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CLARA CUFÍ-CABRÉ AND JAUME LLIBRE

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LIMIT CYCLES OF LINEAR VECTOR FIELDS ON $(\mathbb{S}^2)^m \times \mathbb{R}^n$

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It is well known that linear vector fields defined in $\mathbb{R}^n$ cannot have limit cycles, but this is not the case for linear vector fields defined in other manifolds. We study the existence of limit cycles bifurcating from a continuum of periodic orbits of linear vector fields on manifolds of the form $(\mathbb{S}^2)^m \times \mathbb{R}^n$ when such vector fields are perturbed inside the class of all linear vector fields. The study is done using averaging theory. We also present an open problem about the maximum number of limit cycles of linear vector fields on $(\mathbb{S}^2)^m \times \mathbb{R}^n$.

1. Introduction and statement of the main results

The study of periodic orbits of differential systems plays an important role in the qualitative theory of ordinary differential equations and their applications. A *limit cycle* is defined as a periodic orbit of a differential system which is isolated in the set of all periodic orbits of the system. Among the many works devoted to limit cycles and their applications, we mention [Christopher and Lloyd 1996; Giacomini et al. 1996; Han and Li 2012; Ilyashenko 2002].

It is well known that linear vector fields in $\mathbb{R}^n$ cannot have limit cycles, but this is not the case if one considers linear vector fields in other manifolds different from $\mathbb{R}^n$. The objective of this paper is to study the existence of limit cycles of linear vector fields defined on the manifolds $(\mathbb{S}^2)^n \times \mathbb{R}^n$.

The problem of studying limit cycles of linear vector fields on manifolds different from $\mathbb{R}^n$ was already treated in [Llibre and Zhang 2016], where the authors consider linear vector fields on $\mathbb{S}^m \times \mathbb{R}^n$, and they conjecture that such vector fields may have at most one limit cycle.

Linear autonomous differential systems, namely, systems of the form $\dot{x} = Ax + b$, where A is a $n \times n$ real matrix and b is a vector in $\mathbb{R}^n$, are the easiest systems to study because their solutions can be completely determined (see [Arnold 2006; Sotomayor 1979]), but still they play an important role in the theory of differential

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systems. Thus when a nonlinear differential system has a hyperbolic equilibrium point, the dynamics around that point is determined by the linearization of the vector field at that point (Hartman–Grossman theorem; see [Hartman 1960]).

Linear vector fields having invariant subspaces of periodic orbits can be perturbed inside a concrete class of nonlinear differential systems to obtain limit cycles of these nonlinear systems bifurcating from the periodic orbits of the linear system (see [Ferragut et al. 2007; Llibre and Teixeira 2009; Llibre et al. 2007; 2010]).

Moreover, linear differential systems of the form $\dot{x} = Ax + Bu$, where x are the state variables and u is the control input, are applied in control theory for the modeling of hybrid systems (see [Lafferriere et al. 2001; 1999]). These examples illustrate the importance of linear differential systems.

In this paper we show that linear differential systems can have limit cycles when the manifold where they are defined is different from $\mathbb{R}^n$, and we consider the question of how many limit cycles, at most, a linear vector field can have depending on the manifold where it is defined.

Let M be a smooth connected manifold of dimension n, and let TM be its tangent bundle. A *vector field* on M is a map $X : M \to TM$ such that $X(x) \in T_xM$, where T_xM is the tangent space of M at the point x.

A *linear vector field* in $\mathbb{R}^n$ is a vector field of the form $X(x) = Ax + b$, with $x, b \in \mathbb{R}^n$ and where A is a $n \times n$ real matrix. It is well known linear vector fields on $\mathbb{R}^n$ either do not have periodic orbits or their periodic orbits form a continuum, and therefore they do not have limit cycles.

We consider linear vector fields on some manifolds of the form $(\mathbb{S}^2)^m \times \mathbb{R}^n$, where $\mathbb{S}^2$ denotes the unit two-dimensional sphere. Here the sphere $\mathbb{S}^2$ is parameterized by the coordinates (θ, φ), where $\theta \in [-\pi, \pi)$ denotes the azimuth angle and $\varphi \in [-\pi/2, \pi/2]$ is the polar angle. Hence the curve $\{\varphi = 0\}$ is the equator.

Let $(\theta_1, \varphi_1, \dots, \theta_m, \varphi_m, x_1, \dots, x_n)$ denote the coordinates of the space $(\mathbb{S}^2)^m \times \mathbb{R}^n$. Then we say that a vector field X is *linear* on $M = (\mathbb{S}^2)^m \times \mathbb{R}^n$ if the expression of X in the coordinates $z = (\theta_1, \varphi_1, \dots, \theta_m, \varphi_m, x_1, \dots, x_n) \in M$ is of the form $X(z) = Az + b$, with $b \in M$ and where A is a $(2m+n)\times(2m+n)$ real matrix.

Perrizo [1974] published a paper entitled "ω-linear vector fields on manifolds", but these ω-linear vector fields are very far from the linear differential systems that we study in this paper. More precisely, from Definition 2.2 of [Perrizo 1974] for ω-linear vector fields on a manifold, such a field X requires the existence of a nonlocally constant function that is constant on the orbits of $\tilde{X}$ (using Perrizo's notation), but the existence of limit cycles in our linear vector fields prevents the existence of a such function. So our linear differential systems on the manifolds $(\mathbb{S}^2)^m \times \mathbb{R}^n$ are not ω-linear.

An example in which a linear differential system on the manifold $(\mathbb{S}^2)^m \times \mathbb{R}^n$ has a limit cycle is the following. Take $m = 1, n = 0$ and consider the linear system on the sphere $\mathbb{S}^2$ given by

$$\dot\theta = 1, \quad \dot\varphi = \varphi,$$

for $\theta \in [-\pi, \pi)$ and $\varphi \in (-\pi/2, \pi/2)$, and

$$\dot\theta = 0, \quad \dot\varphi = 0,$$

for $\varphi = \pm\pi/2$. Then, clearly the equator of the sphere $\{\varphi = 0\}$ is the only periodic orbit of the system, and therefore it is a limit cycle.

We consider generic linear perturbations of some linear vector fields on three different manifolds of the form $(\mathbb{S}^2)^m \times \mathbb{R}^n$, and we study whether those families of linear differential systems can have limit cycles.

Let $M = \mathbb{S}^2 \times \mathbb{R}$ and consider the linear differential system in M given by

$$\dot\theta = 1, \quad \dot\varphi = 0, \quad \dot r = r - 1, \tag{1-1}$$

for $r \in \mathbb{R}$, $\theta \in [-\pi, \pi)$ and $\varphi \in (-\pi/2, \pi/2)$, and with $\dot\theta = 0$ on the straight lines $R_1 = \{\varphi = -\pi/2\}$ and $R_2 = \{\varphi = \pi/2\}$.

The solution of system (1-1) is given by

$$\theta(t) = \theta_0 + t, \quad \varphi(t) = \varphi_0, \quad r(t) = (r_0 - 1)e^t + 1.$$

Thus the sphere $\{r = 1\}$ is an invariant manifold with two equilibrium points at the north and the south poles, and is foliated by periodic orbits of period 2π, corresponding to the parallels of the sphere, except at the poles. Moreover the straight lines R_1 and R_2 are invariant.

First we shall study the bifurcation of limit cycles when we perturb system (1-1) inside the class of all linear differential systems, and we shall see that one of the periodic orbits contained in the sphere $\{r = 1\}$ may bifurcate to a limit cycle under certain hypotheses.

We consider the class of differential systems

$$\begin{aligned} \dot\theta &= 1 + \varepsilon(a_0 + a_1\theta + a_2\varphi + a_3 r), \\ \dot\varphi &= \varepsilon(b_0 + b_1\theta + b_2\varphi + b_3 r), \\ \dot r &= r - 1 + \varepsilon(c_0 + c_1\theta + c_2\varphi + c_3 r), \end{aligned} \tag{1-2}$$

where a_i, b_i and c_i, for $i = 0, \dots, 3$, are real numbers and with $\varepsilon > 0$ being a small parameter. Note that this is the more general linear perturbation of system (1-1).

Theorem 1. *For sufficiently small $\varepsilon > 0$ the linear differential system* (1-2) *has a limit cycle bifurcating from a periodic orbit of system* (1-1) *provided that*

$$a_1 b_2 - a_2 b_1 \neq 0.$$

This limit cycle bifurcates from the periodic orbit of system (1-1) *parameterized by*

$(\theta(t), \varphi(t), r(t)) = (\theta_0 + t, \varphi_0, 1)$, *with*

$$\theta_0 = \frac{a_2(b_0 + b_3 + b_1\pi) - b_2(a_0 + a_3 + a_1\pi)}{a_1b_2 - a_2b_1},$$
$$\varphi_0 = \frac{b_1(a_0 + a_3 + a_1\pi) - a_1(b_0 + b_3 + b_2\pi)}{a_1b_2 - a_2b_1}.$$

Theorem 1 is proved in Section 3.

We remark that the existence of the limit cycle for system (1-2) does not depend on the perturbation of the $\dot{r}$ equation.

As an example of the previous result, consider the system

$$\dot{\theta} = 1 + \varepsilon a\varphi, \quad \dot{\varphi} = \varepsilon b\theta, \quad \dot{r} = r - 1, \tag{1-3}$$

with $a, b \in \mathbb{R}$ and $\varepsilon > 0$. In this case the sphere $\{r = 1\}$ is still an invariant manifold. Applying Theorem 1 with $a_2 = a$, $b_1 = b$ and the rest of the coefficients of the perturbation being zero, we find that system (1-3) has a limit cycle bifurcating from the periodic orbit of system (1-1) parameterized by $(\theta(t), \varphi(t), r(t)) = (-\pi + t, 0, 1)$. That is, there is a limit cycle bifurcating from the periodic orbit corresponding to the equator of the sphere $\{r = 1\}$ of system (1-1). This limit cycle is still contained in the sphere $\{r = 1\}$.

Next we consider linear differential systems defined on higher dimensional manifolds. We take $M = \mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{R}$ and

$$\dot{\theta} = 1, \quad \dot{\varphi} = 0, \quad \dot{\nu} = 1, \quad \dot{\phi} = 0, \quad \dot{r} = r - 1, \tag{1-4}$$

for $(\theta, \varphi, \nu, \phi, r) \in M$, with $\theta, \nu \in [-\pi, \pi)$ and $\varphi, \phi \in (-\pi/2, \pi/2)$, and with $\dot{\theta} = 0$ when $\varphi = \pm\pi/2$ and $\dot{\nu} = 0$ when $\phi = \pm\pi/2$.

The general solution of system (1-4) is

$$\theta(t) = \theta_0 + t, \quad \varphi(t) = \varphi_0, \quad \nu(t) = \nu_0 + t, \quad \phi(t) = \phi_0, \quad r(t) = (r_0 - 1)e^t + 1,$$

and thus the product of spheres $\{r = 1\} \cong (\mathbb{S}^2)^2$ is an invariant manifold foliated by periodic orbits of period 2π, except for the four points $\{r = 1, \varphi = \pm\pi/2, \phi = \pm\pi/2\}$, which are equilibrium points.

We consider the most general perturbation of the differential system (1-4) inside the class of all linear differential systems, namely

$$\begin{aligned}
\dot{\theta} &= 1 + \varepsilon(a_0 + a_1\theta + a_2\varphi + a_3\nu + a_4\phi + a_5r),\\
\dot{\varphi} &= \varepsilon(b_0 + b_1\theta + b_2\varphi + b_3\nu + b_4\phi + b_5r)\\
\dot{\nu} &= 1 + \varepsilon(c_0 + c_1\theta + c_2\varphi + c_3\nu + c_4\phi + c_5r),\\
\dot{\phi} &= \varepsilon(d_0 + d_1\theta + d_2\varphi + d_3\nu + d_4\phi + d_5r)\\
\dot{r} &= r - 1 + \varepsilon(e_0 + e_1\theta + e_2\varphi + e_3\nu + e_4\phi + e_5r),
\end{aligned} \tag{1-5}$$

with $a_i, b_i, c_i, d_i, e_i \in \mathbb{R}$ for $i = 0, \dots, 5$, and with $\varepsilon > 0$ being a small parameter. The next result gives sufficient conditions on the coefficients of system (1-5) for there to be a limit cycle bifurcating from a periodic orbit of the unperturbed system.

Theorem 2. *For sufficiently small $\varepsilon > 0$ the differential system* (1-5) *has a limit cycle bifurcating from a periodic orbit of system* (1-4) *provided that*

$$\det \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{pmatrix} \neq 0.$$

This limit cycle bifurcates from the periodic orbit of system (1-4) *parameterized by*

$$(\theta(t), \varphi(t), \nu(t), \phi(t), r(t)) = (\theta_0 + t, \varphi_0, \nu_0 + t, \phi_0, 1),$$

where $(\theta_0, \varphi_0, \nu_0, \phi_0)$ is the unique solution of the linear system

$$\begin{aligned} a_1\theta_0 + a_2\varphi_0 + a_3\nu_0 + a_4\phi_0 &= -a_0 - a_1\pi - a_3\pi - a_5, \\ b_1\theta_0 + b_2\varphi_0 + b_3\nu_0 + b_4\phi_0 &= -b_0 - b_1\pi - b_3\pi - b_5, \\ c_1\theta_0 + c_2\varphi_0 + c_3\nu_0 + c_4\phi_0 &= -c_0 - c_1\pi - c_3\pi - c_5, \\ d_1\theta_0 + d_2\varphi_0 + d_3\nu_0 + d_4\phi_0 &= -d_0 - d_1\pi - d_3\pi - d_5. \end{aligned}$$

Theorem 2 is proved in Section 4.

Finally we consider the linear differential system defined in $M = \mathbb{R}^2 \times \mathbb{S}^2$, for $(x, y, \theta, \varphi) \in \mathbb{R}^2 \times \mathbb{S}^2$, with $\theta \in [-\pi, \pi)$ and $\varphi \in (-\pi/2, \pi/2)$, given by

$$\dot{x} = -y, \quad \dot{y} = x, \quad \dot{\theta} = 1, \quad \dot{\varphi} = 0, \tag{1-6}$$

and with $\dot{\theta} = 0$ in the planes $P_1 = \{\varphi = -\pi/2\}$ and $P_2 = \{\varphi = \pi/2\}$, which are invariant. The general solution of system (1-6) is

$$x(t) = x_0 \cos t - y_0 \sin t, \quad y(t) = x_0 \sin t + y_0 \cos t, \quad \theta(t) = \theta_0 + t, \quad \varphi(t) = \varphi_0,$$

and therefore the whole phase space is filled by periodic orbits of period 2π, except for the two equilibrium points $(x, y, \theta, \varphi) = (0, 0, \theta, -\pi/2)$ and $(x, y, \theta, \phi) = (0, 0, \theta, \pi/2)$.

We consider the most general linear perturbation of system (1-6) and we study the existence of limit cycles bifurcating from the periodic orbits of system (1-6).

Let the perturbed system be

$$\begin{aligned} \dot{x} &= -y + \varepsilon(a_0 + a_1 x + a_2 y + a_3\theta + a_4\varphi), \\ \dot{y} &= x + \varepsilon(b_0 + b_1 x + b_2 y + b_3\theta + b_4\varphi), \\ \dot{\theta} &= 1 + \varepsilon(c_0 + c_1 x + c_2 y + c_3\theta + c_4\varphi), \\ \dot{\varphi} &= \varepsilon(d_0 + d_1 x + d_2 y + d_3\theta + d_4\varphi), \end{aligned} \tag{1-7}$$

where $\varepsilon > 0$ is a small parameter and $a_i, b_i, c_i, d_i \in \mathbb{R}$ for $i = 0, \dots, 4$.

Theorem 3. *For sufficiently small $\varepsilon > 0$ the linear differential system* (1-7) *has a limit cycle bifurcating from a periodic orbit of system* (1-6) *provided that*

$$\det\begin{pmatrix} b_2+a_1 & a_2-b_1 \\ b_1-a_2 & b_2+a_1 \end{pmatrix} \neq 0 \quad \textit{and} \quad \det\begin{pmatrix} c_3 & c_4 \\ d_3 & d_4 \end{pmatrix} \neq 0.$$

This limit cycle bifurcates from the periodic orbit of system (1-6) *passing through the point $(x_0, y_0, \theta_0, \varphi_0)$ where*

$$\begin{aligned} x_0 &= \frac{(2b_2+2a_1)b_3 - 2a_3b_1 + 2a_2a_3}{b_2^2+b_1^2+a_2^2+a_1^2+2a_1b_2-2a_2b_1}, \\ y_0 &= -\frac{(2b_1-2a_2)b_3 + 2a_3b_2 + 2a_1a_3}{b_2^2+b_1^2+a_2^2+a_1^2+2a_1b_2-2a_2b_1}, \\ \theta_0 &= -\frac{(\pi c_3 + c_0)d_4 - \pi c_4 d_3 - c_4 d_0}{c_3d_4 - c_4d_3}, \qquad \varphi_0 = \frac{c_0d_3 - c_3d_0}{c_3d_4 - c_4d_3}. \end{aligned}$$

Theorem 3 is proved in Section 5.

As an example consider the system

$$\dot{x} = -y + \varepsilon a y, \quad \dot{y} = x + \varepsilon b x, \quad \dot{\theta} = 1 + \varepsilon c \varphi, \quad \dot{\varphi} = \varepsilon d \theta, \tag{1-8}$$

with $a, b, c, d \in \mathbb{R}$, and $\varepsilon > 0$. Applying Theorem 3 with $a_2 = a$, $b_1 = b$, $c_4 = c$, $d_3 = d$ and the rest of the coefficients of the perturbation being zero, we obtain that system (1-8) has a limit cycle bifurcating from the periodic orbit of system (1-6) passing through the point $(x_0, y_0, \theta_0, \varphi_0) = (0, 0, -\pi, 0)$, provided $(a-b)cd \neq 0$. That is, here the limit cycle bifurcates from the periodic orbit corresponding to the equator of the invariant sphere $\{x = y = 0\}$ of system (1-6).

The key tool that we use for proving Theorems 1–3 is averaging theory. For a general introduction to this theory, see the books [Sanders et al. 2007; Verhulst 1996]. As one can see in the proofs of Theorems 1–3, our method based on averaging theory can produce at most one limit cycle for the studied systems. Therefore the following open question is natural.

Open question. Let m and n be two nonnegative integers. Is it true that a linear vector field on the manifold $(\mathbb{S}^m)^m \times \mathbb{R}^n$ can have at most one limit cycle?

A similar open question was stated in [Llibre and Zhang 2016] concerning linear vector fields on the manifold $(\mathbb{S}^1)^m \times \mathbb{R}^n$.

2. Basic results on averaging theory

We now state basic results from averaging theory needed for later proofs. Let M be a smooth connected manifold of dimension n, and let $F_0, F_1 : \mathbb{R} \times M \to \mathbb{R}^n$

and $F_2 : \mathbb{R} \times M \times [0, \varepsilon_0) \to \mathbb{R}^n$ be C^2 periodic functions of period T. Given the differential system

$$\dot{x}(t) = F_0(t, x), \tag{2-1}$$

we consider a perturbation of this system of the form

$$\dot{x}(t) = F_0(t, x) + \varepsilon F_1(t, x) + \varepsilon^2 F_2(t, x, \varepsilon). \tag{2-2}$$

The objective is to study the bifurcation of T-periodic solutions of system (2-2) for $\varepsilon > 0$ small enough. A solution to this problem is given by averaging theory.

We assume that there exists $k \le n$ such that $M = M_k \times M_{n-k}$, where M_k is a manifold of dimension k and M_{n-k} is a manifold of dimension $n-k$, and that the unperturbed system, namely system (2-1), contains an open set, $V \subseteq M_k$, such that $\overline{V}$ is filled with periodic solutions all of them with the same period. Such a set is called *isochronous*.

Let $x(t, z, \varepsilon)$ be the solution of system (2-2) such that $x(0, z, \varepsilon) = z$. We write the linearization of the unperturbed system (2-1) along the solution $x(t, z, 0)$ as

$$\dot{y} = D_x F_0(t, x(t, z, 0))y. \tag{2-3}$$

and we let $\mathcal{M}_z(t)$ be the fundamental matrix of the linear differential system (2-3), so $\mathcal{M}_z(0)$ is the $n \times n$ identity matrix. Let $\xi : M = M_k \times M_{n-k} \to M_k$ the projection of M onto its first k coordinates, that is, $\xi(x_1, \dots, x_n) = (x_1, \dots, x_k)$.

The following results give sufficient conditions for the existence of limit cycles for a system of the form (2-2) bifurcating from the periodic orbits of system (2-1).

Theorem 4. *Let $V \subseteq M_k$ be an open and bounded set, and let $\beta_0 : \overline{V} \to M_{n-k}$ be a C^2 function. Assume that*:

(i) $\mathcal{Z} = \{z_\alpha = (\alpha, \beta_0(\alpha)) : \alpha \in \overline{V}\} \subset M$ *and for each $z_\alpha \in \mathcal{Z}$ the solution $x(t, z_\alpha, 0)$ of system* (2-1) *is T-periodic.*

(ii) *For each $z_\alpha \in \mathcal{Z}$, there is a fundamental matrix $\mathcal{M}_{z_\alpha}(t)$ of system* (2-3) *such that the matrix $\mathcal{M}_{z_\alpha}^{-1}(0) - \mathcal{M}_{z_\alpha}^{-1}(T)$ has the $k \times (n-k)$ zero matrix in the upper right corner*, *and a $(n-k) \times (n-k)$ matrix Δ_α in the lower right corner with* $\det(\Delta_\alpha) \neq 0$.

Consider the function $\mathcal{F} : \overline{V} \to \mathbb{R}^k$ defined by

$$\mathcal{F}(\alpha) = \xi\Big(\int_0^T \mathcal{M}_{z_\alpha}^{-1}(t) F_1(t, x(t, z_\alpha, 0))\, dt \Big).$$

If there exists $a \in V$ with $\mathcal{F}(a) = 0$ and with $\det(\mathcal{D}\mathcal{F}(a)) \neq 0$, *then there is a limit cycle $x(t, \varepsilon)$ of period T of system* (2-2) *such that $x(0, \varepsilon) \to z_a$ as $\varepsilon \to 0$.*

The result in Theorem 4 can be found in [Malkin 1956] and [Roseau 1966]. For a shorter proof, see [Buică et al. 2007]. There the result is proved in $\mathbb{R}^n$, but it can be easily extended to a manifold M.

The next result allows us to determine the existence of limit cycles in a system of the form (2-2) in the case when there exists an open set, $V \subset M$, such that for all $z \in \overline{V}$, the solution $x(t, z, 0)$ is T-periodic.

Theorem 5. *Let $V \subseteq M$ be an open and bounded set with $\overline{V} \subseteq M$, and assume that for all $z \in \overline{V}$ the solution $x(t, z, 0)$ of system (2-2) is T-periodic. Consider the function $\mathcal{F} : \overline{V} \to \mathbb{R}^n$ defined by*

$$\mathcal{F}(z) = \int_0^T \mathcal{M}_z^{-1}(t) F_1(t, x(t, z, 0))\, dt.$$

If there exists $a \in V$ with $\mathcal{F}(a) = 0$ and with $\det(\mathcal{D}\mathcal{F}(a)) \neq 0$, then there is a limit cycle $x(t, \varepsilon)$ of period T of system (2-2) such that $x(0, \varepsilon) \to a$ as $\varepsilon \to 0$.

For the proof of Theorem 5 see Corollary 1 of [Buică et al. 2007].

3. Proof of Theorem 1

We use the result from averaging theory given in Theorem 4 to deduce the existence of a limit cycle of system (1-2), for some $\varepsilon > 0$ small enough, bifurcating from a periodic orbit of the same system with $\varepsilon = 0$.

Since the general solution of the differential system (1-1), corresponding to system (1-2) with $\varepsilon = 0$, is given by

$$\theta(t) = \theta_0 + t, \quad \varphi(t) = \varphi_0, \quad r(t) = (r_0 - 1)e^t + 1,$$

it is clear that all the periodic solutions of that system are parameterized by

$$\theta(t) = \theta_0 + t, \quad \varphi(t) = \varphi_0, \quad r(t) = 1,$$

with $(\theta_0, \varphi_0) \in \mathbb{S}^2 \backslash \{\varphi_0 = \pm\pi/2\}$. Then, the periodic solutions all have period 2π and they fill the invariant sphere $\{r = 1\}$ except for the poles, which are equilibrium points.

Therefore, for applying Theorem 4 we take $M = \mathbb{S}^2 \times \mathbb{R}$ and

$$\begin{aligned}
k &= 2, \quad n = 3,\\
M_k &= M_2 = \{(\theta, \varphi, r) \in M : r = 1\} \cong \mathbb{S}^2,\\
x &= (\theta, \varphi, r),\\
\alpha &= (\theta_0, \varphi_0),\\
\beta_0(\alpha) &= \beta_0(\theta_0, \varphi_0) = 1,\\
z_\alpha &= (\alpha, \beta_0(\alpha)) = (\theta_0, \varphi_0, 1),\\
V &= \left\{(\theta, \varphi, r) \in M : r = 1, \varphi \in \left(-\tfrac{\pi}{2} + \delta_0, \tfrac{\pi}{2} - \delta_0\right)\right\},
\end{aligned} \tag{3-1}$$

with $\delta_0 > 0$ small enough that

$$\begin{aligned}
\varphi^* &:= \frac{b_1(a_0+a_3+a_1\pi)-a_1(b_0+b_3+b_2\pi)}{a_1b_2-a_2b_1} \in \left(-\tfrac{\pi}{2}+\delta_0, \tfrac{\pi}{2}-\delta_0\right),\\
\mathcal{Z} &= \overline{V} \times \{r=1\},\\
x(t, z_\alpha, 0) &= (\theta_0+t, \varphi_0, 1),\\
F_0(t,x) &= (1, 0, r-1),\\
F_1(t,x) &= (a_0+a_1\theta+a_2\varphi+a_3r, b_0+b_1\theta+b_2\varphi+b_3r, c_0+c_1\theta+c_2\varphi+c_3r),\\
F_2(t,x,\varepsilon) &= 0,\\
T &= 2\pi,
\end{aligned}$$

where we took $V \subset M_2$ as an open subset that contains the periodic orbit from which a limit cycle bifurcates, as we shall see next.

The fundamental matrix $\mathcal{M}_{z_\alpha}(t)$ with $\mathcal{M}_{z_\alpha}(0) = \mathrm{Id}$ of system (2-3) with F_0 and $x(t, z_\alpha, 0)$ described above is the matrix $\mathcal{M}_{z_\alpha}(t) = \exp(D_x F_0\, t)$, i.e.,

$$\mathcal{M}_{z_\alpha}(t) = \begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & e^t \end{pmatrix}.$$

Note that since F_0 defines a linear differential system, the fundamental matrix $\mathcal{M}_{z_\alpha}(t)$ is independent of the initial conditions z_α. We also have

$$\mathcal{M}^{-1}_{z_\alpha}(0) - \mathcal{M}^{-1}_{z_\alpha}(2\pi) = \begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1-e^{-2\pi} \end{pmatrix},$$

and therefore, all the assumptions in the in the statement of Theorem 4 are satisfied.

With the described setting, the function $\mathcal{F}(\alpha) = \mathcal{F}(\theta_0, \varphi_0)$ from the statement of Theorem 4 associated with system (1-2) is

$$\begin{aligned}
\mathcal{F}(\theta_0, \varphi_0) &= \xi\left(\int_0^{2\pi} \mathcal{M}^{-1}_{z_\alpha}(t) F_1(\theta_0+t, \varphi_0, 1)\, dt\right)\\
&= 2\pi\left(a_0+a_1(\theta_0+\pi)+a_2\varphi_0+b_3, b_0+b_1(\theta_0+\pi)+b_2\varphi_0+b_3\right).
\end{aligned}$$

We have $\det(D\mathcal{F}) = 4\pi^2(a_1b_2-a_2b_1)$. Therefore $\det(D\mathcal{F}) \neq 0$ for all $(\theta_0, \varphi_0) \in V$. Thus, the only solution of $\mathcal{F} = 0$ is given by

$$\begin{aligned}
\theta_0 &= \frac{a_2(b_0+b_3+b_1\pi)-b_2(a_0+a_3+a_1\pi)}{a_1b_2-a_2b_1},\\
\varphi_0 &= \frac{b_1(a_0+a_3+a_1\pi)-a_1(b_0+b_3+b_2\pi)}{a_1b_2-a_2b_1}.
\end{aligned} \tag{3-2}$$

This solution, (θ_0, φ_0), where $\varphi_0 = \varphi^*$, is contained in the set V described in (3-1).

Hence, by Theorem 4, if $\varepsilon > 0$ is small enough, there is a periodic solution, $(\theta(t,\varepsilon), \varphi(t,\varepsilon), r(t,\varepsilon))$, of system (1-3), which is a limit cycle, and such that

$$(\theta(0,\varepsilon), \varphi(0,\varepsilon), r(0,\varepsilon)) \to (\theta_0, \varphi_0, 1),$$

when $\varepsilon \to 0$, and where θ_0 and φ_0 are given in (3-2).

4. Proof of Theorem 2

We use the result from averaging theory given in Theorem 4 to prove that, for some $\varepsilon > 0$ small enough, there exist a limit cycle of system (1-5) bifurcating from a periodic orbit of the same system with $\varepsilon = 0$.

Since the general solution of system (1-5) with $\varepsilon = 0$ (that is, the one of system (1-4)) is

$$\theta(t) = \theta_0 + t, \quad \varphi(t) = \varphi_0, \quad \nu(t) = \nu_0 + t, \quad \phi(t) = \phi_0, \quad r(t) = (r_0 - 1)e^t + 1,$$

we have that all the periodic solutions of that system are

$$\theta(t) = \theta_0 + t, \quad \varphi(t) = \varphi_0, \quad \nu(t) = \nu_0 + t, \quad \phi(t) = \phi_0, \quad r(t) = 1,$$

with $(\theta_0, \varphi_0, \nu_0, \phi_0) \in \mathbb{S}^2 \backslash \{\varphi_0 = \pm\pi/2\} \times \mathbb{S}^2 \backslash \{\varphi_0 = \pm\pi/2\}$. That is, the periodic solutions fill the invariant manifold $\{r = 1\}$ except for the four equilibrium points $\{\varphi = \pm\pi/2, \phi = \pm\pi/2\}$, and they have all period 2π.

For applying Theorem 4 we take $M = (\mathbb{S}^2)^2 \times \mathbb{R}$ and

$$\begin{aligned}
k &= 4, \quad n = 5,\\
M_k &= M_4 = \{\theta, \varphi, \nu, \phi, r \in M : r = 1\} \cong (\mathbb{S}^2)^2,\\
x &= (\theta, \varphi, \nu, \phi, r),\\
\alpha &= (\theta_0, \varphi_0, \nu_0, \phi_0),\\
\beta_0(\alpha) &= \beta_0(\theta_0, \varphi_0, \nu_0, \phi_0) = 1,\\
z_\alpha &= (\alpha, \beta_0(\alpha)) = (\theta_0, \varphi_0, \nu_0, \phi_0, 1),\\
V &= \bigl\{(\theta, \varphi, \nu, \phi, r) \in M : r = 1, \varphi \in \bigl(-\tfrac{\pi}{2} + \delta_0, \tfrac{\pi}{2} - \delta_0\bigr)\bigr\}
\end{aligned} \tag{4-1}$$

with $\delta_0 > 0$ small enough that φ_0, ϕ_0 satisfy (4-2) and

$$\begin{aligned}
\varphi_0, \phi_0 &\in \bigl(-\tfrac{\pi}{2} + \delta_0, \tfrac{\pi}{2} - \delta_0\bigr),\\
\mathcal{Z} &= \overline{V} \times \{r = 1\},\\
x(t, z_\alpha, 0) &= (\theta_0 + t, \varphi_0, \nu_0 + t, \phi_0, 1),\\
F_0(t, x) &= (1, 0, 1, 0, r - 1),
\end{aligned}$$

$$F_1(t,x) = \begin{pmatrix} a_0+a_1\theta+a_2\varphi+a_3\nu+a_4\phi+a_5 r \\ b_0+b_1\theta+b_2\varphi+b_3\nu+b_4\phi+b_5 r \\ c_0+c_1\theta+c_2\varphi+c_3\nu+c_4\phi+c_5 r \\ d_0+d_1\theta+d_2\varphi+d_3\nu+d_4\phi+d_5 r \\ e_0+e_1\theta+e_2\varphi+e_3\nu+e_4\phi+e_5 r \end{pmatrix},$$
$$F_2(t,x,\varepsilon) = 0,$$
$$T = 2\pi,$$

where we chose $V \subset M_4$ as an open subset that contains the periodic orbit from which a limit cycle bifurcates.

The fundamental matrix $\mathcal{M}_{z_\alpha}(t)$ with $\mathcal{M}_{z_\alpha}(0) = \mathrm{Id}$ of system (2-3) with F_0 and $x(t, z_\alpha, 0)$ described above is the matrix $\mathcal{M}_{z_\alpha}(t) = \exp(D_x F_0\, t)$, i.e.,

$$\mathcal{M}_{z_\alpha}(t) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & e^t \end{pmatrix}.$$

We also have

$$\mathcal{M}_{z_\alpha}^{-1}(0) - \mathcal{M}_{z_\alpha}^{-1}(2\pi) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-e^{-2\pi} \end{pmatrix},$$

and therefore, all the assumptions in the statement of Theorem 4 are satisfied.

In this setting, the function $\mathcal{F}(\alpha) = \mathcal{F}(\theta_0, \varphi_0, \nu_0, \phi_0)$ in Theorem 4 associated with system (1-5) is

$$\mathcal{F}(\theta_0, \varphi_0, \nu_0, \phi_0) = \xi\Big(\int_0^{2\pi} \mathcal{M}_{z_\alpha}^{-1}(t) F_1(\theta_0+t, \varphi_0, \nu_0+t, \phi_0, 1)\, dt\Big) = (\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4),$$

with

$$\mathcal{F}_1 = 2\pi(a_0 + a_1\theta_0 + a_1\pi + a_2\varphi_0 + a_3\nu_0 + a_3\pi + a_4\phi_0 + a_5),$$
$$\mathcal{F}_2 = 2\pi(b_0 + b_1\theta_0 + b_1\pi + b_2\varphi_0 + b_3\nu_0 + b_3\pi + b_4\phi_0 + b_5),$$
$$\mathcal{F}_3 = 2\pi(c_0 + c_1\theta_0 + c_1\pi + c_2\varphi_0 + c_3\nu_0 + c_3\pi + c_4\phi_0 + c_5),$$
$$\mathcal{F}_4 = 2\pi(d_0 + d_1\theta_0 + d_1\pi + d_2\varphi_0 + d_3\nu_0 + d_3\pi + d_4\phi_0 + d_5).$$

Also, we have

$$\det(D\mathcal{F}) = 16\pi^4 \det \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{pmatrix} \neq 0,$$

by assumption. The initial conditions $(\theta_0, \varphi_0, \nu_0, \phi_0)$ such that $\mathcal{F}(\theta_0, \varphi_0, \nu_0, \phi_0) = 0$ are the solutions of the linear system

$$\begin{aligned} a_1\theta_0 + a_2\varphi_0 + a_3\nu_0 + a_4\phi_0 &= -a_0 - a_1\pi - a_3\pi - a_5, \\ b_1\theta_0 + b_2\varphi_0 + b_3\nu_0 + b_4\phi_0 &= -b_0 - b_1\pi - b_3\pi - b_5, \\ c_1\theta_0 + c_2\varphi_0 + c_3\nu_0 + c_4\phi_0 &= -c_0 - c_1\pi - c_3\pi - c_5, \\ d_1\theta_0 + d_2\varphi_0 + d_3\nu_0 + d_4\phi_0 &= -d_0 - d_1\pi - d_3\pi - d_5. \end{aligned} \tag{4-2}$$

Since $\det(D\mathcal{F}) \neq 0$, system (4-2) has a unique solution, $(\theta_0, \varphi_0, \nu_0, \phi_0)$, and this solution is contained in the set V described in (4-1).

Hence, by Theorem 4, if $\varepsilon > 0$ is small enough, there is a periodic solution,

$$(\theta(t, \varepsilon), \varphi(t, \varepsilon), \nu(t, \varepsilon), \phi(t, \varepsilon), r(t, \varepsilon)),$$

of system (1-5), which is a limit cycle, and such that

$$(\theta(0, \varepsilon), \varphi(0, \varepsilon), \nu(0, \varepsilon), \phi(0, \varepsilon), r(0, \varepsilon)) \to (\theta_0, \varphi_0, \nu_0, \phi_0, 1),$$

when $\varepsilon \to 0$, and where $\theta_0, \varphi_0, \nu_0$, and ϕ_0 are given by the unique solution of system (4-2).

5. Proof of Theorem 3

Since the general solution of system (1-7) with $\varepsilon = 0$ is given by

$$x(t) = x_0 \cos t - y_0 \sin t, \;\; y(t) = x_0 \sin t + y_0 \cos t, \;\; \theta(t) = \theta_0 + t, \;\; \varphi(t) = \varphi_0, \tag{5-1}$$

the whole phase space is filled by periodic solutions, except from the equilibrium points $(x, y, \theta, \varphi) = (0, 0, \theta, -\pi/2)$ and $(x, y, \theta, \varphi) = (0, 0, \theta, \pi/2)$. Hence, the periodic solutions of the differential system (1-6) fill an open set of the phase space $M = \mathbb{R}^2 \times \mathbb{S}^2$.

To prove Theorem 3 we use the result given in Theorem 5 to deduce that there exists a limit cycle of system (1-7), for some $\varepsilon > 0$ small enough, bifurcating from the periodic orbits of the same system with $\varepsilon = 0$.

To clarify the notation, here we denote the solution $x(t, z, 0)$ from the statement of Theorem 5 by $\boldsymbol{x}(t, z, 0)$, and x will denote the first variable in the phase space.

For applying Theorem 5 we take $M = \mathbb{R}^2 \times \mathbb{S}^2$ and

$$\begin{aligned} \boldsymbol{x} &= (x, y, \theta, \varphi), \\ z &= (x_0, y_0, \theta_0, \varphi_0), \\ \boldsymbol{x}(t, z, 0) &= (x(t), y(t), \theta(t), \varphi(t)) \quad \text{given by (5-1)}, \end{aligned}$$

$$F_0(t,x) = (-y, x, 1, 0),$$

$$F_1(t,x) = \begin{pmatrix} a_0+a_1x+a_2y+a_3\theta+a_4\varphi \\ b_0+b_1x+b_2y+b_3\theta+b_4\varphi \\ c_0+c_1x+c_2y+c_3\theta+c_4\varphi \\ d_0+d_1x+d_2y+d_3\theta+d_4\varphi \end{pmatrix},$$

$$F_2(t,x,\varepsilon) = 0,$$

$$T = 2\pi,$$

$$V = \big\{(x,y,\theta,\varphi) \in M : \|(x,y)\| < 1+\kappa, \varphi \in \big(-\tfrac{\pi}{2}+\delta_0, \tfrac{\pi}{2}-\delta_0\big)\big\}, \tag{5-2}$$

with $\kappa = \dfrac{2\sqrt{a_3^2+b_3^2}}{\sqrt{a_1^2+a_2^2+b_1^2+b_2^2+2a_1b_2-2a_2b_1}}$ and $\delta_0 > 0$ small enough that

$$\varphi^* := \frac{c_0d_3 - c_3d_0}{c_3d_4 - c_4d_3} \in \big(-\tfrac{\pi}{2}+\delta_0, \tfrac{\pi}{2}-\delta_0\big),$$

and where we chose $V \subset M$ as an open subset that contains the periodic orbit from which a limit cycle bifurcates.

The fundamental matrix $\mathcal{M}_z(t)$ of system (2-3) with $\mathcal{M}_z(0) = \mathrm{Id}$ and with F_0 and $\boldsymbol{x}(t,z,0)$ as just described is given by

$$\mathcal{M}_z(t) = \begin{pmatrix} \cos t & -\sin t & 0 & 0 \\ \sin t & \cos t & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Therefore all the assumptions in the statement of Theorem 5 are satisfied.

In this setting the function $\mathcal{F}(z) = \mathcal{F}(x_0, y_0, \theta_0, \varphi_0)$ in Theorem 5 associated with system (1-7), namely,

$$\mathcal{F}(x_0, y_0, \theta_0, \varphi_0) = \int_0^{2\pi} \mathcal{M}_z^{-1}(t) F_1(t, \boldsymbol{x}(t,x,0))\, dt,$$

is given by $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4)$, which after some straightforward computations can be written as

$$\mathcal{F}_1 = (\pi a_2 - \pi b_1)y_0 + (\pi b_2 + \pi a_1)x_0 - 2\pi b_3,$$

$$\mathcal{F}_2 = (\pi b_2 + \pi a_1)y_0 + (\pi b_1 - \pi a_2)x_0 + 2\pi a_3,$$

$$\mathcal{F}_3 = 2\pi c_3\theta_0 + 2\pi c_4\varphi_0 + 2\pi^2 c_3 + 2\pi c_0,$$

$$\mathcal{F}_4 = 2\pi d_3\theta_0 + 2\pi d_4\varphi_0 + 2\pi^2 d_3 + 2\pi d_0.$$

Assuming that

$$\det(\mathcal{DF}) = \det\begin{pmatrix} \pi(b_2+a_1) & \pi(a_2-b_1) & 0 & 0 \\ \pi(b_1-a_2) & \pi(b_2+a_1) & 0 & 0 \\ 0 & 0 & 2\pi c_3 & 2\pi c_4 \\ 0 & 0 & 2\pi d_3 & 2\pi d_4 \end{pmatrix} \neq 0, \tag{5-3}$$

the linear system $(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4) = (0, 0, 0, 0)$ has a unique solution, given by

$$\begin{aligned} x_0 &= \frac{(2b_2+2a_1)b_3 - 2a_3b_1 + 2a_2a_3}{b_2^2+b_1^2+a_2^2+a_1^2+2a_1b_2-2a_2b_1}, \\ y_0 &= -\frac{(2b_1-2a_2)b_3 + 2a_3b_2 + 2a_1a_3}{b_2^2+b_1^2+a_2^2+a_1^2+2a_1b_2-2a_2b_1}, \\ \theta_0 &= -\frac{(\pi c_3 + c_0)d_4 - \pi c_4 d_3 - c_4 d_0}{c_3d_4 - c_4d_3}, \qquad \varphi_0 = \frac{c_0d_3 - c_3d_0}{c_3d_4 - c_4d_3}. \end{aligned} \tag{5-4}$$

This solution $(x_0, y_0, \theta_0, \varphi_0)$, where $\varphi_0 = \varphi^*$, is contained in the set V in (5-2).

The condition (5-3) is clearly satisfied for all $(x_0, y_0, \theta_0, \varphi_0) \in V$ taking into account the assumptions in the statement of Theorem 3.

Hence, by Theorem 5, there is a periodic solution $(x(t,\varepsilon),\ y(t,\varepsilon),\ \theta(t,\varepsilon),\ \varphi(t,\varepsilon))$ of system (1-7), which is a limit cycle, and such that

$$(x(0,\varepsilon), y(0,\varepsilon), \theta(0,\varepsilon), \varphi(0,\varepsilon)) \to (x_0, y_0, \theta_0, \varphi_0)$$

when $\varepsilon \to 0$, and where x_0, y_0, θ_0 and φ_0 are given in (5-4).

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CLARA CUFÍ-CABRÉ

clara.cufi@uab.cat
DEPARTAMENT DE MATEMÀTIQUES
UNIVERSITAT AUTÒNOMA DE BARCELONA
BARCELONA
SPAIN

JAUME LLIBRE

jaume.llibre@uab.cat
DEPARTAMENT DE MATEMÀTIQUES
UNIVERSITAT AUTÒNOMA DE BARCELONA
BARCELONA
SPAIN

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