

*Pacific
Journal of
Mathematics*

**BOUNDARY REGULARITY OF BERGMAN KERNEL
IN HÖLDER SPACE**

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Volume 324 No. 1

May 2023

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Let D be a bounded strictly pseudoconvex domain in \mathbb{C}^n . Assuming $\partial D \in C^{k+3+\alpha}$ where k is a nonnegative integer and $0 < \alpha \leq 1$, we show that (1) the Bergman kernel $B(\cdot, w_0) \in C^{k+\min\{\alpha, 1/2\}}(\bar{D})$, for any $w_0 \in D$ and (2) the Bergman projection on D is a bounded operator from $C^{k+\beta}(\bar{D})$ to $C^{k+\min\{\alpha, \beta/2\}}(\bar{D})$ for any $0 < \beta \leq 1$. Our results both improve and generalize the work of E. Ligocka.

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1. Introduction

The main goal of the paper is to prove the following result.

Theorem 1.1. *Let D be a bounded strictly pseudoconvex domain in \mathbb{C}^n with $C^{k+3+\alpha}$ boundary, where k is a nonnegative integer and $0 < \alpha \leq 1$. Let $B(z, w)$ be the Bergman kernel for D . Then for every $w_0 \in D$, $B(\cdot, w_0) \in C^{k+\min\{\alpha, 1/2\}}(\bar{D})$.*

Earlier, E. Ligocka [1984] showed that if Ω has C^{k+4} boundary for nonnegative integers k , then $B(\cdot, w_0) \in C^{k+1/2}(\bar{D})$. Hence Theorem 1.1 is an improvement and generalization of Ligocka's result to Hölder spaces.

The study of boundary regularity properties of the Bergman projection and Bergman kernel is of fundamental importance in several complex variables, and the subject has found major applications in the theory of biholomorphic mappings and complex geometry, among many other fields. We mention here some brief history for the results on strictly pseudoconvex domains. When the boundary is C^∞ , Kerzman [1972] used the theory of $\bar{\partial}$ -Neumann problem to show that the

MSC2020: primary 32A25; secondary 32T15.

Keywords: Bergman kernel, Bergman projection, strictly pseudoconvex domain.

Bergman kernel function $B(z, w)$ is $C^\infty \times C^\infty(\bar{D} \times \bar{D} \setminus \Delta_{bD})$, where $\Delta_{bD} := \{(z, w) \in bD \times bD, z = w\}$. Soon after, C. Fefferman in his seminal paper [1974] gave a description of the behavior of the Bergman kernel $(z, w) \in bD \times bD$ near its singular set Δ_{bD} , and as an application he proved the now classical Fefferman's mapping theorem, which states that a biholomorphic mapping $F : D_1 \rightarrow D_2$ between two bounded C^∞ strictly pseudoconvex domains D_1, D_2 extends to a C^∞ diffeomorphism $\tilde{F} : \bar{D}_1 \rightarrow \bar{D}_2$. Fefferman's proof was based on the deep properties of the Bergman kernel and Bergman metric on strictly pseudoconvex domains. The analysis however was very difficult and nearly impossible to generalize to other cases. Later on Webster [1979] and Bell and Ligocka [1980] independently found conditions on the boundary behavior of the Bergman kernel that can imply the C^∞ extension of biholomorphic mappings, and consequently they were able to significantly simplify Fefferman's proof.

Phong and Stein [1977] and Ahern and Schneider [1979] independently proved the Hölder estimates for the Bergman projection. In both work the boundary is assumed to be C^∞ and the proof is based on the work of C. Fefferman [1974] and L. Boutet de Monvel and J. Sjöstrand [1976]. Later on, Ligocka [1984] constructed a nonorthogonal projection operator with explicit kernels that "approximates" the Bergman projection operator, and she used it to prove the Hölder estimates assuming boundary is C^{k+4} . Ligocka based off her construction on a similar work done by Kerzman and Stein [1978] for the Szegő projection on C^∞ strictly pseudoconvex domains. The idea is to use the symmetry of the Levi polynomial for the defining function to get a third order cancellation, which then allows one to estimate the singular integrals (see Proposition 3.1). It is also worthwhile to mention that the method of Kerzman, Stein and Ligocka has been used in a number of subsequent works, for example in [Lanzani and Stein 2012; 2013]. For a detailed exposition of the work by Ligocka and Bell and Kerzman, Stein and Ligocka, we refer the reader to the book by M. Range [Range 1986, Chapter VII].

We shall give a variant of Ligocka's method which allows us to prove the estimates in Hölder spaces. Our method also has the advantage that the term on the right-hand side of our integral equation behaves much nicer than the one used by Ligocka, which we now explain. Denote the Bergman projection on D by \mathcal{P} . It is a standard fact that for $w_0 \in D$, one can write $B(\cdot, w_0) = \mathcal{P}\varphi$, where $\varphi = \varphi_{w_0} \in C_c^\infty(D)$ (see Lemma 2.4). Ligocka showed that $\mathcal{P}\varphi$ satisfies an integral equation of the form

$$(1-1) \quad (I + \mathcal{K})\mathcal{P}\varphi = \mathcal{L}^*\varphi.$$

Here \mathcal{L} is a nonorthogonal projection operator mapping $L^2(\Omega)$ into $H^2(\Omega)$, the L^2 Bergman space, \mathcal{L}^* is the adjoint operator of \mathcal{L} , and $\mathcal{K} := \mathcal{L}^* - \mathcal{L}$. It was proved in [Ligocka 1984] that if the boundary is C^{k+4} , then \mathcal{K} is a compact operator

mapping $C^k(\bar{D})$ into $C^{k+1/2}(\bar{D})$, and $\mathcal{L}, \mathcal{L}^*$ map $C^{k+1}(\bar{D})$ (in fact only need derivatives of order k being Lipschitz continuous) into $C^{k+1/2}(\bar{D})$. Hence in particular $\mathcal{L}^*\varphi \in C^{k+1/2}(\bar{D})$. Applying Fredholm theory to the integral (1-1) then shows that $\mathcal{P}\varphi \in C^{k+1/2}(\bar{D})$.

For our proof we shall use the same operators $\mathcal{L}, \mathcal{L}^*, \mathcal{K}$, but instead of considering the integral equation of $\mathcal{P}\varphi$, we show that the following integral equation holds for the function $\mathcal{P}\varphi - \varphi$

$$(1-2) \quad (I + \mathcal{K})(\mathcal{P}\varphi - \varphi) = R(\mathcal{P}\varphi - \varphi),$$

where R is some operator that maps $\mathcal{P}\varphi - \varphi$ to a $C^\infty(\bar{D})$ function, assuming boundary is only C^3 . This is in contrast to the right-hand side of (1-1), where the regularity of $\mathcal{L}^*\varphi$ depends on the regularity of the boundary and the estimate is much more complicated.

Using (1-2), Theorem 1.1 is then an easy consequence of the following compactness result and Fredholm theory.

Proposition 1.2. *Let D be a bounded strictly pseudoconvex domain in \mathbb{C}^n with $C^{k+3+\alpha}$ boundary, where k is a nonnegative integer and $0 < \alpha \leq 1$. Then \mathcal{K} is a bounded operator from $C^k(\bar{D})$ to $C^{k+\min\{\alpha, 1/2\}}(\bar{D})$.*

We remark that Proposition 1.2 is the main estimate of the paper and takes up the majority of the proof.

Using Proposition 1.2 we can also prove the following theorem for the Bergman projection. Similar result has been obtained by Ligočka under the assumption that the boundary is C^{k+4} .

Theorem 1.3. *Let D be a bounded strictly pseudoconvex domain in \mathbb{C}^n with $C^{k+3+\alpha}$ boundary, where k is a nonnegative integer and $0 < \alpha \leq 1$. For $0 < \beta \leq 1$, the Bergman projection \mathcal{P} for the domain D defines a bounded operator from $C^{k+\beta}(\bar{D})$ to $C^{k+\min\{\alpha, \beta/2\}}(\bar{D})$.*

In the special case $\alpha = 1$, we recover Ligočka's result. Note that Theorem 1.1 can also be obtained as a consequence of Theorem 1.3, by the fact that $B = P\varphi$ and setting $\beta = 1$ in Theorem 1.3. However we shall give independent proofs of the two theorems based on Proposition 1.2.

The paper is organized as follows: In Section 2, we prove a simple estimate for Hörmander's $\bar{\partial}$ solution operator on pseudoconvex domains. We also prove a refined version of the regularized defining function introduced in [Gong 2019], which plays an important role in the proof of Proposition 1.2. In Section 3 we follow Ligočka's idea to construct the operators $\mathcal{L}, \mathcal{L}^*, \mathcal{K}$, using the regularized defining function from Section 2. We then prove various estimates for the kernels of $\mathcal{L}, \mathcal{L}^*, \mathcal{K}$. We note that in our proof (Proposition 3.5 and the remark after) that \mathcal{L}

defines a bounded projection operator from $L^2(D)$ to $H^2(D)$, only C^3 boundary regularity is needed.

In Section 4 we will prove Proposition 1.2 and Theorem 1.1. The proof of Proposition 1.2 is split into two parts. In the first part, we prove the case for $k = 0$, i.e., assuming $bD \in C^{3+\alpha}$, $0 < \alpha \leq 1$, we show that \mathcal{K} maps $L^\infty(D)$ boundedly into $C^{\min\{\alpha, 1/2\}}(\bar{D})$. In the second part, we apply the integration by parts techniques from [Ahern and Schneider 1979] to prove the case for $k \geq 1$. We next turn to the proof of Theorem 1.1. First we construct the integral (1-2) using Koppelman's homotopy formula and show that the right-hand side defines a $C^\infty(\bar{D})$ function. Theorem 1.1 then follows easily from Proposition 1.2 and standard Fredholm theory. In Section 5 we prove Theorem 1.3. To this end we show that \mathcal{L} is a bounded operator from $C^{k+\beta}(\bar{D})$ to $C^{k+\beta/2}(\bar{D})$, $0 < \beta \leq 1$, assuming boundary is C^3 .

We now fix some notations used in the paper. The L^2 Bergman space on a domain D is denoted by $H^2(D)$. The Bergman projection and Bergman kernel is denoted by \mathcal{P} and B , respectively. We denote by $C^r(\bar{D})$ the Hölder space of exponent r on D , and $C_c^\infty(D)$ the space of C^∞ functions with compact support in D . For simplicity we write $\|f\|_r := \|f\|_{C^r(\bar{D})}$ when the domain D is clear from context. We write $x \lesssim y$ to mean that $x \leq Cy$ for some constant C independent of x and y . By D^l we mean a differential operator of order l : $D_z^l g(z) = \partial_{z_i}^{\alpha_i} \partial_{\bar{z}_j}^{\beta_j} g(z)$, $\sum_i \alpha_i + \sum_j \beta_j = l$.

2. Preliminaries

Proposition 2.1. *Let D, D' be bounded pseudoconvex domains in \mathbb{C}^n such that $D' \subset D$, and let $l \geq 0$. Suppose φ is a $\bar{\partial}$ -closed $(0, 1)$ form in D , with coefficients in $W^l(D)$. Let $u = S\varphi$, where S is Hörmander's L^2 solution operator which solves $\bar{\partial}u = \varphi$ on D . Then $u \in W^{l+1}(D')$, and*

$$\|u\|_{W^{l+1}(D')} \leq C(\delta/2)^{-l-1} \|\varphi\|_{W^l(D)}, \quad \delta := \text{dist}(D', \partial D),$$

where C is an absolute constant depending only on the domain D ,

Proof. By Hörmander's L^2 estimate [1965], we have $\bar{\partial}u = \varphi$ and

$$(2-1) \quad \|u\|_{L^2(D)} \leq C_0 \|\varphi\|_{L^2(D)},$$

where C_0 is a constant which depends only on the diameter of D . Let $\chi \in C_c^\infty(D)$ be such that $\chi \equiv 1$ on D' . Further, χ satisfies the estimate $|D^\gamma \chi| \leq \delta^{-|\gamma|}$, where $\delta := \text{dist}(D', D)$. We use the following fact: If $v \in L^2(\mathbb{C}^n)$ has compact support and $\bar{\partial}v \in L^2(\mathbb{C}^n)$, then

$$(2-2) \quad \|\partial_{z_i} v\|_{L^2(\mathbb{C}^n)} = \|\partial_{\bar{z}_i} v\|_{L^2(\mathbb{C}^n)}.$$

This can be proved through a simple integration by parts and approximation argument; see [Hörmander 1990, Lemma 4.2.4]. In what follows we let D^l to denote a differential operator of the form $\prod_{i,j=1}^n \partial_{z_i}^{\alpha_i} \partial_{\bar{z}_j}^{\beta_j}$, where $\sum_{i,j=1}^n |\alpha_i| + |\beta_j| = l$, and we use $\bar{\partial}^l$ to denote $\prod_{j=1}^n \partial_{\bar{z}_j}^{\beta_j}$, where $\sum \beta_j = l$. Applying (2-2) repeatedly then gives

$$(2-3) \quad \|D^{l+1}v\|_{L^2(\mathbb{C}^n)} = \|\bar{\partial}^{l+1}v\|_{L^2(\mathbb{C}^n)}$$

for any $v \in L^2(\mathbb{C}^n)$ with compact support and such that $\bar{\partial}^{l+1}v \in L^2(\mathbb{C}^n)$. Applying (2-3) with $v = \chi u$, we get

$$(2-4) \quad \begin{aligned} \|D^{l+1}(\chi u)\|_{L^2(D)} &= \|\bar{\partial}^{l+1}(\chi u)\|_{L^2(D)} \\ &\leq \|(\bar{\partial}^{l+1}\chi)u\|_{L^2(D)} + \sum_{1 \leq s \leq l+1} \|(\bar{\partial}^{l+1-s}\chi)(\bar{\partial}^s u)\|_{L^2(D)}. \end{aligned}$$

By (2-1) and estimates for the derivatives of χ , the first integral is bounded by $C_0\delta^{-(l+1)}\|\varphi\|_{L^2(D)}$. For each integral in the sum, we have for $1 \leq s \leq l+1$

$$\begin{aligned} \|(\bar{\partial}^{l+1-s}\chi)(\bar{\partial}^s u)\|_{L^2(D)} &= \|(\bar{\partial}^{l+1-s}\chi)(\bar{\partial}^{s-1}\bar{\partial}u)\|_{L^2(D)} \\ &\leq \delta^{-(l+1-s)}\|\varphi\|_{W^{s-1}(D)} \leq \delta^{-l}\|\varphi\|_{W^l(D)}. \end{aligned}$$

Now, there are in total $\sum_{k=0}^{l+1} \binom{l+1}{k} = 2^{l+1}$ terms on the right-hand side of (2-4). Thus by combining the estimates we obtain

$$\|D^{l+1}(\chi u)\|_{L^2(D)} \leq C_0 2^{l+1} \delta^{-(l+1)} \|\varphi\|_{W^l(D)} = C_0 (\delta/2)^{-(l+1)} \|\varphi\|_{W^l(D)}.$$

Since $\chi \equiv 1$ on D' , we have

$$\|D^{l+1}u\|_{L^2(D')} \leq \|D^{l+1}(\chi u)\|_{L^2(D)} \leq C_0 (\delta/2)^{-(l+1)} \|\varphi\|_{W^l(D)}. \quad \square$$

We now show the existence of a defining function that is smooth off the boundary and whose derivatives blow up in a controlled way.

Proposition 2.2. *Let D be a bounded domain in \mathbb{R}^N with C^r boundary, $r \geq 3$, and let ρ be a defining function of D of the class C^r , i.e., there exists a \mathcal{U} such that $D \subset \mathcal{U}$, $\nabla \rho \neq 0$ on bD and $D = \{x \in \mathcal{U} : \rho(D) < 0\}$. We denote $|\rho|_r := |\rho|_{C^r(\mathcal{U})}$, where $|\cdot|_{C^r(\mathcal{U})}$ denotes the Hölder r -norm on \mathcal{U} . Then there exists a defining function $\tilde{\rho}$ of D such that:*

- (a) $\tilde{\rho} \in C^r(\mathbb{R}^N) \cap C^\infty(\mathbb{R}^n \setminus bD)$.
- (b) *There exists some $\delta_0 > 0$ such that for any $x \notin bD$ and $0 < \delta(x) := \text{dist}(x, bD) < \delta_0$,*

$$|D^j \tilde{\rho}(x)| \lesssim C_j |\rho|_r (1 + \delta(x)^{r-j}), \quad j = 0, 1, 2, \dots, \delta(x) := \text{dist}(x, bD).$$

(c) *There exists a constant C depending only on the domain D and $|\rho|_3$, and a $\delta_1 > 0$ such that for all $x \in \mathbb{R}^n$ with $\delta(x) < \delta_1$ the following estimate hold*

$$|\widehat{D}^2 \tilde{\rho}(x) - \widehat{D}^2 \rho(x_*)| \lesssim C|x - x_*|, \quad |x_* - x| := \text{dist}(x, bD).$$

Here we use $\widehat{D}^2 \rho$ to denote derivatives of ρ of order 2 and less.

We call $\tilde{\rho}$ a *regularized defining function* of the domain D .

Proof. We will use the argument from [Gong 2019]. Let E_r be the Whitney extension operator for the domain D . By [Gong 2019, Lemma 3.7], $E_r \rho$ is a defining function of D (so that $-E_r \rho$ is a defining function of the domain $(\bar{D})^c$), $E_r \rho \in C^r(\mathbb{R}^N) \cap C^\infty((\bar{D})^c)$ and

$$|D^j E_r \rho(x)| \lesssim C_j |\rho|_r (1 + \delta(x)^{r-j}), \quad j = 0, 1, 2, \dots, x \in \mathbb{R}^n \setminus \bar{D}.$$

Furthermore, for each $x \in \mathbb{R}^n \setminus \bar{D}$ with $0 < \delta(x) < 1$, there exists some constant C depending only on D and $|\rho|_3$ such that $|\widehat{D}^2(E_r \rho)(x) - \widehat{D}^2 \rho(x_*)| \leq C|x - x_*|$, where $x_* := \text{dist}(x, bD)$. Let E'_r be the Whitney extension operator for the domain $(\bar{D})^c$. Then by the same reasoning $\tilde{\rho} := E'_r E_r \rho$ is a defining function of D satisfying $\tilde{\rho} \in C^r(\mathbb{R}^N) \cap C^\infty(\mathbb{R}^n \setminus bD)$, and for all $x \in D$ with $0 < \delta(x) < \delta_1$, the following hold (for $j = 0, 1, 2, \dots, x \in D$)

$$\begin{aligned} |D^j \tilde{\rho}(x)| &\lesssim C'_j |E_r \rho|_r (1 + \delta(x)^{r-j}) \lesssim C''_j |\rho|_r (1 + \delta(x)^{r-j}), \\ |\widehat{D}^2 \tilde{\rho}(x) - \widehat{D}^2 \rho(x_*)| &= |\widehat{D}^2 \tilde{\rho}(x) - \widehat{D}^2(E_r \rho)(x_*)| \leq C'|x - x_*|, \end{aligned}$$

where $|x_* - x| := \text{dist}(x, bD)$. □

We now state a very useful result to prove Hölder estimates, popularly known as the Hardy–Littlewood lemma. For a proof the reader may refer to [Chen and Shaw 2001, p. 345].

Lemma 2.3 (Hardy–Littlewood lemma). *Let D be a bounded domain in \mathbb{R}^N with C^1 boundary. Suppose $g \in C^k(\bar{D})$ and that for some $0 < \beta < 1$ there is a constant C such that*

$$|D^{k+1} g(x)| \leq C \delta(x)^{-1+\beta}, \quad x \in D,$$

where $\delta(x) = \text{dist}(x, bD)$. Then $g \in C^{k+\beta}(\bar{D})$.

The following lemma can be found in [Bell 1993]. We provide the proof for the reader's convenience.

Lemma 2.4. *Let D be a bounded domain $\Omega \subset \mathbb{C}^n$ and let $B(z, w)$ and \mathcal{P} denote the Bergman kernel and the Bergman projection for D , respectively. Given $w_0 \in D$, there exists a function ϕ_{w_0} in $C_c^\infty(D)$ such that*

$$(2-5) \quad \frac{\partial^{|\beta|}}{\partial \bar{w}^\beta} B(z, w_0) = \mathcal{P} \phi_{w_0}^\beta(z), \quad \phi_{w_0}^\beta(z) := (-1)^{|\beta|} \frac{\partial^{|\beta|}}{\partial \bar{z}^\beta} \phi_{w_0}(z),$$

where β is a multiindex.

Proof. Let δ_0 denote the distance from w_0 to bD and let $B_1(\mathbf{0})$ the unit ball in \mathbb{C}^n . Set

$$\phi_{w_0}(z) = \delta_0^{-2n} \phi\left(\frac{z - w_0}{\delta_0}\right), \quad z \in D,$$

where ϕ is a real-valued function in $C_c^\infty(B_1(\mathbf{0}))$ that is radially symmetric about the origin and $\int \phi \, dV = 1$. Clearly, $\phi_{w_0} \in C_c^\infty(D)$. By the property of the Bergman projection and the Bergman kernel, we have

$$\begin{aligned} \mathcal{P}\phi_{w_0}(z) &= \int_D B(z, \zeta) \phi_{w_0}(\zeta) \, dV(\zeta) \\ &= \int_D B(z, \zeta) \delta_0^{-2n} \phi\left(\frac{\zeta - w_0}{\delta_0}\right) \, dV(\zeta) \\ &= \int_{B_1(\mathbf{0})} B(z, \delta_0 \zeta + w_0) \phi(\zeta) \, dV(\zeta) \\ &= \overline{\int_{B_1(\mathbf{0})} B(\delta_0 \zeta + w_0, z) \phi(\zeta) \, dV(\zeta)} \\ &= \overline{B(w_0, z)} \\ &= B(z, w_0), \end{aligned}$$

where we used the fact that B is holomorphic in the first argument and thus both its real and imaginary parts are harmonic functions which satisfy the mean value property. This proves (2-5) for $\beta = 0$. The general case follows similarly by repeating the above calculation and integration by parts. We leave the details to the reader. □

3. Estimates of the kernel

In this section we follow Ligocka’s idea to construct the kernel of the projection operator \mathcal{L} for a strictly pseudoconvex domain. For now we assume the defining function ρ is in the class C^3 .

Suppose a bounded domain $D \subset \mathbb{C}^n$ is given by $D = \{z \in \mathbb{C}^n : \rho(z) < 0\}$. We write

$$D_\delta := \{z \in \mathbb{C}^n : \rho(z) < \delta\}, \quad \delta > 0.$$

We shall sometimes write $D_\delta(z)$ (or $D_\delta(\zeta)$) to indicate that the domain is for the z (or ζ) variable. We now construct the kernel to be used in the integral formula. By setting $\rho' = e^{A\rho} - 1$, for some large A , we see that ρ' is strictly plurisubharmonic in a neighborhood of \bar{D} , and from now on we simply assume ρ satisfies this property.

Define

$$(3-1) \quad F(z, \zeta) = \sum_{j=1}^n \frac{\partial \rho}{\partial \zeta_j}(\zeta)(\zeta_j - z_j) - \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2 \rho}{\partial \zeta_i \partial \zeta_j}(\zeta)(z_i - \zeta_i)(z_j - \zeta_j).$$

By Taylor's formula we have

$$(3-2) \quad \rho(z) = \rho(\zeta) - 2 \operatorname{Re} F(z, \zeta) + \mathcal{L}_\rho(\zeta; z - \zeta) + o(|z - \zeta|^2),$$

where $\mathcal{L}_\rho(\zeta; t)$ is the Levi form of ρ at ζ , i.e., $\mathcal{L}_\rho(\zeta; t) := \sum_{i,j=1}^n \partial^2 \rho / (\partial \zeta_i \partial \bar{\zeta}_j) t_i \bar{t}_j$. Fix some $\varepsilon_0 > 0$ small such that for all $z, \zeta \in D_\delta$, we have $\mathcal{L}_\rho(\zeta; z - \zeta) \geq c|z - \zeta|^2$. It follows from (3-2) that (for $(z, \zeta) \in D_\delta \times D_\delta, |z - \zeta| < \varepsilon_0$)

$$(3-3) \quad \operatorname{Re} F(z, \zeta) \geq \frac{\rho(\zeta) - \rho(z)}{2} + \frac{c}{2}|z - \zeta|^2,$$

$$(3-4) \quad \operatorname{Re} F(z, \zeta) - \rho(\zeta) \geq -\frac{\rho(\zeta) + \rho(z)}{2} + \frac{c}{2}|z - \zeta|^2.$$

Let $\chi(t)$ be a smooth cut-off function such that $\chi(t) \equiv 1$ if $t < \varepsilon_0/4$ and $\chi(t) \equiv 0$ if $t > \varepsilon_0/2$. We define the following global support function:

$$(3-5) \quad G(z, \zeta) = \chi(t)F(z, \zeta) + (1 - \chi(t))|z - \zeta|^2, \quad t = |z - \zeta|.$$

We also define the vector-valued functions $g_0 = (g_0^1, \dots, g_0^n)$ and $g_1 = (g_1^1, \dots, g_1^n)$, where

$$g_0^i(z, \zeta) = \overline{\zeta_i - z_i}, \quad 1 \leq i \leq n,$$

and (for $1 \leq i \leq n$),

$$(3-6) \quad g_1^i(z, \zeta) = \chi(t) \left(\sum_{j=1}^n \frac{\partial \rho}{\partial \zeta_j}(\zeta) + \frac{1}{2} \sum_{j=1}^n \frac{\partial^2 \rho}{\partial \zeta_i \partial \zeta_j}(\zeta)(z_j - \zeta_j) \right) + (1 - \chi(t))(\overline{\zeta_i - z_i}),$$

where $t = |z - \zeta|$. It follows that $\langle g_0, \zeta - z \rangle = |\zeta - z|^2$ and $\langle g_1, \zeta - z \rangle = G(z, \zeta)$. In view of (3-4) and (3-5), there exists some $c > 0$ such that

$$(3-7) \quad \operatorname{Re} G(z, \zeta) - \rho(\zeta) \geq c(-\rho(\zeta) - \rho(z) + |z - \zeta|^2), \quad z, \zeta \in \bar{D},$$

and

$$(3-8) \quad \operatorname{Re} G(z, \zeta) - \rho(\zeta) \geq \frac{1}{4}[-\rho(\zeta) - \rho(z)] + c|z - \zeta|^2, \quad z, \zeta \in D_\delta \times D_\delta.$$

In particular, (3-7) implies

$$(3-9) \quad |G(z, \zeta) - \rho(\zeta)| \gtrsim |\zeta - z|^2, \quad z, \zeta \in D.$$

We note that if the boundary is $C^{k+3+\alpha}$, then $g_1, G \in C^\infty \times C^{k+1+\alpha}(D_\delta(z) \times D_\delta(\zeta))$ and holomorphic in z whenever $|z - \zeta| < \varepsilon_0/4$. Let

$$\omega_\lambda(z, \zeta) = \frac{1}{2\pi\sqrt{-1}} \frac{\langle g_\lambda(z, \zeta), d\zeta \rangle}{\langle g_\lambda, \zeta - z \rangle}, \quad \lambda = 0, 1.$$

The associated Cauchy–Fantappie forms are given by

$$\begin{aligned} \Omega^\lambda &= \omega_\lambda \wedge (\bar{\partial}_{z,\zeta} \omega_\lambda)^{n-1}, & \lambda &= 0, 1. \\ \Omega^{01} &= \omega_0 \wedge \omega_1 \wedge \sum_{k_1+k_2=n-2} (\bar{\partial}_{z,\zeta} \omega_0)^{k_1} \wedge (\bar{\partial}_{z,\zeta} \omega_1)^{k_2}. \end{aligned}$$

We decompose $\Omega^\lambda = \sum_{0 \leq q \leq n} \Omega_{0,q}^\lambda$ and $\Omega^{01} = \sum_{0 \leq q \leq n} \Omega_{0,q}^{01}$, where $\Omega_{0,q}^\lambda$ (resp. $\Omega_{0,q}^{01}$) has type $(0, q)$ in z and type $(n, n - 1 - q)$ type in ζ . The following Koppleman’s formula holds:

$$(3-10) \quad \bar{\partial}_\zeta \Omega_{0,q}^{01} + \bar{\partial}_z \Omega_{0,q-1}^{01} = \Omega_{0,q}^0 - \Omega_{0,q}^1,$$

where we take $\Omega_{0,-1}^{01} \equiv 0$. Write (for $\lambda = 0, 1$)

$$(3-11) \quad \Omega_{0,0}^\lambda(z, \zeta) = \frac{1}{(2\pi\sqrt{-1})^n} \frac{1}{\langle g_\lambda, \zeta - z \rangle^n} \left(\sum_{i=1}^n g_\lambda^i d\zeta_i \right) \wedge \left(\sum_{i,j=1}^n \bar{\partial}_\zeta g_\lambda^i \wedge d\zeta_i \right)^{n-1};$$

$$(3-12) \quad \Omega_{0,0}^{01}(z, \zeta) = \frac{1}{(2\pi\sqrt{-1})^n} \frac{\langle \bar{\zeta} - z, d\zeta \rangle}{|\zeta - z|^2} \wedge \frac{\langle g_1, d\zeta \rangle}{\langle g_1, \zeta - z \rangle} \wedge \sum_{k_1+k_2=n-2} \left(\frac{\langle d\bar{\zeta}, d\zeta \rangle}{|\zeta - z|^2} \right)^{k_1} \wedge \left(\frac{\langle \bar{\partial}_\zeta g_1, d\zeta \rangle}{\langle g_1, \zeta - z \rangle} \right)^{k_2}.$$

Define

$$(3-13) \quad \begin{aligned} N(z, \zeta) &:= \frac{1}{(2\pi\sqrt{-1})^n} \frac{1}{[G(z, \zeta) - \rho(\zeta)]^n} \left(\sum_{i=1}^n g_1^i(z, \zeta) d\zeta_i \right) \\ &\quad \wedge \left(\sum_{i=1}^n \bar{\partial}_\zeta g_1^i(z, \zeta) \wedge d\zeta_i \right)^{n-1} \\ &= C_n \sum_{i=1}^n (-1)^{i-1} \frac{g_1^i(z, \zeta)}{[G(z, \zeta) - \rho(\zeta)]^n} d\zeta \wedge (\bar{\partial}_\zeta g_1^1) \wedge \cdots \\ &\quad \wedge \widehat{(\bar{\partial}_\zeta g_1^i)} \wedge \cdots \wedge (\bar{\partial}_\zeta g_n), \end{aligned}$$

where $\widehat{\eta}$ means η is being excluded. Note that for $\zeta \in bD$, we have $N(z, \zeta) = \Omega_{0,0}^1(z, \zeta)$. Therefore by (3-10) with $q = 0$,

$$(3-14) \quad \Omega_{0,0}^0(z, \zeta) = \bar{\partial}_\zeta \Omega_{0,0}^{01}(z, \zeta) + N(z, \zeta), \quad z \in D_{\varepsilon_0}, \zeta \in bD.$$

Let

$$(3-15) \quad L(z, \zeta)dV(\zeta) := \bar{\partial}_\zeta N(z, \zeta) - S_z(\bar{\partial}_z \bar{\partial}_\zeta N)(z, \zeta),$$

where S_z is Hörmander's operator that solves $\bar{\partial}$ on D_δ . In what follows we write $L = L_0 + L_1$, where

$$L_0 dV(\zeta) = -S_z(\bar{\partial}_z \bar{\partial}_\zeta N)(z, \zeta), \quad L_1 dV(\zeta) = \bar{\partial}_\zeta N(z, \zeta).$$

For each $\zeta \in \bar{D}$, $L(\cdot, \zeta)$ is holomorphic on D . We also note that if $bD \in C^{k+3+\alpha}$, then $\bar{\partial}_\zeta N \in C^\infty \times C^{k+\alpha}(D(z) \times D(\zeta))$. In view of (3-8), (3-13) and the fact that $\bar{\partial}_z G(z, \zeta), \bar{\partial}_z g(z, \zeta) \equiv 0$ for $|z - \zeta| < \varepsilon_0/4$, we see that $\bar{\partial}_z \bar{\partial}_\zeta N(z, \zeta)$ is a well-defined $\bar{\partial}$ -closed $(0, 1)$ form with coefficients in $C^\infty \times C^{k+\alpha}(D_\delta(z) \times D_\delta(\zeta))$, if $\delta > 0$ is sufficiently small. Write

$$(3-16) \quad \begin{aligned} \bar{\partial}_\zeta N(z, \zeta) &= L_1(z, \zeta)dV(\zeta) \\ &= C_n \frac{\bar{\partial}_\zeta [G(z, \zeta) - \rho(\zeta)]}{[G(z, \zeta) - \rho(\zeta)]^{n+1}} \\ &\quad \times \sum_{i=1}^n (-1)^{i-1} g_1^i(z, \zeta) d\zeta \wedge (\bar{\partial}_\zeta g_1^1) \wedge \cdots \wedge \widehat{(\bar{\partial}_\zeta g_1^i)} \wedge \cdots \wedge (\bar{\partial}_\zeta g_1^n) \\ &\quad + \frac{1}{[G(z, \zeta) - \rho(\zeta)]^n} d\zeta \wedge (\bar{\partial}_\zeta g_1^1) \wedge \cdots \wedge (\bar{\partial}_\zeta g_1^n). \end{aligned}$$

In the proof we shall use the following convenient expression from [Ligocka 1984]:

$$(3-17) \quad L_1(z, \zeta) = \frac{\eta(\zeta) + O'(|z - \zeta|)}{[G(z, \zeta) - \rho(\zeta)]^{n+1}}, \quad \eta(\zeta) := c_n \det \begin{vmatrix} \rho(\zeta) & \frac{\partial \rho}{\partial \zeta_i}(\zeta) \\ \frac{\partial \rho}{\partial \zeta_i}(\zeta) & \frac{\partial^2 \rho}{\partial \zeta_i \partial \zeta_j}(\zeta) \end{vmatrix}.$$

Here we note that $\overline{\eta(\zeta)} = \eta(\zeta)$, and $O'(|z - \zeta|)$ is some linear combination of products of $[\widehat{D^3 \rho(\zeta)}](\zeta_i - z_i)$, where $[D^3 \rho(\zeta)]$ denotes products of $\rho(\zeta)$ and $D^k \rho(\zeta)$, $k \leq 3$. In particular, for $l \geq 1$, $O'(|z - \zeta|)$ satisfies the estimates

$$(3-18) \quad \begin{aligned} |O'(|z - \zeta|)| &\lesssim |\rho|_3 |\zeta - z|, \\ |D_z^l O'(|z - \zeta|)| &\lesssim |\rho|_3, \\ |D_\zeta^l O'(|z - \zeta|)| &\lesssim |\rho|_{l+2} + |\rho|_{l+3} |\zeta - z|. \end{aligned}$$

We now define the integral operator

$$(3-19) \quad \mathcal{L}f(z) := \int_D L(z, \zeta)f(\zeta) dV(\zeta) = \int_D [L_0(z, \zeta) + L_1(z, \zeta)]f(\zeta) dV(\zeta),$$

and the associated adjoint operator

$$\mathcal{L}^* f(z) := \int_D \overline{L(\zeta, z)} f(\zeta) dV(\zeta) = \int_D [\overline{L_0(\zeta, z)} + \overline{L_1(\zeta, z)}] f(\zeta) dV(\zeta).$$

In the same way as (3-17), we can also write

$$(3-20) \quad \overline{L_1(\zeta, z)} = \frac{\eta(z) + O''(|z - \zeta|)}{[\overline{G(\zeta, z)} - \rho(z)]^{n+1}}, \quad \eta(z) := c_n \det \begin{vmatrix} \rho(z) & \frac{\partial \rho}{\partial \bar{z}_i}(z) \\ \frac{\partial \rho}{\partial \bar{z}_i}(z) & \frac{\partial^2 \rho}{\partial \bar{z}_i \partial \bar{z}_j}(z) \end{vmatrix}.$$

Here $O''(|z - \zeta|)$ is some linear combination of products of $[\widehat{D^3} \rho(z)](\zeta_i - z_i)$, where $[\widehat{D^3} \rho(z)]$ denotes products of $\rho(z)$ and $D_z^k \rho(z)$, $k \leq 3$. For $l \geq 1$, $O''(|z - \zeta|)$ satisfies the estimate

$$(3-21) \quad \begin{aligned} |O''(|z - \zeta|)| &\lesssim |\rho|_3 |\zeta - z|, \\ |D_\zeta^l O''(|z - \zeta|)| &\lesssim |\rho|_3, \\ |D_z^l O''(|z - \zeta|)| &\lesssim |\rho|_{l+2} + |\rho|_{l+3} |\zeta - z|. \end{aligned}$$

Hence if $bD \in C^{k+3+\alpha}$, then $\overline{L_1(\zeta, z)}$ is $C^{k+\alpha} \times C^\infty(D(z) \times D(\zeta))$.

Let

$$(3-22) \quad K(z, \zeta) := \overline{L(\zeta, z)} - L(z, \zeta) = [\overline{L_0(\zeta, z)} - L_0(z, \zeta)] + [\overline{L_1(\zeta, z)} - L_1(z, \zeta)],$$

and

$$(3-23) \quad \begin{aligned} \mathcal{K}f(z) &:= \int_D K(z, \zeta) f(\zeta) dV(\zeta) \\ &= \int [\overline{L(\zeta, z)} - L(z, \zeta)] f(\zeta) dV(\zeta) \\ &= \mathcal{L}^* f(z) - \mathcal{L}f(z). \end{aligned}$$

For later purpose we note that $\sqrt{-1}\mathcal{K}$ is a self-adjoint operator.

The following cancellation estimate is due to [Kerzman and Stein 1978]. We include a proof here for the reader's convenience.

Proposition 3.1. *Let D be a strictly pseudoconvex domain with a C^3 defining function ρ , with $0 < \alpha < 1$. Let $F(z, \zeta)$ be the function defined by formula (3-1). Then*

$$(3-24) \quad [F(z, \zeta) - \rho(\zeta)] - [\overline{F(\zeta, z)} - \rho(z)] = O(|\zeta - z|^3),$$

where $|O(|\zeta - z|^3)| \lesssim |\rho|_3 |\zeta - z|^3$.

Proof. By (3-1) we have

$$\begin{aligned}
F(z, \zeta) &= \sum_{j=1}^n \frac{\partial \rho}{\partial \zeta_j}(\zeta)(\zeta_j - z_j) - \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2 \rho}{\partial \zeta_i \partial \zeta_j}(\zeta)(z_i - \zeta_i)(z_j - \zeta_j) \\
&= \sum_{j=1}^n \left[\frac{\partial \rho}{\partial z_j}(z) + \sum_{k=1}^n \frac{\partial^2 \rho}{\partial z_j \partial z_k}(z)(\zeta_k - z_k) + \sum_{k=1}^n \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(z)(\overline{\zeta_k - z_k}) \right] (\zeta_j - z_j) \\
&\quad - \sum_{i,j=1}^n \frac{1}{2} \frac{\partial^2 \rho}{\partial z_i \partial z_j}(z)(z_i - \zeta_i)(z_j - \zeta_j) + R(z, \zeta) \\
&= \sum_{j=1}^n \frac{\partial \rho}{\partial z_j}(z)(\zeta_j - z_j) + \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2 \rho}{\partial z_i \partial z_j}(z)(\zeta_i - z_i)(\zeta_j - z_j) \\
&\quad + \sum_{i,j=1}^n \frac{\partial^2 \rho}{\partial z_i \partial \bar{z}_j}(z)(\zeta_i - z_i)(\overline{\zeta_j - z_j}) + R_0(z, \zeta),
\end{aligned}$$

where we did Taylor expansion for the function $\frac{\partial \rho}{\partial \zeta_j}$ at z . Since $\rho \in C^3$, the remainder term R_0 satisfies $|R_0(z, \zeta)| \lesssim |\rho|_3 |\zeta - z|^3$. On the other hand,

$$\overline{F(\zeta, z)} = \sum_{j=1}^n \frac{\partial \rho}{\partial \bar{z}_j}(z)(\overline{z_j - \zeta_j}) - \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2 \rho}{\partial \bar{z}_i \partial \bar{z}_j}(z)(\overline{\zeta_i - z_i})(\overline{\zeta_j - z_j}).$$

Hence

$$\begin{aligned}
F(z, \zeta) - \overline{F(\zeta, z)} &= \sum_{j=1}^n \frac{\partial \rho}{\partial z_j}(z)(\zeta_j - z_j) + \sum_{j=1}^n \frac{\partial \rho}{\partial \bar{z}_j}(z)(\overline{\zeta_j - z_j}) \\
&\quad + \operatorname{Re} \left(\sum_{i,j=1}^n \frac{\partial^2 \rho}{\partial z_i \partial z_j}(z)(\zeta_i - z_i)(\zeta_j - z_j) \right) \\
&\quad + \sum_{i,j=1}^n \frac{\partial^2 \rho}{\partial z_i \partial \bar{z}_j}(z)(\zeta_i - z_i)(\overline{\zeta_j - z_j}) + R_0(z, \zeta),
\end{aligned}$$

where $|R_0(z, \zeta)| \lesssim |\rho|_3 |\zeta - z|^3$. The first four terms on the right-hand side are exactly the first and second order terms in the Taylor polynomial of ρ at z , which is equal to $\rho(\zeta) - \rho(z) + R_1(z, \zeta)$, where $|R_1(z, \zeta)| \lesssim |\rho|_3 |\zeta - z|^3$. Hence

$$F(z, \zeta) - \overline{F(\zeta, z)} = \rho(\zeta) - \rho(z) + R(z, \zeta),$$

where $|R(z, \zeta)| \lesssim |\rho|_3 |\zeta - z|^3$. □

In what follows we shall denote

$$\Phi(z, \zeta) = G(z, \zeta) - \rho(\zeta), \quad \bar{\Phi}(\zeta, z) = \overline{G(\zeta, z)} - \rho(z).$$

Lemma 3.2. *Let D be a bounded strictly pseudoconvex domain with C^3 boundary in \mathbb{C}^n , $n \geq 2$, Let ρ be the defining function of D . Let $0 < \beta \leq 1$. Let $\Theta(z, \zeta)$ denote either $\Phi(z, \zeta)$ or $\Phi(\zeta, z)$:*

(i) *Let $0 < \beta \leq 1$. Then*

$$(3-25) \quad \int_D \frac{dV(\zeta)}{|\zeta - z|^{2-\beta} |\Theta(z, \zeta)|^{n+1}} \lesssim 1 + \delta(z)^{\beta/2-1},$$

where the constant depends only on D .

(ii) *Let $\beta > 0$. Then*

$$(3-26) \quad \int_{B(z, \tau)} \frac{|z - \zeta|^\beta}{|\Theta(z, \zeta)|^{n+1}} dV(\zeta) \lesssim \tau^{\beta/2},$$

where the constant depends only on D .

Proof. First, we show that for each fixed $z \in D$, there exists a small neighborhood U_z and a coordinate chart $\phi_z : U_z \rightarrow \mathbb{R}^{2n}$ with $\phi_z(\zeta) = ((s_1, s_2), t) \in \mathbb{R}^2 \times \mathbb{R}^{2n-2}$ and

$$(3-27) \quad |\Phi(z, \zeta)|, |\Phi(\zeta, z)| \gtrsim \delta(z) + |s_1| + |s_2| + |t|^2, \quad |\zeta - z| \gtrsim |(s_2, t)|.$$

Here $\delta(z) := \text{dist}(z, bD)$ and in the following computation we shall just write δ . We define $s_1(\zeta) = \rho(\zeta)$ and $s_2(\zeta) = \text{Im } \Phi(z, \zeta)$. Recall that $\Phi(z, \zeta) = F(z, \zeta) - \rho(\zeta)$ when z, ζ are close and

$$F(z, \zeta) = \sum_{j=1}^n \frac{\partial \rho}{\partial \zeta_j}(\zeta) (\zeta_j - z_j) + O(|\zeta - z|^2).$$

Hence at $\zeta = z$, we have

$$\begin{aligned} d_\zeta \text{Im } \Phi(z, \zeta) \wedge d_\zeta \rho(\zeta) &= d_\zeta \text{Im } F(z, \zeta) \wedge d_\zeta \rho(\zeta) \\ &= \frac{1}{2\sqrt{-1}} (\partial_\zeta \rho - \bar{\partial}_\zeta \rho) \wedge (\partial_\zeta \rho + \bar{\partial}_\zeta \rho) \\ &= \frac{1}{\sqrt{-1}} \partial \rho \wedge \bar{\partial} \rho \neq 0. \end{aligned}$$

We can then find smooth real-valued functions t_j , $1 \leq j \leq 2n - 2$, with $t_j(\zeta) = 0$ at $\zeta = z$ and

$$d_\zeta \rho(\zeta) \wedge d_\zeta \text{Im } \Phi(z, \zeta) \wedge dt_1(\zeta) \wedge \cdots \wedge dt_{2n-2}(\zeta) \neq 0 \quad \text{at } \zeta = z.$$

By the inverse function theorem, $\phi_z = (s_1, s_2, t)$ defines a C^1 coordinate map in small neighborhood of z .

To prove the first statement in (3-27), we use estimate (3-7) which says that $\operatorname{Re} \Phi(z, \zeta) \gtrsim -\rho(\zeta) - \rho(z) + |\zeta - z|^2$, for all $z, \zeta \in D$. It follows that

$$|\Phi(z, \zeta)| \gtrsim |\operatorname{Re} \Phi(z, \zeta)| + |\operatorname{Im} \Phi(z, \zeta)| \gtrsim \delta(z) + |s_1(\zeta)| + |s_2(\zeta)| + |t(\zeta)|^2.$$

For $\Phi(\zeta, z)$ the argument goes the same: We note that $\Phi(\zeta, z) = F(\zeta, z) - \rho(z)$ when z, ζ are close, and

$$F(\zeta, z) = \sum_{j=1}^n \frac{\partial \rho}{\partial z_j}(z)(z_j - \zeta_j) + O(|\zeta - z|^2).$$

Thus at $\zeta = z$,

$$\begin{aligned} d_\zeta \operatorname{Im} \Phi(\zeta, z) \wedge d_\zeta \rho(\zeta) \Big|_{\zeta=z} &= d_\zeta \operatorname{Im} F(\zeta, z) \wedge d_z \rho(z) \\ &= \frac{1}{2\sqrt{-1}} (\bar{\partial}_z \rho(z) - \partial_z \rho(z)) \wedge (\partial_z \rho(z) + \bar{\partial}_z \rho(z)) \\ &= \frac{1}{\sqrt{-1}} \bar{\partial} \rho \wedge \partial \rho \\ &\neq 0. \end{aligned}$$

The second statement in (3-27) follows from the fact that $s_2(z) = t(z) = 0$. Now, both $\Phi(z, \zeta)$ and $|\zeta - z|$ are bounded below by some positive constant for $\zeta \notin \mathcal{U}_z$. Hence using partition of unity in ζ space, we can bound the integral on the left-hand side of (3-25) by a constant times

$$\int_0^1 \int_0^1 \int_0^1 \frac{t^{2n-3} ds_1 ds_2 dt}{(s_2 + t)^{2-\beta} (\delta + s_1 + s_2 + t^2)^{n+1}} \lesssim \int_0^1 \int_0^1 \frac{rt^{2n-5+\beta} dr dt}{(\delta + r + t^2)^{n+1}} := I,$$

where we used the polar coordinates for (s_1, s_2) with $r = |s|$. We can estimate the integral I by separating into different cases.

Case 1: $\delta > r, t^2$.

$$\begin{aligned} I &\leq \delta^{-(n+1)} \left(\int_0^\delta r dr \right) \left(\int_0^{\sqrt{\delta}} t^{2n-5+\beta} dt \right) \\ &\lesssim 1 + \delta^{-n-1+2+(2n-4+\beta)/2} \\ &= 1 + \delta^{-1+\beta/2}. \end{aligned}$$

Case 2: $r > \delta, t^2$.

$$\begin{aligned} I &\leq \int_\delta^1 r^{-n} \left(\int_0^{\sqrt{r}} t^{2n-5+\beta} dt \right) dr \\ &\lesssim \int_\delta^1 r^{-n+(2n-4+\beta)/2} dr \\ &\lesssim 1 + \delta^{-1+\beta/2}. \end{aligned}$$

Case 3: $t^2 > \delta, r$.

$$\begin{aligned}
I &\leq \int_{\sqrt{\delta}}^1 \left(\int_0^{t^2} r \, dr \right) t^{2n-5+\beta-2n-2} \, dt \\
&\lesssim \int_{\sqrt{\delta}}^1 t^{\beta-3} \, dt \\
&\lesssim 1 + \delta^{-1+\beta/2}.
\end{aligned}$$

Combining the estimates we obtain (3-25).

(ii) Since $|\Theta(z, \zeta)| \gtrsim |z - \zeta|^2$, the integral is bounded by

$$\begin{aligned}
\int_{B_\tau(z)} \frac{|z - \zeta|^\beta}{|\Theta(z, \zeta)|^{n+1}} dV(\zeta) &\leq \int_{B_\tau(z)} \frac{dV(\zeta)}{|z - \zeta|^{2-\beta} |\Theta(z, \zeta)|^n} \\
&\lesssim \int_0^1 \int_0^1 \int_0^1 \frac{t^{2n-3} \, ds_1 \, ds_2 \, dt}{(s_2 + t)^{2-\beta} (\delta + s_1 + s_2 + t^2)^n} \\
&\lesssim \int_{s_1=0}^1 \int_{s_2=0}^\tau \int_{t=0}^\tau \frac{t^{2n-5+\beta} \, ds_1 \, ds_2 \, dt}{(s_1 + s_2 + t^2)^n} \\
&:= I.
\end{aligned}$$

Here we used the fact that $|\zeta - z| \gtrsim (s_2, t)$ and thus $\zeta \in B_\tau(z)$ implies $|s_2|, |t| < \tau$.

We consider several cases.

Case 1: $s_1 > \tau$. The integral is bounded by

$$I \leq \int_\tau^1 \frac{ds_1}{s_1^n} \int_0^\tau ds_2 \int_0^\tau t^{2n-5+\beta} \, dt \lesssim \tau^{-n+1+1+2n-4+\beta} = \tau^{n-2+\beta} \lesssim \tau^\beta.$$

Case 2: $s_1 < \tau$. Then we have $|s| < r$, for $s = (s_1, s_2)$. Divide further into subcases.

If $t^2 > s$, then

$$I \lesssim \int_0^\tau \int_0^\tau \frac{s t^{2n-5+\beta} \, ds \, dt}{(s + t^2)^n} \leq \int_0^\tau \left(\int_0^{t^2} s \, ds \right) t^{2n-5+\beta-2n} \, dt \lesssim \int_0^\tau t^{\beta-1} \, dt \lesssim \tau^\beta.$$

On the other hand, if $t^2 < s$, then

$$\begin{aligned}
I &\lesssim \int_0^\tau \left(\int_0^{\sqrt{s}} t^{2n-3-2+\beta} \, dt \right) \frac{s}{s^n} \\
&\lesssim \int_0^\tau s^{\frac{2n-4+\beta}{2}-n+1} \, ds \\
&\lesssim \int_0^\tau s^{\beta/2-1} \, ds \\
&\lesssim \tau^{\beta/2}.
\end{aligned}$$

□

From the proof of Lemma 3.2, we see that for fixed ζ , we can find a neighborhood U_ζ of ζ and a coordinate chart $\phi_\zeta : U_\zeta \rightarrow \mathbb{R}^{2n}$ with $\phi_\zeta(z) = (s'_1, s'_2, t') \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{2n-2}$.

Indeed, we can set $s'_1(z) = \rho(z)$ and $s'_2(z) = \text{Im } \Phi(z, \zeta)$. At $z = \zeta$,

$$\begin{aligned} d_z \text{Im } \Phi(z, \zeta) \wedge d_z \rho(z) &= d_z \text{Im } F(z, \zeta) \wedge d_z \rho(z) \\ &= \frac{1}{2\sqrt{-1}} (\bar{\partial}_\zeta \rho(\zeta) - \partial_\zeta \rho(\zeta)) \wedge (\partial_\zeta \rho(\zeta) + \bar{\partial}_\zeta \rho(\zeta)) \\ &= \frac{1}{\sqrt{-1}} \bar{\partial} \rho(\zeta) \wedge \partial \rho(\zeta) \\ &\neq 0. \end{aligned}$$

Hence there exists smooth real-valued functions t'_j , $1 \leq j \leq 2n - 2$ with $t'_j(z) = 0$ and

$$d_z \rho(z) \wedge d_z \text{Im } \Phi(z, \zeta) \wedge dt'_1(\zeta) \wedge \cdots \wedge dt'_{2n-2}(\zeta) \neq 0 \quad \text{at } z = \zeta.$$

Consequently (s'_1, s'_2, t') is the desired coordinate chart in the z variable. Now by the same estimate as in the proof of Lemma 3.2, we can prove the following:

Lemma 3.3. *Keeping the assumptions of Lemma 3.2:*

(i) *Let $0 < \beta \leq 1$. Then*

$$(3-28) \quad \int_D \frac{dV(z)}{|\zeta - z|^{2-\beta} |\Theta(z, \zeta)|^{n+1}} \lesssim 1 + \delta(\zeta)^{\beta/2-1}, \quad \delta(\zeta) := \text{dist}(\zeta, bD),$$

where the constant depends only on D .

(ii) *Let $\beta > 0$, and denote by $B_\tau(z)$ the ball of radius τ centered at z . Then*

$$(3-29) \quad \int_{B_\tau(z)} \frac{|z - \zeta|^\beta}{|\Theta(z, \zeta)|^{n+1}} dV(z) \lesssim \tau^{\beta/2},$$

where the constant depends only on D .

Lemma 3.4. *Let D be a bounded strictly pseudoconvex domain with C^3 boundary in \mathbb{C}^n , $n \geq 2$, and let ρ be its defining function. Let $\Theta(z, \zeta)$ denote either $\Phi(z, \zeta)$ or $\Phi(\zeta, z)$. Denote $\delta(z) := \text{dist}(z, bD)$:*

(i) *For $z \in D$,*

$$(3-30) \quad \int_D \frac{dV(\zeta)}{|\Theta(z, \zeta)|^{n+1}} \lesssim 1 + \log \delta(z),$$

where the constant depends only on D .

(ii) *For $z \in D$,*

$$(3-31) \quad \int_{bD} \frac{d\sigma(\zeta)}{|\Theta(z, \zeta)|^n} \lesssim 1 + \log \delta(z),$$

where the constant depends only on D .

Proof. (i) In the proof we shall write $\delta(z)$ simply as δ . For fixed $z \in D$, let $\zeta \mapsto (s_1, s_2, t)$ be the coordinate chart in a neighborhood U_z of z as constructed in the proof of Lemma 3.2. Let χ_0 be a smooth cut-off function such that $\text{supp } \chi_0 \subset E_0(z) := \{\zeta \in D : -\rho(\zeta) - \rho(z) + |z - \zeta| \leq \sigma\}$ and $\chi_0 \equiv 1$ on the set $E_1(z) := \{\zeta \in D : -\rho(\zeta) - \rho(z) + |z - \zeta| \leq \frac{\sigma}{2}\}$. We choose σ sufficiently small such that $E_0(z) \subset U_z$. Then

$$\int_D \frac{dV(\zeta)}{|\Theta(z, \zeta)|^{n+1}} = \int_{D \cap E_0} \frac{\chi_0(\zeta) dV(\zeta)}{|\Theta(z, \zeta)|^{n+1}} + \int_{D \setminus E_1} \frac{(1 - \chi_0(\zeta)) dV(\zeta)}{|\Theta(z, \zeta)|^{n+1}}.$$

In view of (3-7), the second integral is bounded by a constant independent of $z \in D$.

The first integral is bounded by

$$\begin{aligned} \int_{D \cap E_0(z)} \frac{dV(\zeta)}{|\Theta(z, \zeta)|^{n+1}} &\lesssim \int_0^1 \int_0^1 \int_0^1 \frac{t^{2n-3} ds_1 ds_2 dt}{(\delta + s_1 + s_2 + t^2)^{n+1}} \\ &\lesssim \int_0^1 \int_0^1 \frac{t^{2n-3} r dr dt}{(\delta + r + t^2)^{n+1}} \\ &:= I, \end{aligned}$$

where we used the polar coordinates for $r = (s_1, s_2)$. We split the integral into the following cases:

Case 1: $\delta + r \geq t^2$.

$$\begin{aligned} I &\lesssim \int_0^1 \frac{r}{(\delta + r)^{n+1}} \left(\int_0^{\sqrt{\delta+r}} t^{2n-3} dt \right) dr \\ &\lesssim \int_0^1 (\delta + r)^{1-n-1+(2n-2)/2} dr \\ &= \int_0^1 (\delta + r)^{-1} dr \lesssim 1 + \log \delta. \end{aligned}$$

Case 2: $\delta + r \leq t^2$.

$$I \lesssim \int_0^1 r \left(\int_{\sqrt{\delta+r}}^1 \frac{t^{2n-3} dt}{t^{2n+2}} \right) dr \lesssim \int_0^1 (\delta + r)^{-1} dr \lesssim 1 + \log \delta.$$

(ii) Since $s_1(\zeta) = \rho(\zeta) \equiv 0$ for $\zeta \in bD$, for fixed z , there exists some neighborhood U_z of z such that $\zeta \mapsto (s_2, t)$ is a coordinate chart for $\zeta \in bD \cap U_z$. Let χ_0, E_0 be the same as in the proof of (i). We only have to estimate

$$\int_{bD \cap E_0} \frac{\chi_0 d\sigma(\zeta)}{|\Theta(z, \zeta)|^n} \lesssim \int_0^1 \int_0^1 \int_0^1 \frac{t^{2n-3} ds_2 dt}{(\delta + s_2 + t^2)^n} := I.$$

Split the integral into two cases.

Case 1: $\delta + s_2 \geq t^2$.

$$\begin{aligned}
 I &\lesssim \int_0^1 \frac{1}{(\delta + s_2)^n} \left(\int_0^{\sqrt{\delta+s_2}} t^{2n-3} dt \right) ds_2 \\
 &\lesssim \int_0^1 (\delta + s_2)^{-1} ds_2 \\
 &\lesssim 1 + \log \delta.
 \end{aligned}$$

Case 2: $\delta + s_2 \leq t^2$. $I \lesssim \int_0^1 \left(\int_{\sqrt{\delta+s_2}}^1 \frac{t^{2n-3} dt}{t^{2n}} \right) ds_2 \lesssim \int_0^1 (\delta+s_2)^{-1} ds_2 \lesssim 1 + \log \delta.$

□

We now prove the L^2 boundedness of the operator \mathcal{K} , assuming boundary is only C^3 . This result is stated in [Ligocka 1984] assuming the boundary is C^4 , and the proof over there uses a much more general estimate from [Krantz 1976]. We shall instead give a direct proof here.

Proposition 3.5. *Let D be a strictly pseudoconvex domain in \mathbb{C}^n with C^3 boundary, and let \mathcal{K} be the operator given by formula (3-23). Then \mathcal{K} defines a bounded operator from $L^2(D)$ to $L^2(D)$.*

Proof. We shall apply Schur’s test (see for example [Wolff 2003]), which in our case can be formulated as follows. If

$$(3-32) \quad \int_D |K(z, \zeta)| dV(\zeta) \leq A, \quad \text{for each } z,$$

and

$$(3-33) \quad \int_D |K(z, \zeta)| dV(z) \leq B, \quad \text{for each } \zeta,$$

then for $f \in L^2(D)$, \mathcal{K}_f defined by the integral $\int_D K(z, \zeta) f(\zeta) dV(\zeta)$ converges a.e. and there is an estimate

$$\|\mathcal{K}f\|_{L^2(D)} \leq \sqrt{AB} \|f\|_{L^2(D)}.$$

Hence it suffices to prove (3-32) and (3-33). We can write

$$\begin{aligned}
 \int_D |K(z, \zeta)| dV(\zeta) &= \int_D \left| \frac{\eta(z) + O''(|z - \zeta|)}{\bar{\Phi}^{n+1}(\zeta, z)} - \frac{\eta(\zeta) + O'(|z - \zeta|)}{\Phi^{n+1}(z, \zeta)} \right| dV(\zeta) \\
 &\leq J_1 + J_2 + J_3 + J_4,
 \end{aligned}$$

where we denote

$$\begin{aligned}
 J_1 &= \int_D \frac{|\eta(z) - \eta(\zeta)|}{|\Phi(\zeta, z)|^{n+1}} dV(\zeta), & J_2 &= \int_D \eta(\zeta) \left| \frac{1}{\bar{\Phi}^{n+1}(\zeta, z)} - \frac{1}{\Phi^{n+1}(z, \zeta)} \right| dV(\zeta), \\
 J_3 &= \int_D \frac{|O''(|z - \zeta|)|}{|\Phi(\zeta, z)|^{n+1}} dV(\zeta), & J_4 &= \int_D \frac{|O'(|z - \zeta|)|}{|\Phi(z, \zeta)|^{n+1}} dV(\zeta).
 \end{aligned}$$

By the expression for η (3-17), we have $|\eta(z) - \eta(\zeta)| \lesssim |\rho|_3 |\zeta - z|$. We have

$$J_1 \lesssim |\rho|_3 \int_D \frac{|\zeta - z|}{|\Phi(\zeta, z)|^{n+1}} dV(\zeta) \lesssim |\rho|_3,$$

where we applied estimate (3-26) in the last inequality. By estimates (3-9), (3-24) and (3-26), we have

$$\begin{aligned} J_2 &\lesssim |\rho|_2 \int_D \frac{|\Phi^{n+1}(z, \zeta) - \bar{\Phi}^{n+1}(\zeta, z)|}{|\Phi(\zeta, z)|^{n+1} |\Phi^{n+1}(z, \zeta)|} dV(\zeta) \\ &\lesssim |\rho|_2 \int_D |\Phi(z, \zeta) - \overline{\Phi(\zeta, z)}| \left(\frac{|\Phi(z, \zeta)|^n + |\Phi(\zeta, z)|^n}{|\Phi(\zeta, z)|^{n+1} |\Phi^{n+1}(z, \zeta)|} \right) dV(\zeta) \\ &\lesssim |\rho|_3 \left(\int_D \frac{|\zeta - z|^3}{|\Phi(\zeta, z)|^{n+1} |\Phi(z, \zeta)|} dV(\zeta) + \int_D \frac{|\zeta - z|^3}{|\Phi(z, \zeta)|^{n+1} |\Phi(\zeta, z)|} dV(\zeta) \right) \\ &\lesssim |\rho|_3 \left(\int_D \frac{|\zeta - z|}{|\Phi(\zeta, z)|^{n+1}} dV(\zeta) + \int_D \frac{|\zeta - z|}{|\Phi(z, \zeta)|^{n+1}} dV(\zeta) \right) \\ &\lesssim |\rho|_3. \end{aligned}$$

For J_3 , we use estimates (3-18), (3-21) and (3-26):

$$\begin{aligned} J_3 &\lesssim |\rho|_3 \int_D \frac{|\zeta - z|}{|\Phi(\zeta, z)|^{n+1}} dV(\zeta) \lesssim |\rho|_3, \\ J_4 &\lesssim |\rho|_3 \int_D \frac{|\zeta - z|}{|\Phi(z, \zeta)|^{n+1}} dV(\zeta) \lesssim |\rho|_3. \end{aligned}$$

Here we note that all the bounds are uniform in $z \in D$. Hence we have proved (3-33). In a similar way by using estimate (3-29), we can prove (3-32). The proof is now complete. \square

By using Proposition 3.5 and the same argument in [Ligocka 1984], we obtain

Proposition 3.6. *Let D be a strictly pseudoconvex domain in \mathbb{C}^n with C^3 boundary, and let \mathcal{L}, \mathcal{K} be the operators given by formula (3-19) (3-23), respectively. Then the following statements are true:*

- (1) \mathcal{L} is a bounded projection from $L^2(D)$ to $H^2(D)$. In particular, \mathcal{L} is the identity map on $H^2(D)$.
- (2) $\mathcal{P} = \mathcal{L}(I - \mathcal{K})^{-1} = (I + \mathcal{K})^{-1} \mathcal{L}^*$.

It is important to note that unlike the Bergman projection, \mathcal{L} is not an orthogonal projection, namely, $\mathcal{L}g - g$ is not orthogonal to the Bergman space $H^2(D)$.

Lemma 3.7. *Let D be a strictly pseudoconvex domain with C^3 boundary, and let $D_\delta := \{z \in \mathbb{C}^n : \rho(z) < 0\}$:*

(i) *For all $(z, \zeta) \in D_\delta \times D_\delta$ with $|z - \zeta|$ sufficiently small,*

$$(3-34) \quad \left| \sum_{i=1}^n \frac{\partial \Phi(z, \zeta)}{\partial \bar{\zeta}_i} \cdot \frac{\partial \rho}{\partial \zeta_i} \right| > c > 0.$$

(ii) *For each $\zeta_0 \in bD$, there exists a neighborhood $U(\zeta_0)$ and an index $1 \leq j \leq n$ such that $\left| \frac{\partial \rho}{\partial \bar{\zeta}_j}(\zeta) \right| > c > 0$ for all $\zeta \in U(\zeta_0)$. In addition,*

$$(3-35) \quad \frac{\partial \Phi(z, \zeta)}{\partial \zeta_j} \frac{\partial \rho}{\partial \bar{\zeta}_j} - \frac{\partial \Phi(z, \zeta)}{\partial \bar{\zeta}_j} \frac{\partial \rho}{\partial \zeta_j} > c' > 0, \quad \forall (z, \zeta) \in U(\zeta_0) \times U(\zeta_0).$$

(iii) *For all $(z, \zeta) \in D_\delta \times D_\delta$ with $|z - \zeta|$ sufficiently small,*

$$(3-36) \quad \left| \sum_{i=1}^n \frac{\partial \overline{\Phi(\zeta, z)}}{\partial \bar{\zeta}_i} \cdot \frac{\partial \rho}{\partial \zeta_i} \right| > c > 0.$$

(iv) *For each $\zeta_0 \in bD$, there exists a neighborhood $U(\zeta_0)$ and an index $1 \leq j \leq n$ such that $\left| \frac{\partial \rho}{\partial \bar{\zeta}_j}(\zeta) \right| > c > 0$ for all $\zeta \in U(\zeta_0)$. In addition,*

$$(3-37) \quad \frac{\partial \overline{\Phi(\zeta, z)}}{\partial \zeta_j} \frac{\partial \rho}{\partial \bar{\zeta}_j} - \frac{\partial \overline{\Phi(\zeta, z)}}{\partial \bar{\zeta}_j} \frac{\partial \rho}{\partial \zeta_j} > c' > 0, \quad \forall (z, \zeta) \in U(\zeta_0) \times U(\zeta_0).$$

Proof. (i) Compute

$$(3-38) \quad \begin{aligned} \frac{\partial}{\partial \bar{\zeta}_i} [F(z, \zeta) - \rho(\zeta)] &= \frac{\partial}{\partial \bar{\zeta}_i} \left(\sum_{j=1}^n \frac{\partial \rho}{\partial \zeta_j}(\zeta) (\zeta_j - z_j) \right. \\ &\quad \left. - \frac{1}{2} \sum_{j,k=1}^n \frac{\partial^2 \rho}{\partial \zeta_j \partial \bar{\zeta}_k}(\zeta) (z_j - \zeta_j)(z_k - \zeta_k) - \rho(\zeta) \right) \\ &= -\frac{\partial \rho}{\partial \bar{\zeta}_i}(\zeta) + O(|\zeta - z|). \end{aligned}$$

Estimate (3-34) then follows for $|z - \zeta|$ small since $|\nabla \rho(\zeta)| > 0$.

(ii) Since $d\rho(\zeta_0) \neq 0$, there exists some neighborhood $U(\zeta_0)$ and an index i_0 such that $|\frac{\partial \rho}{\partial \bar{\zeta}_{i_0}}(\zeta)| \geq c > 0$ for all $\zeta \in U(\zeta_0)$. We compute

$$\begin{aligned}
 (3-39) \quad \frac{\partial}{\partial \zeta_{i_0}} [F(z, \zeta) - \rho(\zeta)] &= \frac{\partial}{\partial \zeta_{i_0}} \left(\sum_{j=1}^n \frac{\partial \rho}{\partial \zeta_j}(\zeta) (\zeta_j - z_j) \right. \\
 &\quad \left. - \frac{1}{2} \sum_{j,k=1}^n \frac{\partial^2 \rho}{\partial \zeta_j \partial \zeta_k}(\zeta) (z_j - \zeta_j)(z_k - \zeta_k) - \rho(\zeta) \right) \\
 &= \frac{\partial \rho}{\partial \zeta_{i_0}}(\zeta) - \frac{\partial \rho}{\partial \zeta_{i_0}}(\zeta) + O(|\zeta - z|) \\
 &= O(|\zeta - z|).
 \end{aligned}$$

It follows from (3-38) and (3-39) that

$$\frac{\partial [F(z, \zeta) - \rho(\zeta)]}{\partial \zeta_{i_0}} \frac{\partial \rho}{\partial \bar{\zeta}_{i_0}} - \frac{\partial [F(z, \zeta) - \rho(\zeta)]}{\partial \bar{\zeta}_{i_0}} \frac{\partial \rho}{\partial \zeta_{i_0}} = \left| \frac{\partial \rho}{\partial \zeta_{i_0}}(\zeta) \right|^2 + O(|\zeta - z|).$$

Estimate (3-35) then follows if $U(\zeta_0)$ is chosen sufficiently small.

(iii) The proof follows similarly by the fact

$$\begin{aligned}
 (3-40) \quad \frac{\partial}{\partial \bar{\zeta}_i} [\overline{F(\zeta, z)} - \rho(z)] &= \frac{\partial}{\partial \bar{\zeta}_i} \left(\sum_{j=1}^n \frac{\partial \rho}{\partial \bar{z}_j}(z) \overline{(z_j - \zeta_j)} \right. \\
 &\quad \left. - \sum_{j,k=1}^n \frac{\partial^2 \rho}{\partial \bar{z}_j \partial \bar{z}_k}(z) \overline{(\zeta_j - z_j)(\zeta_k - z_k)} - \rho(z) \right) \\
 &= -\frac{\partial \rho}{\partial \bar{z}_i}(z) + O(|\zeta - z|) \\
 &= -\frac{\partial \rho}{\partial \bar{\zeta}_i}(\zeta) + O(|\zeta - z|),
 \end{aligned}$$

where in the last equality we used that $|D\rho(z) - D\rho(\zeta)| \lesssim |\rho|_2 |\zeta - z|$.

(iv) Since $d\rho(\zeta_0) \neq 0$, there exists some neighborhood $U(\zeta_0)$ and an index i_0 such that $|\frac{\partial \rho}{\partial \zeta_{i_0}}(\zeta)| \geq c > 0$ for all $\zeta \in U(\zeta_0)$. Compute

$$\begin{aligned}
 (3-41) \quad \frac{\partial}{\partial \zeta_{i_0}} [\overline{F(\zeta, z)} - \rho(z)] &= \frac{\partial}{\partial \zeta_{i_0}} \left(\sum_{j=1}^n \frac{\partial \rho}{\partial \bar{z}_j}(z) \overline{(z_j - \zeta_j)} \right. \\
 &\quad \left. - \frac{1}{2} \sum_{j,k=1}^n \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(z) \overline{(\zeta_j - z_j)(\zeta_k - z_k)} - \rho(z) \right) \\
 &= 0.
 \end{aligned}$$

It follows from (3-40) and (3-41) that

$$\frac{\partial[\overline{F(\zeta, z)} - \rho(z)]}{\partial \zeta_{i_0}} \frac{\partial \rho}{\partial \bar{\zeta}_{i_0}} - \frac{\partial[\overline{F(\zeta, z)} - \rho(z)]}{\partial \bar{\zeta}_{i_0}} \frac{\partial \rho}{\partial \zeta_{i_0}} = \left| \frac{\partial \rho}{\partial \zeta_{i_0}}(\zeta) \right|^2 + O(|\zeta - z|).$$

Hence estimate (3-37) holds by choosing $U(\zeta_0)$ sufficiently small. \square

Lemma 3.8. *Let D be a bounded strictly pseudoconvex domain with a C^3 defining function ρ , and let $F(z, \zeta)$ be given by (3-1):*

(i) *For each $1 \leq i \leq n$, the following holds for $(z, \zeta) \in D \times D$,*

$$\frac{\partial[F(z, \zeta) - \rho(\zeta)]}{\partial z_i} = -\frac{\partial \rho}{\partial \zeta_i}(\zeta) + O(|\zeta - z|), \quad \frac{\partial[F(z, \zeta) - \rho(\zeta)]}{\partial \bar{z}_i} = O(|\zeta - z|),$$

where $|O(|\zeta - z|)| \lesssim |\rho|_2 |\zeta - z|$.

(ii) *For each $1 \leq i \leq n$, the following holds for $(z, \zeta) \in D_\delta \times D_\delta$,*

$$\frac{\partial[\overline{F(\zeta, z)} - \rho(z)]}{\partial z_i} = -\frac{\partial \rho}{\partial z_i}(z) + O(|\zeta - z|), \quad \frac{\partial[\overline{F(\zeta, z)} - \rho(z)]}{\partial \bar{z}_i} = O(|\zeta - z|),$$

where $|O(|\zeta - z|)| \lesssim |\rho|_3 |\zeta - z|$.

Proof. (i) Using definition of F , we have

$$\begin{aligned} \frac{\partial[F(z, \zeta) - \rho(\zeta)]}{\partial z_i} &= \frac{\partial}{\partial z_i} \left(\sum_{j=1}^n \frac{\partial \rho}{\partial \zeta_j}(\zeta) (\zeta_j - z_j) \right. \\ &\quad \left. - \frac{1}{2} \sum_{j,k=1}^n \frac{\partial^2 \rho}{\partial \zeta_j \partial \zeta_k}(\zeta) (z_j - \zeta_j)(z_k - \zeta_k) - \rho(\zeta) \right) \\ &= -\frac{\partial \rho}{\partial \zeta_i}(\zeta) + O(|\zeta - z|), \end{aligned}$$

$|O(|\zeta - z|)| \lesssim |\rho|_2 |\zeta - z|$, and $\partial[F(z, \zeta) - \rho(\zeta)]/\partial \bar{z}_i = O(|\zeta - z|)$.

$$\begin{aligned} \text{(ii)} \quad \frac{\partial[\overline{F(\zeta, z)} - \rho(z)]}{\partial z_i} &= \frac{\partial}{\partial z_i} \left(\sum_{j=1}^n \frac{\partial \rho}{\partial \bar{z}_j}(z) \overline{(z_j - \zeta_j)} \right. \\ &\quad \left. - \frac{1}{2} \sum_{j,k=1}^n \frac{\partial^2 \rho}{\partial z_j \partial z_k}(z) \overline{(\zeta_j - z_j)(\zeta_k - z_k)} - \rho(z) \right) \\ &= -\frac{\partial \rho}{\partial z_i}(z) + O(|\zeta - z|), \quad |O(|\zeta - z|)| \lesssim |\rho|_3 |\zeta - z|, \end{aligned}$$

and

$$\frac{\partial[\overline{F(\zeta, z)} - \rho(z)]}{\partial \bar{z}_i} = \frac{\partial \rho}{\partial \bar{z}_i} - \frac{\partial \rho}{\partial \bar{z}_i} + O(|\zeta - z|) = O(|\zeta - z|). \quad \square$$

We use the notation:

$$(3-42) \quad Q'(z, \zeta) := \sum_{i=1}^n \frac{\partial \Phi(z, \zeta)}{\partial \bar{\zeta}_i} \frac{\partial \rho}{\partial \zeta_i}, \quad Q''(\zeta, z) := \sum_{i=1}^n \frac{\partial \overline{\Phi(\zeta, z)}}{\partial \bar{\zeta}_i} \cdot \frac{\partial \rho}{\partial \zeta_i},$$

and we write

$$[d\bar{\zeta}]_i = d\bar{\zeta}_1 \wedge \cdots \wedge \widehat{(d\bar{\zeta}_i)} \wedge \cdots \wedge d\bar{\zeta}_n; \quad [d\zeta]_i = d\zeta_1 \wedge \cdots \wedge \widehat{(d\zeta_i)} \wedge \cdots \wedge d\zeta_n.$$

Lemma 3.9. *For all $(z, \zeta) \in D_\delta \times D_\delta$ with $|z - \zeta|$ sufficiently small, the following estimates hold:*

- (i) $|D_z \Phi(z, \zeta) - D_z \overline{\Phi(\zeta, z)}| \lesssim |\rho|_3 |\zeta - z|.$
- (ii) $|D_\zeta \Phi(z, \zeta) - D_\zeta \overline{\Phi(\zeta, z)}| \lesssim |\rho|_3 |\zeta - z|.$
- (iii) $|Q'(z, \zeta) - Q''(\zeta, z)| \lesssim |\rho|_3 |\zeta - z|.$

Proof. This follows immediately from the proof of Lemma 3.7 and Lemma 3.8. \square

We now prove the key integration by parts lemma. This technique was originated by Elgueta [1980] and has been developed and used by Ahern and Schneider [1979], Ligocka [1984], Lieb and Range [1980], and Gong [2019], among others. For our proof we shall mainly follow [Ahern and Schneider 1979]. We mention that integration by parts is not needed for our results with $C^{3+\alpha}$ boundary, and that in the subsequent proof the following lemma will only be applied to domains with $C^{k+3+\alpha}$ boundary, $k \geq 1$.

Lemma 3.10. *Let D be a bounded strictly pseudoconvex domain in \mathbb{C}^n with C^4 boundary. Suppose $u \in C^1(\bar{D})$ and the support of u is contained in some small neighborhood of z . Then the following integration by parts formulae hold:*

- (i) *Here P' is a first order differential operator in ζ variable (see (3-51)):*

$$(3-43) \quad \int_D \frac{u(\zeta) dV(\zeta)}{\Phi^{m+1}(z, \zeta)} = c'_1 \int_{bD} \frac{P'(u)(\zeta) d\sigma(\zeta)}{\Phi^{m-1}(z, \zeta)} + c'_2 \int_D \sum_{i=1}^n \frac{\partial}{\partial \bar{\zeta}_i} \left(\frac{u(\zeta) \frac{\partial \rho}{\partial \zeta_i}(\zeta)}{Q'(z, \zeta)} \right) \frac{dV(\zeta)}{\Phi^m(z, \zeta)}.$$

- (ii) *Here P'' is a first order differential operator in ζ (see (3-55)):*

$$(3-44) \quad \int_D \frac{u(\zeta) dV(\zeta)}{\overline{\Phi}^{m+1}(\zeta, z)} = c''_1 \int_{bD} \frac{P''(u)(\zeta) d\sigma(\zeta)}{\overline{\Phi}^{m-1}(\zeta, z)} + c''_2 \int_D \sum_{i=1}^n \frac{\partial}{\partial \bar{\zeta}_i} \left(\frac{u(\zeta) \frac{\partial \rho}{\partial \zeta_i}(\zeta)}{Q''(\zeta, z)} \right) \frac{dV(\zeta)}{\overline{\Phi}^m(\zeta, z)}.$$

(iii) Here P' , P'' are first order differential operators in ζ . The coefficients of P' (resp. P'') involve derivatives of ρ up to order 3 (resp. order 2):

$$(3-45) \quad \begin{aligned} \int_{bD} \frac{u(\zeta) d\sigma(\zeta)}{\Phi^m(z, \zeta)} &= \int_{bD} \frac{P'(u)(\zeta) d\sigma(\zeta)}{\Phi^{m-1}(z, \zeta)}, \\ \int_{bD} \frac{u(\zeta) d\sigma(\zeta)}{\Phi^m(\zeta, z)} &= \int_{bD} \frac{P''(u)(\zeta) d\sigma(\zeta)}{\Phi^{m-1}(\zeta, z)}. \end{aligned}$$

Proof. In view of (3-5) and (3-8), for each fixed $z \in D$ we have $\Phi(z, \cdot) \in C^1(\bar{D})$. By (3-42), and the assumption that $\rho \in C^3$, we see that $Q', Q'' \in C^1(\bar{D})$. Hence by Stokes' theorem,

$$(3-46) \quad \begin{aligned} \int_D \frac{u(\zeta)}{\Phi^{m+1}(z, \zeta)} dV(\zeta) \\ = -\frac{1}{m} \int_{bD} \frac{u(\zeta)}{Q'(z, \zeta) \Phi^m(z, \zeta)} \sum_{k=1}^n (-1)^{k-1} \frac{\partial \rho}{\partial \zeta_k} [d\bar{\zeta}]_k \wedge d\zeta \\ + \frac{1}{m} \int_D \sum_{k=1}^n \frac{\partial}{\partial \bar{\zeta}_k} \left(\frac{u(\zeta) \frac{\partial \rho}{\partial \zeta_k}(\zeta)}{Q'(z, \zeta)} \right) \frac{1}{\Phi^m(z, \zeta)} dV(\zeta). \end{aligned}$$

To finish the proof we need to apply Stokes' theorem again to the boundary integral. We have on bD ,

$$(3-47) \quad d\rho(\zeta) = \sum_{l=1}^n \left(\frac{\partial \rho}{\partial \zeta_l} d\zeta_l + \frac{\partial \rho}{\partial \bar{\zeta}_l} d\bar{\zeta}_l \right) \equiv 0.$$

Let $\{\chi_\nu\}_{\nu=1}^M$ be a partition of unity of bD subordinate to the cover $\{U_\nu\}_{\nu=1}^M$. We can assume that on U_ν , there exists an index $i = i(\nu)$ such that $\frac{\partial \rho}{\partial \zeta_{i(\nu)}}(\zeta) \neq 0$. By (3-47), we have for $\zeta \in U_\nu \cap bD$:

$$(3-48) \quad \begin{aligned} d_\zeta(\Phi^{-(m-1)}[d\zeta]_i \wedge [d\bar{\zeta}]_i) \\ = -(m-1)\Phi^{-m} \left(\sum_{l=1}^n \frac{\partial \Phi(z, \zeta)}{\partial \zeta_l} d\zeta_l + \frac{\partial \Phi(z, \zeta)}{\partial \bar{\zeta}_l} d\bar{\zeta}_l \right) \wedge [d\zeta]_i \wedge [d\bar{\zeta}]_i \\ = -(m-1)\Phi^{-m} \left[\frac{\partial \Phi(z, \zeta)}{\partial \zeta_i} - \frac{\partial \rho}{\partial \zeta_i} \left(\frac{\partial \rho}{\partial \bar{\zeta}_i} \right)^{-1} \frac{\partial \Phi(z, \zeta)}{\partial \bar{\zeta}_i} \right] d\zeta_i \wedge [d\zeta]_i \wedge [d\bar{\zeta}]_i \\ = -(m-1)\Phi^{-m} (-1)^{i-1} a_i(z, \zeta) d\zeta \wedge [d\bar{\zeta}]_i, \end{aligned}$$

where $i = i(\nu)$ and we set

$$(3-49) \quad a_i(z, \zeta) := \frac{\partial \Phi(z, \zeta)}{\partial \zeta_i} - \frac{\partial \rho}{\partial \zeta_i} \left(\frac{\partial \rho}{\partial \bar{\zeta}_i} \right)^{-1} \frac{\partial \Phi(z, \zeta)}{\partial \bar{\zeta}_i}, \quad \zeta \in U_\nu \cap bD.$$

By assumption, u is supported in a small neighborhood of z . Hence if for some ν , $\text{supp } u \cap U_\nu$ is nonempty, then z must be sufficiently close to U_ν . Hence in view of estimate (3-35) and by shrinking U_ν if necessary, we can assume that $a_i(z, \zeta) \geq c > 0$ for $\zeta \in \text{supp } u \cap U_\nu$. Accordingly,

$$\Phi^{-m} d\zeta \wedge [d\bar{\zeta}]_i = c_m \frac{(-1)^{i-1}}{a_i(z, \zeta)} d_\zeta (\Phi^{-(m-1)} [d\zeta]_i \wedge [d\bar{\zeta}]_i), \quad i = i(\nu), \zeta \in U_\nu \cap bD.$$

Now by (3-47) we can write

$$(3-50) \quad \sum_{k=1}^n (-1)^{k-1} \frac{\partial \rho}{\partial \zeta_k} d\zeta \wedge [d\bar{\zeta}]_k = \varphi_\nu(\zeta) d\zeta \wedge [d\bar{\zeta}]_{i(\nu)}, \quad \zeta \in U_\nu \cap bD,$$

where φ_ν is a linear combination of products of $\frac{\partial \rho}{\partial \zeta_s}$ and $\frac{\partial \rho}{\partial \bar{\zeta}_t}$. Hence for the boundary integral in (3-46) we have

$$\begin{aligned} \int_{bD} \frac{u(\zeta) \chi_\nu(\zeta)}{Q'(z, \zeta) \Phi^m(z, \zeta)} \sum_{k=1}^n (-1)^{k-1} \frac{\partial \rho}{\partial \zeta_k} d\zeta \wedge [d\bar{\zeta}]_k \\ = \int_{bD} \frac{u(\zeta) \chi_\nu(\zeta) \varphi_\nu(\zeta)}{Q'(z, \zeta) \Phi^m(z, \zeta)} d\zeta \wedge [d\bar{\zeta}]_{i(\nu)} \\ = \int_{bD} \frac{u(\zeta) \chi_\nu(\zeta) \varphi_\nu(\zeta)}{(Q' a_{i(\nu)})(z, \zeta)} d_\zeta (\Phi^{-(m-1)} [d\zeta]_{i(\nu)} \wedge [d\bar{\zeta}]_{i(\nu)}), \end{aligned}$$

where the constant is absorbed into φ_ν . By Stokes' theorem, the integral is equal to

$$\int_{bD} d_\zeta \left(\frac{u(\zeta) (\chi_\nu \varphi_\nu)(\zeta)}{(Q' a_{i(\nu)})(z, \zeta)} \right) \Phi^{-(m-1)} [d\zeta]_{i(\nu)} \wedge [d\bar{\zeta}]_{i(\nu)}.$$

Let ψ_ν be the function such that $[d\zeta]_{i(\nu)} \wedge [d\bar{\zeta}]_{i(\nu)} = \psi_\nu(\zeta) d\sigma(\zeta)$. Summing the above expression over ν , the boundary integral in (3-46) can be written as

$$(3-51) \quad \int_{bD} P'(u)(\zeta) \Phi^{-(m-1)} d\sigma(\zeta), \quad P'(u)(\zeta) := \sum_{\nu=1}^M d_\zeta \left(\frac{u(\zeta) (\chi_\nu \varphi_\nu)(\zeta)}{(Q' a_{i(\nu)})(z, \zeta)} \right) \psi_\nu(\zeta).$$

Hence we obtain formula (3-43). This completes the proof of (i).

The proof of (ii) goes similar. By Stokes' theorem we have

$$\begin{aligned} (3-52) \quad \int_D \frac{u(\zeta)}{\bar{\Phi}^{m+1}(\zeta, z)} dV(\zeta) \\ = -\frac{1}{m} \int_{bD} \frac{u(\zeta)}{Q''(\zeta, z) \bar{\Phi}^m(\zeta, z)} \sum_{k=1}^n (-1)^{k-1} \frac{\partial \rho}{\partial \bar{\zeta}_k} [d\bar{\zeta}]_k \wedge d\zeta \\ + \frac{1}{m} \int_D \sum_{k=1}^n \frac{\partial}{\partial \bar{\zeta}_k} \left(\frac{u(\zeta) \frac{\partial \rho}{\partial \zeta_k}(\zeta)}{Q''(\zeta, z)} \right) \frac{1}{\bar{\Phi}^m(\zeta, z)} dV(\zeta). \end{aligned}$$

Let χ_ν , U_ν and $i(\nu)$ be the same as in the proof of (i). By (3-47), we have for $\zeta \in U_\nu \cap bD$:

$$\begin{aligned}
 (3-53) \quad & d_\zeta \bar{\Phi}^{-(m-1)}(\zeta, z) [d\zeta]_i \wedge [d\bar{\zeta}]_i \\
 &= -m \bar{\Phi}^{-m}(\zeta, z) \left(\frac{\partial \bar{\Phi}(\zeta, z)}{\partial \zeta_i} d\zeta_i + \frac{\partial \bar{\Phi}(\zeta, z)}{\partial \bar{\zeta}_i} d\bar{\zeta}_i \right) \wedge [d\zeta]_i \wedge [d\bar{\zeta}]_i \\
 &= -m \bar{\Phi}^{-m}(\zeta, z) \left[\frac{\partial \bar{\Phi}(\zeta, z)}{\partial \zeta_i} - \frac{\partial \rho}{\partial \zeta_i} \left(\frac{\partial \rho}{\partial \bar{\zeta}_i} \right)^{-1} \frac{\partial \bar{\Phi}(\zeta, z)}{\partial \bar{\zeta}_i} \right] d\zeta_i \wedge [d\zeta]_i \wedge [d\bar{\zeta}]_i \\
 &= -m \bar{\Phi}^{-m}(\zeta, z) (-1)^{i-1} b_i(\zeta, z) d\zeta \wedge [d\bar{\zeta}]_i,
 \end{aligned}$$

where $i = i(\nu)$ and we set

$$(3-54) \quad b_i(\zeta, z) := \frac{\partial \bar{\Phi}(\zeta, z)}{\partial \zeta_i} - \frac{\partial \rho}{\partial \zeta_i} \left(\frac{\partial \rho}{\partial \bar{\zeta}_i} \right)^{-1} \frac{\partial \bar{\Phi}(\zeta, z)}{\partial \bar{\zeta}_i}, \quad \zeta \in U_\nu \cap bD.$$

Using estimate (3-37), we may assume that $b_i \geq c > 0$ for $\zeta \in \text{supp } u \cap U_\nu$. It follows that

$$\bar{\Phi}^{-m}(\zeta, z) d\zeta \wedge [d\bar{\zeta}]_i = c_m \frac{(-1)^{i-1}}{b_i(\zeta, z)} d_\zeta (\bar{\Phi}^{-(m-1)}(\zeta, z) [d\zeta]_i \wedge [d\bar{\zeta}]_i),$$

where $i = i(\nu)$, $\zeta \in U_\nu \cap bD$. By (3-50), the boundary integral in (3-52) can be written as

$$\begin{aligned}
 & \int_{bD} \frac{u(\zeta) \chi_\nu(\zeta)}{Q''(z, \zeta) \bar{\Phi}^m(\zeta, z)} \sum_{k=1}^n (-1)^{k-1} \frac{\partial \rho}{\partial \zeta_k} [d\bar{\zeta}]_k \wedge d\zeta \\
 &= \int_{bD} \frac{u(\zeta) \chi_\nu(\zeta) \varphi_\nu(\zeta)}{Q''(z, \zeta) \bar{\Phi}^m(\zeta, z)} d\zeta \wedge [d\bar{\zeta}]_{i(\nu)} \\
 &= \int_{bD} \frac{u(\zeta) \chi_\nu(\zeta) \varphi_\nu(\zeta)}{(Q'' b_{i(\nu)}) (z, \zeta)} d_\zeta (\bar{\Phi}^{-(m-1)}(\zeta, z) [d\zeta]_{i(\nu)} \wedge [d\bar{\zeta}]_{i(\nu)}),
 \end{aligned}$$

where the constant is absorbed into φ_ν . By Stokes' theorem, the integral is equal to

$$\int_{bD} d_\zeta \left(\frac{u(\zeta) (\chi_\nu \varphi_\nu)(\zeta)}{(Q'' b_{i(\nu)}) (z, \zeta)} \right) \bar{\Phi}^{-(m-1)}(\zeta, z) [d\zeta]_{i(\nu)} \wedge [d\bar{\zeta}]_{i(\nu)}.$$

Let ψ_ν be the function such that $[d\zeta]_{i(\nu)} \wedge [d\bar{\zeta}]_{i(\nu)} = \psi_\nu(\zeta) d\sigma(\zeta)$. Summing the above expression over ν , the boundary integral in (3-52) can be written as

$$(3-55) \quad \int_{bD} P''(u)(\zeta) \bar{\Phi}^{-(m-1)}(\zeta, z) d\sigma(\zeta), \quad P''(u)(\zeta) := \sum_{\nu=1}^M d_\zeta \left(\frac{u(\zeta) (\chi_\nu \varphi_\nu)(\zeta)}{(Q'' b_{i(\nu)}) (z, \zeta)} \right) \psi_\nu(\zeta).$$

Hence we obtain formula (3-44).

Finally, the proof of (iii) is clear from the proofs of (i) and (ii). \square

In this section we prove Proposition 1.2 and then use it to prove Theorem 1.1 and Theorem 1.3. First we fix some notations. We will write $|f|_r = |f|_{C^r(\bar{D})}$, where $|\cdot|_{C^r(\bar{D})}$ denotes the Hölder r -norm on D . We also write $\delta(z) := \text{dist}(z, bD)$.

4. Proof of Proposition 1.2 and Theorem 1.1

In this section we prove Theorem 1.1. We begin with Proposition 1.2.

Proof of Proposition 1.2. We shall assume that ρ is a regularized defining function satisfying the properties in Proposition 2.2. In particular, we have $\rho \in C^\infty(\mathbb{C}^n) \cap C^{k+3+\alpha}(\bar{D})$ and

$$(4-1) \quad |D^j \rho(z)| \lesssim C_j |\rho|_{k+2+\alpha} (1 + \delta(z)^{k+3+\alpha-j}), \quad j = 0, 1, 2, \dots$$

We recall the notation

$$\Phi(z, \zeta) := G(z, \zeta) - \rho(\zeta), \quad \overline{\Phi(\zeta, z)} := \overline{G(\zeta, z)} - \rho(z).$$

In view of (3-22), we can write

$$\mathcal{K}f(z) = \mathcal{K}_0 f(z) + \mathcal{K}_1 f(z) := \int_D K_0(z, \zeta) f(\zeta) dV(\zeta) + \int_D K_1(z, \zeta) f(\zeta) dV(\zeta),$$

where $K_0(z, \zeta) := \overline{L_0(\zeta, z)} - L_0(z, \zeta)$ and $K_1(z, \zeta) = \overline{L_1(\zeta, z)} - L_1(z, \zeta)$. We first estimate $\mathcal{K}_0 f$. In view of (3-22), we have

$$\mathcal{K}_0 f(z) = \int_D f(\zeta) (\overline{L_0(\zeta, z)} - L_0(z, \zeta)) dV(\zeta).$$

where $L_0(z, \zeta) dV(\zeta) = S_z(\bar{\partial}_z \bar{\partial}_\zeta N)(z, \zeta)$. As observed earlier, since $bD \in C^{k+3+\alpha}$, the coefficients of $\bar{\partial}_z \bar{\partial}_\zeta N(z, \zeta)$ belong to the class $C^\infty \times C^{k+\alpha}(D_\delta(z) \times D_\delta(\zeta))$. By Proposition 2.1 and the fact that S_z is a linear operator, we see that $L_0(z, \zeta) \in C^\infty \times C^{k+\alpha}(D_\delta(z) \times D_\delta(\zeta))$, which also implies $L_0(\zeta, z) \in C^\infty \times C^{k+\alpha}(D_\delta(\zeta) \times D_\delta(z))$.

Accordingly, we have

$$(4-2) \quad \int_D f(\zeta) L_0(z, \zeta) dV(\zeta) \in C^\infty(\bar{D}), \quad \int_D f(\zeta) \overline{L_0(\zeta, z)} dV(\zeta) \in C^{k+\alpha}(\bar{D}).$$

Here the first statement is clear. We now prove the second statement. By estimate (4-1) and the expression for $L_0(\zeta, z)$, it follows that

$$\left| \int_D f(\zeta) D_z^{k+1} \overline{L_0(\zeta, z)} dV(\zeta) \right| \lesssim |f|_0 (1 + \delta(z)^{-1+\alpha}).$$

Hence by Lemma 2.3, the second integral in (4-2) belongs to $C^{k+\alpha}(\bar{D})$. Thus we have shown $\mathcal{K}_0 f \in C^{k+\alpha}(\bar{D})$.

Next we estimate $\mathcal{K}_1 f$. First we prove for the case $k = 0$, i.e., $\rho \in C^{3+\alpha}$. In view of (3-17), we have

$$\begin{aligned} \mathcal{K}_1 f(z) &= \int_D f(\zeta) (\overline{L_1(\zeta, z)} - L_1(z, \zeta)) dV(\zeta) \\ &= \int_D f(\zeta) \left[\frac{\eta(z) + O''(|z - \zeta|)}{\overline{\Phi}^{n+1}(\zeta, z)} - \frac{\eta(\zeta) + O'(|z - \zeta|)}{\Phi^{n+1}(z, \zeta)} \right] dV(\zeta). \end{aligned}$$

Let χ_0 be a C^∞ cut-off function supported in the set $E_0 := \{(z, \zeta) \in D \times D : |z - \zeta| < \delta_0\}$, and $\chi_0 \equiv 1$ in $\{(z, \zeta) \in D \times D : |z - \zeta| < \delta_0/2\}$, for some $\delta_0 > 0$. From the definition of $G(z, \zeta)$ (see (3-5)), we can choose δ_0 to be sufficiently small such that on the set E_0 , we have $\Phi(z, \zeta) = F(z, \zeta) - \rho(\zeta)$. Write

$$\mathcal{K}_1 f(z) = \mathcal{K}'_1 f(z) + \mathcal{K}''_1 f(z),$$

where

$$\begin{aligned} \mathcal{K}'_1 f(z) &:= \int_D f(\zeta) (\chi_0 K_1)(z, \zeta) dV(\zeta), \\ \mathcal{K}''_1 f(z) &= \int_D f(\zeta) [(1 - \chi_0) K_1](z, \zeta) dV(\zeta), \end{aligned}$$

with $K_1(z, \zeta) = \overline{L_1(\zeta, z)} - L_1(z, \zeta)$. The function $(1 - \chi_0)K_1$ is supported in $E_1 := \{(z, \zeta) \in D \times D : |z - \zeta| \geq \varepsilon_0/2\}$. By estimate (3-7) and the assumption $\rho \in C^{k+3+\alpha}$, we see that $(1 - \chi_0)L_1(z, \zeta) \in C^\infty \times C^{k+\alpha}(\overline{D}(z) \times \overline{D}(\zeta))$ and $(1 - \chi_0)L_1(\zeta, z) \in C^{k+\alpha} \times C^\infty(\overline{D}(z) \times \overline{D}(\zeta))$. By the same argument used to prove (4-2), we can show that $\mathcal{K}''_1 f \in C^{k+\alpha}(\overline{D})$.

It remains to estimate $\mathcal{K}'_1 f$. We will divide the proof into two steps. In the first part, we show that if $bD \in C^{3+\alpha}$, then $\mathcal{K}'_1 f \in C^{\min\{\alpha, 1/2\}}$. In the second part, we use integration by parts to show that if $bD \in C^{k+3+\alpha}$, for $k \geq 1$, then $\mathcal{K}'_1 f \in C^{k+\min\{\alpha, 1/2\}}$.

Case 1: $bD \in C^{3+\alpha}$ Assume now that $bD \in C^{3+\alpha}$. In what follows we will write $D_0 = D_0(z) = \{\zeta \in D : |\zeta - z| \leq \delta_0\}$, and without loss of generality we can assume f is supported in D_0 . Taking z_j derivative we get

$$\begin{aligned} \frac{\partial \mathcal{K}'_1 f}{\partial z_j}(z) &= \int_D f(\zeta) \left(\frac{\frac{\partial}{\partial z_j} [\eta(z) + O''(|z - \zeta|)]}{\overline{\Phi}^{n+1}(\zeta, z)} - \frac{\frac{\partial}{\partial z_j} [\eta(\zeta) + O'(|z - \zeta|)]}{\Phi^{n+1}(z, \zeta)} \right) dV(\zeta) \\ &\quad - (n+1) \int_D f(\zeta) \left(\frac{[\frac{\partial}{\partial z_j} \overline{\Phi}(\zeta, z)](\eta(z) + O''(|z - \zeta|))}{\overline{\Phi}^{n+2}(\zeta, z)} \right. \\ &\quad \left. - \frac{[\frac{\partial}{\partial z_j} \Phi(z, \zeta)](\eta(\zeta) + O'(|z - \zeta|))}{\Phi^{n+2}(z, \zeta)} \right) dV(\zeta) \\ &:= I_1(z) + I_2(z), \end{aligned}$$

where we denote the first and second integral by I_1 and I_2 , respectively. We first estimate I_1 . By (3-18), we have

$$\left| \frac{\partial}{\partial z_j} [\eta(\zeta) + O'(|z - \zeta|)] \right| \lesssim |\rho|_3.$$

Hence by (3-30),

$$(4-3) \quad \int_D \frac{|pa_{z_j}[\eta(\zeta) + O'(|z - \zeta|)]|}{|\Phi^{n+1}(z, \zeta)|} |f(\zeta)| dV(\zeta) \lesssim |f|_0 |\rho|_3 \int_D \frac{dV(\zeta)}{|\Phi(z, \zeta)|^{n+1}} \\ \lesssim |f|_0 |\rho|_3 (1 + \log \delta(z)).$$

On the other hand, using estimate (3-21) we have

$$\left| \frac{\partial}{\partial z_j} [\eta(z) + O''(|z - \zeta|)] \right| \lesssim |D_z^3 \rho(z)| + |D_z^4 \rho(z)| |\zeta - z| \\ \lesssim |\rho|_{3+\alpha} (1 + \delta(z)^{-1+\alpha} |\zeta - z|),$$

where in the last inequality we applied (4-1) with $k = 0$ and $j = 4$.

Thus applying (3-26) and (3-30) we obtain

$$(4-4) \quad \int_D \frac{|\partial_{z_j}[\eta(z) + O''(|z - \zeta|)]|}{|\bar{\Phi}^{n+1}(\zeta, z)|} |f(\zeta)| dV(\zeta) \\ \lesssim |\rho|_{3+\alpha} |f|_0 \left(\int_D \frac{dV(\zeta)}{|\Phi(\zeta, z)|^{n+1}} + \delta(z)^{-1+\alpha} \int_D \frac{|\zeta - z|}{|\Phi(\zeta, z)|^{n+1}} dV(\zeta) \right) \\ \lesssim |\rho|_{3+\alpha} |f|_0 (\log \delta(z) + \delta(z)^{-1+\alpha}) \lesssim |\rho|_{3+\alpha} |f|_0 \delta(z)^{-1+\alpha}.$$

Putting together estimates (4-3) and (4-4), we get

$$(4-5) \quad |I_1(z)| \lesssim |\rho|_{3+\alpha} |f|_0 \delta(z)^{-1+\alpha}, \quad 0 < \alpha < 1.$$

For the integral I_2 , we can write it as $I_2(z) = -(n + 1) \sum_{i=1}^3 J_i(z)$, where

$$J_1(z) = \int_D f(\zeta) \left(\frac{\partial \bar{\Phi}(\zeta, z)}{\partial z_j} - \frac{\partial \Phi(z, \zeta)}{\partial z_j} \right) \frac{[\eta(z) + O''(|z - \zeta|)]}{\bar{\Phi}^{n+2}(\zeta, z)} dV(\zeta); \\ J_2(z) = \int_D f(\zeta) [\eta(z) - \eta(\zeta) + O''(|z - \zeta|) - O'(|z - \zeta|)] \frac{\partial_{z_j} \Phi(z, \zeta)}{\bar{\Phi}^{n+2}(\zeta, z)} dV(\zeta); \\ J_3(z) = \int_D f(\zeta) \left[\frac{\partial \Phi(z, \zeta)}{\partial z_j} \right] [\eta(\zeta) + O'(|z - \zeta|)] \\ \times \left(\frac{1}{\bar{\Phi}^{n+2}(\zeta, z)} - \frac{1}{\Phi^{n+2}(z, \zeta)} \right) dV(\zeta).$$

To estimate J_1 , we note that by Lemma 3.9, for any $\zeta \in \text{supp } f \subset D_0(z) = \{|\zeta - z| < \delta_0\}$,

$$\left| \frac{\partial \overline{\Phi}(\zeta, z)}{\partial z_j} - \frac{\partial \Phi(z, \zeta)}{\partial z_j} \right| \lesssim |\rho|_3 |\zeta - z|.$$

Together with estimates (3-21) and (3-9) we get

$$\begin{aligned} (4-6) \quad |J_1(z)| &\lesssim |f|_0 |\rho|_3 \int_D \frac{|\zeta - z|}{|\overline{\Phi}(\zeta, z)|^{n+2}} dV(\zeta) \\ &\lesssim |f|_0 |\rho|_3 \int_D \frac{dV(\zeta)}{|\zeta - z| |\Phi(\zeta, z)|^{n+1}} \\ &\lesssim |f|_0 |\rho|_3 \delta(z)^{-1/2}, \end{aligned}$$

where in the last inequality we applied estimate (3-25) with $\alpha = 1$. For J_2 , we note that $|\eta(z) - \eta(\zeta)| \lesssim |\rho|_3 |z - \zeta|$. By Lemma 3.8 (i), we have

$$(4-7) \quad \left| \frac{\partial \Phi(z, \zeta)}{\partial z_j} \right| \lesssim |\rho|_1 + |\rho|_2 |\zeta - z| \lesssim |\rho|_2.$$

Applying estimates (3-9), (3-18), (3-21), and Lemma 3.2, we get

$$\begin{aligned} (4-8) \quad |J_2(z)| &\lesssim |f|_0 |\rho|_3 \int_D \frac{|\zeta - z|}{|\Phi(\zeta, z)|^{n+2}} dV(\zeta) \\ &\lesssim |f|_0 |\rho|_3 \int_D \frac{dV(\zeta)}{|\zeta - z| |\Phi(\zeta, z)|^{n+1}} \\ &\lesssim |f|_0 |\rho|_3 \delta(z)^{-1/2}. \end{aligned}$$

For J_3 we use estimate (3-24),

$$\begin{aligned} (4-9) \quad |J_3(z)| &\lesssim |f|_0 |\rho|_3 \int_D |\zeta - z|^3 \left(\frac{|\Phi(z, \zeta)|^{n+1}}{|\overline{\Phi}(\zeta, z)|^{n+2} |\Phi(z, \zeta)|^{n+2}} \right. \\ &\quad \left. + \frac{|\overline{\Phi}(\zeta, z)|^{n+1}}{|\Phi(\zeta, z)|^{n+2} |\Phi(z, \zeta)|^{n+2}} \right) dV(\zeta) \\ &= |f|_0 |\rho|_3 \int_D |\zeta - z|^3 \left(\frac{1}{|\Phi(\zeta, z)|^{n+2} |\Phi(z, \zeta)|} \right. \\ &\quad \left. + \frac{1}{|\Phi(\zeta, z)| |\Phi(z, \zeta)|^{n+2}} \right) dV(\zeta) \\ &\lesssim |f|_0 |\rho|_3 \left(\int_D \frac{|\zeta - z|}{|\Phi(\zeta, z)|^{n+2}} dV(\zeta) + \int_D \frac{|\zeta - z|}{|\Phi(z, \zeta)|^{n+2}} dV(\zeta) \right) \\ &\lesssim |f|_0 |\rho|_3 \left(\int_D \frac{dV(\zeta)}{|\zeta - z| |\Phi(\zeta, z)|^{n+1}} + \int_D \frac{dV(\zeta)}{|\zeta - z| |\Phi(z, \zeta)|^{n+1}} \right) \\ &\lesssim |f|_0 |\rho|_3 \delta(z)^{-1/2}, \end{aligned}$$

where in the last inequality we applied Lemma 3.2 with $\beta = 1$. Hence we have shown that

$$(4-10) \quad |I_2(z)| \lesssim |f|_0 |\rho|_3 \delta(z)^{-1/2}.$$

Combining (4-5) and (4-10), we have

$$\left| \frac{\partial \mathcal{K}'_1 f}{\partial z_j} \right| \lesssim \begin{cases} |f|_0 |\rho|_{3+\alpha} \delta(z)^{-1+\alpha} & \text{if } 0 < \alpha \leq \frac{1}{2}; \\ |f|_0 |\rho|_3 \delta(z)^{-1/2} & \text{if } \frac{1}{2} \leq \alpha \leq 1. \end{cases}$$

In a similar way we can show that $|\partial_{\bar{z}_j} \mathcal{K}'_1 f|$ satisfies the same estimate. It follows by Lemma 2.3 that

$$\mathcal{K}'_1 f \in \begin{cases} C^\alpha(\bar{D}) & \text{if } 0 < \alpha \leq \frac{1}{2}; \\ C^{1/2}(\bar{D}) & \text{if } \frac{1}{2} \leq \alpha \leq 1. \end{cases}$$

This completes the proof for the $k = 0$ case.

Case 2: $bD \in C^{k+3+\alpha}$, $k \geq 1$ We now assume that $\rho \in C^{k+3+\alpha}$, for $k \geq 1$. Taking $k + 1$ derivatives we get

$$(4-11) \quad \begin{aligned} D_z^{k+1} \mathcal{K}'_1 f(z) &= \sum_{\gamma_1 + \gamma_2 \leq k+1} \int_D f(\zeta) [D_z^{\gamma_1} \{\eta(z) + O''(|z-\zeta|)\}] D_z^{\gamma_2} (\bar{\Phi}^{-(n+1)}(\zeta, z)) dV(\zeta) \\ &\quad - \sum_{\gamma_1 + \gamma_2 \leq k+1} \int_D f(\zeta) [D_z^{\gamma_1} \{\eta(\zeta) + O'(|z-\zeta|)\}] D_z^{\gamma_2} (\Phi^{-(n+1)}(z, \zeta)) dV(\zeta) \\ &:= F_1 + F_2, \end{aligned}$$

where we denote the first and second sum in (4-11) by F_1 and F_2 , respectively. We break up into cases.

Case 1: $\gamma_1 = k + 1$. ($\gamma_2 = 0$). By (3-18) and (3-21), we get

$$\begin{aligned} |D_z^{k+1} \{\eta(\zeta) + O'|\zeta - z|\}| &\lesssim |\rho|_{k+3}. \\ |D_z^{k+1} \{\eta(z) + O''|\zeta - z|\}| &\lesssim |\rho|_{k+3} + |\rho|_{k+4} |\zeta - z| \\ &\lesssim |\rho|_{k+3+\alpha} (1 + \delta(z)^{-1+\alpha}) |\zeta - z|, \end{aligned}$$

where for the last inequality we used (4-1) with $j = k + 4$. By doing similar estimate as that for the integral I_1 in the $k = 0$ case, we get

$$|D^\gamma \mathcal{K}'_1 f(z)| \lesssim |\rho|_{k+3+\alpha} |f|_0 \delta(z)^{-1+\alpha}, \quad 0 < \alpha < 1.$$

Case 2: $1 \leq \gamma_2 \leq k$ ($\gamma_1 \leq k$). The term in the sum in (4-11) takes the form

$$(4-12) \quad \int_D f(\zeta) \left[\frac{D_z^{\gamma_1} \{\eta(z) + O''(|z - \zeta|)\}}{\bar{\Phi}^{n+1+\tau}(\zeta, z)} S_\tau''(z, \zeta) \right] dV(\zeta) \\ - \int_D f(\zeta) \left[\frac{D_z^{\gamma_1} \{\eta(\zeta) + O'(|z - \zeta|)\}}{\Phi^{n+1+\tau}(z, \zeta)} S_\tau'(z, \zeta) \right] dV(\zeta),$$

where $\tau \leq \gamma_2 \leq k$ and $S_\tau''(z, \zeta)$ is some linear combination of products of $D_z^l \bar{\Phi}(\zeta, z)$, $l \leq k$, and S_τ' is some linear combination of products of $D_z^l \Phi(z, \zeta)$, $l \leq k$.

It is convenient to recall the notation:

$$(4-13) \quad Q'(z, \zeta) = \sum_{i=1}^n \frac{\partial \Phi(z, \zeta)}{\partial \bar{\zeta}_i} \frac{\partial \rho}{\partial \zeta_i}, \quad Q''(z, \zeta) = \sum_{i=1}^n \frac{\partial \overline{\Phi(\zeta, z)}}{\partial \bar{\zeta}_i} \cdot \frac{\partial \rho}{\partial \zeta_i},$$

and for $|\zeta - z|$ small, we have

$$(4-14) \quad \Phi(z, \zeta) = F(z, \zeta) - \rho(\zeta) \\ = \sum_{j=1}^n \frac{\partial \rho}{\partial \zeta_j}(\zeta) (\zeta_j - z_j) - \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2 \rho}{\partial \zeta_i \partial \zeta_j}(\zeta) (z_i - \zeta_i)(z_j - \zeta_j) - \rho(\zeta);$$

$$(4-15) \quad \Phi(\zeta, z) = F(\zeta, z) - \rho(z) \\ = \sum_{j=1}^n \frac{\partial \rho}{\partial z_j}(z) (z_j - \zeta_j) - \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2 \rho}{\partial z_i \partial z_j}(z) (\zeta_i - z_i)(\zeta_j - z_j) - \rho(z).$$

For the first integral in (4-12), we apply integration by parts formulae (3-43) and (3-45) iteratively until the integral becomes, for $\mu_0, \eta_0 \leq k$, a linear combination of

$$(4-16) \quad \int_{bD} \frac{[D^{\mu_0} f(\zeta)] W_1''(z, \zeta)}{\bar{\Phi}^n(\zeta, z)} dV(\zeta), \quad \int_D \frac{[D^{\eta_0} f(\zeta)] W_2''(z, \zeta)}{\bar{\Phi}^{n+1}(\zeta, z)} dV(\zeta),$$

where W_1'', W_2'' are some linear combinations of products of

$$D_\zeta^{\mu_1} D_z^{\gamma_1} \{\eta(z) + O''(|z - \zeta|)\}, \quad D_\zeta^{\mu_2} [(Q'')^{-1}], \quad D_\zeta^{\mu_3+1} D_z^l \overline{\Phi(\zeta, z)}, \quad D_\zeta^{\mu_4+1} \rho(\zeta),$$

for $l \leq k$, and $\mu_i \geq 0$ satisfies $\sum_{i=1}^4 \mu_i \leq k$. Now we have $|D_\zeta^{\mu_1} D_z^{\gamma_1} \{\eta(z) + O''(|z - \zeta|)\}| \leq C_k |\rho|_{k+2}$ (since $\gamma_1 \leq k, \mu_1 \leq k$), $|D_\zeta^{\mu_2} [(Q'')^{-1}(z, \zeta)]| \leq C_k$, $|D_\zeta^{\mu_3+1} D_z^l \overline{\Phi(\zeta, z)}| \leq C_k |\rho|_{k+1}$ (since $l \leq k$). Hence the integrals in (4-16) and thus the first integral in (4-12) can be bounded by

$$(4-17) \quad |f|_k |\rho|_{k+3} \left(\int_{bD} \frac{d\sigma(\zeta)}{|\Phi(\zeta, z)|^n} + \int_D \frac{dV(\zeta)}{|\Phi(\zeta, z)|^{n+1}} \right) \\ \lesssim |f|_k |\rho|_{k+3} (1 + \log \delta(z)),$$

where we applied Lemma 3.4. For the second integral in (4-12), we apply formulae (3-44) and (3-45) iteratively until the integral becomes a linear combination of

$$(4-18) \quad \int_{bD} \frac{D^{\mu_0} f(\zeta) W'_1(z, \zeta)}{\Phi^n(z, \zeta)} dV(\zeta), \quad \int_D \frac{D^{\eta_0} f(\zeta) W'_2(z, \zeta)}{\Phi^{n+1}(z, \zeta)} dV(\zeta),$$

for $\mu_0, \eta_0 \leq k$. Here W'_1 and W'_2 are linear combinations of products of

$$D_\zeta^{\mu_1} D_z^{\gamma_1} \{\eta(\zeta) + O'(|z - \zeta|)\}, \quad D_\zeta^{\mu_2} [(Q')^{-1}], \quad D_\zeta^{\mu_3+1} D_z^l \Phi(z, \zeta), \quad D_\zeta^{\mu_4+1} \rho(\zeta),$$

where $l \leq k$ and $\mu_i \geq 0$ satisfies $\sum_{i=1}^4 \mu_i \leq k$. We have $|D_\zeta^{\mu_1} D_z^{\gamma_1} \{\eta(\zeta) + O'(|z - \zeta|)\}| \leq C_k |\rho|_{\mu_1+3} \leq |\rho|_{k+3}$ (since $\gamma_1, \mu_1 \leq k$), $|D_\zeta^{\mu_2} [(Q')^{-1}]| \lesssim |\rho|_{\mu_2+3} \lesssim |\rho|_{k+3}$, and $|D_\zeta^{\mu_3+1} D_z^l \Phi(z, \zeta)| \lesssim C_k |\rho|_{\mu_3+3} \lesssim |\rho|_{k+3}$. It follows that the integrals in (4-18) and hence the second integral in (4-12) is bounded by

$$(4-19) \quad |f|_k |\rho|_{k+3} \left(\int_{bD} \frac{d\sigma(\zeta)}{|\Phi(z, \zeta)|^n} + \int_D \frac{dV(\zeta)}{|\Phi(z, \zeta)|^{n+1}} \right) \lesssim |f|_k |\rho|_{k+3} (1 + \log \delta(z)).$$

Combining (4-17) and (4-19), we get for this case

$$|D^\gamma \mathcal{K}'_1 f(z)| \lesssim |\rho|_{k+3} |f|_k (1 + \log \delta(z)).$$

Case 3: $\gamma_2 = k + 1$ ($\gamma_1 = 0$). Applying integration by parts formulae (3-43), (3-44) and (3-45) iteratively to $F_1(z)$ in (4-11) yields a linear combination of

$$(4-20) \quad \int_{bD} \frac{D^{\eta_0} f(\zeta) R''_0(z, \zeta)}{\bar{\Phi}^{n+1}(\zeta, z)} dV(\zeta), \quad \int_D \frac{D^{\mu_0} f(\zeta) R''_1(z, \zeta)}{\bar{\Phi}^{n+2}(\zeta, z)} dV(\zeta),$$

for $\eta_0, \mu_0 \leq k$. Similarly we apply integration by parts to $F_2(z)$ until it becomes a linear combination of

$$(4-21) \quad \int_{bD} \frac{D^{\eta_0} f(\zeta) R'_0(z, \zeta)}{\Phi^{n+1}(z, \zeta)} dV(\zeta), \quad \int_D \frac{D^{\mu_0} f(\zeta) R'_1(z, \zeta)}{\Phi^{n+2}(z, \zeta)} dV(\zeta).$$

Here $R''_0(z, \zeta)$ and $R''_1(z, \zeta)$ are some linear combination of products of

$$D_\zeta^{\mu_1} (\eta(z) + O''(|z - \zeta|)), \quad D_\zeta^{\mu_2} [(Q'')^{-1}], \quad D_\zeta^{\mu_3+1} \bar{\Phi}(\zeta, z), \\ D_\zeta^{\mu_4} D_z \bar{\Phi}(\zeta, z), \quad D_\zeta^{\mu_5+1} \rho(\zeta),$$

and $R'_0(z, \zeta)$ and $R'_1(z, \zeta)$ are some linear combination of the products of

$$(4-22) \quad D_\zeta^{\mu_1} (\eta(\zeta) + O'(|z - \zeta|)), \quad D_\zeta^{\mu_2} [(Q')^{-1}], \quad D_\zeta^{\mu_3+1} \Phi(z, \zeta), \\ D_\zeta^{\mu_4} D_z \Phi(z, \zeta), \quad D_\zeta^{\mu_5+1} \rho(\zeta),$$

where $0 \leq \mu_i \leq k$ for $0 \leq i \leq 5$, and $\sum_{i=0}^5 \mu_i \leq k$. There are five subcases to consider:

Subcase 1: $\gamma_2 = k + 1$, $\mu_0, \mu_1, \mu_2, \mu_3 \leq k - 1$. Then we do integration by parts one more time to the integrals in (4-21) and the resulting integrals become

$$(4-23) \quad \int_{bD} \frac{D^{\tilde{\eta}_0} f(\zeta) R'_0(z, \zeta)}{\Phi^n(z, \zeta)} dV(\zeta), \quad \int_D \frac{D^{\tilde{\mu}_0} f(\zeta) R'_1(z, \zeta)}{\Phi^{n+1}(z, \zeta)} dV(\zeta),$$

where $\tilde{\eta}_0, \tilde{\mu}_0 \leq k$, and \tilde{R}'_0 and \tilde{R}'_1 are linear combinations of products of

$$D^{\tilde{\mu}_1}(\eta(\zeta) + O'(|z - \zeta|)), \quad D^{\tilde{\mu}_2}[(Q')^{-1}], \quad D^{\tilde{\mu}_3+1}\Phi(z, \zeta), \\ D^{\tilde{\mu}_4}D_z\Phi(z, \zeta), \quad D^{\tilde{\mu}_5+1}\rho(\zeta),$$

with $\tilde{\mu}_0, \tilde{\mu}_1, \tilde{\mu}_2, \tilde{\mu}_3 \leq k$ and $\sum_i \tilde{\mu}_i \leq k + 1$. Then

$$|D^{\tilde{\mu}_1}(\eta(\zeta) + O'(|z - \zeta|))| \lesssim |\rho|_{k+3}.$$

In view of (3-38) and (3-39), we have

$$(4-24) \quad |D^l_\zeta Q'(z, \zeta)| = D^l_\zeta \left(\sum_{i=1}^n \frac{\partial \Phi(z, \zeta)}{\partial \bar{\zeta}_i} \frac{\partial \rho}{\partial \zeta_i} \right) \lesssim |\rho|_{l+2} + |\rho|_{l+3} |\zeta - z| \lesssim |\rho|_{l+3},$$

and similarly $|D^l_\zeta \Phi(z, \zeta)| \lesssim |\rho|_{l+3}$. Hence for $\tilde{\mu}_2, \tilde{\mu}_3 \leq k$, we have $|D^{\tilde{\mu}_2}[(Q')^{-1}]|, |D^{\tilde{\mu}_3+1}\Phi(z, \zeta)| \lesssim |\rho|_{k+3}$. Putting together the estimates, it follows that the integrals in (4-23) and thus in (4-21) satisfy

$$\left| \int_{bD} \frac{D^{\tilde{\mu}_0} f(\zeta) R'_0(z, \zeta)}{\Phi^n(z, \zeta)} dV(\zeta) \right| \lesssim |f|_k |\rho|_{k+3} \int_{bD} \frac{d\sigma(\zeta)}{|\Phi(z, \zeta)|^n} \\ \lesssim |f|_k |\rho|_{k+3} (1 + \log \delta(z)); \\ \left| \int_D \frac{D^{\tilde{\mu}_0} f(\zeta) R'_1(z, \zeta)}{\Phi^{n+1}(z, \zeta)} dV(\zeta) \right| \lesssim |f|_k |\rho|_{k+3} \int_D \frac{dV(\zeta)}{|\Phi(z, \zeta)|^{n+1}} \\ \lesssim |f|_k |\rho|_{k+3} (1 + \log \delta(z)).$$

We can obtain similar estimates for the integrals in (4-20), where the proof is easier since the functions R''_0 and R''_1 are C^∞ in ζ . In conclusion we have shown that in this case

$$|D^\gamma_z \mathcal{K}'_1 f(z)| \lesssim |\rho|_{k+3} |f|_k (1 + \log \delta(z)).$$

Subcase 2: $\gamma_2 = k + 1$, $\mu_1 = k$. Again we shall only estimate (4-21) as a similar procedure can be applied to (4-20). The integrals in (4-21) can be written as

$$(4-25) \quad \int_{bD} \frac{f(\zeta) D^k_\zeta [\eta(\zeta) + O'|z - \zeta|] R'_0(z, \zeta)}{\Phi^{n+1}(z, \zeta)} d\sigma(\zeta), \\ \int_D \frac{f(\zeta) D^k_\zeta [\eta(\zeta) + O'|z - \zeta|] R'_1(z, \zeta)}{\Phi^{n+2}(z, \zeta)} dV(\zeta),$$

where R'_0 and R'_1 are some linear combination of the products of

$$(Q')^{-1}, \quad D_z \Phi(z, \zeta), \quad D_\zeta \Phi(z, \zeta), \quad D\rho(\zeta).$$

We now estimate the domain integral in (4-25) which can be written as $B_1 + B_2$, where

$$(4-26) \quad \begin{aligned} B_1(z) &= \int_D \frac{f(\zeta) D_\zeta^k \eta(\zeta) R'_1(z, \zeta)}{\Phi^{n+2}(z, \zeta)} dV(\zeta), \\ B_2(z) &= \int_D \frac{f(\zeta) D_\zeta^k [O'(|z - \zeta|)] R'_1(z, \zeta)}{\Phi^{n+2}(z, \zeta)} dV(\zeta). \end{aligned}$$

We apply integration by parts formulae (3-43) and (3-45) to B_1 so that

$$\begin{aligned} B_1(z) &= \int_{bD} \frac{D_\zeta^{\tilde{\nu}_0} f(\zeta) D_\zeta^{k+\tilde{\nu}_1} \eta(\zeta) \tilde{R}'_{10}(z, \zeta)}{\Phi^n(z, \zeta)} d\sigma(\zeta) \\ &\quad + \int_D \frac{D_\zeta^{\tilde{\mu}_0} f(\zeta) D_\zeta^{k+\tilde{\mu}_1} \eta(\zeta) \tilde{R}'_{11}(z, \zeta)}{\Phi^{n+1}(z, \zeta)} dV(\zeta). \end{aligned}$$

Here $\tilde{\nu}_0, \tilde{\mu}_0, \tilde{\mu}_1 \leq 1$. $\tilde{R}'_{10}(z, \zeta)$ and $\tilde{R}'_{11}(z, \zeta)$ are linear combinations of the products of

$$\widehat{D}_\zeta (Q')^{-1}, \quad \widehat{D}_\zeta D_z \Phi(z, \zeta), \quad \widehat{D}_\zeta^2 \Phi(z, \zeta), \quad \widehat{D}_\zeta^2 \rho(\zeta).$$

In particular $|\tilde{R}'_{10}(z, \zeta)|, |\tilde{R}'_{11}(z, \zeta)| \lesssim |\rho|_4 \lesssim |\rho|_{k+3}$ ($k \geq 1$). It follows from (4-26) that

$$\begin{aligned} |B_1(z)| &\lesssim |f|_1 |\rho|_4 \left(\int_{bD} \frac{d\sigma(\zeta)}{|\Phi(z, \zeta)|^n} + \int_D \frac{dV(\zeta)}{|\Phi(z, \zeta)|^{n+1}} \right) \\ &\lesssim |f|_1 |\rho|_{k+3} (1 + \log \delta(z)), \quad k \geq 1. \end{aligned}$$

For B_2 , we use estimate (3-18),

$$D_\zeta^k [O'(|z - \zeta|)] = g_1(z, \zeta) + g_2(z, \zeta),$$

where $|g_1(z, \zeta)| \lesssim |\rho|_{k+2}$ and $|g_2(z, \zeta)| \lesssim |\rho|_{k+3} |\zeta - z|$. Write

$$(4-27) \quad B_2(z) = \int_D \frac{f(\zeta) (g_1 R'_1)(z, \zeta)}{\Phi^{n+2}(z, \zeta)} dV(\zeta) + \int_D \frac{f(\zeta) (g_2 R'_1)(z, \zeta)}{\Phi^{n+2}(z, \zeta)} dV(\zeta).$$

The second integral is bounded in absolute value by (up to a constant)

$$|f|_0 |\rho|_{k+3} \int_D \frac{|\zeta - z|}{|\Phi(z, \zeta)|^{n+2}} dV(\zeta) \lesssim |f|_0 |\rho|_{k+3} \int_D \frac{dV(\zeta)}{|\zeta - z| |\Phi(z, \zeta)|^{n+1}} \lesssim \delta(z)^{-1/2}.$$

For the first integral in (4-27) we apply integration by parts and the resulting integral is bounded up to a constant by

$$(4-28) \quad |f|_1 |\rho|_{k+3} \left(\int_{bD} \frac{d\sigma(\zeta)}{\Phi^n(z, \zeta)} + \int_D \frac{dV(\zeta)}{\Phi^{n+1}(z, \zeta)} \right) \lesssim |f|_1 |\rho|_{k+3} (1 + \log \delta(z)).$$

This shows that $|B_2(z)| \lesssim |f|_1 |\rho|_{k+3} \delta(z)^{-1/2}$. Combining the estimates we have shown that the domain integral in (4-25) is bounded by $C|f|_1 |\rho|_{k+3} \delta(z)^{-1/2}$. The estimate for the boundary integral in (4-25) is similar and we leave the details to the reader. In summary we have in this case

$$|D_z^\nu \mathcal{K}'_1 f(z)| \lesssim |\rho|_{k+3} |f|_1 \delta(z)^{-1/2}.$$

Subcase 3: $\gamma_2 = k + 1$, $\mu_2 = k$ in (4-22). From (4-13) we can write out Q' as

$$Q'(z, \zeta) = \sum_{i=1}^n \frac{\partial \Phi(z, \zeta)}{\partial \bar{\zeta}_i} \frac{\partial \rho}{\partial \zeta_i} = \sum_{i=1}^n \left(-\frac{\partial \rho}{\partial \bar{\zeta}_i} + O(|\zeta - z|) \right) \frac{\partial \rho}{\partial \zeta_i},$$

with

$$O(|\zeta - z|) \sim \widehat{D}_\zeta^3 \rho(\zeta) (\zeta_i - z_i).$$

In view of (3-38) and (3-39), we can write $D_\zeta^k [(Q')^{-1}] = Y_1(z, \zeta) + Y_2(z, \zeta)$, where $|Y_1(z, \zeta)| \lesssim |\rho|_{k+2}$ and $|Y_2(z, \zeta)| \lesssim |\rho|_{k+3} |z - \zeta|$. The integrals in (4-21) have the form

$$(4-29) \quad \int_{bD} \frac{f(\zeta) D_\zeta^k [(Q')^{-1}] W_0(z, \zeta)}{\Phi^{n+1}(z, \zeta)} d\sigma(\zeta), \\ \int_D \frac{f(\zeta) D_\zeta^k [(Q')^{-1}] W_1(z, \zeta)}{\Phi^{n+2}(z, \zeta)} dV(\zeta).$$

Here W_0 and W_1 are some linear combinations of the products of $D_\zeta \rho$, $D_z \Phi(z, \zeta)$, $D_\zeta \Phi(z, \zeta)$ and $\eta(\zeta) + O'(|z - \zeta|)$. For the domain integral in (4-29) we have

$$(4-30) \quad \int_D \frac{f(\zeta) D_\zeta^k [(Q')^{-1}] W_1(z, \zeta)}{\Phi^{n+2}(z, \zeta)} dV(\zeta) \\ = \int_D \frac{f(\zeta) [Y_1 W_1](z, \zeta)}{\Phi^{n+2}(z, \zeta)} dV(\zeta) + \int_D \frac{f(\zeta) [Y_2 W_1](z, \zeta)}{\Phi^{n+2}(z, \zeta)} dV(\zeta).$$

For the first term we use integration by parts. Since $|D_\zeta Y_1(z, \zeta)| \lesssim |\rho|_{k+3}$, the resulting integral is bounded by the expression (4-28). For the second term in (4-30)

we estimate directly

$$\begin{aligned} \left| \int_D \frac{f(\zeta)[Y_2 W_1](z, \zeta)}{\Phi^{n+2}(z, \zeta)} dV(\zeta) \right| &\lesssim |f|_0 |\rho|_{k+3} \int_D \frac{|\zeta - z|}{|\Phi(z, \zeta)|^{n+2}} dV(\zeta) \\ &\lesssim |f|_0 |\rho|_{k+3} \int_D \frac{dV(\zeta)}{|\zeta - z| |\Phi(z, \zeta)|^{n+1}} \\ &\lesssim |f|_0 |\rho|_{k+3} \delta(z)^{-1/2}, \end{aligned}$$

where in the last inequality we applied estimate (3-25) with $\beta = 1$. Thus the absolute value of the domain integral in (4-29) is bounded up to constant by $|\rho|_{k+3} |f|_1 \delta(z)^{-1/2}$. We can similarly show the same bound for the boundary integral in (4-29). Hence in this case

$$|D_z^\gamma \mathcal{K}'_1 f(z)| \lesssim |\rho|_{k+3} |f|_1 \delta(z)^{-1/2}.$$

Subcase 4: $\gamma_2 = k + 1$, $\mu_3 = k$ in (4-22). Then the integrals in (4-21) take the form

$$\int_{bD} \frac{f(\zeta) D_\zeta^{k+1} \Phi(z, \zeta) W_0(z, \zeta)}{\Phi^{n+1}(z, \zeta)} dV(\zeta), \quad \int_D \frac{f(\zeta) D_\zeta^{k+1} \Phi(z, \zeta) W_1(z, \zeta)}{\Phi^{n+2}(z, \zeta)} dV(\zeta),$$

where W_0, W_1 are some linear combinations of the products of $D_\zeta \rho(\zeta)$, $D_z \Phi(z, \zeta)$, $D_\zeta \Phi(z, \zeta)$ and $\eta(\zeta) + O'(|z - \zeta|)$. As in the subcase 3 we can write $D_\zeta^{k+1} \Phi(z, \zeta) = Y_1 + Y_2$, where $|D_\zeta^{k+1} Y_1(z, \zeta)| \lesssim |\rho|_{k+2}$ and $|D_\zeta^{k+1} Y_2(z, \zeta)| \lesssim |\rho|_{k+3} |\zeta - z|$. The rest of the estimates are the same as in Subcase 3.

Subcase 5: $\gamma_2 = k + 1$, $\mu_0 = k$. Then the integrals in (4-20) can be written as

$$\int_{bD} \frac{D^k f(\zeta) A''_0(z, \zeta)}{\overline{\Phi}^{n+1}(\zeta, z)} d\sigma(\zeta), \quad \int_D \frac{D^k f(\zeta) A''_1(z, \zeta)}{\overline{\Phi}^{n+2}(\zeta, z)} dV(\zeta),$$

where A''_0 and A''_1 are some linear combination of products of

$$\eta(z) + O''(|z - \zeta|), \quad (Q'')^{-1}, \quad D_\zeta \overline{\Phi(\zeta, z)}, \quad D_z \overline{\Phi(\zeta, z)}, \quad D_\zeta \rho(\zeta).$$

Likewise, the integrals in (4-21) can be written as

$$\int_{bD} \frac{D^k f(\zeta) A'_0(z, \zeta)}{\Phi^{n+1}(z, \zeta)} d\sigma(\zeta), \quad \int_D \frac{D^k f(\zeta) A'_1(z, \zeta)}{\Phi^{n+2}(z, \zeta)} dV(\zeta),$$

where A'_0 and A'_1 are linear combination of products of

$$\eta(\zeta) + O'(|z - \zeta|), \quad (Q')^{-1}, \quad D_\zeta \Phi(z, \zeta), \quad D_z \Phi(z, \zeta), \quad D_\zeta \rho(\zeta),$$

with coefficients identical to the linear combination A_0'' and A_1'' , respectively. In view of (4-11), it suffices to estimate the difference

$$\int_{bD} D^k f(\zeta) \left(\frac{A_0''(z, \zeta)}{\overline{\Phi}^{n+1}(\zeta, z)} - \frac{A_0'(z, \zeta)}{\Phi^{n+1}(z, \zeta)} \right) d\sigma(\zeta),$$

$$\int_D D^k f(\zeta) \left(\frac{A_1''(z, \zeta)}{\overline{\Phi}^{n+2}(\zeta, z)} - \frac{A_1'(z, \zeta)}{\Phi^{n+2}(z, \zeta)} \right) dV(\zeta).$$

We shall again estimate only the domain integral as the proof for the boundary integral is similar. By the expression for η and Lemma 3.9, we have

$$|\eta(z) - \eta(\zeta)| \lesssim |\rho|_3 |\zeta - z|, \quad |D_\zeta \overline{\Phi}(\zeta, z) - D_\zeta \Phi(z, \zeta)| \lesssim |\rho|_3 |\zeta - z|$$

$$|D_z \overline{\Phi}(\zeta, z) - D_z \Phi(z, \zeta)| \lesssim |\rho|_3 |\zeta - z|, \quad |Q''(z, \zeta) - Q'(z, \zeta)| \lesssim |\rho|_3 |\zeta - z|.$$

By procedure similar to the estimates of the I_2 integral in the $k = 0$ case, we can prove the following estimate:

$$\left| \int_D D^k f(\zeta) \left(\frac{A_1''(z, \zeta)}{\overline{\Phi}^{n+2}(\zeta, z)} - \frac{A_1'(z, \zeta)}{\Phi^{n+2}(z, \zeta)} \right) dV(\zeta) \right| \lesssim |\rho|_3 |f|_k \delta(z)^{-1/2}.$$

Consequently we conclude that in this case

$$|D_z^\gamma \mathcal{K}'_1 f(z)| \lesssim |\rho|_3 |f|_k \delta(z)^{-1/2}.$$

Finally combining the results from all cases we have shown that

$$|D_z^{k+1} \mathcal{K}'_1 f(z)| \lesssim \begin{cases} |f|_k |\rho|_{k+3+\alpha} \delta(z)^{-1+\alpha} & \text{if } 0 < \alpha \leq \frac{1}{2}; \\ |f|_k |\rho|_{k+3} \delta(z)^{-1/2} & \text{if } \frac{1}{2} \leq \alpha \leq 1. \end{cases}$$

By Lemma 2.3, $\mathcal{K}'_1 f \in C^{k+\min\{\alpha, 1/2\}}(\overline{D})$. Combined with earlier estimates for $\mathcal{K}''_1 f$ and $\mathcal{K}_0 f$, the proof of Proposition 1.2 is now complete. \square

Proposition 4.1. *Let D be a strictly pseudoconvex domain with C^3 boundary. Let f be a function in $C^1(\overline{\Omega})$ such that $\bar{\partial} f \in C^1(\overline{\Omega})$. Then the following formula holds:*

$$f(z) = \mathcal{L}f(z) + \int_D S_z(\bar{\partial}_z \bar{\partial}_\zeta N)(z, \cdot) \wedge f - \int_D N(z, \cdot) \wedge \bar{\partial} f$$

$$+ \int_{bD} \Omega_{0,0}^{01}(z, \cdot) \wedge \bar{\partial} f + \int_D \Omega_{0,0}^0(z, \cdot) \wedge \bar{\partial} f, \quad z \in D.$$

Here N and \mathcal{L} are given by formulae (3-13) (3-19).

Proof. Starting with the Bochner–Martinelli formula, see for example [Chen and Shaw 2001, Theorem 2.2.1],

$$f(z) = \int_{bD} \Omega_{0,0}^0(z, \zeta) \wedge f(\zeta) + \int_D \Omega_{0,0}^0(z, \zeta) \wedge \bar{\partial} f, \quad z \in D.$$

By (3-14), (3-15) and Stokes' theorem, we have

$$\begin{aligned}
f(z) &= \int_{bD} N(z, \cdot) \wedge f + \int_{bD} \bar{\partial}_\zeta \Omega_{0,0}^{01}(z, \cdot) \wedge f + \int_D \Omega_{0,0}^0(z, \zeta) \wedge \bar{\partial} f \\
&= \int_D \bar{\partial}_\zeta N(z, \cdot) \wedge f - \int_D N(z, \cdot) \wedge \bar{\partial} f + \int_{bD} \Omega_{0,0}^{01}(z, \cdot) \wedge \bar{\partial} f \\
&\quad + \int_D \Omega_{0,0}^0(z, \zeta) \wedge \bar{\partial} f \\
&= \mathcal{L}f(z) + \int_D S_z(\bar{\partial}_z \bar{\partial}_\zeta N)(z, \cdot) \wedge f - \int_D N(z, \cdot) \wedge \bar{\partial} f \\
&\quad + \int_{bD} \Omega_{0,0}^{01}(z, \cdot) \wedge \bar{\partial} f + \int_D \Omega_{0,0}^0(z, \zeta) \wedge \bar{\partial} f. \quad \square
\end{aligned}$$

Proposition 4.2. *Let D be a bounded strictly pseudoconvex domain with $C^{k+3+\alpha}$ boundary, with $0 < \alpha \leq 1$. Suppose f is orthogonal to the Bergman space $H^2(D)$, is C^∞ in D and is holomorphic in $D \setminus D_{-\delta}$, for some $\delta > 0$. Then*

$$f \in C^{k+\min\{\alpha, 1/2\}}(\bar{D}).$$

Here we recall the notation.

$$D_{-\delta} := \{z \in D : \rho(z) < -\delta\}.$$

Proof. Let \mathcal{P} be the Bergman projection for D . By assumption $\mathcal{P}f \equiv 0$. By Proposition 3.6, $\mathcal{L}^*f = (I + \mathcal{K})\mathcal{P}f \equiv 0$, which implies that

$$\mathcal{K}f = \mathcal{L}^*f - \mathcal{L}f = -\mathcal{L}f.$$

Consequently by Proposition 4.1 and the assumption that $\bar{\partial}f \equiv 0$ on bD ,

$$\begin{aligned}
(4-31) \quad f(z) &= -\mathcal{K}f(z) + \int_D S_z(\bar{\partial}_z \bar{\partial}_\zeta N)(z, \cdot) \wedge f - \int_D N(z, \cdot) \wedge \bar{\partial} f \\
&\quad + \int_{bD} \Omega_{0,0}^{01}(z, \cdot) \wedge \bar{\partial} f + \int_D \Omega_{0,0}^0(z, \cdot) \wedge \bar{\partial} f \\
&= -\mathcal{K}f(z) + \int_D S_z(\bar{\partial}_z \bar{\partial}_\zeta N)(z, \cdot) \wedge f - \int_D N(z, \cdot) \wedge \bar{\partial} f \\
&\quad + \int_D \Omega_{0,0}^0(z, \cdot) \wedge \bar{\partial} f,
\end{aligned}$$

for $z \in D$. Here the kernels $\Omega_{0,0}^0$ and N are given by formulae (3-11) and (3-13) on D :

$$(4-32) \quad N(z, \zeta) = \frac{1}{(2\pi\sqrt{-1})^n} \frac{1}{[G(z, \zeta) - \rho(\zeta)]^n} \\ \times \left(\sum g_1^i(z, \zeta) d\zeta_i \right) \wedge \left(\sum_i \bar{\partial}_\zeta g_1^i(z, \zeta) \wedge d\zeta_i \right)^{n-1}, \\ \Omega_{0,0}^0(z, \zeta) = \frac{1}{(2\pi\sqrt{-1})^n} \frac{1}{|\zeta - z|^{2n}} \\ \times \sum_{i=1}^n (\bar{\zeta}_i - \bar{z}_i) d\zeta_i \wedge \left(\sum_{j=1}^n (d\bar{\zeta}_j - d\bar{z}_j) \wedge d\zeta_j \right)^{n-1},$$

where G and g_1 are given by expressions (3-5) and (3-6). We can rewrite (4-31) as

$$(4-33) \quad f + \mathcal{K}f = h := h_1 + h_2 + h_3,$$

where we denote

$$h_1(z) := \int_D S_z(\bar{\partial}_z \bar{\partial}_\zeta N)(z, \cdot) \wedge f, \\ h_2(z) := - \int_D N(z, \cdot) \wedge \bar{\partial} f, \\ h_3(z) := \int_D \Omega_{0,0}^0(z, \cdot) \wedge \bar{\partial} f.$$

We show that each h_i defines a function in $C^\infty(\bar{D})$. By the first statement in (4-2), we have $h_1 \in C^\infty(\bar{D})$. For h_2 , note that the functions $G(z, \zeta)$, $g_1(z, \zeta)$ are C^∞ in z , and the following estimate (see (3-8)) holds

$$G(z, \zeta) - \rho(\zeta) \geq c(-\rho(z) - \rho(\zeta) + |z - \zeta|^2), \quad z, \zeta \in \bar{D}.$$

In particular, for $\zeta \in \text{supp}(\bar{\partial} f)$, i.e., $\zeta \in D_{-\delta}$, the function $G(z, \zeta) - \rho(\zeta)$ is bounded below by some positive constant for all $z \in \bar{D}$. Hence in view of (4-32), $h_2 \in C^\infty(\bar{D})$. To see that $h_3 \in C^\infty(\bar{D})$, we note that by assumption $\bar{\partial} f \in C_c^\infty(D)$, and the argument is done using integration by parts.

Now, by Proposition 1.2, \mathcal{K} is a compact operator on the Banach space $C^k(\bar{D})$. Thus by the Fredholm alternative, either $I + \mathcal{K}$ is invertible or $\ker(I + \mathcal{K})$ is nonempty. Suppose $f \in \ker(I + \mathcal{K})$; then $f = -\mathcal{K}f$ and $\sqrt{-1}\mathcal{K}f = -\sqrt{-1}f$. If $f \neq 0$, this would imply that $-\sqrt{-1}$ is an eigenvalue of the operator $\sqrt{-1}\mathcal{K}$, which is impossible since $\sqrt{-1}\mathcal{K}$ is self-adjoint and have only real eigenvalues. Therefore we conclude that $f \equiv 0$, and $\ker(I + \mathcal{K}) = \emptyset$. This implies $I + \mathcal{K}$ is an invertible operator on the space $C^k(\bar{D})$.

Applying this to (4-33) and $h \in C^\infty(\bar{\Omega})$, we obtain $f \in C^k(\bar{D})$. By Proposition 1.2, we have $\mathcal{K}f \in C^{k+\min\{\alpha, 1/2\}}(\bar{D})$. Hence $f = -\mathcal{K}f + h \in C^{k+\min\{\alpha, 1/2\}}(\bar{D})$. \square

We can now finally prove Theorem 1.1.

Proof of Theorem 1.1. Fix $w_0 \in D$, we can write the $B(\cdot, w_0) = \mathcal{P}\varphi$, where $\varphi \in C_c^\infty(D)$. Applying Proposition 4.2 to $f = \mathcal{P}\varphi - \varphi$ we get $\mathcal{P}\varphi - \varphi \in C^{k+\min\{\alpha, 1/2\}}(\bar{D})$. Hence $\mathcal{P}\varphi \in C^{k+\min\{\alpha, 1/2\}}(\bar{D})$. \square

5. Proof of Theorem 1.3

In this section we prove Theorem 1.3, which will also follow from Proposition 1.2. First we need an approximation lemma.

Lemma 5.1. *Let D be a bounded Lipschitz domain in \mathbb{R}^N . Suppose $f \in C^{k+\beta}(\bar{D})$, where k is a nonnegative integer and $0 < \beta < 1$. Then there exists a family $\{f_\varepsilon\}_{\varepsilon>0} \in C^\infty(D) \cap C^{k+\beta}(\bar{D})$ such that f_ε converges to f uniformly as $\varepsilon \rightarrow 0$. Furthermore, $|f_\varepsilon|_{k+\beta}$ is uniformly bounded by $|f|_{k+\beta}$.*

Remark 5.2. Let f_ε be constructed as above. It follows from [Shi 2023, Proposition 2.3] that f_ε converges to f in $|\cdot|_\tau$, for any $0 \leq \tau < k + \beta$.

Proof. It suffices to take D as a special Lipschitz domain of the form $\omega = \{x \in \mathbb{R}^N : x_N > \psi(x_1, \dots, x_{n-1}), |\psi|_{L^\infty} \leq C\}$, as the general case follows by standard partition of unity argument. There exists some cone K such that for any $x \in \omega$, $x + K \subseteq \omega$. Let ϕ be a C^∞ with compact supported in $-K$ and such that $\phi \geq 0$ and $\int_{\mathbb{R}^N} \phi = 1$. Let $\phi_\varepsilon = \frac{1}{\varepsilon^n} \phi(\frac{x}{\varepsilon})$. Then we can define for $x \in \omega$ the function

$$f_\varepsilon(x) = f * \phi_\varepsilon(x) = \int_{-K} f(x - \varepsilon y) \phi(y) dV(y).$$

It is clear that $f_\varepsilon \in C^\infty(D)$ and

$$\begin{aligned} |f_\varepsilon(x) - f(x)| &= \left| \int_{-K} [f(x - \varepsilon y) - f(x)] \phi(y) dV(y) \right| \\ &\leq \int_{-K} |f(x - \varepsilon y) - f(x)| \phi(y) dV(y) \\ &\leq |f|_\beta \varepsilon^\beta \int_{-K} y^\alpha \phi(y) dV(y) \\ &\lesssim |f|_\beta \varepsilon^\beta. \end{aligned}$$

Hence f_ε converges to f uniformly in ω . Let $x_1, x_2 \in \omega$. Then for all $\varepsilon > 0$,

$$\begin{aligned} |f_\varepsilon(x_1) - f_\varepsilon(x_2)| &= \left| \int_{-K} [f(x_1 - \varepsilon y) - f(x_2 - \varepsilon y)] \phi(y) dV(y) \right| \\ &\leq |f|_\beta |x_1 - x_2|^\beta. \end{aligned}$$

Accordingly $|f_\varepsilon|_\beta$ is uniformly bounded by $|f|_\beta$. This proves the case $k = 0$. For $k \geq 1$ the proof is similar and we leave the details to the reader. \square

Lemma 5.3. *Let D be a strictly pseudoconvex domain in \mathbb{C}^n with C^{k+3} boundary, where k is a nonnegative integer. Then for any $\beta > 0$,*

$$|\mathcal{L}f|_k \lesssim |f|_{k+\beta}.$$

Proof. Write $D_z^k \mathcal{L}f(z)$ as a linear combination of

$$(5-1) \quad \int_D f(\zeta) \frac{W(z, \zeta)}{\Phi^{n+1+\mu}(z, \zeta)} dV(\zeta), \quad \mu \leq k,$$

where $W(z, \zeta)$ is some linear combination of products of

$$D_z^{\tau_1}[l(\zeta) + O'(|z - \zeta|)], \quad D_z^{\tau_2}\Phi(z, \zeta), \quad \mu_1, \mu_2 \leq k.$$

Applying integration by parts formulae (3-43) and (3-45) iteratively to the integral (5-1) until it can be written as a linear combination of

$$(5-2) \quad \int_{bD} D^{\eta_0} f(\zeta) \frac{W_0(z, \zeta)}{\Phi^{n+1}(z, \zeta)} d\sigma(\zeta), \quad \int_D D^{\mu_0} f(\zeta) \frac{W_1(z, \zeta)}{\Phi^{n+1}(z, \zeta)} dV(\zeta),$$

where $\eta_0, \mu_0 \leq k$. Here W_0 and W_1 are some linear combination of

$$(5-3) \quad D_\zeta^{\mu_1} D_z^{\tau_1}[l(\zeta) + O'(|z - \zeta|)], \quad D_\zeta^{\mu_2}[(Q')^{-1}], \quad D_\zeta^{\mu_3+1}\Phi(z, \zeta), \\ D_\zeta^{\mu_4} D_z^{\tau_2}\Phi(z, \zeta), \quad D_\zeta^{\mu_5+1}\rho(\zeta),$$

with $\mu_i \leq k$, $0 \leq i \leq 5$ and $\sum_{i=0}^5 \mu_i \leq k$. We shall only estimate the domain integral in (5-2), as the proof of the boundary integral is similar. In view of (5-3), we can write $W_1(z, \zeta) = Y_1(z, \zeta) + Y_2(z, \zeta)$, where $|Y_1(z, \zeta)| \lesssim |\rho|_{k+2}$, and $|Y_2(z, \zeta)| \lesssim |\rho|_{k+3}|z - \zeta|$. Write

$$\int_D D^{\mu_0} f(\zeta) \frac{W_1(z, \zeta)}{\Phi^{n+1}(z, \zeta)} dV(\zeta) \\ = \int_D D^{\mu_0} f(\zeta) \frac{Y_1(z, \zeta)}{\Phi^{n+1}(z, \zeta)} dV(\zeta) + \int_D D^{\mu_0} f(\zeta) \frac{Y_2(z, \zeta)}{\Phi^{n+1}(z, \zeta)} dV(\zeta).$$

The Y_2 integral is bounded by

$$\left| \int_D D^{\mu_0} f(\zeta) \frac{Y_2(z, \zeta)}{\Phi^{n+1}(z, \zeta)} dV(\zeta) \right| \lesssim |\rho|_{k+3} |f|_k \int_D \frac{|z - \zeta|}{|\Phi(z, \zeta)|^{n+1}} \lesssim |\rho|_{k+3} |f|_k,$$

where we used (3-26). For the Y_1 integral we use the assumption that $f \in C^{k+\beta}$, $\beta > 0$,

$$\begin{aligned} & \int_D D^{\mu_0} f(\zeta) \frac{Y_1(z, \zeta)}{\Phi^{n+1}(z, \zeta)} dV(\zeta) \\ &= \int_D [D^{\mu_0} f(\zeta) - D^{\mu_0} f(z)] \frac{Y_1(z, \zeta)}{\Phi^{n+1}(z, \zeta)} dV(\zeta) + D^{\mu_0} f(z) \int_D \frac{Y_1(z, \zeta)}{\Phi^{n+1}(z, \zeta)} dV(\zeta). \end{aligned}$$

The first integral on the right-hand side is bounded by

$$|\rho|_{k+2} |f|_{k+\beta} \int_D \frac{|\zeta - z|^\beta dV(\zeta)}{|\Phi^{n+1}(z, \zeta)|} \lesssim |\rho|_{k+2} |f|_{k+\beta}.$$

For the other integral, since Y_1 involves derivatives of ρ up to order $k+2$, we can apply integration by parts and take one more derivative of ρ against ζ . The resulting integrals are bounded by $|\rho|_{k+3}$ up to a constant. Summing up the estimates we have

$$\left| \int_D D^{\mu_0} f(\zeta) \frac{Y_1(z, \zeta)}{\Phi^{n+1}(z, \zeta)} dV(\zeta) \right| \lesssim |\rho|_{k+3} |f|_{k+\beta}.$$

Consequently this shows that $|D_z^k \mathcal{L}f(z)| \lesssim |\rho|_{k+3} |f|_{k+\beta}$, finishing the proof. \square

We are now ready to prove Theorem 1.3.

Proposition 5.4. *Let D be a strictly pseudoconvex domain in \mathbb{C}^n . Let k be a nonnegative integer, and $0 < \alpha, \beta \leq 1$:*

- (i) *Suppose $bD \in C^{k+3}$. Then \mathcal{L} defines a bounded operator from $C^{k+\beta}(\bar{D})$ to $C^{k+\beta/2}(\bar{D})$.*
- (ii) *Suppose $bD \in C^{k+3+\alpha}$. Then $\mathcal{P}, \mathcal{L}^*$ define bounded operators from $C^{k+\beta}(\bar{D})$ to $C^{k+\min\{\alpha, \beta/2\}}(\bar{D})$.*

Proof. (i) We first prove the statement for \mathcal{L} and we begin by considering the case $k=0$. Assume first that $0 < \beta < 1$. Let $f \in C^\beta(\bar{D})$ and $\{f_\varepsilon\}_{\varepsilon>0}$ be the functions constructed in Lemma 5.1. In particular, we have:

- (1) $f_\varepsilon \in C^\infty(D) \cap C^\beta(\bar{D})$;
- (2) $|f_\varepsilon - f|_\eta \rightarrow 0$, for any $0 \leq \eta < \beta$ (Remark 5.2).

We claim that for each $\mathcal{L}f_\varepsilon \in C^{\beta/2}(\bar{D})$ with $|\mathcal{L}f_\varepsilon|_{\beta/2}$ uniformly bounded by some constant C_0 . Assuming the claim holds, then for any $z_1, z_2 \in D$, we have

$$\begin{aligned} (5-4) \quad & |\mathcal{L}f(z_1) - \mathcal{L}f(z_2)| \\ & \leq |\mathcal{L}f(z_1) - \mathcal{L}f_\varepsilon(z_1)| + |\mathcal{L}f_\varepsilon(z_1) - \mathcal{L}f_\varepsilon(z_2)| + |\mathcal{L}f(z_2) - \mathcal{L}f_\varepsilon(z_2)| \\ & \leq 2|\mathcal{L}(f - f_\varepsilon)|_0 + |\mathcal{L}f_\varepsilon|_{\beta/2} |z_1 - z_2|^{\beta/2} \\ & \leq 2|\mathcal{L}(f - f_\varepsilon)|_0 + C_0 |z_1 - z_2|^{\beta/2}. \end{aligned}$$

Now, given a function $g \in C^\eta(\bar{D})$ with $\eta > 0$, using the reproducing property of \mathcal{L} we have

$$(5-5) \quad \begin{aligned} |\mathcal{L}g(z)| &= \left| \int_D [g(\zeta) - g(z)]L(z, \zeta) dV(\zeta) + g(z) \right| \\ &\lesssim |\rho|_2 |g|_\eta \int_D \frac{|\zeta - z|^\eta}{|\Phi(z, \zeta)|^{n+1}} dV(\zeta) + |g|_0 \\ &\lesssim (|\rho|_2 + 1) |g|_\eta, \end{aligned}$$

where in the last inequality we applied Lemma 3.2.

Applying (5-5) with $g = f - f_\varepsilon$ and using property (2) from above, we get $|\mathcal{L}(f - f_\varepsilon)|_0 \rightarrow 0$ as $\varepsilon \rightarrow 0$. It follows from (5-4) that $|\mathcal{L}f(z_1) - \mathcal{L}f(z_2)| \leq C_0 |z_1 - z_2|^{\beta/2}$. This shows that $\mathcal{L}f \in C^{\beta/2}(\bar{D})$.

It remains to prove the claim, namely, $|\mathcal{L}f_\varepsilon|_{\beta/2}$ is bounded by some constant C_0 independent of ε . To this end, we will show that $|\mathcal{L}f_\varepsilon|_{\beta/2} \leq C'_0 |f_\varepsilon|_\beta$, where C'_0 depends only on $|\rho|_3$. Since $|f_\varepsilon|_\beta \leq |f|_\beta$, this proves the claim.

For $f_\varepsilon \in C^\infty(D) \cap C^\beta(\bar{D})$, we have

$$f_\varepsilon(z) - \mathcal{L}f_\varepsilon(z) = \int_D [f_\varepsilon(z) - f_\varepsilon(\zeta)]L(z, \zeta) dV(\zeta),$$

where we used the reproducing property of kernel L : $\int_D L(z, \zeta) dV(\zeta) \equiv 1$. Then

$$\frac{\partial f_\varepsilon}{\partial z_i}(z) - \frac{\partial \mathcal{L}f_\varepsilon}{\partial z_i}(z) = \int_D \frac{\partial f_\varepsilon}{\partial z_i}(z) L(z, \zeta) dV(\zeta) + \int_D [f_\varepsilon(z) - f_\varepsilon(\zeta)] \frac{\partial L}{\partial z_i}(z, \zeta) dV(\zeta).$$

The first term on each side cancels out, which leaves us with

$$\begin{aligned} &\frac{\partial \mathcal{L}}{\partial z_i}(z, \zeta) dV(\zeta) \\ &= \int_D [f_\varepsilon(\zeta) - f_\varepsilon(z)] \frac{\partial L}{\partial z_i}(z, \zeta) dV(\zeta) \\ &= \int_D [f_\varepsilon(\zeta) - f_\varepsilon(z)] \\ &\quad \times \left[\frac{\partial_{z_i} [l(\zeta) + O'(|z - \zeta|)]}{\Phi^{n+1}(z, \zeta)} - (n+1) \frac{[l(\zeta) + O'(|z - \zeta|)] \partial_{z_i} \Phi(z, \zeta)}{\Phi^{n+2}(z, \zeta)} \right] dV(\zeta). \end{aligned}$$

For \bar{z}_i derivatives we have a similar expression. By estimate (3-18) and (3-9), we obtain

$$\begin{aligned} |\nabla \mathcal{L}f_\varepsilon(z)| &\lesssim |\rho|_3 |f_\varepsilon|_\beta \left(\int_D \frac{|z - \zeta|^\beta}{|\Phi(z, \zeta)|^{n+1}} dV(\zeta) + \int_D \frac{|z - \zeta|^\beta}{|\Phi(z, \zeta)|^{n+2}} dV(\zeta) \right) \\ &\lesssim |\rho|_3 |f_\varepsilon|_\beta \left(\int_D \frac{|z - \zeta|^\beta}{|\Phi(z, \zeta)|^{n+1}} dV(\zeta) + \int_D \frac{dV(\zeta)}{|\zeta - z|^{2-\beta} \Phi(z, \zeta)^{n+1}} \right) \\ &\lesssim |\rho|_3 |f_\varepsilon|_\beta (1 + \delta(z)^{-1+\beta/2}), \end{aligned}$$

where in the last step we applied Lemma 3.2. It follows by Hardy–Littlewood lemma that $\mathcal{L}f_\varepsilon \in C^{\beta/2}(\bar{D})$ and $|\mathcal{L}f_\varepsilon|_{\beta/2}$ is bounded by $C'_0|f_\varepsilon|_\beta$, where C'_0 depends only on $|\rho|_3$. Combined with the earlier argument, this proves (i) for $k = 0$ and $0 < \beta < 1$. If $k = 0$ and $\beta = 1$, we can repeat the above proof without doing the approximation, obtaining in the end

$$|\nabla \mathcal{L}f(z)| \lesssim |\rho|_3 |f|_1 (1 + \delta(z))^{-1/2}, \quad z \in D.$$

Hence by Hardy–Littlewood lemma, $\mathcal{L}f \in C^{1/2}(\bar{D})$.

Next we consider the case $k \geq 1$. Suppose $f \in C^{k+\beta}(\bar{D})$, for $0 < \beta < 1$. As before we first construct $\{f_\varepsilon\}_{\varepsilon>0}$ such that

- (1) $f_\varepsilon \in C^\infty(D) \cap C^{k+\beta}(\bar{D})$;
- (2) $|f_\varepsilon - f|_\eta \rightarrow 0$, for any $0 \leq \eta < k + \beta$.

We claim that $|\mathcal{L}f_\varepsilon|_{k+\beta/2}$ is bounded uniformly by some constant C_0 . Assuming the validity of the claim, for $z_1, z_2 \in D$ and $\ell \leq k$, we have

$$\begin{aligned} (5-6) \quad |D^\ell \mathcal{L}f(z_1) - D^\ell \mathcal{L}f(z_2)| &\leq |D^\ell \mathcal{L}f(z_1) - D^\ell \mathcal{L}f_\varepsilon(z_1)| \\ &\quad + |D^\ell \mathcal{L}f_\varepsilon(z_1) - D^\ell \mathcal{L}f_\varepsilon(z_2)| \\ &\quad + |D^\ell \mathcal{L}f(z_2) - D^\ell \mathcal{L}f_\varepsilon(z_2)| \\ &\leq 2|\mathcal{L}(f - f_\varepsilon)|_\ell + |\mathcal{L}f_\varepsilon|_{k+\beta/2} |z_1 - z_2|^{\beta/2} \\ &\leq 2|\mathcal{L}(f - f_\varepsilon)|_k + C_0 |z_1 - z_2|^{\beta/2}. \end{aligned}$$

As before we want to show that

$$|\mathcal{L}(f - f_\varepsilon)|_k \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0.$$

Here the estimate is more subtle since $D\mathcal{L}g = \mathcal{L}Dg$ does not hold and thus one cannot estimate as easily as in (5-5). Instead we apply Lemma 5.3 to get

$$(5-7) \quad |\mathcal{L}(f - f_\varepsilon)|_k \lesssim |f - f_\varepsilon|_{k+\tau}, \quad \text{for any } \tau > 0.$$

By property (2) above, we have $|f - f_\varepsilon|_\eta \rightarrow 0$ for any $\eta < k + \beta$. Hence (5-7) implies $|\mathcal{L}(f - f_\varepsilon)|_k \rightarrow 0$. Letting $\varepsilon \rightarrow 0$ in (5-6), we get $\mathcal{L}f \in C^{k+\beta/2}(\bar{D})$, which proves the reduction.

To finish the proof it remains to show that there exists a constant $C'_0 > 0$ (which we will show depends only on $|\rho|_{k+\beta}$) such that $|\mathcal{L}f_\varepsilon|_{k+\beta/2} \leq C'_0|f_\varepsilon|_{k+\beta}$. Then by

Lemma 5.1 we get $|\mathcal{L}f_\varepsilon|_{k+\beta/2} \leq C'_0|f_\varepsilon|_{k+\beta} \leq C'_0|f|_{k+\beta}$. We have

$$\begin{aligned} & D_z^{k+1}[f_\varepsilon(z) - \mathcal{L}f_\varepsilon(z)] \\ &= D_z^{k+1} \int_D [f_\varepsilon(z) - f_\varepsilon(\zeta)]L(z, \zeta) dV(\zeta) \\ &= \int_D D_z^{k+1} f_\varepsilon(z)L(z, \zeta) dV(\zeta) + \int_D \sum_{\substack{\gamma_1+\gamma_2=k+1 \\ 1 \leq \gamma_2 \leq k}} D_z^{\gamma_1} f_\varepsilon(z) D_z^{\gamma_2} L(z, \zeta) dV(\zeta) \\ &\quad + \int_D (f_\varepsilon(z) - f_\varepsilon(\zeta)) D_z^{k+1} L(z, \zeta) dV(\zeta). \end{aligned}$$

The first integral is equal to $D_z^{k+1} f(z)$. Hence

$$\begin{aligned} D_z^{k+1} \mathcal{L}f_\varepsilon(z) &= - \int_D \sum_{\substack{\gamma_1+\gamma_2=k+1 \\ 1 \leq \gamma_2 \leq k}} D_z^{\gamma_1} f_\varepsilon(z) D_z^{\gamma_2} L(z, \zeta) dV(\zeta) \\ &\quad + \int_D [f_\varepsilon(\zeta) - f_\varepsilon(z)] D_z^{k+1} L(z, \zeta) dV(\zeta) \\ &:= I_1 + I_2, \end{aligned}$$

where we denote the first and second integral by I_1 and I_2 , respectively. For I_1 , we can write it as a linear combination of integrals of the form

$$(5-8) \quad D_z^{\mu_0} f_\varepsilon(z) \int_D \frac{W(z, \zeta)}{\Phi^{n+1+\mu_1}(z, \zeta)} dV(\zeta), \quad \mu_0, \mu_1 \leq k,$$

where W is some linear combination of $D_z^{\mu_2}[l(\zeta) + O'(|z - \zeta|)]$ and $D_z^{\mu_3} \Phi(z, \zeta)$ with $\mu_2, \mu_3 \leq k$. We apply integration by parts formulae (3-43) and (3-44) iteratively to the integral in (5-8) until it can be written as a linear combination of

$$(5-9) \quad \int_{bD} \frac{W_0(z, \zeta)}{\Phi^n(z, \zeta)} d\sigma(\zeta), \quad \int_D \frac{W_1(z, \zeta)}{\Phi^{n+1}(z, \zeta)} dV(\zeta).$$

Here W_0, W_1 are linear combinations of products of

$$\begin{aligned} & D_\zeta^{\tau_1} D_{z_i}^{\mu_2} [l(\zeta) + O'(|z - \zeta|)], \quad D_\zeta^{\tau_2} [(Q')^{-1}], \quad D_\zeta^{\tau_3+1} \Phi(z, \zeta), \\ & D_\zeta^{\tau_4} D_z^{\mu_3} \Phi(z, \zeta), \quad D_\zeta^{\tau_5+1} \rho(\zeta), \end{aligned}$$

with $\tau_i \leq k, 1 \leq i \leq 5$. Note that all these quantities are bounded by some constant multiple of $|\rho|_{k+3}$. It follows that the integrals in (5-9) and hence I_1 is bounded by

$$\begin{aligned} (5-10) \quad |I_1| &\lesssim |f_\varepsilon|_k |\rho|_{k+3} \left(\int_{bD} \frac{d\sigma(\zeta)}{|\Phi(z, \zeta)|^n} + \int_D \frac{dV(\zeta)}{|\Phi(z, \zeta)|^{n+1}} \right) \\ &\lesssim |f_\varepsilon|_k |\rho|_{k+3} (1 + \log \delta(z)), \end{aligned}$$

where we applied Lemma 3.4. The integral I_2 can be written as a linear combination of integrals of the form

$$(5-11) \quad \int_D \frac{[f_\varepsilon(\zeta) - f_\varepsilon(z)]W(z, \zeta)}{\Phi^{n+2+\mu}(z, \zeta)} dV(\zeta), \quad \mu \leq k.$$

Here $W(z, \zeta)$ is some linear combination of

$$D_z^{\tau_0}[l(\zeta) + O'(|z - \zeta|)], \quad D_z^{\tau_1}\Phi(z, \zeta), \quad \tau_0, \tau_1 \leq k + 1.$$

If $\mu \leq k - 1$ we can integrate by parts and estimate just like I_1 to show that $|I_2| \lesssim |f_\varepsilon|_k |\rho|_{k+3} (1 + \log|\delta(z)|)$. If $\mu = k$, we apply integration by parts formulae (3-43) and (3-45) until the integral (5-11) can be expressed as a linear combination of integrals of the form

$$(5-12) \quad \int_{bD} \frac{D_\zeta^{\eta_0} f_\varepsilon(\zeta) A_0(z, \zeta)}{\Phi^{n+1}(z, \zeta)} d\sigma(\zeta), \quad \int_D \frac{D^{\mu_0} f_\varepsilon(\zeta) A_1(z, \zeta)}{\Phi^{n+2}(z, \zeta)} dV(\zeta),$$

for $\eta_0, \mu_0 \leq k$. Here A_0, A_1 are linear combination of products of

$$(5-13) \quad D_\zeta^{\mu_1}[l(\zeta) + O'(|z - \zeta|)], \quad D_\zeta^{\mu_2}[(Q')^{-1}], \quad D_\zeta^{\mu_3+1}\Phi(z, \zeta), \\ D_\zeta^{\mu_4}D_z\Phi(z, \zeta), \quad D_\zeta^{\mu_5+1}\rho(\zeta),$$

where $\mu_i \leq k$, and $\sum_{i=0}^5 \mu_i = k$. We now use the fact that $\rho \in C^\infty(D) \cap C^{k+3}(\bar{D})$ satisfies the estimate

$$|D_z^j \rho(z)| \lesssim C_j |\rho|_{k+3} (1 + \delta(z)^{k+3-j}), \quad j = 0, 1, 2, \dots$$

We shall only estimate the domain integral in (5-12), as the estimate for the boundary integral is similar. In view of (5-13) we can write $A_1(z, \zeta) = X_1(z, \zeta) + X_2(z, \zeta)$, where $|X_1(z, \zeta)| \lesssim |\rho|_{k+2}$ and $|X_2(z, \zeta)| \lesssim |\rho|_{k+3} |\zeta - z|$. Write

$$\int_D \frac{D^{\mu_0} f_\varepsilon(\zeta) A_1(z, \zeta)}{\Phi^{n+2}(z, \zeta)} dV(\zeta) \\ = \int_D \frac{D^{\mu_0} f_\varepsilon(\zeta) X_1(z, \zeta)}{\Phi^{n+2}(z, \zeta)} dV(\zeta) + \int_D \frac{D^{\mu_0} f_\varepsilon(\zeta) X_2(z, \zeta)}{\Phi^{n+2}(z, \zeta)} dV(\zeta).$$

By estimates (3-9) and (3-25), we see that

$$\left| \int_D \frac{D^{\mu_0} f_\varepsilon(\zeta) X_2(z, \zeta)}{\Phi^{n+2}(z, \zeta)} dV(\zeta) \right| \lesssim |\rho|_{k+3} |f_\varepsilon|_k \int_D \frac{|\zeta - z|}{|\Phi(z, \zeta)|^{n+2}} dV(\zeta) \\ \lesssim |\rho|_{k+3} |f_\varepsilon|_k \int_D \frac{dV(\zeta)}{|\zeta - z| |\Phi(z, \zeta)|^{n+1}} \\ \lesssim |\rho|_{k+3} |f_\varepsilon|_k (1 + \delta(z)^{-1/2}), \quad \mu_0 \leq k.$$

On the other hand, we can write

$$(5-14) \quad \int_D \frac{D^{\mu_0} f_\varepsilon(\zeta) X_1(z, \zeta)}{\Phi^{n+2}(z, \zeta)} dV(\zeta) \\ = \int_D \frac{[D_\zeta^{\mu_0} f_\varepsilon(\zeta) - D_z^{\mu_0} f_\varepsilon(z)] X_1(z, \zeta)}{\Phi^{n+2}(z, \zeta)} dV(\zeta) \\ + D_z^{\mu_0} f_\varepsilon(z) \int_D \frac{X_1(z, \zeta)}{\Phi^{n+2}(z, \zeta)} dV(\zeta).$$

Since $f_\varepsilon \in C^{k+\beta}(\bar{D})$, the first integral on the right-hand side above is bounded up to a constant by

$$|\rho|_{k+2} |f_\varepsilon|_{k+\beta} \int_D \frac{|\zeta - z|^\beta}{|\Phi(z, \zeta)|^{n+2}} dV(\zeta) \lesssim |\rho|_{k+2} |f_\varepsilon|_{k+\beta} \int_D \frac{dV(\zeta)}{|\zeta - z|^{2-\beta} |\Phi(z, \zeta)|^{n+1}} \\ \lesssim |\rho|_{k+2} |f_\varepsilon|_{k+\beta} (1 + \delta(z)^{-1+\beta/2}).$$

For the second integral on the right-hand side of (5-14), we can integrate by parts and bound the resulting expression by

$$|\rho|_{k+3} |f_\varepsilon|_k \left(\int_{bD} \frac{d\sigma(\zeta)}{|\Phi(z, \zeta)|^n} + \int_D \frac{dV(\zeta)}{|\Phi(z, \zeta)|^{n+1}} \right) \lesssim |\rho|_{k+3} |f_\varepsilon|_k (1 + \log \delta(z)).$$

Hence we have shown that

$$|I_2| \lesssim |f_\varepsilon|_{k+\beta} |\rho|_{k+3} (1 + \delta(z)^{-1+\beta/2}).$$

Combined with the estimate (5-10) for I_1 , this shows that

$$|D_z^{k+1} \mathcal{L} f_\varepsilon(z)| \lesssim |f_\varepsilon|_{k+\beta} |\rho|_{k+3} (1 + \delta(z)^{-1+\beta/2}).$$

By Lemma 2.3, $\mathcal{L} f_\varepsilon \in C^{k+\beta/2}(\bar{D})$ and $|\mathcal{L} f_\varepsilon|_{k+\beta/2} \leq C'_0 |f_\varepsilon|_{k+\beta}$ where C'_0 depends only on $|\rho|_{k+3}$. This proves the claim and hence the case when $0 < \beta < 1$. Finally if $\beta = 1$, the same proof works without the use of the approximation.

(ii) From Proposition 1.2 we know that $\mathcal{K}f \in C^{k+\min\{\alpha, 1/2\}}(\bar{D})$ if $f \in C^k(\bar{D})$ (and in particular if $f \in C^{k+\beta}(\bar{D})$ for $0 < \beta \leq 1$). By (i), $\mathcal{L}f \in C^{k+\beta/2}(\bar{D})$. Since $\mathcal{L}^* f = \mathcal{K}f + \mathcal{L}f$, and $\min\{\alpha, \frac{1}{2}, \beta/2\} = \min\{\alpha, \beta/2\}$, we have

$$\mathcal{L}^* f \in C^{k+\min\{\alpha, \beta/2\}}(\bar{D}).$$

Finally by the integral equation $(I + \mathcal{K})Pf = \mathcal{L}^* f$, and the fact that $I + \mathcal{K}$ is invertible in the space $C^k(\bar{D})$, we get $Pf \in C^k(\bar{D})$ and thus $\mathcal{K}Pf \in C^{k+\min\{\alpha, 1/2\}}(\bar{D})$ by Proposition 1.2. Therefore $Pf = -\mathcal{K}Pf + \mathcal{L}^* f \in C^{k+\min\{\alpha, \beta/2\}}(\bar{D})$. \square

Acknowledgements

The author would like to thank the anonymous referee for many valuable suggestions that improve the exposition of the paper.

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Received August 7, 2022. Revised February 8, 2023.

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
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The Pacific Journal of Mathematics (ISSN 1945-5844 electronic, 0030-8730 printed) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

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Volume 324 No. 1 May 2023

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