-45\zeta^5 + 52\zeta^4 - 38\zeta^3 + 3\zeta^2 - \zeta + 19)t^{11} + (68\zeta^5 + 51\zeta^4 + 114\zeta^3 + 24\zeta^2 + 95\zeta + 63)t^{10} + (116\zeta^5 + 121\zeta^4 + 80\zeta^3 + 56\zeta^2 + 124\zeta + 65)t^9 + (149\zeta^5 - 3\zeta^4 - 3\zeta^3 + 149\zeta^2 + 19)t^8 + (-68\zeta^5 - 44\zeta^4 - 3\zeta^3 - 8\zeta^2 - 124\zeta - 59)t^7 + (-71\zeta^5 + 19\zeta^4 - 44\zeta^3 - 27\zeta^2 - 95\zeta - 32)t^6 + (4\zeta^5 - 37\zeta^4 + 53\zeta^3 - 44\zeta^2 + \zeta + 20)t^5 + (27\zeta^5 + 76\zeta^4 - 32\zeta^3 + 42\zeta^2 + 35\zeta + 42)t^4 + (-5\zeta^5 - 26\zeta^4 + 5\zeta^3 - 5\zeta^2 - 2\zeta - 22)t^3 + (\zeta^5 + 6\zeta^4 + \zeta^3 + 4\zeta^2 + 2\zeta + 7)t^2 - t$

$(K_1, \chi_4^{\beta\alpha})$

$$-t^{15} + (2\zeta^5 + \zeta^4 + 2\zeta^3 + \zeta^2 - \zeta + 2)t^{14} + (-5\zeta^5 - 2\zeta^4 - 3\zeta^3 - 6\zeta^2 - 2\zeta - 9)t^{13} + (10\zeta^5 + 4\zeta^4 + 9\zeta^2 + 20)t^{12} + (-35\zeta^5 - 36\zeta^4 - 30\zeta^3 - 35\zeta^2 - 4\zeta - 10)t^{11} + (44\zeta^5 - 10\zeta^4 + 8\zeta^3 + 47\zeta^2 + 52\zeta + 85)t^{10} + (-57\zeta^5 - 17\zeta^4 - 63\zeta^3 + 29\zeta^2 - 27\zeta + 11)t^9 + (7\zeta^5 + 38\zeta^4 + 38\zeta^3 + 7\zeta^2 - 59)t^8 + (56\zeta^5 - 36\zeta^4 + 10\zeta^3 - 30\zeta^2 + 27\zeta + 38)t^7 + (-5\zeta^5 - 44\zeta^4 - 62\zeta^3 - 8\zeta^2 - 52\zeta + 33)t^6 + (-31\zeta^5 - 26\zeta^4 - 32\zeta^3 - 31\zeta^2 + 4\zeta - 6)t^5 + (9\zeta^5 + 4\zeta^3 + 10\zeta^2 + 20)t^4 + (-4\zeta^5 - \zeta^4 - 3\zeta^2 + 2\zeta - 7)t^3 + (2\zeta^5 + 3\zeta^4 + 2\zeta^3 + 3\zeta^2 + \zeta + 3)t^2 - t$$

(K_2, χ_0^{α})

$$t^{15} + (-\theta - 2)t^{14} + (-2\theta - 1)t^{13} + (3\theta + 3)t^{12} + (-13\theta - 22)t^{11} + (-15\theta - 5)t^{10} + (25\theta + 13)t^9 - 82t^8 + (-25\theta - 12)t^7 + (15\theta + 10)t^6 + (13\theta - 9)t^5 - 3\theta t^4 + (2\theta + 1)t^3 + (\theta - 1)t^2 + t$$

(K_2, χ_0^{β})

$$t^{15} + (-4\theta - 7)t^{14} + (16\theta + 15)t^{13} + (-41\theta - 26)t^{12} + (55\theta + 5)t^{11} + (-20\theta - 18)t^{10} + (-25\theta + 114)t^9 - 292t^8 + (25\theta + 139)t^7 + (20\theta + 2)t^6 + (-55\theta - 50)t^5 + (41\theta + 15)t^4 + (-16\theta - 1)t^3 + (4\theta - 3)t^2 + t$$

$(K_2, \chi_1^{\alpha\beta})$

$$t^{15} + (-3\zeta^5 + 3\zeta^4 - 2\zeta^3 + \zeta^2 - 4)t^{14} + (4\zeta^5 - 12\zeta^4 + 6\zeta^3 - 13\zeta^2 + \zeta)t^{13} + (23\zeta^4 + 9\zeta^3 + 30\zeta^2 - 4\zeta + 17)t^{12} + (-49\zeta^5 - 17\zeta^4 - 50\zeta^3 - 46\zeta^2 - 33\zeta - 13)t^{11} + (-48\zeta^5 + 5\zeta^4 + 67\zeta^3 - 34\zeta^2 + 87\zeta - 36)t^{10} + (164\zeta^5 + 69\zeta^4 + 127\zeta^3 + 39\zeta^2 + 83\zeta + 75)t^9 + (173\zeta^5 + 32\zeta^4 + 32\zeta^3 + 173\zeta^2 + 166)t^8 + (-44\zeta^5 + 44\zeta^4 - 14\zeta^3 + 81\zeta^2 - 83\zeta - 8)t^7 + (-121\zeta^5 - 20\zeta^4 - 82\zeta^3 - 135\zeta^2 - 87\zeta - 123)t^6 + (-13\zeta^5 - 17\zeta^4 + 16\zeta^3 - 16\zeta^2 + 33\zeta + 20)t^5 + (34\zeta^5 + 13\zeta^4 + 27\zeta^3 + 4\zeta^2 + 4\zeta + 21)t^4 + (-14\zeta^5 + 5\zeta^4 - 13\zeta^3 + 3\zeta^2 - \zeta - 1)t^3 + (\zeta^5 - 2\zeta^4 + 3\zeta^3 - 3\zeta^2 - 4)t^2 + t$$

$(K_2, \chi_1^{\beta\alpha})$

$$t^{15} + (-\zeta^5 - 2\zeta^4 + \zeta^2 - 3\zeta - 7)t^{14} + (4\zeta^5 + 8\zeta^4 - 4\zeta^3 - 4\zeta^2 + 17\zeta + 28)t^{13} + (-\zeta^5 - 20\zeta^4 + 21\zeta^3 + 30\zeta^2 - 52\zeta - 78)t^{12} + (-10\zeta^5 + 38\zeta^4 - 51\zeta^3 - 88\zeta^2 + 122\zeta + 187)t^{11} + (81\zeta^5 - 15\zeta^4 + 87\zeta^3 + 205\zeta^2 - 155\zeta - 358)t^{10} + (-256\zeta^5 - 31\zeta^4 - 157\zeta^3 - 312\zeta^2 + 91\zeta + 487)t^9 + (434\zeta^5 + 146\zeta^4 + 146\zeta^3 + 434\zeta^2 - 430)t^8 + (-403\zeta^5 - 248\zeta^4 - 122\zeta^3 - 347\zeta^2 - 91\zeta + 396)t^7 + (360\zeta^5 + 242\zeta^4 + 140\zeta^3 + 236\zeta^2 + 155\zeta - 203)t^6 + (-210\zeta^5 - 173\zeta^4 - 84\zeta^3 - 132\zeta^2 - 122\zeta + 65)t^5 + (82\zeta^5 + 73\zeta^4 + 32\zeta^3 + 51\zeta^2 + 52\zeta - 26)t^4 + (-21\zeta^5 - 21\zeta^4 - 9\zeta^3 - 13\zeta^2 - 17\zeta + 11)t^3 + (4\zeta^5 + 3\zeta^4 + \zeta^3 + 2\zeta^2 + 3\zeta - 4)t^2 + t$$

$$
\begin{aligned}
&(K_3,\chi_0^{\alpha}) && t^{15}+(\theta-3)t^{14}+(-3\theta-1)t^{13}+(-2\theta-22)t^{12}+(-73\theta-8)t^{11}\\
&=(K_3,\chi_2^{\alpha\beta}) && \quad +(10\theta+239)t^{10}+(362\theta+223)t^{9}-675t^{8}+(-362\theta-139)t^{7}\\
& && \quad +(-10\theta+229)t^{6}+(73\theta+65)t^{5}+(2\theta-20)t^{4}+(3\theta+2)t^{3}+(-\theta-4)t^{2}+t\\
&(K_3,\chi_0^{\beta}) && t^{15}-7t^{14}+(-2\theta+17)t^{13}+(6\theta-32)t^{12}+(-26\theta+26)t^{11}\\
& && \quad +(24\theta+8)t^{10}+(40\theta+83)t^{9}-178t^{8}+(-40\theta+43)t^{7}+(-24\theta-16)t^{6}\\
& && \quad +(26\theta+52)t^{5}+(-6\theta-38)t^{4}+(2\theta+19)t^{3}-7t^{2}+t\\
&(K_3,\chi_3^{\beta\alpha}) && t^{15}+(-\zeta^5+3\zeta^4+2\zeta^3+2\zeta^2+4\zeta-3)t^{14}\\
& && \quad +(18\zeta^5+\zeta^4+3\zeta^3+3\zeta^2-4\zeta+11)t^{13}\\
& && \quad +(-33\zeta^5-17\zeta^4-26\zeta^3-21\zeta^2-11\zeta-60)t^{12}\\
& && \quad +(-5\zeta^5-52\zeta^4-16\zeta^3-3\zeta^2-56\zeta+45)t^{11}\\
& && \quad +(-14\zeta^5+48\zeta^4+66\zeta^3-18\zeta^2+59\zeta-5)t^{10}\\
& && \quad +(106\zeta^5+89\zeta^4-10\zeta^3+109\zeta^2+18\zeta+101)t^{9}\\
& && \quad +(-133\zeta^5-123\zeta^4-123\zeta^3-133\zeta^2-212)t^{8}\\
& && \quad +(91\zeta^5-28\zeta^4+71\zeta^3+88\zeta^2-18\zeta+83)t^{7}\\
& && \quad +(-77\zeta^5+7\zeta^4-11\zeta^3-73\zeta^2-59\zeta-64)t^{6}\\
& && \quad +(53\zeta^5+40\zeta^4+4\zeta^3+51\zeta^2+56\zeta+101)t^{5}\\
& && \quad +(-10\zeta^5-15\zeta^4-6\zeta^3-22\zeta^2+11\zeta-49)t^{4}\\
& && \quad +(7\zeta^5+7\zeta^4+5\zeta^3+22\zeta^2+4\zeta+15)t^{3}\\
& && \quad +(-2\zeta^5-2\zeta^4-\zeta^3-5\zeta^2-4\zeta-7)t^{2}+t
\end{aligned}
$$

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