Pacific Journal of Mathematics

POLYNOMIAL CONDITIONS AND HOMOLOGY OF FI-MODULES

CIHAN BAHRAN

Volume 324 No. 2 June 2023

PACIFIC JOURNAL OF MATHEMATICS
Vol. 324, No. 2, 2023
https://doi.org/10.2140/pjm.2023.324.207

POLYNOMIAL CONDITIONS AND HOMOLOGY OF FI-MODULES

CIHAN BAHRAN

We identify two recursively defined polynomial conditions for FI-modules in the literature. We characterize these conditions using homological invariants of FI-modules (namely the local degree and regularity, together with the stable degree) and clarify their relationship. For one of these conditions, we give improved twisted homological stability ranges for the symmetric groups. As another application, we improve the representation stability ranges for congruence subgroups with respect to the action of an appropriate linear group by a factor of 2 in its slope.

1. Introduction

There are (at least) two classes of papers that deal in some depth with **FI**-modules:

(1) In papers such as [3; 4; 6; 11; 14] the **FI**-module is the central object of study. They attach *homological invariants* to an **FI**-module by means such as **FI**-homology or local cohomology, and study the relationship of these invariants both with the stabilization behavior of the **FI**-module and/or between each other.

(2) Papers such as [12; 13; 16; 18; 22] might be thought of as stability machines. The sequence $\{\mathfrak{S}_n\}$ of symmetric groups is but one of many sequences of groups they deal with and **FI**-modules arise as the suitable notion of *coefficient systems* for $\{\mathfrak{S}_n\}$. They declare a coefficient system to be *polynomial* with certain parameters in a *recursive* fashion: there is a base case, and above that, being polynomial demands a related coefficient system to be polynomial with some of the parameters lowered.

The main objective of this paper is to characterize the polynomial conditions in (2) for **FI**-modules by the homological invariants in (1).

The author was supported in part by TÜBİTAK 119F422.
MSC2020: primary 18A25, 20J06; secondary 11F75.
Keywords: FI-modules, homological stability, representation stability, congruence subgroups.

Notation. We write **FI** for the category of finite sets and injections. An **FI**-module is a functor $V : \mathbf{FI} \to \mathbb{Z}\text{-Mod}$ and given a finite set S, we write V_S for its evaluation; given an injection of finite sets $\alpha : S \hookrightarrow T$, we write $V_\alpha : V_S \to V_T$ for its induced map. For $n \in \mathbb{N}$ we set $V_n := V_{\{1,\ldots,n\}}$. We write **FI**-Mod for the category of **FI**-modules. Throughout, our notation for **FI**-modules will be consistent with [6] and [1].

Degree and torsion. Given an **FI**-module W, we write

$$\deg(W) := \min\{d \geq -1 : W_S = 0 \text{ for } |S| > d\} \in \{-1, 0, 1, 2, 3, \ldots\} \cup \{\infty\}.$$

An **FI**-module V is *torsion* if for every finite set S and $x \in V_S$, there exists an injection $\alpha : S \hookrightarrow T$ such that $V_\alpha(x) = 0 \in V_T$. We write

$$\mathrm{H}^0_{\mathfrak{m}} : \mathbf{FI}\text{-Mod} \to \mathbf{FI}\text{-Mod}$$

for the functor which assigns an **FI**-module its largest torsion **FI**-submodule, and write

$$h^0(V) := \deg(\mathrm{H}^0_{\mathfrak{m}}(V)).$$

Shift and derivative functors. Given any **FI**-module V, we write ΣV for the composition

$$\mathbf{FI} \xrightarrow{-\sqcup\{*\}} \mathbf{FI} \xrightarrow{V} \mathbb{Z}\text{-Mod}$$

and call it the *shift functor*. It receives a natural transformation from the identity functor $\mathrm{id}_{\mathbf{FI}\text{-Mod}}$, whose cokernel

$$\Delta := \mathrm{coker}(\mathrm{id}_{\mathbf{FI}\text{-Mod}} \to \Sigma)$$

we call the *derivative functor*.

Stable degree. For an **FI**-module V, we set

$$\delta(V) := \min\{r \geq -1 : \Delta^{r+1}(V) \text{ is torsion}\} \in \{-1, 0, 1, \ldots\} \cup \{\infty\}$$

and call it the *stable degree* of V. In both polynomial conditions for **FI**-modules we shall consider, the stable degree will be in analogy with the usual degree of a polynomial. Also see [6, Proposition 2.14].

First polynomial condition and local degree. Suppose f is a function in $n \in \mathbb{N}$ which is equal to a polynomial of degree $\leq r$ in the range $n \geq L$. We can consider its *discrete derivative* Δf, which is the function

$$\Delta f(n) := f(n+1) - f(n).$$

Note that Δf is equal to a polynomial of degree $\leq r - 1$ in the same range $n \geq L$. The first polynomial condition we treat for **FI**-modules is a categorification of this recursion. See [22, Section 4.4 and Remark 4.19] for references to similar definitions in the literature.

Definition 1.1. For every pair of integers $r \geq -1,\ L \geq 0$, we define a class of **FI**-modules $\mathbf{Poly}_1(r, L)$ recursively via

$$\mathbf{Poly}_1(r, L) := \begin{cases} \{V \in \mathbf{FI}\text{-Mod} : \deg(V) \leq L-1\} & \text{if } r = -1, \\ \left\{ V \in \mathbf{FI}\text{-Mod} : \begin{array}{l} h^0(V) \leq L-1 \text{ and} \\ \Delta V \in \mathbf{Poly}_1(r-1, L) \end{array} \right\} & \text{if } r \geq 0. \end{cases}$$

Remark 1.2. Let V be an **FI**-module and $r \geq -1,\ L \geq 0$ be integers. The following can be seen to be equivalent by inspection:

- $V \in \mathbf{Poly}_1(r, L)$.
- In the sense of [21, Definition 4.10],[1] V has degree r at L.
- In the sense of [13, Definition 3.24] and [16, Definition 7.1], V has polynomial degree $\leq r$ in ranks $> L-1$.

Local cohomology and local degree. The functor $\mathrm{H}^0_{\mathfrak{m}}$ defined above is left exact. For each $j \geq 0$, we write $\mathrm{H}^j_{\mathfrak{m}} := \mathrm{R}^j \mathrm{H}^0_{\mathfrak{m}}$ for the j-th right derived functor of $\mathrm{H}^0_{\mathfrak{m}}$, and write

$$h^j(V) := \deg(\mathrm{H}^j_{\mathfrak{m}}(V)) \in \{-1, 0, 1, \dots\} \cup \{\infty\},$$
$$h^{\max}(V) := \max\{h^j(V) : j \geq 0\} \in \{-1, 0, 1, \dots\} \cup \{\infty\}$$

for every **FI**-module V. We call $h^{\max}(V)$ the *local degree* of V.

Our first main result is that the stable degree $\delta(V)$ and the local degree $h^{\max}(V)$ together characterize the first polynomial condition.

Theorem A. *For every pair of integers $r \geq -1,\ L \geq 0$, we have*

$$\mathbf{Poly}_1(r, L) = \{V \in \mathbf{FI}\text{-Mod} : \delta(V) \leq r \text{ and } h^{\max}(V) \leq L-1\}.$$

Second polynomial condition and regularity. The second polynomial condition we shall treat is, perhaps deceivingly, very similar to the first one. In fact the confusion between the two and the resulting need to clarify was what prompted this paper.

Definition 1.3. For every pair of integers $r \geq -1,\ M \geq 0$, we define a class of **FI**-modules $\mathbf{Poly}_2(r, M)$ recursively via

$$\mathbf{Poly}_2(r, M) := \begin{cases} \{V \in \mathbf{FI}\text{-Mod} : \deg(V) \leq M-1\} & \text{if } r = -1, \\ \left\{ V \in \mathbf{FI}\text{-Mod} : \begin{array}{l} h^0(V) \leq M-1 \text{ and} \\ \Delta V \in \mathbf{Poly}_2(r-1, \max\{0, M-1\}) \end{array} \right\} & \text{if } r \geq 0. \end{cases}$$

Remark 1.4. Let V be an **FI**-module and $r \geq -1,\ M \geq 0$ be integers. The following can be seen to be equivalent by inspection:

- $V \in \mathbf{Poly}_2(r, M)$.
- In the sense of [22, Definition 4.10], V has degree r at M.

[1] Note that [21] is an early preprint version of the published [22]. The authors switched from Definition 1.1 to Definition 1.3 in between.

- In the sense of [12, Definition 2.40],[2] V has polynomial degree $\leq r$ in ranks $> M-1$.
- In the sense of [18, Definition 1.6], V is polynomial of degree r starting at M.

FI-homology and regularity. Regard the functor $\mathrm{H}_0^{\mathbf{FI}} : \mathbf{FI}\text{-}\mathsf{Mod} \to \mathbf{FI}\text{-}\mathsf{Mod}$ defined via
$$\mathrm{H}_0^{\mathbf{FI}}(V)_S := \operatorname{coker}\big(\textstyle\bigoplus_{T \subsetneq S} V_T \to V_S\big)$$
for every finite set S, which is right exact. For each $i \geq 0$, we write $\mathrm{H}_i^{\mathbf{FI}} := \mathrm{L}_i \mathrm{H}_0^{\mathbf{FI}}$ for its i-th left derived functor, and write
$$t_i(V) := \deg(\mathrm{H}_i^{\mathbf{FI}}(V)) \in \{-1, 0, 1, \dots\} \cup \{\infty\},$$
$$\operatorname{reg}(V) := \max\{t_i(V) - i : i \geq 1\} \in \{-2, -1, 0, 1, \dots\} \cup \{\infty\}$$
for every **FI**-module V. We say that V is *generated in degrees* $\leq g$ if $t_0(V) \leq g$, and that V is *presented in finite degrees* if $t_0(V)$ and $t_1(V)$ are both finite. We call $\operatorname{reg}(V)$ the *regularity* of V.

Our second main result is that the stable degree $\delta(V)$ and the regularity $\operatorname{reg}(V)$ together characterize the second polynomial condition.

Theorem B. *For every pair of integers $r \geq -1$, $M \geq 0$, we have*
$$\mathbf{Poly}_2(r, M) = \{V \in \mathbf{FI}\text{-}\mathsf{Mod} : \delta(V) \leq r \;\; \textit{and} \;\; \operatorname{reg}(V) \leq M-1\}.$$

Twisted homological stability with* FI*-module coefficients. For any **FI**-module V and homological degree $k \geq 0$, there is a sequence of maps
$$\mathrm{H}_k(\mathfrak{S}_0; V_0) \to \mathrm{H}_k(\mathfrak{S}_1; V_1) \to \mathrm{H}_k(\mathfrak{S}_2; V_2) \to \cdots$$
between the homology groups of the symmetric groups twisted by V_n's. For the stabilization of this sequence, recently Putman [18, Theorems A and A′] established explicit ranges for the class $\mathbf{Poly}_2(r, M)$ in terms of r, M. We give ranges for the class $\mathbf{Poly}_1(r, L)$ in terms of r, L.

Theorem C. *Let V be an* **FI***-module and $r, L \geq 0$ be integers such that $V \in \mathbf{Poly}_1(r, L)$. Then for every $k \geq 0$, the map $\mathrm{H}_k(\mathfrak{S}_n; V_n) \to \mathrm{H}_k(\mathfrak{S}_{n+1}; V_{n+1})$ is*
$$\textit{an isomorphism for } n \geq \begin{cases} 2k+r+1 & \textit{if } L = 0, \\ 2k+r+\left\lfloor \frac{L+1}{2} \right\rfloor + 2 & \textit{if } 1 \leq L \leq 2r-2, \\ \max\{2k+2r+1, L\} & \textit{if } L \geq \max\{1, 2r-1\}, \end{cases}$$
$$\textit{and a surjection for } n \geq \begin{cases} 2k+r & \textit{if } L = 0, \\ 2k+r+\left\lfloor \frac{L+1}{2} \right\rfloor + 1 & \textit{if } 1 \leq L \leq 2r-2, \\ \max\{2k+2r, L\} & \textit{if } L \geq \max\{1, 2r-1\}. \end{cases}$$

[2]Although the *terminology* used for the polynomial conditions in [13], [16] and [12] are the same, the first two use Definition 1.1 while the latter uses Definition 1.3.

Remark 1.5. Under the same hypotheses with Theorem C, Theorem 5.1 of [21] establishes

- an isomorphism for $n \geq \max\{2L+1, 2k+2r+2\}$,
- and a surjection for $n \geq \max\{2L+1, 2k+2r\}$.

The ranges in Theorem C are improvements over these.

$\mathbf{SL}_n^{\mathfrak{U}}$-stability ranges for congruence subgroups. For every ring R, the assignment $n \mapsto \mathrm{GL}_n(R)$ defines an **FI**-group (a functor from **FI** to the category of groups), for which we write $\mathrm{GL}_\bullet(R)$. If I is an ideal of R, as the kernel of the mod-I reduction we get a smaller **FI**-group

$$\mathrm{GL}_\bullet(R, I) := \ker\bigl(\mathrm{GL}_\bullet(R) \to \mathrm{GL}_\bullet(R/I)\bigr)$$

called the *I-congruence subgroup* of $\mathrm{GL}_\bullet(R)$. For each $k \geq 0$ and abelian group $\mathcal{A}$, taking the k-th homology with coefficients in $\mathcal{A}$ defines an **FI**-module

$$\mathrm{H}_k(\mathrm{GL}_\bullet(R, I); \mathcal{A}).$$

We wish to extend the $\mathfrak{S}_n$-action on $\mathrm{H}_k(\mathrm{GL}_n(R, I); \mathcal{A})$ to an action of a linear group and formulate representation stability over it, in accordance with [17, fifth Remark, page 990].

Special linear group with respect to a subgroup of the unit group. For a commutative ring A and a subgroup $\mathfrak{U} \leq A^\times$, we write

$$\mathrm{SL}_n^{\mathfrak{U}}(A) := \{f \in \mathrm{GL}_n(A) : \det(f) \in \mathfrak{U}\},$$

so that we interpolate between $\mathrm{SL}_n(A) \leq \mathrm{SL}_n^{\mathfrak{U}}(A) \leq \mathrm{GL}_n(A)$ as we vary $1 \leq \mathfrak{U} \leq A^\times$. Note that we are using the notation in [19], whereas in [13] and [12] this group is denoted $\mathrm{GL}_n^{\mathfrak{U}}(A)$.

Hypothesis 1.6. In the triple (R, I, n_0), we have a commutative ring R, an ideal I of R, and an integer $n_0 \in \mathbb{N}$ such that the mod-I reduction

$$\mathrm{SL}_n(R) \to \mathrm{SL}_n(R/I)$$

for the special linear group is surjective for every $n \geq n_0$.

Stable rank of a ring. Let R be a nonzero unital (associative) ring. A column vector $\boldsymbol{v} \in \mathrm{Mat}_{m\times 1}(R)$ of size m is *unimodular* if there is a row vector $\boldsymbol{u} \in \mathrm{Mat}_{1\times m}(R)$ such that $\boldsymbol{uv} = 1$. Writing $I_r \in \mathrm{Mat}_{r\times r}(R)$ for the identity matrix of size r, we say a column vector $\boldsymbol{v}$ of size m is *reducible* if there exists $A \in \mathrm{Mat}_{(m-1)\times m}(R)$ with block form $A = [I_{m-1} \mid \boldsymbol{x}]$ such that the column vector $A\boldsymbol{v}$ (of size $m-1$) is unimodular. We write $\mathbf{st\text{-}rank}(R) \leq s$ if every unimodular column vector of size $> s$ is reducible.

Remark 1.7. We make a few observations about Hypothesis 1.6.

(1) It is straightforward to check that the triple (R, I, n_0) satisfies Hypothesis 1.6 if and only if setting $\mathfrak{U} := \{x + I : x \in R^\times\}$, there is a short exact sequence

$$1 \to \mathrm{GL}_n(R, I) \to \mathrm{GL}_n(R) \to \mathrm{SL}_n^{\mathfrak{U}}(R/I) \to 1$$

of groups in the range $n \geq n_0$ where the epimorphism is the mod-I reduction. Consequently, for every $n \geq n_0$ and any coefficients $\mathcal{A}$, the conjugation $\mathrm{GL}_n(R)$-action on the homology groups $\mathrm{H}_\star(\mathrm{GL}_n(R, I), \mathcal{A})$ descends to an $\mathrm{SL}_n^{\mathfrak{U}}(R/I)$-action. It is this action for which we will obtain an improved representation stability range.

(2) For a Dedekind domain R and any ideal I of R, the triple $(R, I, 0)$ satisfies Hypothesis 1.6; see [7, page 2].

(3) If $\mathrm{SL}_n(R/I)$ is generated by elementary matrices for $n \geq n_0$, then (R, I, n_0) satisfies Hypothesis 1.6.

(4) If the K-group $\mathrm{SK}_1(R/I) = 0$ (equivalently, the natural map $\mathrm{K}_1(R/I) \to (R/I)^\times$ is an isomorphism) and $\textbf{st-rank}(R/I) \leq s < \infty$, then by (3) and [10, 4.3.8, page 172], the triple $(R, I, s + 1)$ satisfies Hypothesis 1.6.

Theorem D. *Let I be a proper ideal in a commutative ring R and $s, n_0 \in \mathbb{N}$, so*

- $\textbf{st-rank}(R) \leq s$, *and*
- *the triple (R, I, n_0) satisfies Hypothesis 1.6 with $n_0 \leq 2s + 3$.*

Then writing

$$\mathfrak{U} := \{x + I : x \in R^\times\}, \quad \mathcal{G}_n := \mathrm{SL}_n^{\mathfrak{U}}(R/I)$$

for every homological degree $k \geq 1$ and abelian group $\mathcal{A}$, there is a coequalizer diagram

$$\mathrm{Ind}_{\mathcal{G}_{n-2}}^{\mathcal{G}_n} \mathrm{H}_k(\mathrm{GL}_{n-2}(R, I); \mathcal{A}) \rightrightarrows \mathrm{Ind}_{\mathcal{G}_{n-1}}^{\mathcal{G}_n} \mathrm{H}_k(\mathrm{GL}_{n-1}(R, I); \mathcal{A}) \to \mathrm{H}_k(\mathrm{GL}_n(R, I); \mathcal{A})$$

of $\mathbb{Z}\mathcal{G}_n$-modules whenever

$$n \geq \begin{cases} 2s + 5 & \text{if } k = 1, \\ 4k + 2s + 2 & \text{if } k \geq 2. \end{cases}$$

Remark 1.8. The best stable ranges established previously in the literature under the assumptions (with $n_0 = 0$) of Theorem D are due to Miller, Patzt and Petersen [12, proof of Theorem 1.4, page 46]: they obtained the conclusion of Theorem D in the range $n \geq 8k + 4s + 9$.

2. Homological algebra of FI-modules

Regularity in terms of local cohomology. We first recall a characterization of the regularity by Nagpal, Sam and Snowden [14].

Theorem 2.1 [14, Theorem 1.1, Remark 1.3]. *Let V be an* **FI***-module presented in finite degrees which is not* $\mathrm{H}_0^{\mathbf{FI}}$*-acyclic. Then*

$$\operatorname{reg}(V) = \max\{h^j(V) + j : \mathrm{H}^j_{\mathfrak{m}}(V) \neq 0\} = \max\{h^j(V) + j : h^j(V) \geq 0\}.$$

Remark 2.2. Under the hypotheses of Theorem 2.1, by [1, Theorem 2.4] and [6, Corollary 2.13], we have

$$\varnothing \neq \{j : \mathrm{H}^j_{\mathfrak{m}}(V) \neq 0\} = \{j : h^j(V) \geq 0\} \subseteq \{0, \dots, \delta(V) + 1\}.$$

Corollary 2.3. *Let V be a nonzero* **FI***-module with* $\deg(V) < \infty$. *Then V is presented in finite degrees*, *and*

$$h^j(V) = \begin{cases} \deg(V) = \operatorname{reg}(V) & \text{if } j = 0, \\ -1, & \text{otherwise.} \end{cases}$$

Proof. V is certainly generated in degrees $\leq \deg(V)$ and also $h^0(V) \leq \deg(V)$. Thus by [1, Proposition 2.5] and [20, Theorem A], V is presented in finite degrees. Now V and the complex $0 \to V \to 0 \to 0 \to \cdots$ satisfy the assumptions of [6, Theorem 2.10] and hence

$$\mathrm{H}^j_{\mathfrak{m}}(V) = \begin{cases} V & \text{if } j = 0, \\ 0, & \text{otherwise.} \end{cases}$$

The rest follows from Theorem 2.1. □

The derivative and local cohomology. In this section, we investigate the relationship between the local cohomology of an **FI**-module and that of its derivative.

We write $K := \ker(\mathrm{id}_{\mathbf{FI}\text{-}\mathsf{Mod}} \to \Sigma)$ so that we have an exact sequence

$$0 \to K \to \mathrm{id}_{\mathbf{FI}\text{-}\mathsf{Mod}} \to \Sigma \to \Delta \to 0$$

of functors **FI**-Mod → **FI**-Mod.

Lemma 2.4. *For every* **FI***-module V, we have* $\deg(KV) = h^0(V)$.

Proof. Since KV is a torsion submodule of V, we have $KV \subseteq \mathrm{H}^0_{\mathfrak{m}}(V)$ and hence

$$\deg(KV) \leq \deg(\mathrm{H}^0_{\mathfrak{m}}(V)) = h^0(V).$$

There is nothing to show when $h^0(V) = -1$, so we consider two cases.

Case 1. $h^0(V) = \infty$. To show $\deg(KV) = \infty$, we will show that for every $d \in \mathbb{N}$ we have $\deg(KV) \geq d$. Because $h^0(V) \geq d$, there exists a torsion element $x \in V_S - \{0\}$ of V with $|S| \geq d$. Because x is torsion, the set

$$\{|T| : V_\iota(x) = 0 \text{ for some } \iota : S \hookrightarrow T\} \subseteq \mathbb{N}$$

is nonempty and hence has a least element, say N. Noting $N > d$, let A be a finite set of size $N-1$ and $f : S \hookrightarrow A$ so by the minimality of N we have $0 \neq V_f(x) \in (KV)_A$ and $\deg(KV) \geq |A| = N-1 \geq d$.

Case 2. $0 \leq d := h^0(V) < \infty$. We pick a torsion element $x \in V_S - \{0\}$ with $|S| = d$ and we claim that $V_\iota(x) = 0$ for the embedding $\iota : S \hookrightarrow S \sqcup \{\star\}$. There is a finite set T and an injection $f : S \hookrightarrow T$ such that $V_f(x) = 0$. As $x \neq 0$, f cannot be an isomorphism so $|T| > |S|$ and $f = g \circ \iota$ for some injection $g : S \sqcup \{\star\} \hookrightarrow T$. As $V_g(V_\iota(x)) = 0$, the element $V_\iota(x)$ is torsion but it lies in degree $d+1$, forcing $V_\iota(x) = 0$ and hence $x \in KV$, showing $\deg(KV) \geq d$. □

Proposition 2.5. *Given an* **FI**-*module* V, *the following are equivalent*:

(1) *V is presented in finite degrees.*

(2) *$h^0(V) < \infty$ and ΔV is presented in finite degrees.*

Proof. Assume (1). Then by [8, Theorem 1] ΣV is presented in finite degrees, and hence so are KV and ΔV by [20, Theorem B] and [1, Proposition 2.5]. We have $h^0(V) < \infty$ by [20, Theorem A].

Conversely, assume (2). By [6, Proposition 2.9, part (4)], $u := \delta(\Delta V) < \infty$, so $\Delta^{u+2} V = \Delta^{u+1} \Delta V$ is torsion. Also, by applying the implication (1) $\Rightarrow$ (2) to ΔV and iterating it, $\Delta^{u+2} V$ is presented in finite degrees. Being a torsion **FI**-module generated in finite degrees, $\Delta^{u+2} V$ has finite degree, say d. Therefore by [3, Proposition 4.6], V is generated in degrees $\leq u+d+2$. We conclude by [20, Theorem A]. □

Proposition 2.6. *Given an* **FI**-*module* V, *the following hold*:

(1) *If $h^0(V) < \infty$, then there is a long exact sequence*

$$0 \longrightarrow KV \longrightarrow \mathrm{H}^0_{\mathfrak{m}}(V) \longrightarrow \Sigma \mathrm{H}^0_{\mathfrak{m}}(V) \longrightarrow \mathrm{H}^0_{\mathfrak{m}}(\Delta V) \longrightarrow \cdots$$
$$\longrightarrow \mathrm{H}^j_{\mathfrak{m}}(V) \longrightarrow \Sigma \mathrm{H}^j_{\mathfrak{m}}(V) \longrightarrow \mathrm{H}^j_{\mathfrak{m}}(\Delta V) \longrightarrow \mathrm{H}^{j+1}_{\mathfrak{m}}(V) \longrightarrow \Sigma \mathrm{H}^{j+1}_{\mathfrak{m}}(V) \longrightarrow \mathrm{H}^{j+1}_{\mathfrak{m}}(\Delta V) \longrightarrow \cdots$$

(2) *If V is presented in finite degrees*, (1) *holds such that every* **FI**-*module in the sequence has finite degree.*

Proof. For (1), note that KV is certainly generated in degrees

$$\leq \deg(KV) = h^0(V) < \infty$$

by Lemma 2.4. Thus by [20, Theorem A] (see [1, Proposition 2.5]), KV is presented in finite degrees. Therefore, [6, Theorem 2.10] applies to KV and the complex $0 \to KV \to 0 \to 0 \to \dots$ and hence

$$\mathrm{H}^j_{\mathfrak{m}}(KV) = \begin{cases} KV & \text{if } j = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Now applying $\mathrm{H}^0_{\mathfrak{m}}$ to the short exact sequence

$$0 \to KV \to V \to V/KV \to 0,$$

the associated long exact sequence gives a short exact sequence

$$0 \to KV \to \mathrm{H}^0_{\mathfrak{m}}(V) \to \mathrm{H}^0_{\mathfrak{m}}(V/KV) \to 0$$

and isomorphisms

$$\mathrm{H}^j_{\mathfrak{m}}(V) \cong \mathrm{H}^j_{\mathfrak{m}}(V/KV)$$

for every $j \geq 1$. Using these isomorphisms after applying $\mathrm{H}^0_{\mathfrak{m}}$ to the short exact sequence

$$0 \to V/KV \to \Sigma V \to \Delta V \to 0,$$

the associated long exact sequence will almost have the desired form, except we need to splice it in the beginning and interchange the order of the shift functor Σ with local cohomology $\mathrm{H}^\star_{\mathfrak{m}}$ in the middle column. To see $\Sigma \circ \mathrm{H}^\star_{\mathfrak{m}} = \mathrm{H}^\star_{\mathfrak{m}} \circ \Sigma$, first note that $\Sigma : \mathbf{FI}\text{-}\mathsf{Mod} \to \mathbf{FI}\text{-}\mathsf{Mod}$

- is exact,
- has an exact left adjoint [9, Theorem 4],
- satisfies $\Sigma \circ \mathrm{H}^0_{\mathfrak{m}} = \mathrm{H}^0_{\mathfrak{m}} \circ \Sigma$.

Consequently, given an **FI**-module U and an injective resolution $0 \to U \to I^\star$, applying Σ we get an injective resolution $0 \to \Sigma U \to \Sigma I^\star$ of ΣU, and hence

$$\mathrm{H}^j_{\mathfrak{m}}(\Sigma U) = \mathrm{H}^j(\mathrm{H}^0_{\mathfrak{m}}(\Sigma I^\star)) = \mathrm{H}^j(\Sigma \mathrm{H}^0_{\mathfrak{m}}(I^\star)) = \Sigma \mathrm{H}^j(\mathrm{H}^0_{\mathfrak{m}}(I^\star)) = \Sigma \mathrm{H}^j_{\mathfrak{m}}(U)$$

for every $j \geq 0$, naturally in U.

For (2), assume V is presented in finite degrees. Then $\deg(KV) = h^0(V) < \infty$ (so we have the long exact sequence from (1)) and ΔV is presented in finite degrees by Lemma 2.4 and Proposition 2.5. Now invoke [6, Theorem 2.10] for V and ΔV. □

Corollary 2.7. *For every* **FI***-module V presented in finite degrees, the following hold*:

(1) *For every $j \geq 0$, we have $h^j(\Delta V) \leq \max\{h^j(V) - 1, h^{j+1}(V)\}$.*

(2) *For every $j \geq 1$, we have $h^j(V) \leq \max\{h^{j-1}(\Delta V), h^j(\Delta V)\}$.*

Proof. By Proposition 2.6, for every $j \geq 0$ we have

$$\begin{aligned} h^j(\Delta V) &= \deg \mathrm{H}^j_{\mathfrak{m}}(\Delta V) \\ &\leq \max\{\deg \Sigma \mathrm{H}^j_{\mathfrak{m}}(V), \deg \mathrm{H}^{j+1}_{\mathfrak{m}}(V)\} = \max\{\deg \Sigma \mathrm{H}^j_{\mathfrak{m}}(V), h^{j+1}(V)\}. \end{aligned}$$

If $\mathrm{H}^j_{\mathfrak{m}}(V) \neq 0$, then

$$\deg \Sigma \mathrm{H}^j_{\mathfrak{m}}(V) = \deg \mathrm{H}^j_{\mathfrak{m}}(V) - 1 = h^j(V) - 1$$

and (1) follows. If $\mathrm{H}^j_{\mathfrak{m}}(V) = 0$, then $\mathrm{H}^j_{\mathfrak{m}}(\Delta V)$ embeds in $\mathrm{H}^{j+1}_{\mathfrak{m}}(V)$ and (1) again follows.

To prove (2), fix $j \geq 1$ and set $N := \max\{h^{j-1}(\Delta V), h^j(\Delta V)\}$ so for every $n > N$, by Proposition 2.6 we have an isomorphism

$$\mathrm{H}^j_{\mathfrak{m}}(V)_n \cong \Sigma \mathrm{H}^j_{\mathfrak{m}}(V)_n = \mathrm{H}^j_{\mathfrak{m}}(V)_{n+1}.$$

But $\mathrm{H}^j_{\mathfrak{m}}(V)$ has finite degree, therefore the above isomorphisms in the entire range $n > N$ have to be between zero modules so that $h^j(V) = \deg \mathrm{H}^j_{\mathfrak{m}}(V) \leq N$. □

Critical index and the regularity of derivative. In this section, we introduce the notion of critical index for an **FI**-module and use it to study how regularity interacts with the derivative functor.

Definition 2.8. For an **FI**-module V presented in finite degrees which is not $\mathrm{H}^{\mathbf{FI}}_0$-acyclic, we define its *critical index* as

$$\mathrm{crit}(V) := \min\{j : h^j(V) \geq 0 \ \text{ and } \ h^j(V) + j = \mathrm{reg}(V)\}.$$

Remark 2.9. Let V be as in Definition 2.8 and set $\gamma := \mathrm{crit}(V)$, $\rho := \mathrm{reg}(V)$. The following will not be needed in our arguments but we note them for context.

(1) By Theorem 2.1 and [6, Theorem 2.10], the set of indices

$$\{j : h^j(V) \geq 0 \ \text{ and } \ h^j(V) + j = \rho\}$$

is a nonempty subset of $\{0, \ldots, \delta(V) + 1\}$; thus $0 \leq \gamma \leq \delta(V) + 1$.

(2) Although they do not give it a name, Nagpal, Sam and Snowden [14, Definition 3.3] use the critical index: their invariant ν satisfies

$$\nu(\mathrm{H}^{\mathbf{FI}}_i(V)_{i+\rho}) = i + \gamma$$

for $i \gg 0$ [14, Proposition 4.3].

(3) It is possible that $h^\gamma(V) < h^{\max}(V)$. To see this, let us start by an exact sequence

$$(\lozenge) \qquad 0 \to Z \to A \to B \to W \to 0$$

of **FI**-modules presented in finite degrees where A, B are $\mathrm{H}_0^{\mathbf{FI}}$-acyclic, $\deg(W) = 0$. Breaking this into two short exact sequences, the associated long exact sequences for $\mathrm{H}_{\mathfrak{m}}^*$ yields

$$\mathrm{H}_{\mathfrak{m}}^j(Z) \cong \mathrm{H}_{\mathfrak{m}}^{j-2}(W)$$

for every $j \geq 0$. In particular, $h^2(Z) = 0$ and $h^j(Z) = -1$ if $j \neq 2$. Now setting $V := Z \oplus T$ with $\deg(T) = 1$, we get

$$h^j(V) = \begin{cases} 1 & \text{if } j = 0, \\ 0 & \text{if } j = 2, \\ -1, & \text{otherwise} \end{cases}$$

so that $\operatorname{reg}(V) = \operatorname{crit}(V) = 2$, $h^2(V) = 0$, but $h^{\max}(V) = 1$.

Proposition 2.10. *Let V be an* **FI***-module presented in finite degrees which is not* $\mathrm{H}_0^{\mathbf{FI}}$*-acyclic. Then*:

(1) $\operatorname{reg}(\Delta V) \leq \operatorname{reg}(V) - 1$.

(2) *If* $\operatorname{crit}(V) \geq 1$, *then* $0 \leq \operatorname{reg}(\Delta V) = \operatorname{reg}(V) - 1$ *and* $\operatorname{crit}(\Delta V) = \operatorname{crit}(V) - 1$.

Proof. Note that ΔV is presented in finite degrees by Proposition 2.5. Set $\rho := \operatorname{reg}(V)$ and $\gamma := \operatorname{crit}(V)$.

Assume ΔV is $\mathrm{H}_0^{\mathbf{FI}}$-acyclic. Then by [1, Theorem 2.4], $h^j(\Delta V) = -1$ for every $j \geq 0$, and hence by part (2) of Corollary 2.7 we have $h^j(V) = -1$ for every $j \geq 1$, forcing $\gamma = 0$. Therefore the condition $\operatorname{reg}(\Delta V) < 0$ (which is equivalent to ΔV being $\mathrm{H}_0^{\mathbf{FI}}$-acyclic by [1, Corollary 2.9]) implies $\gamma = 0$. In this case, we further have

$$\operatorname{reg}(\Delta V) < 0 \leq h^0(V) = \rho$$

by [1, Theorem 2.4] and (1) follows.

Next, assume ΔV is not $\mathrm{H}_0^{\mathbf{FI}}$-acyclic (hence neither is V, see the discussion in [6, Section 2.3]). Let us write

$$J(V) := \{j \geq 0 : h^j(V) \geq 0\}.$$

Let $j \in J(\Delta V)$. By part (1) of Corollary 2.7, we have either $0 \leq h^j(\Delta V) \leq h^j(V) - 1$ or $0 \leq h^j(\Delta V) \leq h^{j+1}(V)$. In the former case, we have $j \in J(V)$ and

$$h^j(\Delta V) + j \leq h^j(V) + j - 1 \leq \rho - 1$$

by Theorem 2.1, and in the latter case, we have $j + 1 \in J(V)$ and

$$h^j(\Delta V) + j \leq h^{j+1}(V) + j + 1 - 1 \leq \rho - 1$$

by Theorem 2.1. Yet another application of Theorem 2.1 now yields $\operatorname{reg}(\Delta V) \leq \rho - 1$, which is exactly (1). To prove (2), we further assume that $\gamma \geq 1$. We claim that

$$h^{\gamma-1}(\Delta V) + \gamma - 1 = \rho - 1.$$

To that end, by Proposition 2.6 we have an exact sequence

$$\Sigma \mathrm{H}^{\gamma-1}_{\mathfrak{m}}(V) \to \mathrm{H}^{\gamma-1}_{\mathfrak{m}}(\Delta V) \to \mathrm{H}^{\gamma}_{\mathfrak{m}}(V) \to \Sigma \mathrm{H}^{\gamma}_{\mathfrak{m}}(V),$$

which we evaluate at a finite set of size $\rho - \gamma$ to get an exact sequence

$$\mathrm{H}^{\gamma-1}_{\mathfrak{m}}(V)_{\rho-\gamma+1} \to \mathrm{H}^{\gamma-1}_{\mathfrak{m}}(\Delta V)_{\rho-\gamma} \to \mathrm{H}^{\gamma}_{\mathfrak{m}}(V)_{\rho-\gamma} \to \mathrm{H}^{\gamma}_{\mathfrak{m}}(V)_{\rho-\gamma+1}$$

of $\mathfrak{S}_{\rho-\gamma}$-modules. Here:

- $\mathrm{H}^{\gamma-1}_{\mathfrak{m}}(V)_{\rho-\gamma+1} = 0$, because by the definition of critical index we have

$$h^{\gamma-1}(V) + \gamma - 1 < \rho, \quad \deg(\mathrm{H}^{\gamma-1}_{\mathfrak{m}}(V)) < \rho - \gamma + 1.$$

- $\mathrm{H}^{\gamma}_{\mathfrak{m}}(V)_{\rho-\gamma} \neq 0$ and $\mathrm{H}^{\gamma}_{\mathfrak{m}}(V)_{\rho-\gamma+1} = 0$, because by the definition of critical index we have

$$h^{\gamma}(V) \geq 0 \quad \text{and} \quad h^{\gamma}(V) + \gamma = \rho, \quad \deg(\mathrm{H}^{\gamma}_{\mathfrak{m}}(V)) = \rho - \gamma.$$

Therefore we conclude that

$$\mathrm{H}^{\gamma-1}_{\mathfrak{m}}(\Delta V)_{\rho-\gamma} \neq 0, \quad h^{\gamma-1}(\Delta V) \geq 0 \quad \text{and} \quad h^{\gamma-1}(\Delta V) + \gamma - 1 \geq \rho - 1.$$

On the other hand, part (1) and Theorem 2.1 give the reverse inequality to the above, establishing our claim, the equation $\operatorname{reg}(\Delta V) = \rho - 1$ and the inequality $\operatorname{crit}(\Delta V) \leq \gamma - 1$. To see in fact $\operatorname{crit}(\Delta V) = \gamma - 1$, we can take $0 \leq j < \gamma - 1$ and evaluate the exact sequence

$$\Sigma \mathrm{H}^{j}_{\mathfrak{m}}(V) \to \mathrm{H}^{j}_{\mathfrak{m}}(\Delta V) \to \mathrm{H}^{j+1}_{\mathfrak{m}}(V)$$

at a finite set of size $\rho - 1 - j$ to get

$$0 = \mathrm{H}^{j}_{\mathfrak{m}}(V)_{\rho-j} \to \mathrm{H}^{j}_{\mathfrak{m}}(\Delta V)_{\rho-1-j} \to \mathrm{H}^{j+1}_{\mathfrak{m}}(V)_{\rho-(j+1)} = 0$$

and conclude $h^{j}(\Delta V) + j < \rho - 1$, as desired. □

Identifying the polynomial conditions. In this section we prove Theorems A and B. Because **FI**-modules being presented in finite degrees is such a common assumption, we first incorporate it as a redundant hypothesis in Theorems 2.12 and 2.13, and then remove this redundancy using Theorem 2.11.

Theorem 2.11. *For an* **FI**-*module V with $\delta(V) < \infty$, the following are equivalent*:

(1) $\operatorname{reg}(V) < \infty$.

(2) $h^{\max}(V) < \infty$.

(3) *V is presented in finite degrees.*

Proof. (3) ⇒ (1): Immediate from [3, Theorem A].

(1) ⇒ (2): We write:

- **FB** for the category of finite sets and bijections.
- $\mathrm{Ind}_{\mathbf{FB}}^{\mathbf{FI}}$ for the left adjoint of the restriction functor $\mathrm{Res}_{\mathbf{FB}}^{\mathbf{FI}} : \mathbf{FI}\text{-Mod} \to \mathbf{FB}\text{-Mod}$.
- $W := \mathrm{H}_0^{\mathbf{FI}}(V)$.

Then there is a short exact sequence

$$0 \to K \to \mathrm{Ind}_{\mathbf{FB}}^{\mathbf{FI}}(W) \to V \to 0 \tag{†}$$

for some **FI**-module K. Here $\mathrm{Ind}_{\mathbf{FB}}^{\mathbf{FI}}(W)$ is $\mathrm{H}_0^{\mathbf{FI}}$-acyclic [3, Lemma 2.3]. Moreover, the $\mathrm{H}_0^{\mathbf{FI}}$-image of the epimorphism in (†) is the identity map $\mathrm{H}_0^{\mathbf{FI}}(V) \to \mathrm{H}_0^{\mathbf{FI}}(V)$. Thus applying $\mathrm{H}_0^{\mathbf{FI}}$ to (†), the associated long exact sequence splits into isomorphisms

$$\mathrm{H}_{i+1}^{\mathbf{FI}}(V) \cong \mathrm{H}_i^{\mathbf{FI}}(K)$$

for every $i \geq 0$. In particular, we have

$$t_i(K) = t_{i+1}(V) < \mathrm{reg}(V) + i + 1 < \infty$$

for every $i \geq 0$, so K is presented in finite degrees. Consequently $h^{\max}(K) < \infty$ by [6, Proposition 2.9, part (4)] and [6, Theorem 2.10]. We will be done once we show

$$\mathrm{H}_{\mathfrak{m}}^{*}(\mathrm{Ind}_{\mathbf{FB}}^{\mathbf{FI}}(W)) = 0,$$

because applying $\mathrm{H}_{\mathfrak{m}}^{0}$ to (†), the long exact sequence yields $h^{\max}(V) = h^{\max}(K)$. The last claim follows from W being the direct product of **FB**-modules each supported in a single degree, and the functors $\mathrm{Ind}_{\mathbf{FB}}^{\mathbf{FI}}$, $\mathrm{H}_{\mathfrak{m}}^{*}$ commuting with direct products (for instance, the former via [5, Definition 2.2.2] and the latter via [11, Definition 5.4]) together with [1, Theorem 2.4].

(2) $\Rightarrow$ (3): We employ induction on $\delta(V)$: if $\delta(V) = -1$, then $V = \mathrm{H}_{\mathfrak{m}}^{0}(V)$ is torsion and so

$$\deg(V) = h^0(V) \leq h^{\max}(V) < \infty.$$

Thus V is presented in finite degrees by Corollary 2.3. Next, assume $\delta(V) \geq 0$. We can apply Proposition 2.6 to V to conclude $h^{\max}(\Delta V) < \infty$. We also have $\delta(\Delta V) \leq \delta(V) - 1$, therefore ΔV is presented in finite degrees by the induction hypothesis. We conclude by applying Proposition 2.5. □

Theorem 2.12. *For every pair of integers $r \geq -1$, $L \geq 0$, we have*

$$\mathbf{Poly}_1(r, L) = \left\{ V \in \mathbf{FI}\text{-Mod} : \begin{array}{l} V \text{ is presented in finite degrees}, \\ \delta(V) \leq r, \text{ and } h^{\max}(V) \leq L - 1 \end{array} \right\}.$$

Proof. We fix $L \geq 0$ and employ induction on r. For the base case $r = -1$, we first let $V \in \mathbf{Poly}_1(-1, L)$, that is, $\deg(V) \leq L - 1$. Then V is torsion so $\delta(V) = -1$, and by Corollary 2.3 V is presented in finite degrees with $h^{\max}(V) \leq L - 1$. Conversely, suppose V is presented in finite degrees, $\delta(V) \leq -1$, and $h^{\max}(V) \leq L-1$. Then V is torsion, so $\mathrm{H}^0_{\mathfrak{m}}(V) = V$ has degree $\leq L-1$.

For the inductive step, fix $r \geq 0$ and assume that we have

$$\mathbf{Poly}_1(r-1, L) = \left\{ U \in \mathbf{FI}\text{-}\mathsf{Mod} : \begin{array}{l} U \text{ is presented in finite degrees,} \\ \delta(U) \leq r-1, \text{ and } h^{\max}(U) \leq L-1 \end{array} \right\}.$$

Next, let $V \in \mathbf{Poly}_1(r, L)$, so by Definition 1.1, $h^0(V) \leq L-1$ and

$$\Delta V \in \mathbf{Poly}_1(r-1, L).$$

By the induction hypothesis, we conclude the following.

- ΔV is presented in finite degrees: it follows that V is presented in finite degrees by Proposition 2.5.
- $\delta(\Delta V) \leq r-1$: this means $\Delta^r \Delta V = \Delta^{r+1} V$ is torsion, so $\delta(V) \leq r$.
- $h^{\max}(\Delta V) \leq L-1$: by part (2) of Corollary 2.7, we have $h^{\max}(V) \leq L-1$.

Conversely, let V be an **FI**-module which is presented in finite degrees, $\delta(V) \leq r$, and $h^{\max}(V) \leq L-1$. We observe:

- $\Delta^{r+1} V = \Delta^r \Delta V$ is torsion, so $\delta(\Delta V) \leq r-1$.
- By Proposition 2.5, ΔV is presented in finite degrees.
- $h^{\max}(\Delta V) \leq L-1$ by part (1) of Corollary 2.7.

Therefore by the induction hypothesis, we get $\Delta V \in \mathbf{Poly}_1(r-1, L)$ and hence $V \in \mathbf{Poly}_1(r, L)$ by Definition 1.1. □

Proof of Theorem A. Immediate from Theorems 2.12 and 2.11. □

Theorem 2.13. *For every pair of integers $r \geq -1$, $M \geq 0$, we have*

$$\mathbf{Poly}_2(r, M) = \left\{ V \in \mathbf{FI}\text{-}\mathsf{Mod} : \begin{array}{l} V \textit{ is presented in finite degrees,} \\ \delta(V) \leq r, \textit{ and } \operatorname{reg}(V) \leq M-1 \end{array} \right\}.$$

Proof. We fix $M \geq 0$ and employ induction on r. For the base case $r = -1$, we first let $V \in \mathbf{Poly}_2(-1, M)$, that is, $\deg(V) \leq M-1$. Then V is torsion so $\delta(V) = -1$, and by Corollary 2.3 V is presented in finite degrees with $\operatorname{reg}(V) \leq M-1$. Conversely, suppose V is presented in finite degrees, $\delta(V) \leq -1$, and $\operatorname{reg}(V) \leq M-1$. Then V is torsion, so $\mathrm{H}^0_{\mathfrak{m}}(V) = V$ has degree $\leq M-1$ by Theorem 2.1.

For the inductive step, fix $r \geq 0$ and assume that for every $M' \geq 0$ we have

$$\mathbf{Poly}_2(r-1, M') = \left\{ U \in \mathbf{FI}\text{-}\mathsf{Mod} : \begin{array}{l} U \text{ is presented in finite degrees,} \\ \delta(U) \leq r-1, \text{ and } \operatorname{reg}(U) \leq M'-1 \end{array} \right\}.$$

Next, fix $M \geq 0$ and let $V \in \mathbf{Poly}_2(r, M)$, so by Definition 1.3, $h^0(V) \leq M-1$ and

$$\Delta V \in \mathbf{Poly}_2(r-1, \max\{0, M-1\}).$$

By the induction hypothesis, we conclude the following.

- ΔV is presented in finite degrees: it follows that V is presented in finite degrees by Proposition 2.5.
- $\delta(\Delta V) \leq r-1$: this means $\Delta^r \Delta V = \Delta^{r+1} V$ is torsion, so $\delta(V) \leq r$.
- $\mathrm{reg}(\Delta V) \leq \max\{-1, M-2\}$.

Three possibilities arise:

(1) V is $\mathrm{H}_0^{\mathbf{FI}}$-acyclic. Then $\mathrm{reg}(V) = -2 \leq M-1$.

(2) V is not $\mathrm{H}_0^{\mathbf{FI}}$-acyclic and $\mathrm{crit}(V) = 0$. Here the definition of critical index immediately yields

$$\mathrm{reg}(V) = h^0(V) \leq M-1.$$

(3) V is not $\mathrm{H}_0^{\mathbf{FI}}$-acyclic and $\mathrm{crit}(V) \geq 1$. Part (2) of Proposition 2.10 yields

$$1 \leq \mathrm{reg}(V) = \mathrm{reg}(\Delta V) + 1 \leq \max\{0, M-1\}.$$

Hence $M-1 > 0$ and $\mathrm{reg}(V) \leq M-1$.

Conversely, let V be an **FI**-module which is presented in finite degrees, $\delta(V) \leq r$, and $\mathrm{reg}(V) \leq M-1$ (in particular, $h^0(V) \leq M-1$ by Theorem 2.1). We observe:

- $\Delta^{r+1} V = \Delta^r \Delta V$ is torsion, so $\delta(\Delta V) \leq r-1$.
- By Proposition 2.5, ΔV is presented in finite degrees.
- Either V is $\mathrm{H}_0^{\mathbf{FI}}$-acyclic and hence so is ΔV (see the discussion in [6, Section 2.3]) and $\mathrm{reg}(\Delta V) = -2$, or V is not $\mathrm{H}_0^{\mathbf{FI}}$-acyclic so that

$$0 \leq t_1(V) - 1 \leq \mathrm{reg}(V) \leq M-1$$

by [1, Corollary 2.9], and $\mathrm{reg}(\Delta V) \leq M-2$ by part (1) of Proposition 2.10. In both cases we have $\mathrm{reg}(\Delta V) \leq \max\{-1, M-2\}$.

Therefore the induction hypothesis yields $\Delta V \in \mathbf{Poly}_2(r-1, \max\{0, M-1\})$. We also have $h^0(V) \leq M-1$, so $V \in \mathbf{Poly}_2(r, M)$ by Definition 1.3. □

Proof of Theorem B. Immediate from Theorems 2.13 and 2.11. □

Twisted homological stability.

Proof of Theorem C. By Theorem 2.12, V is presented in finite degrees, $\delta(V) \leq r$, and $h^{\max}(V) \leq L-1$. Hence by [1, Theorem 2.6], the triple $(V, L-1, r)$ satisfies [1, Hypothesis 1.2]. Noting that

$$r > \left\lceil \tfrac{L-1}{2} \right\rceil \quad \text{if and only if } L < 2r,$$

by [1, Theorem C] we have

$$\operatorname{reg}(V) \le \begin{cases} -2 & \text{if } L = 0, \\ L & \text{if } L \ge \max\{1, 2r\}, \\ r + \lfloor \frac{L+1}{2} \rfloor & \text{if } 1 \le L < 2r. \end{cases}$$

Thus by Theorem B, we have

$$V \in \begin{cases} \mathbf{Poly}_2(r, 0) & \text{if } L = 0, \\ \mathbf{Poly}_2(r, L+1) & \text{if } L \ge \max\{1, 2r\}, \\ \mathbf{Poly}_2\big(r, r + \lfloor \frac{L+1}{2} \rfloor + 1\big) & \text{if } 1 \le L < 2r. \end{cases}$$

Consequently by [18, Theorem A], for every $k \ge 0$ the map

$$\mathrm{H}_k(\mathfrak{S}_n; V_n) \to \mathrm{H}_k(\mathfrak{S}_{n+1}; V_{n+1})$$

is an isomorphism for

$$n \ge \begin{cases} 2k + r + 1 & \text{if } L = 0, \\ 2k + L + 2 & \text{if } L \ge \max\{1, 2r\}, \\ 2k + r + \lfloor \frac{L+1}{2} \rfloor + 2 & \text{if } 1 \le L < 2r \end{cases}$$

and a surjection for

$$n \ge \begin{cases} 2k + r & \text{if } L = 0, \\ 2k + L + 1 & \text{if } L \ge \max\{1, 2r\}, \\ 2k + r + \lfloor \frac{L+1}{2} \rfloor + 1 & \text{if } 1 \le L < 2r. \end{cases}$$

It remains to improve the bounds in the case $L \ge \max\{1, 2r-1\}$ to

- $n \ge \max\{2k + 2r + 1, L\}$ for the isomorphism range,
- $n \ge \max\{2k + 2r, L\}$ for the surjection range.

To that end, we induct on r. For the base case $r = 0$, by [1, Theorem 2.11] there is an $\mathrm{H}_0^{\mathbf{FI}}$-acyclic I with $\delta(I) \le 0$ and a map $V \to I$ which is an isomorphism in degrees $\ge L$. As ΔI is torsion but also is $\mathrm{H}_0^{\mathbf{FI}}$-acyclic, we have $\Delta I = KI = 0$, in other words $I \to I$ is an isomorphism. Thus I_n is the same trivial $\mathfrak{S}_n$-representation for every $n \ge 0$ (namely the abelian group I_0 with the trivial $\mathfrak{S}_n$-action). Now by [15, Corollary 6.7], for every $k \ge 0$ the map

$$\mathrm{H}_k(\mathfrak{S}_n; I_0) \to \mathrm{H}_k(\mathfrak{S}_{n+1}; I_0)$$

is an isomorphism for $n \ge 2k$. Thus for every $k \ge 0$, the map

$$\mathrm{H}_k(\mathfrak{S}_n; V_n) \to \mathrm{H}_k(\mathfrak{S}_{n+1}; V_{n+1})$$

is an isomorphism for $n \ge \max\{2k, L\}$ (which is better than what the base case demands: an isomorphism for $n \ge \max\{2k+1, L\}$ and a surjection for $n \ge \max\{2k, L\}$).

Next, take $r \geq 1$ and assume that every **FI**-module $U \in \mathbf{Poly}_1(r, L-1)$, that is, by Theorem 2.12, every U presented in finite degrees with $\delta(U) \leq r-1$ and $h^{\max}(U) \leq L-1$ satisfies[3] the following: for every $k \geq 0$ the map

$$\mathrm{H}_k(\mathfrak{S}_n; U_n) \to \mathrm{H}_k(\mathfrak{S}_{n+1}; U_{n+1})$$

is an isomorphism for

$$n \geq \max\{2k+2r-1, L\}$$

and a surjection for

$$n \geq \max\{2k+2r-2, L\}.$$

In particular by [6, Proposition 2.9, part (7)], this applies to

$$U := \operatorname{coker}(V \to \Sigma^L V).$$

In degrees $n \geq L$, writing $I := \Sigma^L V$, we have a short exact sequence

$$0 \to V_n \to I_n \to U_n \to 0$$

of $\mathfrak{S}_n$-modules, and the associated long exact sequence in $\mathrm{H}_*(\mathfrak{S}_n; -)$ maps to that of $\mathrm{H}_*(\mathfrak{S}_{n+1}; -)$. More precisely, suppressing the symmetric groups in the homology notation, there is a commutative diagram

$$\begin{array}{ccccccccc}
\mathrm{H}_{k+1}(I_n) & \longrightarrow & \mathrm{H}_{k+1}(U_n) & \longrightarrow & \mathrm{H}_k(V_n) & \longrightarrow & \mathrm{H}_k(I_n) & \longrightarrow & \mathrm{H}_k(U_n) \\
\downarrow{\scriptstyle \mu_{k+1}} & & \downarrow{\scriptstyle \nu_{k+1}} & & \downarrow{\scriptstyle \lambda_k} & & \downarrow{\scriptstyle \mu_k} & & \downarrow{\scriptstyle \nu_k} \\
\mathrm{H}_{k+1}(I_{n+1}) & \longrightarrow & \mathrm{H}_{k+1}(U_{n+1}) & \longrightarrow & \mathrm{H}_k(V_{n+1}) & \longrightarrow & \mathrm{H}_k(I_{n+1}) & \longrightarrow & \mathrm{H}_k(U_{n+1})
\end{array}$$

of abelian groups with exact rows. We observe:

- As I is $\mathrm{H}_0^{\mathbf{FI}}$-acyclic and $\delta(I) \leq r$, then $I \in \mathbf{Poly}_1(r, 0)$ and so for every $k \geq 0$ the map μ_k is an isomorphism for $n \geq 2k+r+1$ and a surjection for $n \geq 2k+r$.
- By the induction hypothesis on U, for every $k \geq 0$ the map ν_k is an isomorphism for $n \geq \max\{2k+2r-1, L\}$ and a surjection for $n \geq \max\{2k+2r-2, L\}$.

Therefore we have:

- By the five-lemma, λ_k is an isomorphism provided that ν_{k+1} and μ_k are isomorphisms, μ_{k+1} is surjective, and ν_k is injective: these are guaranteed in the range $n \geq \max\{2k+2r+1, L\}$ (noting $2(k+1)+r \leq 2(k+1)+2r-1$ because $r \geq 1$).
- By one of the four-lemmas, λ_k is surjective provided that ν_{k+1} and μ_k are surjective, and ν_k is injective: these are guaranteed in the range $n \geq \max\{2k+2r, L\}$. □

[3] Here the inequality $L \geq \max\{1, 2(r-1)-1\}$ is guaranteed as we are assuming $L \geq \max\{1, 2r-1\}$.

3. Application to congruence subgroups

Proof of Theorem D. By [1, Theorem 4.15], we have

- $\delta(\mathrm{H}_k(\mathrm{GL}_\bullet(R, I); \mathcal{A})) \le 2k$, and
- $\mathrm{reg}(\mathrm{H}_k(\mathrm{GL}_\bullet(R, I); \mathcal{A})) \le \begin{cases} 2s+3 & \text{if } k = 1, \\ 4k+2s & \text{if } k \ge 2. \end{cases}$

We now consider the groupoid $\mathcal{G} := \mathrm{SL}^{\mathfrak{U}}(R/I)$ in order to follow the argument and notation in [12, proof of Theorem 1.4], with the following adjustment: Declare a new **FI**-module V via

$$V_S := \begin{cases} \mathrm{H}_k(\mathrm{GL}_S(R, I); \mathcal{A}) & \text{if } |S| \ge n_0, \\ 0 & \text{if } |S| < n_0, \end{cases}$$

so that by part (1) of Remark 1.7, V extends to a $U\mathcal{G}$-module. Note that as an **FI**-module by construction there is a short exact sequence

$$0 \to V \to \mathrm{H}_k(\mathrm{GL}_S(R, I); \mathcal{A}) \to T \to 0,$$

with $\deg(T) \le n_0 - 1 \le 2s+2$. Invoking Corollary 2.3, applying $\mathrm{H}^0_{\mathfrak{m}}$ the associated long exact sequence here yields

$$\begin{aligned} &h^0(V) \le h^0(\mathrm{H}_k(\mathrm{GL}_\bullet(R, I); \mathcal{A})), \\ &h^1(V) \le \max\{\deg T, h^1(\mathrm{H}_k(\mathrm{GL}_\bullet(R, I); \mathcal{A}))\}, \\ &h^j(V) = h^j(\mathrm{H}_k(\mathrm{GL}_\bullet(R, I); \mathcal{A})) \quad \text{if } j \ge 2. \end{aligned}$$

Thus if $h^j(V) \ge 0$ for $j \ne 1$, we have $h^j(\mathrm{H}_k(\mathrm{GL}_\bullet(R, I); \mathcal{A})) \ge 0$ and hence

$$h^j(V) + j \le h^j(\mathrm{H}_k(\mathrm{GL}_\bullet(R, I); \mathcal{A})) + j \le \begin{cases} 2s+3 & \text{if } k = 1, \\ 4k+2s & \text{if } k \ge 2 \end{cases}$$

by Theorem 2.1 applied to $\mathrm{H}_k(\mathrm{GL}_\bullet(R, I); \mathcal{A})$. If $h^1(V) \ge 0$, there are two possibilities:

- $h^1(\mathrm{H}_k(\mathrm{GL}_\bullet(R, I); \mathcal{A}) \le \deg T$. Then $h^1(V) + 1 \le \deg T + 1 \le 2s+3$.
- $h^1(\mathrm{H}_k(\mathrm{GL}_\bullet(R, I); \mathcal{A}) > \deg T$. Then $h^1(\mathrm{H}_k(\mathrm{GL}_\bullet(R, I); \mathcal{A})) \ge 0$ and hence by Theorem 2.1,

$$h^1(V) + 1 \le h^1(\mathrm{H}_k(\mathrm{GL}_\bullet(R, I); \mathcal{A})) + 1 \le \begin{cases} 2s+3 & \text{if } k = 1, \\ 4k+2s & \text{if } k \ge 2. \end{cases}$$

Applying Theorem 2.1 to V now, we get

$$\mathrm{reg}(V) \le \begin{cases} 2s+3 & \text{if } k = 1, \\ 4k+2s & \text{if } k \ge 2. \end{cases}$$

By [6, Proposition 3.3] we also have $\delta(V) \leq 2k$. Thus by Theorem B and Remark 1.4, and in the sense of [12, Definition 2.40], V has polynomial degree $\leq 2k$

$$\text{in ranks} > \begin{cases} 2s+3 & \text{if } k=1, \\ 4k+2s & \text{if } k \geq 2. \end{cases}$$

(1) By [12, Remark 2.42], V has the same polynomial degree and rank bounds as a $U\mathcal{G}$-module.

(2) Noting that $\mathbf{st\text{-}rank}(R/I) \leq s$ as well [2, Lemma 4.1], by [12, Proposition 2.13], the category $U\mathcal{G}$ satisfies $\mathbf{H3}(2, s+1)$.

Therefore by [12, Theorem 3.11], we have

$$\widetilde{\mathrm{H}}_i^{\mathcal{G}}(V)_n = 0 \quad \text{for } n > \begin{cases} \max\{2s+i+4, s+2i+3\} & \text{if } k=1, \\ \max\{4k+2s+i+1, 2k+s+2i+1\} & \text{if } k \geq 2, \end{cases}$$

and in particular

$$\widetilde{\mathrm{H}}_{-1}^{\mathcal{G}}(V)_n = 0 \quad \text{for } n > \begin{cases} 2s+3 & \text{if } k=1, \\ 4k+2s & \text{if } k \geq 2, \end{cases}$$

$$\widetilde{\mathrm{H}}_0^{\mathcal{G}}(V)_n = 0 \quad \text{for } n > \begin{cases} 2s+4 & \text{if } k=1, \\ 4k+2s+1 & \text{if } k \geq 2. \end{cases}$$

Noting that the definitions of $\widetilde{\mathrm{H}}_*^{\mathcal{G}}$ in [13, Definition 3.14] and [12, Definition 2.9] are consistent with each other, the vanishing above corresponds to a coequalizer diagram of the form

$$\mathrm{Ind}_{\mathcal{G}_{n-2}}^{\mathcal{G}_n} V_{n-2} \rightrightarrows \mathrm{Ind}_{\mathcal{G}_{n-1}}^{\mathcal{G}_n} V_{n-1} \to V_n$$

of $\mathbb{Z}\mathcal{G}_n$-modules whenever

$$n \geq \begin{cases} 2s+5 & \text{if } k=1, \\ 4k+2s+2 & \text{if } k \geq 2 \end{cases}$$

by [13, Remark 3.16]. In this range, we have $n-2 \geq 2s+3 \geq n_0$, so that

$$V_j = \mathrm{H}_k(\mathrm{GL}_j(R, I); \mathcal{A}) \quad \text{for } j \in \{n-2, n-1, n\}. \qquad \square$$

References

[1] C. Bahran, “Regularity and stable ranges of FI-modules”, preprint, 2022. arXiv 2203.06698

[2] H. Bass, “K-theory and stable algebra”, *Inst. Hautes Études Sci. Publ. Math.* **22** (1964), 5–60. MR Zbl

[3] T. Church and J. S. Ellenberg, “Homology of FI-modules”, *Geom. Topol.* **21**:4 (2017), 2373–2418. MR Zbl

[4] T. Church, J. S. Ellenberg, B. Farb, and R. Nagpal, “FI-modules over Noetherian rings”, *Geom. Topol.* **18**:5 (2014), 2951–2984. MR Zbl

[5] T. Church, J. S. Ellenberg, and B. Farb, "FI-modules and stability for representations of symmetric groups", *Duke Math. J.* **164**:9 (2015), 1833–1910. MR Zbl

[6] T. Church, J. Miller, R. Nagpal, and J. Reinhold, "Linear and quadratic ranges in representation stability", *Adv. Math.* **333** (2018), 1–40. MR Zbl

[7] P. L. Clark, "A note on Euclidean order types", *Order* **32**:2 (2015), 157–178. MR Zbl

[8] W. L. Gan, "A long exact sequence for homology of FI-modules", *New York J. Math.* **22** (2016), 1487–1502. MR Zbl

[9] W. L. Gan, "On the negative-one shift functor for FI-modules", *J. Pure Appl. Algebra* **221**:5 (2017), 1242–1248. MR Zbl

[10] A. J. Hahn and O. T. O'Meara, *The classical groups and K-theory*, Grundlehren der Math. Wissenschaften **291**, Springer, 1989. MR Zbl

[11] L. Li and E. Ramos, "Depth and the local cohomology of $\mathbb{FI}_G$-modules", *Adv. Math.* **329** (2018), 704–741. MR Zbl

[12] J. Miller, P. Patzt, and D. Petersen, "Representation stability, secondary stability, and polynomial functors", preprint, 2019. arXiv 1910.05574

[13] J. Miller, P. Patzt, and J. C. H. Wilson, "Central stability for the homology of congruence subgroups and the second homology of Torelli groups", *Adv. Math.* **354** (2019), art. id. 106740. MR Zbl

[14] R. Nagpal, S. V. Sam, and A. Snowden, "Regularity of FI-modules and local cohomology", *Proc. Amer. Math. Soc.* **146**:10 (2018), 4117–4126. MR Zbl

[15] M. Nakaoka, "Decomposition theorem for homology groups of symmetric groups", *Ann. of Math.* (2) **71** (1960), 16–42. MR Zbl

[16] P. Patzt, "Central stability homology", *Math. Z.* **295**:3-4 (2020), 877–916. MR Zbl

[17] A. Putman, "Stability in the homology of congruence subgroups", *Invent. Math.* **202**:3 (2015), 987–1027. MR Zbl

[18] A. Putman, "A new approach to twisted homological stability, with applications to congruence subgroups", preprint, 2021. arXiv 2109.14015

[19] A. Putman and S. V. Sam, "Representation stability and finite linear groups", *Duke Math. J.* **166**:13 (2017), 2521–2598. MR Zbl

[20] E. Ramos, "On the degree-wise coherence of $\mathbb{FI}_G$-modules", *New York J. Math.* **23** (2017), 873–895. MR Zbl

[21] O. Randal-Williams and N. Wahl, "Homological stability for automorphism groups", preprint, 2015. arXiv 1409.3541v3

[22] O. Randal-Williams and N. Wahl, "Homological stability for automorphism groups", *Adv. Math.* **318** (2017), 534–626. MR Zbl

Received October 11, 2022. Revised January 10, 2023.

CIHAN BAHRAN
DEPARTMENT OF MATHEMATICS
BOĞAZIÇI UNIVERSITY
ISTANBUL
TURKEY
cihanbahran@gmail.com

PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

msp.org/pjm

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2023 is US $605/year for the electronic version, and $820/year for print and electronic.
Subscriptions, requests for back issues and changes of subscriber address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 1945-5844 electronic, 0030-8730 printed) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFlow® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

PACIFIC JOURNAL OF MATHEMATICS

Volume 324 No. 2 June 2023