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We compute the cohomological Brauer groups of twists of weighted projective spaces and weighted projective stacks, generalizing Gabber's computation of the Brauer group of Brauer–Severi varieties. A key ingredient in our proof is a description of the Brauer group of toric varieties due to DeMeyer, Ford, Miranda (1993).

1. Introduction

Weighted projective spaces and stacks are a natural generalization of projective space that often arise in the construction of certain moduli spaces. For example, the moduli space of cubic surfaces is isomorphic to $\mathbb{P}(1, 2, 3, 4, 5)$, see, e.g., [13, Section 9.4.5]. Over a field k of characteristic not 2 or 3, the moduli stack of elliptic curves $\mathcal{M}_{1,1,k}$ is isomorphic to an open substack of $\mathcal{P}_k(4, 6)$.

To recall the construction of weighted projective spaces and stacks, let $n \geq 1$ and let $\rho = (\rho_0, \dots, \rho_n)$ be an $(n+1)$-tuple of positive integers. For any field k, we may define an equivalence relation on $k^{n+1} \setminus \{(0, \dots, 0)\}$ by

$$(x_0, \dots, x_n) \sim (u^{\rho_0} x_0, \dots, u^{\rho_n} x_n)$$

for all units $u \in k^\times$. The *weighted projective space* $\mathbb{P}_{\mathbb{Z}}(\rho)$ is the scheme-theoretic quotient of this action; it is the scheme whose k-rational points correspond to the equivalence classes of this equivalence relation. Taking the stack-theoretic quotient of the above action gives the *weighted projective stack* $\mathcal{P}_{\mathbb{Z}}(\rho)$ associated to ρ. If every ρ_i is equal to 1, then $\mathbb{P}_{\mathbb{Z}}(\rho)$ and $\mathcal{P}_{\mathbb{Z}}(\rho)$ are isomorphic to the (unweighted) projective space $\mathbb{P}^n$.

In this paper, we are interested in the cohomological Brauer groups of étale twists of weighted projective spaces and weighted projective stacks. For any scheme S, we denote $\mathrm{Br}'(S) := \mathrm{H}^2_{\text{ét}}(S, \mathbb{G}_m)_{\text{tors}}$ the *cohomological Brauer group* of S. In the unweighted case, an étale twist of projective space is called a *Brauer–Severi scheme*; it is well known that to every Brauer–Severi scheme $f : X \to S$ there is an associated class $[X] \in \mathrm{Br}'(S)$ and that the pullback of $[X]$ to $\mathrm{Br}'(X)$ is trivial.

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A theorem of Gabber states that the induced map $\mathrm{Br}'(S)/\langle [X] \rangle \to \mathrm{Br}'(X)$ is in fact an isomorphism.

Theorem 1.1 (Gabber [18, Chapter II, Theorem 2]). *Let S be a scheme and let $f : X \to S$ be a Brauer–Severi scheme. Then the sequence*

$$\Gamma(S, \underline{\mathbb{Z}}) \to \mathrm{Br}'(S) \xrightarrow{f^*} \mathrm{Br}'(X) \to 0 \tag{1.1.1}$$

is exact, where the first map sends $1 \mapsto [X]$.

The purpose of this paper is to extend the above theorem to include weighted projective spaces and stacks.

Theorem 1.2. *Let $f_X : X \to S$ be a morphism of schemes such that there exists an étale surjection $S' \to S$ such that $X \times_S S' \simeq \mathbb{P}_{S'}(\rho)$. Then there is a natural Brauer class $[X] \in \mathrm{Br}'(S)$ associated to X, and the sequence*

$$\Gamma(S, \underline{\mathbb{Z}}) \to \mathrm{Br}'(S) \xrightarrow{f_X^*} \mathrm{Br}'(X) \to 0 \tag{1.2.1}$$

is exact, where the first map sends $1 \mapsto [X]$.

Theorem 1.3. *Let S be a scheme, let $f_{\mathcal{X}} : \mathcal{X} \to S$ be a morphism of algebraic stacks such that there exists an étale surjection $S' \to S$ such that $\mathcal{X} \times_S S' \simeq \mathcal{P}_{S'}(\rho)$. Then there is a natural Brauer class $[\mathcal{X}] \in \mathrm{Br}'(S)$ associated to $\mathcal{X}$, and the sequence*

$$\Gamma(S, \underline{\mathbb{Z}}) \to \mathrm{Br}'(S) \xrightarrow{f_{\mathcal{X}}^*} \mathrm{Br}'(\mathcal{X}) \to 0 \tag{1.3.1}$$

is exact, where the first map sends $1 \mapsto [\mathcal{X}]$. If $\pi : \mathcal{X} \to X$ denotes the coarse moduli space of $\mathcal{X}$, then X satisfies the hypothesis of Theorem 1.2, and pullback by π induces a commutative diagram

$$\begin{array}{ccccccc} \Gamma(S, \underline{\mathbb{Z}}) & \longrightarrow & \mathrm{Br}'(S) & \xrightarrow{f_{\mathcal{X}}^*} & \mathrm{Br}'(\mathcal{X}) & \longrightarrow & 0 \\ {\scriptstyle \times\,\mathrm{lcm}(\rho)}\uparrow & & \| & & \uparrow{\scriptstyle \pi^*} & & \\ \Gamma(S, \underline{\mathbb{Z}}) & \longrightarrow & \mathrm{Br}'(S) & \xrightarrow[f_X^*]{} & \mathrm{Br}'(X) & \longrightarrow & 0 \end{array} \tag{1.3.2}$$

where the rows are (1.2.1) and (1.3.1) and the leftmost vertical map denotes multiplication-by-$\mathrm{lcm}(\rho)$.

1.4. ***Outline of the paper.*** To prove Theorems 1.2 and 1.3, we show that $\boldsymbol{R}^1 f_* \mathbb{G}_m \simeq \underline{\mathbb{Z}}$ and $\boldsymbol{R}^2 f_* \mathbb{G}_m = 0$ and apply the Leray spectral sequence to the morphism f. For the claim that $\boldsymbol{R}^2 f_* \mathbb{G}_m = 0$, a deformation theory argument of Mathur (personal communication, 2019) which uses a Tannaka duality result of Hall and Rydh [22], reduces us to the case where S is the spectrum of a field. Here, the proofs of Theorems 1.2 and 1.3 require different approaches (indeed, a Deligne–Mumford stack $\mathcal{X}$ and its coarse moduli space X may have nonisomorphic (Picard groups and) Brauer groups in general).

As we recall in Section 3.2, a weighted projective space $\mathbb{P}(\rho)$ is a toric variety. In Section 3, we use the results of DeMeyer, Ford and Miranda [11] on the Brauer group of toric varieties to compute the Brauer group of $\mathbb{P}(\rho)$ over an algebraically closed field; taking the prime-to-p limit of dilations of the toric variety reduces us to computing the p-torsion when each weight ρ_i is a power of p. In Section 5 we prove Theorem 1.3, for which the key observation turns out to be that the $\mathbb{G}_m$-action on $\mathbb{A}^{n+1}$ extends to an action of the multiplicative monoid $\mathbb{A}^1$ on $\mathbb{A}^{n+1}$.

2. Weighted projective spaces

In this section, we recall basic facts about weighted projective spaces (in particular, regarding their Picard group Lemma 2.8 and cohomology of line bundles Lemma 2.9) which will be used in the proof of Theorem 1.2. For general background on weighted projective spaces, we refer to [12; 29].

2.1. For a weight vector $\rho = (\rho_0, \dots, \rho_n)$, the *weighted projective space* associated to ρ is

$$\mathbb{P}_{\mathbb{Z}}(\rho) := \operatorname{Proj} \mathbb{Z}[t_0, \dots, t_n],$$

where $\mathbb{Z}[t_0, \dots, t_n]$ has the $\mathbb{Z}$-grading defined by $\deg(t_i) = \rho_i$. We set

$$\mathbb{P}_S(\rho) := \mathbb{P}_{\mathbb{Z}}(\rho) \times_{\operatorname{Spec} \mathbb{Z}} S$$

for any scheme S. By [14, Lemme (2.1.6), Proposition (2.4.7)], the weighted projective space $\mathbb{P}_{\mathbb{Z}}(\rho)$ is projective. Thus, if S is quasicompact and admits an ample line bundle, then the same is true for $\mathbb{P}_S(\rho)$; hence in this case $\operatorname{Br} = \operatorname{Br}'$ for $\mathbb{P}_S(\rho)$ by a theorem of Gabber [9] (i.e., the Azumaya Brauer group coincides with the cohomological Brauer group).

2.2. Suppose a positive integer d divides all ρ_i and set $\rho/d := (\rho_0/d, \dots, \rho_n/d)$. Then there is a natural isomorphism $\mathbb{P}_{\mathbb{Z}}(\rho) \simeq \mathbb{P}_{\mathbb{Z}}(\rho/d)$ by [14, Proposition (2.4.7)(i)], and under this isomorphism $\mathcal{O}_{\mathbb{P}_{\mathbb{Z}}(\rho/d)}(\ell)$ corresponds to $\mathcal{O}_{\mathbb{P}_{\mathbb{Z}}(\rho)}(d\ell)$ for all $\ell \in \mathbb{Z}$.

If ρ, σ are two weight vectors such that one is a permutation of the other, then the corresponding weighted projective spaces $\mathbb{P}_{\mathbb{Z}}(\rho)$, $\mathbb{P}_{\mathbb{Z}}(\sigma)$ are isomorphic. The converse is not true in general, but is true if ρ, σ satisfy a certain "normalization" condition.

Definition 2.3 (normalized weight vectors [2, Section 2]). We say that $\rho = (\rho_0, \dots, \rho_n)$ satisfies (N) if, for all $0 \le i \le n$, we have $\gcd(\{\rho_j\}_{j \ne i}) = 1$.

Lemma 2.4 [2, Section 8]. *Let ρ, σ be two weight vectors satisfying* (N). *We have $\mathbb{P}_{\mathbb{Z}}(\rho) \simeq \mathbb{P}_{\mathbb{Z}}(\sigma)$ if and only if ρ is a permutation of σ.*

By Lemma 2.5 below, every weight vector ρ has an associated normalized weight vector ρ' such that ρ' satisfies (N) and $\mathbb{P}_{\mathbb{Z}}(\rho) \simeq \mathbb{P}_{\mathbb{Z}}(\rho')$. Thus in Theorem 1.2 we may always assume that our weight vector ρ satisfies (N).

Lemma 2.5 (reduction of weights [10, Proposition 1.3; 12, Section 1.3.1; 2, Sections 1.3, 1.4]). *Suppose* $\gcd(\rho) = 1$. *Define the constants*

$$d_i := \gcd(\{\rho_j\}_{j\neq i}), \quad s_i := \operatorname{lcm}(\{d_j\}_{j\neq i}), \quad s := \operatorname{lcm}(s_0, \ldots, s_n),$$
$$\rho_i' := \rho_i / s_i, \quad \rho' := (\rho_0', \ldots, \rho_n')$$

and let $R' := \mathbb{Z}[t_0', \ldots, t_n']$ *be the ring with the* $\mathbb{Z}$*-grading determined by* $\deg(t_i') = \rho_i'$. *The ring homomorphism* $R' \to R$ *sending* $t_i' \mapsto t_i^{d_i}$ (*which multiplies the degree by* s) *induces an isomorphism*

$$\varphi : \mathbb{P}_{\mathbb{Z}}(\rho) \to \mathbb{P}_{\mathbb{Z}}(\rho')$$

of schemes. We have

$$\operatorname{lcm}(\rho) = s \cdot \operatorname{lcm}(\rho') \tag{2.5.1}$$

since $v_p(\operatorname{lcm}(\rho)) = \alpha_{i_0}$ *and* $v_p(\operatorname{lcm}(\rho')) = \alpha_{i_0} - \alpha_{i_{n-1}}$ *for any prime* p, *in the notation of* [2, Section 1.2].

For any integer ℓ, *there exists a unique pair* $(b_i(\ell), c_i(\ell)) \in \mathbb{Z}^2$ *satisfying* $0 \le b_i(\ell) < d_i$ *and* $\ell = b_i(\ell)\rho_i + c_i(\ell)\, d_i$; *set* $\ell' := \ell - \sum_{i=0}^n b_i(\ell)\rho_i$. *The multiplication-by-*$(t_0^{b_0(\ell)} \cdots t_n^{b_n(\ell)})$ *map* $R(\ell') \to R(\ell)$ *induces an isomorphism* $\mathcal{O}_{\mathbb{P}_{\mathbb{Z}}(\rho)}(\ell') \simeq \mathcal{O}_{\mathbb{P}_{\mathbb{Z}}(\rho)}(\ell)$ *of* $\mathcal{O}_{\mathbb{P}_{\mathbb{Z}}(\rho)}$*-modules. Furthermore* ℓ' *is divisible by* s *and we obtain an isomorphism*

$$\mathcal{O}_{\mathbb{P}_{\mathbb{Z}}(\rho')}(\ell'/s) \simeq \varphi_*(\mathcal{O}_{\mathbb{P}_{\mathbb{Z}}(\rho)}(\ell))$$

of $\mathcal{O}_{\mathbb{P}_{\mathbb{Z}}(\rho)}$*-modules. In particular, we have*

$$\varphi^*(\mathcal{O}_{\mathbb{P}_{\mathbb{Z}}(\rho')}(\ell)) \simeq \mathcal{O}_{\mathbb{P}_{\mathbb{Z}}(\rho)}(s\ell) \tag{2.5.2}$$

for all $\ell \in \mathbb{Z}$ *since* $b_i(s\ell) = 0$.

Remark 2.6. By Lemma 2.5, all weighted projective lines $\mathbb{P}_{\mathbb{Z}}(q_0, q_1)$ are isomorphic to $\mathbb{P}^1_{\mathbb{Z}}$; thus, for Theorem 1.2, we may assume $n \ge 2$.

Lemma 2.7. *The sheaf* $\mathcal{O}_{\mathbb{P}_{\mathbb{Z}}(\rho)}(r)$ *is reflexive for any* $r \in \mathbb{Z}$. *If* ρ *satisfies* (*N*), *the sheaf* $\mathcal{O}_{\mathbb{P}_{\mathbb{Z}}(\rho)}(r)$ *is invertible if and only if* $\operatorname{lcm}(\rho)$ *divides* r.

Lemma 2.8 (Picard group of $\mathbb{P}(\rho)$ [2, Section 6.1]). *For any connected locally Noetherian scheme* S, *the map*

$$\mathbb{Z} \oplus \operatorname{Pic}(S) \to \operatorname{Pic}(\mathbb{P}_S(\rho))$$

sending

$$(\ell, \mathcal{L}) \mapsto \mathcal{O}_{\mathbb{P}_S(\rho)}(\ell \cdot \operatorname{lcm}(\rho)) \otimes f_S^* \mathcal{L}$$

is an isomorphism. (*See also* [26, Section 6].)

Proof. By Section 2.2, we may assume $\gcd(\rho) = 1$. In [2] the desired claim is proved assuming that ρ satisfies (N). If ρ does not satisfy (N), then we conclude using (2.5.1) and (2.5.2). □

Lemma 2.9 (cohomology of $\mathcal{O}_{\mathbb{P}(\rho)}(\ell)$ [10, Section 3]). *Let A be a ring and set $X := \mathbb{P}_A(\rho)$.*

(1) *For $\ell \geq 0$, the A-module $\mathrm{H}^0(X, \mathcal{O}_X(\ell))$ is free with basis consisting of monomials $t_0^{e_0} \cdots t_n^{e_n}$ such that $e_0, \ldots, e_n \in \mathbb{Z}_{\geq 0}$ and $\rho_0 e_0 + \cdots + \rho_n e_n = \ell$.*

(2) *For $\ell < 0$, the A-module $\mathrm{H}^n(X, \mathcal{O}_X(\ell))$ is free with basis consisting of monomials $t_0^{e_0} \cdots t_n^{e_n}$ such that $e_0, \ldots, e_n \in \mathbb{Z}_{<0}$ and $\rho_0 e_0 + \cdots + \rho_n e_n = \ell$.*

(3) *If $(i, \ell) \notin (\{0\} \times \mathbb{Z}_{\geq 0}) \cup (\{n\} \times \mathbb{Z}_{<0})$, then $\mathrm{H}^i(X, \mathcal{O}_X(\ell)) = 0$.*

(4) *For any A-module M and any (i, ℓ), the canonical map*

$$\mathrm{H}^i(X, \mathcal{O}_X(\ell)) \otimes_A M \to \mathrm{H}^i(X, \mathcal{O}_X(\ell) \otimes_A M)$$

is an isomorphism.

Remark 2.10. The projection $\mathbb{P}_{\mathbb{Z}}(\rho) \to \operatorname{Spec} \mathbb{Z}$ is a flat morphism of relative dimension n, and its geometric fibers are normal. By [12, Section 1.3.3(iii)], we have that $\mathbb{P}_S(\rho) \to S$ is smooth if and only if $\mathbb{P}_S(\rho) \simeq \mathbb{P}^n_S$. If ρ satisfies (N), then Lemma 2.4 implies that $\mathbb{P}_S(\rho) \simeq \mathbb{P}^n_S$ if and only if $\rho = (1, \ldots, 1)$.

3. Over an algebraically closed field

In this section, we prove Lemma 3.1 (i.e., Theorem 1.2 when $S = \operatorname{Spec} k$ for an algebraically closed field k). We will consider arbitrary fields in Lemma 4.1, and generalize from fields to (strictly henselian) local rings in Lemma 4.3.

Lemma 3.1. *If k is an algebraically closed field, then $\mathrm{H}^2_{\acute{e}t}(\mathbb{P}_k(\rho), \mathbb{G}_m) = 0$.*

Proof (outline of argument). In Section 3.2, we recall how to construct a fan Δ such that $\mathbb{P}_k(\rho)$ is isomorphic to the toric variety $X = X(\Delta)$. In Section 3.3, we recall a result of DeMeyer, Ford and Miranda giving an isomorphism

$$\mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_m) \simeq \check{\mathrm{H}}^2(\mathfrak{U}, \mathbb{G}_m),$$

where $\mathfrak{U}$ denotes the standard affine open cover of the toric variety X (corresponding to the maximal cones of the fan Δ). We show (in Section 3.4 and Lemma 3.5) that it suffices to show that the p-torsion vanishes, when each weight in $\rho = (\rho_0, \ldots, \rho_n)$ is a power of p. In Sections 3.7–3.9, we define a double complex $A^{\bullet,\bullet}$ such that the spectral sequence $\{\mathrm{E}_\bullet^{\bullet,\bullet}\}$ corresponding to the horizontal filtration on $A^{\bullet,\bullet}$ satisfies $\check{\mathrm{H}}^p(\mathfrak{U}, \mathbb{G}_m) \simeq \mathrm{E}_2^{p,0}$ for all p. We compute $\mathrm{E}_2^{2,0}$ in Sections 3.10 and 3.11. □

3.2. ***Presentation as a toric variety.*** We recall from [17, Section 2.2; 8, Example 3.1.17] how to view a weighted projective space as a toric variety (i.e., what the fan is).

Let $\mathrm{U} \in \mathrm{GL}_{n+1}(\mathbb{Z})$ be an invertible matrix which has ρ as its first row (using the Euclidean algorithm, do column operations on ρ to reduce to $(1, 0, \dots, 0)$, then apply the inverse column operations in the reverse order on the identity matrix id_{n+1}); let $\mathrm{Y} \in \mathrm{Mat}_{(n+1)\times n}(\mathbb{Z})$ be the matrix obtained by removing the leftmost column of U^{-1}; let $\boldsymbol{v}_0, \dots, \boldsymbol{v}_n \in \mathbb{Z}^n$ be the rows of Y; then $\mathbb{P}(\rho)$ is isomorphic to the toric variety associated to the fan Δ whose maximal cones are generated by the n-element subsets of $\{\boldsymbol{v}_0, \dots, \boldsymbol{v}_n\}$.

3.3. ***Reduce to computing the subgroup of Zariski-locally trivial Brauer classes.*** Let Δ' be a nonsingular subdivision of Δ, and let X' be the toric variety associated to Δ'. The morphism of fans $\Delta' \to \Delta$ gives rise to a morphism of toric varieties $X' \to X$ which is a resolution of singularities for X. As in [11], we set

$$\mathrm{H}^2(K/X_{\text{ét}}, \mathbb{G}_m) := \ker(\mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_m) \to \mathrm{H}^2_{\text{ét}}(K, \mathbb{G}_m)),$$

since X' is regular, the restriction $\mathrm{H}^2_{\text{ét}}(X', \mathbb{G}_m) \to \mathrm{H}^2_{\text{ét}}(K, \mathbb{G}_m)$ is injective; hence there is an exact sequence

$$0 \to \mathrm{H}^2(K/X_{\text{ét}}, \mathbb{G}_m) \to \mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_m) \to \mathrm{H}^2_{\text{ét}}(X', \mathbb{G}_m)$$

of abelian groups. Here X' is a smooth, proper, geometrically connected, rational k-scheme; hence $\mathrm{H}^2_{\text{ét}}(X', \mathbb{G}_m) = 0$ by birational invariance of the Brauer group (see [21, corollaire 7.3] in characteristic 0 and [6, Corollary 5.2.6] in general); thus it remains to compute $\mathrm{H}^2(K/X_{\text{ét}}, \mathbb{G}_m)$. By [11, 4.3, 5.1], there are natural isomorphisms

$$\check{\mathrm{H}}^2(\mathfrak{U}, \mathbb{G}_m) \simeq \mathrm{H}^2_{\text{zar}}(X, \mathbb{G}_m) \simeq \mathrm{H}^2(K/X_{\text{ét}}, \mathbb{G}_m), \tag{3.3.1}$$

where $\mathfrak{U} = \{U_{\sigma_0}, \dots, U_{\sigma_n}\}$ is the Zariski cover of X corresponding to the set of maximal cones of Δ.

3.4. ***Limit of dilations.*** Let A be a ring and let X be the toric variety (over A) associated to a fan Δ of cones in $\mathsf{N}_{\mathbb{Q}}$. For any positive integer d, the multiplication-by-d map $\times d : \mathsf{N} \to \mathsf{N}$ induces a finite A-morphism

$$\theta_d : X \to X,$$

which is equivariant for the d-th power map on tori. This is called a *dilation* [7, Section 6] (or *toric Frobenius* [23, Remark 4.14]). For a cone σ of Δ, this is the A-algebra endomorphism of $\Gamma(U_\sigma, \mathcal{O}_{U_\sigma}) = A[\sigma^\vee \cap \mathsf{M}]$ sending $\chi^{\mathsf{m}} \mapsto \chi^{d\mathsf{m}}$ for $\mathsf{m} \in \sigma^\vee \cap \mathsf{M}$. If σ is a smooth cone, then $\theta_d : U_\sigma \to U_\sigma$ is flat for any d.

We view $\mathbb{N}$ as a category whose objects correspond to positive integers $m \in \mathbb{N}$ and there is a morphism $m_1 \to m_2$ if m_1 divides m_2. Let $S \subset \mathbb{N}$ be a multiplicatively closed subset; there is a functor $S^{\mathrm{op}} \to (\mathrm{Sch})$ sending $m \mapsto X$ and $\{m_1 \to m_2\} \mapsto \theta_{m_2/m_1}$; the limit

$$X^{1/S} := \varprojlim(\theta_{m_2/m_1} : X \to X)$$

of the resulting projective system is representable by a scheme since all the transition maps are affine. The scheme $X^{1/S}$ is isomorphic to the monoid scheme obtained by the usual construction with the finite free $\mathbb{Z}$-module N and its dual M replaced by the $S^{-1}\mathbb{Z}$-module $S^{-1}\mathsf{N}$ and its dual $S^{-1}\mathsf{M} = \mathrm{Hom}_{S^{-1}\mathbb{Z}}(S^{-1}\mathsf{N}, S^{-1}\mathbb{Z})$. More precisely, set

$$U_\sigma^{1/S} := \mathrm{Spec}\, A[\sigma^\vee \cap S^{-1}\mathsf{M}];$$

for any face τ of σ, the canonical map $U_\tau^{1/S} \to U_\sigma^{1/S}$ is an open immersion; then $U_{\sigma_1}^{1/S}$ and $U_{\sigma_2}^{1/S}$ are glued along the common open subscheme $U_{\sigma_1\cap\sigma_2}^{1/S}$.

If A is reduced, then we have

$$\Gamma(U_\sigma, \mathbb{G}_m) = (A[\sigma^\vee \cap \mathsf{M}])^\times = A^\times \cdot (\sigma^\perp \cap \mathsf{M}) \tag{3.4.1}$$

for any cone $\sigma \in \Delta$; hence, by (3.3.1), the pullback

$$\theta_d^* : \mathrm{H}^p_{\mathrm{zar}}(X, \mathbb{G}_m) \to \mathrm{H}^p_{\mathrm{zar}}(X, \mathbb{G}_m)$$

is multiplication-by-d. In the limit, we obtain a natural isomorphism

$$S^{-1}(\mathrm{H}^p_{\mathrm{zar}}(X, \mathbb{G}_m)) \simeq \mathrm{H}^p_{\mathrm{zar}}(X^{1/S}, \mathbb{G}_m) \tag{3.4.2}$$

of $S^{-1}\mathbb{Z}$-modules.

Lemma 3.5. *Let d be a positive integer dividing ρ_i, and set $\rho' := (\rho'_0, \dots, \rho'_n)$ where $\rho'_i := \rho_i/d$ and $\rho'_j := \rho_j$ for $j \neq i$. If $d \in S$, then $\mathbb{P}_{\mathbb{Z}}(\rho)^{1/S} \simeq \mathbb{P}_{\mathbb{Z}}(\rho')^{1/S}$.*

Proof. As in Section 3.2, let $\mathrm{U}, \mathrm{U}' \in \mathrm{GL}_{n+1}(\mathbb{Z})$ be invertible matrices whose first rows are ρ and ρ', respectively. Let $\mathrm{U}^\circ \in \mathrm{GL}_{n+1}(S^{-1}\mathbb{Z})$ be the matrix obtained by dividing the i-th column of U by d; then $(\mathrm{U}^\circ)^{-1}$ is obtained by multiplying the i-th row of U^{-1} by d; this does not change the cones since we are just replacing v'_i by $\frac{1}{d}\mathrm{v}'_i$. Set $\mathrm{V} := \mathrm{U}' \cdot (\mathrm{U}^\circ)^{-1} \in \mathrm{GL}_{n+1}(S^{-1}\mathbb{Z})$; since the first rows of U°, U' are the same, the matrix V has the form

$$\mathrm{V} = \begin{bmatrix} 1 & \mathbf{0} \\ \mathrm{V}' & \mathrm{V}'' \end{bmatrix}$$

for some $\mathrm{V}' \in \mathrm{Mat}_{n\times 1}(S^{-1}\mathbb{Z})$ and $\mathrm{V}'' \in \mathrm{GL}_n(S^{-1}\mathbb{Z})$. Let $\mathrm{Y}^\circ, \mathrm{Y}' \in \mathrm{Mat}_{(n+1)\times n}(S^{-1}\mathbb{Z})$ be the matrices obtained by removing the leftmost column of $(\mathrm{U}^\circ)^{-1}$, $(\mathrm{U}')^{-1}$ respectively; then $(\mathrm{U}')^{-1} \cdot \mathrm{V} = (\mathrm{U}^\circ)^{-1}$ implies $\mathrm{Y}' \cdot \mathrm{V}'' = \mathrm{Y}^\circ$; then $\mathrm{V}'' : S^{-1}\mathsf{N} \to S^{-1}\mathsf{N}$ induces the desired isomorphism $\mathbb{P}_{\mathbb{Z}}(\rho)^{1/S} \to \mathbb{P}_{\mathbb{Z}}(\rho')^{1/S}$. □

3.6. We show that $\mathrm{H}^2_{\mathrm{zar}}(X, \mathbb{G}_m) = 0$ by showing that the localization

$$\mathrm{H}^2_{\mathrm{zar}}(X, \mathbb{G}_m) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$$

is 0 for every prime p. By (3.4.2) and Lemma 3.5, we may thus assume that

$$\rho = (1, p^{e_1}, \dots, p^{e_n})$$

for some nonnegative integers $e_1 \le \cdots \le e_n$. In this case, in Section 3.2 we may take the first row of U to be ρ and the other rows to coincide with the identity id_{n+1}, so that

$$\mathrm{Y} = \begin{bmatrix} -p^{e_1} & \cdots & -p^{e_n} \\ 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} \tag{3.6.1}$$

and thus $\boldsymbol{v}_0 = (-p^{e_1}, \dots, -p^{e_n})$ and $\boldsymbol{v}_i$ is the i-th standard basis vector of $\mathbb{Z}^n$.

3.7. ***Definition of*** $\mathsf{A}^{\bullet,\bullet}$**.** For convenience, we set $[n] := \{0, 1, \dots, n\}$; we will use I to denote a subset of $[n]$. We construct a double complex

$$\big(\{\mathsf{A}^{p,q}\}, \{\mathsf{d}_{\mathrm{v}}^{p,q} : \mathsf{A}^{p,q} \to \mathsf{A}^{p,q+1}\}, \{\mathsf{d}_{\mathrm{h}}^{p,q} : \mathsf{A}^{p,q} \to \mathsf{A}^{p+1,q}\}\big)$$

as follows: for $-1 \le p \le n$, we set

$$\mathsf{A}^{p,1} = \bigoplus_{|I|=n-p} \mathbb{Z}^{n-p}, \qquad \mathsf{A}^{p,0} = \bigoplus_{|I|=n-p} \mathbb{Z}^{n} \tag{3.7.1}$$

and $\mathsf{A}^{p,q} = 0$ if $(p, q) \notin \{-1, \dots, n\} \times \{0, 1\}$.

For the vertical differential $\mathsf{d}_{\mathrm{v}}^{p,0} : \mathsf{A}^{p,0} \to \mathsf{A}^{p,1}$, the I-th component (with $|I| = n - p$) of this map is the group homomorphism $\mathbb{Z}^n \to \mathbb{Z}^{n-p}$ whose corresponding matrix has rows $\boldsymbol{v}_i$ for $i \in I$.

The horizontal differentials $\mathsf{d}_{\mathrm{h}}^{p,q}$ are defined with the sign conventions as follows: if $I = \{i_0, \dots, i_{n-p-1}\} \subset [n]$ is a subset of size $|I| = n - p$ and I' is obtained by removing the i-th element of I (where $0 \le i \le n - p - 1$), then the restriction from the I-th to I'-th components has sign $(-1)^i$.

The subcomplex of $A^{\bullet,\bullet}$ obtained by restricting to $p \ge 1$ is isomorphic to the morphism of Čech complexes

$$\check{\mathrm{C}}^{\bullet}(\Delta, \mathcal{M}) \to \check{\mathrm{C}}^{\bullet}(\Delta, \mathcal{SF}),$$

in the notation of [11, (5.0.1)].

3.8. ***Diagram of*** $\mathsf{A}^{\bullet,\bullet}$**.** Here is a diagram of the double complex $\mathsf{A}^{\bullet,\bullet}$:

$$
\begin{array}{ccccccccccccc}
\mathsf{A}^{-1,1} & \xrightarrow{\mathsf{d}_{\mathrm{h}}^{-1,1}} & \mathsf{A}^{0,1} & \xrightarrow{\mathsf{d}_{\mathrm{h}}^{0,1}} & \mathsf{A}^{1,1} & \xrightarrow{\mathsf{d}_{\mathrm{h}}^{1,1}} & \mathsf{A}^{2,1} & \longrightarrow & \cdots & \longrightarrow & \mathsf{A}^{n-1,1} & \xrightarrow{\mathsf{d}_{\mathrm{h}}^{n-1,1}} & \mathsf{A}^{n,1} \\
\uparrow{\scriptstyle \mathsf{d}_{\mathrm{v}}^{-1,0}} & & \uparrow{\scriptstyle \mathsf{d}_{\mathrm{v}}^{0,0}} & & \uparrow{\scriptstyle \mathsf{d}_{\mathrm{v}}^{1,0}} & & \uparrow{\scriptstyle \mathsf{d}_{\mathrm{v}}^{2,0}} & & & & \uparrow{\scriptstyle \mathsf{d}_{\mathrm{v}}^{n-1,0}} & & \uparrow{\scriptstyle \mathsf{d}_{\mathrm{v}}^{n,0}} \\
\mathsf{A}^{-1,0} & \xrightarrow[\mathsf{d}_{\mathrm{h}}^{-1,1}]{} & \mathsf{A}^{0,0} & \xrightarrow[\mathsf{d}_{\mathrm{h}}^{0,1}]{} & \mathsf{A}^{1,0} & \xrightarrow[\mathsf{d}_{\mathrm{h}}^{1,1}]{} & \mathsf{A}^{2,0} & \longrightarrow & \cdots & \longrightarrow & \mathsf{A}^{n-1,0} & \xrightarrow[\mathsf{d}_{\mathrm{h}}^{n-1,0}]{} & \mathsf{A}^{n,0}
\end{array}
$$

For a weighted projective surface (i.e., $n=2$), this looks like

$$
\begin{array}{ccccccc}
\mathbb{Z}^3 & \longrightarrow & \mathbb{Z}^2\oplus\mathbb{Z}^2\oplus\mathbb{Z}^2 & \longrightarrow & \mathbb{Z}^1\oplus\mathbb{Z}^1\oplus\mathbb{Z}^1 & \longrightarrow & 0 \\
\uparrow{\scriptstyle \begin{bmatrix} \boldsymbol{v}_0 \\ \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \end{bmatrix}} & & \uparrow{\scriptstyle \begin{bmatrix} \boldsymbol{v}_0 \\ \boldsymbol{v}_1 \end{bmatrix}\begin{bmatrix} \boldsymbol{v}_0 \\ \boldsymbol{v}_2 \end{bmatrix}\begin{bmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \end{bmatrix}} & & \uparrow{\scriptstyle [\boldsymbol{v}_0]\,[\boldsymbol{v}_1]\,[\boldsymbol{v}_2]} & & \uparrow \\
\mathbb{Z}^2 & \longrightarrow & \mathbb{Z}^2\oplus\mathbb{Z}^2\oplus\mathbb{Z}^2 & \longrightarrow & \mathbb{Z}^2\oplus\mathbb{Z}^2\oplus\mathbb{Z}^2 & \longrightarrow & \mathbb{Z}^2
\end{array}
$$

3.9. Let $\boldsymbol{C}_n^\bullet$ be the complex with $\boldsymbol{C}_n^k = \mathbb{Z}^{\binom{n}{k}}$ and such that the differentials $\boldsymbol{C}_n^k \to \boldsymbol{C}_n^{k+1}$ have sign conventions as above. Then $\boldsymbol{C}_n^\bullet$ is isomorphic to a direct sum of shifts of $\mathrm{id}:\mathbb{Z}\to\mathbb{Z}$, and hence is exact. The complex $\mathsf{A}^{\bullet,0}$ is isomorphic to the direct sum $(\boldsymbol{C}_{n+1}^\bullet)^n$, and hence is exact. The complex $\mathsf{A}^{\bullet,1}$ is isomorphic to the direct sum $(\boldsymbol{C}_{n-1}^\bullet)^{n+1}$, and hence is exact. Let

$$
\big(\{\mathrm{E}_r^{p,q}\}, \{\mathrm{d}_r^{p,q} : \mathrm{E}_r^{p,q} \to \mathrm{E}_r^{p+r,q-r+1}\}\big) \tag{3.9.1}
$$

denote the spectral sequence corresponding to the horizontal filtration on $\mathsf{A}^{\bullet,\bullet}$, so that $\mathrm{E}_0^{p,q} = \mathsf{A}^{p,q}$ and $\mathrm{d}_0^{p,q} = \mathsf{d}_{\mathrm{v}}^{p,q}$. Then there is a natural isomorphism

$$
\mathrm{E}_2^{p,0} \simeq \check{\mathrm{H}}^p(\mathfrak{U}, \mathbb{G}_m),
$$

where $\mathfrak{U}$ is the Zariski open cover of X corresponding to the maximal cones of Δ. Since there are only two nonzero rows, the differentials

$$
\mathrm{d}_2^{p,1} : \mathrm{E}_2^{p,1} \to \mathrm{E}_2^{p+2,0}
$$

are isomorphisms for all p. We are interested in $\check{\mathrm{H}}^2(\mathfrak{U}, \mathbb{G}_m) \simeq \mathrm{E}_2^{2,0} \simeq \mathrm{E}_2^{0,1}$.

3.10. For the differential $\mathsf{d}_{\mathrm{v}}^{0,0} : \mathsf{A}^{0,0} \to \mathsf{A}^{0,1}$, the I-th component (with $|I| = n$) of this map is the group homomorphism $\mathbb{Z}^n \to \mathbb{Z}^n$ whose corresponding matrix is obtained by removing the i-th rows from Y (3.6.1) for $i \notin I$; hence

$$
\mathrm{E}_1^{0,1} \simeq \bigoplus_{i\in[n]} \mathbb{Z}/(p^{e_i}), \tag{3.10.1}
$$

where a generator of the i-th component $\mathbb{Z}/(p^{e_i})$ is given by the image of the first standard basis vector of $\mathbb{Z}^n$ (see (3.7.1)).

For the differential $\mathrm{d}_{\mathrm{v}}^{1,0} : \mathsf{A}^{1,0} \to \mathsf{A}^{1,1}$, the I-th component (with $|I| = n-1$) of this map is the group homomorphism $\mathbb{Z}^n \to \mathbb{Z}^{n-1}$ whose corresponding matrix is obtained by removing the i-th rows from Y (3.6.1) for $i \notin I$; hence

$$\mathrm{E}_1^{1,1} \simeq \bigoplus_{i_1 < i_2} \mathbb{Z}/(p^{\min\{e_{i_1}, e_{i_2}\}}) \tag{3.10.2}$$

where a generator of the i-th component $\mathbb{Z}/(p^{\min\{e_{i_1}, e_{i_2}\}})$ is given by the image of the first standard basis vector of $\mathbb{Z}^{n-1}$ (see (3.7.1)).

3.11. We compute $\mathrm{E}_2^{0,1} = \ker \mathrm{d}_1^{0,1} / \operatorname{im} \mathrm{d}_1^{-1,1}$ in (3.9.1). With identifications as in (3.10.1) and (3.10.2), the image of $(x_0, x_1, \ldots, x_n) \in \mathrm{E}_1^{0,1}$ under the differential $\mathrm{d}_1^{0,1} : \mathrm{E}_1^{0,1} \to \mathrm{E}_1^{1,1}$ has (i_1, i_2)-th coordinate $(-1)^{i_1} x_{i_1} + (-1)^{i_2 - 1} x_{i_2}$. Suppose $(x_0, x_1, \ldots, x_n) \in \ker \mathrm{d}_1^{0,1}$; using the differential $\mathrm{d}_1^{-1,1} : \mathrm{E}_1^{-1,1} \to \mathrm{E}_1^{0,1}$, we may assume that $x_n = 0$ in $\mathbb{Z}/(p^{e_n})$. Since $e_{n-1} \le e_n$, the condition

$$(-1)^{n-1} x_{n-1} + (-1)^{n-1} x_n = 0$$

in $\mathbb{Z}/(p^{e_{n-1}})$ forces $x_{n-1} = 0$ in $\mathbb{Z}/(p^{e_{n-1}})$. Using downward induction on i, we conclude that $x_i = 0$ in $\mathbb{Z}/(p^{e_i})$ for all i. Thus we have $\mathrm{E}_2^{0,1} = 0$.

Remark 3.12 (assumptions on the base field). In [11], there are two implicit assumptions regarding the base field k:

(1) It is assumed that k is algebraically closed, as we do throughout Section 3 (this assumption will be removed in Lemma 4.2). This is used to conclude that all closed points are k-points and to identify the henselization and the strict henselization at a closed point of a variety. In the proof of Lemma 4.1, the reference to [31, Chapter VI, Section 14, Theorem 32, p. 92] (in showing that an affine toric variety is analytically normal) requires k to be perfect (here we may also use [24, (33.I) Theorem 79]).

(2) It is assumed that k has characteristic 0. This is used to conclude that (5.1.1) is a split surjection; we only use their Lemmas 4.3 and 5.1, which do not depend on the characteristic of k. (There are potential subtleties when considering the Brauer group of (affine) toric varieties in positive characteristic; for example, if k is an algebraically closed field of characteristic p, the Brauer group of $\mathbb{A}_k^2$ has nontrivial p-torsion by [4, Theorem 7.5]. These classes are not cup products since $\mathrm{H}_{\mathrm{fppf}}^1(\mathbb{A}_k^2, \mu_p) = 0$.)

4. Over a general base scheme

In this section, we prove Theorem 1.2 for an arbitrary scheme S (see Section 4.4). This is a Leray spectral sequence argument for the structure morphism $f : X \to S$. For this, we show that $\boldsymbol{R}^1 f_* \mathbb{G}_m = \underline{\mathbb{Z}}$ and $\boldsymbol{R}^2 f_* \mathbb{G}_m = 0$ (see Lemma 4.3).

We first generalize Lemma 3.1 to arbitrary fields:

Lemma 4.1. *For any field k, the pullback map*

$$\mathrm{H}^2_{ét}(\operatorname{Spec} k, \mathbb{G}_m) \to \mathrm{H}^2_{ét}(\mathbb{P}_k(\rho), \mathbb{G}_m)$$

is an isomorphism.

Proof. Let $P \in (\mathbb{P}_k(\rho))(k)$ be a k-point, and let $\alpha \in \mathrm{H}^2_{\text{ét}}(\mathbb{P}_k(\rho), \mathbb{G}_m)$ be a Brauer class such that $\alpha_P = 0$ in $\mathrm{H}^2_{\text{ét}}(\operatorname{Spec} k, \mathbb{G}_m)$. It suffices to show that $\alpha = 0$; this follows from Lemma 4.2, whose hypotheses are satisfied by Lemma 3.1. □

Lemma 4.2. *Let A be a local ring, set $X := \mathbb{P}_A(\rho)$, let $P \in X(A)$ be an A-rational point and let $\alpha \in \mathrm{H}^2_{ét}(X, \mathbb{G}_m)$ be a class such that $\alpha_P = 0$. If there exists a finite faithfully flat A-algebra A' such that $\alpha_{A'} = 0$, then $\alpha = 0$.*

Proof. Let $\mathcal{G} \to X$ be the $\mathbb{G}_m$-gerbe corresponding to α. Since $\mathcal{G}_{A'}$ is trivial, there is a 1-twisted line bundle $\mathcal{L}'$ on $\mathcal{G}_{A'}$; set $A'' := A' \otimes_A A'$ and $A''' := A' \otimes_A A' \otimes_A A'$; then there exists a line bundle L'' on $X_{A''}$ such that $L''|_{\mathcal{G}_{A''}} \simeq (p_1^*\mathcal{L}')^{-1} \otimes p_2^*\mathcal{L}'$; this line bundle L'' satisfies $p_{13}^* L'' \simeq p_{23}^* L'' \otimes p_{12}^* L''$; hence L'' is trivial since $p_{12}^*, p_{13}^*, p_{23}^* : \operatorname{Pic}(X_{A''}) \to \operatorname{Pic}(X_{A'''})$ are the same maps $\mathbb{Z} \to \mathbb{Z}$ (see Lemma 2.8). Choose an isomorphism $\varphi : p_1^*\mathcal{L}' \to p_2^*\mathcal{L}'$ of $\mathcal{O}_{\mathcal{G}_{A''}}$-modules; the isomorphisms $p_{13}^*\varphi$ and $p_{23}^*\varphi \circ p_{12}^*\varphi$ differ by an element $u_\alpha \in \Gamma(X_{A'''}, \mathbb{G}_m) \simeq \Gamma(A''', \mathbb{G}_m)$. Since $\mathcal{G}|_P$ is trivial, we may refine the finite flat cover $A \to A'$ if necessary so that u_α is the coboundary of some $u_\beta \in \Gamma(X_{A''}, \mathbb{G}_m)$. After modifying φ by this u_β, we have that the descent datum $(\mathcal{L}', \varphi)$ gives a 1-twisted line bundle on $\mathcal{G}$. □

We use deformation theory of twisted sheaves to deduce Theorem 1.2 over strictly henselian local rings:

Lemma 4.3. *Let A be a strictly henselian local ring. Then $\mathrm{H}^2_{ét}(\mathbb{P}_A(\rho), \mathbb{G}_m) = 0$.*

Proof. This proof is an argument of Siddharth Mathur (personal communication, 2019). By standard limit techniques, we may assume that A is the strict henselization of a localization of a finite type $\mathbb{Z}$-algebra; in particular, A is excellent [20, Corollary 5.6(iii)]. Let $\mathfrak{m}$ be the maximal ideal of A and let $k := A/\mathfrak{m}$ be the residue field.

We first consider the case when A is complete. Set $X := \mathbb{P}_A(\rho)$ and let $\pi : \mathcal{G} \to X$ be a $\mathbb{G}_m$-gerbe corresponding to a class $[\mathcal{G}] \in \mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_m)$. The class $[\mathcal{G}]$ is trivial if and only if π admits a section. We have that $\mathcal{G}_0$ is a $\mathbb{G}_m$-gerbe over $X_0 = \mathbb{P}_k(\rho)$, which is a trivial gerbe by Lemma 4.1 since k is separably closed. For $\ell \in \mathbb{N}$, set $X_\ell := X \times_{\operatorname{Spec} A} \operatorname{Spec} A/\mathfrak{m}^{\ell+1}$ and $\mathcal{G}_\ell := \mathcal{G} \times_X X_\ell$. We have equivalences of categories

$$\begin{aligned} \operatorname{Mor}(X, \mathcal{G}) &\overset{1}{\simeq} \operatorname{Hom}_{r\otimes,\simeq}(\mathsf{Coh}(\mathcal{G}), \mathsf{Coh}(X)) \\ &\overset{2}{\simeq} \operatorname{Hom}_{r\otimes,\simeq}(\mathsf{Coh}(\mathcal{G}), \varprojlim \mathsf{Coh}(X_\ell)) \\ &\overset{3}{\simeq} \varprojlim \operatorname{Hom}_{r\otimes,\simeq}(\mathsf{Coh}(\mathcal{G}), \mathsf{Coh}(X_\ell)) \\ &\overset{1}{\simeq} \varprojlim \operatorname{Mor}(X_\ell, \mathcal{G}), \end{aligned}$$

where the equivalences marked 1 are by [22, Theorem 1.1] (here we use that A is excellent), the equivalence marked 2 is Grothendieck existence [15, Scholie 5.1.4], the equivalence marked 3 is [22, Lemma 3.8].

It remains now to construct a compatible system of morphisms $X_\ell \to \mathcal{G}$. A morphism $X_\ell \to \mathcal{G}$ over $\mathbb{P}_A(\rho)$ corresponds to a 1-twisted line bundle on $\mathcal{G}_\ell$; the obstruction to lifting a line bundle via $\mathcal{G}_\ell \to \mathcal{G}_{\ell+1}$ lies in $\mathrm{H}^2_{\text{ét}}(\mathcal{G}_\ell, \mathfrak{m}^\ell \mathcal{O}_{\mathcal{G}_\ell})$. Since $\mathcal{G} \to X$ is a cohomologically affine morphism and the diagonal of $\mathcal{G}$ is affine, the pullback $\mathrm{H}^2_{\text{ét}}(X_\ell, \mathfrak{m}^\ell \mathcal{O}_{X_\ell}) \to \mathrm{H}^2_{\text{ét}}(\mathcal{G}_\ell, \mathfrak{m}^\ell \mathcal{O}_{\mathcal{G}_\ell})$ is an isomorphism by [3, Remark 3.5] and a Leray spectral sequence argument; by Lemma 2.9, we have $\mathrm{H}^2_{\text{ét}}(X_\ell, \mathfrak{m}^\ell \mathcal{O}_{X_\ell}) = 0$.

In general, if A is not complete, we use Artin approximation to descend a 1-twisted line bundle from $\mathcal{G}^\wedge$ to $\mathcal{G}$. □

4.4. ***Proof of Theorem 1.2.*** Set $f := f_X$. The Leray spectral sequence associated to the map f and sheaf $\mathbb{G}_m$ is of the form

$$\mathrm{E}_2^{p,q} = \mathrm{H}^p_{\text{ét}}(S, \boldsymbol{R}^q f_* \mathbb{G}_m) \Rightarrow \mathrm{H}^{p+q}_{\text{ét}}(X, \mathbb{G}_m) \tag{4.4.1}$$

with differentials $\mathrm{d}_2^{p,q} : \mathrm{E}_2^{p,q} \to \mathrm{E}_2^{p+2,q-1}$. For any strictly henselian local ring A, we have $\mathrm{H}^2_{\text{ét}}(\mathbb{P}_A(\rho), \mathbb{G}_m) = 0$ by Lemma 4.3, and hence $\boldsymbol{R}^2 f_* \mathbb{G}_m = 0$ since its stalks vanish. The sheaf $\boldsymbol{R}^1 f_* \mathbb{G}_m$ is the sheaf associated to $T \mapsto \operatorname{Pic}(X_T)$; by Lemma 2.8, every line bundle on $\mathbb{P}_T(\rho)$ is, locally on T, isomorphic to one pulled back from $\mathbb{P}_{\mathbb{Z}}(\rho)$; hence $\boldsymbol{R}^1 f_* \mathbb{G}_m$ is isomorphic to the constant sheaf $\underline{\mathbb{Z}}$. Hence we have an exact sequence

$$\mathrm{H}^0_{\text{ét}}(S, \underline{\mathbb{Z}}) \xrightarrow{\dagger} \mathrm{H}^2_{\text{ét}}(S, \mathbb{G}_m) \xrightarrow{f^*} \mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_m) \to \mathrm{H}^1_{\text{ét}}(S, \underline{\mathbb{Z}}) \tag{4.4.2}$$

and we may argue as in [30, Section 4.3] to show that f^* restricts to a surjection on the torsion subgroups, inducing an exact sequence (1.2.1) as desired. □

Remark 4.5 (the Brauer class of a twist of weighted projective space). By the argument in Section 4.4, the map $\Gamma(S, \underline{\mathbb{Z}}) \to \mathrm{Br}'(S)$ in (1.2.1) corresponds to the differential $\mathrm{d}_2^{0,1}$ in the Leray spectral sequence for $f : X \to S$. The Brauer class $[X] \in \mathrm{Br}'(S)$ is defined to be the image of $1 \in \Gamma(S, \underline{\mathbb{Z}})$ under $\dagger$ in (4.4.2). We have the following alternative description of $[X]$. Let $R := \mathbb{Z}[t_0, \dots, t_n]$ be the

$\mathbb{Z}$-graded ring with $\deg(t_i) = \rho_i$, and let $\mathrm{Aut}_{\mathrm{gr.alg.}}(R)$ denote the group sheaf sending a scheme T to the set of $\mathbb{Z}$-graded $\mathcal{O}_T$-algebra automorphisms of $R \otimes_{\mathbb{Z}} \mathcal{O}_T$. By [2, Section 8], we have an exact sequence

$$1 \to \mathbb{G}_m \to \mathrm{Aut}_{\mathrm{gr.alg.}}(R) \to \mathrm{Aut}_{\mathrm{sch}}(\mathbb{P}_{\mathbb{Z}}(\rho)) \to 1$$

of sheaves of groups for the étale topology on the category of schemes, where the image of $\mathbb{G}_m$ is contained in the center of $\mathrm{Aut}_{\mathrm{gr.alg.}}(R)$. By definition, X is an $\mathrm{Aut}_{\mathrm{sch}}(\mathbb{P}_{\mathbb{Z}}(\rho))$-torsor over S, and the class of $[X]$ under the coboundary map

$$\mathrm{H}^1_{\text{ét}}\big(S, \mathrm{Aut}_{\mathrm{sch}}(\mathbb{P}_{\mathbb{Z}}(\rho))\big) \to \mathrm{H}^2_{\text{ét}}(S, \mathbb{G}_m)$$

is the desired Brauer class.

Alternatively, fix an étale surjection $S' \to S$ and set $S'' := S' \times_S S'$ and $S''' := S' \times_S S' \times_S S'$; the choice of an isomorphism $X \times_S S' \simeq \mathbb{P}_{S'}(\rho)$ yields an automorphism $\varphi : \mathbb{P}_{S''}(\rho) \to \mathbb{P}_{S''}(\rho)$ satisfying the cocycle condition $p_{13}^*\varphi = p_{23}^*\varphi \circ p_{12}^*\varphi$ over S'''. Choose $\ell \gg 0$ so that $\mathcal{O}_{\mathbb{P}(\rho)}(\ell)$ is very ample; fixing a $\mathbb{Z}$-basis of $\Gamma(\mathbb{P}_{\mathbb{Z}}(\rho), \mathcal{O}_{\mathbb{P}_{\mathbb{Z}}(\rho)}(\ell))$ gives an invertible matrix $\varphi^\sharp \in \mathrm{GL}_r(\Gamma(S'', \mathcal{O}_{S''}))$, where $r = \mathrm{rank}_{\mathbb{Z}}\, \Gamma(\mathbb{P}_{\mathbb{Z}}(\rho), \mathcal{O}_{\mathbb{P}_{\mathbb{Z}}(\rho)}(\ell))$; here the invertible matrices

$$p_{13}^*\varphi^\sharp,\ p_{12}^*\varphi^\sharp \cdot p_{23}^*\varphi^\sharp \in \mathrm{GL}_r(\Gamma(S''', \mathcal{O}_{S'''}))$$

differ by a unit $u \in \Gamma(S''', \mathbb{G}_m)$, which is the desired class in $\mathrm{H}^2_{\text{ét}}(S, \mathbb{G}_m)$. In other words, given a $\mathbb{Z}$-graded algebra automorphism of R, it restricts to a $\mathbb{Z}$-graded algebra automorphism of its ℓ-th Veronese subring $R^{(\ell)} := \bigoplus_{i \geq 0} R_{i\ell}$, which restricts to an abelian group automorphism of R_ℓ and thus a $\mathbb{Z}$-graded algebra automorphism of the standard graded algebra $\mathrm{Sym}^\bullet_{\mathbb{Z}} R_\ell \simeq \mathbb{Z}[t'_1, \ldots, t'_r]$; the induced group homomorphism $\mathrm{Aut}_{\mathrm{gr.alg.}}(R) \to \mathrm{Aut}_{\mathrm{gr.alg.}}(\mathrm{Sym}^\bullet_{\mathbb{Z}} R_\ell)$ induces a commutative diagram of exact sequences which we may use to compare the two constructions above.

Remark 4.6 (comparison to the argument of Gabber). Gabber [18] computes the Brauer group of Brauer–Severi schemes over an arbitrary base scheme by combining the following two facts to reduce to the $\mathbb{P}^1$ case:

(1) Suppose $Y \to X$ is a closed immersion locally defined by a regular sequence, and let $B \to X$ be the blowup of X at Y; then $\mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_m) \to \mathrm{H}^2_{\text{ét}}(B, \mathbb{G}_m)$ is injective.

(2) The blowup of $\mathbb{P}^n$ at a point is a $\mathbb{P}^1$-bundle over $\mathbb{P}^{n-1}$.

In our case, we may ask whether the analogous statement to (2) holds — namely, whether a (weighted) blowup of $\mathbb{P}(\rho)$ at a (torus-invariant) local complete intersection subscheme is isomorphic to a $\mathbb{P}(\rho')$-bundle over $\mathbb{P}(\rho'')$ for some ρ', ρ'' such that $|\rho| - 1 = |\rho'| - 1 + |\rho''| - 1$. Indeed, the blowup of the weighted projective surface $\mathbb{P}(1, 1, q_2)$ at its unique singular point gives the q_2-th Hirzebruch surface $\mathbb{F}_{q_2}$ (see [12, Section 1.2.3; 19]). Such a result for arbitrary ρ would give an alternative proof of Theorem 1.2. This seems unlikely, however, as it (with Remark 2.6) would imply that every weighted projective surface $\mathbb{P}(\rho_0, \rho_1, \rho_2)$ is a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$, which has Picard group $\mathbb{Z}^2$, but $\mathbb{P}(2, 3, 5)$ has three isolated singular points and blowing up these points increases the Picard rank by 3.

5. Weighted projective stacks

In this section we prove Theorem 1.3 (see Section 5.9).

5.1. Let $\rho = (\rho_0, \dots, \rho_n)$ be a weight vector, and consider the $\mathbb{G}_m$-action on $\mathbb{A}^{n+1}$ sending $u \cdot (t_0, \dots, t_n) \mapsto (u^{\rho_0} t_0, \dots, u^{\rho_n} t_n)$. The *weighted projective stack* associated to ρ is the quotient stack

$$\mathcal{P}_{\mathbb{Z}}(\rho) := [(\mathbb{A}^{n+1}_{\mathbb{Z}} \setminus \{0\})/\mathbb{G}_m]$$

for this action. For any scheme S, we denote the base change of $\mathcal{P}_{\mathbb{Z}}(\rho)$ to S by $\mathcal{P}_S(\rho) := \mathcal{P}_{\mathbb{Z}}(\rho) \times_{\operatorname{Spec} \mathbb{Z}} S$.

The weighted projective stack $\mathcal{P}_{\mathbb{Z}}(\rho)$ admits a natural morphism

$$\pi_\rho : \mathcal{P}_{\mathbb{Z}}(\rho) \to \mathbb{P}_{\mathbb{Z}}(\rho)$$

to the weighted projective space $\mathbb{P}_{\mathbb{Z}}(\rho)$, which is a coarse moduli space morphism [1, Section 2.1]. Since $\mathcal{P}_{\mathbb{Z}}(\rho)$ is smooth for any ρ, the morphism π_ρ is not an isomorphism if $\rho \neq (1, \dots, 1)$.

Lemma 5.2. *For any field k, the pullback map*

$$\mathrm{H}^2_{ét}(\operatorname{Spec} k, \mathbb{G}_m) \to \mathrm{H}^2_{ét}(\mathcal{P}_k(\rho), \mathbb{G}_m)$$

is an isomorphism.

Proof. We have a descent spectral sequence

$$\mathrm{E}_1^{p,q} = \mathrm{H}^q_{\text{ét}}(\mathbb{G}_{m,k}^{\times p} \times_k (\mathbb{A}^{n+1}_k \setminus \{0\}), \mathbb{G}_m) \Longrightarrow \mathrm{H}^{p+q}_{\text{ét}}(\mathcal{P}_k(\rho), \mathbb{G}_m) \tag{5.2.1}$$

with differentials $\mathrm{d}_1^{p,q} : \mathrm{E}_1^{p,q} \to \mathrm{E}_1^{p+1,q}$. Each $\mathbb{G}_{m,k}^{\times p} \times_k (\mathbb{A}^{n+1}_k \setminus \{0\})$ is an open subscheme of $\mathbb{A}^{n+p+1}_k$, hence has trivial Picard group; hence $\mathrm{E}_1^{p,1} = 0$ for all p. The pullback $\mathrm{B}\mathbb{G}_{m,k} \to \mathcal{P}_k(\rho)$ induces an isomorphism of complexes $\mathrm{H}^0_{\text{ét}}(\mathbb{G}_{m,k}^{\times \bullet}, \mathbb{G}_m) \to \mathrm{E}_1^{\bullet,0}$; hence, by the proof of [30, Lemma 4.2], we have $\mathrm{E}_2^{2,0} = 0$.

It remains to compute $\mathrm{E}_2^{0,2}$, which is isomorphic to the equalizer of the two pullback maps

$$a^*, p_2^* : \mathrm{H}^2_{\text{ét}}(\mathbb{A}^{n+1}_k \setminus \{0\}, \mathbb{G}_m) \rightrightarrows \mathrm{H}^2_{\text{ét}}(\mathbb{G}_m \times_k (\mathbb{A}^{n+1}_k \setminus \{0\}), \mathbb{G}_m)$$

corresponding to the action map and second projection, respectively; by purity for the Brauer group (see Gabber [16] and Česnavičius [5]), this is isomorphic to the equalizer of

$$a^*, p_2^* : \mathrm{H}^2_{\text{ét}}(\mathbb{A}^{n+1}_k, \mathbb{G}_m) \rightrightarrows \mathrm{H}^2_{\text{ét}}(\mathbb{G}_m \times_k \mathbb{A}^{n+1}_k, \mathbb{G}_m),$$

and also to the equalizer of

$$a^*, p_2^* : \mathrm{H}^2_{\text{ét}}(\mathbb{A}^{n+1}_k, \mathbb{G}_m) \rightrightarrows \mathrm{H}^2_{\text{ét}}(\mathbb{A}^1_k \times_k \mathbb{A}^{n+1}_k, \mathbb{G}_m)$$

since the restriction

$$\mathrm{H}^2_{\text{ét}}(\mathbb{A}^1_k \times_k \mathbb{A}^{n+1}_k, \mathbb{G}_m) \to \mathrm{H}^2_{\text{ét}}(\mathbb{G}_m \times_k \mathbb{A}^{n+1}_k, \mathbb{G}_m)$$

is injective. With coordinates $\mathbb{A}^1_k = \operatorname{Spec} k[u]$, let $f : \mathbb{A}^{n+1}_k \to \mathbb{A}^1_k \times_k \mathbb{A}^{n+1}_k$ be the morphism of k-schemes obtained by setting $u = 0$; note that $p_2 f = \mathrm{id}$ and af factors through $\operatorname{Spec} k$. Let $\alpha \in \mathrm{H}^2_{\text{ét}}(\mathbb{A}^{n+1}_k, \mathbb{G}_m)$ be a Brauer class such that $a^*\alpha = p_2^*\alpha$ in $\mathrm{H}^2_{\text{ét}}(\mathbb{A}^1_k \times_k \mathbb{A}^{n+1}_k, \mathbb{G}_m)$; then $f^*a^*\alpha = f^*p_2^*\alpha = \alpha$; hence α is in the image of $\mathrm{H}^2_{\text{ét}}(\operatorname{Spec} k, \mathbb{G}_m)$. □

Lemma 5.3 [26, Corollary 4.3]. *For any connected scheme S, the map*

$$\mathbb{Z} \oplus \operatorname{Pic}(S) \to \operatorname{Pic}(\mathcal{P}_S(\rho))$$

sending

$$(\ell, \mathcal{L}) \mapsto \mathcal{O}_{\mathcal{P}_S(\rho)}(\ell) \otimes \pi_S^* \mathcal{L}$$

is an isomorphism.

Lemma 5.4 (cohomology of $\mathcal{O}_{\mathcal{P}(\rho)}(\ell)$ [25, Proposition 2.5]). *Let A be a ring and set $X := \mathcal{P}_A(\rho)$.*

(1) *For $\ell \geq 0$, the A-module $\mathrm{H}^0(X, \mathcal{O}_X(\ell))$ is free with basis consisting of monomials $t_0^{e_0} \cdots t_n^{e_n}$ such that $e_0, \dots, e_n \in \mathbb{Z}_{\geq 0}$ and $\rho_0 e_0 + \cdots + \rho_n e_n = \ell$.*

(2) *For $\ell < 0$, the A-module $\mathrm{H}^n(X, \mathcal{O}_X(\ell))$ is free with basis consisting of monomials $t_0^{e_0} \cdots t_n^{e_n}$ such that $e_0, \dots, e_n \in \mathbb{Z}_{<0}$ and $\rho_0 e_0 + \cdots + \rho_n e_n = \ell$.*

(3) *If $(i, \ell) \notin (\{0\} \times \mathbb{Z}_{\geq 0}) \cup (\{n\} \times \mathbb{Z}_{<0})$, then $\mathrm{H}^i(X, \mathcal{O}_X(\ell)) = 0$.*

(4) *For any A-module M and any (i, ℓ), the canonical map*

$$\mathrm{H}^i(X, \mathcal{O}_X(\ell)) \otimes_A M \to \mathrm{H}^i(X, \mathcal{O}_X(\ell) \otimes_A M)$$

is an isomorphism.

Lemma 5.5. *Let A be a strictly henselian local ring. Then $\mathrm{H}^2_{\acute{e}t}(\mathcal{P}_A(\rho), \mathbb{G}_m) = 0$.*

Proof. The proof is the same as that of Lemma 4.3 with the following modifications: for the triviality of the gerbe $\mathcal{G}_0$ over the special fiber, we use Lemma 5.2; to obtain the equivalence marked 2, we use Grothendieck existence for stacks [28, Theorem 1.4] (using that $\mathcal{P}(\rho)$ is proper [25, Proposition 2.1]); to conclude that $\mathrm{H}^2_{\text{ét}}(X_\ell, \mathfrak{m}^\ell \mathcal{O}_{X_\ell}) = 0$, we use Lemma 5.4. □

Lemma 5.6. *Let*

$$\pi_\rho : \mathcal{P}_{\mathbb{Z}}(\rho) \to \mathbb{P}_{\mathbb{Z}}(\rho)$$

denote the coarse moduli space morphism. For any $\ell \in \mathbb{Z}$*, there is a canonical* $\mathcal{O}_{\mathcal{P}_{\mathbb{Z}}(\rho)}$*-linear map*

$$\pi_\rho^*(\mathcal{O}_{\mathbb{P}_{\mathbb{Z}}(\rho)}(\ell)) \to \mathcal{O}_{\mathcal{P}_{\mathbb{Z}}(\rho)}(\ell), \tag{5.6.1}$$

which is an isomorphism if ℓ *is divisible by* $\operatorname{lcm}(\rho)$.

Proof. Set $R := \mathbb{Z}[t_0, \dots, t_n]$ with the $\mathbb{Z}$-grading determined by $\deg(t_i) = \rho_i$. The restriction of (5.6.1) to the open substack $[(\operatorname{Spec} R[t_i^{-1}])/\mathbb{G}_m]$ corresponds to the graded homomorphism

$$R(\ell)[t_i^{-1}]_0 \otimes_{R[t_i^{-1}]_0} R[t_i^{-1}] \to R(\ell)[t_i^{-1}] \tag{5.6.2}$$

of $\mathbb{Z}$-graded $R[t_i^{-1}]$-modules; the m-th component of (5.6.2) is isomorphic to the $R[t_i^{-1}]_0$-linear map

$$R[t_i^{-1}]_\ell \otimes_{R[t_i^{-1}]_0} R[t_i^{-1}]_m \to R[t_i^{-1}]_{\ell+m} \tag{5.6.3}$$

induced by multiplication.

If ℓ is divisible by ρ_i, then the multiplication-by-t_i^{ℓ/ρ_i} map $R[t_i^{-1}] \to R[t_i^{-1}](\ell)$ is an isomorphism of $\mathbb{Z}$-graded $R[t_i^{-1}]$-modules, thus (5.6.3) is an isomorphism for all $m \in \mathbb{Z}$, in other words the restriction of (5.6.1) to $[(\operatorname{Spec} R[t_i^{-1}])/\mathbb{G}_m]$ is an isomorphism. □

Lemma 5.7. *The pullback*

$$\pi_\rho^* : \operatorname{Pic}(\mathbb{P}_{\mathbb{Z}}(\rho)) \to \operatorname{Pic}(\mathcal{P}_{\mathbb{Z}}(\rho))$$

is multiplication by $\operatorname{lcm}(\rho)$.

Proof. We have that $\operatorname{Pic}(\mathbb{P}_{\mathbb{Z}}(\rho)) \simeq \mathbb{Z}$ is generated by the class of $\mathcal{O}_{\mathbb{P}_{\mathbb{Z}}(\rho)}(\operatorname{lcm}(\rho))$ and that $\operatorname{Pic}(\mathcal{P}_{\mathbb{Z}}(\rho)) \simeq \mathbb{Z}$ is generated by the class of $\mathcal{O}_{\mathcal{P}_{\mathbb{Z}}(\rho)}(\operatorname{lcm}(1))$ by Lemma 2.8 and Lemma 5.3, respectively. We have the desired claim by Lemma 5.6. □

Remark 5.8. There exist ρ, ℓ for which the natural map (5.6.1) is not an isomorphism. For example, in case $\rho = (1, 2)$ and $\ell = 1$, the element $t_0 \in R[t_0^{-1}]_2$ is not in the image of the map (5.6.3) for $m = 1$ and $i = 0$. We have $\mathcal{O}_{\mathbb{P}(\rho)}(1) \simeq \mathcal{O}_{\mathbb{P}(\rho)}$, and the pullback (5.6.1) is multiplication by $t_1 \in \Gamma(\mathcal{P}(\rho), \mathcal{O}_{\mathcal{P}(\rho)}(1))$; see Lemma 2.5 for details. Furthermore, the natural map $\mathcal{O}_{\mathbb{P}(\rho)}(1) \otimes \mathcal{O}_{\mathbb{P}(\rho)}(1) \to \mathcal{O}_{\mathbb{P}(\rho)}(2)$ is not an isomorphism; here [14, Proposition 2.5.13] does not apply since R is not generated in degree 1. (See also [10, Exemple 4.8; 12, Section 1.5.3].)

5.9. *Proof of Theorem 1.3.* The proof of the exactness of (1.3.1) is the same as in Section 4.4 with the following modifications: to show $\boldsymbol{R}^2 f_* \mathbb{G}_m = 0$, we use Lemma 5.5; to show $\boldsymbol{R}^1 f_* \mathbb{G}_m \simeq \underline{\mathbb{Z}}$, we use Lemma 5.3.

For any faithfully flat morphism $S' \to S$, the pullback $\pi_{S'} : \mathcal{X} \times_S S' \to X \times_S S'$ is a coarse moduli space morphism. Since $\mathcal{X} \times_S S' \simeq \mathcal{P}_{S'}(\rho)$, we have $X \times_S S' \simeq \mathcal{P}_{S'}(\rho)$.

We have a morphism between Leray spectral sequences for $\mathcal{X}$ and X induced by pullback via π, from which we obtain the vertical maps in (1.3.2). The description of the left vertical arrow in (1.3.2) follows from Lemma 5.7. □

Remark 5.10. It should be possible to describe the Brauer class $[\mathcal{X}] \in \mathrm{Br}'(S)$ in a similar way to Remark 4.6, using Noohi's description of the automorphism 2-group of weighted projective stacks in [27].

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