# *Pacific Journal of Mathematics*

Volume 324 No. 2 June 2023

PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

msp.org/pjm



See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2023 is US $605/year for the electronic version, and $820/year for print and electronic.
Subscriptions, requests for back issues and changes of subscriber address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 1945-5844 electronic, 0030-8730 printed) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFlow® from Mathematical Sciences Publishers.

PUBLISHED BY
mathematical sciences publishers
nonprofit scientific publishing
http://msp.org/

PACIFIC JOURNAL OF MATHEMATICS
Vol. 324, No. 2, 2023
https://doi.org/10.2140/pjm.2023.324.207

# POLYNOMIAL CONDITIONS AND HOMOLOGY OF FI-MODULES

CIHAN BAHRAN

**We identify two recursively defined polynomial conditions for FI-modules in the literature. We characterize these conditions using homological invariants of FI-modules (namely the local degree and regularity, together with the stable degree) and clarify their relationship. For one of these conditions, we give improved twisted homological stability ranges for the symmetric groups. As another application, we improve the representation stability ranges for congruence subgroups with respect to the action of an appropriate linear group by a factor of 2 in its slope.**

## 1. Introduction

There are (at least) two classes of papers that deal in some depth with **FI**-modules:

(1) In papers such as [3; 4; 6; 11; 14] the **FI**-module is the central object of study. They attach *homological invariants* to an **FI**-module by means such as **FI**-homology or local cohomology, and study the relationship of these invariants both with the stabilization behavior of the **FI**-module and/or between each other.

(2) Papers such as [12; 13; 16; 18; 22] might be thought of as stability machines. The sequence $\{\mathfrak{S}_n\}$ of symmetric groups is but one of many sequences of groups they deal with and **FI**-modules arise as the suitable notion of *coefficient systems* for $\{\mathfrak{S}_n\}$. They declare a coefficient system to be *polynomial* with certain parameters in a *recursive* fashion: there is a base case, and above that, being polynomial demands a related coefficient system to be polynomial with some of the parameters lowered.

The main objective of this paper is to characterize the polynomial conditions in (2) for **FI**-modules by the homological invariants in (1).

The author was supported in part by TÜBİTAK 119F422.
*MSC2020:* primary 18A25, 20J06; secondary 11F75.
*Keywords:* FI-modules, homological stability, representation stability, congruence subgroups.

***Notation.*** We write **FI** for the category of finite sets and injections. An **FI**-module is a functor $V : \mathbf{FI} \to \mathbb{Z}\text{-}\mathsf{Mod}$ and given a finite set $S$, we write $V_S$ for its evaluation; given an injection of finite sets $\alpha : S \hookrightarrow T$, we write $V_\alpha : V_S \to V_T$ for its induced map. For $n \in \mathbb{N}$ we set $V_n := V_{\{1,\dots,n\}}$. We write **FI**-Mod for the category of **FI**-modules. Throughout, our notation for **FI**-modules will be consistent with [6] and [1].

***Degree and torsion.*** Given an **FI**-module $W$, we write

$$\deg(W) := \min\{d \geq -1 : W_S = 0 \text{ for } |S| > d\} \in \{-1, 0, 1, 2, 3, \dots\} \cup \{\infty\}.$$

An **FI**-module $V$ is *torsion* if for every finite set $S$ and $x \in V_S$, there exists an injection $\alpha : S \hookrightarrow T$ such that $V_\alpha(x) = 0 \in V_T$. We write

$$\mathrm{H}^0_{\mathfrak{m}} : \mathbf{FI}\text{-}\mathsf{Mod} \to \mathbf{FI}\text{-}\mathsf{Mod}$$

for the functor which assigns an **FI**-module its largest torsion **FI**-submodule, and write

$$h^0(V) := \deg(\mathrm{H}^0_{\mathfrak{m}}(V)).$$

***Shift and derivative functors.*** Given any **FI**-module $V$, we write $\Sigma V$ for the composition

$$\mathbf{FI} \xrightarrow{-\sqcup\{*\}} \mathbf{FI} \xrightarrow{V} \mathbb{Z}\text{-}\mathsf{Mod}$$

and call it the *shift functor*. It receives a natural transformation from the identity functor $\mathrm{id}_{\mathbf{FI}\text{-}\mathsf{Mod}}$, whose cokernel

$$\Delta := \operatorname{coker}(\mathrm{id}_{\mathbf{FI}\text{-}\mathsf{Mod}} \to \Sigma)$$

we call the *derivative functor*.

***Stable degree.*** For an **FI**-module $V$, we set

$$\delta(V) := \min\{r \geq -1 : \Delta^{r+1}(V) \text{ is torsion}\} \in \{-1, 0, 1, \dots\} \cup \{\infty\}$$

and call it the *stable degree* of $V$. In both polynomial conditions for **FI**-modules we shall consider, the stable degree will be in analogy with the usual degree of a polynomial. Also see [6, Proposition 2.14].

***First polynomial condition and local degree.*** Suppose $f$ is a function in $n \in \mathbb{N}$ which is equal to a polynomial of degree $\leq r$ in the range $n \geq L$. We can consider its *discrete derivative* $\Delta f$, which is the function

$$\Delta f(n) := f(n+1) - f(n).$$

Note that $\Delta f$ is equal to a polynomial of degree $\leq r - 1$ in the same range $n \geq L$. The first polynomial condition we treat for **FI**-modules is a categorification of this recursion. See [22, Section 4.4 and Remark 4.19] for references to similar definitions in the literature.

**Definition 1.1.** For every pair of integers $r \geq -1,\ L \geq 0$, we define a class of **FI**-modules $\mathbf{Poly}_1(r, L)$ recursively via

$$\mathbf{Poly}_1(r, L) := \begin{cases} \{V \in \mathbf{FI}\text{-}\mathsf{Mod} : \deg(V) \leq L - 1\} & \text{if } r = -1, \\ \left\{ V \in \mathbf{FI}\text{-}\mathsf{Mod} : \begin{array}{l} h^0(V) \leq L - 1 \text{ and} \\ \Delta V \in \mathbf{Poly}_1(r-1, L) \end{array} \right\} & \text{if } r \geq 0. \end{cases}$$

**Remark 1.2.** Let $V$ be an **FI**-module and $r \geq -1,\ L \geq 0$ be integers. The following can be seen to be equivalent by inspection:

- $V \in \mathbf{Poly}_1(r, L)$.
- In the sense of [21, Definition 4.10],[1] $V$ has degree $r$ at $L$.
- In the sense of [13, Definition 3.24] and [16, Definition 7.1], $V$ has polynomial degree $\leq r$ in ranks $> L - 1$.

***Local cohomology and local degree.*** The functor $\mathrm{H}^0_{\mathfrak{m}}$ defined above is left exact. For each $j \geq 0$, we write $\mathrm{H}^j_{\mathfrak{m}} := \mathrm{R}^j \mathrm{H}^0_{\mathfrak{m}}$ for the $j$-th right derived functor of $\mathrm{H}^0_{\mathfrak{m}}$, and write

$$h^j(V) := \deg(\mathrm{H}^j_{\mathfrak{m}}(V)) \in \{-1, 0, 1, \ldots\} \cup \{\infty\},$$
$$h^{\max}(V) := \max\{h^j(V) : j \geq 0\} \in \{-1, 0, 1, \ldots\} \cup \{\infty\}$$

for every **FI**-module $V$. We call $h^{\max}(V)$ the *local degree* of $V$.

Our first main result is that the stable degree $\delta(V)$ and the local degree $h^{\max}(V)$ together characterize the first polynomial condition.

**Theorem A.** *For every pair of integers* $r \geq -1,\ L \geq 0$, *we have*

$$\mathbf{Poly}_1(r, L) = \{V \in \mathbf{FI}\text{-}\mathsf{Mod} : \delta(V) \leq r \textit{ and } h^{\max}(V) \leq L - 1\}.$$

***Second polynomial condition and regularity.*** The second polynomial condition we shall treat is, perhaps deceivingly, very similar to the first one. In fact the confusion between the two and the resulting need to clarify was what prompted this paper.

**Definition 1.3.** For every pair of integers $r \geq -1,\ M \geq 0$, we define a class of **FI**-modules $\mathbf{Poly}_2(r, M)$ recursively via

$$\mathbf{Poly}_2(r, M) := \begin{cases} \{V \in \mathbf{FI}\text{-}\mathsf{Mod} : \deg(V) \leq M - 1\} & \text{if } r = -1, \\ \left\{ V \in \mathbf{FI}\text{-}\mathsf{Mod} : \begin{array}{l} h^0(V) \leq M - 1 \text{ and} \\ \Delta V \in \mathbf{Poly}_2(r-1, \max\{0, M-1\}) \end{array} \right\} & \text{if } r \geq 0. \end{cases}$$

**Remark 1.4.** Let $V$ be an **FI**-module and $r \geq -1,\ M \geq 0$ be integers. The following can be seen to be equivalent by inspection:

- $V \in \mathbf{Poly}_2(r, M)$.
- In the sense of [22, Definition 4.10], $V$ has degree $r$ at $M$.

[1] Note that [21] is an early preprint version of the published [22]. The authors switched from Definition 1.1 to Definition 1.3 in between.

- In the sense of [12, Definition 2.40],[2] $V$ has polynomial degree $\leq r$ in ranks $> M-1$.
- In the sense of [18, Definition 1.6], $V$ is polynomial of degree $r$ starting at $M$.

**FI-*homology and regularity.*** Regard the functor $\mathrm{H}_0^{\mathbf{FI}} : \mathbf{FI}\text{-}\mathsf{Mod} \to \mathbf{FI}\text{-}\mathsf{Mod}$ defined via

$$\mathrm{H}_0^{\mathbf{FI}}(V)_S := \operatorname{coker}\big(\textstyle\bigoplus_{T \subsetneq S} V_T \to V_S\big)$$

for every finite set $S$, which is right exact. For each $i \geq 0$, we write $\mathrm{H}_i^{\mathbf{FI}} := \mathrm{L}_i \mathrm{H}_0^{\mathbf{FI}}$ for its $i$-th left derived functor, and write

$$t_i(V) := \deg(\mathrm{H}_i^{\mathbf{FI}}(V)) \in \{-1, 0, 1, \ldots\} \cup \{\infty\},$$
$$\operatorname{reg}(V) := \max\{t_i(V) - i : i \geq 1\} \in \{-2, -1, 0, 1, \ldots\} \cup \{\infty\}$$

for every **FI**-module $V$. We say that $V$ is *generated in degrees* $\leq g$ if $t_0(V) \leq g$, and that $V$ is *presented in finite degrees* if $t_0(V)$ and $t_1(V)$ are both finite. We call $\operatorname{reg}(V)$ the *regularity* of $V$.

Our second main result is that the stable degree $\delta(V)$ and the regularity $\operatorname{reg}(V)$ together characterize the second polynomial condition.

**Theorem B.** *For every pair of integers $r \geq -1$, $M \geq 0$, we have*

$$\mathbf{Poly}_2(r, M) = \{V \in \mathbf{FI}\text{-}\mathsf{Mod} : \delta(V) \leq r \ \textit{and} \ \operatorname{reg}(V) \leq M-1\}.$$

***Twisted homological stability with* FI*-module coefficients.*** For any **FI**-module $V$ and homological degree $k \geq 0$, there is a sequence of maps

$$\mathrm{H}_k(\mathfrak{S}_0; V_0) \to \mathrm{H}_k(\mathfrak{S}_1; V_1) \to \mathrm{H}_k(\mathfrak{S}_2; V_2) \to \cdots$$

between the homology groups of the symmetric groups twisted by $V_n$'s. For the stabilization of this sequence, recently Putman [18, Theorems A and A′] established explicit ranges for the class $\mathbf{Poly}_2(r, M)$ in terms of $r, M$. We give ranges for the class $\mathbf{Poly}_1(r, L)$ in terms of $r, L$.

**Theorem C.** *Let $V$ be an* **FI***-module and $r, L \geq 0$ be integers such that $V \in \mathbf{Poly}_1(r, L)$. Then for every $k \geq 0$, the map $\mathrm{H}_k(\mathfrak{S}_n; V_n) \to \mathrm{H}_k(\mathfrak{S}_{n+1}; V_{n+1})$ is*

$$\textit{an isomorphism for } n \geq \begin{cases} 2k+r+1 & \textit{if } L = 0, \\ 2k+r+\left\lfloor \frac{L+1}{2} \right\rfloor + 2 & \textit{if } 1 \leq L \leq 2r-2, \\ \max\{2k+2r+1, L\} & \textit{if } L \geq \max\{1, 2r-1\}, \end{cases}$$

$$\textit{and a surjection for } n \geq \begin{cases} 2k+r & \textit{if } L = 0, \\ 2k+r+\left\lfloor \frac{L+1}{2} \right\rfloor + 1 & \textit{if } 1 \leq L \leq 2r-2, \\ \max\{2k+2r, L\} & \textit{if } L \geq \max\{1, 2r-1\}. \end{cases}$$

[2] Although the *terminology* used for the polynomial conditions in [13], [16] and [12] are the same, the first two use Definition 1.1 while the latter uses Definition 1.3.

**Remark 1.5.** Under the same hypotheses with Theorem C, Theorem 5.1 of [21] establishes

- an isomorphism for $n \geq \max\{2L+1, 2k+2r+2\}$,
- and a surjection for $n \geq \max\{2L+1, 2k+2r\}$.

The ranges in Theorem C are improvements over these.

**$\mathbf{SL}_n^{\mathfrak{U}}$*-stability ranges for congruence subgroups.*** For every ring $R$, the assignment $n \mapsto \mathrm{GL}_n(R)$ defines an **FI**-group (a functor from **FI** to the category of groups), for which we write $\mathrm{GL}_\bullet(R)$. If $I$ is an ideal of $R$, as the kernel of the mod-$I$ reduction we get a smaller **FI**-group

$$\mathrm{GL}_\bullet(R, I) := \ker\bigl(\mathrm{GL}_\bullet(R) \to \mathrm{GL}_\bullet(R/I)\bigr)$$

called the *$I$-congruence subgroup* of $\mathrm{GL}_\bullet(R)$. For each $k \geq 0$ and abelian group $\mathcal{A}$, taking the $k$-th homology with coefficients in $\mathcal{A}$ defines an **FI**-module

$$\mathrm{H}_k(\mathrm{GL}_\bullet(R, I); \mathcal{A}).$$

We wish to extend the $\mathfrak{S}_n$-action on $\mathrm{H}_k(\mathrm{GL}_n(R, I); \mathcal{A})$ to an action of a linear group and formulate representation stability over it, in accordance with [17, fifth Remark, page 990].

***Special linear group with respect to a subgroup of the unit group.*** For a commutative ring $A$ and a subgroup $\mathfrak{U} \leq A^\times$, we write

$$\mathrm{SL}_n^{\mathfrak{U}}(A) := \{f \in \mathrm{GL}_n(A) : \det(f) \in \mathfrak{U}\},$$

so that we interpolate between $\mathrm{SL}_n(A) \leq \mathrm{SL}_n^{\mathfrak{U}}(A) \leq \mathrm{GL}_n(A)$ as we vary $1 \leq \mathfrak{U} \leq A^\times$. Note that we are using the notation in [19], whereas in [13] and [12] this group is denoted $\mathrm{GL}_n^{\mathfrak{U}}(A)$.

**Hypothesis 1.6.** In the triple $(R, I, n_0)$, we have a commutative ring $R$, an ideal $I$ of $R$, and an integer $n_0 \in \mathbb{N}$ such that the mod-$I$ reduction

$$\mathrm{SL}_n(R) \to \mathrm{SL}_n(R/I)$$

for the special linear group is surjective for every $n \geq n_0$.

***Stable rank of a ring.*** Let $R$ be a nonzero unital (associative) ring. A column vector $\boldsymbol{v} \in \mathrm{Mat}_{m\times 1}(R)$ of size $m$ is *unimodular* if there is a row vector $\boldsymbol{u} \in \mathrm{Mat}_{1\times m}(R)$ such that $\boldsymbol{uv} = 1$. Writing $I_r \in \mathrm{Mat}_{r\times r}(R)$ for the identity matrix of size $r$, we say a column vector $\boldsymbol{v}$ of size $m$ is *reducible* if there exists $A \in \mathrm{Mat}_{(m-1)\times m}(R)$ with block form $A = [I_{m-1} \mid \boldsymbol{x}]$ such that the column vector $A\boldsymbol{v}$ (of size $m-1$) is unimodular. We write $\mathbf{st}\text{-}\mathbf{rank}(R) \leq s$ if every unimodular column vector of size $> s$ is reducible.

**Remark 1.7.** We make a few observations about Hypothesis 1.6.

(1) It is straightforward to check that the triple $(R, I, n_0)$ satisfies Hypothesis 1.6 if and only if setting $\mathfrak{U} := \{x + I : x \in R^\times\}$, there is a short exact sequence

$$1 \to \mathrm{GL}_n(R, I) \to \mathrm{GL}_n(R) \to \mathrm{SL}_n^{\mathfrak{U}}(R/I) \to 1$$

of groups in the range $n \geq n_0$ where the epimorphism is the mod-$I$ reduction. Consequently, for every $n \geq n_0$ and any coefficients $\mathcal{A}$, the conjugation $\mathrm{GL}_n(R)$-action on the homology groups $\mathrm{H}_\star(\mathrm{GL}_n(R, I), \mathcal{A})$ descends to an $\mathrm{SL}_n^{\mathfrak{U}}(R/I)$-action. It is this action for which we will obtain an improved representation stability range.

(2) For a Dedekind domain $R$ and any ideal $I$ of $R$, the triple $(R, I, 0)$ satisfies Hypothesis 1.6; see [7, page 2].

(3) If $\mathrm{SL}_n(R/I)$ is generated by elementary matrices for $n \geq n_0$, then $(R, I, n_0)$ satisfies Hypothesis 1.6.

(4) If the $K$-group $\mathrm{SK}_1(R/I) = 0$ (equivalently, the natural map $\mathrm{K}_1(R/I) \to (R/I)^\times$ is an isomorphism) and $\textbf{st-rank}(R/I) \leq s < \infty$, then by (3) and [10, 4.3.8, page 172], the triple $(R, I, s+1)$ satisfies Hypothesis 1.6.

**Theorem D.** *Let $I$ be a proper ideal in a commutative ring $R$ and $s, n_0 \in \mathbb{N}$, so*

- $\textbf{st-rank}(R) \leq s$, *and*
- *the triple $(R, I, n_0)$ satisfies Hypothesis 1.6 with $n_0 \leq 2s + 3$.*

*Then writing*

$$\mathfrak{U} := \{x + I : x \in R^\times\}, \quad \mathcal{G}_n := \mathrm{SL}_n^{\mathfrak{U}}(R/I)$$

*for every homological degree $k \geq 1$ and abelian group $\mathcal{A}$, there is a coequalizer diagram*

$$\mathrm{Ind}_{\mathcal{G}_{n-2}}^{\mathcal{G}_n} \mathrm{H}_k(\mathrm{GL}_{n-2}(R, I); \mathcal{A}) \rightrightarrows \mathrm{Ind}_{\mathcal{G}_{n-1}}^{\mathcal{G}_n} \mathrm{H}_k(\mathrm{GL}_{n-1}(R, I); \mathcal{A}) \to \mathrm{H}_k(\mathrm{GL}_n(R, I); \mathcal{A})$$

*of $\mathbb{Z}\mathcal{G}_n$-modules whenever*

$$n \geq \begin{cases} 2s + 5 & \text{if } k = 1, \\ 4k + 2s + 2 & \text{if } k \geq 2. \end{cases}$$

**Remark 1.8.** The best stable ranges established previously in the literature under the assumptions (with $n_0 = 0$) of Theorem D are due to Miller, Patzt and Petersen [12, proof of Theorem 1.4, page 46]: they obtained the conclusion of Theorem D in the range $n \geq 8k + 4s + 9$.

## 2. Homological algebra of FI-modules

***Regularity in terms of local cohomology.*** We first recall a characterization of the regularity by Nagpal, Sam and Snowden [14].

**Theorem 2.1** [14, Theorem 1.1, Remark 1.3]. *Let $V$ be an* **FI**-*module presented in finite degrees which is not* $\mathrm{H}_0^{\mathbf{FI}}$-*acyclic. Then*

$$\operatorname{reg}(V) = \max\{h^j(V) + j : \mathrm{H}^j_{\mathfrak{m}}(V) \neq 0\} = \max\{h^j(V) + j : h^j(V) \geq 0\}.$$

**Remark 2.2.** Under the hypotheses of Theorem 2.1, by [1, Theorem 2.4] and [6, Corollary 2.13], we have

$$\varnothing \neq \{j : \mathrm{H}^j_{\mathfrak{m}}(V) \neq 0\} = \{j : h^j(V) \geq 0\} \subseteq \{0, \ldots, \delta(V) + 1\}.$$

**Corollary 2.3.** *Let $V$ be a nonzero* **FI**-*module with* $\deg(V) < \infty$. *Then $V$ is presented in finite degrees*, *and*

$$h^j(V) = \begin{cases} \deg(V) = \operatorname{reg}(V) & \text{if } j = 0, \\ -1, & \text{otherwise.} \end{cases}$$

*Proof.* $V$ is certainly generated in degrees $\leq \deg(V)$ and also $h^0(V) \leq \deg(V)$. Thus by [1, Proposition 2.5] and [20, Theorem A], $V$ is presented in finite degrees. Now $V$ and the complex $0 \to V \to 0 \to 0 \to \cdots$ satisfy the assumptions of [6, Theorem 2.10] and hence

$$\mathrm{H}^j_{\mathfrak{m}}(V) = \begin{cases} V & \text{if } j = 0, \\ 0, & \text{otherwise.} \end{cases}$$

The rest follows from Theorem 2.1. □

***The derivative and local cohomology.*** In this section, we investigate the relationship between the local cohomology of an **FI**-module and that of its derivative.

We write $K := \ker(\mathrm{id}_{\mathbf{FI}\text{-}\mathsf{Mod}} \to \Sigma)$ so that we have an exact sequence

$$0 \to K \to \mathrm{id}_{\mathbf{FI}\text{-}\mathsf{Mod}} \to \Sigma \to \Delta \to 0$$

of functors **FI**-Mod → **FI**-Mod.

**Lemma 2.4.** *For every* **FI**-*module $V$, we have* $\deg(KV) = h^0(V)$.

*Proof.* Since $KV$ is a torsion submodule of $V$, we have $KV \subseteq \mathrm{H}^0_{\mathfrak{m}}(V)$ and hence

$$\deg(KV) \leq \deg(\mathrm{H}^0_{\mathfrak{m}}(V)) = h^0(V).$$

There is nothing to show when $h^0(V) = -1$, so we consider two cases.

**Case 1**. $h^0(V) = \infty$. To show $\deg(KV) = \infty$, we will show that for every $d \in \mathbb{N}$ we have $\deg(KV) \geq d$. Because $h^0(V) \geq d$, there exists a torsion element $x \in V_S - \{0\}$ of $V$ with $|S| \geq d$. Because $x$ is torsion, the set

$$\{|T| : V_\iota(x) = 0 \text{ for some } \iota : S \hookrightarrow T\} \subseteq \mathbb{N}$$

is nonempty and hence has a least element, say $N$. Noting $N > d$, let $A$ be a finite set of size $N-1$ and $f : S \hookrightarrow A$ so by the minimality of $N$ we have $0 \neq V_f(x) \in (KV)_A$ and $\deg(KV) \geq |A| = N-1 \geq d$.

**Case 2**. $0 \leq d := h^0(V) < \infty$. We pick a torsion element $x \in V_S - \{0\}$ with $|S| = d$ and we claim that $V_\iota(x) = 0$ for the embedding $\iota : S \hookrightarrow S \sqcup \{\star\}$. There is a finite set $T$ and an injection $f : S \hookrightarrow T$ such that $V_f(x) = 0$. As $x \neq 0$, $f$ cannot be an isomorphism so $|T| > |S|$ and $f = g \circ \iota$ for some injection $g : S \sqcup \{\star\} \hookrightarrow T$. As $V_g(V_\iota(x)) = 0$, the element $V_\iota(x)$ is torsion but it lies in degree $d+1$, forcing $V_\iota(x) = 0$ and hence $x \in KV$, showing $\deg(KV) \geq d$. $\square$

**Proposition 2.5.** *Given an* **FI**-*module* $V$, *the following are equivalent*:

(1) *$V$ is presented in finite degrees.*

(2) *$h^0(V) < \infty$ and $\Delta V$ is presented in finite degrees.*

*Proof.* Assume (1). Then by [8, Theorem 1] $\Sigma V$ is presented in finite degrees, and hence so are $KV$ and $\Delta V$ by [20, Theorem B] and [1, Proposition 2.5]. We have $h^0(V) < \infty$ by [20, Theorem A].

Conversely, assume (2). By [6, Proposition 2.9, part (4)], $u := \delta(\Delta V) < \infty$, so $\Delta^{u+2} V = \Delta^{u+1} \Delta V$ is torsion. Also, by applying the implication (1) $\Rightarrow$ (2) to $\Delta V$ and iterating it, $\Delta^{u+2} V$ is presented in finite degrees. Being a torsion **FI**-module generated in finite degrees, $\Delta^{u+2} V$ has finite degree, say $d$. Therefore by [3, Proposition 4.6], $V$ is generated in degrees $\leq u+d+2$. We conclude by [20, Theorem A]. $\square$

**Proposition 2.6.** *Given an* **FI**-*module* $V$, *the following hold*:

(1) *If $h^0(V) < \infty$, then there is a long exact sequence*

$$0 \longrightarrow KV \longrightarrow \mathrm{H}^0_{\mathfrak{m}}(V) \longrightarrow \Sigma \mathrm{H}^0_{\mathfrak{m}}(V) \longrightarrow \mathrm{H}^0_{\mathfrak{m}}(\Delta V) \longrightarrow \cdots$$
$$\longrightarrow \mathrm{H}^j_{\mathfrak{m}}(V) \longrightarrow \Sigma \mathrm{H}^j_{\mathfrak{m}}(V) \longrightarrow \mathrm{H}^j_{\mathfrak{m}}(\Delta V) \longrightarrow \mathrm{H}^{j+1}_{\mathfrak{m}}(V) \longrightarrow \Sigma \mathrm{H}^{j+1}_{\mathfrak{m}}(V) \longrightarrow \mathrm{H}^{j+1}_{\mathfrak{m}}(\Delta V) \longrightarrow \cdots$$

(2) *If $V$ is presented in finite degrees*, (1) *holds such that every* **FI**-*module in the sequence has finite degree.*

*Proof.* For (1), note that $KV$ is certainly generated in degrees

$$\leq \deg(KV) = h^0(V) < \infty$$

by Lemma 2.4. Thus by [20, Theorem A] (see [1, Proposition 2.5]), $KV$ is presented in finite degrees. Therefore, [6, Theorem 2.10] applies to $KV$ and the complex $0 \to KV \to 0 \to 0 \to \dots$ and hence

$$\mathrm{H}^j_{\mathfrak{m}}(KV) = \begin{cases} KV & \text{if } j = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Now applying $\mathrm{H}^0_{\mathfrak{m}}$ to the short exact sequence

$$0 \to KV \to V \to V/KV \to 0,$$

the associated long exact sequence gives a short exact sequence

$$0 \to KV \to \mathrm{H}^0_{\mathfrak{m}}(V) \to \mathrm{H}^0_{\mathfrak{m}}(V/KV) \to 0$$

and isomorphisms

$$\mathrm{H}^j_{\mathfrak{m}}(V) \cong \mathrm{H}^j_{\mathfrak{m}}(V/KV)$$

for every $j \geq 1$. Using these isomorphisms after applying $\mathrm{H}^0_{\mathfrak{m}}$ to the short exact sequence

$$0 \to V/KV \to \Sigma V \to \Delta V \to 0,$$

the associated long exact sequence will almost have the desired form, except we need to splice it in the beginning and interchange the order of the shift functor $\Sigma$ with local cohomology $\mathrm{H}^\star_{\mathfrak{m}}$ in the middle column. To see $\Sigma \circ \mathrm{H}^\star_{\mathfrak{m}} = \mathrm{H}^\star_{\mathfrak{m}} \circ \Sigma$, first note that $\Sigma : \mathbf{FI}\text{-}\mathsf{Mod} \to \mathbf{FI}\text{-}\mathsf{Mod}$

- is exact,
- has an exact left adjoint [9, Theorem 4],
- satisfies $\Sigma \circ \mathrm{H}^0_{\mathfrak{m}} = \mathrm{H}^0_{\mathfrak{m}} \circ \Sigma$.

Consequently, given an **FI**-module $U$ and an injective resolution $0 \to U \to I^\star$, applying $\Sigma$ we get an injective resolution $0 \to \Sigma U \to \Sigma I^\star$ of $\Sigma U$, and hence

$$\mathrm{H}^j_{\mathfrak{m}}(\Sigma U) = \mathrm{H}^j(\mathrm{H}^0_{\mathfrak{m}}(\Sigma I^\star)) = \mathrm{H}^j(\Sigma \mathrm{H}^0_{\mathfrak{m}}(I^\star)) = \Sigma \mathrm{H}^j(\mathrm{H}^0_{\mathfrak{m}}(I^\star)) = \Sigma \mathrm{H}^j_{\mathfrak{m}}(U)$$

for every $j \geq 0$, naturally in $U$.

For (2), assume $V$ is presented in finite degrees. Then $\deg(KV) = h^0(V) < \infty$ (so we have the long exact sequence from (1)) and $\Delta V$ is presented in finite degrees by Lemma 2.4 and Proposition 2.5. Now invoke [6, Theorem 2.10] for $V$ and $\Delta V$. □

**Corollary 2.7.** *For every* **FI**-*module $V$ presented in finite degrees, the following hold*:

(1) *For every $j \geq 0$, we have $h^j(\Delta V) \leq \max\{h^j(V) - 1, h^{j+1}(V)\}$.*

(2) *For every $j \geq 1$, we have $h^j(V) \leq \max\{h^{j-1}(\Delta V), h^j(\Delta V)\}$.*

*Proof.* By Proposition 2.6, for every $j \geq 0$ we have

$$\begin{aligned} h^j(\Delta V) &= \deg \mathrm{H}^j_{\mathfrak{m}}(\Delta V) \\ &\leq \max\{\deg \Sigma \mathrm{H}^j_{\mathfrak{m}}(V), \deg \mathrm{H}^{j+1}_{\mathfrak{m}}(V)\} = \max\{\deg \Sigma \mathrm{H}^j_{\mathfrak{m}}(V), h^{j+1}(V)\}. \end{aligned}$$

If $\mathrm{H}^j_{\mathfrak{m}}(V) \neq 0$, then

$$\deg \Sigma \mathrm{H}^j_{\mathfrak{m}}(V) = \deg \mathrm{H}^j_{\mathfrak{m}}(V) - 1 = h^j(V) - 1$$

and (1) follows. If $\mathrm{H}^j_{\mathfrak{m}}(V) = 0$, then $\mathrm{H}^j_{\mathfrak{m}}(\Delta V)$ embeds in $\mathrm{H}^{j+1}_{\mathfrak{m}}(V)$ and (1) again follows.

To prove (2), fix $j \geq 1$ and set $N := \max\{h^{j-1}(\Delta V), h^j(\Delta V)\}$ so for every $n > N$, by Proposition 2.6 we have an isomorphism

$$\mathrm{H}^j_{\mathfrak{m}}(V)_n \cong \Sigma \mathrm{H}^j_{\mathfrak{m}}(V)_n = \mathrm{H}^j_{\mathfrak{m}}(V)_{n+1}.$$

But $\mathrm{H}^j_{\mathfrak{m}}(V)$ has finite degree, therefore the above isomorphisms in the entire range $n > N$ have to be between zero modules so that $h^j(V) = \deg \mathrm{H}^j_{\mathfrak{m}}(V) \leq N$. □

***Critical index and the regularity of derivative.*** In this section, we introduce the notion of critical index for an **FI**-module and use it to study how regularity interacts with the derivative functor.

**Definition 2.8.** For an **FI**-module $V$ presented in finite degrees which is not $\mathrm{H}^{\mathbf{FI}}_0$-acyclic, we define its *critical index* as

$$\mathrm{crit}(V) := \min\{j : h^j(V) \geq 0 \text{ and } h^j(V) + j = \mathrm{reg}(V)\}.$$

**Remark 2.9.** Let $V$ be as in Definition 2.8 and set $\gamma := \mathrm{crit}(V)$, $\rho := \mathrm{reg}(V)$. The following will not be needed in our arguments but we note them for context.

(1) By Theorem 2.1 and [6, Theorem 2.10], the set of indices

$$\{j : h^j(V) \geq 0 \text{ and } h^j(V) + j = \rho\}$$

is a nonempty subset of $\{0, \ldots, \delta(V) + 1\}$; thus $0 \leq \gamma \leq \delta(V) + 1$.

(2) Although they do not give it a name, Nagpal, Sam and Snowden [14, Definition 3.3] use the critical index: their invariant $\nu$ satisfies

$$\nu(\mathrm{H}^{\mathbf{FI}}_i(V)_{i+\rho}) = i + \gamma$$

for $i \gg 0$ [14, Proposition 4.3].

(3) It is possible that $h^{\gamma}(V) < h^{\max}(V)$. To see this, let us start by an exact sequence

$$(\lozenge) \qquad 0 \to Z \to A \to B \to W \to 0$$

of **FI**-modules presented in finite degrees where $A$, $B$ are $\mathrm{H}_0^{\mathbf{FI}}$-acyclic, $\deg(W)=0$. Breaking this into two short exact sequences, the associated long exact sequences for $\mathrm{H}_{\mathfrak{m}}^*$ yields

$$\mathrm{H}_{\mathfrak{m}}^j(Z) \cong \mathrm{H}_{\mathfrak{m}}^{j-2}(W)$$

for every $j \geq 0$. In particular, $h^2(Z)=0$ and $h^j(Z)=-1$ if $j \neq 2$. Now setting $V := Z \oplus T$ with $\deg(T)=1$, we get

$$h^j(V) = \begin{cases} 1 & \text{if } j=0, \\ 0 & \text{if } j=2, \\ -1, & \text{otherwise} \end{cases}$$

so that $\operatorname{reg}(V)=\operatorname{crit}(V)=2$, $h^2(V)=0$, but $h^{\max}(V)=1$.

**Proposition 2.10.** *Let $V$ be an* **FI***-module presented in finite degrees which is not $\mathrm{H}_0^{\mathbf{FI}}$-acyclic. Then*:

(1) $\operatorname{reg}(\Delta V) \leq \operatorname{reg}(V)-1$.

(2) *If* $\operatorname{crit}(V) \geq 1$, *then* $0 \leq \operatorname{reg}(\Delta V) = \operatorname{reg}(V)-1$ *and* $\operatorname{crit}(\Delta V) = \operatorname{crit}(V)-1$.

*Proof.* Note that $\Delta V$ is presented in finite degrees by Proposition 2.5. Set $\rho := \operatorname{reg}(V)$ and $\gamma := \operatorname{crit}(V)$.

Assume $\Delta V$ is $\mathrm{H}_0^{\mathbf{FI}}$-acyclic. Then by [1, Theorem 2.4], $h^j(\Delta V)=-1$ for every $j \geq 0$, and hence by part (2) of Corollary 2.7 we have $h^j(V)=-1$ for every $j \geq 1$, forcing $\gamma=0$. Therefore the condition $\operatorname{reg}(\Delta V)<0$ (which is equivalent to $\Delta V$ being $\mathrm{H}_0^{\mathbf{FI}}$-acyclic by [1, Corollary 2.9]) implies $\gamma=0$. In this case, we further have

$$\operatorname{reg}(\Delta V) < 0 \leq h^0(V) = \rho$$

by [1, Theorem 2.4] and (1) follows.

Next, assume $\Delta V$ is not $\mathrm{H}_0^{\mathbf{FI}}$-acyclic (hence neither is $V$, see the discussion in [6, Section 2.3]). Let us write

$$J(V) := \{j \geq 0 : h^j(V) \geq 0\}.$$

Let $j \in J(\Delta V)$. By part (1) of Corollary 2.7, we have either $0 \leq h^j(\Delta V) \leq h^j(V)-1$ or $0 \leq h^j(\Delta V) \leq h^{j+1}(V)$. In the former case, we have $j \in J(V)$ and

$$h^j(\Delta V)+j \leq h^j(V)+j-1 \leq \rho-1$$

by Theorem 2.1, and in the latter case, we have $j+1 \in J(V)$ and

$$h^j(\Delta V)+j \leq h^{j+1}(V)+j+1-1 \leq \rho-1$$

by Theorem 2.1. Yet another application of Theorem 2.1 now yields $\operatorname{reg}(\Delta V) \leq \rho-1$, which is exactly (1). To prove (2), we further assume that $\gamma \geq 1$. We claim that

$$h^{\gamma-1}(\Delta V)+\gamma-1 = \rho-1.$$

To that end, by Proposition 2.6 we have an exact sequence

$$\Sigma \mathrm{H}^{\gamma-1}_{\mathfrak{m}}(V) \to \mathrm{H}^{\gamma-1}_{\mathfrak{m}}(\Delta V) \to \mathrm{H}^{\gamma}_{\mathfrak{m}}(V) \to \Sigma \mathrm{H}^{\gamma}_{\mathfrak{m}}(V),$$

which we evaluate at a finite set of size $\rho - \gamma$ to get an exact sequence

$$\mathrm{H}^{\gamma-1}_{\mathfrak{m}}(V)_{\rho-\gamma+1} \to \mathrm{H}^{\gamma-1}_{\mathfrak{m}}(\Delta V)_{\rho-\gamma} \to \mathrm{H}^{\gamma}_{\mathfrak{m}}(V)_{\rho-\gamma} \to \mathrm{H}^{\gamma}_{\mathfrak{m}}(V)_{\rho-\gamma+1}$$

of $\mathfrak{S}_{\rho-\gamma}$-modules. Here:

- $\mathrm{H}^{\gamma-1}_{\mathfrak{m}}(V)_{\rho-\gamma+1} = 0$, because by the definition of critical index we have
$$h^{\gamma-1}(V) + \gamma - 1 < \rho, \quad \deg(\mathrm{H}^{\gamma-1}_{\mathfrak{m}}(V)) < \rho - \gamma + 1.$$
- $\mathrm{H}^{\gamma}_{\mathfrak{m}}(V)_{\rho-\gamma} \neq 0$ and $\mathrm{H}^{\gamma}_{\mathfrak{m}}(V)_{\rho-\gamma+1} = 0$, because by the definition of critical index we have
$$h^{\gamma}(V) \geq 0 \quad \text{and} \quad h^{\gamma}(V) + \gamma = \rho, \quad \deg(\mathrm{H}^{\gamma}_{\mathfrak{m}}(V)) = \rho - \gamma.$$

Therefore we conclude that

$$\mathrm{H}^{\gamma-1}_{\mathfrak{m}}(\Delta V)_{\rho-\gamma} \neq 0, \quad h^{\gamma-1}(\Delta V) \geq 0 \quad \text{and} \quad h^{\gamma-1}(\Delta V) + \gamma - 1 \geq \rho - 1.$$

On the other hand, part (1) and Theorem 2.1 give the reverse inequality to the above, establishing our claim, the equation $\operatorname{reg}(\Delta V) = \rho - 1$ and the inequality $\operatorname{crit}(\Delta V) \leq \gamma - 1$. To see in fact $\operatorname{crit}(\Delta V) = \gamma - 1$, we can take $0 \leq j < \gamma - 1$ and evaluate the exact sequence

$$\Sigma \mathrm{H}^{j}_{\mathfrak{m}}(V) \to \mathrm{H}^{j}_{\mathfrak{m}}(\Delta V) \to \mathrm{H}^{j+1}_{\mathfrak{m}}(V)$$

at a finite set of size $\rho - 1 - j$ to get

$$0 = \mathrm{H}^{j}_{\mathfrak{m}}(V)_{\rho-j} \to \mathrm{H}^{j}_{\mathfrak{m}}(\Delta V)_{\rho-1-j} \to \mathrm{H}^{j+1}_{\mathfrak{m}}(V)_{\rho-(j+1)} = 0$$

and conclude $h^{j}(\Delta V) + j < \rho - 1$, as desired. □

***Identifying the polynomial conditions.*** In this section we prove Theorems A and B. Because **FI**-modules being presented in finite degrees is such a common assumption, we first incorporate it as a redundant hypothesis in Theorems 2.12 and 2.13, and then remove this redundancy using Theorem 2.11.

**Theorem 2.11.** *For an* **FI**-*module $V$ with $\delta(V) < \infty$, the following are equivalent*:

(1) $\operatorname{reg}(V) < \infty$.

(2) $h^{\max}(V) < \infty$.

(3) *$V$ is presented in finite degrees.*

*Proof.* (3) ⇒ (1): Immediate from [3, Theorem A].

(1) ⇒ (2): We write:

- **FB** for the category of finite sets and bijections.
- $\mathrm{Ind}_{\mathbf{FB}}^{\mathbf{FI}}$ for the left adjoint of the restriction functor $\mathrm{Res}_{\mathbf{FB}}^{\mathbf{FI}} : \mathbf{FI}\text{-Mod} \to \mathbf{FB}\text{-Mod}$.
- $W := \mathrm{H}_0^{\mathbf{FI}}(V)$.

Then there is a short exact sequence

$$0 \to K \to \mathrm{Ind}_{\mathbf{FB}}^{\mathbf{FI}}(W) \to V \to 0 \tag{$\dagger$}$$

for some **FI**-module $K$. Here $\mathrm{Ind}_{\mathbf{FB}}^{\mathbf{FI}}(W)$ is $\mathrm{H}_0^{\mathbf{FI}}$-acyclic [3, Lemma 2.3]. Moreover, the $\mathrm{H}_0^{\mathbf{FI}}$-image of the epimorphism in (†) is the identity map $\mathrm{H}_0^{\mathbf{FI}}(V) \to \mathrm{H}_0^{\mathbf{FI}}(V)$. Thus applying $\mathrm{H}_0^{\mathbf{FI}}$ to (†), the associated long exact sequence splits into isomorphisms

$$\mathrm{H}_{i+1}^{\mathbf{FI}}(V) \cong \mathrm{H}_i^{\mathbf{FI}}(K)$$

for every $i \geq 0$. In particular, we have

$$t_i(K) = t_{i+1}(V) < \mathrm{reg}(V) + i + 1 < \infty$$

for every $i \geq 0$, so $K$ is presented in finite degrees. Consequently $h^{\max}(K) < \infty$ by [6, Proposition 2.9, part (4)] and [6, Theorem 2.10]. We will be done once we show

$$\mathrm{H}_{\mathfrak{m}}^{*}(\mathrm{Ind}_{\mathbf{FB}}^{\mathbf{FI}}(W)) = 0,$$

because applying $\mathrm{H}_{\mathfrak{m}}^0$ to (†), the long exact sequence yields $h^{\max}(V) = h^{\max}(K)$. The last claim follows from $W$ being the direct product of **FB**-modules each supported in a single degree, and the functors $\mathrm{Ind}_{\mathbf{FB}}^{\mathbf{FI}}$, $\mathrm{H}_{\mathfrak{m}}^{*}$ commuting with direct products (for instance, the former via [5, Definition 2.2.2] and the latter via [11, Definition 5.4]) together with [1, Theorem 2.4].

(2) ⇒ (3): We employ induction on $\delta(V)$: if $\delta(V) = -1$, then $V = \mathrm{H}_{\mathfrak{m}}^0(V)$ is torsion and so

$$\deg(V) = h^0(V) \leq h^{\max}(V) < \infty.$$

Thus $V$ is presented in finite degrees by Corollary 2.3. Next, assume $\delta(V) \geq 0$. We can apply Proposition 2.6 to $V$ to conclude $h^{\max}(\Delta V) < \infty$. We also have $\delta(\Delta V) \leq \delta(V) - 1$, therefore $\Delta V$ is presented in finite degrees by the induction hypothesis. We conclude by applying Proposition 2.5. □

**Theorem 2.12.** *For every pair of integers $r \geq -1$, $L \geq 0$, we have*

$$\mathbf{Poly}_1(r, L) = \left\{ V \in \mathbf{FI}\text{-Mod} : \begin{array}{l} V \text{ is presented in finite degrees,} \\ \delta(V) \leq r, \text{ and } h^{\max}(V) \leq L-1 \end{array} \right\}.$$

*Proof.* We fix $L \geq 0$ and employ induction on $r$. For the base case $r = -1$, we first let $V \in \mathbf{Poly}_1(-1, L)$, that is, $\deg(V) \leq L - 1$. Then $V$ is torsion so $\delta(V) = -1$, and by Corollary 2.3 $V$ is presented in finite degrees with $h^{\max}(V) \leq L - 1$. Conversely, suppose $V$ is presented in finite degrees, $\delta(V) \leq -1$, and $h^{\max}(V) \leq L-1$. Then $V$ is torsion, so $\mathrm{H}^0_{\mathfrak{m}}(V) = V$ has degree $\leq L-1$.

For the inductive step, fix $r \geq 0$ and assume that we have

$$\mathbf{Poly}_1(r-1, L) = \left\{ U \in \mathbf{FI}\text{-}\mathsf{Mod} : \begin{array}{l} U \text{ is presented in finite degrees,} \\ \delta(U) \leq r-1, \text{ and } h^{\max}(U) \leq L-1 \end{array} \right\}.$$

Next, let $V \in \mathbf{Poly}_1(r, L)$, so by Definition 1.1, $h^0(V) \leq L-1$ and

$$\Delta V \in \mathbf{Poly}_1(r-1, L).$$

By the induction hypothesis, we conclude the following.

- $\Delta V$ is presented in finite degrees: it follows that $V$ is presented in finite degrees by Proposition 2.5.
- $\delta(\Delta V) \leq r-1$: this means $\Delta^r \Delta V = \Delta^{r+1} V$ is torsion, so $\delta(V) \leq r$.
- $h^{\max}(\Delta V) \leq L-1$: by part (2) of Corollary 2.7, we have $h^{\max}(V) \leq L-1$.

Conversely, let $V$ be an **FI**-module which is presented in finite degrees, $\delta(V) \leq r$, and $h^{\max}(V) \leq L-1$. We observe:

- $\Delta^{r+1} V = \Delta^r \Delta V$ is torsion, so $\delta(\Delta V) \leq r-1$.
- By Proposition 2.5, $\Delta V$ is presented in finite degrees.
- $h^{\max}(\Delta V) \leq L-1$ by part (1) of Corollary 2.7.

Therefore by the induction hypothesis, we get $\Delta V \in \mathbf{Poly}_1(r-1, L)$ and hence $V \in \mathbf{Poly}_1(r, L)$ by Definition 1.1. □

*Proof of Theorem A.* Immediate from Theorems 2.12 and 2.11. □

**Theorem 2.13.** *For every pair of integers $r \geq -1$, $M \geq 0$, we have*

$$\mathbf{Poly}_2(r, M) = \left\{ V \in \mathbf{FI}\text{-}\mathsf{Mod} : \begin{array}{l} V \textit{ is presented in finite degrees,} \\ \delta(V) \leq r, \textit{ and } \operatorname{reg}(V) \leq M-1 \end{array} \right\}.$$

*Proof.* We fix $M \geq 0$ and employ induction on $r$. For the base case $r = -1$, we first let $V \in \mathbf{Poly}_2(-1, M)$, that is, $\deg(V) \leq M-1$. Then $V$ is torsion so $\delta(V) = -1$, and by Corollary 2.3 $V$ is presented in finite degrees with $\operatorname{reg}(V) \leq M-1$. Conversely, suppose $V$ is presented in finite degrees, $\delta(V) \leq -1$, and $\operatorname{reg}(V) \leq M-1$. Then $V$ is torsion, so $\mathrm{H}^0_{\mathfrak{m}}(V) = V$ has degree $\leq M-1$ by Theorem 2.1.

For the inductive step, fix $r \geq 0$ and assume that for every $M' \geq 0$ we have

$$\mathbf{Poly}_2(r-1, M') = \left\{ U \in \mathbf{FI}\text{-}\mathsf{Mod} : \begin{array}{l} U \text{ is presented in finite degrees,} \\ \delta(U) \leq r-1, \text{ and } \operatorname{reg}(U) \leq M'-1 \end{array} \right\}.$$

Next, fix $M \geq 0$ and let $V \in \mathbf{Poly}_2(r, M)$, so by Definition 1.3, $h^0(V) \leq M-1$ and

$$\Delta V \in \mathbf{Poly}_2(r-1, \max\{0, M-1\}).$$

By the induction hypothesis, we conclude the following.

- $\Delta V$ is presented in finite degrees: it follows that $V$ is presented in finite degrees by Proposition 2.5.
- $\delta(\Delta V) \leq r-1$: this means $\Delta^r \Delta V = \Delta^{r+1} V$ is torsion, so $\delta(V) \leq r$.
- $\operatorname{reg}(\Delta V) \leq \max\{-1, M-2\}$.

Three possibilities arise:

(1) $V$ is $\mathrm{H}_0^{\mathbf{FI}}$-acyclic. Then $\operatorname{reg}(V) = -2 \leq M-1$.

(2) $V$ is not $\mathrm{H}_0^{\mathbf{FI}}$-acyclic and $\operatorname{crit}(V) = 0$. Here the definition of critical index immediately yields

$$\operatorname{reg}(V) = h^0(V) \leq M-1.$$

(3) $V$ is not $\mathrm{H}_0^{\mathbf{FI}}$-acyclic and $\operatorname{crit}(V) \geq 1$. Part (2) of Proposition 2.10 yields

$$1 \leq \operatorname{reg}(V) = \operatorname{reg}(\Delta V) + 1 \leq \max\{0, M-1\}.$$

Hence $M-1 > 0$ and $\operatorname{reg}(V) \leq M-1$.

Conversely, let $V$ be an **FI**-module which is presented in finite degrees, $\delta(V) \leq r$, and $\operatorname{reg}(V) \leq M-1$ (in particular, $h^0(V) \leq M-1$ by Theorem 2.1). We observe:

- $\Delta^{r+1} V = \Delta^r \Delta V$ is torsion, so $\delta(\Delta V) \leq r-1$.
- By Proposition 2.5, $\Delta V$ is presented in finite degrees.
- Either $V$ is $\mathrm{H}_0^{\mathbf{FI}}$-acyclic and hence so is $\Delta V$ (see the discussion in [6, Section 2.3]) and $\operatorname{reg}(\Delta V) = -2$, or $V$ is not $\mathrm{H}_0^{\mathbf{FI}}$-acyclic so that

$$0 \leq t_1(V) - 1 \leq \operatorname{reg}(V) \leq M-1$$

by [1, Corollary 2.9], and $\operatorname{reg}(\Delta V) \leq M-2$ by part (1) of Proposition 2.10. In both cases we have $\operatorname{reg}(\Delta V) \leq \max\{-1, M-2\}$.

Therefore the induction hypothesis yields $\Delta V \in \mathbf{Poly}_2(r-1, \max\{0, M-1\})$. We also have $h^0(V) \leq M-1$, so $V \in \mathbf{Poly}_2(r, M)$ by Definition 1.3. $\square$

*Proof of Theorem B*. Immediate from Theorems 2.13 and 2.11. $\square$

***Twisted homological stability.***

*Proof of Theorem C*. By Theorem 2.12, $V$ is presented in finite degrees, $\delta(V) \leq r$, and $h^{\max}(V) \leq L-1$. Hence by [1, Theorem 2.6], the triple $(V, L-1, r)$ satisfies [1, Hypothesis 1.2]. Noting that

$$r > \left\lceil \tfrac{L-1}{2} \right\rceil \quad \text{if and only if } L < 2r,$$

by [1, Theorem C] we have

$$\operatorname{reg}(V) \le \begin{cases} -2 & \text{if } L = 0, \\ L & \text{if } L \ge \max\{1, 2r\}, \\ r + \left\lfloor \frac{L+1}{2} \right\rfloor & \text{if } 1 \le L < 2r. \end{cases}$$

Thus by Theorem B, we have

$$V \in \begin{cases} \mathbf{Poly}_2(r, 0) & \text{if } L = 0, \\ \mathbf{Poly}_2(r, L+1) & \text{if } L \ge \max\{1, 2r\}, \\ \mathbf{Poly}_2\big(r, r + \left\lfloor \frac{L+1}{2} \right\rfloor + 1\big) & \text{if } 1 \le L < 2r. \end{cases}$$

Consequently by [18, Theorem A], for every $k \ge 0$ the map

$$\mathrm{H}_k(\mathfrak{S}_n; V_n) \to \mathrm{H}_k(\mathfrak{S}_{n+1}; V_{n+1})$$

is an isomorphism for

$$n \ge \begin{cases} 2k + r + 1 & \text{if } L = 0, \\ 2k + L + 2 & \text{if } L \ge \max\{1, 2r\}, \\ 2k + r + \left\lfloor \frac{L+1}{2} \right\rfloor + 2 & \text{if } 1 \le L < 2r \end{cases}$$

and a surjection for

$$n \ge \begin{cases} 2k + r & \text{if } L = 0, \\ 2k + L + 1 & \text{if } L \ge \max\{1, 2r\}, \\ 2k + r + \left\lfloor \frac{L+1}{2} \right\rfloor + 1 & \text{if } 1 \le L < 2r. \end{cases}$$

It remains to improve the bounds in the case $L \ge \max\{1, 2r - 1\}$ to

- $n \ge \max\{2k + 2r + 1, L\}$ for the isomorphism range,
- $n \ge \max\{2k + 2r, L\}$ for the surjection range.

To that end, we induct on $r$. For the base case $r = 0$, by [1, Theorem 2.11] there is an $\mathrm{H}_0^{\mathbf{FI}}$-acyclic $I$ with $\delta(I) \le 0$ and a map $V \to I$ which is an isomorphism in degrees $\ge L$. As $\Delta I$ is torsion but also is $\mathrm{H}_0^{\mathbf{FI}}$-acyclic, we have $\Delta I = KI = 0$, in other words $I \to I$ is an isomorphism. Thus $I_n$ is the same trivial $\mathfrak{S}_n$-representation for every $n \ge 0$ (namely the abelian group $I_0$ with the trivial $\mathfrak{S}_n$-action). Now by [15, Corollary 6.7], for every $k \ge 0$ the map

$$\mathrm{H}_k(\mathfrak{S}_n; I_0) \to \mathrm{H}_k(\mathfrak{S}_{n+1}; I_0)$$

is an isomorphism for $n \ge 2k$. Thus for every $k \ge 0$, the map

$$\mathrm{H}_k(\mathfrak{S}_n; V_n) \to \mathrm{H}_k(\mathfrak{S}_{n+1}; V_{n+1})$$

is an isomorphism for $n \ge \max\{2k, L\}$ (which is better than what the base case demands: an isomorphism for $n \ge \max\{2k+1, L\}$ and a surjection for $n \ge \max\{2k, L\}$).

Next, take $r \geq 1$ and assume that every **FI**-module $U \in \mathbf{Poly}_1(r, L-1)$, that is, by Theorem 2.12, every $U$ presented in finite degrees with $\delta(U) \leq r-1$ and $h^{\max}(U) \leq L-1$ satisfies[3] the following: for every $k \geq 0$ the map

$$\mathrm{H}_k(\mathfrak{S}_n; U_n) \to \mathrm{H}_k(\mathfrak{S}_{n+1}; U_{n+1})$$

is an isomorphism for

$$n \geq \max\{2k+2r-1, L\}$$

and a surjection for

$$n \geq \max\{2k+2r-2, L\}.$$

In particular by [6, Proposition 2.9, part (7)], this applies to

$$U := \mathrm{coker}(V \to \Sigma^L V).$$

In degrees $n \geq L$, writing $I := \Sigma^L V$, we have a short exact sequence

$$0 \to V_n \to I_n \to U_n \to 0$$

of $\mathfrak{S}_n$-modules, and the associated long exact sequence in $\mathrm{H}_*(\mathfrak{S}_n; -)$ maps to that of $\mathrm{H}_*(\mathfrak{S}_{n+1}; -)$. More precisely, suppressing the symmetric groups in the homology notation, there is a commutative diagram

$$\begin{tikzcd}
\mathrm{H}_{k+1}(I_n) \arrow[r] \arrow[d, "\mu_{k+1}"'] & \mathrm{H}_{k+1}(U_n) \arrow[r] \arrow[d, "\nu_{k+1}"'] & \mathrm{H}_k(V_n) \arrow[r] \arrow[d, "\lambda_k"'] & \mathrm{H}_k(I_n) \arrow[r] \arrow[d, "\mu_k"'] & \mathrm{H}_k(U_n) \arrow[d, "\nu_k"'] \\
\mathrm{H}_{k+1}(I_{n+1}) \arrow[r] & \mathrm{H}_{k+1}(U_{n+1}) \arrow[r] & \mathrm{H}_k(V_{n+1}) \arrow[r] & \mathrm{H}_k(I_{n+1}) \arrow[r] & \mathrm{H}_k(U_{n+1})
\end{tikzcd}$$

of abelian groups with exact rows. We observe:

- As $I$ is $\mathrm{H}_0^{\mathbf{FI}}$-acyclic and $\delta(I) \leq r$, then $I \in \mathbf{Poly}_1(r, 0)$ and so for every $k \geq 0$ the map $\mu_k$ is an isomorphism for $n \geq 2k+r+1$ and a surjection for $n \geq 2k+r$.
- By the induction hypothesis on $U$, for every $k \geq 0$ the map $\nu_k$ is an isomorphism for $n \geq \max\{2k+2r-1, L\}$ and a surjection for $n \geq \max\{2k+2r-2, L\}$.

Therefore we have:

- By the five-lemma, $\lambda_k$ is an isomorphism provided that $\nu_{k+1}$ and $\mu_k$ are isomorphisms, $\mu_{k+1}$ is surjective, and $\nu_k$ is injective: these are guaranteed in the range $n \geq \max\{2k+2r+1, L\}$ (noting $2(k+1)+r \leq 2(k+1)+2r-1$ because $r \geq 1$).
- By one of the four-lemmas, $\lambda_k$ is surjective provided that $\nu_{k+1}$ and $\mu_k$ are surjective, and $\nu_k$ is injective: these are guaranteed in the range $n \geq \max\{2k+2r, L\}$. $\square$

[3] Here the inequality $L \geq \max\{1, 2(r-1)-1\}$ is guaranteed as we are assuming $L \geq \max\{1, 2r-1\}$.

## 3. Application to congruence subgroups

*Proof of Theorem D.* By [1, Theorem 4.15], we have

- $\delta(\mathrm{H}_k(\mathrm{GL}_\bullet(R,I);\mathcal{A})) \le 2k$, and
- $\mathrm{reg}(\mathrm{H}_k(\mathrm{GL}_\bullet(R,I);\mathcal{A})) \le \begin{cases} 2s+3 & \text{if } k=1, \\ 4k+2s & \text{if } k \ge 2. \end{cases}$

We now consider the groupoid $\mathcal{G} := \mathrm{SL}^{\mathfrak{U}}(R/I)$ in order to follow the argument and notation in [12, proof of Theorem 1.4], with the following adjustment: Declare a new **FI**-module $V$ via

$$V_S := \begin{cases} \mathrm{H}_k(\mathrm{GL}_S(R,I);\mathcal{A}) & \text{if } |S| \ge n_0, \\ 0 & \text{if } |S| < n_0, \end{cases}$$

so that by part (1) of Remark 1.7, $V$ extends to a $U\mathcal{G}$-module. Note that as an **FI**-module by construction there is a short exact sequence

$$0 \to V \to \mathrm{H}_k(\mathrm{GL}_S(R,I);\mathcal{A}) \to T \to 0,$$

with $\deg(T) \le n_0 - 1 \le 2s+2$. Invoking Corollary 2.3, applying $\mathrm{H}^0_{\mathfrak{m}}$ the associated long exact sequence here yields

$$\begin{aligned} h^0(V) &\le h^0(\mathrm{H}_k(\mathrm{GL}_\bullet(R,I);\mathcal{A})), \\ h^1(V) &\le \max\{\deg T, h^1(\mathrm{H}_k(\mathrm{GL}_\bullet(R,I);\mathcal{A}))\}, \\ h^j(V) &= h^j(\mathrm{H}_k(\mathrm{GL}_\bullet(R,I);\mathcal{A})) \quad \text{if } j \ge 2. \end{aligned}$$

Thus if $h^j(V) \ge 0$ for $j \ne 1$, we have $h^j(\mathrm{H}_k(\mathrm{GL}_\bullet(R,I);\mathcal{A})) \ge 0$ and hence

$$h^j(V) + j \le h^j(\mathrm{H}_k(\mathrm{GL}_\bullet(R,I);\mathcal{A})) + j \le \begin{cases} 2s+3 & \text{if } k=1, \\ 4k+2s & \text{if } k \ge 2 \end{cases}$$

by Theorem 2.1 applied to $\mathrm{H}_k(\mathrm{GL}_\bullet(R,I);\mathcal{A})$. If $h^1(V) \ge 0$, there are two possibilities:

- $h^1(\mathrm{H}_k(\mathrm{GL}_\bullet(R,I);\mathcal{A}) \le \deg T$. Then $h^1(V) + 1 \le \deg T + 1 \le 2s+3$.
- $h^1(\mathrm{H}_k(\mathrm{GL}_\bullet(R,I);\mathcal{A}) > \deg T$. Then $h^1(\mathrm{H}_k(\mathrm{GL}_\bullet(R,I);\mathcal{A})) \ge 0$ and hence by Theorem 2.1,
$$h^1(V) + 1 \le h^1(\mathrm{H}_k(\mathrm{GL}_\bullet(R,I);\mathcal{A})) + 1 \le \begin{cases} 2s+3 & \text{if } k=1, \\ 4k+2s & \text{if } k \ge 2. \end{cases}$$

Applying Theorem 2.1 to $V$ now, we get

$$\mathrm{reg}(V) \le \begin{cases} 2s+3 & \text{if } k=1, \\ 4k+2s & \text{if } k \ge 2. \end{cases}$$

By [6, Proposition 3.3] we also have $\delta(V) \le 2k$. Thus by Theorem B and Remark 1.4, and in the sense of [12, Definition 2.40], $V$ has polynomial degree $\le 2k$

$$\text{in ranks} > \begin{cases} 2s+3 & \text{if } k=1, \\ 4k+2s & \text{if } k \ge 2. \end{cases}$$

(1) By [12, Remark 2.42], $V$ has the same polynomial degree and rank bounds as a $U\mathcal{G}$-module.

(2) Noting that $\mathbf{st\text{-}rank}(R/I) \le s$ as well [2, Lemma 4.1], by [12, Proposition 2.13], the category $U\mathcal{G}$ satisfies $\mathbf{H3}(2, s+1)$.

Therefore by [12, Theorem 3.11], we have

$$\widetilde{\mathrm{H}}_i^{\mathcal{G}}(V)_n = 0 \quad \text{for } n > \begin{cases} \max\{2s+i+4,\, s+2i+3\} & \text{if } k=1, \\ \max\{4k+2s+i+1,\, 2k+s+2i+1\} & \text{if } k \ge 2, \end{cases}$$

and in particular

$$\widetilde{\mathrm{H}}_{-1}^{\mathcal{G}}(V)_n = 0 \quad \text{for } n > \begin{cases} 2s+3 & \text{if } k=1, \\ 4k+2s & \text{if } k \ge 2, \end{cases}$$

$$\widetilde{\mathrm{H}}_0^{\mathcal{G}}(V)_n = 0 \quad \text{for } n > \begin{cases} 2s+4 & \text{if } k=1, \\ 4k+2s+1 & \text{if } k \ge 2. \end{cases}$$

Noting that the definitions of $\widetilde{\mathrm{H}}_*^{\mathcal{G}}$ in [13, Definition 3.14] and [12, Definition 2.9] are consistent with each other, the vanishing above corresponds to a coequalizer diagram of the form

$$\mathrm{Ind}_{\mathcal{G}_{n-2}}^{\mathcal{G}_n} V_{n-2} \rightrightarrows \mathrm{Ind}_{\mathcal{G}_{n-1}}^{\mathcal{G}_n} V_{n-1} \to V_n$$

of $\mathbb{Z}\mathcal{G}_n$-modules whenever

$$n \ge \begin{cases} 2s+5 & \text{if } k=1, \\ 4k+2s+2 & \text{if } k \ge 2 \end{cases}$$

by [13, Remark 3.16]. In this range, we have $n-2 \ge 2s+3 \ge n_0$, so that

$$V_j = \mathrm{H}_k(\mathrm{GL}_j(R, I); \mathcal{A}) \quad \text{for } j \in \{n-2, n-1, n\}. \qquad \square$$

## References


[1] C. Bahran, "Regularity and stable ranges of FI-modules", preprint, 2022. arXiv 2203.06698

[2] H. Bass, "$K$-theory and stable algebra", *Inst. Hautes Études Sci. Publ. Math.* **22** (1964), 5–60. MR Zbl

[3] T. Church and J. S. Ellenberg, "Homology of FI-modules", *Geom. Topol.* **21**:4 (2017), 2373–2418. MR Zbl

[4] T. Church, J. S. Ellenberg, B. Farb, and R. Nagpal, "FI-modules over Noetherian rings", *Geom. Topol.* **18**:5 (2014), 2951–2984. MR Zbl

[5] T. Church, J. S. Ellenberg, and B. Farb, "FI-modules and stability for representations of symmetric groups", *Duke Math. J.* **164**:9 (2015), 1833–1910. MR Zbl

[6] T. Church, J. Miller, R. Nagpal, and J. Reinhold, "Linear and quadratic ranges in representation stability", *Adv. Math.* **333** (2018), 1–40. MR Zbl

[7] P. L. Clark, "A note on Euclidean order types", *Order* **32**:2 (2015), 157–178. MR Zbl

[8] W. L. Gan, "A long exact sequence for homology of FI-modules", *New York J. Math.* **22** (2016), 1487–1502. MR Zbl

[9] W. L. Gan, "On the negative-one shift functor for FI-modules", *J. Pure Appl. Algebra* **221**:5 (2017), 1242–1248. MR Zbl

[10] A. J. Hahn and O. T. O'Meara, *The classical groups and $K$-theory*, Grundlehren der Math. Wissenschaften **291**, Springer, 1989. MR Zbl

[11] L. Li and E. Ramos, "Depth and the local cohomology of $\mathbb{FI}_G$-modules", *Adv. Math.* **329** (2018), 704–741. MR Zbl

[12] J. Miller, P. Patzt, and D. Petersen, "Representation stability, secondary stability, and polynomial functors", preprint, 2019. arXiv 1910.05574

[13] J. Miller, P. Patzt, and J. C. H. Wilson, "Central stability for the homology of congruence subgroups and the second homology of Torelli groups", *Adv. Math.* **354** (2019), art. id. 106740. MR Zbl

[14] R. Nagpal, S. V. Sam, and A. Snowden, "Regularity of FI-modules and local cohomology", *Proc. Amer. Math. Soc.* **146**:10 (2018), 4117–4126. MR Zbl

[15] M. Nakaoka, "Decomposition theorem for homology groups of symmetric groups", *Ann. of Math.* (2) **71** (1960), 16–42. MR Zbl

[16] P. Patzt, "Central stability homology", *Math. Z.* **295**:3-4 (2020), 877–916. MR Zbl

[17] A. Putman, "Stability in the homology of congruence subgroups", *Invent. Math.* **202**:3 (2015), 987–1027. MR Zbl

[18] A. Putman, "A new approach to twisted homological stability, with applications to congruence subgroups", preprint, 2021. arXiv 2109.14015

[19] A. Putman and S. V. Sam, "Representation stability and finite linear groups", *Duke Math. J.* **166**:13 (2017), 2521–2598. MR Zbl

[20] E. Ramos, "On the degree-wise coherence of $\mathbb{FI}_G$-modules", *New York J. Math.* **23** (2017), 873–895. MR Zbl

[21] O. Randal-Williams and N. Wahl, "Homological stability for automorphism groups", preprint, 2015. arXiv 1409.3541v3

[22] O. Randal-Williams and N. Wahl, "Homological stability for automorphism groups", *Adv. Math.* **318** (2017), 534–626. MR Zbl


Received October 11, 2022. Revised January 10, 2023.

CIHAN BAHRAN
DEPARTMENT OF MATHEMATICS
BOĞAZIÇI UNIVERSITY
ISTANBUL
TURKEY
cihanbahran@gmail.com

PACIFIC JOURNAL OF MATHEMATICS
Vol. 324, No. 2, 2023
https://doi.org/10.2140/pjm.2023.324.227

# A LIFT OF WEST'S STACK-SORTING MAP TO PARTITION DIAGRAMS

JOHN M. CAMPBELL

**We introduce a lifting of West's stack-sorting map $s$ to partition diagrams, which are combinatorial objects indexing bases of partition algebras. Our lifting $\mathscr{S}$ of $s$ is such that $\mathscr{S}$ behaves in the same way as $s$ when restricted to diagram basis elements in the order-$n$ symmetric group algebra as a diagram subalgebra of the partition algebra $\mathscr{P}_n^\xi$. We then introduce a lifting of the notion of 1-stack-sortability, using our lifting of $s$. By direct analogy with Knuth's famous result that a permutation is 1-stack-sortable if and only if it avoids the pattern 231, we prove a related pattern-avoidance property for partition diagrams, as opposed to permutations, according to what we refer to as *stretch-stack-sortability*.**

## 1. Introduction

For a permutation $p$ in the symmetric group $S_n$, we write

$$p = LnR, \tag{1}$$

letting $p$ be denoted as a string or tuple given by the entries of the bottom row of the two-line notation for $p$. We then let *West's stack-sorting map* $s$ [6; 7; 8; 9; 10; 11; 12; 13; 14] (compare [27]) be defined recursively so that

$$s(p) = s(L)\, s(R)\, n \tag{2}$$

and so that $s$ maps permutations to permutations and sends the empty permutation to itself. Our notation and terminology concerning this mapping are mainly based on references such as [6; 7; 8; 9; 10; 11; 12; 13; 14]. In this article, we introduce a lifting of $s$ so as to allow combinatorial objects known as *partition diagrams* as input.

The problem of generalizing West's stack-sorting map has been considered in a number of different contexts. Notably, the stack-sorting map $s$ has been

*MSC2020:* primary 05A05; secondary 05E16.
*Keywords:* stack-sorting, partition diagram, permutation, permutation pattern, partition monoid.

generalized to Coxeter groups [12] and to words [13], and Cerbai, Claesson, and Ferrari generalized

$$s\colon S_n \to S_n \tag{3}$$

to a function $s_\sigma\colon S_n \to S_n$ for a permutation pattern $\sigma$ [5], so that $s = s_{21}$, with a related generalization of $s$ having been given by Defant and Zheng in [14]. Since $s$ is defined on permutations, it is natural to consider generalizing this mapping using "permutation-like" combinatorial objects. The bases of the partition algebra $\mathscr{P}_n^\xi$ generalize the bases of the order-$n$ symmetric group algebra in ways that are of interest from both a combinatorial and an algebraic perspective, and this leads us to consider how partition diagrams, which index the bases of $\mathscr{P}_n^\xi$, may be used to generalize West's stack-sorting map. We generalize, in this article, $s\colon S_n \to S_n$ to a mapping $\mathscr{S}\colon \mathscr{P}_n \to \mathscr{P}_n$ from the partition monoid $\mathscr{P}_n$ to itself such that $\mathscr{S}$ restricted to permuting diagrams behaves in the same way as $s$. We then apply our lifting $\mathscr{S}$ to determine an analogue of a famous result due to Knuth [20, Section 2.2.1] on 1-stack-sortable permutations.

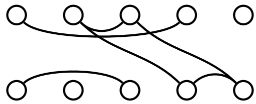
The representation theory of $\mathscr{P}_n^\xi$ is intimately linked with that for the $n!$-dimensional symmetric group algebra. Indeed, there is so much about the representation theory of $\mathscr{P}_n^\xi$ and associated combinatorial properties of $\mathscr{P}_n^\xi$ that are directly derived from or otherwise based on the representation theory of the algebra $\operatorname{span}(S_n)$ and associated combinatorics [2; 3; 17; 18; 22; 23; 25]. The Schur–Weyl duality given by how the bases of $\mathscr{P}_n^\xi$ generalize the bases of $\operatorname{span}(S_n)$ is of importance in both combinatorial representation theory and in the areas of statistical mechanics in which partition algebras had originally been defined by Martin [21; 22; 23; 24] and Jones [19]. The foregoing considerations are representative of the extent to which partition diagrams generalize permutations in a way that is of much significance in both mathematics and physics. This motivates our lifting West's stack-sorting map so as to allow partition diagrams, in addition to permutations, as input.

***Preliminaries.*** To be consistent with the notation in references as in [6; 7; 8; 9; 10; 11; 12; 13; 14] for indexing symmetric groups and maximal elements in permutations, as in (2) and (3), we let the order of a given partition algebra/monoid be denoted as in the order of the symmetric group shown in (3). In particular, following [16], we let the partition monoid of a given order be denoted as $\mathscr{P}_n$, and we let the corresponding partition algebra with a complex parameter $\xi$ be denoted as $\mathscr{P}_n^\xi$. These structures are defined as follows.

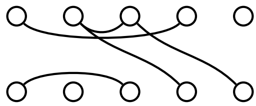
We let $\mathscr{P}_n$ consist of set-partitions $\mu$ of $\{1, 2, \dots, n, 1', 2', \dots, n'\}$ that we denote with a two-line notation by analogy with permutations, by aligning nodes labeled with $1, 2, \dots, n$ into an upper row and nodes labeled with $1', 2', \dots, n'$ into a bottom row, and by forming any graph $G$ such that the set of components of $G$

equals $\mu$. Graphs of this form, denoted in the manner we have specified, are referred to as *partition diagrams*, and two such partition diagrams are considered to be the same if the connected components are the same in both cases. The components of the graph $G$ given as before are referred to as *blocks*.

**Example 1.** The set-partition $\{\{1, 4\}, \{2, 3, 4', 5'\}, \{5\}, \{1', 3'\}, \{2'\}\}$ may be denoted as



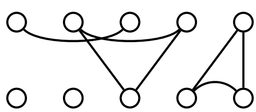
or, equivalently, as



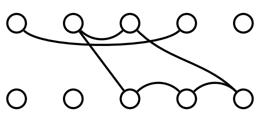
Following [18], we let the underlying field for partition algebras and symmetric group algebras be $\mathbb{C}$, as $\mathbb{C}$ being algebraically closed is of direct relevance in terms of the study of the semisimple structure for partition algebras [18]. We may let the symmetric group algebra of order $n$ be denoted by taking the linear span $\mathrm{span}(S_n)$, or, more explicitly, $\mathrm{span}_{\mathbb{C}}\{\sigma : \sigma \in S_n\}$, of the symmetric group $S_n$.

For two partition diagrams $d_1$ and $d_2$, we place $d_1$ on top of $d_2$ so that the bottom nodes of $d_1$ overlap with the top nodes of $d_2$, then we remove the central row in this concatenation $d_1 * d_2$ in such a way so as to preserve the relation given by topmost nodes being in the same component as bottommost nodes in $d_1 * d_2$. We then let $d_1 \circ d_2$ denote the graph thus obtained from this concatenation. This is the underlying multiplicative operation for partition monoids.

**Example 2.** Borrowing an example from [16], we let $d_1$ be as in Example 1, and we let $d_2$ be as below:



We may verify that the monoid product $d_1 \circ d_2$, in this case, is as below [16]:



For a complex parameter $\xi$, we endow the $\mathbb{C}$-span of $\mathscr{P}_n$ with a multiplicative binary operation as $d_1 d_2 = \xi^{\ell} d_1 \circ d_2$, where $\ell$ denotes the number of components contained entirely in the middle row of $d_1 * d_2$. By extending this binary operation linearly, this gives us the underlying multiplicative operation for the structure known as the partition algebra, which is denoted as $\mathscr{P}_n^{\xi}$. The set of all elements in $\mathscr{P}_n$, as elements in $\mathscr{P}_n^{\xi}$, is referred to as the *diagram basis* of $\mathscr{P}_n^{\xi}$.

Although the algebraic structure of $\mathscr{P}_n^\xi$ will not be used directly in this article, it is useful to define $\mathscr{P}_n^\xi$ explicitly, since we are to heavily make use of an algebra homomorphism defined on partition algebras and introduced in [4], and it is often convenient to use notation and terminology associated with $\mathscr{P}_n^\xi$ or its algebraic structure, e.g., by referring to the diagram basis of $\mathscr{P}_n^\xi$.

For a partition diagram $\pi$, the *propagation number* of $\pi$ refers to the number of blocks in $\pi$ containing at least one vertex in the top row and at least on vertex in the bottom row. So, there is a clear bijection between the set of all permutations in $S_n$ and the set of all diagrams in $\mathscr{P}_n$ that are of propagation number $n$. In our lifting West's stack-sorting map to the partition monoid, to be consistent with two-line notation for permutations and with (1), we let a permutation

$$p\colon \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\} \tag{4}$$

be in correspondence with the partition diagram $\pi$ given by the set-partition

$$\big\{\{1, p(1)'\}, \{2, p(2)'\}, \ldots, \{n, p(n)'\}\big\}, \tag{5}$$

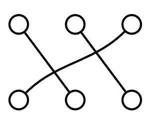
although it is common to instead let a permutation $p$ be mapped to the partition diagram obtained by reflecting $\pi$ vertically.

## 2. A lift of West's stack-sorting map

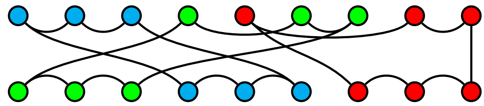
As Defant and Kravitz explain in [13], there is a matter of ambiguity in the problem of lifting the mapping $s$ so as to allow words involving repeated characters. We encounter similar kinds of problems in terms of the problem of lifting $s$ so as to allow partition diagrams, as opposed to permutations written as permutation diagrams, as the argument. This is illustrated below.

**Example 3.** According to the embedding indicated in (5), we let the permutation $\left(\begin{smallmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{smallmatrix}\right)$ be written as

(6)



If we visualize the block $\{2, 3'\}$ being removed and placed on the right of a resultant configuration, by analogy with how the permutation (1) gets mapped to the right-hand side of (2) under the application of West's stack-sorting map, there is a clear way how this removal "splits" the permutation diagram in (6) into a left and a right configuration by direct analogy with (1), and part of the purpose of our procedure in the next subsection is to formalize this idea in a way that may be extended to all partition diagrams. In contrast, if we remove, for example, the rightmost block of

(7)


where the above coloring is to distinguish the blocks in the partition diagram depicted, then the block highlighted in green is not strictly to the left of the rightmost block, since there are green nodes to the left and right of a red node in the upper row.

To lift the recursive definition for $s$ indicated in (2) so as to allow members of $\mathscr{P}_n$ as the input for an extension or lifting of $s$, and to deal with the matter of ambiguity explained in Example 3, we would want to mimic (2) by allowing the possibility of "middle" configurations, as opposed to the left-right dichotomy indicated in (1) and (2). We formalize this idea below.

***A procedure for stack-sorting partition diagrams.*** Define $\mathscr{S}_n \colon \mathscr{P}_n \to \mathscr{P}_n$ according to the following procedure, for an arbitrary partition diagram $\pi$ in $\mathscr{P}_n$. We order the bottom nodes of a given partition diagram in $\mathscr{P}_n$ in the natural way, with $1' < 2' < \cdots < n'$. For the sake of convenience, we may write $\mathscr{S} = \mathscr{S}_n$.

(1) If there are no propagating blocks in $\pi$, skip the below steps involving propagating blocks.

(2) Take the largest bottom node of $\pi$ that is part of a propagating block $B$. This propagating block separates $\pi$ into three (possibly empty) classes of configurations according to how the top nodes of $\pi$ are separated by removing $B$. Explicitly, we define $\mathscr{L}$, $\mathscr{M}_1$, $\mathscr{M}_2, \ldots, \mathscr{M}_\mu$, and $\mathscr{R}$ as follows, by direct analogy with (1). We may denote $\mathscr{L}$, $\mathscr{M}_i$, and $\mathscr{R}$ as diagrams in $\mathscr{P}_n$. The blocks of $\mathscr{L}$ (resp. $\mathscr{R}$) consist of one of the following:

- Any blocks of $\pi$ with upper nodes that are all strictly to the left (resp. right) of all of the upper nodes of $B$.
- Any nonpropagating blocks of $\pi$ on the bottom row of $\pi$ with nodes that are all strictly to the left (resp. right) of all of the lower nodes of $B$.
- Singleton blocks.

The blocks of expressions of the form $\mathscr{M}_i$ consist either of

- any blocks of $\pi$ that do not satisfy either of the first two bullet points listed above, or
- singleton blocks.

We order $\mathscr{M}_1 < \mathscr{M}_2 < \cdots < \mathscr{M}_\mu$ according to the ordering of the minimal elements, subject the ordering whereby $1' < 2' < \cdots < n' < 1 < 2 < \cdots < n$. We then let $B'$ denote the partition diagram obtained from $B$ by adding any singleton nodes so as to form a partition diagram in $\mathscr{P}_n$. With this setup, we write

$$\mathscr{S}(\pi) = \mathscr{S}(\mathscr{L}) \odot \mathscr{S}(\mathscr{M}_1) \odot \cdots \odot \mathscr{S}(\mathscr{M}_\mu) \odot \mathscr{S}(\mathscr{R}) \odot B', \tag{8}$$

where the associative binary relation $\odot$ is to later be defined.

(3) Repeatedly apply the above step wherever possible, i.e., to the expressions in or derived from (8) given by $\mathscr{S}$ evaluated at a partition diagram.

(4) The above steps yield a $\odot$-product of expressions of the forms

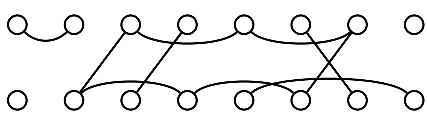
$$\mathscr{S}(N_1), \ldots, \mathscr{S}(N_{m_1}) \quad \text{and} \quad B'_1, \ldots, B'_{m_2}, \tag{9}$$

not necessarily in this order, for nonpropagating diagrams $N_i$, and where each expression of the form $B'_j$ is a partition diagram with exactly one propagating block and with singleton blocks anywhere else. The factors of the aforementioned $\odot$-product indicated in (9) are ordered in the following way: $B'_j$ is the $j$-th factor of the form $B'_\kappa$ appearing in this $\odot$-product, and $\mathscr{S}(N_i)$ is the $i$-th factor of the form $\mathscr{S}(N_\kappa)$ appearing in this $\odot$-product. The operation $\odot$ indicates that the following is to be applied. We label the top nodes in the propagating block in $B'_1$ from left to right with consecutive integers starting with 1, and we then label the top nodes in the propagating block in $B'_2$ with consecutive integers (starting with 1 plus the number of top nodes in the propagating block in $B'_1$), and we continue in this manner. We then continue with this labeling, by labeling any nonpropagating blocks of size greater than 1 in the top row of the $N_i$-expressions, in order of the nodes as they appear among consecutive $N_i$-diagrams. If there are any unused labels for the top row, label singleton blocks in the upper row with these leftover labels. However, for the bottom nodes of any nonsingleton blocks from $\pi$, we let these bottom nodes keep their original labelings (and if necessary we add in singleton blocks in a bottom row to form a partition diagram based on the preceding steps).

(5) As indicated above, the above steps produce an element in $\mathscr{P}_n$. We set $\mathscr{S}(\pi)$ to be this element.

**Example 4.** Let $\pi$ denote the partition diagram

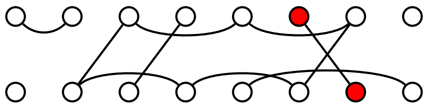
(10)



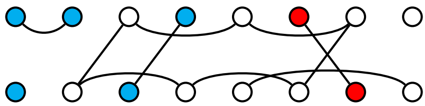
in $\mathscr{P}_8$ corresponding to the set-partition

$$\{\{1, 2\}, \{3, 5, 7, 2', 4', 6'\}, \{4, 3'\}, \{6, 7'\}, \{8\}, \{1'\}, \{5', 8'\}\}.$$

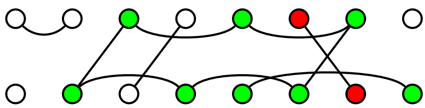
The largest bottom node of $\pi$ that is part of a propagating block $B$, in this case, is $7'$, and $B$ is $\{6, 7'\}$. For the sake of clarity, we highlight this block in the following manner:



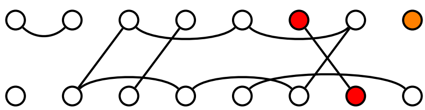
The blocks of $\mathscr{L}$ are highlighted in cyan as below (with the propagating block $B$ again highlighted in a different color for the sake of clarity).



We find that there is only one expressions of the form $\mathscr{M}_i$, and we may denote this expression as $\mathscr{M}$. The blocks of $\mathscr{M}$ are highlighted in green as below.

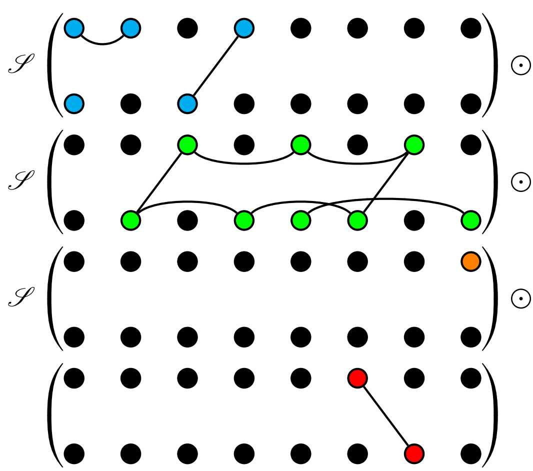


We see that $\mathscr{R}$ consists of only one block, which is highlighted in orange, as below.



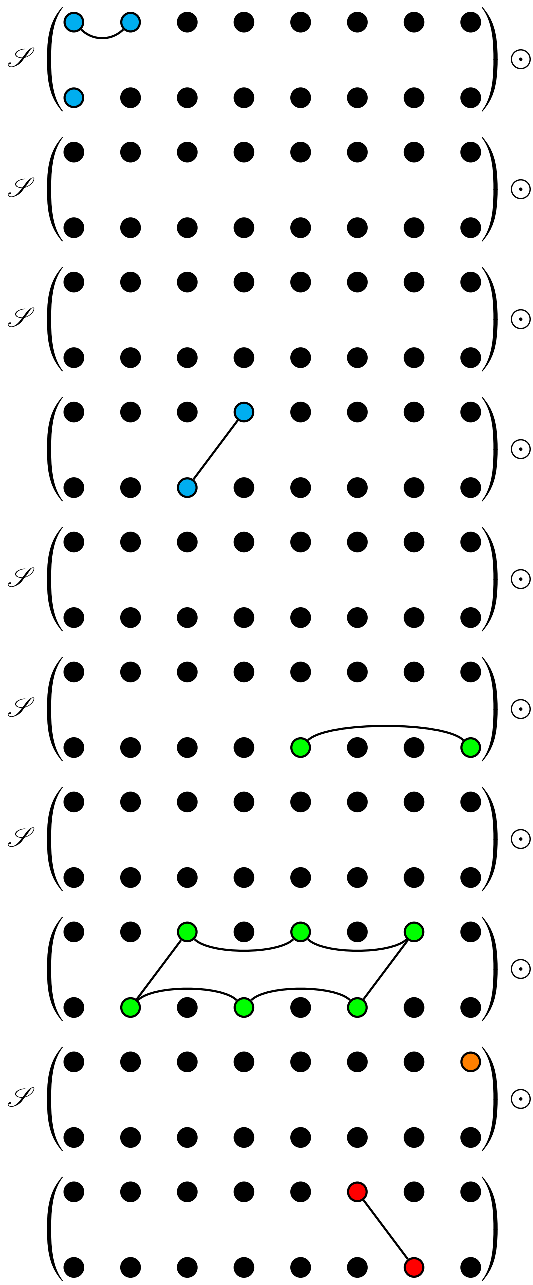
So, according to our lifting of West's stack-sorting map, we obtain the following, where singleton nodes colored black indicate that these singleton nodes have been "added in" according to the above given procedure.

$\mathscr{S}(\pi) =$



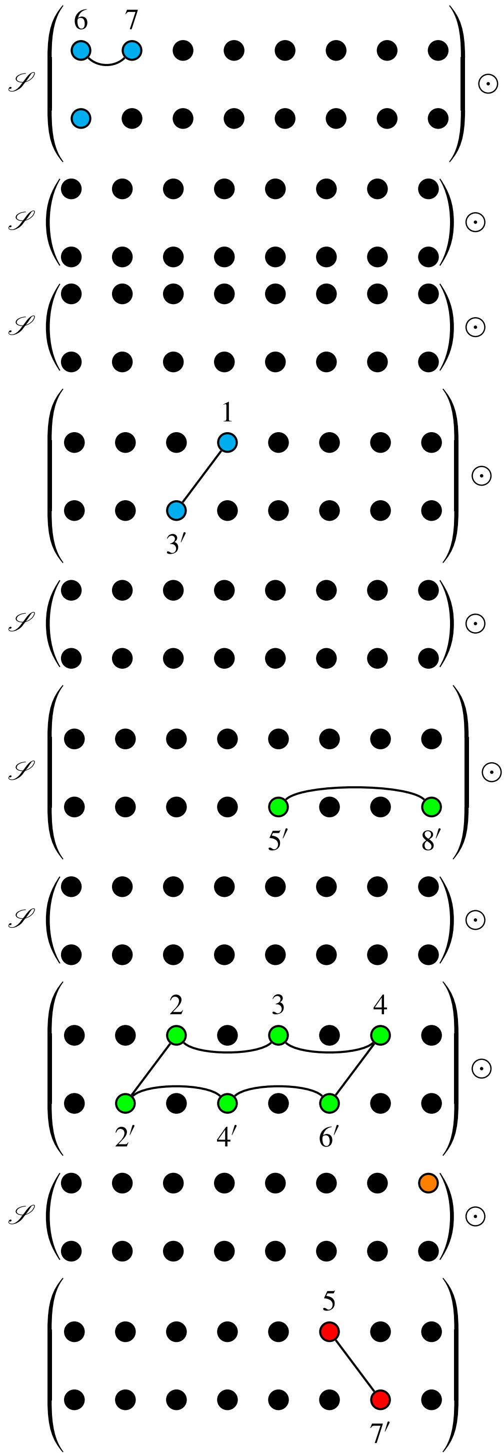
Now, let us repeat the given procedure to the first $\mathscr{S}$-factor on the right-hand side of the above equality, and then to the resultant $\mathscr{S}$-factor involving an argument with a block of size 6. This results in the following equality:

$\mathscr{S}(\pi) =$

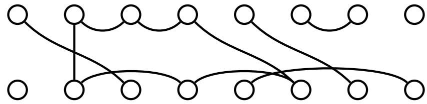


So, we have determined a $\odot$-product of expressions of the forms indicated in (9). Now, apply the labeling indicated as follows, according to our procedure.

$\mathscr{S}(\pi) =$



So, this shows us how the partition diagram in (10) gets mapped to



according to our sorting map.

***Sorting permuting diagrams.*** Since we claim that our sorting map $\mathscr{S}\colon \mathscr{P}_n \to \mathscr{P}_n$ lifts (3), it would be appropriate to formalize and prove this claim, as in Theorem 5 below and our proof of this theorem. We proceed to briefly review some preliminaries concerning Theorem 5.

Our convention for mapping permutations to permutation diagrams, as indicated in (5), is to be used consistently throughout our article. Again, this convention is consistent with the notation for West's stack-sorting map indicated in (1), since it mimics two-line notation for permutations. We are to extend, as below, this embedding so as to be applicable to expressions as in $L$ and $R$ in the permutation decomposition shown in (1).

The *rook algebra* is a subalgebra of $\mathscr{P}_n^{\xi}$ that is spanned by partial permutations. Partial permutations are diagrams that consist of blocks of size 1 and blocks of size 2 that consist of a vertex in the top row and a vertex in the bottom row [17]. Following [28], the expression $\mathrm{Rd}_k$ denotes the set of all rook $k$-diagrams. Similarly, for each $r \in \mathbb{Z}$ satisfying $0 \leq r \leq k$, the expression $\mathrm{Rd}_k[r]$ denotes the set of rook $k$-diagrams with precisely $r$ singleton vertices in each row. Given a permutation decomposition of the form indicated in (1), we can identify $L$ (resp. $R$) with a partial permutation such that the primed elements in any 2-blocks in this partial permutation are the primed versions of any numbers in $L$ (resp. $R$) and in such a way so as to agree with the two-line notation indicated in (1). Explicitly, if $L$ is empty, we let it be mapped to the partition diagram in $\mathscr{P}_n$ consisting of singleton blocks, and if we write

$$L = l_1 l_2 \cdots l_{\ell(L)},$$

we may identify $L$ with the partition diagram with 2-blocks of the forms

$$\{p^{-1}(l_1), l_1'\}, \{p^{-1}(l_2), l_2'\}, \ldots, \{p^{-1}(l_{\ell(L)}), l_{\ell(L)}'\}$$

and with singleton blocks everywhere else, and similarly for $R$. This agrees with our convention indicated in (5) for embedding $S_n$ into $\mathscr{P}_n$.

Again with reference to the permutation decomposition in (1) we may identify the expression $n$ with a partial permutation and in a similar fashion as above, with the understanding that this expression is part of the concatenation in (1). Explicitly, we would identify $n$ with the partition diagram given by the set-partition

$$\{\{p^{-1}(n), n'\}\} \cup \tag{11}$$

$$\{\{x\} : x \in \mathbb{N},\ 1 \leq x \leq n,\ x \neq p^{-1}(n)\} \cup \tag{12}$$

$$\{\{x\} : x \in \mathbb{N},\ 1 \leq x \leq n,\ x \neq n\}. \tag{13}$$

As illustrated in Example 4, for permutation diagrams $\pi_1$ and $\pi_2$, if $\pi_2$ consists entirely of singleton nodes then $\mathscr{S}(\pi_1) \odot \mathscr{S}(\pi_2) = \mathscr{S}(\pi_2) \odot \mathscr{S}(\pi_1) = \mathscr{S}(\pi_1)$. This algebraic property is to be used in our proof of Theorem 5.

**Theorem 5.** *Let $p$ be a permutation in $S_n$, and let $\pi$ denote the corresponding partition diagram according to (5). Then $\mathscr{S}(\pi)$ equals the partition diagram corresponding to $s(p)$.*

*Proof.* As above, we let $p \in S_n$, writing $p\colon \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$. As above, we let $\pi$ denote the partition diagram corresponding to (5), i.e., so that the set of blocks of $\pi$ is (5). With respect to the notation in (8), the expression $\mathscr{S}(\mathscr{L})$ (resp. $\mathscr{S}(\mathscr{R})$) is equal to $\mathscr{S}$ evaluated at a diagram obtained from $\pi$ by taking any blocks consisting of a bottom node that is labeled the primed version of a number to the left (resp. right) of $n$ in the sense indicated in (1), i.e., a number in $L$ (resp. $R$) according to the notation in (1) for a permutation $p \in S_n$. Since $\pi$ is a permutation diagram, any $\mathscr{M}$-expression consists entirely of singleton nodes. So, the product in (8) reduces to

$$\mathscr{S}(\pi) = \mathscr{S}(\mathscr{L}) \odot \mathscr{S}(\mathscr{R}) \odot B'. \tag{14}$$

As indicated above, $\mathscr{L}$ (resp. $\mathscr{R}$) is precisely the partition diagram given by embedding $L$ (resp. $R$) into $\mathscr{P}_n$. Similarly, $B'$ is precisely the partition diagram given by embedding the factor $n$ in (1), in the manner indicated in (11)–(13). So, letting it be understood that the factors in (1) and the left-hand side of (1) may be identified with their respective embeddings, we find that the $\odot$-product in (14) may be written as

$$\mathscr{S}(p) = \mathscr{S}(L) \odot \mathscr{S}(R) \odot n.$$

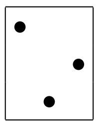
By comparing both sides of the above decomposition with both sides of the decomposition in (2), an inductive argument provides us with the desired result. $\square$

Since we have lifted West's stack-sorting map so as to allow partition diagrams in addition to permuting diagrams as input, this leads us to consider how fundamental properties concerning the $s$-map (3) may be "translated" according to our lifting. In particular, we are led to desire to generalize Knuth's classic result that a permutation is 1-stack-sortable if and only if it avoids the pattern 231.

The intricate combinatorial structures and behaviors associated with our lifting $\mathscr{S}$ of West's stack-sorting map are investigated in Section 3 below. Such investigations are inspired by the extent of mathematical interest concerning the beautiful combinatorics, at both a structural and enumerative level, associated with an important variant of the $s$-map referred to as *pop-stack sorting* for permutations, with reference to the work of Asinowski et. al. [1].

## 3. Pattern avoidance

We again let $p \in S_n$ be written as a mapping in the manner indicated in (4). In the context of the study of pattern avoidance in permutations, it is often desirable to denote $p$ using an $n \times n$ grid in the Cartesian plane. For example, if we begin by letting a permutation $p \in S_3$ be denoted in the manner indicated in (1), writing $p = 312$, we may let this be denoted using the $n \times n$ grid below.



More generally, we may denote a permutation

$$p = p(1)p(2)\cdots p(n)$$

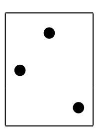
with an $n \times n$ array such that an $(i, j)$-entry is nonempty if and only if $(i, j)$ is of the form $(k, n - p^{-1}(k) + 1)$ for some index $k$, and where an $(k, n - p^{-1}(k) + 1)$-entry is highlighted in some way, as above. We say that a permutation $p = p(1)p(2)\cdots p(n)$ contains the pattern 231 if there exist indices $k_1$, $k_2$, and $k_3$ such that $k_1 < k_2 < k_3$ and such that the entries

(15) $$(k_1, n - p^{-1}(k_1) + 1),$$

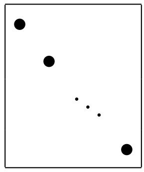
(16) $$(k_2, n - p^{-1}(k_2) + 1),$$

(17) $$(k_3, n - p^{-1}(k_3) + 1),$$

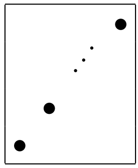
satisfy the following in the $n \times n$ array we use to denote $p$: Point (15) is strictly below (16) and is strictly above point (17). In other words, the $n \times n$ array corresponding to $p$ contains a pattern that is equivalent, in the sense that we have specified, to the following configuration:



A permutation is said to *avoid* the pattern 231 if it does not contain this pattern. In a similar fashion, we may characterize or define a permutation that avoids the pattern 12 as a *decreasing* permutation, i.e., a permutation of the following form:



Correspondingly, an *increasing* permutation is of the form shown below, i.e., it is an identity permutation:



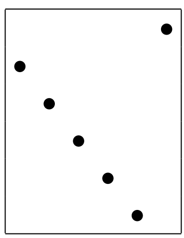
A permutation $p$ is said to be *$t$-stack-sortable* if

$$\underbrace{s\circ s\circ\cdots\circ s}_{t}(p)$$

is increasing; see [9; 13] for related research that has inspired much about this article.

The concept of a 231-avoiding permutation is of considerable interest for the purposes of this article, so it is worthwhile to illustrate a permutation of this form. In this regard, we see that the permutation corresponding to

(18)



avoids the pattern 231. So, according to Knuth's characterization of 1-stack-sortable permutations, we would expect the permutation illustrated in (18) to be 1-stack-sortable. It is worthwhile for our purposes to illustrate this.

**Example 6.** Being consistent with the notation in (1), the permutation $p \in S_6$ depicted in (18) may be written as 543216. According to the recursion shown in (2) for West's stack-sorting map, we obtain

$$\begin{aligned} s(p) = s(543216) = s(54321)6 = s(4321)56 \\ = s(321)456 \\ = s(21)3456 = s(1)23456 = 123456. \end{aligned}$$

So, since $s(p)$ is the identity permutation in $S_6$, we have that $p$ is 1-stack-sortable.

The main goal of our current section is to generalize the following groundbreaking result due to Knuth, which, as indicated in [9], was the starting point for the study of both stack-sorting and pattern avoidance in permutations.

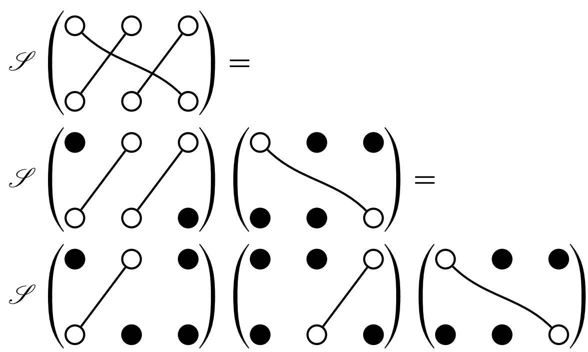
**Theorem 7** [20]. *A permutation is* 1*-stack sortable if and only if it avoids* 231 (*compare* [9]).

So, in view of our lifting $\mathscr{S}$ of the mapping $s$, what would be an appropriate way of generalizing Theorem 7 according to the mapping $\mathscr{S}$? In particular, how can the concept of "1-stack-sortability" be translated in a meaningful way so as to be applicable with respect to $\mathscr{S}$? To answer these questions, we are to make use of a morphism on partition algebras that we had previously applied in a representation-theoretic context [4].

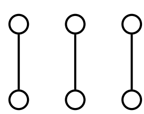
***Stretch morphisms.*** To illustrate the problem of determining a suitable analogue of Theorem 7 according to our lifting $\mathscr{S}$ of West's stack-sorting map, let us consider $\mathscr{S}$ evaluated at a permuting diagram that is 1-stack-sortable.

**Example 8.** We see that the permutation 312 avoids the pattern 231. So, let us consider $\mathscr{S}$ evaluated at the permutation diagram corresponding to 312, as below.

(19)

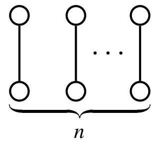


Following through with the procedure introduced in Section 2, we obtain that identity permutation diagram in $\mathscr{P}_3$ shown below.



From Theorem 5 together with Example 8, given a partition diagram $\pi \in \mathscr{P}_n$, to generalize the notion of 1-stack-sortability in an applicable and meaningful way, it would be appropriate to use a condition whereby $\mathscr{S}(\pi)$ belongs to a fixed class of generalizations of the multiplicative identity element

(20)



in $\mathscr{P}_n$. This leads us to apply, as below, a morphism on partition algebras introduced in a representation-theoretic context in [4].

Since our lifting of West's stack-sorting map applies to partition diagrams in general, as opposed to, say, rook diagrams, it would be appropriate to allow nonrook diagrams in our lifting of the concept of 1-stack sortability. However, our lifting of $s\colon S_n \to S_n$ is such that it preserves the sizes of blocks in a partition diagram (but not necessarily the arrangement or ordering of the blocks). So, it would be appropriate to let a partition diagram be mapped, under $\mathscr{S}$, to a generalization of (20) allowing for the possibility of blocks other than 2-blocks. In this regard, we are to employ what is referred to as the *Stretch* morphism for partition algebras, as introduced in [4].

Let $S$ be a finite set of natural numbers. Let $\pi$ be a set-partition of $S \cup S'$. Let $k$ be a natural number such that $k \geq \max(S)$. Following [4], we write $\delta_k(\pi)$ to denote the diagram basis element in $\mathscr{P}_k^{\xi}$ corresponding to the set-partition given by adding blocks of the form $\{i, i'\}$ to $\pi$, where $i$ is a natural number such that $i \notin S$ and $i \leq k$. Again, following [4], we let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{\ell(\alpha)})$ be a set-composition of a finite set of natural numbers (referring to [4] for details on combinatorial terms related to Stretch morphisms), and we write $m = \ell(\alpha)$, and set $k \geq \max\bigl(\bigcup \alpha\bigr)$. As in [4], we define

$$\text{Stretch}_{\alpha,k}\colon \mathscr{P}_m^{\xi} \to \mathscr{P}_k^{\xi} \tag{21}$$

in the following manner.

Let $d_\pi$ be a member of the diagram basis of $\mathscr{P}_m^{\xi}$. Let $\pi = \{\pi_1, \pi_2, \dots, \pi_{\ell(\pi)}\}$. Then

$$\text{Stretch}_{\alpha,k}(d_\pi) = \delta_k\Biggl(\Biggl\{ \bigcup_{\substack{i \in \pi_j \\ i \text{ is unprimed}}} \alpha_i \cup \bigcup_{i' \in \pi_j} \alpha_i' : 1 \leq j \leq \ell(\pi) \Biggr\}\Biggr). \tag{22}$$

We may extend this definition linearly as in [4] so as to obtain an algebra homomorphism, but this is not important for our purposes.

An interesting feature concerning the problem of generalizing 1-stack-sortability by generalizing (3) so as to allow partition diagrams as the arguments of a lifting/extension of the $s$-map may be explained as follows. By enforcing different conditions on how (20) should be generalized in order to generalize 1-stack-sortability, say, in the context of a given application, this gives rise to different families of combinatorial objects in the study of the classification of partition diagrams that reduce to a specified generalization of (20), under the application of $\mathscr{S}$.

**Definition 9.** We say that a partition diagram $\pi$ is *stretch-stack-sortable* if $\mathscr{S}(\pi)$ is the evaluation of a Stretch map on an identity permutation diagram.

To begin with, we illustrate the effect of applying (22) to the multiplicative identity elements in a partition algebra.

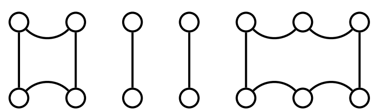
**Example 10.** With regard to the notation in (22), we let $d_\pi$ be the partition diagram corresponding to

$$\pi = \big\{\{1, 1'\}, \{2, 2'\}, \{3, 3'\}, \{4, 4'\}\big\}. \tag{23}$$

Define $\alpha$ as the set-composition $(\{1, 2\}, \{3\}, \{5, 6, 7\}, \{4\})$. Observe that the length of $\alpha$ equals the order of $d_\pi$, in accordance with the definition in (22). Let us consider the $j = 3$ case within the family of the argument of $\delta_k$ indicated in (22), letting the elements in $\pi$ be ordered in the manner we have written these elements in (23). So, the element corresponding to this $j = 3$ case is

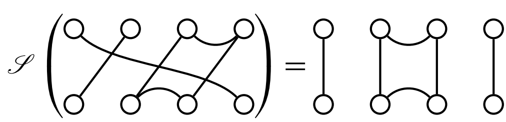
$$\bigcup_{\substack{i\in\pi_3 \\ i \text{ is unprimed}}} \alpha_i \cup \bigcup_{i'\in\pi_3} \alpha_i' = \alpha_3 \cup \alpha_3' = \{5, 6, 7, 5', 6', 7'\}.$$

Continuing similarly with respect to the other possible values for the $j$-index, we obtain that $\mathrm{Stretch}_{\alpha,7}$ evaluated at $d_\pi$ is equal to



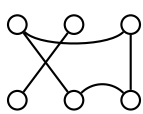
Now, let us consider an illustration of a partition diagram satisfying the conditions of Definition 9.

**Example 11.** According to the procedure in Section 2, we may verify the evaluation



So, the argument of $\mathscr{S}$, in this case, is stretch-stack-sortable.

**Example 12.** We also find that



it stretch-stack-sortable.

Examples 11 and 12 illustrate the problem of classifying stretch-stack-sortable partition diagrams. The importance of diagram algebras in both the study of algebraic groups and in the field of knot theory provides a source of further motivation concerning our interest in lifting and generalizing the notions of stack-sortability and pattern avoidance in permutations to basis elements in diagram algebras. The following theorem appears to be the first direct step in this direction. The situation indicated via the bullet points given in Theorem 13 formalizes how the notion of avoiding the pattern 231 may be lifted from permutations to partition diagrams, in a way that is directly applicable to lifting West's stack-sorting map.

**Theorem 13.** *A partition diagram $\pi$ is stretch-stack-sortable if and only if*:

(1) *The only blocks of $\pi$ are propagating.*

(2) *For each block of $\pi$, it has the same number of upper and lower vertices.*

(3) *For each block of $\pi$, all of its lower nodes are consecutive as primed integers.*

(4) *Letting the blocks of $\pi$ be denoted in the form $C_i \cup D'_i$ for indices $i$, and where $C_i$ and $D'_i$ respectively denote the set of upper and lower nodes of a given block, and writing*

$$D'_1 < D'_2 < \cdots, \tag{24}$$

*according to the consecutive primed integers labeling $D'_i$, the situation indicated via the following bullet points cannot occur.*

- $D'_{i_1} < D'_{i_2} < D'_{i_3}$.
- *By applying the procedure used to define $\mathscr{S}$, at the stage in this application when $D'_{i_3}$ contains the greatest nonsingleton primed nodes, one of the following situations occurs*:
- *either $D'_{i_2}$ is part of an $\mathscr{L}$-configuration and $D'_{i_1}$ is part of an $\mathscr{R}$-configuration, or*
- *$D'_{i_2}$ is part of an $\mathscr{L}$-configuration and $D'_{i_1}$ is part of an $\mathscr{M}$-configuration, or*
- *$D'_{i_2}$ is part of an $\mathscr{M}$-configuration and $D'_{i_1}$ is part of an $\mathscr{R}$-configuration, or*
- *$D'_{i_2}$ is part of an $\mathscr{M}_j$-configuration and $D'_{i_1}$ is part of an $\mathscr{M}_k$-configuration, with $j < k$.*

*Proof.* ($\Rightarrow$) Let $\pi$ be a stretch-stack-sortable partition diagram. The mapping $\mathscr{S}$ is such that the number of blocks in a partition diagram $\mu$ with $u$ upper nodes and $l$ lower nodes is equal to the number of blocks in $\mathscr{S}(\mu)$ with $u$ upper nodes and $l$ lower nodes. So, $\pi$ cannot have any nonpropagating blocks, because, otherwise, $\pi$ would have a nonpropagating block, contradicting that $\pi$ is stretch-stack-sortable. By using the same property of $\mathscr{S}$ that we have previously indicated, we have that each block of $\pi$ is such that it has the same number of upper and lower vertices.

Let $B$ be a block of $\pi$, and, by way of contradiction, suppose that it is not the case that all of its lower nodes are consecutive as primed integers. According to the procedure used to define $\mathscr{S}$, in $\mathscr{S}(\pi)$, there would have to be a new block $N$ such that the labels for the lower nodes of $N$ are the same as the labels of the lower nodes of $B$ and such that the upper nodes of $N$ are consecutive integers. So, there would be a block $N$ in $\mathscr{S}(\pi)$ with consecutive integers labeling the upper nodes of $N$ and such that the lower nodes of $N$ do not form a set of consecutive primed integers. However, this is impossible for the stretch of an identity diagram. Now, by way of contradiction, suppose that the situation indicated via the above six bullet points occurs. In any out of the four possibilities corresponding to the last four bullet points, the removal of the block containing $D'_{i_3}$ has the effect of separating the partition diagram containing $D'_{i_1}$ and $D'_{i_2}$ in such a way so that in the $\odot$-product shown in (8), the block containing $D'_{i_1}$ will be in an argument of $\mathscr{S}$ strictly to the left of the factor in the $\odot$-product given by $\mathscr{S}$ evaluated at an expression involving $D'_{i_2}$. Consequently, the ordering of the positions of the bottom nodes of $D'_{i_1}$ and $D'_{i_2}$ will be reversed, but then the top nodes corresponding to $D'_{i_2}$ will be labeled with integers strictly smaller than the top nodes corresponding to $D'_{i_1}$ in the evaluation of $\mathscr{S}(\pi)$, giving us a crossing in the partition diagram $\mathscr{S}(\pi)$ that would be impossible in the stretch of an identity partition diagram.

($\Leftarrow$) Conversely, suppose that a partition diagram $\pi \in \mathscr{P}_n$ satisfies all four of the conditions listed above. We apply $\mathscr{S}$ to $\pi$, so as to obtain a decomposition of the form indicated in (8). Adopting notation from (8), we may deduce that the following properties hold, again working under the assumption that the four conditions in the theorem under consideration hold:

(i) The expression $B'$ consists, apart from singleton blocks, of a single propagating block with the same number of upper and lower nodes, and such that the lower nodes are labeled with consecutive primed integers ending with $n'$.

(ii) If we were to keep the labelings for the lower nodes as in $\pi'$, then the labeling for the lower nonsingleton blocks in the $\odot$-decomposition in (8) would be in the same order.

This latter property may be verified by a case analysis of the form suggested by the last four out of the six bullet points given above. Repeating this argument inductively, and mimicking notation from (9), we find that $\mathscr{S}(\pi)$ may be written as an $\odot$-product of the form

$$\mathscr{B}'_1 \odot \mathscr{B}'_2 \odot \cdots \odot \mathscr{B}'_{m_2}, \tag{25}$$

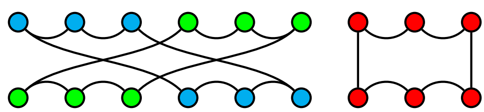
where each expression of the form $B'_i$ consists of, apart from singleton blocks, only one propagating block, and this propagating block has the same number of top nodes as bottom nodes and has consecutive primed integers as its bottom nodes; moreover, the bottom nodes according to the ordering in (25) form the sequence

$$1' < 2' < \cdots < n'.$$

So, a direct application of the labeling according to the definition for the $\odot$-product gives us a stretch of an identity partition diagram. $\square$

**Example 14.** If we return to the partition diagram in $\mathscr{P}_9$ that is illustrated in (7) within Example 3, we find that there is a propagating block containing the largest bottom node $9'$. Being consistent with our notation in (24), we write $D'_1 = \{1', 2', 3'\}$, $D'_2 = \{4', 5', 6'\}$, and $D'_3 = \{7', 8', 9'\}$, and we let the corresponding blocks in (7) be denoted as $C_1 \cup D'_1$, $C_2 \cup D'_2$, and $C_3 \cup D'_3$. We have that $D'_1 < D'_2 < D'_3$, but when we apply the procedure used to define $\mathscr{S}$ to the partition diagram in (7), as we remove the block $C_3 \cup D'_3$, we find that $D'_2$ is part of an $\mathscr{L}$-configuration and $D'_1$ is part of an $\mathscr{M}$-configuration, which is one of the forbidden patterns indicated in Theorem 13. So, according to Theorem 13, the partition diagram in (7) should not be stretch-stack-sortable, and we may verify that $\mathscr{S}$ evaluated at this same diagram in (7) is equal to

(26)



being consistent with the coloring in (7). We see that the partition diagram in (26) is not a stretch of any identity partition diagram.

## 4. Future work and applications

In regard to Knuth's famous result that the number of 1-stack-sortable permutations in $S_n$ is equal to the $n$-th Catalan number $\frac{1}{n+1}\binom{2n}{n}$, it would be desirable to obtain a meaningfully similar result for counting the elements in $\mathscr{P}_n$ satisfying the conditions in Theorem 13, i.e., the number of stretch-stack-sortable elements in $\mathscr{P}_n$. We leave this as an open problem.

An advantage of our lifting of West's stack-sorting map may be described as follows. Since our mapping $\mathscr{S}$ may be evaluated at an arbitrary partition diagram, this allows us to apply and experiment with different ways of generalizing stack-sortability, based on different kinds of partition diagram generalizations of identity permutations. We may obtain many further combinatorial results, relative to our main results, through the use of variants and generalizations of Definition 9. For example, what kinds of bijective and enumerative results can we obtain, relative to the theorems introduced in this article, if we instead consider partition diagrams that get mapped, under the application of $\mathscr{S}$, to stretches of rook diagrams with

2-blocks of the form $\{i, i'\}$? What if nonrectangular and nonsingleton blocks are allowed?

Many remarkable results and applications have concerned preimages under West's stack-sorting map, as in the work of Defant, Engen, and Miller in [15], for example. What kinds of applications and combinatorial results can we obtain concerning preimages under our lifting of $s$?

Since our definition for stretch-stack-sortability is a lifting of 1-stack-sortability, this raises the question as to how the notion of $t$-stack-sortability may be generalized, according to our procedure from Section 2. The study of $t$-stack-sortability is widely known to be much more difficult even for the $t = 2$ case, relative to the $t = 1$ case; see [26, Section 1.2.3], for example. In consideration as to Zeilberger's renowned proof [29] of West's conjecture that the number of 2-stack-sortable permutations in $S_n$ is

$$\frac{2(3n)!}{(n+1)!(2n+1)!},$$

this inspires the determination of an analogue of the above indicated enumerative result, using a lifting of 2-stack-sortability via our mapping $\mathscr{S}$.

## Acknowledgements

The author wants to thank Mike Zabrocki for having described to the author (and introduced to the author) the partition algebra morphisms of the form indicated in (22) and how such morphisms may be applied in the representation theory for partition algebras. The author is very thankful for the referee feedback that has been provided.

## References

[1] A. Asinowski, C. Banderier, S. Billey, B. Hackl, and S. Linusson, "Pop-stack sorting and its image: permutations with overlapping runs", *Acta Math. Univ. Comenian.* (*N.S.*) **88**:3 (2019), 395–402. MR

[2] G. Benkart and T. Halverson, "Partition algebras and the invariant theory of the symmetric group", pp. 1–41 in *Recent trends in algebraic combinatorics*, Assoc. Women Math. Ser. **16**, Springer, Cham, 2019. MR Zbl

[3] G. Benkart and T. Halverson, "Partition algebras $\mathsf{P}_k(n)$ with $2k > n$ and the fundamental theorems of invariant theory for the symmetric group $\mathsf{S}_n$", *J. Lond. Math. Soc.* (2) **99**:1 (2019), 194–224. MR Zbl

[4] J. M. Campbell, "Alternating submodules for partition algebras, rook algebras, and rook-Brauer algebras", preprint, 2022, available at https://hal.science/hal-03850236. To appear in *J. Pure Appl. Algebra*.

[5] G. Cerbai, A. Claesson, and L. Ferrari, "Stack sorting with restricted stacks", *J. Combin. Theory Ser. A* **173** (2020), art. id. 105230. MR Zbl

[6] C. Defant, "Descents in $t$-sorted permutations", *J. Comb.* **11**:3 (2020), 511–526. MR Zbl

[7] C. Defant, "Polyurethane toggles", *Electron. J. Combin.* **27**:2 (2020), art. id. P2.46. MR Zbl

[8] C. Defant, "Stack-sorting preimages of permutation classes", *Sém. Lothar. Combin.* **82** (2020), art. id. B82b. MR Zbl

[9] C. Defant, "Enumeration of stack-sorting preimages via a decomposition lemma", *Discrete Math. Theor. Comput. Sci.* **22**:2 (2021), art. id. 3. MR Zbl

[10] C. Defant, "Fertility monotonicity and average complexity of the stack-sorting map", *European J. Combin.* **93** (2021), art. id. 103276. MR Zbl

[11] C. Defant, "Highly sorted permutations and Bell numbers", *Enumer. Comb. Appl.* **1**:1 (2021), art. id. S2R6. MR Zbl

[12] C. Defant, "Stack-sorting for Coxeter groups", *Comb. Theory* **2**:1 (2022), art. id. 18. MR Zbl

[13] C. Defant and N. Kravitz, "Stack-sorting for words", *Australas. J. Comb.* **77** (2020), 51–68. Zbl

[14] C. Defant and K. Zheng, "Stack-sorting with consecutive-pattern-avoiding stacks", *Adv. Appl. Math.* **128** (2021), art. id. 102192. Zbl

[15] C. Defant, M. Engen, and J. A. Miller, "Stack-sorting, set partitions, and Lassalle's sequence", *J. Combin. Theory Ser. A* **175** (2020), art. id. 105275. MR Zbl

[16] J. East, "Generators and relations for partition monoids and algebras", *J. Algebra* **339** (2011), 1–26. MR Zbl

[17] T. Halverson and T. N. Jacobson, "Set-partition tableaux and representations of diagram algebras", *Algebr. Comb.* **3**:2 (2020), 509–538. MR Zbl

[18] T. Halverson and A. Ram, "Partition algebras", *European J. Combin.* **26**:6 (2005), 869–921. MR Zbl

[19] V. F. R. Jones, "The Potts model and the symmetric group", pp. 259–267 in *Subfactors* (Kyuzeso, 1993), World Sci. Publ., River Edge, NJ, 1994. MR Zbl

[20] D. E. Knuth, *The art of computer programming*, *I*: *Fundamental algorithms*, vol. 1, Addison-Wesley, Reading, MA, 1973. MR

[21] P. Martin, *Potts models and related problems in statistical mechanics*, Series on Advances in Statistical Mechanics **5**, World Scientific Publishing Co., Teaneck, NJ, 1991. MR Zbl

[22] P. Martin, "Temperley–Lieb algebras for nonplanar statistical mechanics — the partition algebra construction", *J. Knot Theory Ramifications* **3**:1 (1994), 51–82. MR

[23] P. Martin, "The structure of the partition algebras", *J. Algebra* **183**:2 (1996), 319–358. MR Zbl

[24] P. P. Martin, "The partition algebra and the Potts model transfer matrix spectrum in high dimensions", *J. Phys. A* **33**:19 (2000), 3669–3695. MR Zbl

[25] P. Martin and D. Woodcock, "On central idempotents in the partition algebra", *J. Algebra* **217**:1 (1999), 156–169. MR Zbl

[26] V. Vatter, *The enumeration and structure of permutation classes*, Ph.D. thesis, Rutgers University, New Brunswick, NJ, 2006, available at https://sites.math.rutgers.edu/~zeilberg/Theses/VinceVatterThesis.pdf.

[27] J. West, *Permutations with forbidden subsequences and stack-sortable permutations*, Ph.D. thesis, Massachusetts Institute of Technology, Cambridge, MA, 1990, available at https://dspace.mit.edu/handle/1721.1/13641. MR

[28] Z. K. Xiao, "On tensor spaces for rook monoid algebras", *Acta Math. Sin.* (*Engl. Ser.*) **32**:5 (2016), 607–620. MR Zbl

[29] D. Zeilberger, "A proof of Julian West's conjecture that the number of two-stack-sortable permutations of length $n$ is $2(3n)!/((n+1)!(2n+1)!)$", *Discrete Math.* **102**:1 (1992), 85–93. MR Zbl

Received December 11, 2022. Revised April 15, 2023.

JOHN M. CAMPBELL
DEPARTMENT OF MATHEMATICS
TORONTO METROPOLITAN UNIVERSITY
TORONTO, ON
CANADA
jmaxwellcampbell@gmail.com

PACIFIC JOURNAL OF MATHEMATICS
Vol. 324, No. 2, 2023
https://doi.org/10.2140/pjm.2023.324.249

# LIMIT CYCLES OF LINEAR VECTOR FIELDS ON $(\mathbb{S}^2)^m \times \mathbb{R}^n$

CLARA CUFÍ-CABRÉ AND JAUME LLIBRE

**It is well known that linear vector fields defined in $\mathbb{R}^n$ cannot have limit cycles, but this is not the case for linear vector fields defined in other manifolds. We study the existence of limit cycles bifurcating from a continuum of periodic orbits of linear vector fields on manifolds of the form $(\mathbb{S}^2)^m \times \mathbb{R}^n$ when such vector fields are perturbed inside the class of all linear vector fields. The study is done using averaging theory. We also present an open problem about the maximum number of limit cycles of linear vector fields on $(\mathbb{S}^2)^m \times \mathbb{R}^n$.**

## 1. Introduction and statement of the main results

The study of periodic orbits of differential systems plays an important role in the qualitative theory of ordinary differential equations and their applications. A *limit cycle* is defined as a periodic orbit of a differential system which is isolated in the set of all periodic orbits of the system. Among the many works devoted to limit cycles and their applications, we mention [Christopher and Lloyd 1996; Giacomini et al. 1996; Han and Li 2012; Ilyashenko 2002].

It is well known that linear vector fields in $\mathbb{R}^n$ cannot have limit cycles, but this is not the case if one considers linear vector fields in other manifolds different from $\mathbb{R}^n$. The objective of this paper is to study the existence of limit cycles of linear vector fields defined on the manifolds $(\mathbb{S}^2)^n \times \mathbb{R}^n$.

The problem of studying limit cycles of linear vector fields on manifolds different from $\mathbb{R}^n$ was already treated in [Llibre and Zhang 2016], where the authors consider linear vector fields on $\mathbb{S}^m \times \mathbb{R}^n$, and they conjecture that such vector fields may have at most one limit cycle.

Linear autonomous differential systems, namely, systems of the form $\dot{x} = Ax + b$, where $A$ is a $n \times n$ real matrix and $b$ is a vector in $\mathbb{R}^n$, are the easiest systems to study because their solutions can be completely determined (see [Arnold 2006; Sotomayor 1979]), but still they play an important role in the theory of differential

This work was supported by the Ministerio de Ciencia, Innovación y Universidades, Agencia Estatal de Investigación grants PID2019-104658GB-I00 and BES-2017-081570, and the H2020 European Research Council grant MSCA-RISE-2017-777911.
*MSC2020:* 34A30, 34C25, 34C29.
*Keywords:* limit cycle, periodic orbit, isochronous center, averaging method.

systems. Thus when a nonlinear differential system has a hyperbolic equilibrium point, the dynamics around that point is determined by the linearization of the vector field at that point (Hartman–Grossman theorem; see [Hartman 1960]).

Linear vector fields having invariant subspaces of periodic orbits can be perturbed inside a concrete class of nonlinear differential systems to obtain limit cycles of these nonlinear systems bifurcating from the periodic orbits of the linear system (see [Ferragut et al. 2007; Llibre and Teixeira 2009; Llibre et al. 2007; 2010]).

Moreover, linear differential systems of the form $\dot{x} = Ax + Bu$, where $x$ are the state variables and $u$ is the control input, are applied in control theory for the modeling of hybrid systems (see [Lafferriere et al. 2001; 1999]). These examples illustrate the importance of linear differential systems.

In this paper we show that linear differential systems can have limit cycles when the manifold where they are defined is different from $\mathbb{R}^n$, and we consider the question of how many limit cycles, at most, a linear vector field can have depending on the manifold where it is defined.

Let $M$ be a smooth connected manifold of dimension $n$, and let $TM$ be its tangent bundle. A *vector field* on $M$ is a map $X : M \to TM$ such that $X(x) \in T_x M$, where $T_x M$ is the tangent space of $M$ at the point $x$.

A *linear vector field* in $\mathbb{R}^n$ is a vector field of the form $X(x) = Ax + b$, with $x, b \in \mathbb{R}^n$ and where $A$ is a $n \times n$ real matrix. It is well known linear vector fields on $\mathbb{R}^n$ either do not have periodic orbits or their periodic orbits form a continuum, and therefore they do not have limit cycles.

We consider linear vector fields on some manifolds of the form $(\mathbb{S}^2)^m \times \mathbb{R}^n$, where $\mathbb{S}^2$ denotes the unit two-dimensional sphere. Here the sphere $\mathbb{S}^2$ is parameterized by the coordinates $(\theta, \varphi)$, where $\theta \in [-\pi, \pi)$ denotes the azimuth angle and $\varphi \in [-\pi/2, \pi/2]$ is the polar angle. Hence the curve $\{\varphi = 0\}$ is the equator.

Let $(\theta_1, \varphi_1, \ldots, \theta_m, \varphi_m, x_1, \ldots, x_n)$ denote the coordinates of the space $(\mathbb{S}^2)^m \times \mathbb{R}^n$. Then we say that a vector field $X$ is *linear* on $M = (\mathbb{S}^2)^m \times \mathbb{R}^n$ if the expression of $X$ in the coordinates $z = (\theta_1, \varphi_1, \ldots, \theta_m, \varphi_m, x_1, \ldots, x_n) \in M$ is of the form $X(z) = Az + b$, with $b \in M$ and where $A$ is a $(2m+n) \times (2m+n)$ real matrix.

Perrizo [1974] published a paper entitled "$\omega$-linear vector fields on manifolds", but these $\omega$-linear vector fields are very far from the linear differential systems that we study in this paper. More precisely, from Definition 2.2 of [Perrizo 1974] for $\omega$-linear vector fields on a manifold, such a field $X$ requires the existence of a nonlocally constant function that is constant on the orbits of $\tilde{X}$ (using Perrizo's notation), but the existence of limit cycles in our linear vector fields prevents the existence of a such function. So our linear differential systems on the manifolds $(\mathbb{S}^2)^m \times \mathbb{R}^n$ are not $\omega$-linear.

An example in which a linear differential system on the manifold $(\mathbb{S}^2)^m \times \mathbb{R}^n$ has a limit cycle is the following. Take $m = 1, n = 0$ and consider the linear system on the sphere $\mathbb{S}^2$ given by

$$\dot{\theta} = 1, \quad \dot{\varphi} = \varphi,$$

for $\theta \in [-\pi, \pi)$ and $\varphi \in (-\pi/2, \pi/2)$, and

$$\dot{\theta} = 0, \quad \dot{\varphi} = 0,$$

for $\varphi = \pm\pi/2$. Then, clearly the equator of the sphere $\{\varphi = 0\}$ is the only periodic orbit of the system, and therefore it is a limit cycle.

We consider generic linear perturbations of some linear vector fields on three different manifolds of the form $(\mathbb{S}^2)^m \times \mathbb{R}^n$, and we study whether those families of linear differential systems can have limit cycles.

Let $M = \mathbb{S}^2 \times \mathbb{R}$ and consider the linear differential system in $M$ given by

$$\dot{\theta} = 1, \quad \dot{\varphi} = 0, \quad \dot{r} = r - 1, \tag{1-1}$$

for $r \in \mathbb{R}$, $\theta \in [-\pi, \pi)$ and $\varphi \in (-\pi/2, \pi/2)$, and with $\dot{\theta} = 0$ on the straight lines $R_1 = \{\varphi = -\pi/2\}$ and $R_2 = \{\varphi = \pi/2\}$.

The solution of system (1-1) is given by

$$\theta(t) = \theta_0 + t, \quad \varphi(t) = \varphi_0, \quad r(t) = (r_0 - 1)e^t + 1.$$

Thus the sphere $\{r = 1\}$ is an invariant manifold with two equilibrium points at the north and the south poles, and is foliated by periodic orbits of period $2\pi$, corresponding to the parallels of the sphere, except at the poles. Moreover the straight lines $R_1$ and $R_2$ are invariant.

First we shall study the bifurcation of limit cycles when we perturb system (1-1) inside the class of all linear differential systems, and we shall see that one of the periodic orbits contained in the sphere $\{r = 1\}$ may bifurcate to a limit cycle under certain hypotheses.

We consider the class of differential systems

$$\begin{aligned} \dot{\theta} &= 1 + \varepsilon(a_0 + a_1\theta + a_2\varphi + a_3 r), \\ \dot{\varphi} &= \varepsilon(b_0 + b_1\theta + b_2\varphi + b_3 r), \\ \dot{r} &= r - 1 + \varepsilon(c_0 + c_1\theta + c_2\varphi + c_3 r), \end{aligned} \tag{1-2}$$

where $a_i$, $b_i$ and $c_i$, for $i = 0, \ldots, 3$, are real numbers and with $\varepsilon > 0$ being a small parameter. Note that this is the more general linear perturbation of system (1-1).

**Theorem 1.** *For sufficiently small $\varepsilon > 0$ the linear differential system (1-2) has a limit cycle bifurcating from a periodic orbit of system (1-1) provided that*

$$a_1 b_2 - a_2 b_1 \neq 0.$$

*This limit cycle bifurcates from the periodic orbit of system (1-1) parameterized by*

$(\theta(t), \varphi(t), r(t)) = (\theta_0 + t, \varphi_0, 1)$, *with*

$$\theta_0 = \frac{a_2(b_0 + b_3 + b_1\pi) - b_2(a_0 + a_3 + a_1\pi)}{a_1 b_2 - a_2 b_1},$$
$$\varphi_0 = \frac{b_1(a_0 + a_3 + a_1\pi) - a_1(b_0 + b_3 + b_2\pi)}{a_1 b_2 - a_2 b_1}.$$

Theorem 1 is proved in Section 3.

We remark that the existence of the limit cycle for system (1-2) does not depend on the perturbation of the $\dot{r}$ equation.

As an example of the previous result, consider the system

$$\dot{\theta} = 1 + \varepsilon a\varphi, \quad \dot{\varphi} = \varepsilon b\theta, \quad \dot{r} = r - 1, \tag{1-3}$$

with $a, b \in \mathbb{R}$ and $\varepsilon > 0$. In this case the sphere $\{r = 1\}$ is still an invariant manifold. Applying Theorem 1 with $a_2 = a$, $b_1 = b$ and the rest of the coefficients of the perturbation being zero, we find that system (1-3) has a limit cycle bifurcating from the periodic orbit of system (1-1) parameterized by $(\theta(t), \varphi(t), r(t)) = (-\pi + t, 0, 1)$. That is, there is a limit cycle bifurcating from the periodic orbit corresponding to the equator of the sphere $\{r = 1\}$ of system (1-1). This limit cycle is still contained in the sphere $\{r = 1\}$.

Next we consider linear differential systems defined on higher dimensional manifolds. We take $M = \mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{R}$ and

$$\dot{\theta} = 1, \quad \dot{\varphi} = 0, \quad \dot{\nu} = 1, \quad \dot{\phi} = 0, \quad \dot{r} = r - 1, \tag{1-4}$$

for $(\theta, \varphi, \nu, \phi, r) \in M$, with $\theta, \nu \in [-\pi, \pi)$ and $\varphi, \phi \in (-\pi/2, \pi/2)$, and with $\dot{\theta} = 0$ when $\varphi = \pm\pi/2$ and $\dot{\nu} = 0$ when $\phi = \pm\pi/2$.

The general solution of system (1-4) is

$$\theta(t) = \theta_0 + t, \quad \varphi(t) = \varphi_0, \quad \nu(t) = \nu_0 + t, \quad \phi(t) = \phi_0, \quad r(t) = (r_0 - 1)e^t + 1,$$

and thus the product of spheres $\{r = 1\} \cong (\mathbb{S}^2)^2$ is an invariant manifold foliated by periodic orbits of period $2\pi$, except for the four points $\{r = 1, \varphi = \pm\pi/2, \phi = \pm\pi/2\}$, which are equilibrium points.

We consider the most general perturbation of the differential system (1-4) inside the class of all linear differential systems, namely

$$\begin{aligned}
\dot{\theta} &= 1 + \varepsilon(a_0 + a_1\theta + a_2\varphi + a_3\nu + a_4\phi + a_5 r),\\
\dot{\varphi} &= \varepsilon(b_0 + b_1\theta + b_2\varphi + b_3\nu + b_4\phi + b_5 r)\\
\dot{\nu} &= 1 + \varepsilon(c_0 + c_1\theta + c_2\varphi + c_3\nu + c_4\phi + c_5 r),\\
\dot{\phi} &= \varepsilon(d_0 + d_1\theta + d_2\varphi + d_3\nu + d_4\phi + d_5 r)\\
\dot{r} &= r - 1 + \varepsilon(e_0 + e_1\theta + e_2\varphi + e_3\nu + e_4\phi + e_5 r),
\end{aligned} \tag{1-5}$$

with $a_i, b_i, c_i, d_i, e_i \in \mathbb{R}$ for $i = 0, \dots, 5$, and with $\varepsilon > 0$ being a small parameter. The next result gives sufficient conditions on the coefficients of system (1-5) for there to be a limit cycle bifurcating from a periodic orbit of the unperturbed system.

**Theorem 2.** *For sufficiently small $\varepsilon > 0$ the differential system (1-5) has a limit cycle bifurcating from a periodic orbit of system (1-4) provided that*

$$\det \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{pmatrix} \neq 0.$$

*This limit cycle bifurcates from the periodic orbit of system (1-4) parameterized by*

$$(\theta(t), \varphi(t), \nu(t), \phi(t), r(t)) = (\theta_0 + t, \varphi_0, \nu_0 + t, \phi_0, 1),$$

*where $(\theta_0, \varphi_0, \nu_0, \phi_0)$ is the unique solution of the linear system*

$$\begin{aligned} a_1\theta_0 + a_2\varphi_0 + a_3\nu_0 + a_4\phi_0 &= -a_0 - a_1\pi - a_3\pi - a_5, \\ b_1\theta_0 + b_2\varphi_0 + b_3\nu_0 + b_4\phi_0 &= -b_0 - b_1\pi - b_3\pi - b_5, \\ c_1\theta_0 + c_2\varphi_0 + c_3\nu_0 + c_4\phi_0 &= -c_0 - c_1\pi - c_3\pi - c_5, \\ d_1\theta_0 + d_2\varphi_0 + d_3\nu_0 + d_4\phi_0 &= -d_0 - d_1\pi - d_3\pi - d_5. \end{aligned}$$

Theorem 2 is proved in Section 4.

Finally we consider the linear differential system defined in $M = \mathbb{R}^2 \times \mathbb{S}^2$, for $(x, y, \theta, \varphi) \in \mathbb{R}^2 \times \mathbb{S}^2$, with $\theta \in [-\pi, \pi)$ and $\varphi \in (-\pi/2, \pi/2)$, given by

$$\dot{x} = -y, \quad \dot{y} = x, \quad \dot{\theta} = 1, \quad \dot{\varphi} = 0, \tag{1-6}$$

and with $\dot{\theta} = 0$ in the planes $P_1 = \{\varphi = -\pi/2\}$ and $P_2 = \{\varphi = \pi/2\}$, which are invariant. The general solution of system (1-6) is

$$x(t) = x_0 \cos t - y_0 \sin t, \quad y(t) = x_0 \sin t + y_0 \cos t, \quad \theta(t) = \theta_0 + t, \quad \varphi(t) = \varphi_0,$$

and therefore the whole phase space is filled by periodic orbits of period $2\pi$, except for the two equilibrium points $(x, y, \theta, \varphi) = (0, 0, \theta, -\pi/2)$ and $(x, y, \theta, \phi) = (0, 0, \theta, \pi/2)$.

We consider the most general linear perturbation of system (1-6) and we study the existence of limit cycles bifurcating from the periodic orbits of system (1-6).

Let the perturbed system be

$$\begin{aligned} \dot{x} &= -y + \varepsilon(a_0 + a_1 x + a_2 y + a_3\theta + a_4\varphi), \\ \dot{y} &= x + \varepsilon(b_0 + b_1 x + b_2 y + b_3\theta + b_4\varphi), \\ \dot{\theta} &= 1 + \varepsilon(c_0 + c_1 x + c_2 y + c_3\theta + c_4\varphi), \\ \dot{\varphi} &= \varepsilon(d_0 + d_1 x + d_2 y + d_3\theta + d_4\varphi), \end{aligned} \tag{1-7}$$

where $\varepsilon > 0$ is a small parameter and $a_i, b_i, c_i, d_i \in \mathbb{R}$ for $i = 0, \dots, 4$.

**Theorem 3.** *For sufficiently small $\varepsilon > 0$ the linear differential system (1-7) has a limit cycle bifurcating from a periodic orbit of system (1-6) provided that*

$$\det\begin{pmatrix} b_2+a_1 & a_2-b_1 \\ b_1-a_2 & b_2+a_1 \end{pmatrix} \neq 0 \quad \text{and} \quad \det\begin{pmatrix} c_3 & c_4 \\ d_3 & d_4 \end{pmatrix} \neq 0.$$

*This limit cycle bifurcates from the periodic orbit of system (1-6) passing through the point $(x_0, y_0, \theta_0, \varphi_0)$ where*

$$x_0 = \frac{(2b_2+2a_1)b_3 - 2a_3b_1 + 2a_2a_3}{b_2^2+b_1^2+a_2^2+a_1^2+2a_1b_2-2a_2b_1},$$
$$y_0 = -\frac{(2b_1-2a_2)b_3 + 2a_3b_2 + 2a_1a_3}{b_2^2+b_1^2+a_2^2+a_1^2+2a_1b_2-2a_2b_1},$$
$$\theta_0 = -\frac{(\pi c_3 + c_0)d_4 - \pi c_4d_3 - c_4d_0}{c_3d_4 - c_4d_3}, \qquad \varphi_0 = \frac{c_0d_3 - c_3d_0}{c_3d_4 - c_4d_3}.$$

Theorem 3 is proved in Section 5.

As an example consider the system

$$\dot{x} = -y + \varepsilon a y, \quad \dot{y} = x + \varepsilon b x, \quad \dot{\theta} = 1 + \varepsilon c \varphi, \quad \dot{\varphi} = \varepsilon d \theta, \tag{1-8}$$

with $a, b, c, d \in \mathbb{R}$, and $\varepsilon > 0$. Applying Theorem 3 with $a_2 = a$, $b_1 = b$, $c_4 = c$, $d_3 = d$ and the rest of the coefficients of the perturbation being zero, we obtain that system (1-8) has a limit cycle bifurcating from the periodic orbit of system (1-6) passing through the point $(x_0, y_0, \theta_0, \varphi_0) = (0, 0, -\pi, 0)$, provided $(a-b)cd \neq 0$. That is, here the limit cycle bifurcates from the periodic orbit corresponding to the equator of the invariant sphere $\{x = y = 0\}$ of system (1-6).

The key tool that we use for proving Theorems 1–3 is averaging theory. For a general introduction to this theory, see the books [Sanders et al. 2007; Verhulst 1996]. As one can see in the proofs of Theorems 1–3, our method based on averaging theory can produce at most one limit cycle for the studied systems. Therefore the following open question is natural.

**Open question.** Let $m$ and $n$ be two nonnegative integers. Is it true that a linear vector field on the manifold $(\mathbb{S}^m)^m \times \mathbb{R}^n$ can have at most one limit cycle?

A similar open question was stated in [Llibre and Zhang 2016] concerning linear vector fields on the manifold $(\mathbb{S}^1)^m \times \mathbb{R}^n$.

## 2. Basic results on averaging theory

We now state basic results from averaging theory needed for later proofs. Let $M$ be a smooth connected manifold of dimension $n$, and let $F_0, F_1 : \mathbb{R} \times M \to \mathbb{R}^n$

and $F_2 : \mathbb{R} \times M \times [0, \varepsilon_0) \to \mathbb{R}^n$ be $C^2$ periodic functions of period $T$. Given the differential system

$$\dot{x}(t) = F_0(t, x), \tag{2-1}$$

we consider a perturbation of this system of the form

$$\dot{x}(t) = F_0(t, x) + \varepsilon F_1(t, x) + \varepsilon^2 F_2(t, x, \varepsilon). \tag{2-2}$$

The objective is to study the bifurcation of $T$-periodic solutions of system (2-2) for $\varepsilon > 0$ small enough. A solution to this problem is given by averaging theory.

We assume that there exists $k \le n$ such that $M = M_k \times M_{n-k}$, where $M_k$ is a manifold of dimension $k$ and $M_{n-k}$ is a manifold of dimension $n-k$, and that the unperturbed system, namely system (2-1), contains an open set, $V \subseteq M_k$, such that $\overline{V}$ is filled with periodic solutions all of them with the same period. Such a set is called *isochronous*.

Let $x(t, z, \varepsilon)$ be the solution of system (2-2) such that $x(0, z, \varepsilon) = z$. We write the linearization of the unperturbed system (2-1) along the solution $x(t, z, 0)$ as

$$\dot{y} = D_x F_0(t, x(t, z, 0))y. \tag{2-3}$$

and we let $\mathcal{M}_z(t)$ be the fundamental matrix of the linear differential system (2-3), so $\mathcal{M}_z(0)$ is the $n \times n$ identity matrix. Let $\xi : M = M_k \times M_{n-k} \to M_k$ the projection of $M$ onto its first $k$ coordinates, that is, $\xi(x_1, \dots, x_n) = (x_1, \dots, x_k)$.

The following results give sufficient conditions for the existence of limit cycles for a system of the form (2-2) bifurcating from the periodic orbits of system (2-1).

**Theorem 4.** *Let $V \subseteq M_k$ be an open and bounded set, and let $\beta_0 : \overline{V} \to M_{n-k}$ be a $C^2$ function. Assume that*:

(i) *$\mathcal{Z} = \{z_\alpha = (\alpha, \beta_0(\alpha)) : \alpha \in \overline{V}\} \subset M$ and for each $z_\alpha \in \mathcal{Z}$ the solution $x(t, z_\alpha, 0)$ of system (2-1) is T-periodic.*

(ii) *For each $z_\alpha \in \mathcal{Z}$, there is a fundamental matrix $\mathcal{M}_{z_\alpha}(t)$ of system (2-3) such that the matrix $\mathcal{M}^{-1}_{z_\alpha}(0) - \mathcal{M}^{-1}_{z_\alpha}(T)$ has the $k \times (n-k)$ zero matrix in the upper right corner, and a $(n-k) \times (n-k)$ matrix $\Delta_\alpha$ in the lower right corner with* $\det(\Delta_\alpha) \neq 0$.

*Consider the function $\mathcal{F} : \overline{V} \to \mathbb{R}^k$ defined by*

$$\mathcal{F}(\alpha) = \xi\Big(\int_0^T \mathcal{M}^{-1}_{z_\alpha}(t) F_1(t, x(t, z_\alpha, 0))\, dt\Big).$$

*If there exists $a \in V$ with $\mathcal{F}(a) = 0$ and with* $\det(\mathcal{D}\mathcal{F}(a)) \neq 0$, *then there is a limit cycle $x(t, \varepsilon)$ of period $T$ of system (2-2) such that $x(0, \varepsilon) \to z_a$ as $\varepsilon \to 0$.*

The result in Theorem 4 can be found in [Malkin 1956] and [Roseau 1966]. For a shorter proof, see [Buică et al. 2007]. There the result is proved in $\mathbb{R}^n$, but it can be easily extended to a manifold $M$.

The next result allows us to determine the existence of limit cycles in a system of the form (2-2) in the case when there exists an open set, $V \subset M$, such that for all $z \in \overline{V}$, the solution $x(t, z, 0)$ is $T$-periodic.

**Theorem 5.** *Let $V \subseteq M$ be an open and bounded set with $\overline{V} \subseteq M$, and assume that for all $z \in \overline{V}$ the solution $x(t, z, 0)$ of system (2-2) is $T$-periodic. Consider the function $\mathcal{F} : \overline{V} \to \mathbb{R}^n$ defined by*

$$\mathcal{F}(z) = \int_0^T \mathcal{M}_z^{-1}(t) F_1(t, x(t, z, 0))\, dt.$$

*If there exists $a \in V$ with $\mathcal{F}(a) = 0$ and with $\det(\mathcal{D}\mathcal{F}(a)) \neq 0$, then there is a limit cycle $x(t, \varepsilon)$ of period $T$ of system (2-2) such that $x(0, \varepsilon) \to a$ as $\varepsilon \to 0$.*

For the proof of Theorem 5 see Corollary 1 of [Buică et al. 2007].

## 3. Proof of Theorem 1

We use the result from averaging theory given in Theorem 4 to deduce the existence of a limit cycle of system (1-2), for some $\varepsilon > 0$ small enough, bifurcating from a periodic orbit of the same system with $\varepsilon = 0$.

Since the general solution of the differential system (1-1), corresponding to system (1-2) with $\varepsilon = 0$, is given by

$$\theta(t) = \theta_0 + t, \quad \varphi(t) = \varphi_0, \quad r(t) = (r_0 - 1)e^t + 1,$$

it is clear that all the periodic solutions of that system are parameterized by

$$\theta(t) = \theta_0 + t, \quad \varphi(t) = \varphi_0, \quad r(t) = 1,$$

with $(\theta_0, \varphi_0) \in \mathbb{S}^2 \backslash \{\varphi_0 = \pm\pi/2\}$. Then, the periodic solutions all have period $2\pi$ and they fill the invariant sphere $\{r = 1\}$ except for the poles, which are equilibrium points.

Therefore, for applying Theorem 4 we take $M = \mathbb{S}^2 \times \mathbb{R}$ and

$$\begin{aligned}
k &= 2, \quad n = 3,\\
M_k &= M_2 = \{(\theta, \varphi, r) \in M : r = 1\} \cong \mathbb{S}^2,\\
x &= (\theta, \varphi, r),\\
\alpha &= (\theta_0, \varphi_0),\\
\beta_0(\alpha) &= \beta_0(\theta_0, \varphi_0) = 1,\\
z_\alpha &= (\alpha, \beta_0(\alpha)) = (\theta_0, \varphi_0, 1),\\
V &= \left\{(\theta, \varphi, r) \in M : r = 1, \varphi \in \left(-\tfrac{\pi}{2} + \delta_0, \tfrac{\pi}{2} - \delta_0\right)\right\},
\end{aligned} \tag{3-1}$$

with $\delta_0 > 0$ small enough that

$$
\begin{aligned}
\varphi^* &:= \frac{b_1(a_0+a_3+a_1\pi)-a_1(b_0+b_3+b_2\pi)}{a_1b_2-a_2b_1} \in \left(-\tfrac{\pi}{2}+\delta_0,\, \tfrac{\pi}{2}-\delta_0\right),\\
\mathcal{Z} &= \overline{V} \times \{r=1\},\\
x(t, z_\alpha, 0) &= (\theta_0+t, \varphi_0, 1),\\
F_0(t,x) &= (1, 0, r-1),\\
F_1(t,x) &= (a_0+a_1\theta+a_2\varphi+a_3r,\, b_0+b_1\theta+b_2\varphi+b_3r,\, c_0+c_1\theta+c_2\varphi+c_3r),\\
F_2(t,x,\varepsilon) &= 0,\\
T &= 2\pi,
\end{aligned}
$$

where we took $V \subset M_2$ as an open subset that contains the periodic orbit from which a limit cycle bifurcates, as we shall see next.

The fundamental matrix $\mathcal{M}_{z_\alpha}(t)$ with $\mathcal{M}_{z_\alpha}(0) = \mathrm{Id}$ of system (2-3) with $F_0$ and $x(t, z_\alpha, 0)$ described above is the matrix $\mathcal{M}_{z_\alpha}(t) = \exp(D_x F_0\, t)$, i.e.,

$$
\mathcal{M}_{z_\alpha}(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^t \end{pmatrix}.
$$

Note that since $F_0$ defines a linear differential system, the fundamental matrix $\mathcal{M}_{z_\alpha}(t)$ is independent of the initial conditions $z_\alpha$. We also have

$$
\mathcal{M}_{z_\alpha}^{-1}(0) - \mathcal{M}_{z_\alpha}^{-1}(2\pi) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1-e^{-2\pi} \end{pmatrix},
$$

and therefore, all the assumptions in the in the statement of Theorem 4 are satisfied.

With the described setting, the function $\mathcal{F}(\alpha) = \mathcal{F}(\theta_0, \varphi_0)$ from the statement of Theorem 4 associated with system (1-2) is

$$
\begin{aligned}
\mathcal{F}(\theta_0, \varphi_0) &= \xi\left(\int_0^{2\pi} \mathcal{M}_{z_\alpha}^{-1}(t) F_1(\theta_0+t, \varphi_0, 1)\, dt\right)\\
&= 2\pi\left(a_0+a_1(\theta_0+\pi)+a_2\varphi_0+b_3,\, b_0+b_1(\theta_0+\pi)+b_2\varphi_0+b_3\right).
\end{aligned}
$$

We have $\det(D\mathcal{F}) = 4\pi^2(a_1b_2-a_2b_1)$. Therefore $\det(D\mathcal{F}) \neq 0$ for all $(\theta_0, \varphi_0) \in V$. Thus, the only solution of $\mathcal{F}=0$ is given by

$$
\begin{aligned}
\theta_0 &= \frac{a_2(b_0+b_3+b_1\pi)-b_2(a_0+a_3+a_1\pi)}{a_1b_2-a_2b_1},\\
\varphi_0 &= \frac{b_1(a_0+a_3+a_1\pi)-a_1(b_0+b_3+b_2\pi)}{a_1b_2-a_2b_1}.
\end{aligned}
\tag{3-2}
$$

This solution, $(\theta_0, \varphi_0)$, where $\varphi_0 = \varphi^*$, is contained in the set $V$ described in (3-1).

Hence, by Theorem 4, if $\varepsilon > 0$ is small enough, there is a periodic solution, $(\theta(t,\varepsilon), \varphi(t,\varepsilon), r(t,\varepsilon))$, of system (1-3), which is a limit cycle, and such that

$$(\theta(0,\varepsilon), \varphi(0,\varepsilon), r(0,\varepsilon)) \to (\theta_0, \varphi_0, 1),$$

when $\varepsilon \to 0$, and where $\theta_0$ and $\varphi_0$ are given in (3-2).

## 4. Proof of Theorem 2

We use the result from averaging theory given in Theorem 4 to prove that, for some $\varepsilon > 0$ small enough, there exist a limit cycle of system (1-5) bifurcating from a periodic orbit of the same system with $\varepsilon = 0$.

Since the general solution of system (1-5) with $\varepsilon = 0$ (that is, the one of system (1-4)) is

$$\theta(t) = \theta_0 + t, \quad \varphi(t) = \varphi_0, \quad \nu(t) = \nu_0 + t, \quad \phi(t) = \phi_0, \quad r(t) = (r_0 - 1)e^t + 1,$$

we have that all the periodic solutions of that system are

$$\theta(t) = \theta_0 + t, \quad \varphi(t) = \varphi_0, \quad \nu(t) = \nu_0 + t, \quad \phi(t) = \phi_0, \quad r(t) = 1,$$

with $(\theta_0, \varphi_0, \nu_0, \phi_0) \in \mathbb{S}^2 \backslash \{\varphi_0 = \pm\pi/2\} \times \mathbb{S}^2 \backslash \{\varphi_0 = \pm\pi/2\}$. That is, the periodic solutions fill the invariant manifold $\{r = 1\}$ except for the four equilibrium points $\{\varphi = \pm\pi/2, \phi = \pm\pi/2\}$, and they have all period $2\pi$.

For applying Theorem 4 we take $M = (\mathbb{S}^2)^2 \times \mathbb{R}$ and

$$\begin{aligned}
k &= 4, \quad n = 5,\\
M_k &= M_4 = \{\theta, \varphi, \nu, \phi, r \in M : r = 1\} \cong (\mathbb{S}^2)^2,\\
x &= (\theta, \varphi, \nu, \phi, r),\\
\alpha &= (\theta_0, \varphi_0, \nu_0, \phi_0),\\
\beta_0(\alpha) &= \beta_0(\theta_0, \varphi_0, \nu_0, \phi_0) = 1,\\
z_\alpha &= (\alpha, \beta_0(\alpha)) = (\theta_0, \varphi_0, \nu_0, \phi_0, 1),\\
V &= \left\{(\theta, \varphi, \nu, \phi, r) \in M : r = 1, \varphi \in \left(-\tfrac{\pi}{2} + \delta_0, \tfrac{\pi}{2} - \delta_0\right)\right\}
\end{aligned} \tag{4-1}$$

with $\delta_0 > 0$ small enough that $\varphi_0, \phi_0$ satisfy (4-2) and

$$\begin{aligned}
\varphi_0, \phi_0 &\in \left(-\tfrac{\pi}{2} + \delta_0, \tfrac{\pi}{2} - \delta_0\right),\\
\mathcal{Z} &= \overline{V} \times \{r = 1\},\\
x(t, z_\alpha, 0) &= (\theta_0 + t, \varphi_0, \nu_0 + t, \phi_0, 1),\\
F_0(t, x) &= (1, 0, 1, 0, r - 1),
\end{aligned}$$

$$F_1(t,x) = \begin{pmatrix} a_0+a_1\theta+a_2\varphi+a_3\nu+a_4\phi+a_5 r \\ b_0+b_1\theta+b_2\varphi+b_3\nu+b_4\phi+b_5 r \\ c_0+c_1\theta+c_2\varphi+c_3\nu+c_4\phi+c_5 r \\ d_0+d_1\theta+d_2\varphi+d_3\nu+d_4\phi+d_5 r \\ e_0+e_1\theta+e_2\varphi+e_3\nu+e_4\phi+e_5 r \end{pmatrix},$$

$$F_2(t,x,\varepsilon) = 0,$$

$$T = 2\pi,$$

where we chose $V \subset M_4$ as an open subset that contains the periodic orbit from which a limit cycle bifurcates.

The fundamental matrix $\mathcal{M}_{z_\alpha}(t)$ with $\mathcal{M}_{z_\alpha}(0) = \mathrm{Id}$ of system (2-3) with $F_0$ and $x(t, z_\alpha, 0)$ described above is the matrix $\mathcal{M}_{z_\alpha}(t) = \exp(D_x F_0\, t)$, i.e.,

$$\mathcal{M}_{z_\alpha}(t) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & e^t \end{pmatrix}.$$

We also have

$$\mathcal{M}_{z_\alpha}^{-1}(0) - \mathcal{M}_{z_\alpha}^{-1}(2\pi) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-e^{-2\pi} \end{pmatrix},$$

and therefore, all the assumptions in the statement of Theorem 4 are satisfied.

In this setting, the function $\mathcal{F}(\alpha) = \mathcal{F}(\theta_0, \varphi_0, \nu_0, \phi_0)$ in Theorem 4 associated with system (1-5) is

$$\mathcal{F}(\theta_0, \varphi_0, \nu_0, \phi_0) = \xi\Big( \int_0^{2\pi} \mathcal{M}_{z_\alpha}^{-1}(t) F_1(\theta_0+t, \varphi_0, \nu_0+t, \phi_0, 1)\, dt \Big) = (\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4),$$

with

$$\begin{aligned} \mathcal{F}_1 &= 2\pi(a_0 + a_1\theta_0 + a_1\pi + a_2\varphi_0 + a_3\nu_0 + a_3\pi + a_4\phi_0 + a_5), \\ \mathcal{F}_2 &= 2\pi(b_0 + b_1\theta_0 + b_1\pi + b_2\varphi_0 + b_3\nu_0 + b_3\pi + b_4\phi_0 + b_5), \\ \mathcal{F}_3 &= 2\pi(c_0 + c_1\theta_0 + c_1\pi + c_2\varphi_0 + c_3\nu_0 + c_3\pi + c_4\phi_0 + c_5), \\ \mathcal{F}_4 &= 2\pi(d_0 + d_1\theta_0 + d_1\pi + d_2\varphi_0 + d_3\nu_0 + d_3\pi + d_4\phi_0 + d_5). \end{aligned}$$

Also, we have

$$\det(D\mathcal{F}) = 16\pi^4 \det \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{pmatrix} \neq 0,$$

by assumption. The initial conditions $(\theta_0, \varphi_0, \nu_0, \phi_0)$ such that $\mathcal{F}(\theta_0, \varphi_0, \nu_0, \phi_0) = 0$ are the solutions of the linear system

$$
\begin{aligned}
a_1\theta_0 + a_2\varphi_0 + a_3\nu_0 + a_4\phi_0 &= -a_0 - a_1\pi - a_3\pi - a_5,\\
b_1\theta_0 + b_2\varphi_0 + b_3\nu_0 + b_4\phi_0 &= -b_0 - b_1\pi - b_3\pi - b_5,\\
c_1\theta_0 + c_2\varphi_0 + c_3\nu_0 + c_4\phi_0 &= -c_0 - c_1\pi - c_3\pi - c_5,\\
d_1\theta_0 + d_2\varphi_0 + d_3\nu_0 + d_4\phi_0 &= -d_0 - d_1\pi - d_3\pi - d_5.
\end{aligned}
\tag{4-2}
$$

Since $\det(D\mathcal{F}) \neq 0$, system (4-2) has a unique solution, $(\theta_0, \varphi_0, \nu_0, \phi_0)$, and this solution is contained in the set $V$ described in (4-1).

Hence, by Theorem 4, if $\varepsilon > 0$ is small enough, there is a periodic solution,

$$(\theta(t, \varepsilon), \varphi(t, \varepsilon), \nu(t, \varepsilon), \phi(t, \varepsilon), r(t, \varepsilon)),$$

of system (1-5), which is a limit cycle, and such that

$$(\theta(0, \varepsilon), \varphi(0, \varepsilon), \nu(0, \varepsilon), \phi(0, \varepsilon), r(0, \varepsilon)) \to (\theta_0, \varphi_0, \nu_0, \phi_0, 1),$$

when $\varepsilon \to 0$, and where $\theta_0, \varphi_0, \nu_0$, and $\phi_0$ are given by the unique solution of system (4-2).

## 5. Proof of Theorem 3

Since the general solution of system (1-7) with $\varepsilon = 0$ is given by

$$x(t) = x_0 \cos t - y_0 \sin t,\ \ y(t) = x_0 \sin t + y_0 \cos t,\ \ \theta(t) = \theta_0 + t,\ \ \varphi(t) = \varphi_0, \tag{5-1}$$

the whole phase space is filled by periodic solutions, except from the equilibrium points $(x, y, \theta, \varphi) = (0, 0, \theta, -\pi/2)$ and $(x, y, \theta, \varphi) = (0, 0, \theta, \pi/2)$. Hence, the periodic solutions of the differential system (1-6) fill an open set of the phase space $M = \mathbb{R}^2 \times \mathbb{S}^2$.

To prove Theorem 3 we use the result given in Theorem 5 to deduce that there exists a limit cycle of system (1-7), for some $\varepsilon > 0$ small enough, bifurcating from the periodic orbits of the same system with $\varepsilon = 0$.

To clarify the notation, here we denote the solution $x(t, z, 0)$ from the statement of Theorem 5 by $\boldsymbol{x}(t, z, 0)$, and $x$ will denote the first variable in the phase space.

For applying Theorem 5 we take $M = \mathbb{R}^2 \times \mathbb{S}^2$ and

$$
\begin{aligned}
\boldsymbol{x} &= (x, y, \theta, \varphi),\\
z &= (x_0, y_0, \theta_0, \varphi_0),\\
\boldsymbol{x}(t, z, 0) &= (x(t), y(t), \theta(t), \varphi(t)) \quad \text{given by (5-1)},
\end{aligned}
$$

$$F_0(t,x) = (-y, x, 1, 0),$$

$$F_1(t,x) = \begin{pmatrix} a_0+a_1x+a_2y+a_3\theta+a_4\varphi \\ b_0+b_1x+b_2y+b_3\theta+b_4\varphi \\ c_0+c_1x+c_2y+c_3\theta+c_4\varphi \\ d_0+d_1x+d_2y+d_3\theta+d_4\varphi \end{pmatrix},$$

$$F_2(t,x,\varepsilon) = 0,$$

$$T = 2\pi,$$

$$V = \left\{(x,y,\theta,\varphi) \in M : \|(x,y)\| < 1+\kappa, \varphi \in \left(-\tfrac{\pi}{2}+\delta_0, \tfrac{\pi}{2}-\delta_0\right)\right\}, \tag{5-2}$$

with $\kappa = \dfrac{2\sqrt{a_3^2+b_3^2}}{\sqrt{a_1^2+a_2^2+b_1^2+b_2^2+2a_1b_2-2a_2b_1}}$ and $\delta_0 > 0$ small enough that

$$\varphi^* := \frac{c_0\,d_3 - c_3\,d_0}{c_3\,d_4 - c_4\,d_3} \in \left(-\tfrac{\pi}{2}+\delta_0, \tfrac{\pi}{2}-\delta_0\right),$$

and where we chose $V \subset M$ as an open subset that contains the periodic orbit from which a limit cycle bifurcates.

The fundamental matrix $\mathcal{M}_z(t)$ of system (2-3) with $\mathcal{M}_z(0) = \mathrm{Id}$ and with $F_0$ and $\boldsymbol{x}(t,z,0)$ as just described is given by

$$\mathcal{M}_z(t) = \begin{pmatrix} \cos t & -\sin t & 0 & 0 \\ \sin t & \cos t & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Therefore all the assumptions in the statement of Theorem 5 are satisfied.

In this setting the function $\mathcal{F}(z) = \mathcal{F}(x_0, y_0, \theta_0, \varphi_0)$ in Theorem 5 associated with system (1-7), namely,

$$\mathcal{F}(x_0, y_0, \theta_0, \varphi_0) = \int_0^{2\pi} \mathcal{M}_z^{-1}(t) F_1(t, \boldsymbol{x}(t,x,0))\, dt,$$

is given by $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4)$, which after some straightforward computations can be written as

$$\begin{aligned}
\mathcal{F}_1 &= (\pi a_2 - \pi b_1)y_0 + (\pi b_2 + \pi a_1)x_0 - 2\pi b_3, \\
\mathcal{F}_2 &= (\pi b_2 + \pi a_1)y_0 + (\pi b_1 - \pi a_2)x_0 + 2\pi a_3, \\
\mathcal{F}_3 &= 2\pi c_3\theta_0 + 2\pi c_4\varphi_0 + 2\pi^2 c_3 + 2\pi c_0, \\
\mathcal{F}_4 &= 2\pi d_3\theta_0 + 2\pi d_4\varphi_0 + 2\pi^2 d_3 + 2\pi d_0.
\end{aligned}$$

Assuming that

$$\det(\mathcal{DF}) = \det \begin{pmatrix} \pi(b_2+a_1) & \pi(a_2-b_1) & 0 & 0 \\ \pi(b_1-a_2) & \pi(b_2+a_1) & 0 & 0 \\ 0 & 0 & 2\pi c_3 & 2\pi c_4 \\ 0 & 0 & 2\pi d_3 & 2\pi d_4 \end{pmatrix} \neq 0, \tag{5-3}$$

the linear system $(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4) = (0,0,0,0)$ has a unique solution, given by

$$\begin{aligned} x_0 &= \frac{(2b_2+2a_1)b_3 - 2a_3b_1 + 2a_2a_3}{b_2^2+b_1^2+a_2^2+a_1^2+2a_1b_2-2a_2b_1}, \\ y_0 &= -\frac{(2b_1-2a_2)b_3 + 2a_3b_2 + 2a_1a_3}{b_2^2+b_1^2+a_2^2+a_1^2+2a_1b_2-2a_2b_1}, \\ \theta_0 &= -\frac{(\pi c_3 + c_0)d_4 - \pi c_4 d_3 - c_4 d_0}{c_3d_4 - c_4d_3}, \qquad \varphi_0 = \frac{c_0d_3 - c_3d_0}{c_3d_4 - c_4d_3}. \end{aligned} \tag{5-4}$$

This solution $(x_0, y_0, \theta_0, \varphi_0)$, where $\varphi_0 = \varphi^*$, is contained in the set $V$ in (5-2).

The condition (5-3) is clearly satisfied for all $(x_0, y_0, \theta_0, \varphi_0) \in V$ taking into account the assumptions in the statement of Theorem 3.

Hence, by Theorem 5, there is a periodic solution $(x(t,\varepsilon), y(t,\varepsilon), \theta(t,\varepsilon), \varphi(t,\varepsilon))$ of system (1-7), which is a limit cycle, and such that

$$(x(0,\varepsilon), y(0,\varepsilon), \theta(0,\varepsilon), \varphi(0,\varepsilon)) \to (x_0, y_0, \theta_0, \varphi_0)$$

when $\varepsilon \to 0$, and where $x_0$, $y_0$, $\theta_0$ and $\varphi_0$ are given in (5-4).

## References


[Arnold 2006] V. I. Arnold, *Ordinary differential equations*, Springer, 2006. MR

[Buică et al. 2007] A. Buică, J.-P. Françoise, and J. Llibre, "Periodic solutions of nonlinear periodic differential systems with a small parameter", *Commun. Pure Appl. Anal.* **6**:1 (2007), 103–111. MR Zbl

[Christopher and Lloyd 1996] C. J. Christopher and N. G. Lloyd, "Small-amplitude limit cycles in polynomial Liénard systems", *NoDEA Nonlinear Differential Equations Appl.* **3**:2 (1996), 183–190. MR Zbl

[Ferragut et al. 2007] A. Ferragut, J. Llibre, and M. A. Teixeira, "Hyperbolic periodic orbits from the bifurcation of a four-dimensional nonlinear center", *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **17**:6 (2007), 2159–2167. MR Zbl

[Giacomini et al. 1996] H. Giacomini, J. Llibre, and M. Viano, "On the nonexistence, existence and uniqueness of limit cycles", *Nonlinearity* **9**:2 (1996), 501–516. MR Zbl

[Han and Li 2012] M. Han and J. Li, "Lower bounds for the Hilbert number of polynomial systems", *J. Differential Equations* **252**:4 (2012), 3278–3304. MR Zbl

[Hartman 1960] P. Hartman, "A lemma in the theory of structural stability of differential equations", *Proc. Amer. Math. Soc.* **11** (1960), 610–620. MR Zbl

[Ilyashenko 2002] Y. Ilyashenko, "Centennial history of Hilbert's 16th problem", *Bull. Amer. Math. Soc.* (*N.S.*) **39**:3 (2002), 301–354. MR Zbl

[Laferriere et al. 1999] G. Laferriere, G. J. Pappas, and S. Yovine, "A new class of decidable hybrid systems", pp. 137–151 in *Hybrid Systems*: *Computation and Control*, edited by F. W. Vaandrager and J. H. van Schuppen, Lecture Notes in Computer Science **1569**, Springer, Berlin, 1999. Zbl

[Lafferriere et al. 2001] G. Lafferriere, G. J. Pappas, and S. Yovine, "Symbolic reachability computation for families of linear vector fields", *J. Symbolic Comput.* **32**:3 (2001), 231–253. MR Zbl

[Llibre and Teixeira 2009] J. Llibre and M. A. Teixeira, "Limit cycles bifurcating from a two-dimensional isochronous cylinder", *Appl. Math. Lett.* **22**:8 (2009), 1231–1234. MR Zbl

[Llibre and Zhang 2016] J. Llibre and X. Zhang, "Limit cycles of linear vector fields on manifolds", *Nonlinearity* **29**:10 (2016), 3120–3131. MR Zbl

[Llibre et al. 2007] J. Llibre, M. A. Teixeira, and J. Torregrosa, "Limit cycles bifurcating from a $k$-dimensional isochronous center contained in $\mathbb{R}^n$ with $k \leq n$", *Math. Phys. Anal. Geom.* **10**:3 (2007), 237–249. MR Zbl

[Llibre et al. 2010] J. Llibre, R. M. Martins, and M. A. Teixeira, "Periodic orbits, invariant tori, and cylinders of Hamiltonian systems near integrable ones having a return map equal to the identity", *J. Math. Phys.* **51**:8 (2010), art. id. 082704. MR Zbl

[Malkin 1956] I. G. Malkin, *Some problems of the theory of nonlinear oscillations*, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1956. In Russian. MR Zbl

[Perrizo 1974] W. Perrizo, "$\omega$-linear vector fields on manifolds", *Trans. Amer. Math. Soc.* **196** (1974), 289–312. MR Zbl

[Roseau 1966] M. Roseau, *Vibrations non linéaires et théorie de la stabilité*, Springer Tracts in Natural Philosophy **8**, Springer, 1966. MR Zbl

[Sanders et al. 2007] J. A. Sanders, F. Verhulst, and J. Murdock, *Averaging methods in nonlinear dynamical systems*, 2nd ed., Applied Mathematical Sciences **59**, Springer, 2007. MR

[Sotomayor 1979] J. Sotomayor, *Lições de equações diferenciais ordinárias*, Projeto Euclides **11**, Instituto de Matemática Pura e Aplicada, Rio de Janeiro, 1979. MR Zbl

[Verhulst 1996] F. Verhulst, *Nonlinear differential equations and dynamical systems*, 2nd ed., Springer, 1996. MR Zbl

Received July 13, 2022. Revised January 13, 2023.

CLARA CUFÍ-CABRÉ
clara.cufi@uab.cat
DEPARTAMENT DE MATEMÀTIQUES
UNIVERSITAT AUTÒNOMA DE BARCELONA
BARCELONA
SPAIN

JAUME LLIBRE
jaume.llibre@uab.cat
DEPARTAMENT DE MATEMÀTIQUES
UNIVERSITAT AUTÒNOMA DE BARCELONA
BARCELONA
SPAIN

PACIFIC JOURNAL OF MATHEMATICS
Vol. 324, No. 2, 2023
https://doi.org/10.2140/pjm.2023.324.265

# HOROSPHERICAL COORDINATES OF LATTICE POINTS IN HYPERBOLIC SPACES: EFFECTIVE COUNTING AND EQUIDISTRIBUTION

TAL HORESH AND AMOS NEVO

**We establish effective counting results for lattice points in families of domains in real, complex and quaternionic hyperbolic spaces of any dimension. The domains we focus on are defined as product sets with respect to an Iwasawa decomposition. Several natural diophantine problems can be reduced to counting lattice points in such domains. These include equidistribution of the ratio of the length of the shortest solution $(x, y)$ to the gcd equation $bx - ay = 1$ relative to the length of $(a, b)$, where $(a, b)$ ranges over primitive vectors in a disc whose radius increases, the natural analog of this problem in imaginary quadratic number fields, as well as equidistribution of integral solutions to the diophantine equation defined by an integral Lorentz form in three or more variables. We establish an effective rate of convergence for these equidistribution problems, depending on the size of the spectral gap associated with a suitable lattice subgroup in the isometry group of the relevant hyperbolic space. The main result underlying our discussion amounts to establishing effective joint equidistribution for the horospherical component and the radial component in the Iwasawa decomposition of lattice elements.**



## 1. Introduction and statement of main results

Our goal in the present paper is to establish effective counting and equidistribution results for Iwasawa components of elements of lattice subgroups of isometry groups

*MSC2020:* 22E30, 22E40.

*Keywords:* hyperbolic spaces, horospherical coordinates, equidistribution of lattice points, spectral gap.

of (real, complex or quaternionic) hyperbolic spaces. The problem is an instance of counting problems in which one seeks to study the asymptotic behavior of the number of lattice orbit points in some expanding family of regions in hyperbolic space, going beyond the classical problem of counting in hyperbolic balls. Since counting the points of a lattice orbit in regions of hyperbolic space reduces to counting the elements of the lattice subgroup in suitable lifted regions of the group of isometries $G$, we will focus on counting in the group itself, rather than in the symmetric space.

The domains that we consider are product sets in the Iwasawa coordinates on $G$. We write the Iwasawa decomposition in the form $G = NAK$, where $K$ is maximal compact, $A \cong \mathbb{R}$, and $N$ is the unipotent subgroup that stabilizes an ideal boundary point which we denote $\{\infty\}$. The map $N \times A \times K \to G$ given by $(n, a, k) \mapsto nak$ is a diffeomorphism, so these are indeed coordinates on $G$.

Let $G$ denote a nonexceptional simple Lie group of real rank one with finite center; namely, locally isomorphic to one of the following: $\mathrm{SO}(1, n)$, $\mathrm{SU}(1, n)$, or $\mathrm{SP}(1, n)$ for some $n \geq 1$. The corresponding rank 1 symmetric spaces $G/K$ are, respectively, the real hyperbolic space $\mathbf{H}^n_{\mathbb{R}}$, the complex hyperbolic space $\mathbf{H}^n_{\mathbb{C}}$ and the quaternionic hyperbolic space $\mathbf{H}^n_{\mathbb{H}}$. The group $G$ acts on the corresponding space by isometries of a Riemannian distance, which we will refer to as the "hyperbolic distance" and denote by $d(\cdot\,, \cdot)$. The remaining rank one simple Lie group is $\mathrm{F}_{4(-20)}$, which corresponds to the octonionic hyperbolic plane $\mathbf{H}^2_{\mathbb{O}}$; we shall not consider this case.

A Haar measure $\mu$ on $G$ is given in the Iwasawa coordinates as follows. We parametrize $A$ by $A = \{a_t : t \in \mathbb{R}\}$, where $a_t = \exp tH_1$, and $H_1 \in \mathfrak{a} = \mathrm{Lie}(A)$ is the element satisfying $\alpha(H_1) = 1$, where $\alpha$ is the unique short positive root.

Let $\mu_K$ denote a Haar probability measure on $K$. The subgroup $N$ is parametrized by a euclidean space of the appropriate dimension, see Table 1 for the different cases, and a Haar measure on $N$ is the Lebesgue measure on this underlying euclidean space. A Haar measure on $G$ with respect to the Iwasawa coordinates is given by

$$\mu = \mu_N \times \frac{dt}{e^{2\rho t}} \times \mu_K, \tag{1-1}$$

where $\rho$ is a positive parameter that depends on the group $G$. The Iwasawa component subgroups, symmetric spaces and Haar measure of the rank 1 groups are summarized in Table 1 in Section 4A below.

Let $\Gamma \subset G$ be a lattice subgroup. We consider lattice points whose $N$ and $K$ components lie in given bounded subsets $\Psi \subset N$ and $\Phi \subseteq K$, and study their asymptotic behavior as their $A$-component $a_t$ ranges over $(-\infty, 0]$ and tends to $-\infty$.

Define the family $\{R_T(\Psi, \Phi)\}_{T>0}$, where

$$R_T(\Psi, \Phi) := \Psi A_{[-T,0]}\Phi = \{na_tk : n \in \Psi,\ t \in [-T, 0],\ k \in \Phi\}$$

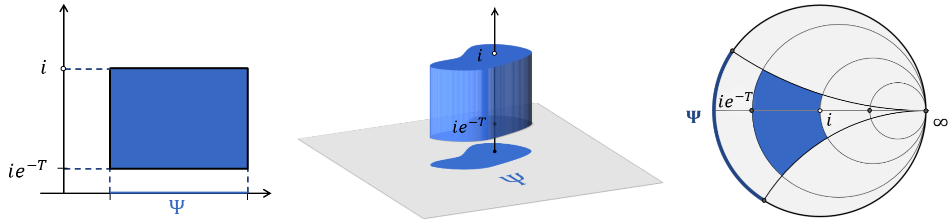



**Figure 1.** The domains $R_T(\Psi, \Phi)$ projected to the real hyperbolic plane in the upper half-plane model (left), the real hyperbolic 3-space in the upper half-space model (middle) and the real hyperbolic plane in the unit disc model (right).

as $T \to \infty$; see Figure 1. According to (1-1), the volume of these domains equals

$$\mu(R_T(\Psi, \Phi)) = \frac{1}{2\rho} \cdot \mu_N(\Psi)\mu_K(\Phi)(e^{2\rho T} - 1),$$

while the volume of $NA_{[0,\infty)}K$ is finite. We shall require that the domains $\Psi \subset N$ and $\Phi \subseteq K$ are *nice*: bounded, embedded smooth submanifolds of full dimension whose boundaries are piecewise smooth — namely, a finite union of smooth submanifolds of codimension 1. We allow the case where only some of these boundary submanifolds are included in the nice set, while others are not, e.g., a half open rectangle, two of whose edges are included and the remaining two are not.

We can now formulate our main result, namely an effective solution to the lattice point counting problem in the sets $R_T(\Psi, \Phi)$, for an arbitrary lattice.

**Theorem 1.1.** *Fix any Iwasawa decomposition in $G$, let $\Psi \subset N$ and $\Phi \subseteq K$ be nice domains, and consider the family $R_T(\Psi, \Phi)$ as defined above. For any lattice $\Gamma < G$, there exists a parameter $\kappa = \kappa(\Gamma) < 1$ (defined explicitly in (4-2)) such that for $T > 0$,*

$$\begin{aligned}\#(R_T(\Psi, \Phi) \cap \Gamma) &= \frac{\mu(R_T(\Psi, \Phi))}{\mu(G/\Gamma)} + O\big(\mu(R_T(\Psi, \Phi))^{\kappa} \cdot \log \mu(R_T(\Psi, \Phi))\big) \\ &= \frac{\mu_N(\Psi)\mu_K(\Phi)}{\mu(G/\Gamma)} \cdot \frac{e^{2\rho T}}{2\rho} + O_{\Gamma,\Psi,\Phi}(T(e^{2\rho T})^{\kappa}).\end{aligned}$$

For example, for the lattice $\mathrm{SL}_2(\mathbb{Z})$ in $\mathrm{SL}_2(\mathbb{R})$, we will see below that one can take $\kappa(\mathrm{SL}_2(\mathbb{Z})) = \frac{7}{8}$.

This case has of course received considerable attention in the past, and we will describe it further at the end of the next section.

We now formulate our two main corollaries. The first is an effective ratio theorem for lattice points in the sets $R_T(\Psi, \Phi)$, and the second effective joint equidistribution of the $N$ and $K$ components in the Iwasawa decomposition, as follows.

For $H \in \{N, A, K\}$, we denote the projection to the $H$-component by

$$\pi_H : G = NAK \to H.$$

**Corollary 1.2.** (1) *Let $\Psi, \Psi' \subset N$ and $\Phi, \Phi' \subseteq K$ be nice sets, and let $\Gamma < G$ be any lattice. Then the denominator in the following ratio is eventually positive and*

$$\frac{\#(\Gamma \cap R_T(\Psi', \Phi'))}{\#(\Gamma \cap R_T(\Psi, \Phi))} = \frac{\mu_N(\Psi')\mu_K(\Phi')}{\mu_N(\Psi)\mu_K(\Phi)} + O(T(e^{2\rho T})^{-(1-\kappa)}),$$

*where the implied constant depends on $\Psi, \Psi', \Phi, \Phi'$ and $\kappa = \kappa(\Gamma) < 1$ is the exponent associated with $\Gamma$ appearing in Theorem 1.1.*

(2) *The set of $N$-components and $K$-components become jointly effectively equidistributed in $\Psi \times \Phi$ with respect to $\mu_N \times \mu_K$ as $T \to \infty$. Namely, for every Lipschitz function $\psi$ defined on $\Psi$, and Lipschitz function $\phi$ defined on $\Phi$,*

$$\frac{1}{\#(\Gamma \cap \Psi A_{[-T,0]}\Phi)} \sum_{\gamma \in \Psi A_{[-T,0]}\Phi} \psi(\pi_N(\gamma))\phi(\pi_K(\gamma))$$
$$= \frac{1}{\mu_N(\Psi)} \int_N \psi(n)\, d\mu_N(n) \cdot \frac{1}{\mu_K(\Phi)} \int_K \phi(k)\, d\mu_K(k) + O(Te^{-2\rho(1-\kappa)T}),$$

*where the constant depends on the functions $\psi$ and $\phi$, and the sets $\Psi$ and $\Phi$.*

The proofs of Theorem 1.1 and Corollary 1.2 are in Section 4.

**Remark 1.3.** When the lattice in question is cocompact, the (unbounded) cuspidal strip $\Psi A_{(0,\infty)}\Phi$ may contain infinitely many lattice points, despite its bounded volume. One expects that this fact should not change the overall asymptotics, but the irregularity caused by this cuspidal strip requires further consideration, and this is the reason why we have decided to consider here the domains $R_T(\Psi, \Phi)$ which are truncated at some fixed height, say $t = 0$. These domains are the natural ones to consider in the context of lattices with a cusp, because they account for *all but finitely many* lattice points in the strip $\Psi A_{(-\infty,\infty)}\Phi$, provided the lattice orbit points $\gamma \cdot z$ all have bounded height; namely, that the $A$-component $a_t$ of $\gamma \cdot z$ satisfies $t \le C(\Gamma, z)$. This property amounts to a generalized form of the Shimizu lemma, and it holds in all the specific examples we will consider below, namely real hyperbolic spaces of arbitrary dimension.

**Remark 1.4.** Iwasawa decomposition of a Lie group is used in one of two conventions: $G = NAK$ or $G = KAN$. Our results are phrased with respect to the first option, but the corresponding statements with respect to the $KAN$ decomposition may be easily deduced. Indeed, the $KAN$ coordinates of $g \in G$ are obtained from the $NAK$ coordinates of $g^{-1}$: $g^{-1} = nak$ implies $g = k^{-1}a^{-1}n^{-1}$. In particular, the Haar measure with respect to the $KAN$ coordinates is $\mu_K \times e^{2\rho t}\,dt \times \mu_N$, and the statement of Theorem 1.1 is replaced by

$$\#\Gamma \cap (\Phi A_{[0,T]}\Psi) = \frac{\mu_N(\Psi)\mu_K(\Phi)}{\mu(G/\Gamma)} \cdot \frac{e^{2\rho T}}{2\rho} + O(T(e^{2\rho T})^{\kappa}),$$

for $\Phi \subset K,\ \Psi \subset N,\ \kappa$ as in Theorem 1.1, and $T > 0$.

**Remark 1.5.** Theorem 1.1 is formulated for a family of domains in $G$ itself, rather than in the symmetric space, and this makes it possible to analyze the distribution of the $K$-components of the lattice elements. As we shall see below, equidistribution of the $K$-components plays a key role in a number of applications, including angular equidistribution of shortest solutions to the gcd equation in $\mathbb{Z}^2$ and in imaginary quadratic number fields. The connection between the problem of equidistribution of the norms of the shortest solutions of the gcd equation in $\mathbb{Z}^2$ and the equidistribution of Iwasawa $N$-components in $\mathrm{SL}_2(\mathbb{Z})$ was first pointed out by Risager and Rudnick [2009], and has motivated the approach pursued in the present paper. We will first formulate and prove some applications of Theorem 1.1 and Corollary 1.2 and then comment further on the history of this problem.

## 2. Iwasawa components and diophantine problems

**2A.** ***Distribution of shortest solutions of the gcd equation.*** We begin with applications related to certain arithmetic lattices in $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{SL}_2(\mathbb{C})$. In what follows, the norm we refer to is the euclidean norm on $\mathbb{R}^2$ or $\mathbb{C}^2$, denoted by $\|\cdot\|$.

For every primitive integral vector $v = (a, b)$, let $w_v$ denote the shortest integral vector that completes $v$ to a (positively oriented) basis of $\mathbb{Z}^2$, namely, the shortest solution to the gcd equation $bx - ay = 1$. Let $\theta_v$ denote the angle between $w_v$ and $v$, where $0 < \theta_v < \pi$. We say that $v$ is *positive* if $\theta_v$ is acute, and *negative* if $\theta_v$ is obtuse (Figure 3).

Let $\Theta \subseteq S^1$ be a subarc of the unit circle and $|\Theta|$ its length, $0 < |\Theta| \le 2\pi$. Let $\mathcal{S}_\Theta$ denote the corresponding sector of the plane $\mathbb{R}^2$ (see Figure 2), and

$$\mathcal{S}_\Theta(R) = \mathcal{S}_\Theta \cap B_R \quad \text{where } B_R = \{v \in \mathbb{R}^2\,;\ \|v\| \le R\}.$$

In the case of the lattice $\mathrm{SL}_2(\mathbb{Z})$, Corollary 1.2 has the following geometric interpretation.

**Theorem 2.1.** *For every primitive integral vector $v = (a, b)$, let $w_v$ and $\theta_v$ be as above.*

(1) *The number of primitive vectors $v$ in $\mathcal{S}_\Theta(R)$ is given by*

$$\frac{3}{\pi^2}|\Theta| R^2 + O(R^{7/4}\log R).$$

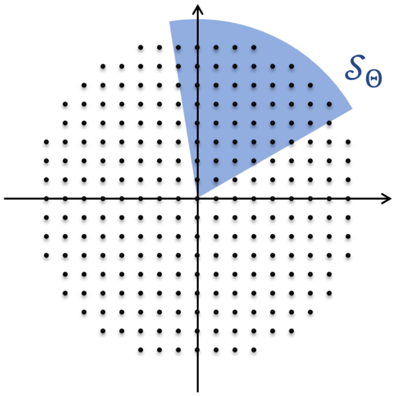



**Figure 2.** $\mathbb{Z}^2$-points contained in the sector $\mathcal{S}_\Theta$.

(2) *The ratios* $\|w_v\|/\|v\|$ *of the length of the shortest solution* $w_v$ *relative to the length of* $v$, *for* $v \in \mathcal{S}_\Theta(R)$, *become effectively equidistributed in the interval* $\left[0, \frac{1}{2}\right]$ *as* $R \to \infty$. *The rate of convergence for a Lipschitz function* $f$ *is* $O_{f,\Theta}(R^{-1/4} \cdot \log R)$.

(3) *The number of positive primitive vectors in* $\mathcal{S}_\Theta(R)$ *is*

$$\frac{3}{2\pi^2}|\Theta|R^2 + O(R^{7/4}\log R),$$

*and the same formula holds for the number of negative primitive vectors in* $\mathcal{S}_\Theta(R)$.

(4) *Part (2) holds when* $v$ *is restricted to positive vectors only, or to negative vectors only.*

**Remark 2.2.** Note that part (1) counts the number of primitive integral vectors of norm at most $R$ in any sector in the plane, with error estimate given by $R^{7/4}\log R$. This result is of course not new, and in fact our estimate falls short of a much better estimate for this problem that follows from the work of Selberg and Good; see [Selberg 1991; Good 1983; 1984]. We refer to Section 3C for a more detailed discussion.

*Proof of Theorem 2.1.* If $v = (a, b) \in \mathbb{Z}^2$ is primitive, it can be completed to countably many matrices in $\mathrm{SL}_2(\mathbb{Z})$, representing the different integral solutions to the equation $bx - ay = 1$. The $NAK$ components of these integral matrices encode the vector $v$ and the different solutions to $bx - ay = 1$ as follows. By the Iwasawa decomposition of $\mathrm{SL}_2(\mathbb{R})$ if $(x, y)$ is such a solution, the corresponding matrix in

$\mathrm{SL}_2(\mathbb{Z})$ has $NAK$ decomposition

$$\begin{pmatrix} x & y \\ a & b \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & \frac{xa+yb}{a^2+b^2} \\ & 1 \end{pmatrix}}_{N\text{-component}} \underbrace{\begin{pmatrix} \frac{1}{\sqrt{a^2+b^2}} & \\ & \sqrt{a^2+b^2} \end{pmatrix}}_{A\text{-component}} \underbrace{\frac{1}{\sqrt{a^2+b^2}} \begin{pmatrix} b & -a \\ a & b \end{pmatrix}}_{K\text{-component}}.$$

The subgroups $N$, $A$, and $K$ of $\mathrm{SL}_2(\mathbb{R})$ are identified with $\mathbb{R}$, $\mathbb{R}$, and $S^1$ respectively through $\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) \leftrightarrow x$, $\left(\begin{smallmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{smallmatrix}\right) \leftrightarrow t$, and $\left(\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right) \leftrightarrow \theta$. We choose here the Haar measure $\mu_K$ on the circle to have total mass $2\pi$. Note that the $A$ and $K$ components of the Iwasawa decomposition depend only on the vector $v$: the $A$-component $\left(\begin{smallmatrix} 1/\|v\| & 0 \\ 0 & \|v\| \end{smallmatrix}\right)$ encodes the norm of $v$, and the $K$-component $\left(\begin{smallmatrix} v^{\perp}/\|v\| \\ v/\|v\| \end{smallmatrix}\right)$ is the rotation matrix by an angle of $\theta + \frac{1}{2}\pi$, where $\theta$ is the angle between the positive $x$-axis and $v$ (counterclockwise). The $N$-component depends on the specific solution $(x, y)$, namely the upper row of the matrix. If $w := (x, y)$, then the $N$-component is $\left(\begin{smallmatrix} 1 & \langle w, v\rangle/\|v\|^2 \\ 0 & 1 \end{smallmatrix}\right)$, namely its $N$-coordinate is given by the projection of $w$ to the line $\mathrm{span}\{v\}$, divided by the norm of $v$.

The different solutions to $bx - ay = 1$ are $\{(x + ma, y + mb) : m \in \mathbb{Z}\}$, and they correspond to matrices $\left(\begin{smallmatrix} x+ma & y+mb \\ a & b \end{smallmatrix}\right)$ whose $N$-coordinate is

$$\frac{(x+ma)a + (y+mb)b}{a^2+b^2} = m + \frac{xa+yb}{a^2+b^2} = m + \frac{\langle w, v\rangle}{\|v\|^2},$$

namely all the integral translations of $\langle w, v\rangle/\|v\|^2$, the $N$-coordinate appearing above.

Observe that among all the integral matrices that correspond to $v$, the one whose $N$-coordinate is minimal — i.e., in the interval $\left[-\frac{1}{2}, \frac{1}{2}\right)$ — is the one that corresponds to the shortest solution to $bx - ay = 1$, namely, the one whose upper row has minimal norm. This is because the integral solutions are the integral points on the affine line $\mathrm{span}\{v\} + w$ (where $w$ is any fixed solution of $bx - ay = 1$) which is parallel to $\mathrm{span}\{v\}$. Hence, when decomposing $\mathbb{R}^2$ as $\mathrm{span}\{v\} \oplus \mathrm{span}\{v^{\perp}\}$, all of these solutions have the same $v^{\perp}$ component, and the shortest integral solution is the one with the shortest $v$-component (namely, the shortest projection on $\mathrm{span}\{v\}$). The shortest integral solution $w_v$ corresponds to the matrix $\left(\begin{smallmatrix} w_v \\ v \end{smallmatrix}\right) = \left(\begin{smallmatrix} x_v & y_v \\ a & b \end{smallmatrix}\right)$, which we denote by $\gamma_v$.

We conclude that the set $\left\{\gamma \in \mathrm{SL}_2(\mathbb{Z}) : \pi_N(\gamma) \in \left[-\frac{1}{2}, \frac{1}{2}\right)\right\}$ is in one-to-one correspondence $\gamma_v \leftrightarrow v$ with the set of primitive integral vectors. Furthermore, for each $\gamma_v$, $e^{-\pi_A(\gamma_v)/2}$ is the length of $v$, and the rotation angle determined by $\pi_K(\gamma_v)$ is in $\frac{1}{2}\pi + \Theta$ if and only if $v \in S_\Theta$.

Consequently, the sets

$$\left\{\gamma \in \Gamma \,;\, \pi_N(\gamma) \in \left[-\tfrac{1}{2}, \tfrac{1}{2}\right),\ -T < \pi_A(\gamma) \le 0,\ \pi_K(\gamma) \in \tfrac{1}{2}\pi + \Theta\right\}$$

and

$$\left\{ v \in \mathbb{R}^2 \,;\, v \text{ is a primitive integral vector, } \|v\| < e^{T/2},\ \frac{v}{\|v\|} \in \Theta \right\}$$

are in one-to-one correspondence.

Now apply part (1) of Corollary 1.2 with $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, the interval $A_{(-T,0]} \subset A \cong \mathbb{R}$, $\Phi = \frac{1}{2}\pi + \Theta \subset S^1 \cong K$ and $\Psi = \left[-\frac{1}{2}, \frac{1}{2}\right) \subset \mathbb{R} \cong N$. Noting that $\rho = \frac{1}{2}$ for $\mathrm{SL}_2(\mathbb{R})$, the volume of the domains in question is $|\Theta| \cdot (e^T - 1)/v_\Gamma$ where $v_\Gamma$ is the covolume of $\mathrm{SL}_2(\mathbb{Z})$ in $\mathrm{SL}_2(\mathbb{R})$, which is $2\zeta(2) = \frac{1}{3}\pi^2$ (with respect to the Haar measure chosen). The domain $\mathcal{S}_\Theta(R) = \mathcal{S}_\Theta \cap B_R$ is defined by $R^2 = e^T$ and hence the main term of the volume is

$$\frac{3}{\pi^2}|\Theta| R^2 = \frac{6}{\pi^2}\frac{|\Theta|}{2\pi}(\pi R^2).$$

Note that when $|\Theta| = 2\pi$ the number of primitive lattice points of norm at most $R$ has main term $|B_R|/\zeta(2) = 6R^2/\pi$.

Furthermore, we note $\kappa(\mathrm{SL}_2(\mathbb{Z})) = \frac{7}{8}$ (see Section 4A). Hence the main factor in the error estimate is $\exp\left(\frac{7}{8}T\right) = R^{7/4}$. This proves part (1) of Theorem 2.1.

To prove part (2), recall that $\theta_v$ denotes the angle between $w_v$ and $v$, so that the $N$-component of $\gamma_v$ is given by

$$\pi_N(\gamma_v) = \frac{x_v a + y_v b}{a^2 + b^2} = \frac{\langle w_v, v\rangle}{\|v\|^2} = \frac{\|w_v\| \cos(\theta_v)}{\|v\|}. \tag{2-1}$$

Since

$$1 = \det\begin{pmatrix} x_v & y_v \\ a & b \end{pmatrix} = \det\begin{pmatrix} w_v \\ v \end{pmatrix} = \|w_v\|\,\|v\|\,|\sin(\theta_v)|,$$

and $\|w_v\| \geq 1$, it follows that $|\sin(\theta_v)| = O(\|v\|^{-1})$, or equivalently $1 - |\cos(\theta_v)| = O(\|v\|^{-2})$. From part (2) of Corollary 1.2 applied to $\Psi = \left[-\frac{1}{2}, \frac{1}{2}\right)$ and $\Phi = \frac{1}{2}\pi + \Theta$, we have that for primitive vectors $v$ in $\mathcal{S}_\Theta(R)$, the $N$-components of $\gamma_v$ become effectively equidistributed in $\left[-\frac{1}{2}, \frac{1}{2}\right)$ as $R \to \infty$, at rate $O(R^{-1/4} \cdot \log R)$. It follows that their absolute values become effectively equidistributed in $\left[0, \frac{1}{2}\right]$ at the same rate. These absolute values are

$$|\pi_N(\gamma_v)| = \frac{\|w_v\|}{\|v\|} \cdot |\cos(\theta_v)| = \frac{\|w_v\|}{\|v\|}(1 + O(\|v\|^{-2})) = \frac{\|w_v\|}{\|v\|} + O(\|v\|^{-2});$$

where the last equality follows since the ratio $\|w_v\|/\|v\|$ is bounded. This follows from (2-1), since $|\pi_N(\gamma_v)| \leq \frac{1}{2}$ and $|\cos\theta_v| \geq 1 - C/\|v\|^2$.

To prove part (2) it remains to show that the values $\|w_v\|/\|v\|$ for $v \in \mathcal{S}_\Theta(R)$ are also effectively equidistributed in $\left[0, \frac{1}{2}\right]$ at rate $O(R^{-1/4} \cdot \log R)$ as $R \to \infty$. Let $P_R$ denote the primitive integral vectors $v \in \mathcal{S}_\Theta(R)$, and note that by part (1) we have $|P_R| \geq R^2$ for $R \geq R_1$. Let $f$ be a Lipschitz function on $\left[0, \frac{1}{2}\right]$. Then for

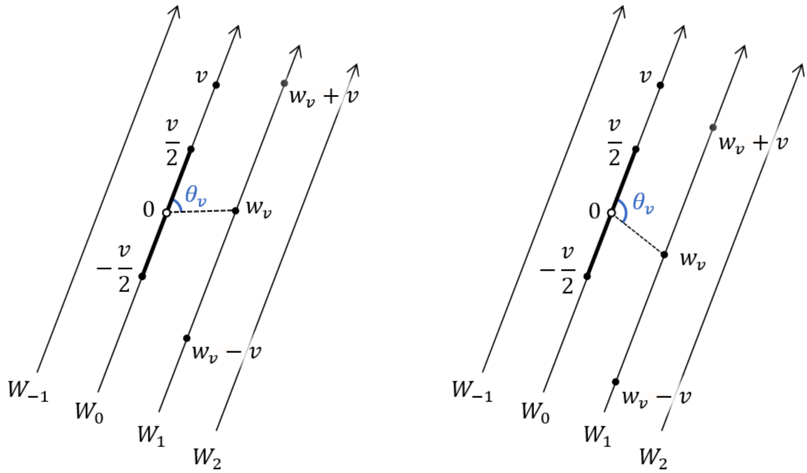



**Figure 3.** Integral vectors $v$, $w_v$ and angle $\theta_v$ for $\theta_v$ acute and $v$ positive (left) and $\theta_v$ obtuse and $v$ negative (right). This figure also depicts the lines $W_m = \{w : \det\binom{w}{v} = m\}$ for $m \in \mathbb{Z}$, where $W_0 = \operatorname{span}\{v\}$ and $w_v$ is the shortest integral vector in $W_1$.

$R \geq R_1$,

$$\frac{1}{|P_R|}\sum_{v\in P_R}\left|f\left(\frac{\|w_v\|}{\|v\|}\right) - f(|\pi_N(\gamma_v)|)\right| \leq \frac{C_f}{|P_R|}\sum_{v\in P_R}\left|\frac{\|w_v\|}{\|v\|} - |\pi_N(\gamma_v)|\right|$$
$$\leq \frac{C_f}{R^2}\sum_{v\in B_R}\frac{C}{\|v\|^2} = \frac{C_f C}{R^2}\sum_{n=1}^{R^2}\sum_{\substack{v\in B_R\\ \|v\|^2=n}}\frac{1}{n} \leq \frac{C_f C}{R^2}\sum_{n=1}^{R^2}\frac{r_2(n)}{n} = O_{f,\Theta}(R^{-2}\log R).$$

Here $r_2(n)$ is the number of representations of an integer $n$ as a sum of two squares, and the last estimate follows using Abel's partial summation formula. Since $\sum_{k=1}^{K} r_2(k) = \pi K + O(\sqrt{K})$,

$$\sum_{n=1}^{R^2}\frac{r_2(n)}{n} = \frac{1}{R^2+1}\sum_{n=1}^{R^2} r_2(n) + \sum_{n=1}^{R^2}(\pi n + O(\sqrt{n}))\left(\frac{1}{n} - \frac{1}{n+1}\right) = O(\log R).$$

Since the $N$-components equidistribute at rate $R^{-1/4}\log R$, and the term just estimated vanishes faster, this proves part (2) of Theorem 2.1.

By the expression (2-1) for the $N$-component of $\gamma_v$, the vector $v$ is positive if and only if $\cos\theta_v > 0$, i.e., if and only if $\pi_N(\gamma_v) > 0$. Similarly, $v$ is negative if and only if $\pi_N(\gamma_v) < 0$. Thus, by applying Corollary 1.2 to $\Psi_+ = \left[0, \frac{1}{2}\right]$, we obtain part (2) for the positive vectors, and similarly when $\Psi_- = \left[-\frac{1}{2}, 0\right]$, we obtain part (2) for the negative vectors. This proves part (3), and part (4) follows by applying the argument used in the proof of part (1) to the choices $\Psi_-$ or $\Psi_+$ as the set of $N$-components. □

**2B. *The gcd equation in imaginary quadratic number fields.*** Theorem 2.1 extends to rings of integers in imaginary quadratic number fields, as follows. Let $d$ be a negative squarefree integer, and let $\mathcal{O}_d$ denote the ring of integers in the quadratic number field $\mathbb{Q}[\sqrt{d}]$. The ring $\mathcal{O}_d$ is a lattice in $\mathbb{C}$, which is a rectangular lattice when $d$ is congruent to 1 or 3 modulo 4, and an "isosceles-triangular" lattice when $d$ is congruent to 1 modulo 4. The Dirichlet fundamental domain $\mathcal{D}_d$ of the lattice $\mathcal{O}_d$ is defined as the (closed) set of all points whose distance to 0 is less than or equal to their distance to any other lattice point. The *strict* Dirichlet fundamental domain $\mathcal{D}'_d$ is defined as the interior of $\mathcal{D}_d$ together with a union of intervals contained in the boundary $\partial\mathcal{D}_d$, chosen so that every orbit has a unique point in $\mathcal{D}'_d$. We let $[0, r_d]$ denote the image of the norm function $\|\cdot\| : \mathcal{D}_d \to \mathbb{R}$, and let $\nu_d$ denote the probability measure on $[0, r_d]$ which is the distribution of the norm of a random uniform point in $\mathcal{D}_d$. Equivalently $\nu_d$ is the push-forward of Lebesgue measure on $\mathcal{D}_d$, normalized to have total mass 1. Note that $\nu_d$ is equivalent but not equal to Lebesgue measure on the interval.

We refer to $v = (\alpha, \beta)$ in $\mathcal{O}_d^2$ as *primitive* if the ideals $\langle\alpha\rangle$ and $\langle\beta\rangle$ are coprime; namely, if there exists a solution $(\xi, \eta)$ in $\mathcal{O}_d^2$ to $\beta\xi - \alpha\eta = 1$. Consider a shortest vector that completes $v$ to a basis of $\mathcal{O}_d^2$, namely, a shortest $\mathcal{O}_d$-integral solution to the equation $\beta\xi - \alpha\eta = 1$. We will presently explain how to specify a unique such shortest solution, which we denote by $w_v$.

Let $\Theta \subseteq S^3$ be a spherical cap in the unit sphere, let $\mathcal{S}_\Theta$ denote the corresponding sector of $\mathbb{R}^4 \cong \mathbb{C}^2$, and $\mathcal{S}_\Theta(R) = \mathcal{S}_\Theta \cap B_R$ where $B_R = \{v \in \mathbb{C}^2 ; \|v\| \le R\}$.

**Theorem 2.3.** *Consider primitive vectors $v = (\alpha, \beta) \in \mathcal{O}_d^2$ in $\mathcal{S}_\Theta(R) \subset \mathbb{C}^2$.*

(1) *The number of primitive vectors in $\mathcal{S}_\Theta(R)$ is given by $c_d|\Theta|R^4 + O(R^{4\kappa_d}\log R)$, where $c_d$ is a positive constant depending only on $d$.*

(2) *The ratios $\|w_v\|/\|v\|$ of the length of the shortest solution $w_v$ relative to the length of $v$, for $v \in \mathcal{S}_\Theta(R)$, become effectively equidistributed in the interval $[0, r_d]$ as $R \to \infty$, with respect to the measure $\nu_d$. The rate of convergence for a Lipschitz function $f$ is $O_{f,\Theta}(R^{-4(1-\kappa_d)} \cdot \log R)$.*

*Here $\kappa_d$ is the exponent that corresponds to the lattice $\mathrm{PSL}_2(\mathcal{O}_d)$ of $\mathrm{PSL}_2(\mathbb{C})$ in Theorem 1.1.*

Observe that when $\mathcal{O}_d$ is a euclidean domain with respect to the usual norm, namely when $d \in \{-1, -2, -3, -7, -11\}$, and $\alpha$ and $\beta$ are coprime, the equation $\beta\xi - \alpha\eta = 1$ is their gcd equation. Then $w_v$ is the shortest solution to the gcd equation defined by $v$, as in the case of $\mathbb{Z}$.

The arguments in the proof of Theorem 2.1 can be applied, with some modifications, to prove Theorem 2.3. In the present context, Corollary 1.2 is applied for the Iwasawa components of the lattice $\mathrm{SL}_2(\mathcal{O}_d)$ in $\mathrm{SL}_2(\mathbb{C})$. We briefly describe the necessary adjustments, see [Elstrodt et al. 1998] for further details.

*Proof.* Recall the Iwasawa decomposition of $\mathrm{SL}_2(\mathbb{C})$ consists of the subgroups

$$\begin{aligned} N &= \left\{ \begin{pmatrix} 1 & z \\ & 1 \end{pmatrix} : z \in \mathbb{C} \right\}, \\ A &= \left\{ \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} : t \in \mathbb{R} \right\}, \\ K &= \left\{ \begin{pmatrix} \bar{b} & -\bar{a} \\ a & b \end{pmatrix} : |a|^2 + |b|^2 = 1 \right\} = \mathrm{SU}(2). \end{aligned}$$

Clearly, $K$ is isomorphic to the unit sphere $S^3$ in $\mathbb{C}^2$.

A primitive pair $(\alpha, \beta) \in \mathcal{O}_d^2$ can be completed to a matrix $\left(\begin{smallmatrix} \xi & \eta \\ \alpha & \beta \end{smallmatrix}\right)$ in $\mathrm{SL}_2(\mathcal{O}_d)$, and the Iwasawa coordinates of such a matrix are

$$\begin{pmatrix} \xi & \eta \\ \alpha & \beta \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & \frac{\xi\bar{\alpha}+\eta\bar{\beta}}{\|\alpha\|^2+\|\beta\|^2} \\ & 1 \end{pmatrix}}_{N\text{-component}} \underbrace{\begin{pmatrix} \frac{1}{\sqrt{\|\alpha\|^2+\|\beta\|^2}} & \\ & \sqrt{\|\alpha\|^2+\|\beta\|^2} \end{pmatrix}}_{A\text{-component}} \underbrace{\frac{1}{\sqrt{\|\alpha\|^2+\|\beta\|^2}} \begin{pmatrix} \bar{\beta} & -\bar{\alpha} \\ \alpha & \beta \end{pmatrix}}_{K\text{-component}}.$$

The $A$ and $K$ components encode the vector $v$: the $A$-component encodes its norm, and the bottom row of the $K$-component encodes the unit vector $v/\|v\|$ in the unit sphere $S^3$. The $N$-component encodes the upper row of the matrix: if $w = (\xi, \eta)$, this component equals $\langle w, v\rangle/\|v\|^2$. The set of solutions to $\beta\xi - \alpha\eta = 1$ is $\{(\xi + m\alpha, \eta + m\beta) : m \in \mathcal{O}_d\}$, and the matrices in $\mathrm{SL}_2(\mathcal{O}_d)$ that correspond to these solutions differ only by their $N$-components; these components are

$$\begin{pmatrix} 1 & \frac{(\xi+m\alpha)\bar{\alpha}+(\eta+m\beta)\bar{\beta}}{\|\alpha\|^2+\|\beta\|^2} \\ & 1 \end{pmatrix} = \begin{pmatrix} 1 & m + \frac{\xi\bar{\alpha}+\eta\bar{\beta}}{\|\alpha\|^2+\|\beta\|^2} \\ & 1 \end{pmatrix} = \begin{pmatrix} 1 & m + \frac{\langle w,v\rangle}{\|v\|^2} \\ & 1 \end{pmatrix},$$

where $m \in \mathcal{O}_d$. Using $v$ and $v^\perp$ as orthogonal coordinates in the plane $\mathbb{C}^2$, by the same argument that was used in the real case (Theorem 2.1), the shortest $\mathcal{O}_d$-integral solution $w_v = (\xi_v, \eta_v)$ to $\beta\xi - \alpha\eta = \det\left(\begin{smallmatrix} w \\ v \end{smallmatrix}\right) = 1$ corresponds to the matrix $\gamma_v \in \mathrm{SL}_2(\mathcal{O}_d)$ whose $N$-coordinate has minimal norm.

Clearly, the set of $N$-components $\{m + \langle w, v\rangle/\|v\|^2 : m \in \mathcal{O}_d\}$ is the orbit of $\langle w, v\rangle/\|v\|^2$ under translations by the lattice $\mathcal{O}_d$ in $\mathbb{C}$. Hence there is a unique element in this orbit in every $\mathcal{O}_d$-integral translation of the *strict* Dirichlet fundamental domain $\mathcal{D}'_d$. The representative which is of minimal norm is the one that lies in $\mathcal{D}'_d$ itself. Thus, $\gamma_v$ is the unique matrix in $\mathrm{SL}_2(\mathcal{O}_d)$, among the matrices whose bottom row is $v$, whose $N$-component lies in $\mathcal{D}'_d$.

Let $\mathrm{s}(v)$ and $\mathrm{c}(v)$ be such that

$$1 = \det\begin{pmatrix} \xi_v & \eta_v \\ \alpha & \beta \end{pmatrix} = \det\begin{pmatrix} w_v \\ v \end{pmatrix} = \|w_v\|\|v\| \cdot \mathrm{s}(v),$$

and $\langle w_v, v\rangle = \|w_v\|\|v\| \cdot \mathrm{c}(v)$; these are the analogs for $\sin\theta_v$ and $\cos\theta_v$: see (2-1).

Taking the absolute value squared of the previous two equations, the identity

$$|\mathrm{s}(v)|^2 + |\mathrm{c}(v)|^2 = 1,$$

is easily seen to be equivalent to $1 + |\langle w_v, v\rangle|^2 = \|w_v\|^2\|v\|^2$, and this equation is a consequence of $\beta\xi_v - \alpha\eta_v = 1$, as can be verified directly (starting with $\langle w_v, v\rangle = \xi_v\bar{\alpha} + \eta_v\bar{\beta}$).

In particular, when $|v|$ is large, $\mathrm{s}(v)$ is small (since $\|w_v\|$ is bounded below), and in fact $\mathrm{s}(v) = O(\|v\|^{-1})$. Now $|\mathrm{s}(v)|^2 = 1 - |\mathrm{c}(v)|^2 \geq 1 - |\mathrm{c}(v)|$ so $1 - |\mathrm{c}(v)| = O(\|v\|^{-2})$, and

$$\frac{|\langle w_v, v\rangle|}{\|v\|^2} = \frac{\|w_v\|}{\|v\|} \cdot |\mathrm{c}(v)| = \frac{\|w_v\|}{\|v\|}(1 + O(\|v\|^{-2})).$$

As above, the last estimate follows since $|\langle w_v, v\rangle|/\|v\|^2$ is the $N$-component which is in $\mathcal{D}'_d$ and hence bounded, and $|\mathrm{c}(v)| \geq 1 - C/\|v\|^2$.

The proof now proceeds in a manner analogous to the proof of Theorem 2.3. Namely, we apply Corollary 1.2 to $\Gamma = \mathrm{SL}_2(\mathcal{O}_d)$, the interval $A_{[-T,0]}$ in $A$ with $e^{T/2} = R$, and to $\Psi = \mathcal{D}'_d$, $\Phi = \Theta$. Using also that $\rho = 1$ for $\mathrm{SL}_2(\mathbb{C})$, the measure of $R_T(\Phi, \Psi)$ here is $c'_d|\Theta| \cdot e^{2T}$. Therefore, the error estimate for the number of lattice points in the set is bounded by a multiple of $e^{2T\kappa_d}$.

We conclude that the number of primitive lattice points $(\alpha, \beta) \in \mathcal{S}_\Theta(R)$ is then given by $c_d|\Theta|R^4 + O(R^{4\kappa_d}\log R)$. This completes the proof of part (1) of Theorem 2.3.

To prove part (2), note by Corollary 1.2, part (2), that the $N$-components of $\gamma_v$, where $v \in \mathcal{S}_\Theta(R)$, equidistribute effectively in $\Psi$ with respect to the Lebesgue measure, at the rate $O_{f,\Theta}(R^{-4(1-\kappa_d)} \cdot \log R)$ for any Lipschitz function $f$. Therefore, their norms effectively equidistribute in $[0, r_d]$ with respect to $\nu_d$, at the same rate. To show that the ratios $\|w_v\|/\|v\|$, with $v \in \mathcal{S}_\Theta(R)$, also have that property, we consider the difference. Let $Q_R = \{v = (\alpha, \beta) \in \mathcal{S}_\Theta(R),\ \alpha, \beta \in \mathcal{O}_d \text{ coprime}\}$. By part (1) of the theorem, $Q_R$ satisfies $|Q_R| \geq R^4/\tilde{c}_d$. Then

$$\frac{1}{|Q_R|}\sum_{v\in Q_R}\left| f\left(\frac{\|w_v\|}{\|v\|}\right) - f(|\pi_N(\gamma_v)|)\right| \leq \frac{C_f}{|Q_R|}\sum_{v\in Q_R}\left|\frac{\|w_v\|}{\|v\|} - |\pi_N(\gamma_v)|\right|$$

$$\leq \frac{\tilde{c}_d C_f}{R^4}\sum_{v\in B_R}\frac{C}{\|v\|^2} = \frac{\tilde{c}_d C_f C}{R^4}\sum_{n=1}^{R^2}\sum_{\substack{v\in B_R \\ \|v\|^2=n}}\frac{1}{n} = \frac{\tilde{c}_d C_f C}{R^4}\sum_{n=1}^{R^2}\frac{r_4(n)}{n} = O_{f,\Theta}(R^{-2}).$$

Here $r_4(n)$ is the number of representations of an integer $n$ as a sum of four squares. To establish this estimate, note that $\sum_{k=1}^{K} r_4(k) = |B_1|K^2 + O(K^{3/2})$, and using Abel's partial summation formula,

$$\sum_{n=1}^{R^2} \frac{r_4(n)}{n} = \frac{1}{R^2+1} \sum_{n=1}^{R^2} r_4(n) + \sum_{n=1}^{R^2} (|B_1|n^2 + O(n^{3/2}))\Big(\frac{1}{n} - \frac{1}{n+1}\Big) = O(R^2).$$

Finally, since the $N$-components equidistribute at rate $R^{-4(1-\kappa_d)} \log R$, and the term just estimated vanishes faster, the conclusion of part (2) holds. □

## 3. Further diophantine and geometric applications

Let us now demonstrate some further applications of Theorem 1.1 and Corollary 1.2.

**3A.** ***Lifts of horospheres.*** Let $\Gamma$ be a nonuniform lattice in $G$, with a cusp at the point $\sigma$ at the boundary of the associated hyperbolic space. Let $H_\sigma$ be the unipotent subgroup in $G$ which stabilizes $\sigma$ (in particular, it is conjugate to $N$). We consider the case in which $\Gamma \cap H_\sigma$ is a lattice in $H_\sigma$. Let $\mathcal{H}$ be a horosphere in the hyperbolic space of $G$ which is based at $\sigma$; in other words, $\mathcal{H}$ is an orbit of $H_\sigma$. Observe that $\mathcal{H}$ projects to a closed horosphere $\overline{\mathcal{H}}$ in the space $\Gamma \backslash G$. Let $B_T(z)$ denote a hyperbolic ball of radius $T$ that is centered at $z$, and let $N(T)$ denote the number of horospheres of the form $\gamma\mathcal{H}$, with $\gamma \in \Gamma$, that meet the ball $B_T(z)$. Eskin and McMullen [1993, Theorem 7.2] have considered the counting function $N(T)$ and discussed the case of $G = \mathrm{PSL}_2(\mathbb{R})$. This problem can be formulated for a simple Lie group of any real rank, see [Mohammadi and Salehi Golsefidy 2014]; we will provide an effective estimate for real rank 1.

**Theorem 3.1.** *Let $\Gamma < G$ be a nonuniform lattice, and let $\sigma$, $H_\sigma$, $\mathcal{H}$ be as above. If $\Gamma \cap H_\sigma$ is a lattice in $H_\sigma$, then*

$$N(T) = \frac{\mathrm{Vol}_{G/\Gamma}(\overline{\mathcal{H}})}{\mu(G/\Gamma)} \cdot \frac{e^{2\rho T}}{2\rho} + O_{z,\Gamma,\mathcal{H}}(T(e^{2\rho T})^{\kappa}),$$

*where $\kappa = \kappa(\Gamma)$ is the exponent associated with $\Gamma$.*

*Proof.* Assume first that $z = i$ (where $i$ is the point stabilized by $K$) and that $\sigma = \infty$ (where $\infty$ is the point stabilized by $N$), or equivalently $H_\sigma = N$ and $N \cap \Gamma$ is a lattice in $N$. Then $\mathcal{H}$ is a horizontal horosphere, i.e., it is orthogonal to the geodesic $A \cdot i$, and we may write $\mathcal{H} = Na_y \cdot i$ for some $y \in \mathbb{R}$. Then the set of horospheres $\gamma\mathcal{H}$ that meet the ball $B_T(i)$ is in one-to-one correspondence with the elements of the set

$$\{\gamma N : d(i, \gamma\mathcal{H}) < T\} = \{\gamma N : d(i, \gamma N a_y \cdot i) < T\}.$$

We write the elements of $\Gamma$ in their $KAN$ coordinates, and denote $\gamma = k_\gamma a_{t(\gamma)} n_\gamma$. Then

$$\begin{aligned}\{\gamma N : d(i, k_\gamma a_{t(\gamma)} n_\gamma N a_y \cdot i) < T\} &= \{\gamma N : d(i, a_{t(\gamma)} N a_y \cdot i) < T\}\\ &= \{\gamma N : d(i, a_{t(\gamma)} N \cdot a_{-t(\gamma)} a_{t(\gamma)} \cdot a_y \cdot i) < T\}\\ &= \{\gamma N : d(i, N a_{t(\gamma)+y} \cdot i) < T\}\\ &= \{\gamma N : d(i, a_{t(\gamma)+y} \cdot i) < T\},\end{aligned}$$

where the last equality follows since the horosphere $N a_{y+t(\gamma)} \cdot i$ is orthogonal to the geodesic $A \cdot i$, and thus the point nearest to $i$ on this horosphere is its meeting point with the geodesic, $a_{y+t(\gamma)} \cdot i$.

Now, $d(i, a_{t(\gamma)+y} \cdot i) = |t(\gamma) + y|$, so $d(i, a_{t(\gamma)+y} \cdot i) < T$ if and only if $-T - y \le t(\gamma) \le T - y$. Moreover, the cosets $\gamma N$ are in one-to-one correspondence with the lattice elements $\gamma = k_\gamma a_{t(\gamma)} n_\gamma$ such that $n_\gamma \in \Psi(\Gamma)$, for a choice $\Psi(\Gamma)$ of a fundamental domain for $\Gamma \cap N$ in $N$. Then,

$$\begin{aligned}N(T) &= \#\{\gamma = k_\gamma a_{t(\gamma)} n_\gamma : n_\gamma \in \Psi(\Gamma),\ -T - y \le t(\gamma) \le T - y\}\\ &= \#\Gamma \cap (K A_{[-T-y, T-y]} \Psi(\Gamma)).\end{aligned}$$

Now the desired result follows from Theorem 1.1 and Remark 1.4:

$$\begin{aligned}N(T) &= \frac{\mu_N(\Psi(\Gamma))}{\mu(G/\Gamma)} \cdot \frac{e^{2\rho(T-y)}}{2\rho} + O((T-y)(e^{2\rho(T-y)})^\kappa)\\ &= \frac{\mu_N(\Psi(\Gamma))}{\mu(G/\Gamma)} \cdot e^{-2\rho y} \cdot \frac{e^{2\rho T}}{2\rho} + O(T(e^{2\rho(T-y)})^\kappa)\\ &= \frac{\mathrm{Vol}(\overline{\mathcal{H}})}{\mu(G/\Gamma)} \cdot \frac{e^{2\rho T}}{2\rho} + O(T(e^{2\rho T})^\kappa).\end{aligned}$$

The foregoing result is valid for any lattice $\Gamma$ satisfying that $N \cap \Gamma$ is a lattice in $N$. Recall that $S = AN$ stabilizes $\infty$, normalizes $N$, and acts transitively on the symmetric space. For any $z$ we can choose $s \in S$ such that $s \cdot z = i$, and then $sNs^{-1} \cap s\Gamma s^{-1} = N \cap \Gamma^s$ is a lattice in $N^s = N$ and the previous result applies to $\Gamma^s$. Writing $s = a_s n_s$, we have

$$d(i, s\gamma s^{-1} N a_y \cdot i) = d(s^{-1} \cdot i, \gamma s^{-1} N s n_s^{-1} a_s^{-1} a_y \cdot i) = d(z, \gamma N a_{y-s} \cdot i),$$

and we can conclude that the result holds for the lattice $\Gamma$ and any choice of origin $z$.

Finally, if the cusp $\sigma$ of a lattice $\Gamma'$ is arbitrary, we can conjugate the lattice $\Gamma'$ and obtain a lattice $\Gamma$ containing a lattice in $N$, for which the result applies with any choice of origin. This implies its validity for the lattice $\Gamma'$, with any choice of origin. $\square$

**3B.** ***Diophantine equation associated with a Lorentz form.*** When $G = \mathrm{SO}^0(1, n)$, the elements of the subgroups $A$ and $N$ of $G$ can also be given explicitly as

$$a_t = \begin{pmatrix} \cosh t & 0 & \sinh t \\ 0 & I_{n-1} & 0 \\ \sinh t & 0 & \cosh t \end{pmatrix} \tag{3-1}$$

and

$$n_v = \begin{pmatrix} 1 + \frac{1}{2}\|v\|^2 & v^* & -\frac{1}{2}\|v\|^2 \\ v & I_{n-1} & -v \\ \frac{1}{2}\|v\|^2 & v^* & 1 - \frac{1}{2}\|v\|^2 \end{pmatrix}, \tag{3-2}$$

see, e.g., [Faraut 1983, pp. 373 and 375].

The explicit $N$ and $A$ Iwasawa components of a given $g \in G$ can be deduced as follows.

**Claim 3.2.** *Let*

$$g = \begin{pmatrix} g_{0,0} & \cdots & g_{0,n} \\ \vdots & & \vdots \\ g_{n,0} & \cdots & g_{n,n} \end{pmatrix} \in G.$$

*If* $g = n_v a_t k$, *then*

$$e^t = (g_{0,0} - g_{n,0})^{-1} \quad \textit{and} \quad v = \frac{1}{g_{0,0} - g_{n,0}} \begin{pmatrix} g_{1,0} \\ \vdots \\ g_{n-1,0} \end{pmatrix}.$$

*Proof.* On the one hand,

$$g \cdot i = \begin{pmatrix} g_{0,0} & \cdots & g_{0,n} \\ \vdots & & \vdots \\ g_{n,0} & \cdots & g_{n,n} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} g_{0,0} \\ \vdots \\ g_{n,0} \end{pmatrix}.$$

On the other hand,

$$g \cdot i = n_v a_t k \cdot i = n_v a_t \cdot i,$$

where

$$n_v a_t \cdot i = \begin{pmatrix} 1 + \frac{1}{2}\|v\|^2 & v^* & -\frac{1}{2}\|v\|^2 \\ v & I_{n-1} & -v \\ \frac{1}{2}\|v\|^2 & v^* & 1 - \frac{1}{2}\|v\|^2 \end{pmatrix} \begin{pmatrix} \cosh t & 0 & \sinh t \\ 0 & I_{n-1} & 0 \\ \sinh t & 0 & \cosh t \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \cosh t + \frac{1}{2} e^{-t}\|v\|^2 \\ e^{-t} \cdot v \\ \sinh t + \frac{1}{2} e^{-t}\|v\|^2 \end{pmatrix}.$$

Namely,

$$\begin{pmatrix} g_{0,0} \\ \vdots \\ g_{n,0} \end{pmatrix} = \begin{pmatrix} \cosh t + \frac{1}{2}e^{-t}\|v\|^2 \\ e^{-t} \cdot v \\ \sinh t + \frac{1}{2}e^{-t}\|v\|^2 \end{pmatrix}.$$

In particular,

$$g_{0,0} - g_{n,0} = \cosh t - \sinh t = e^{-t},$$

and

$$\begin{pmatrix} g_{1,0} \\ \vdots \\ g_{n-1,0} \end{pmatrix} = e^{-t} \cdot v.$$

The expression for $e^t$ and $v$ easily follows from the foregoing equations. □

$\mathrm{SO}^0(1, n)$ is the isometry group of the Lorentz form, and each element $g = (g_{i,j})_{0 \le i,j \le n}$ satisfies $g_{0,0} > 0$ and $g_{0,0}^2 - g_{1,0}^2 - \cdots - g_{n,0}^2 = 1$. Namely, the first column of $g$ satisfies the Lorentz equation

$$x_0^2 - x_1^2 - \cdots - x_n^2 = 1, \quad x_0 > 0. \tag{3-3}$$

Conversely, every such vector $(x_0, \ldots, x_n)$ appears as the first column of a matrix $(g_{i,j})$ belonging to the connected component of the isometry groups of the Lorentz form.

By Claim 3.2 above, the $N$ and $A$ components of $g$ depend only on the first column of $g$. Hence, Corollary 1.2 concerning the equidistribution of the $N$-components as the $A$-components approach $\infty$ can be used to study the behavior of the corresponding parameters of solutions to the Lorentz equation (3-3).

For every $\underline{x} = (x_0, \ldots, x_n) \in \mathbb{R}^{n+1}$ that satisfies (3-3) with $x_0 > 0$, note that $x_0 > x_n$ and define the height function $h(\underline{x}) = \log(1/(x_0 - x_n))$ (corresponding to the $A$-component). Define also the vector $v(\underline{x}) = (1/(x_0 - x_n))(x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$ (corresponding to the $N$-component). Assume $\Psi \subset \mathbb{R}^{n-1}$ is nice, and consider the family $R_T(\Psi, K) \subset G$. By applying Corollary 1.2 (part (2)) to the lattice $\mathrm{SO}^0(1, n)(\mathbb{Z})$ in $\mathrm{SO}^0(1, n)$, we conclude:

**Corollary 3.3.** *Let $S_T$ denote the set of integral solutions $\underline{x}$ to the Lorentz equation (3-3) with $-T \le h(\underline{x}) \le 0$. The rational vectors $v(\underline{x})$, $\underline{x} \in S_T$ become effectively equidistributed in $\Psi$ as $T \to \infty$, at rate $O(e^{2\rho T(\kappa_\Gamma - 1)})$.*

**3C.** ***Equidistribution of Iwasawa coordinates: history of the problem.*** The problem of joint equidistribution of the Iwasawa coordinates of nonuniform lattices has quite a long history. We summarize the relevant results we are aware of as follows.

*The analysis by Selberg and Good of the* $\mathrm{SL}_2(\mathbb{R})$ *case.* Consider a nonuniform lattice $\Gamma \subset \mathrm{SL}_2(\mathbb{R}) = NAK$, with a maximal parabolic subgroup $\Gamma_\infty$ equal to $N(\mathbb{Z})$, so that its fundamental domain in $N$ is (say) $[0, 1]$. The joint equidistribution (in $[0, 1] \times K$) of the $N$ component and the $K$ components of the lattice elements was established by Selberg [1991]. This result was also proved by Good [1983, Corollary p. 119; 1984, p. 101].

In particular, the problem of counting the number of points $\gamma \in \Gamma/\Gamma_\infty$ with $\mathrm{Im}(\gamma i) \geq T^{-1}$ corresponds to taking $\Psi$ to be a fundamental domain of $\Gamma_\infty \subset N$ in Theorem 1.1, and $\Phi = K$, and amounts to establishing equidistribution for the real parts of the orbit points. This problem was considered in detail by Good [1983, §11], using a thorough analysis of generalized Kloosterman sums. The error term in this problem stated by Good, who attributes it also to Selberg, is superior to the one which follows from Theorem 1.1 in this case. For a tempered lattice $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ as above, the error exponent stated by Good [1983, §11, Theorem 4] amounts to $\frac{2}{3}$, as opposed to $\frac{7}{8}$ which follows from Theorem 1.1.

Good's analysis [1983] can be elaborated further and used to derive effective joint equidistribution of both the $N$ and the $K$ components of lattice elements, which again will be superior to Theorem 1.1 in this case. Apparently, this elaboration has not been carried out in that paper and it is not clear what is the error estimate it provides.

The analysis of Selberg and Good is based on intricate estimates in harmonic analysis of automorphic forms and generalized Kloosterman sums, and was confined to the $\mathrm{SL}_2(\mathbb{R})$ case only. We note that Selberg [1991, §3] raises specifically the problem of extending the joint equidistribution statement to higher-dimensional hyperbolic spaces. This problem is given an effective solution in Corollary 1.2 of the present paper. A superior error estimate could in principle be derived by extending the analysis of Selberg and Good to higher-dimensional hyperbolic spaces, but judging by [Good 1983], this is quite a formidable task.

*From equidistribution of real parts of lattice orbits to the gcd equation in* $\mathbb{Z}^2$. The problem of analyzing the distribution of the shortest solution to the gcd equation in $\mathbb{Z}^2$ was considered by Dinaburg and Sinai [1990] who measured the size of the shortest solution by the maximum norm, and used the theory of continued fractions. It was subsequently observed by Risager and Rudnick [2009] that when the size of the smallest solution is measured using the euclidean norm, equidistribution of shortest solutions is equivalent to the problem of equidistribution of real parts of the points in the orbit of $i$ under $\mathrm{SL}_2(\mathbb{Z})$ in the upper half-plane, and the latter result has already been established by Good [1983]. This elegant observation has provided the motivation for the present paper. Following Risager and Rudnick, Truelsen [2013] has established a quantitative form for the equidistribution of real parts for any lattice with a standard cusp in $\mathrm{SL}_2(\mathbb{R})$. In particular, Truelsen established a rate of convergence in the equidistribution of the lengths of shortest solutions of the gcd equation, which is $R^{-1/4+\epsilon}$ in the case of tempered lattices. Theorem 2.1 provides the error estimate of $R^{-1/4}\log R$, and shows that effective equidistribution at this rate is valid with $v$ varying in any given fixed angular sector of radius $R$. Note that Truelsen's results are confined to lattices in $\mathrm{SL}_2(\mathbb{R})$, and do not consider lattices in $\mathrm{SL}_2(\mathbb{C})$, for example.

*Counting in general domains.* Chamizo [2011] and Truelsen [2013] have established some further lattice point counting results for a variety of other families of sets in the upper half-plane. Theorem 1.1 also allows for a large collection of families of increasing domains in hyperbolic spaces, namely $R_T(\Psi,\Phi)$, with $\Psi\subset N$ and $\Phi\subset K$ any nice sets. These families are also called bisectors associated with the decomposition $G=NAK$. Mohammadi and Oh [2015, Theorem 7.21] have considered the problem of counting the points of a Zariski-dense geometrically finite discrete subgroup $\Lambda$ in such sets. Under some restrictions they have obtained an asymptotic estimate for the count, with unspecified rate. A generalization of this discussion to the case of discrete geometrically finite groups in general real, complex and quaternionic hyperbolic spaces was established by Kim [2015].

*Lifts of horospheres in hyperbolic space.* Eskin and McMullen [1993] have raised the problem of counting the number of lifts of a closed horosphere $\mathcal{H}$ in $G/\Gamma$ which intersect a ball of radius $T$ in hyperbolic space, and have established the main term for this counting problem. As shown in Theorem 3.1, in the case of hyperbolic space, the problem amounts to counting the points of a nonuniform lattice lying in the sets $R_T(\Psi,K)$, where $\Psi\subset\mathbb{R}^{n-1}$ is the fundamental domain of $\Gamma_\infty=\Gamma\cap N$, and thus is solved effectively by Theorem 1.1.

The problem can be formulated also for higher-rank symmetric space, and the main term of the asymptotics for counting such lifts has been established recently by Mohammadi and Salehi Golsefidy [2014]. Further work on the subject has been recently carried out by Dabbs, Kelly and Li in [Dabbs et al. 2016], where an effective count for the lift of horospheres in certain higher rank locally symmetric spaces was established.

*Local statistics of the Iwasawa $N$-component.* Marklof and Vinogradov [2018] have recently considered, among other things, the projection of lattice orbit points to a neighborhood of a horizontal horosphere tending to the boundary, namely the sets given by $R_T(\Psi,K)\setminus R_{T-c}(\Psi,K)$. Theorem 1.1 implies without difficulty, as we shall see below, an effective counting result for these sets as well. In their work, they have analyzed the local statistics of the Iwasawa $N$-components in $\Psi$ as $T\to\infty$. This problem is more delicate than just the equidistribution of the $N$-component, and it was established by them in real hyperbolic space of any dimension, but not in an effective form.

## 4. Proof of the main theorem

**4A.** ***A spectral method for counting lattice points.*** In the discussion of the present section, $G$ is a connected almost simple Lie group, not necessarily of real rank 1. The lattice point counting method in the family of domains $\{\mathcal{B}_T\} \subset G$ that we will use [Gorodnik and Nevo 2010; 2012] has two ingredients: a spectral estimate and a regularity property. The crucial spectral estimate requires bounding the norm of the averaging operators defined by $\mathcal{B}_T$ in the representation on $L^2_0(\Gamma\backslash G)$. Let us recall the fact that there exists $m \in \mathbb{N}$ such that the unitary representation of $G$ in $L^2_0(\Gamma\backslash G)$, when taken to the $m$-th tensor power, is weakly contained in the regular representation of $G$. The essential property of such $m$ is that $m \geq \frac{1}{2}p$, where $p$ satisfies that the $K$-finite matrix coefficients of $\pi^0_{\Gamma\backslash G}$ are in $L^{p+\zeta}(G)$ for every $\zeta > 0$. We define $m(\Gamma)$ to be the least even integer with this property if $p > 2$, or 1 if $p = 2$; see [Gorodnik and Nevo 2012, Definition 3.1]. One of the remarkable features of harmonic analysis on simple Lie groups is that then for any measurable set of positive finite measure $B$ in $G$, if we denote by $\beta$ the Haar uniform measure on $B$ divided by $\mu(B)$, the estimate

$$\|\pi^0_{\Gamma\backslash G}(\beta)\| \leq C_{G,\zeta} \cdot \mu(B)^{-1/(2m(\Gamma))+\zeta} \quad \text{for every } \zeta > 0 \tag{4-1}$$

holds [Nevo 1998]. Thus, $m(\Gamma)$ measures the size of the spectral gap in $L^2(\Gamma\backslash G)$. The lattice $\Gamma$ is called *tempered* if the representation $\pi^0_{\Gamma\backslash G}$ is already weakly contained in regular representation, namely if $m(\Gamma) = 1$.

We now turn to the second ingredient, which is the Lipschitz property of the domains $\mathcal{B}_T$.

**Definition 4.1** [Gorodnik and Nevo 2012]. Let $G$ be a Lie group with Haar measure $m_G$. Assume $\{\mathcal{B}_T\} \subset G$ is a family of bounded measurable sets such that $\mu(\mathcal{B}_T) \to \infty$ as $T \to \infty$. Let $\mathcal{O}_\epsilon \subset G$ be the image of a euclidean ball of radius $\epsilon$ in the Lie algebra under the exponential map. Denote

$$\mathcal{B}^+_T(\epsilon) := \mathcal{O}_\epsilon \mathcal{B}_T \mathcal{O}_\epsilon = \bigcup_{u,v \in \mathcal{O}_\epsilon} u\mathcal{B}_T v \quad \text{and} \quad \mathcal{B}^-_T(\epsilon) := \bigcap_{u,v \in \mathcal{O}_\epsilon} u\mathcal{B}_T v.$$

The family $\{\mathcal{B}_T\}$ is *Lipschitz well-rounded* if there exist $\epsilon_0 > 0$ and $T_0 \geq 0$ such that for every $0 < \epsilon \leq \epsilon_0$ and $T \geq T_0$,

$$\mu(\mathcal{B}^+_T(\epsilon)) \leq (1 + C\epsilon)\mu(\mathcal{B}^-_T(\epsilon)),$$

where $C > 0$ is a constant that does not depend on $\epsilon$ or $T$.

The concept of well-roundedness appeared first in [Duke et al. 1993; Eskin and McMullen 1993], and was used later also in [Gorodnik and Weiss 2007]. Lipschitz well-roundedness was applied extensively in [Gorodnik and Nevo 2012].

**Theorem 4.2** [Gorodnik and Nevo 2012]. *Let $G$ be a connected almost simple Lie group with Haar measure $m_G$, and let $\Gamma < G$ be a lattice. Assume $\{\mathcal{B}_T\} \subset G$ is a family of bounded measurable sets which satisfies $\mu(\mathcal{B}_T) \to \infty$ as $T \to \infty$. If the family $\{\mathcal{B}_T\}$ is Lipschitz well-rounded, then*

$$\#(\mathcal{B}_T \cap \Gamma) = \frac{1}{\mu(G/\Gamma)}\mu(\mathcal{B}_T) + O(\mu(\mathcal{B}_T) \cdot E(T)^{1/(1+\dim G)})$$

*as $T \to \infty$, where $\mu(G/\Gamma)$ is the measure of a fundamental domain of $\Gamma$ in $G$, and $E(T)$ is (a bound on) the rate of decay of operator norm $\|\pi^0_{\Gamma\backslash G}(\beta_T)\|$.*

Note that the above theorem applies to every lattice $\Gamma$.

When using the estimate (4-1) for $\|\pi^0_{\Gamma\backslash G}(\beta_T)\|$, the error term obtained in Theorem 4.2 is

$$O(\mu(\mathcal{B}_T)^{\kappa(\Gamma)+\zeta}) \quad \text{for every } \zeta > 0,$$

where

$$\kappa(\Gamma) = 1 - \frac{1}{2m(\Gamma)(1+\dim G)} \in (0, 1). \tag{4-2}$$

In our case, where $G$ is of real rank 1 and the family of domains is $R_T(\Psi, \Phi)$, the estimate (4-1) may be very slightly improved so that the error term is given by

$$O\big((\log \mu(R_T(\Psi, \Phi)) \cdot \mu(R_T(\Psi, \Phi)))^{\kappa(\Gamma)}\big),$$

as we now explain. Assume that a set $B \subset G$ of positive finite measure satisfies that

$$\mu(K \cdot B \cdot K) \le \text{const} \cdot \mu(B).$$

This property is called $K$-radializability; see [Gorodnik and Nevo 2010, Definition 3.21]. When $B$ is radializable, then it is a consequence of the spectral transfer principle [Nevo 1998] and of estimates of the Harish-Chandra function in real rank 1 that

$$\|\pi^0_{\Gamma\backslash G}(\beta)\| \le C_G \cdot (\log \mu(B))^{\frac{1}{m(\Gamma)}} \cdot \mu(B)^{-\frac{1}{2m(\Gamma)}} \le C_G \cdot (\log \mu(B)) \cdot \mu(B)^{-\frac{1}{2m(\Gamma)}};$$

see [Gorodnik and Nevo 2010, Proposition 5.9; Nevo 1998, §2.2, Theorem 6]. The sets $R_T(\Psi, \Phi)$ are indeed radializable, with constant that depends on $\Xi$ and $\Phi$ but does not depend on $T$. In particular, when $\beta_T$ are the probability measures that correspond to $R_T = R_T(\Psi, \Phi)$, then

$$E(T) = \|\pi^0_{\Gamma\backslash G}(\beta_T)\| \le C_G \cdot (\log \mu(R_T)) \cdot \mu(R_T)^{-\frac{1}{2m(\Gamma)}},$$

as claimed.

From the above discussion it follows that *in order to prove Theorem 1.1, it suffices to show that the family $\{R_T(\Psi, \Phi)\}$ is Lipschitz well-rounded.*

| $G$ | $\mathrm{SO}^0(1,n)$ | $\mathrm{SU}(1,n)$ | $\mathrm{SP}(1,n)$ | $\mathrm{F}_{4(-20)}$ |
|---|---|---|---|---|
| $\mathbb{K}$ | $\mathbb{R}$ | $\mathbb{C}$ | $\mathbb{H}$ | $\mathbb{O}$ |
| $N$ (as a manifold) | $\mathbb{R}^{n-1}$ | $\mathbb{C}^{n-1} \oplus \mathbb{R}$ | $\mathbb{H}^{n-1} \oplus \mathbb{R}^3$ | $\mathbb{O} \oplus \mathbb{R}^7$ |
| $K$ | $\mathrm{SO}(n)$ | $\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(n))$ | $\mathrm{SP}(1) \times \mathrm{SP}(n)$ | $\mathrm{Spin}(9)$ |
| $G/K$ | $\mathbf{H}^n_{\mathbb{R}}$ | $\mathbf{H}^n_{\mathbb{C}}$ | $\mathbf{H}^n_{\mathbb{H}}$ | $\mathbf{H}^2_{\mathbb{O}}$ |
| $(p,q)$ | $(n-1,0)$ | $(2n-2,1)$ | $(4n-4,3)$ | $(8,7)$ |
| $dt/e^{2\rho t}$ | $dt/e^{(n-1)t}$ | $dt/e^{2nt}$ | $dt/e^{(4n+2)t}$ | $dt/e^{22t}$ |

**Table 1.** Simple rank 1 Lie groups: Iwasawa subgroups, symmetric spaces and Haar measure $\mu = \mu_N \times (dt/e^{2\rho t}) \times \mu_K$.

**4B.** ***Lipschitz property for Iwasawa coordinates in the negative direction of A.*** In order to show that the family $R_T(\Psi, \Phi)$ is Lipschitz well-rounded, we introduce coordinates on $N$, in addition to the parametrization we have already introduced above for $A$, namely $A = \{a_t : t \in \mathbb{R}\}$, where $a_t = \exp tH_1$, and $H_1 \in \mathfrak{a} = \mathrm{Lie}(A)$ is the element satisfying $\alpha(H_1) = 1$, with $\alpha$ the unique short positive root.

Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ be the (possibly noncommutative) field over which the matrices in $G$ are defined, and $n$ the dimension (over $\mathbb{K}$) of the corresponding hyperbolic space. The group $N$ is of Heisenberg type (see [Cowling et al. 1991; 1998]), and in particular it is parametrized by the space $\mathbb{K}^n \oplus \Im(\mathbb{K})$, where $\Im(\mathbb{K})$ is the subspace of "pure imaginary" numbers in $\mathbb{K}$, namely of elements $w$ such that $w + \mathrm{conj}(w) = 0$. A parametrization may be chosen such that

$$N = \{n_{v,z} : v \in \mathbb{K}^n, z \in \Im(\mathbb{K})\},$$

with the group multiplication

$$n_{v_1,z_1} n_{v_2,z_2} = n_{v_1+v_2, z_1+z_2+\Im(\langle v_2, v_1\rangle)},$$

where $\langle v_2, v_1\rangle = v_1^* v_2$. The subspaces $\mathbb{K}^n$ and $\Im(\mathbb{K})$ correspond to subsets of $N$ that are invariant under conjugation by $A$, and specifically,

$$a_t n_{v,z} a_{-t} = n_{e^t v, e^{2t} z}. \tag{4-3}$$

As a result, if $p := \dim_{\mathbb{R}}(\mathbb{K}^n)$ and $q := \dim_{\mathbb{R}}(\Im(\mathbb{K})) = \dim_{\mathbb{R}}(\mathbb{K}) - 1$, then $\mu_N$ is the Lebesgue measure on $\mathbb{R}^{p+q}$, and the parameter $\rho$ that appears in (1-1) for the Haar measure equals $\frac{1}{2}(p+2q)$.

Let $\overline{N}$ denote the opposite unipotent group, namely the one that corresponds to the negative roots

$$a_t \bar{n}_{v,z} a_{-t} = n_{e^{-t} v, e^{-2t} z}. \tag{4-4}$$

On the subgroups $H \in \{A, K\}$ we consider the metric $d_H$ induced by the riemannian metric on $G$. We denote by $K_{(\phi,\delta)}$ the open ball in $K$ with center $\phi \in K$ and radius $\delta$, and by $A_{(t,\delta)}$ the open ball in $A$, with center $t$ and radius $\delta$ (these are simply the elements that correspond to the interval $(t-\delta, t+\delta)$, since $d_A$ is the euclidean metric on $\mathbb{R}$). On the product space $N = \mathbb{R}^p \times \mathbb{R}^q$ we let $d_N$ denote the maximum of the two euclidean metrics $d_N^{(1)}, d_N^{(2)}$ on the components $\mathbb{K}^n \cong \mathbb{R}^p$ and $\Im(\mathbb{K}) \cong \mathbb{R}^q$, and let $N_{(v,\delta_1)\times(z,\delta_2)}$ be the domain in $N$ parametrized by the product of open euclidean balls in $\mathbb{K}^n \cong \mathbb{R}^p$ and $\Im(\mathbb{K}) \cong \mathbb{R}^q$ with centers $v, z$ and radii $\delta_1, \delta_2$ respectively. When a ball is centered at the identity we omit the center and denote it by $K_{(\delta)}$, $A_{(\delta)}$, and $N_{(\delta_1)\times(\delta_2)}$.

In what follows, $\|\cdot\|_{\mathrm{CK}} = \|\cdot\|$ is the Cartan–Killing norm on the Lie algebra $\mathrm{Lie}(G)$ of $G$, and $\|\cdot\|_{\mathrm{op}}$ is the norm on the space of linear operators on $\mathrm{Lie}(G)$.

**Lemma 4.3.** *Let $G \subset \mathrm{SL}_N(\mathbb{R})$ be a connected semisimple linear Lie group. Let $\mathcal{O}_\epsilon = \exp(B_\epsilon)$, where $B_\epsilon = \{X \in \mathrm{Lie}(G) : \|X\|_{CK} < \epsilon\}$. For every $g \in G$, the following inclusion holds*:

$$g\mathcal{O}_\epsilon g^{-1} \subseteq \mathcal{O}_{\epsilon\cdot\|\mathrm{Ad}\, g\|_{\mathrm{op}}} = \exp\{X \in \mathrm{Lie}(G) : \|X\|_{\mathrm{CK}} < \epsilon \cdot \|\mathrm{Ad}\, g\|_{\mathrm{op}}\}.$$

*Proof.* The operator norm is defined by

$$\|\mathrm{Ad}\, g\|_{\mathrm{op}} = \max_{\|X\|\le 1} \|\mathrm{Ad}\, g(X)\| = \max_{\|X\|\le 1} \|gXg^{-1}\|.$$

Therefore $\mathrm{Ad}\, g(B_\epsilon) \subset \mathrm{Lie}(G)$ is contained in a ball of radius

$$\max_{X\in B_\epsilon} \|\mathrm{Ad}\, g(X)\| = \max_{X\in B_\epsilon} \|gXg^{-1}\| = \max_{X\in\epsilon B_1} \|gXg^{-1}\| = \max_{X\in B_1} \|g\epsilon Xg^{-1}\| = \epsilon\|\mathrm{Ad}\, g\|_{\mathrm{op}}.$$

Since the exponential function $e^X : M_N(\mathbb{R}) \to M_N(\mathbb{R})$ has a convergent power series expansion at every point $X$, it follows that $ge^X g^{-1} = e^{gXg^{-1}}$ for every $g \in \mathrm{GL}_N(\mathbb{R})$.

Therefore

$$\begin{aligned} g\mathcal{O}_\epsilon g^{-1} = g\exp(B_\epsilon)g^{-1} = \exp(gB_\epsilon g^{-1}) = \exp(\mathrm{Ad}\, g(B_\epsilon)) \\ \subseteq \exp(B_{\epsilon\cdot\|\mathrm{Ad}\, g\|_{\mathrm{op}}}) = \mathcal{O}_{\epsilon\cdot\|\mathrm{Ad}\, g\|_{\mathrm{op}}}. \quad \square \end{aligned}$$

Let $M$ denote the centralizer of $A$ in $K$. We will use that there exists $\delta_0 > 0$ and positive constants $c_1, c_2, c_1', c_2'$ such that for all $0 < \delta < \delta_0$,

$$\mathcal{O}_\delta \subseteq N_{(c_1\delta)\times(c_1\delta)} A_{(c_1\delta)} K_{(c_1\delta)} \subseteq \mathcal{O}_{c_1'c_1\delta}, \tag{4-5}$$

$$\mathcal{O}_\delta \subseteq N_{(c_2\delta)\times(c_2\delta)} A_{(c_2\delta)} M_{(c_2\delta)} \overline{N}_{(c_2\delta)\times(c_2\delta)} \subseteq \mathcal{O}_{c_2'c_2\delta}, \tag{4-6}$$

the latter being the Bruhat coordinates on a neighborhood of the identity in $G$.

The foregoing inclusions are consequences of the following fact. Let there be given a decomposition of a Lie algebra as a direct sum of Lie subalgebras, and let there be given any basis of the Lie algebra. Then there exists a closed ball of fixed size centered at 0 in the Lie algebra satisfying the following property. The map assigning the canonical coordinates associated with the first decomposition (of either the first or the second kind) to the canonical coordinates associated with the given basis (of either the first or the second kind), is an invertible smooth map with all its derivatives (and the derivatives of its inverse) bounded. In particular, it is a bi-Lipschitz map.

**Proposition 4.4** (effective Iwasawa decomposition). *Let $n_{v,z} \in N$, $\phi \in K$, $a_t \in A$ with $t \leq 0$. There exists $\epsilon_1 > 0$ such that for every $0 < \epsilon < \epsilon_1$ there are positive constants $C'_N, C''_N, C_A, C_K$ that depend only on $n_{v,z}$ and $\phi$ (in particular, independent of $t$) such that*

$$\mathcal{O}_\epsilon \cdot n_{v,z} a_t \phi \cdot \mathcal{O}_\epsilon \subseteq N_{(v, C'_N \epsilon) \times (z, C''_N \epsilon)} A_{(t, C_A \epsilon)} K_{(\phi, C_K \epsilon)}.$$

*Furthermore, when $n_{v,z}$ varies over a compact set $\Psi$, and $\phi$ varies over $K$, these constants can be taken to be uniform.*

*Proof.* Observe that

$$N_{(\delta_1) \times (\delta_2)} N_{(\rho_1) \times (\rho_2)} \subseteq N_{(\delta_1 + \rho_1) \times (\delta_2 + \rho_2 + \rho_1 \delta_1)}$$

and

$$n_{v,z} N_{(\rho_1) \times (\rho_2)} \subseteq N_{(v, \rho_1) \times (z, \rho_2 + \|v\| \rho_1)}. \tag{4-7}$$

In particular,

$$\begin{aligned} n_{v,z} N_{(\delta_1) \times (\delta_2)} N_{(\rho_1) \times (\rho_2)} &\subseteq n_{v,z} N_{(\delta_1 + \rho_1) \times (\delta_2 + \rho_2 + \rho_1 \delta_1)} \\ &\subseteq N_{(v, \delta_1 + \rho_1) \times (z, \delta_2 + \rho_2 + \rho_1 \delta_1 + \|v\| (\delta_1 + \rho_1))}. \end{aligned} \tag{4-8}$$

Finally, note that

$$K_{(\delta)} \phi \subset K_{(\phi, \delta)}. \tag{4-9}$$

*Step 1: Right perturbations.* We show that

$$n_{v,z} a_t \phi \cdot \mathcal{O}_\epsilon \subseteq N_{(v, r_1 \epsilon) \times (z, r_2 \epsilon)} A_{(t, r_3 \epsilon)} K_{(\phi, r_4 \epsilon)}$$

where $r_i = r_i(v, z)$ is independent of $t \leq 0$. Recall $\|\mathrm{Ad}\, \phi\| = 1$. Then, by Lemma 4.3,

$$\begin{aligned} n_{v,z} a_t \phi \cdot \mathcal{O}_\epsilon \subseteq n_{v,z} a_t \mathcal{O}_\epsilon \phi &\overset{(4\text{-}5)}{\subseteq} n_{v,z} a_t (N_{(c_1 \epsilon) \times (c_1 \epsilon)} A_{(c_1 \epsilon)} K_{(c_1 \epsilon)}) \phi \\ &\overset{(4\text{-}3)}{\subseteq} n_{v,z} N_{(e^t c_1 \epsilon) \times (e^{2t} c_1 \epsilon)} a_t A_{(c_1 \epsilon)} K_{(c_1 \epsilon)} \phi \\ &\overset{(4\text{-}7),(4\text{-}9)}{\subseteq} N_{(v, c_1 e^t \epsilon) \times (z, c_1 e^{2t} \epsilon + c_1 \|v\| e^t \epsilon)} A_{(t, c_1 \epsilon)} K_{(\phi, c_1 \epsilon)} \\ &\subseteq N_{(v, c_1 \epsilon) \times (z, c_1 \epsilon + c_1 \|v\| \epsilon)} A_{(t, c_1 \epsilon)} K_{(\phi, c_1 \epsilon)}, \end{aligned}$$

the latter inclusion holding since $e^t \leq 1$.

*Step 2: Left perturbations.* We show that

$$\mathcal{O}_\epsilon \cdot n_{v,z} a_t \phi \subseteq N_{(v,\ell_1\epsilon)\times(z,\ell_2\epsilon)} A_{(t,\ell_3\epsilon)} K_{(\phi,\ell_4\epsilon)}$$

where $\ell_i = \ell_i(v,z)$ is independent of $t \le 0$.

Denote $\eta = \|\mathrm{Ad}\, n_{v,z}^{-1}\|_{\mathrm{op}}$. By Lemma 4.3,

$$\mathcal{O}_\epsilon \cdot n_{v,z} a_t \phi \subseteq n_{v,z} \mathcal{O}_{\eta\epsilon} a_t \phi.$$

(We note that we will apply this argument below to $n_{v,z} \in \Psi \subset N$ for a fixed bounded set $\Psi$).

Assume $0 < \epsilon < \min\{1, \delta_0/\eta\}$. Then

$$\begin{aligned} n_{v,z}\mathcal{O}_{\eta\epsilon} a_t \phi &\overset{(4\text{-}6)}{\subseteq} n_{v,z}(N_{(c_2\eta\epsilon)\times(c_2\eta\epsilon)} A_{(c_2\eta\epsilon)} M_{(c_2\eta\epsilon)} \overline{N}_{(c_2\eta\epsilon)\times(c_2\eta\epsilon)}) a_t \phi \\ &\overset{(4\text{-}4)}{\subseteq} n_{v,z} N_{(c_2\eta\epsilon)\times(c_2\eta\epsilon)} A_{(c_2\eta\epsilon)} a_t M_{(c_2\eta\epsilon)} \overline{N}_{(c_2 e^t \eta\epsilon)\times(c_2 e^{2t}\eta\epsilon)} \phi \\ &\subseteq n_{v,z} N_{(c_2\eta\epsilon)\times(c_2\eta\epsilon)} A_{(c_2\eta\epsilon)} a_t (\mathcal{O}_{c_2' c_2 \eta\epsilon}) \phi, \end{aligned}$$

the latter inclusion coming from the second inclusion in (4-6), since $1 \ge e^t \ge e^{2t}$ and $M_{(c_2\eta\epsilon)} \overline{N}_{(c_2 e^t\eta\epsilon)\times(c_2 e^{2t}\eta\epsilon)} \subset \mathcal{O}_{c_2' c_2\eta\epsilon}$. By the first inclusion in (4-5), provided $0 < \epsilon < \delta_0/(c_2' c_2 \eta)$, this is included in

$$n_{v,z} N_{(c_2\eta\epsilon)\times(c_2\eta\epsilon)} A_{(t,c_2\eta\epsilon)} (N_{(c_1c_2'c_2\eta\epsilon)\times(c_1c_2'c_2\eta\epsilon)} A_{(c_1c_2'c_2\eta\epsilon)} K_{(c_2'c_1c_2\eta\epsilon)})\phi,$$

and by (4-3) and (4-9), the above set is included in

$$n_{v,z} N_{(c_2\eta\epsilon)\times(c_2\eta\epsilon)} N_{(e^{t+c_2\eta\epsilon}\cdot c_1c_2'c_2\eta\epsilon)\times(e^{2(t+c_2\eta\epsilon)}\cdot c_1c_2'c_2\eta\epsilon)} A_{(t,c_2\eta\epsilon)} A_{(c_1c_2'c_2\eta\epsilon)} K_{(\phi,c_1c_2'c_2\eta\epsilon)}.$$

Hence by (4-8), using $e^t \le 1$, for every $\epsilon < \epsilon_1$ where $\epsilon_1$ satisfies the two foregoing conditions, we obtain that the above set is included in

$$N_{(v,(1+c_1c_2'e^{c_2\eta})c_2\eta\epsilon)\times(z,(1+c_1c_2'e^{2c_2\eta\epsilon}(1+c_2\eta\epsilon)+\|v\|(1+e^{2c_2\eta\epsilon}c_1c_2')c_2\eta\epsilon))} A_{(t,(1+c_1c_2')c_2\eta\epsilon)} K_{(\phi,c_1c_2'c_2\eta\epsilon)}.$$

*Step 3: Combining left and right perturbations.* Let $g := n_{v,z} a_t \phi$ with $t \le 0$ and let $\epsilon < \epsilon_1$. Fix positive constants $\bar{\ell}_i = \max\{\ell_i(v',z') : n_{v',z'} \in \pi_N(g \cdot \mathcal{O}_1)\}$, which are uniform, namely independent of $t$. Since $g \cdot \mathcal{O}_\epsilon \subset g \cdot \mathcal{O}_1$, it follows from Step 2 that for every

$$g_0 = n_{v_0,z_0} a_{t_0} \phi_0 \in g \cdot \mathcal{O}_\epsilon,$$

it holds that

$$\mathcal{O}_\epsilon \cdot g_0 \subset N_{(v_0,\bar{\ell}_1\epsilon)\times(z_0,\bar{\ell}_2\epsilon)} A_{(t_0,\bar{\ell}_3\epsilon)} K_{(\phi_0,\bar{\ell}_4\epsilon)}.$$

But, as was shown in Step 1, $d_N^{(1)}(v_0,v) < r_1\epsilon$, $d_N^{(2)}(z_0,z) < r_2\epsilon$, $d_A(t_0,t) < r_3\epsilon$ and $d_K(\phi_0,\phi) < r_4\epsilon$. Then by the triangle inequality for the metrics $d_N^{(1)}, d_N^{(2)}, d_A, d_K$,

$$\mathcal{O}_\epsilon \cdot g \cdot \mathcal{O}_\epsilon \subset N_{(v,r_1\epsilon+\bar{\ell}_1\epsilon)\times(z,r_2\epsilon+\bar{\ell}_2\epsilon)} A_{(t,r_3\epsilon+\bar{\ell}_3\epsilon)} K_{(\phi,r_4\epsilon+\bar{\ell}_4\epsilon)}. \qquad \square$$

**4C. *Lipschitz regularity of the domains $R_T(\Psi, \Phi)$*.** Recall that we wish to show that the family $\{R_T(\Psi, \Phi)\}_{T>0}$ is Lipschitz well-rounded (Definition 4.1). Since we have already established the Lipschitz property for the Iwasawa coordinates in the negative direction of $A$, all that remains is to bound the quotient of the measures of $R_T(\Psi, \Phi)^+(\epsilon)$ and $R_T(\Psi, \Phi)^-(\epsilon)$, which we perform below.

*Proof of Theorem 1.1.* Throughout this proof, it will be convenient to parametrize $N$ as $\mathbb{R}^{p+q}$ instead of $\mathbb{R}^p \oplus \mathbb{R}^q$. We will write $n_{\underline{x}}$ instead of $n_{v,z}$, and $N_{(\underline{x},\delta)} = N_{(v,\delta),(z,\delta)}$ for a ball of radius $\delta$ centered at $\underline{x} = (v, z)$. It will also be convenient to denote $\mu_A = dt/e^{2\rho t}$, and then $\mu = \mu_N \times \mu_A \times \mu_K$.

The proof will proceed by showing that there exists $\epsilon_0 > 0$, which will be described explicitly below, such that for $0 < \epsilon < \epsilon_0$, $R_T(\Psi, \Phi)^+(\epsilon)$ is contained in a product set of the form $\Psi^+ A_{-T^+,S^+} \Phi^+$, and $R_T(\Psi, \Phi)^-(\epsilon)$ contains a product set of the form $\Psi^- A_{-T^-,S^-} \Phi^-$, with the following property. The ratio of the measure of the three components of $\Psi^+ A_{-T^+,S^+} \Phi^+$ to the corresponding components of $\Psi^- A_{-T^-,S^-} \Phi^-$ is bounded by $1 + C\epsilon$, for $0 < \epsilon < \epsilon_0$, $T \geq T_0$. It then follows immediately that

$$\frac{\mu(R_T(\Psi, \Phi)^+(\epsilon))}{\mu(R_T(\Psi, \Phi)^-(\epsilon))} \leq 1 + C'\epsilon.$$

To construct the sets alluded to above, recall first that for every $H \in \{N, A, K\}$, $\xi \in H$ and $\delta > 0$, $H_{(\xi,\delta)}$ denotes the open ball of radius $\delta$ centered at $\xi$ with respect to the metric $d_H$ on $H$. By Proposition 4.4 there exist positive constants $C_N$, $C_A$, $C_K$ that depend on $\Psi$ and $\Phi$ alone such that for every $\underline{x} \in \Psi$, $\phi \in \Phi$, $0 < \epsilon < \epsilon_1$ and $t \leq 0$,

$$\mathcal{O}_\epsilon \cdot n_{\underline{x}} a_t k_\phi \cdot \mathcal{O}_\epsilon \subseteq N_{(\underline{x}, C_N\epsilon)} A_{(t, C_A\epsilon)} K_{(\phi, C_K\epsilon)}.$$

We now claim that the inclusions

$$\pi_H(R_T(\Psi, \Phi)^+(\epsilon)) \subseteq \bigcup_{\xi \in \Xi} H_{(\xi, C_H\epsilon)}, \tag{4-10}$$

$$\pi_H(R_T(\Psi, \Phi)^-(\epsilon)) \supseteq \bigcup_{\xi \in \Xi} H_{(\xi, C_H\epsilon)} \setminus \bigcup_{\xi \in \partial\Xi} H_{(\xi, C_H\epsilon)} \tag{4-11}$$

hold for every $H \in \{N, A, K\}$ and the corresponding $\Xi \in \{\Psi, [-T, 0], \Phi\}$ in $H$. The sets appearing in the right-hand side of the first inclusion are the sets $\Psi^+$, $[-T^+, S^+]$, $\Phi^+$, and the sets appearing in the right-hand side of the second inclusion are the sets $\Psi^-$, $[-T^-, S^-]$, $\Phi^-$, alluded to above, for $H = N, A, K$.

Note that the set on the right-hand side of (4-11) is the set of points in $\Xi$ whose distance from the complement of $\Xi$ is at least $C_H\epsilon$. Namely it is the set of points such that an open $C_H\epsilon$-ball centered around them is fully contained in $\Xi = \pi_H(R_T(\Psi, \Phi))$. This follows from the following fact. If $\xi \in \Xi$, $\xi' \notin \Xi$ and the distance between them is less than some $\eta > 0$, then $\xi$ has distance less than $\eta$

from some point in $\partial\Xi$. Conversely, if $\xi\in\Xi$ has distance less than $\eta$ from $\partial\Xi$, then it has distance less than $\eta$ to some point $\xi'\notin\Xi$.

The inclusion (4-10) is a straightforward consequence of Proposition 4.4. To prove the inclusion (4-11), we note that

$$\begin{aligned} \text{(4-12)}\quad g\in R_T(\Psi,\Phi)^-(\epsilon) &\iff g\in uR_T(\Psi,\Phi)v && \forall u,v\in\mathcal{O}_\epsilon\\ &\iff ugv\in R_T(\Psi,\Phi) && \forall u,v\in\mathcal{O}_\epsilon\\ &\iff \pi_H(ugv)\in\pi_H(R_T(\Psi,\Phi)) && \forall u,v\in\mathcal{O}_\epsilon,\ \forall H, \end{aligned}$$

since $R_T(\Psi,\Phi)$ is a product set, and $\mathcal{O}_\epsilon=\mathcal{O}_\epsilon^{-1}$.

Now consider $g$ such that $\pi_H(g)\in\bigcup_{\xi\in\Xi}H_{(\xi,C_H\epsilon)}\setminus\bigcup_{\xi\in\partial\Xi}H_{(\xi,C_H\epsilon)}$. Then as just noted above, $\pi_H(g)\in\Xi$ and $H_{(\pi_H(g),C_H\epsilon)}\subset\Xi$, and therefore by the version of Proposition 4.4 stated just before (4-10),

$$\pi_H(ugv)\in H_{(\pi_H(g),C_H\epsilon)}\subset\pi_H(R_T(\Psi,\Phi)).$$

Thus every such $g$ is contained in $R_T(\Psi,\Phi)^-(\epsilon)$ by (4-12). Letting $g$ range over the product set, we conclude that $\Psi^-A_{-T^-,S^-}\Phi^-\subseteq R_T(\Psi,\Phi)^-(\epsilon)$, as stated.

As to the volume estimate, we begin with the $N$-component. Since $\Psi$ is assumed to be nice, there exists a constant $\alpha_1$ which depends on $\Psi$ and $C_N$ such that

$$\mu_N\biggl(\bigcup_{\underline{x}\in\partial\Psi}N_{(\underline{x},C_N\epsilon)}\biggr)\le\alpha_1\epsilon.$$

Note that $\bigcup_{\underline{x}\in\Psi}N_{(\underline{x},C_N\epsilon)}\subset\Psi\cup\bigcup_{\underline{x}\in\partial\Psi}N_{(\underline{x},C_N\epsilon)}$. Therefore, by definition of $\Psi^+$ in (4-10),

$$\mu_N(\Psi^+)\le\mu_N\biggl(\bigcup_{\underline{x}\in\Psi}N_{(\underline{x},C_N\epsilon)}\biggr)\le\mu_N(\Psi)+\mu_N\biggl(\bigcup_{\underline{x}\in\partial\Psi}N_{(\underline{x},C_N\epsilon)}\biggr)\le\mu_N(\Psi)+\alpha_1\epsilon.$$

On the other hand, by (4-11),

$$\begin{aligned}\mu_N(\Psi^-)&\ge\mu_N\biggl(\biggl(\bigcup_{\underline{x}\in\Psi}N_{(\underline{x},C_N\epsilon)}\cup\bigcup_{\underline{x}\in\partial\Psi}N_{(\underline{x},C_N\epsilon)}\biggr)\setminus\bigcup_{\underline{x}\in\partial\Psi}N_{(\underline{x},C_N\epsilon)}\biggr)\\ &\ge\mu_N(\Psi)-\mu_N\biggl(\bigcup_{\underline{x}\in\partial\Psi}N_{(\underline{x},C_N\epsilon)}\biggr)\\ &\ge\mu_N(\Psi)-\alpha_1\epsilon.\end{aligned}$$

By assuming $\epsilon$ is small enough such that $\alpha_1\epsilon\le\frac{1}{2}\mu_N(\Psi)$, the last two inequalities imply

$$\frac{\mu_N(\Psi^+)}{\mu_N(\Psi^-)}-1\le\frac{\mu_N(\Psi)+\alpha_1\epsilon-(\mu_N(\Psi)-\alpha_1\epsilon)}{\frac{1}{2}\mu_N(\Psi)}=\frac{2\alpha_1}{\frac{1}{2}\mu_N(\Psi)}\cdot\epsilon.$$

The same considerations apply for $\Phi \subseteq K$. Since it is also assumed to be nice, there exists $\alpha_2 > 0$ that depends on $\partial\Phi$ and $C_K$ such that

$$\mu_K(\Phi^+) = \mu_K\Bigg(\bigcup_{k\in\partial\Phi} K_{(k,C_K\epsilon)}\Bigg) \le \alpha_2\epsilon,$$

and, similarly to the $N$ case, by assuming $\alpha_2\epsilon \le \frac{1}{2}\mu_K(\Phi)$,

$$\frac{\mu_K(\Phi^+)}{\mu_K(\Phi^-)} - 1 \le \frac{2\alpha_2}{\frac{1}{2}\mu_K(\Phi)} \cdot \epsilon.$$

Finally, for the $A$-component, it follows from (4-10) and (4-11) that

$$\pi_A(R_T(\Psi,\Phi)^+(\epsilon)) \subseteq [-T - C_A\epsilon, 0 + C_A\epsilon] = [-T^+, S^+]$$

and

$$\pi_A(R_T(\Psi,\Phi)^-(\epsilon)) \supseteq [-T + C_A\epsilon, 0 - C_A\epsilon] = [-T^-, S^-].$$

Clearly,

$$\mu_A([-T^+, S^+]) \le \int_{-T-C_A\epsilon}^{tC_A\epsilon} \frac{dt}{e^{2\rho t}} = \frac{1}{2\rho}(e^{2\rho(T+C_A\epsilon)} - e^{-2\rho C_A\epsilon})$$

and

$$\mu_A([-T^-, S^-]) \ge \int_{-T+C_A\epsilon}^{-C_A\epsilon} \frac{dt}{e^{2\rho t}} = \frac{1}{2\rho}(e^{2\rho(T-C_A\epsilon)} - e^{2\rho C_A\epsilon}).$$

As a result,

$$\begin{aligned}\frac{\mu_A([-T^+, S^+])}{\mu_A([-T^-, S^-])} - 1 &\le \frac{e^{2\rho(T+C_A\epsilon)} - e^{-2\rho C_A\epsilon} - (e^{2\rho(T-C_A\epsilon)} - e^{2\rho C_A\epsilon})}{e^{2\rho(T-C_A\epsilon)} - e^{2\rho C_A\epsilon}} \\ &= \frac{(e^{2\rho T}+1)}{e^{2\rho T}} \cdot \frac{(e^{2\rho C_A\epsilon} - e^{-2\rho C_A\epsilon})}{e^{-2\rho C_A\epsilon} - e^{-2\rho T}e^{2\rho C_A\epsilon}}.\end{aligned}$$

For $\epsilon \le (4\rho C_A)^{-1}$ and $T \ge 2\rho^{-1}$ it holds that $e^{2\rho C_A\epsilon} - e^{-2\rho C_A\epsilon} \le 3\cdot 2\rho C_A\epsilon$ and $e^{-2\rho C_A\epsilon} - e^{-2\rho T}e^{2\rho C_A\epsilon} \ge \frac{1}{2}$; therefore,

$$\frac{\mu_A([-T^+, S^+])}{\mu_A([-T^-, S^-])} - 1 \le 2\cdot\frac{6\rho C_A\epsilon}{1/2} = 24\rho C_A\epsilon.$$

Now since

$$\frac{\mu(R_T(\Psi,\Phi)^+(\epsilon))}{\mu(R_T(\Psi,\Phi)^-(\epsilon))} \le \frac{\mu_N(\Psi^+)\mu_A([-T^+, S^+])\mu_K(\Phi^+)}{\mu_N(\Psi^-)\mu_A([-T^-, S^-])\mu_K(\Phi^-)},$$

by choosing $T_0 = 2\rho^{-1}$ and $\epsilon_0 = \min\{\epsilon_1, \mu_N(\Psi)/(2\alpha_1), \mu_K(\Phi)/(2\alpha_2), 1/(4\rho C_A)\}$ we conclude that the family $\{R_T(\Psi,\Phi)\}_{T>T_0}$ is Lipschitz well-rounded for $0 < \epsilon < \epsilon_0$, and by Theorem 4.2 (and the discussion in Section 4A) we are done. □

**4D.** ***Proof of Corollary 1.2.*** Let $\Psi$, $\Psi'$, $\Phi$, $\Phi'$ and $\kappa$ as in the statement of the corollary. By Theorem 1.1, the denominator in the following ratio is eventually positive and the following estimate holds:

$$\frac{\#(\Gamma \cap R_T(\Psi', \Phi'))}{\#(\Gamma \cap R_T(\Psi, \Phi))} = \frac{\mu_N(\Psi')\mu_K(\Phi')e^{2\rho T} + O(Te^{2\rho\kappa T})}{\mu_N(\Psi)\mu_K(\Phi)e^{2\rho T} + O(Te^{2\rho\kappa T})}$$
$$= \frac{\mu_N(\Psi')\mu_K(\Phi')}{\mu_N(\Psi)\mu_K(\Phi)} + O(T(e^{2\rho T})^{-(1-\kappa)}).$$

The limit of the foregoing expression is $(\mu_N(\Psi')\mu_K(\Phi'))/(\mu_N(\Psi)\mu_K(\Phi))$ as $T \to \infty$, since $\kappa < 1$. The implied constant depends on $\Psi$, $\Psi'$, $\Phi$, $\Phi'$.

Let now $\psi$ and $\phi$ be nonnegative Lipschitz functions with positive integral, with $\psi$ defined on $\Psi$, and $\phi$ defined on $\Phi$. We also view $\psi$ as a (measurable bounded) function on $N$ by defining it to be zero outside $\Psi$, and we extend $\phi$ to $K$ similarly. Let $R_T(\psi, \phi)$ be the measure on $G$ whose density with respect to Haar measure on $G$ (written in Iwasawa coordinates as in (1-1)) is given by the function $D_T(na_tk) = \psi(n)\chi_{[-T,0]}(a_t)\phi(k)$. Equivalently, the measure is given by the following formula: for $F \in C_c(G)$,

$$R_T(\psi, \phi)(F) = \int_\Psi \int_{-T}^0 \int_\Phi F(na_tk)\psi(n)\phi(k)\, d\mu_N(n)\, \frac{dt}{e^{2\rho t}}\, d\mu_K(k).$$

The family of measures $R_T(\psi, \phi)$ is Lipschitz well-rounded, in the following sense. Defining

$$D_T^{+,\epsilon}(g) = \sup_{u,v\in\mathcal{O}_\epsilon} D_T(ugv), \qquad D_T^{-,\epsilon}(g) = \inf_{u,v\in\mathcal{O}_\epsilon} D_T(ugv),$$

we have

$$\int_G D_T^{+,\epsilon}(g)\, d\mu(g) \le (1 + C\epsilon)\int_G D_T^{-,\epsilon}(g)\, d\mu(g).$$

The family $R_T(\psi, \phi)$ satisfies a weighted version of the lattice point counting result which the sets $R_T(\Psi, \Phi)$ satisfy, namely

$$\sum_{\gamma\in\Gamma} D_T(\gamma) = \int_G D_T(g)d\mu(g) + O_{\phi,\psi}\Big(\Big(\int_G D_T(g)d\mu(g)\Big)^{\kappa(\Gamma)} \cdot \log \int_G D_T(g)d\mu(g)\Big),$$

so that in the present case,

$$\sum_{\gamma\in\Gamma} \psi(\pi_N(\gamma))\chi_{[-T,0]}(\pi_A(\gamma))\phi(\pi_K(\gamma))$$
$$= e^{2\rho T}\int_N \psi(n)\, d\mu_N(n) \cdot \int_K \phi(k)\, d\mu_K(k) + O_{\phi,\psi}(Te^{2\rho T\kappa(\Gamma)}).$$

The proof of the weighted version of the lattice point problem stated above under the assumption of Lipschitz well-roundedness is a straightforward modification of the arguments that appear in [Gorodnik and Nevo 2012]. The fact that when $\psi$ and

$\phi$ are Lipschitz functions on $N$ and $K$ the measures $R_T(\psi, \phi)$ defined above are Lipschitz well-rounded is a straightforward modification of the arguments in the present paper. Note that it suffices to consider nonnegative Lipschitz functions on $N$ and $K$, and the case of general Lipschitz functions follows, since $\max(f, 0)$ and $\max(-f, 0)$ are nonnegative Lipschitz functions and $f$ is their difference. Finally, the statement of Corollary 1.2 part (2) follows by considering a Lipschitz function $\psi$ defined on $\Psi \subset N$, a Lipschitz function $\phi$ defined on $\Phi \subset K$, defining $D_T$ using $\psi$ and $\phi$, and estimating the ratios as

$$\frac{\sum_{\gamma\in\Gamma} D_T(\gamma)}{\sum_{\gamma\in\Gamma} \chi_{R_T(\Psi,\Phi)}(\gamma)} = \frac{1}{\mu_N(\Psi)}\int_N \psi(n)\,d\mu_N(n)\cdot\frac{1}{\mu_K(\Phi)}\int_K \phi(k)\,d\mu_K(k) + O(Te^{-2\rho(1-\kappa)T}),$$

where the implied constant depends on $\Phi$, $\Psi$, $\phi$, $\psi$. □

## Acknowledgments

The authors thank the referee for important comments which led to significant improvements is the presentation of several results in the paper. They also thank Ami Paz for preparing the figures for this paper. Horesh thanks Ami Paz and Yakov Karasik for helpful discussions. Nevo thanks John Parker and Rene Rühr for providing some very useful references. Nevo is supported by ISF Grant No. 2095/15.

## References

[Chamizo 2011] F. Chamizo, "Hyperbolic lattice point problems", *Proc. Amer. Math. Soc.* **139**:2 (2011), 451–459. MR Zbl

[Cowling et al. 1991] M. Cowling, A. H. Dooley, A. Korányi, and F. Ricci, "$H$-type groups and Iwasawa decompositions", *Adv. Math.* **87**:1 (1991), 1–41. MR Zbl

[Cowling et al. 1998] M. Cowling, A. Dooley, A. Korányi, and F. Ricci, "An approach to symmetric spaces of rank one via groups of Heisenberg type", *J. Geom. Anal.* **8**:2 (1998), 199–237. MR Zbl

[Dabbs et al. 2016] K. Dabbs, M. Kelly, and H. Li, "Effective equidistribution of translates of maximal horospherical measures in the space of lattices", *J. Mod. Dyn.* **10** (2016), 229–254. MR Zbl

[Dinaburg and Sinai 1990] E. I. Dinaburg and Y. G. Sinai, "The statistics of the solutions of the integer equation $ax - by = \pm 1$", *Funktsional. Anal. i Prilozhen.* **24**:3 (1990), 1–8. MR Zbl

[Duke et al. 1993] W. Duke, Z. Rudnick, and P. Sarnak, "Density of integer points on affine homogeneous varieties", *Duke Math. J.* **71**:1 (1993), 143–179. MR Zbl

[Elstrodt et al. 1998] J. Elstrodt, F. Grunewald, and J. Mennicke, *Groups acting on hyperbolic space: harmonic analysis and number theory*, Springer, 1998. MR Zbl

[Eskin and McMullen 1993] A. Eskin and C. McMullen, "Mixing, counting, and equidistribution in Lie groups", *Duke Math. J.* **71**:1 (1993), 181–209. MR Zbl

[Faraut 1983] J. Faraut, "Analyse harmonique sur les paires de Guelfand et les espaces hyperboliques", pp. 315–446 in *Analyse harmonique* (Nancy, France, 1980), edited by J.-L. Clerc et al., Les Cours du CIMPA, Nice, 1983.

[Good 1983] A. Good, *Local analysis of Selberg's trace formula*, Lecture Notes in Mathematics **1040**, Springer, 1983. MR Zbl

[Good 1984] A. Good, "Analytical and arithmetical methods in the theory of Fuchsian groups", pp. 86–103 in *Number theory* (Noordwijkerhout, The Netherlands, 1983), edited by H. Jager, Lecture Notes in Math. **1068**, Springer, 1984. MR Zbl

[Gorodnik and Nevo 2010] A. Gorodnik and A. Nevo, *The ergodic theory of lattice subgroups*, Annals of Mathematics Studies **172**, Princeton University Press, 2010. MR Zbl

[Gorodnik and Nevo 2012] A. Gorodnik and A. Nevo, "Counting lattice points", *J. Reine Angew. Math.* **663** (2012), 127–176. MR Zbl

[Gorodnik and Weiss 2007] A. Gorodnik and B. Weiss, "Distribution of lattice orbits on homogeneous varieties", *Geom. Funct. Anal.* **17**:1 (2007), 58–115. MR Zbl

[Kim 2015] I. Kim, "Counting, mixing and equidistribution of horospheres in geometrically finite rank one locally symmetric manifolds", *J. Reine Angew. Math.* **704** (2015), 85–133. MR Zbl

[Marklof and Vinogradov 2018] J. Marklof and I. Vinogradov, "Directions in hyperbolic lattices", *J. Reine Angew. Math.* **740** (2018), 161–186. MR Zbl

[Mohammadi and Oh 2015] A. Mohammadi and H. Oh, "Matrix coefficients, counting and primes for orbits of geometrically finite groups", *J. Eur. Math. Soc.* (*JEMS*) **17**:4 (2015), 837–897. MR Zbl

[Mohammadi and Salehi Golsefidy 2014] A. Mohammadi and A. Salehi Golsefidy, "Translate of horospheres and counting problems", *Amer. J. Math.* **136**:5 (2014), 1301–1346. MR Zbl

[Nevo 1998] A. Nevo, "Spectral transfer and pointwise ergodic theorems for semi-simple Kazhdan groups", *Math. Res. Lett.* **5**:3 (1998), 305–325. MR Zbl

[Risager and Rudnick 2009] M. S. Risager and Z. Rudnick, "On the statistics of the minimal solution of a linear Diophantine equation and uniform distribution of the real part of orbits in hyperbolic spaces", pp. 187–194 in *Spectral analysis in geometry and number theory*, edited by M. Kotani et al., Contemp. Math. **484**, Amer. Math. Soc., Providence, RI, 2009. MR Zbl

[Selberg 1991] A. Selberg, "Abstracts of lectures given by A. Selberg at Tel Aviv University", 1991, available at http://publications.ias.edu/node/2482.

[Truelsen 2013] J. L. Truelsen, "Effective equidistribution of the real part of orbits on hyperbolic surfaces", *Proc. Amer. Math. Soc.* **141**:2 (2013), 505–514. MR Zbl

Received October 24, 2021. Revised December 29, 2022.

TAL HORESH
IST AUSTRIA
KLOSTERNEUBURG
AUSTRIA
talhoresh@gmail.com

AMOS NEVO
DEPARTMENT OF MATHEMATICS
TECHNION — ISRAEL INSTITUTE OF TECHNOLOGY
HAIFA
ISRAEL
anevo@tx.technion.ac.il

PACIFIC JOURNAL OF MATHEMATICS
Vol. 324, No. 2, 2023
https://doi.org/10.2140/pjm.2023.324.295

# BOUNDED RICCI CURVATURE AND POSITIVE SCALAR CURVATURE UNDER RICCI FLOW

KLAUS KRÖNCKE, TOBIAS MARXEN AND BORIS VERTMAN

**We consider a Ricci de Turck flow of spaces with isolated conical singularities, which preserves the conical structure along the flow. We establish that a given initial regularity of Ricci curvature is preserved along the flow. Moreover under additional assumptions, positivity of scalar curvature is preserved under such a flow, mirroring the standard property of Ricci flow on compact manifolds. The analytic difficulty is the a priori low regularity of scalar curvature at the conical tip along the flow, so that the maximum principle does not apply. We view this work as a first step toward studying positivity of the curvature operator along the singular Ricci flow.**



## 1. Introduction and statement of main results

Consider a compact smooth Riemannian manifold $(M, g)$ without boundary. Its Ricci flow is a smooth family of metrics $(g(t))_{t\geq 0}$ such that

$$\partial_t g(t) = -2\,\mathrm{Ric}(g(t)), \quad g(0) = g, \tag{1-1}$$

where $\mathrm{Ric}(g(t))$ denotes the Ricci curvature tensor of $g(t)$. Due to diffeomorphism invariance of the Ricci tensor, this evolution equation fails to be strongly parabolic. One overcomes this problem by adding an additional term to the equation which

Partial support by DFG Priority Programme "Geometry at Infinity".
*MSC2020:* primary 53E20; secondary 53C25, 58J35.
*Keywords:* Ricci flow, positive scalar curvature, conical singularities.

breaks the diffeomorphism invariance. This leads to the equivalent and analytically more convenient Ricci de Turck flow

$$\partial_t g(t) = -2\,\mathrm{Ric}(g(t)) + \mathcal{L}_{W(t)} g(t), \quad g(0) = g, \tag{1-2}$$

where $W(t)$ is the de Turck vector field defined in terms of the Christoffel symbols for the metrics $g(t)$ and a background metric $h$

$$W(t)^k = g(t)^{ij}(\Gamma^k_{ij}(g(t)) - \Gamma^k_{ij}(h)). \tag{1-3}$$

In the applications, the background metric $h$ is usually taken as the initial metric $g$, or as its small Ricci flat perturbation. The de Turck vector field defines a one-parameter family of diffeomorphisms $(\phi(t))_{t\geq 0}$ and if $(g(t))_{t\geq 0}$ is a solution to the Ricci de Turck flow (1-2), then the pullback $(\phi(t)^* g(t))_{t\geq 0}$ is a solution to the Ricci flow (1-1).

On singular spaces, the de Turck vector field may point towards the singular stratum and thus lengths of the corresponding integral curves may not be bounded from below away from zero. Hence the one-parameter family of diffeomorphisms $(\phi(t))_{t\geq 0}$ may not exist for positive times. Therefore, in the singular setting (1-1) and (1-2) are generally not equivalent and we study the latter flow.

Our work establishes an interesting property of the flow, namely that a given initial regularity of the Ricci curvature is preserved along the flow.

**Theorem 1.1.** *The singular Ricci de Turck flow preserves the initial regularity of the Ricci curvature. In particular, if the initial metric has bounded Ricci curvature, the flow remains of bounded Ricci curvature as well.*

We are not able to deduce an analogous result for the scalar curvature. The reason is that the norm of the Ricci tensor appears as the reaction term in the evolution equation of the scalar curvature. Thus, unbounded Ricci curvature at the singularity pushes the scalar curvature to infinity after infinitesimal time. In contrast, the evolution equation on the Ricci curvature is tensorial which allows more flexibility. However, we are able to prove a different property of the scalar curvature along the Ricci flow which is well known in the smooth compact case.

**Theorem 1.2.** *An admissible Riemannian manifold with isolated conical singularities and positive scalar curvature admits a singular Ricci de Turck flow preserving the singular structure and under the additional assumption of strong tangential stability, positivity of scalar curvature along the flow.*

The present work is a continuation of a research program on singular Ricci flow, that preserves the initial singular structure, that has seen several recent advances. In the setting of surfaces with conical singularities, singular Ricci flow has been studied by Mazzeo, Rubinstein and Sesum [16] and Yin [23]. The Yamabe flow, which coincides with the Ricci flow in the two-dimensional setting, has been studied in general dimension on spaces with edge singularities by Bahuaud and Vertman in [1] and [2]. In the setting of Kähler manifolds, Kähler–Ricci flow on spaces with edge singularities appears in the recent results on the Calabi–Yau conjecture on Fano manifolds; see Donaldson [6], Tian [20], see also Jeffres, Mazzeo and Rubinstein [9]. Kähler–Ricci flow in case of isolated conical singularities has been addressed by Chen and Wang [4], Wang [22], as well as Liu and Zhang [15].

Ricci flow on singular spaces of general dimension, without the Kähler condition, does not reduce to a scalar equation and has been studied in [21] by Vertman. Subsequently, Kröncke and Verman [13; 14] established stability and studied solitons and Perelman entropies of this flow in the special case of isolated conical singularities. The present work is a continuation of this research program.

Let us point out that Ricci flow preserving the initial singular structure, is not the only possible way to evolve a singular metric. In fact, Giesen and Topping [7; 8] construct a solution to the Ricci flow on surfaces with singularities, which becomes instantaneously complete. Alternatively, Simon [19] constructs Ricci flow in dimension two and three that smoothens out the singularity.

## 2. Geometric preliminaries on conical manifolds

We begin with a definition of spaces with isolated conical singularities. We point out that part of our analysis in fact applies to spaces with nonisolated conical singularities, the so-called edges.

**Definition 2.1.** Let $M$ be the open interior of a compact smooth manifold $\overline{M}$ with boundary $F := \partial M$. Let $x$ be a boundary defining function and $\overline{\mathcal{C}(F)}$ a tubular neighborhood of the boundary with open interior $\mathcal{C}(F) := (0,1)_x \times F$. An incomplete Riemannian metric $g$ on $M$ with an isolated conical singularity is a smooth metric on $M$ satisfying

$$g \restriction \mathcal{C}(F) = dx^2 + x^2 g_F + h =: \bar{g} + h,$$

where the higher order term $h$ has the following asymptotics at $x = 0$. Let $\bar{g} = dx^2 + x^2 g_F$ denote the exact conical part of the metric $g$ over $\mathcal{C}(F)$ and $\nabla_{\bar{g}}$ the corresponding Levi Civita connection. Then we require that for some $\gamma > 0$ and all integers $k \in \mathbb{N}_0$ the pointwise norm

$$|x^k \nabla_{\bar{g}}^k h|_{\bar{g}} = O(x^\gamma), \quad x \to 0. \tag{2-1}$$

**Remark 2.2.** We emphasize here that we do not assume that the higher order term $h$ is smooth up to $x = 0$ and do not restrict the order $\gamma > 0$ to be an integer. In that sense the notion of conical singularities in the present discussion is more general than the classical notion of conical singularities where $h$ is usually assumed to be smooth up to $x = 0$ with $\gamma = 1$. This minor generalization is necessary, since the

Ricci de Turck flow, which will be introduced below, preserves a conical singularity only up to a higher order term $h$ as above.

We call $(M, g)$ a manifold with an isolated conical singularity, or a *conical manifold* for short. The definition extends directly to conical manifolds with finitely many isolated conical singularities. Since the analytic arguments are local in nature, we may assume without loss of generality that $M$ has a single conical singularity only.

Let $(z) = (z_1, \dots, z_n)$ be local coordinates on $F$, where $n = \dim F$. Then $(x, z)$ are local coordinates on the conical neighborhood $\mathcal{C}(F) \subset M$. A b-vector field is by definition a smooth vector field on $\overline{M}$ which is tangent to the boundary $\partial M = F$. The b-vector fields form a Lie algebra, denoted by $\mathcal{V}_b$. In the local coordinates $(x, z)$, the algebra $\mathcal{V}_b$ is locally, near $\partial M$, generated by

$$\left\{ x\frac{\partial}{\partial x}, \partial_z = \left( \frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n} \right) \right\},$$

with smooth coefficients on $\overline{M}$. The b-tangent bundle ${}^bTM$ is defined by requiring that the b-vector fields form a spanning set of sections for it, i.e., $\mathcal{V}_b = C^\infty(\overline{M}, {}^bTM)$. The b-cotangent bundle ${}^bT^*M$ is the dual bundle and locally, near $\partial M$, generated by the one-forms

$$\left\{ \frac{dx}{x}, dz_1, \dots, dz_n \right\}. \tag{2-2}$$

Note that the differential form $\frac{dx}{x}$ is singular in the usual sense, but smooth as a section of ${}^bT^*M$. Extend the radial function $x$ of the cone $\mathcal{C}(F)$ to a nowhere vanishing smooth function $x : \overline{M} \to [0, 2]$. Then we define the incomplete b-tangent space ${}^{ib}TM$ by asking that $C^\infty(\overline{M}, {}^{ib}TM) := x^{-1} C^\infty(\overline{M}, {}^bTM)$. The dual bundle, the incomplete b-cotangent bundle ${}^{ib}T^*M$, is related to its complete counterpart ${}^bT^*M$ by $C^\infty(\overline{M}, {}^{ib}T^*M) = x C^\infty(\overline{M}, {}^bT^*M)$, and is locally generated by

$$\{dx, x dz_1, \dots, x dz_n\}. \tag{2-3}$$

## 3. Weighted Hölder spaces on conical manifolds

In this section we review definitions from [21, Section 1.3] in the case of isolated conical singularities. Let $(M, g)$ be a manifold with isolated conical singularities.

**Definition 3.1.** Let $d_M(p, p')$ denote the distance between two points $p, p' \in M$ with respect to the conical metric $g$. In terms of the local coordinates $(x, z)$ over the singular neighborhood $\mathcal{C}(F)$, the distance can be estimated up to a constant uniformly from above and below by the distance function of the model cone

$$d((x, z), (x', z')) = (|x - x'|^2 + (x + x')^2 |z - z'|^2)^{1/2}.$$

The Hölder space $\mathcal{C}^{\alpha}_{\mathrm{ie}}(M \times [0,T])$, $\alpha \in [0,1)$, is by definition the set of functions $u(p,t)$ that are continuous on $\overline{M} \times [0,T]$ with finite $\alpha$-th Hölder norm

$$\|u\|_{\alpha} := \|u\|_{\infty} + \sup\left( \frac{|u(p,t) - u(p',t')|}{d_M(p,p')^{\alpha} + |t-t'|^{\alpha/2}} \right) < \infty, \tag{3-1}$$

with supremum taken over all $p, p' \in M$ and $t, t' \in [0,T]$ with $p \neq p'$ and $t \neq t'$.[1]

We now extend the notion of Hölder spaces to sections of the vector bundle $S := \mathrm{Sym}^2({}^{ib}T^*M)$ of symmetric 2-tensors. Note that the Riemannian metric $g$ induces a fiberwise inner product on $S$, which we also denote by $g$.

**Definition 3.2.** The Hölder space $\mathcal{C}^{\alpha}_{\mathrm{ie}}(M \times [0,T], S)$ is by definition the set of all sections $\omega$ of $S$ which are continuous on $\overline{M} \times [0,T]$, such that for any local orthonormal frame $\{s_j\}$ of $S$, the scalar functions $g(\omega, s_j)$ are in $\mathcal{C}^{\alpha}_{\mathrm{ie}}(M \times [0,T])$.

The $\alpha$-th Hölder norm of $\omega$ is defined using a partition of unity $\{\phi_j\}_{j \in J}$ subordinate to a cover of local trivializations of $S$, with a local orthonormal frame $\{s_{jk}\}$ over $\mathrm{supp}(\phi_j)$ for each $j \in J$. We put

$$\|\omega\|^{(\phi,s)}_{\alpha} := \sum_{j \in J} \sum_{k} \|g(\phi_j \, \omega, s_{jk})\|_{\alpha}. \tag{3-2}$$

Different choices of $(\{\phi_j\}, \{s_{jk}\})$ lead to equivalent norms so that we may drop the upper index $(\phi, s)$ from notation. Next we come to weighted and higher order Hölder spaces of $S$-sections.

**Definition 3.3.** (1) The (hybrid) weighted Hölder space for $\gamma \in \mathbb{R}$ is

$$\mathcal{C}^{\alpha}_{\mathrm{ie}}(M \times [0,T], S)_{\gamma} := x^{\gamma}\, \mathcal{C}^{\alpha}_{\mathrm{ie}}(M \times [0,T], S) \cap x^{\gamma+\alpha} \mathcal{C}^{0}_{\mathrm{ie}}(M \times [0,T], S)$$

with Hölder norm $\|\omega\|_{\alpha,\gamma} := \|x^{-\gamma}\, \omega\|_{\alpha} + \|x^{-\gamma-\alpha}\, \omega\|_{\infty}$.

(2) The weighted higher order Hölder spaces are defined by

$$\mathcal{C}^{\mathrm{k},\alpha}_{\mathrm{ie}}(M \times [0,T], S)_{\gamma} := \{\omega \in \mathcal{C}^{\alpha}_{\mathrm{ie},\gamma} \mid \{\mathcal{V}_b^j \circ (x^2 \partial_t)^l\}\, \omega \in \mathcal{C}^{\alpha}_{\mathrm{ie},\gamma} \text{ for all } j + 2l \leq k\},$$

for any $\gamma \in \mathbb{R}$ and $k \in \mathbb{N}$. For any $\gamma > -\alpha$ and $k \in \mathbb{N}$ we also define

$$\mathcal{C}^{\mathrm{k},\alpha}_{\mathrm{ie}}(M \times [0,T], S)^{b}_{\gamma} := \{\omega \in \mathcal{C}^{\alpha}_{\mathrm{ie}} \mid \{\mathcal{V}_b^j \circ (x^2 \partial_t)^l\}\, \omega \in \mathcal{C}^{\alpha}_{\mathrm{ie},\gamma} \text{ for all } j + 2l \leq k\}.$$

The corresponding Hölder norms are defined using a finite cover of coordinate charts trivializing $S_0$ and a subordinate partition of unity $\{\phi_j\}_{j \in J}$. By a slight abuse of notation, we identify $\mathcal{V}_b$ with its finite family of generators over each coordinate chart. Writing $\mathcal{D} := \{\mathcal{V}_b^j \circ (x^2 \partial_t)^l \mid j + 2l \leq k\}$ the Hölder norm on $\mathcal{C}^{\mathrm{k},\alpha}_{\mathrm{ie}}(M \times [0,T], S)_{\gamma}$ is then given by

$$\|\omega\|_{k+\alpha,\gamma} = \sum_{j \in J} \sum_{X \in \mathcal{D}} \|X(\phi_j\, \omega)\|_{\alpha,\gamma} + \|\omega\|_{\alpha,\gamma}. \tag{3-3}$$

[1]We can assume without loss of generality that the tuples $(p, p')$ are always taken from within the same coordinate patch of a given atlas.

For the Hölder norm on $\mathcal{C}^{\mathrm{k},\alpha}_{\mathrm{ie}}(M \times [0,T], S)^b_\gamma$ replace in (3-3) $\|\omega\|_{\alpha,\gamma}$ by $\|\omega\|_\alpha$.

The Hölder norms for different choices of coordinate charts, the subordinate partition of unity or vector fields $\mathcal{V}_b$ are equivalent. Analogously we also consider time-independent Hölder spaces, which are denoted in the same way with $[0,T]$ deleted from notation above.

The vector bundle $S$ decomposes into a direct sum of subbundles $S = S_0 \oplus S_1$, where the subbundle $S_0 = \mathrm{Sym}^2_0({}^{ib}T^*M)$ is the space of trace-free (with respect to the fixed metric $g$) symmetric 2-tensors, and $S_1$ is the space of pure trace (with respect to the fixed metric $g$) symmetric 2-tensors. Definition 3.3 extends verbatim to subbundles $S_0$ and $S_1$.

**Remark 3.4.** The spaces presented here are slightly different from the spaces originally introduced in [21]. There, in case of $S_1$-sections, higher order weighted Hölder spaces were defined in terms of $x^\gamma\, \mathcal{C}^\alpha_{\mathrm{ie}}$ instead of $\mathcal{C}^\alpha_{\mathrm{ie},\gamma}$. Here, we present a more unified definition, which will become much more convenient below. The arguments of [21] still carry over to yield regularity statements in these unified Hölder spaces.

**Definition 3.5.** Let $(M, g)$ be a compact conical manifold, $\gamma_0, \gamma_1 \in \mathbb{R}$. We define the following spaces:

(1) If $(M, g)$ is not an orbifold, we set
$$\mathcal{H}^{k,\alpha}_{\gamma_0,\gamma_1}(M \times [0,T], S) := \mathcal{C}^{\mathrm{k},\alpha}_{\mathrm{ie}}(M \times [0,T], S_0)_{\gamma_0} \oplus \mathcal{C}^{\mathrm{k},\alpha}_{\mathrm{ie}}(M \times [0,T], S_1)^b_{\gamma_1}.$$
(2) If $(M, g)$ is an orbifold, we set
$$\mathcal{H}^{k,\alpha}_{\gamma_0,\gamma_1}(M \times [0,T], S) := \mathcal{C}^{\mathrm{k},\alpha}_{\mathrm{ie}}(M \times [0,T], S_0)^b_{\gamma_0} \oplus \mathcal{C}^{\mathrm{k},\alpha}_{\mathrm{ie}}(M \times [0,T], S_1)^b_{\gamma_1}.$$

Note that e.g., for the nonorbifold case, the different choice of spaces for the $S_0$ and $S_1$ components, simply ensures that the $S_1$ component is not (!) included in $x^\gamma \mathcal{C}^0_{\mathrm{ie}}$ for some positive weight $\gamma$. In case of orbifold singularities, this restriction is imposed on both the $S_0$ and $S_1$ components. We can now impose regularity assumptions on our initial data $(M, g)$. Definitions 3.2 and 3.3 extend naturally to associated bundles of ${}^{ib}TM$. Also we write $\mathcal{H}^{k,\alpha}_\gamma \equiv \mathcal{H}^{k,\alpha}_{\gamma,\gamma}$.

**Definition 3.6.** Let $\alpha \in [0, 1)$, $k \in \mathbb{N}_0$ and $\gamma > 0$. A conical manifold $(M, g)$ is $(\alpha, \gamma, k)$ Hölder regular if the following two conditions are satisfied:

(i) The $(0, 4)$ curvature tensor $\mathrm{Rm} \in \mathcal{C}^{\mathrm{k},\alpha}_{\mathrm{ie}}(M \times [0,T], \otimes^4\, {}^{ib}T^*M)_{-2}$.

(ii) The Ricci curvature tensor $\mathrm{Ric} \in \mathcal{H}^{k,\alpha}_{-2+\gamma}(M \times [0,T], S)$.

We continue under that assumption from now on.

**Remark 3.7.** The asymptotics of the Ricci curvature tensor, as a section of $S$, is generically $O(x^{-2})$ as $x \to 0$. Hence $(\alpha, \gamma, k)$ Hölder regularity with $\gamma > 0$ in particular implies that the exact conical part $(\mathcal{C}(F), \bar{g})$ must be Ricci-flat. This is equivalent to $(F^n, g_F)$ being Einstein with Einstein constant $n-1$. Moreover, the weight $\gamma$ corresponds to the weight in (2-1).

## 4. Lichnerowicz Laplacian and tangential stability

Consider the right hand side $-2\,\mathrm{Ric}(g(t)) + \mathcal{L}_{W(t)}g(t)$ of the Ricci de Turck flow equation (1-2). Write $W(t) = W(g(t), h)$ to indicate precisely what the de Turck vector field depends on. Choose $h = g$ as the background metric. Then, replacing $g(t) = g + s\,\omega$ for some symmetric 2-tensor $\omega$, the linearization of the right-hand side of (1-2) is given by

$$\frac{d}{ds}(-2\,\mathrm{Ric}(g+s\,\omega) + \mathcal{L}_{W(g+s\,\omega,g)}(g+s\,\omega))|_{s=0} = -\Delta_L\,\omega, \tag{4-1}$$

where $\Delta_L$ is an elliptic operator, which is known as the Lichnerowicz Laplacian of $g$, acting on symmetric 2-tensors by

$$\Delta_L\,\omega_{ij} = \Delta\,\omega_{ij} - 2g^{pq}\,\mathrm{Rm}^r_{qij}\,\omega_{rp} + g^{pq}R_{ip}\,\omega_{qj} + g^{pq}R_{jp}\,\omega_{iq},$$

where $\Delta$ denotes the rough Laplacian, and $\mathrm{Rm}^r_{qij}$ and $R_{ij}$ denote the components of the $(1,3)$-Riemann curvature tensor and the Ricci tensor, respectively. Near the conical singularity $\Delta_L$ can be written as follows. We choose local coordinates $(x, z)$ over the singular neighborhood $\mathcal{C}(F) = (0,1)_x \times F$. Consider a decomposition of compactly supported smooth sections $C_0^\infty(\mathcal{C}(F), S \restriction \mathcal{C}(F))$

$$\begin{aligned} C_0^\infty(\mathcal{C}(F), S \restriction \mathcal{C}(F)) &\to C_0^\infty((0,1), C^\infty(F) \times \Omega^1(F) \times \mathrm{Sym}^2(T^*F)),\\ \omega &\mapsto (\omega(\partial_x,\partial_x), \omega(\partial_x,\cdot\,), \omega(\,\cdot\,,\cdot\,)), \end{aligned} \tag{4-2}$$

where $\Omega^1(F)$ denotes differential 1-forms on $F$. Under such a decomposition, the Lichnerowicz Laplace operator $\Delta_L$ associated to the singular Riemannian metric $g$ attains the following form over $\mathcal{C}(F)$

$$\Delta_L = -\frac{\partial^2}{\partial x^2} - \frac{n}{x}\frac{\partial}{\partial x} + \frac{\square_L}{x^2} + \mathcal{O}, \tag{4-3}$$

where $\square_L$ is a differential operator on $C^\infty(F) \times \Omega^1(F) \times \mathrm{Sym}^2(T^*F)$, depending only on the exact conical part $\bar{g}$, and the higher order term is $\mathcal{O} \in x^{-2+\gamma}\,\mathcal{V}_b^2$ with Hölder regular coefficients.

***Tangential stability.*** We can now impose a central analytic condition, under which existence of singular Ricci de Turck flow has been established in [21].

**Definition 4.1.** $(F, g_F)$ is called (strictly) tangentially stable if the tangential operator $\square_L$ of the Lichnerowicz Laplacian on its cone restricted to tracefree tensors is nonnegative (resp. strictly positive).

Tangential stability (in fact a smaller lower bound $\square_L \geq -((n-1)/2)^2$ would suffice) has a straight forward implication: for $\omega \in C_0^\infty(\mathcal{C}(F), S)$ we find

$$\Delta_L\,\omega = \left(\frac{\partial}{\partial x} + \frac{1}{x}\left(\sqrt{\square_L + \left(\frac{n-1}{2}\right)^2} + \frac{n-1}{2}\right)\right)^t \circ \left(\frac{\partial}{\partial x} + \frac{1}{x}\left(\sqrt{\square_L + \left(\frac{n-1}{2}\right)^2} + \frac{n-1}{2}\right)\right)\omega\,. \tag{4-4}$$

Thus, tangential stability in particular implies that $\Delta_L$, acting on compactly supported smooth sections, has a lower bound, i.e., there exists some $C \in \mathbb{R}$ such that

$$\Delta_L \restriction C_0^\infty(M, S) \geq C. \tag{4-5}$$

For convenience of the reader we shall add here a complete characterization of tangential stability, obtained by Kröncke and Vertman [13]. Recall that by Remark 3.7, the assumption of $(\alpha, \gamma, k)$ Hölder regularity implies that $(F, g_F)$ is Einstein with Einstein constant $(n-1)$.

**Theorem 4.2.** *Let $(F, g_F)$, $n \geq 3$ be a compact Einstein manifold with constant $(n-1)$. We write $\Delta_E$ for its Einstein operator, and denote the Laplace Beltrami operator by $\Delta$. Then $(F, g_F)$ is tangentially stable if and only if* $\operatorname{Spec}(\Delta_E|_{TT}) \geq 0$ *and* $\operatorname{Spec}(\Delta) \setminus \{0\} \cap (n, 2(n+1)) = \varnothing$. *Similarly, $(M, g)$ is strictly tangentially stable if and only if* $\operatorname{Spec}(\Delta_E|_{TT}) > 0$ *and* $\operatorname{Spec}(\Delta) \setminus \{0\} \cap [n, 2(n+1)] = \varnothing$.

There are plenty of examples, where (strict) tangential stability is satisfied. Any spherical space form is tangentially stable. [13] also provides a detailed list of strict tangentially stable Einstein manifolds that are symmetric spaces. Note that $\mathbb{S}^n$ is tangentially stable but not strictly tangentially stable.

**Theorem 4.3.** *Let $(F^n, g_F)$, $n \geq 2$ be a closed Einstein manifold with constant $(n-1)$, which is a symmetric space of compact type. If it is a simple Lie group $G$, it is strictly tangentially stable if $G$ is one of the following spaces*:

$$\mathrm{Spin}(p)(p \geq 6, p \neq 7), \quad \mathrm{E}_6, \quad \mathrm{E}_7, \quad \mathrm{E}_8, \quad \mathrm{F}_4. \tag{4-6}$$

*If the cross section is a rank-1 symmetric space of compact type $G/K$, $(M, g)$ is strictly tangentially stable if $G$ is one of the following real Grassmannians*

$$\begin{aligned} &\frac{\mathrm{SO}(2q+2p+1)}{\mathrm{SO}(2q+1)\times\mathrm{SO}(2p)}\ (p \geq 2, q \geq 1), && \frac{\mathrm{SO}(8)}{\mathrm{SO}(5)\times\mathrm{SO}(3)},\\ &\frac{\mathrm{SO}(2p)}{\mathrm{SO}(p)\times\mathrm{SO}(p)}\quad (p \geq 4), && \frac{\mathrm{SO}(2p+2)}{\mathrm{SO}(p+2)\times\mathrm{SO}(p)}\ (p \geq 4),\\ &\frac{\mathrm{SO}(2p)}{\mathrm{SO}(2p-q)\times\mathrm{SO}(q)}\ (p-2 \geq q \geq 3), \end{aligned} \tag{4-7}$$

*or one of the following spaces*:

$$\begin{array}{lll} \mathrm{SU}(2p)/\mathrm{SO}(p)\ (n \geq 6), & \mathrm{E}_6/[\mathrm{Sp}(4)/\{\pm I\}], & \mathrm{E}_6/\mathrm{SU}(2)\cdot\mathrm{SU}(6), \\ \mathrm{E}_7/[\mathrm{SU}(8)/\{\pm I\}], & \mathrm{E}_7/\mathrm{SO}(12)\cdot\mathrm{SU}(2), & \mathrm{E}_8/\mathrm{SO}(16), \\ \mathrm{E}_8/\mathrm{E}_7\cdot\mathrm{SU}(2), & \mathrm{F}_4/Sp(3)\cdot\mathrm{SU}(2). & \end{array} \tag{4-8}$$

***Self-adjoint extensions of the Lichnerowicz Laplacian.*** We can now study the self-adjoint closed extensions of $\Delta_L$ acting on compactly supported sections $C_0^\infty(M, S)$. We write $L^2(M, S)$ for the completion of $C_0^\infty(M, S)$ with respect to the $L^2$-norm $(\cdot\,,\cdot)_{L^2}$ defined by $g$. The maximal closed extension of $\Delta_L$ in $L^2(M, S)$ is defined by the following domain

$$\mathcal{D}(\Delta_{L,\max}) := \{\omega \in L^2(M, S) \mid \Delta_L\,\omega \in L^2(M, S)\}, \tag{4-9}$$

where $\Delta_L\,\omega$ is defined distributionally. The minimal closed extension of $\Delta_L$ in $L^2(M, S)$ is obtained as the domain of the graph closure of $\Delta_L$ acting on $C_0^\infty(M, S)$

$$\mathcal{D}(\Delta_{L,\min}) := \big\{\omega \in \mathcal{D}(\Delta_{L,\max}) \mid \exists (\omega_n)_{n\in\mathbb{N}} \subset C_0^\infty(M, S) : \\ \omega_n \xrightarrow{n\to\infty} \omega,\ \Delta_L\,\omega_n \xrightarrow{n\to\infty} \Delta_L\,\omega \text{ in } L^2(M, S)\big\}.$$

Let $(\lambda, \omega_\lambda)$ be the set of eigenvalues and corresponding eigentensors of the tangential operator $\square_L$. Assuming tangential stability, we have $\lambda \geq 0$, and hence we define

$$\nu(\lambda) := \sqrt{\lambda + \left(\frac{n-1}{2}\right)^2}. \tag{4-10}$$

Standard arguments, see, e.g., [10], show that for each $\omega \in \mathcal{D}(\Delta_{L,\max})$ there exist constants $c_\lambda^\pm$, $\nu(\lambda) \in [0, 1)$, depending only on $\omega$, such that $\omega$ admits a partial asymptotic expansion as $x \to 0$

$$\omega = \sum_{\nu(\lambda)=0} (c_\lambda^+(\omega)x^{-(n-1)/2} + c_\lambda^-(\omega)x^{-(n-1)/2}\log(x)) \cdot \omega_\lambda \\ + \sum_{\nu(\lambda)\in(0,1)} (c_\lambda^+(\omega)x^{\nu(\lambda)-(n-1)/2} + c_\lambda^-(\omega)x^{-\nu(\lambda)-(n-1)/2}) \cdot \omega_\lambda + \widetilde{\omega}, \tag{4-11}$$

for $\widetilde{\omega} \in \mathcal{D}(\Delta_{L,\min})$. All self-adjoint extensions for $\Delta_L$ can be classified by algebraic conditions on the coefficients in the asymptotic expansion (4-11), see, e.g., Kirsten, Loya and Park [10, Proposition 3.3]. The Friedrichs self-adjoint extension of $\Delta_L$ on $C_0^\infty(M, S) \subset L^2(M, S)$ is given by the domain

$$\mathcal{D}(\Delta_L) := \{\omega \in \mathcal{D}(\Delta_{L,\max}) \mid c_\lambda^-(\omega) = 0 \text{ for } \nu(\lambda) \in [0, 1)\}. \tag{4-12}$$

Note that if $n \geq 3$, then tangential stability implies that all $\nu(\lambda) \geq 1$. Hence the minimal and maximal domains coincide and we find

$$\mathcal{D}(\Delta_L) = \mathcal{D}(\Delta_{L,\max}) = \mathcal{D}(\Delta_{L,\min}), \quad \text{for } n \geq 3. \tag{4-13}$$

We close the section with an observation by Friedrichs and Stone, see Riesz and Nagy [18, Theorem on page 330], compare the corresponding statement in our previous work [13, Proposition 2.2].

**Proposition 4.4.** *Assume that $(M, g)$ is tangentially stable, so that by (4-5) the Lichnerowicz Laplacian $\Delta_L$ with domain $C_0^\infty(M, S)$ is bounded from below by a constant $C \in \mathbb{R}$.*[2] *Then the Friedrichs self-adjoint extension of the Lichnerowicz Laplacian $\Delta_L$ is bounded from below by $C$ as well.*

## 5. Existence and regularity of singular Ricci de Turck flow

The main result of [21, Theorem 4.1], see also [13, Theorem 1.2], is existence and regularity of singular Ricci de Turck flow. Despite a slight difference of Hölder spaces used here and in [21], we still conclude from [21, Theorem 4.1] that the heat operator of the Friedrichs extension $\Delta_L$ maps

$$e^{-t\Delta_L} : \mathcal{H}^{k,\alpha}_{-2+\gamma_0,-2+\gamma_1}(M \times [0,T], S) \to \mathcal{H}^{k+2,\alpha}_{\gamma_0,\gamma_1}(M \times [0,T], S).$$

Therefore existence and regularity obtained in [21, Theorem 4.1], as well as in [13, Theorem 1.2] for a different choice of a background metric, still hold in our spaces.

**Theorem 5.1.** *Let $(M, g)$ be a conical manifold, which is tangentially stable and $(\alpha, k+1, \gamma)$ Hölder regular. In case the conical singularity is not orbifold, we assume strict tangential stability. Let the background metric be either equal to $g$ or a conical Ricci flat metric, in which case $g_0$ is assumed to be a sufficiently small perturbation of $\widetilde{g}$ in $\mathcal{H}^{k+2,\alpha}_{\gamma,\gamma}(M, S)$. Then there exists some $T > 0$, such that the Ricci de Turck flow (1-2), starting at $g$ admits a solution $g(\cdot) \in \mathcal{H}^{k+2,\alpha}_{\gamma_0,\gamma_1}(M \times [0,T], S)$ for some $\gamma_0, \gamma_1 \in (0, \gamma)$ sufficiently small.*

Let us now explain in what sense the flow preserves the conical singularity. Given an admissible perturbation $g$ of the conical metric $g_0$, the pointwise trace of $g$ with respect to $g_0$, denoted as $\mathrm{tr}_{g_0} g$ is by definition of admissibility an element of the Hölder space $\mathcal{C}^{\mathrm{k},\alpha}_{\mathrm{ie}}(M, S_1)^b_\gamma$, restricting at $x = 0$ to a constant function $(\mathrm{tr}_{g_0} g)(0) = u_0 > 0$. Setting $\tilde{x} := \sqrt{u_0} \cdot x$, the admissible perturbation $g = g_0 + h$ attains the form

$$g = d\tilde{x}^2 + \tilde{x}^2 g_F + \tilde{h},$$

[2] The case of $C = 0$ is commonly referred to as linear stability in the literature.

where $|\tilde{h}|_g = O(x^\gamma)$ as $x \to 0$. Note that the leading part of the admissible perturbation $g$ near the conical singularity differs from the leading part of the admissible metric $g_0$ only by scaling.

We now wish to specify conditions on the weights $\gamma_0, \gamma_1$. Let us write $\square'_L$ for the tangential operator of the Lichnerowicz Laplacian $\Delta_L$ acting on trace-free tensors $S_0$. The tangential operator of $\Delta_L$ acting on the pure trace tensors $S_1$ is simply the Laplace Beltrami operator $\Delta_F$ of $(F, g_F)$. We set $u_0 := \min(\operatorname{Spec} \square'_L \backslash \{0\}) > 0$ and $u_1 := \min(\operatorname{Spec} \Delta_F \backslash \{0\}) > 0$. In the orbifold case we set $u := \min\{u_0, u_1\}$.[3] Define

$$\mu_0 := \sqrt{u_0 + \frac{n-1}{2}} - \frac{n-1}{2}, \quad \mu_1 := \sqrt{u_1 + \frac{n-1}{2}} - \frac{n-1}{2}.$$

Then the admissible choice of weights $(\gamma_0, \gamma_1)$ (in the orbifold case set $\gamma = \gamma_0 = \gamma_1$) and the Hölder exponent $\alpha$ is given by the following restrictions:

$$\begin{aligned} &\gamma_0 \in (0, \mu_0), \quad \gamma_0 \le 2\gamma_1, \quad \gamma_0 < \gamma, \\ &\gamma_1 \in (0, \mu_1), \quad \gamma_1 \le \gamma_0, \quad \gamma_1 < \gamma, \\ &\alpha \in (0, \mu_0 - \gamma_0) \cap (0, \mu_1 - \gamma_1). \end{aligned} \tag{5-1}$$

**Remark 5.2.** Note that in case of $u_0, u_1 > n$, i.e., $\square_L > n$, and assuming additionally $\gamma > 1$, we may choose $\gamma_0, \gamma_1 > 1$ satisfying (5-1). This stronger condition is studied in the Appendix, where a list of examples is provided.

We close the section by pointing out regularity of the de Turck vector field. As before we may define weighted higher order Hölder spaces of the incomplete $b$-cotangent bundle ${}^{ib}T^*M$. By Theorem 5.1, see [21, Section 6],

$$W(t) \in \mathcal{C}^{k+1,\alpha}_{\text{ie}}(M \times [0, T], {}^{ib}T^*M)_{-1+\bar{\gamma}}, \tag{5-2}$$

where $\bar{\gamma} := \min\{\gamma_0, \gamma_1\}$, and $\bar{\gamma} = \gamma$ in the orbifold case. As explained in the previous Remark 5.2, assuming $\square_L > n$ we can choose $\bar{\gamma} > 1$, so that existence time of the integral curves of $W(t)$ is positive, uniformly bounded away from zero. In this case we may pass from the Ricci de Turck flow back to Ricci flow, which is generally not clear in the singular setting.

## 6. Evolution of curvatures under Ricci de Turck flow

The results of this section are well-known in the setting of smooth compact manifolds where the Ricci flow is defined. Evolution equations are usually proven by studying evolution of curvatures under the Ricci flow, and then the corresponding equations follow for the Ricci de Turck flow by applying the corresponding diffeomorphisms. In the singular setting there might be no globally well-defined diffeomorphism to

[3] See [21, Remark 2.5].

go from the Ricci de Turck flow back to the Ricci flow, as the de Turck vector field may be pointing toward the singularity. Thus we need to establish the evolution of Ricci curvature under Ricci de Turck flow directly without passing back to the Ricci flow. The evolution of the scalar curvature is then a direct consequence.

*Notation.* Let $g(t), t \in [0, T]$ be a Ricci de Turck flow of Riemannian metrics on $M$. We denote by $\nabla$, $\Delta$, $\Delta_L$ and $\Gamma$ the covariant derivative, the Laplace Beltrami operator, the Lichnerowicz Laplacian and the Christoffel symbols, respectively, defined with respect to the metric $g(t)$. We let $|\cdot|$ be the norm with respect to $g(t)$. We denote the Riemann curvature tensor by Rm, the Ricci tensor by Ric and the scalar curvature by $R$, and we use the following conventions:

$$\begin{aligned}\mathrm{Rm}(X, Y)Z &:= \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]}Z,\\ \mathrm{Rm}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^k} &= R^l_{ijk}\frac{\partial}{\partial x^l}.\end{aligned}$$

Here and below we use the Einstein summation convention and sum over repeated upper and lower indices. We lower the upper index to the fourth slot $R_{ijkl} := g_{pl} R^p_{ijk}$. Then the Ricci tensor Ric and the scalar curvature $R$ are given by

$$\mathrm{Ric}_{jk} \equiv R_{jk} := R^i_{ijk} = g^{il} R_{ijkl}, \quad R := g^{jk} R_{jk}.$$

**Theorem 6.1.** *Let $M$ be a smooth manifold. Let $g(t)$, $t \in [0, T]$ be a solution of the Ricci de Turck flow with initial metric $g$ and reference metric $h$, i.e.*,

$$\begin{cases} \frac{\partial}{\partial t} g_{ij}(x, t) = (-2R_{ij} + \nabla_i W_j + \nabla_j W_i)(x, t), & (x, t) \in M \times [0, T], \\ g_{ij}(x, 0) = g_{ij}(x), & x \in M, \end{cases} \tag{6-1}$$

*where $W(t)^k = g(t)^{ij}(\Gamma^k_{ij}(g(t)) - \Gamma^k_{ij}(h))$ is the de Turck vector field. Then the Ricci tensor* Ric *evolves by*

$$(\partial_t + \Delta_L)R_{jk} = \nabla_m R_{jk} W^m + R_{jm}\nabla_k W^m + R_{km}\nabla_j W^m. \tag{6-2}$$

*Proof.* By [5, Lemma 3.5] the evolution of the Ricci curvature is given by

$$\partial_t R_{jk} = \tfrac{1}{2} g^{pq}(\nabla_q\nabla_j a_{kp} + \nabla_q\nabla_k a_{jp} - \nabla_p\nabla_q a_{jk} - \nabla_j\nabla_k a_{pq}), \tag{6-3}$$

where the time-dependent $(0, 2)$-tensor $a$ is the time-derivative of the metric, $a_{ij} := \frac{\partial}{\partial t} g_{ij}$. Using the Ricci de Turck flow equation $\frac{\partial}{\partial t} g_{ij} = -2R_{ij} + \nabla_i W_j + \nabla_j W_i$ we obtain

$$\begin{aligned}\partial_t R_{jk} = &-g^{pq}(\nabla_q\nabla_j R_{kp} + \nabla_q\nabla_k R_{jp} - \nabla_p\nabla_q R_{jk} - \nabla_j\nabla_k R_{pq})\\ &+\tfrac{1}{2} g^{pq}\big(\nabla_q\nabla_j\nabla_k W_p + \nabla_q\nabla_j\nabla_p W_k + \nabla_q\nabla_k\nabla_j W_p + \nabla_q\nabla_k\nabla_p W_j\\ &\quad - \nabla_p\nabla_q\nabla_j W_k - \nabla_p\nabla_q\nabla_k W_j - \nabla_j\nabla_k\nabla_p W_q - \nabla_j\nabla_k\nabla_q W_p\big).\end{aligned} \tag{6-4}$$

The first line on the right-hand side is the time-derivative of the Ricci tensor when the metric evolves by Ricci flow. Since it is known [5, Lemma 6.9] that under Ricci flow the time-derivative of the Ricci tensor equals the Lichnerowicz Laplacian of the Ricci tensor, i.e., $\partial_t R_{jk} = -\Delta_L R_{jk}$, it follows that

$$\Delta_L R_{jk} = g^{pq}(\nabla_q \nabla_j R_{kp} + \nabla_q \nabla_k R_{jp} - \nabla_p \nabla_q R_{jk} - \nabla_j \nabla_k R_{pq}). \tag{6-5}$$

To simplify the second and third line on the right-hand side, we compute several terms by commuting covariant derivatives. We have

$$\begin{aligned} \nabla_p \nabla_j \nabla_p W_k &= \nabla_p(\nabla_q \nabla_j W_k - R^m_{jqk} W_m) \\ &= \nabla_p \nabla_q \nabla_j W_k - \nabla_p R^m_{jqk} W_m - R^m_{jqk} \nabla_p W_m. \end{aligned} \tag{6-6}$$

Hence by exchanging $j$ and $k$

$$\nabla_p \nabla_k \nabla_p W_j = \nabla_p \nabla_q \nabla_k W_j - \nabla_p R^m_{kqj} W_m - R^m_{kqj} \nabla_p W_m. \tag{6-7}$$

Furthermore,

$$\begin{aligned} &\nabla_p \nabla_j \nabla_k W_q \\ &= \nabla_j \nabla_p \nabla_k W_q - R^m_{pjk} \nabla_m W_q - R^m_{pjq} \nabla_k W_m \\ &= \nabla_j(\nabla_k \nabla_p W_q - R^m_{pkq} W_m) - R^m_{pjk} \nabla_m W_q - R^m_{pjq} \nabla_k W_m \\ &= \nabla_j \nabla_k \nabla_p W_q - \nabla_j R^m_{pkq} W_m - R^m_{pkq} \nabla_j W_m - R^m_{pjk} \nabla_m W_q - R^m_{pjq} \nabla_k W_m. \end{aligned} \tag{6-8}$$

By exchanging $j$ and $k$ and commuting covariant derivatives we have

$$\begin{aligned} \nabla_p \nabla_k \nabla_j W_q &= \nabla_k \nabla_j \nabla_p W_q - \nabla_k R^m_{pjq} W_m - R^m_{pjq} \nabla_k W_m \\ &\qquad - R^m_{pkj} \nabla_m W_q - R^m_{pkq} \nabla_j W_m \\ &= \nabla_j \nabla_k \nabla_p W_q - R^m_{kjp} \nabla_m W_q - R^m_{kjq} \nabla_p W_m - \nabla_k R^m_{pjq} W_m \\ &\qquad - R^m_{pjq} \nabla_k W_m - R^m_{pkj} \nabla_m W_q - R^m_{pkq} \nabla_j W_m. \end{aligned} \tag{6-9}$$

Now plugging (6-5), (6-6), (6-7), (6-8), (6-9) into (6-4) we obtain

$$\begin{aligned} \partial_t R_{jk} = -\Delta_L R_{jk} - \tfrac{1}{2} g^{pq}\big(&\nabla_p R^m_{jqk} W_m + R^m_{jqk} \nabla_p W_m + \nabla_p R^m_{kqj} W_m \\ &+ R^m_{kqj} \nabla_p W_m + \nabla_j R^m_{pkq} W_m + R^m_{pkq} \nabla_j W_m \\ &+ R^m_{pjk} \nabla_m W_q + R^m_{pjq} \nabla_k W_m + R^m_{kjp} \nabla_m W_q \\ &+ R^m_{kjq} \nabla_p W_m + \nabla_k R^m_{pjq} W_m + R^m_{pjq} \nabla_k W_m \\ &+ R^m_{pkj} \nabla_m W_q + R^m_{pkq} \nabla_j W_m\big). \end{aligned} \tag{6-10}$$

Simplifying and rearranging terms yields

$$
\begin{aligned}
\text{(6-11)}\quad \partial_t R_{jk} = -\Delta_L R_{jk} &- \tfrac{1}{2} g^{pq} \nabla_q W_m (R^m_{jpk} + R^m_{kpj} + R^m_{kjp}) \\
&- \tfrac{1}{2} g^{pq} \nabla_m W_q (R^m_{pjk} + R^m_{kjp} + R^m_{pkj}) \\
&- \tfrac{1}{2} g^{pq} W_m (\nabla_p R^m_{jqk} + \nabla_p R^m_{kqj} + \nabla_j R^m_{pkq} + \nabla_k R^m_{pjq}) \\
&- g^{pq} (R^m_{pkq} \nabla_j W_m + R^m_{pjq} \nabla_k W_m).
\end{aligned}
$$

The second line on the right-hand side is

$$
\begin{aligned}
\text{(6-12)}\quad & -\tfrac{1}{2} g^{pq} \nabla_q W_m (R^m_{jpk} + R^m_{kpj} + R^m_{kjp}) - \tfrac{1}{2} g^{pq} \nabla_m W_q (R^m_{pjk} + R^m_{kjp} + R^m_{pkj}) \\
&= -\tfrac{1}{2} \nabla^p W^m (R_{jpkm} + R_{kpjm} + R_{kjpm}) - \tfrac{1}{2} \nabla^m W^p (R_{pjkm} + R_{kjpm} + R_{pkjm}) \\
&= -\tfrac{1}{2} \nabla^p W^m (R_{jpkm} + R_{kpjm} + R_{kjpm}) - \tfrac{1}{2} \nabla^p W^m (R_{mjkp} + R_{kjmp} + R_{mkjp}) \\
&= -\tfrac{1}{2} \nabla^p W^m (R_{jpkm} + R_{kpjm} + R_{kjpm} - R_{kpjm} - R_{kjpm} - R_{jpkm}) \\
&= 0,
\end{aligned}
$$

where in the penultimate line we used the symmetries of the curvature tensor. The fourth line on the right-hand side is given by

$$
\text{(6-13)}\qquad -g^{pq} (R^m_{pkq} \nabla_j W_m + R^m_{pjq} \nabla_k W_m) = R_{km} \nabla_j W^m + R_{jm} \nabla_k W^m.
$$

To compute the third line on the right-hand side, we first observe that by the symmetries of the curvature tensor and the second Bianchi identity

$$
\text{(6-14)}\qquad \nabla_p R_{jqkm} = \nabla_p R_{kmjq} = -\nabla_m R_{pkjq} - \nabla_k R_{mpjq}.
$$

Moreover, exchanging $j$ and $k$ yields

$$
\text{(6-15)}\qquad \nabla_p R_{kqjm} = -\nabla_m R_{pjkq} - \nabla_j R_{mpkq}.
$$

Now using (6-14) and (6-15) we obtain for the third line on the right-hand side

$$
\begin{aligned}
\text{(6-16)}\quad & -\tfrac{1}{2} g^{pq} W_m (\nabla_p R^m_{jqk} + \nabla_p R^m_{kqj} + \nabla_j R^m_{pkq} + \nabla_k R^m_{pjq}) \\
&= -\tfrac{1}{2} g^{pq} W^m (-\nabla_m R_{pkjq} - \nabla_k R_{mpjq} - \nabla_m R_{pjkq} \\
&\qquad\qquad - \nabla_j R_{mpkq} + \nabla_j R_{pkqm} + \nabla_k R_{pjqm}) \\
&= -\tfrac{1}{2} W^m (-\nabla_m R_{kj} + \nabla_k R_{mj} - \nabla_m R_{jk} + \nabla_j R_{mk} - \nabla_j R_{km} - \nabla_k R_{jm}) \\
&= W^m \nabla_m R_{jk}.
\end{aligned}
$$

Plugging (6-12), (6-13) and (6-16) into (6-11) we finally obtain the claimed evolution equation of the Ricci tensor

$$
\partial_t R_{jk} = -\Delta_L R_{jk} + \nabla_m R_{jk} W^m + R_{jm} \nabla_k W^m + R_{km} \nabla_j W^m. \qquad \square
$$

**Corollary 6.2.** *Let $M$ be a smooth manifold. Let $g(t)$, $t \in [0, T]$ be a solution of the Ricci de Turck flow with initial metric $g$ and reference metric $h$, i.e.,*

$$\begin{cases} \frac{\partial}{\partial t} g_{ij}(x,t) = (-2R_{ij} + \nabla_i W_j + \nabla_j W_i)(x,t), & (x,t) \in M \times [0,T], \\ g_{ij}(x,0) = g_{ij}(x), & x \in M, \end{cases} \tag{6-17}$$

*where $W(t)^k = g(t)^{ij}(\Gamma^k_{ij}(g(t)) - \Gamma^k_{ij}(h))$ is the de Turck vector field. Then the scalar curvature $R$ evolves by*

$$(\partial_t + \Delta)R = \langle W, \nabla R\rangle + 2|\mathrm{Ric}|^2. \tag{6-18}$$

*Proof.* The scalar curvature is given by $R = g^{jk}R_{jk}$. Since

$$\partial_t g^{jk} = -g^{jp}g^{kq}\partial_t g_{pq} = -g^{jp}g^{kq}(-2R_{pq} + \nabla_p W_q + \nabla_q W_p)$$

and by the evolution equation for the Ricci tensor (6-2) we thus obtain

$$\begin{aligned} \partial_t R &= -g^{jp}g^{kq}(-2R_{pq} + \nabla_p W_q + \nabla_q W_p)R_{jk} \\ &\qquad + g^{jk}(-\Delta_L R_{jk} + \nabla_m R_{jk} W^m + R_{jm}\nabla_k W^m + R_{km}\nabla_j W^m) \\ &= 2|\mathrm{Ric}|^2 - 2g^{jp}g^{kq}\nabla_p W_q R_{jk} - g^{jk}\Delta_L R_{jk} + \nabla_m R W^m + 2g^{jk}R_{jm}\nabla_k V^m \\ &= -\Delta R + \nabla_m R W^m + 2|\mathrm{Ric}|^2, \end{aligned}$$

where in the last step we used $g^{jk}\Delta_L R_{jk} = \Delta R$. □

**Remark 6.3.** Note that in the proof we didn't use the special form of $W$, we just used that $W$ is a (time-dependent) one-form.

**Remark 6.4.** Note also that, although the computations above are local, it is not possible to locally go from the Ricci de Turck flow back to the Ricci flow. If the de Turck vector field points towards the singularity, such that there exists a sequence of points $p_n \in M$ where the integral curve of the de Turck vector field starting at $p_n$ has maximal existence time $t_{\max}(p_n) \to 0$, the local diffeomorphisms generated by the de Turck vector field are not defined near $p_n$ for times $t_{\max}(p_n) < t \le T$. As the times $t_{\max}(p_n)$ become arbitrarily small, it is also not possible to replace $T$ by a smaller time $0 < S < T$ and obtain the evolution equations on $[0, S]$.

## 7. Regularity of Ricci curvature along the Ricci de Turck flow

Our aim in this section is to improve the a priori low regularity of the Ricci curvature along the singular Ricci de Turck flow, as noted in [21, Theorem 8.1]. Consider an $(\alpha, k+1, \gamma)$-Hölder regular conical manifold $(M, g)$, satisfying tangential stability, with singular Ricci de Turck flow $g(\,\cdot\,) \in \mathcal{H}^{k,\alpha}_{\gamma_0,\gamma_1}(M \times [0,T], S)$, i.e., decomposing $g(t) = (1+u)g + \omega$ into trace and trace-free parts with respect to the initial metric $g$, we have

$$(\omega, u) \in \mathcal{C}^{k+2,\alpha}_{\mathrm{ie}}(M \times [0,T], S_0)_{\gamma_0} \oplus \mathcal{C}^{k+2,\alpha}_{\mathrm{ie}}(M \times [0,T], S_1)^b_{\gamma_1}.$$

Recall the following transformation rule for the Ricci curvature tensor under conformal transformations (setting $1+u = e^{2\phi}$ and noting that $\dim M = n+1$)

$$\mathrm{Ric}((1+u)g) = \mathrm{Ric}(g) - (n-1)(\nabla\partial\phi - \partial\phi\cdot\partial\phi) + (\Delta_L\phi - (n-1)\|\nabla\phi\|^2)g. \tag{7-1}$$

From here we conclude that $\mathrm{Ric}((1+u)g) - \mathrm{Ric}(g) \in \mathcal{C}^{k,\alpha}_{\mathrm{ie}}(M \times [0,T], S)_{-2+\gamma_1}$. Now consider $\mathrm{Ric}((1+u)g+\omega) - \mathrm{Ric}((1+u)g)$, which is an intricate combination of $a$ and $\omega$, involving their second order $x^{-2}\,\mathcal{V}^2$ derivatives. Hence that difference lies in $\mathcal{C}^{k,\alpha}_{\mathrm{ie}}(M \times [0,T], S)_{-2+\bar{\gamma}}$ with $\bar{\gamma} := \min\{\gamma_0, \gamma_1\}$. We conclude

$$\begin{aligned} \mathrm{Ric}(g) &\in \mathcal{C}^{k+1,\alpha}_{\mathrm{ie}}(M \times [0,T], S)_{-2+\gamma}, \\ \mathrm{Ric}(g(t)) &\in \mathcal{C}^{k,\alpha}_{\mathrm{ie}}(M \times [0,T], S)_{-2+\bar{\gamma}}, \qquad t > 0. \end{aligned} \tag{7-2}$$

In particular, e.g., if $\mu_0, \mu_1 \le 2 \le \gamma$, then $\bar{\gamma} \le 2 \le \gamma$. In that case, the initial Ricci curvature $\mathrm{Ric}(g)$ is bounded as a section of $S$, while for positive times $\mathrm{Ric}(g(t))$ is singular as a section of $S$. In this section we improve this low regularity result.

***Expansion of the Lichnerowicz Laplacian.*** Below we fix the following notation: A superscript "$\sim$" indicates that the quantity is taken with respect to the initial metric $g(0)$. For example, $\widetilde{\nabla}$ and $\widetilde{\Delta_L}$ refer to the covariant derivative and the Lichnerowicz Laplacian with respect to $g(0)$. Otherwise the quantities are defined with respect to the flow $g(t)$.

**Theorem 7.1.** *Consider an $(\alpha, k+1, \gamma)$-Hölder regular conical manifold $(M, g)$, satisfying tangential stability, with singular Ricci de Turck flow $g(\,\cdot\,) \in \mathcal{H}^{k+2,\alpha}_{\gamma_0,\gamma_1}(M \times [0,T], S)$, i.e., decomposing $g(t) = ag + \omega$ into trace and trace-free parts with respect to the initial metric $g$. Then there exist*

$$\begin{aligned} A &\in \mathcal{C}^{k,\alpha}_{\mathrm{ie}}(M \times [0,T], S)_{\bar{\gamma}}, \\ B &\in \mathcal{C}^{k,\alpha}_{\mathrm{ie}}(M \times [0,T], S)_{-1+\bar{\gamma}}, \\ C &\in \mathcal{C}^{k,\alpha}_{\mathrm{ie}}(M \times [0,T], S)_{-2+\bar{\gamma}}, \end{aligned}$$

*such that for any (time-dependent) symmetric 2-tensor $c$*

$$\Delta_L c = a^{-1/2}\widetilde{\Delta} a^{-1/2} c + (A \cdot (x^{-1}\,\mathcal{V}_b)^2 + B \cdot x^{-1}\,\mathcal{V}_b + C)c. \tag{7-3}$$

*Proof.* Consider the rough Laplacian $\Delta = -g^{ij}\nabla_i\nabla_j$ first. We start by writing out $\Delta$ in local coordinates and later on turn to the Lichnerowicz Laplacian $\Delta_L$. We compute for any symmetric 2-tensor $c$

$$
\begin{aligned}
(\Delta c)_{kl} &= -g^{ij}\nabla_i\nabla_j c_{kl} \\
&= -g^{ij}\big(\partial_i\partial_j c_{kl} - \partial_i\Gamma^m_{jk}c_{ml} - \Gamma^m_{jk}\partial_i c_{ml} - \partial_i\Gamma^m_{jl}c_{km} - \Gamma^m_{jl}\partial_i c_{km} \\
&\qquad - \Gamma^m_{ij}\partial_m c_{kl} + \Gamma^m_{ij}\Gamma^p_{mk}c_{pl} + \Gamma^m_{ij}\Gamma^p_{ml}c_{kp} - \Gamma^m_{ik}\partial_j c_{ml} + \Gamma^m_{ik}\Gamma^p_{jm}c_{pl} \\
&\qquad + \Gamma^m_{ik}\Gamma^p_{jl}c_{mp} - \Gamma^m_{il}\partial_j c_{km} + \Gamma^m_{jk}\Gamma^p_{il}c_{mp} + \Gamma^m_{il}\Gamma^p_{jm}c_{kp}\big).
\end{aligned} \tag{7-4}
$$

Our goal is to obtain an expansion for each term on the right-hand side of (7-4). The metric $g(t)$ has the form $g_{ij} = a\tilde{g}_{ij} + \omega_{ij}$, with inverse (see [21, page 28])

$$
g^{ij} = a^{-1}\tilde{g}^{ij} - a^{-2}\tilde{g}^{il}\tilde{g}^{jp}\,\omega_{pl} + a^{-2}g^{jl}\tilde{g}^{ir}\tilde{g}^{pq}\,\omega_{lp}\,\omega_{rq}\,. \tag{7-5}
$$

Hence the first term on the right-hand side of (7-4) is given by

$$
-g^{ij}\partial_i\partial_j c_{kl} = -a^{-1}\tilde{g}^{ij}\partial_i\partial_j c_{kl} - (-a^{-2}\tilde{g}^{is}\tilde{g}^{jp}\,\omega_{ps} + a^{-2}g^{js}\tilde{g}^{ir}\tilde{g}^{pq}\,\omega_{sp}\,\omega_{rq})\partial_i\partial_j c_{kl}.
$$

Now we study the asymptotics of the last expression at the conical singularity. This is analogous to the discussion in [21, pages 28–29]. Consider the local coordinates $(z_0, \dots, z_n)$ of $\mathcal{C}(F)$ with $z_0 = x$ and $(z_1, \dots, z_n)$ being local coordinates of $F$. An upper index $i = 0$ does not contribute any singular factor of $x$ due to the structure of the inverse $g^{-1}$. A lower index $i = 0$ indicates a differentiation by $\partial_x \in x^{-1}\,\mathcal{V}_b$ Hence an index $i = 0$ (as a combination of a lower and upper index) contributes $x^{-1}\,\mathcal{V}_b$ up to a term of type $A$.

Similarly, an upper index $i > 0$ contributes a singular factor $x^{-1}$ due to the structure of the inverse $g^{-1}$. A lower index $i > 0$ indicates a differentiation by $\partial_{z_j} \in \mathcal{V}_b$ Hence an index $i > 0$ (as a combination of a lower and upper index) contributes $x^{-1}\,\mathcal{V}_b$ up to a term of type $A$. Hence in total we find

$$
-g^{ij}\partial_i\partial_j c_{kl} = -a^{-1}\tilde{g}^{ij}\partial_i\partial_j c_{kl} + A\cdot(x^{-1}\,\mathcal{V}_b)^2 c_{kl}, \tag{7-6}
$$

for some $A \in \mathcal{C}^{k,\alpha}_{\mathrm{ie}}(M \times [0, T], S)_{\bar{\gamma}}$. In order to study the remaining terms in (7-4), involving Christoffel symbols, we note that the Christoffel symbols $\Gamma^k_{ij}$ with respect to the metric $g(t)$ are related to the Christoffel symbols $\widetilde{\Gamma}^k_{ij}$ of the initial metric $\tilde{g}$ as follows

$$
\begin{aligned}
\Gamma^k_{ij} = \widetilde{\Gamma}^k_{ij} &+ \tfrac{1}{2}a^{-1}\tilde{g}^{km}\big(\partial_i a\tilde{g}_{jm} + \partial_j a\tilde{g}_{im} - \partial_m a\tilde{g}_{ij} + \partial_i\,\omega_{jm} + \partial_j\,\omega_{im} - \partial_m\,\omega_{ij}\big) \\
&+ \tfrac{1}{2}\big(-a^{-2}\tilde{g}^{kl}\tilde{g}^{mp}\,\omega_{pl} + a^{-2}g^{ml}\tilde{g}^{kr}\tilde{g}^{pq}\,\omega_{lp}\,\omega_{rq}\big) \\
&\qquad \times \big(a(\partial_i\tilde{g}_{jm} + \partial_j\tilde{g}_{im} - \partial_m\tilde{g}_{ij}) + \partial_i a\tilde{g}_{jm} + \partial_j a\tilde{g}_{im} \\
&\qquad - \partial_m a\tilde{g}_{ij} + \partial_i\,\omega_{jm} + \partial_j\,\omega_{im} - \partial_m\,\omega_{ij}\big)
\end{aligned} \tag{7-7}
$$

This allows us to conclude by counting upper and lower indices as above that for any $\{Y_\ell\}_\ell$ with $Y_0 := \partial_x$ and $Y_i := x^{-1}\partial_{z_i}$ for $i = 1, \dots, n$, we have

$$
(\Delta c)(Y_k, Y_l) = a^{-1}(\widetilde{\Delta}c)(Y_k, Y_l) + (A\cdot(x^{-1}\,\mathcal{V}_b)^2 + B\cdot x^{-1}\,\mathcal{V}_b + C)c(Y_k, Y_l), \tag{7-8}
$$

for some terms $A$, $B$, $C$ as in the statement of the theorem. Computing further

$$\Delta = a^{-1/2}\widetilde{\Delta}a^{-1/2} + +(A\cdot(x^{-1}\mathcal{V}_b)^2 + B'\cdot x^{-1}\mathcal{V}_b + C'), \tag{7-9}$$

where $B' = B - a^{-1/2}(\widetilde{\nabla}a^{-1/2})$, $C' = C - a^{-1/2}(\widetilde{\Delta}a^{-1/2})$. Note that the terms $B'$, $C'$ still have the same Hölder regularity as $B$, $C$. Now we turn to the Lichnerowicz Laplacian. Recall that it is given by

$$\Delta_L c_{kl} = \Delta c_{kl} - 2g^{pq}R^r_{qkl}c_{rp} + g^{pq}R_{kp}c_{ql} + g^{pq}R_{lp}c_{kq}.$$

To deal with the extra terms involving the Riemann curvature tensor and the Ricci curvature on the right-hand side, we note the following formula for the components of the curvature tensor in local coordinates

$$R^l_{ijk} = \partial_i\Gamma^l_{jk} - \partial_j\Gamma^l_{ik} + \Gamma^p_{jk}\Gamma^l_{ip} - \Gamma^p_{ik}\Gamma^l_{jp}, \tag{7-10}$$

which also leads to a formula for the components of the Ricci tensor

$$R_{jk} = R^i_{ijk} = \partial_i\Gamma^i_{jk} - \partial_j\Gamma^i_{ik} + \Gamma^p_{jk}\Gamma^i_{ip} - \Gamma^p_{ik}\Gamma^i_{jp}. \tag{7-11}$$

These extra terms can be treated using (7-7), which leads to the claimed expansion for the Lichnerowicz Laplacian. □

**Corollary 7.2.** *Under the conditions of Theorem 7.1 we conclude*

$$(\partial_t + a^{-1/2}\widetilde{\Delta}a^{-1/2})\,\mathrm{Ric} =: X(\mathrm{Ric}) \in \mathcal{C}^{k-1,\alpha}_{\mathrm{ie}}(M\times[0,T], S)_{-4+2\bar{\gamma}}. \tag{7-12}$$

*Proof.* Recall the evolution (6-2) of Ricci curvature along the Ricci de Turck flow

$$(\partial_t + \Delta_L)\,\mathrm{Ric}_{jk} = \nabla_m \mathrm{Ric}_{jk} W^m + \mathrm{Ric}_{jm}\nabla_k W^m + \mathrm{Ric}_{km}\nabla_j W^m, \tag{7-13}$$

By (5-2) and (7-2) we conclude

$$(\partial_t + \Delta_L)\,\mathrm{Ric} \in \mathcal{C}^{k-1,\alpha}_{\mathrm{ie}}(M\times[0,T], S)_{-4+2\bar{\gamma}}.$$

The statement now follows from Theorem 7.1 and (7-2). □

***Mapping properties of the heat parametrix for $L := a^{1/2}\widetilde{\Delta}a^{-1/2}$.*** We first point out the following relation between $L = a^{1/2}\widetilde{\Delta}a^{-1/2}$ and the Lichnerowicz Laplace operator $\widetilde{\Delta_L}$

$$\begin{aligned} L &= a^{1/2}\widetilde{\Delta}a^{-1/2} \\ &= \widetilde{\Delta} + a^{1/2}(\widetilde{\Delta}a^{-1/2}) + a^{1/2}(\widetilde{\nabla}a^{-1/2})\widetilde{\nabla} \\ &= \widetilde{\Delta}_L + (B\cdot\widetilde{\nabla} + C) \\ &=: \widetilde{\Delta}_L + P, \end{aligned} \tag{7-14}$$

where $B$ and $C$ are terms with same regularity as in Theorem 7.1. The heat operator for $\widetilde{\Delta_L}$ has been studied in [21, Theorem 3.1], which asserts that for $n \geq 3$, any

$\gamma \in (0, 2+\min\{\mu_0, \mu_1\})$ and $\alpha \in (0, 1)$ sufficiently small, the following is a bounded mapping (we simplify notation by writing $\mathcal{H}^{k,\alpha}_{\gamma} \equiv \mathcal{H}^{k,\alpha}_{\gamma,\gamma}$)

$$e^{-t\widetilde{\Delta_L}} : \mathcal{H}^{k,\alpha}_{-4+\gamma}(M \times [0, T], S) \to \mathcal{H}^{k+2,\alpha}_{-2+\gamma}(M \times [0, T], S). \tag{7-15}$$

Since by (7-14), $L$ and $\widetilde{\Delta_L}$ differ only by lower order terms $P$, the heat operator construction in [21] carries over to get a heat parametrix for $L$ as well. Thus, exactly as in (7-15), we find

$$H := e^{-tL} : \mathcal{H}^{k,\alpha}_{-4+\gamma}(M \times [0, T], S) \to \mathcal{H}^{k+2,\alpha}_{-2+\gamma}(M \times [0, T], S). \tag{7-16}$$

The following mapping properties are needed later in this section, and follow by similar arguments as in [21, Theorem 3.1] and (7-15).

**Lemma 7.3.** *Consider smooth functions $\rho \in C^\infty(\overline{M})$ smooth up to the boundary, $\eta \in C^\infty_0(M)$ smooth with compact support away from the conical singularity. Let $\nu \in C^\infty_0(M, TM)$ be a smooth 1-form and $\omega \in C^\infty_0(M, {}^{ib}T^*M \otimes {}^{ib}T^*M)$ a smooth $(0, 2)$-tensor, both with compact support away from the conical singularity. Let $H$ be the heat parametrix of $L = a^{1/2}\widetilde{\Delta} a^{-1/2}$. Then*

$$\begin{aligned}
E_0 &:= \eta H\rho : \mathcal{H}^{k,\alpha}_{-4+\gamma}(M \times [0, T], S) \to \mathcal{H}^{k+2,\alpha}_{-2+\gamma}(M \times [0, T], S),\\
\widetilde{E}_0 &:= \nu \otimes H\rho : \mathcal{H}^{k,\alpha}_{-4+\gamma}(M \times [0, T], S) \to \mathcal{H}^{k+2,\alpha}_{-2+\gamma}(M \times [0, T], {}^{ib}T^*M \otimes S),\\
\widetilde{\widetilde{E}}_0 &:= \omega \otimes H\rho : \mathcal{H}^{k,\alpha}_{-4+\gamma}(M \times [0, T], S) \to \mathcal{H}^{k+2,\alpha}_{-2+\gamma}(M \times [0, T], {}^{ib}T^*M^{\otimes^2} \otimes S),
\end{aligned}$$

*with $\|E_0\| \to 0$, $\|\widetilde{E}_0\| \to 0$ and $\|\widetilde{\widetilde{E}}_0\| \to 0$ as $T \to 0$.*

The next result is a straightforward consequence of Lemma 7.3.

**Corollary 7.4.** *In the notation of Lemma 7.3 we find*

$$\begin{aligned}
F_0 &:= \nu\otimes\nabla H\rho : \mathcal{H}^{k,\alpha}_{-4+\gamma}(M\times[0, T], S) \to \mathcal{H}^{k+1,\alpha}_{-3+\gamma}(M\times[0, T], {}^{ib}T^*M^{\otimes^2}\otimes S),\\
G_0 &:= g^{ij}\nu_i\nabla_j H\rho : \mathcal{H}^{k,\alpha}_{-4+\gamma}(M\times[0, T], S) \to \mathcal{H}^{k+1,\alpha}_{-4+\gamma}(M\times[0, T], S),
\end{aligned}$$

*with $\|F_0\| \to 0$, $\|G_0\| \to 0$ as $T \to 0$.*

*Proof.* By Lemma 7.3 we have

$$\nabla(\nu \otimes H\rho) : \mathcal{H}^{k,\alpha}_{-4+\gamma}(M \times [0, T], S) \to \mathcal{H}^{k+1,\alpha}_{-3+\gamma}(M \times [0, T], {}^{ib}T^*M^{\otimes^2} \otimes S),$$

with $\|\nabla(\nu \otimes H\rho)\| \to 0$ as $T \to 0$. On the other hand

$$\nabla(\nu \otimes H\rho) = \nabla\nu \otimes H\rho + \nu \otimes \nabla H\rho.$$

Hence the mapping properties of $F_0$ follow by applying Lemma 7.3 to $\nabla\nu \otimes H\rho$. The mapping properties of $G_0$ follow from the ones of $F_0$, as $G_0$ is simply $F_0$ composed with a contraction. $\square$

***Heat operator parametrix for $a^{-1/2}\widetilde{\Delta}a^{-1/2}$.*** Our goal in this section is the existence of inverses $\mathcal{Q}$ and $\mathcal{R}$ for the parabolic operator $P := \partial_t + a^{-1}L$, where $a$ is the pure trace part of $g(t)$ with respect to the initial metric $g$ and is positive uniformly bounded away from zero for short time. These parametrices are constructed out of the heat parametrix $H$ for $L = a^{1/2}\widetilde{\Delta}a^{-1/2}$, using (7-16), Lemma 7.3 and Corollary 7.4. Our main result in this subsection is as follows.

**Theorem 7.5.** *Let $n \geq 3$. Consider any $\gamma \in (0, 2 + \min\{\mu_0, \mu_1\})$ and $\alpha \in (0, 1)$ sufficiently small, such that (7-15) holds. Then there exists $T_0 > 0$ sufficiently small and a bounded linear map*

$$\mathcal{Q} : \mathcal{H}^{k,\alpha}_{-4+\gamma}(M \times [0, T_0], S) \to \mathcal{H}^{k+2,\alpha}_{-2+\gamma}(M \times [0, T_0], S),$$

*such that if $f \in \mathcal{H}^{k,\alpha}_{-4+\gamma}(M \times [0, T_0], S)$, then $u = \mathcal{Q}f$ solves the initial value problem*

$$(\partial_t + a^{-1}L)u = f, \quad u(\,\cdot\,, 0) = 0.$$

The proof of that theorem will occupy the rest of this subsection. Before we proceed, let us note an immediate consequence.

**Theorem 7.6.** *Let $n \geq 3$. Consider any $\gamma \in (0, 2 + \min\{\mu_0, \mu_1\})$ and $\alpha \in (0, 1)$ sufficiently small, such that (7-15) holds. Then there exists $T_0 > 0$ sufficiently small and a bounded linear map*

$$\mathcal{R} : \mathcal{H}^{k+2,\alpha}_{-2+\gamma}(M, S) \to \mathcal{H}^{k+2,\alpha}_{-2+\gamma}(M \times [0, T_0], S),$$

*such that if $u_0 \in \mathcal{H}^{k+2,\alpha}_{-2+\gamma}(M, S)$, then $u = \mathcal{R}u_0$ solves the initial value problem*

$$(\partial_t + a^{-1}L)u = 0, \quad u(\,\cdot\,, 0) = u_0.$$

*Proof.* We have $Lu_0 \in \mathcal{H}^{k,\alpha}_{-4+\gamma}(M, S)$ and thus $a^{-1}Lu_0 \in \mathcal{H}^{k,\alpha}_{-4+\gamma}(M \times [0, T_0], S)$. We set $\mathcal{R}u_0 := u_0 + \mathcal{Q}(a^{-1}Lu_0)$, which yields the desired solution to the initial value problem above. $\square$

*Construction of a boundary parametrix.* We now begin the proof of Theorem 7.5. The proof idea is to construct boundary and interior parametrices, and glue them together to an approximate solution to $(\partial_t + a^{-1}L)$. Provided the error is sufficiently "nice", we can obtain a solution $\mathcal{Q}$ by a Neumann series argument. We follow the analytic path outlined in the work by Vertman jointly with Bahuaud [2].

We start by introducing the notion of a *reference covering* which is a special case of a covering considered in [2]. Let $U = [0, 1)_x \times (-1, 1)^n_z$ be a model half-cube. For each $p \in \partial M$, there is a coordinate chart $\Phi_p : U \to W_p$ onto an open neighborhood $W_p \subset \mathcal{C}(F) \subset \overline{M}$, centered around $p_i$. Due to compactness of $\partial M$, there are finitely many such charts $\{W_i, \Phi_{p_i}\}_{i=1}^N$ covering $\mathcal{C}(F)$. Together with

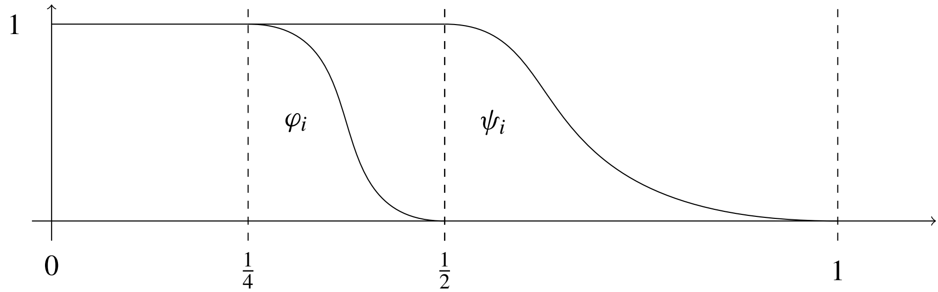



**Figure 1.** The cutoff functions $\varphi_i$, $\psi_i$.

the open set $W_0 = \overline{M} \backslash \big([0, \frac{1}{2}) \times F\big)$ we obtain an open cover of $\overline{M}$ that we call a reference covering.

Let $\sigma : \mathbb{R}^+ \to [0, 1]$ be a smooth cutoff function with $\sigma(s) = 1$ for $s \leq \frac{1}{4}$ and $\sigma(s) = 0$ for $s \geq \frac{1}{2}$. Denote by $(x, z) \in \Phi_{p_i}^{-1}(W_i)$ the local coordinates on the coordinate chart $W_i$, centered around $p_i \in \partial M$. Then we define

$$\varphi_i(\Phi_{p_i}(x, z)) := \sigma(x)\sigma(\|z\|), \quad \psi_i(\Phi_{p_i}(x, z)) := \sigma\left(\frac{x}{2}\right)\sigma\left(\frac{\|z\|}{2}\right).$$

Notice that $\varphi_i, \psi_i \in C^\infty(\widetilde{M})$ are both identically 1 near $(0, 0)$, and $\psi_i \equiv 1$ on $\operatorname{supp} \varphi_i$. Furthermore, $\varphi_i, \psi_i \in \mathrm{C}^\alpha_{\mathrm{ie}}(M)$, since they are constant near the cone points. These functions are illustrated in the radial direction in Figure 1.

Next, for $\varepsilon \in (0, 1)$ to be specified later, and any $p = z_0 \in W_i \cap \partial M$ we set

$$\widetilde{\varphi}_{i,p}(x, z) := \varphi_i\left(\frac{x}{\varepsilon}, z - z_0\right), \quad \widetilde{\psi}_{i,p}(x, z) := \psi_i\left(\frac{x}{\varepsilon}, z - z_0\right).$$

This defines smooth cutoff functions $\widetilde{\varphi}_{i,p}, \widetilde{\psi}_{i,p} \in C^\infty(\overline{M})$ which again lie in $\mathrm{C}^{k,\alpha}_{\mathrm{ie}}(M)$, with the Hölder norms bounded by $\varepsilon^{-\alpha}\|\varphi_i\|_{\mathrm{C}^{k,\alpha}_{\mathrm{ie}}(M)}$ and $\varepsilon^{-\alpha}\|\psi_i\|_{\mathrm{C}^{k,\alpha}_{\mathrm{ie}}(M)}$, respectively

$$\text{(7-17)} \qquad \|\widetilde{\varphi}_{i,p}\|_{\mathrm{C}^{k,\alpha}_{\mathrm{ie}}(M)} \leq \varepsilon^{-\alpha}\|\varphi_i\|_{\mathrm{C}^{k,\alpha}_{\mathrm{ie}}(M)}, \quad \|\widetilde{\psi}_{i,p}\|_{\mathrm{C}^{k,\alpha}_{\mathrm{ie}}(M)} \leq \varepsilon^{-\alpha}\|\psi_i\|_{\mathrm{C}^{k,\alpha}_{\mathrm{ie}}(M)}.$$

Let $L_i$ be the lattice of points in the coordinate chart $W_i$ of the form $\{(0, w) \mid w \in \mathbb{Z}^n\}$. By construction, every point on $\partial M$ lies in the support of at most a fixed number (independent of $\varepsilon$) of functions $\{\widetilde{\psi}_{i,p} \mid i = 1, \dots, N, p \in L_i\}$. Furthermore, let $F_\varepsilon = F_\varepsilon(x)$ be a cutoff function with $\operatorname{supp} F_\varepsilon \subset \{x \geq \varepsilon/4\} \subset M$, which is identically 1 on the set $\{x \geq \varepsilon\}$. Let $G_\varepsilon(x) = F_\varepsilon\big(\frac{x}{2}\big)$. For any $p \in L_i$ we normalize

$$\text{(7-18)} \qquad \varphi_{i,p} := \frac{\widetilde{\varphi}_{i,p}}{G_\varepsilon + \sum_k \sum_{q \in L_k} \widetilde{\varphi}_{k,q}}, \quad \psi_{i,p} := \frac{\widetilde{\psi}_{i,p}}{F_\varepsilon + \sum_k \sum_{q \in L_k} \widetilde{\varphi}_{k,q}}.$$

By (7-17), these functions again lie in $C^{\alpha}_{\mathrm{ie}}(M)$, with

$$\begin{aligned} \|\varphi_{i,p}\|_{C^{k,\alpha}_{\mathrm{ie}}(M)} &\le \text{const} \cdot \varepsilon^{-\alpha} \|\varphi_i\|_{C^{k,\alpha}_{\mathrm{ie}}(M)} \le \text{const} \cdot \varepsilon^{-\alpha}, \\ \|\psi_{i,p}\|_{C^{k,\alpha}_{\mathrm{ie}}(M)} &\le \text{const} \cdot \varepsilon^{-\alpha} \|\psi_i\|_{C^{k,\alpha}_{\mathrm{ie}}(M)} \le \text{const} \cdot \varepsilon^{-\alpha}. \end{aligned} \tag{7-19}$$

Finally note that due to normalization, for any $0 < \varepsilon < 1$, the functions

$$\Phi := \sum_i \sum_{p \in L_i} \varphi_{i,p}, \quad \Psi := \sum_i \sum_{p \in L_i} \psi_{i,p}$$

are smooth cutoff functions in $C^\infty(\overline{M})$ which are identically 1 in a neighborhood of the cone singularity. We can now introduce an approximate boundary parametrix. Consider any $f \in \mathcal{H}^{k,\alpha}_{-4+\gamma}(M \times [0, T_0], S)$ and denote by $H_p$ the heat kernel of $a(p,0)^{-1}L$,[4] for any $p \in \partial M$ (note this is simply a rescaling of $L$ by a positive constant). Then we define our boundary parametrix by

$$Q_b f := \sum_{j=1}^{N} \sum_{p \in L_j} \psi_{j,p} H_p[\varphi_{j,p} f]. \tag{7-20}$$

**Lemma 7.7.** *The solution $u_p := \psi_{j,p} H_p[\varphi_{j,p} f]$ satisfies*

$$(\partial_t + a^{-1}L)u_p = \varphi_{j,p} f + E^0_{j,p} f + E^1_{j,p} f,$$

*where the operators $E^0_{j,p}$ and $E^1_{j,p}$ are as follows*:

(i) *$E^0_{j,p}$ is a bounded linear map on $\mathcal{H}^{k,\alpha}_{-4+\gamma}(M \times [0, T], S)$, and there exists a constant $C > 0$ independent of $j$, $p$, such that for $T/\varepsilon^2 < 1$*

$$\|E^0_{j,p} f\| \le C(\varepsilon^\gamma + T^{\alpha/2}) \|\varphi_{j,p} f\|,$$

(ii) *$E^1_{j,p}$ is a bounded linear map on $\mathcal{H}^{k,\alpha}_{-4+\gamma}(M \times [0, T], S)$, with operator norm satisfying $\|E^1_{j,p}\| \to 0$ as $T \to 0^+$.*

*Proof.* We simplify notation by omitting the subscripts on $\varphi$, $\psi$ and $E^0$, $E^1$. Then we compute in view of (7-14)

$$\begin{aligned} (\partial_t + a^{-1}L)u_p &= (\partial_t + a^{-1}L)(\psi H_p[\varphi f]) \\ &= \psi \partial_t H_p[\varphi f] + a^{-1}\big(\widetilde{\Delta}\psi \cdot H_p[\varphi f] + 2\tilde{g}^{ij} \widetilde{\nabla}_i \psi (\widetilde{\nabla}_j H_p[\varphi f] \\ &\qquad + B \cdot H_p[\varphi f]) + \psi L H_p[\varphi f]\big) \\ &= \psi(\partial_t + a^{-1}(p,0)L) H_p[\varphi f] + \psi(a^{-1} - a^{-1}(p,0)) L H_p[\varphi f] \\ &\qquad + a^{-1}\widetilde{\Delta}\psi \cdot H_p[\varphi f] + a^{-1} 2\tilde{g}^{ij} \widetilde{\nabla}_i \psi (\widetilde{\nabla}_j H_p[\varphi f] + B \cdot H_p[\varphi f]) \\ &=: \psi\varphi f + E^0 f + E^1 f, \end{aligned}$$

[4] Recall that $g(p,t) = a(p,t) g(p) \oplus \omega(p,t)$.

where the operators $E^0$ and $E^1$ are explicitly given by

$$E^0 f := \psi(a^{-1} - a^{-1}(p,0)) L H_p[\varphi f],$$
$$E^1 f := a^{-1}\widetilde{\Delta}\psi \cdot H_p[\varphi f] + 2a^{-1}\tilde{g}^{ij}\widetilde{\nabla}_i \psi(\widetilde{\nabla}_j H_p[\varphi f] + B \cdot H_p[\varphi f]).$$

Since $\psi \equiv 1$ on $\operatorname{supp}\varphi$, any derivative of $\psi$ vanishes on a neighborhood of the conical singularity. Hence the claimed mapping properties of $E^1$ follow from Lemma 7.3 and Corollary 7.4. What is left is establishing the mapping properties of $E^0$. Writing out definitions of the various Hölder norms, we find

$$\|E^0 f\|_{\mathcal{H}^{k,\alpha}_{-4+\gamma}} \le C\|\psi(a^{-1} - a^{-1}(p,0))\|_{\mathcal{C}^{k,\alpha}_{\mathrm{ie}}} \cdot \|L H_p[\varphi f]\|_{\mathcal{H}^{k,\alpha}_{-4+\gamma}}. \tag{7-21}$$

We write $p = (0, z_0)$. For any $(x,z)$ in the support of $\psi$ we obtain

$$\begin{aligned} |a^{-1}(x,z,t) - a^{-1}(0,z_0,0)| &\le |x|^\gamma \|x^{-\gamma}(x\partial_x)a^{-1}\|_\infty + t^{\alpha/2}\|a^{-1}\|_\alpha \\ &\le |x|^\gamma \|x^{-\gamma}(x\partial_x)a\|_\infty + t^{\alpha/2}\|a\|_\alpha \\ &\le C(d_M(p,q)^\gamma + t^{\alpha/2})\|a\|_{\mathcal{C}^{1,\alpha}_{\mathrm{ie}}(M\times[0,T],S_1)^b_\gamma} \\ &\le C(\varepsilon^\gamma + T^{\alpha/2})\|a\|_{\mathcal{C}^{1,\alpha}_{\mathrm{ie}}(M\times[0,T],S_1)^b_\gamma}, \end{aligned}$$

where we used that $a(0,\cdot,t)$ is constant in $z$, and $\partial_x a^{-1} = -\partial_x a \cdot a^{-2}$. Recall the notation $\mathcal{D}$ of Definition 3.3. Then for any partial differential operator $X \in \mathcal{D}$ of order at least 1, we compute similar to above for any $(x,z) \in \operatorname{supp}\psi$

$$\begin{aligned} |X(a^{-1}(x,z,t) - a^{-1}(0,z_0,0))| = |Xa^{-1}(x,z,t)| &\le |x|^\gamma \|x^{-\gamma} X a^{-1}\|_\infty \\ &\le C\varepsilon^\gamma \|a\|_{\mathcal{C}^{1,\alpha}_{\mathrm{ie}}(M\times[0,T],S_1)^b_\gamma} \\ &\le C(\varepsilon^\gamma + T^{\frac{\alpha}{2}})\|a\|_{\mathcal{C}^{1,\alpha}_{\mathrm{ie}}(M\times[0,T],S_1)^b_\gamma}. \end{aligned}$$

Analogous estimates hold for Hölder differences. Noting (7-19), we conclude

$$\|\psi(a^{-1} - a^{-1}(p,0))\|_{\mathcal{C}^{k,\alpha}_{\mathrm{ie}}} \le C\varepsilon^{-\alpha}(\varepsilon^\gamma + T^{\alpha/2})\|a\|_{\mathcal{C}^{k+1,\alpha}_{\mathrm{ie}}(M\times[0,T],S_1)^b_\gamma}. \tag{7-22}$$

Now we estimate $LH_p[\varphi f]$. We first note that it vanishes identically at $t = 0$. This is obtained from the following mapping properties of the heat operator

$$H_p \circ \varphi : \mathcal{H}^{k,\alpha}_{-4+\gamma}(M\times[0,T],S) \to \mathcal{H}^{k+2,\alpha}_{-2+\gamma}(M\times[0,T],S),$$
$$H_p \circ \varphi : \mathcal{H}^{k,\alpha}_{-4+\gamma}(M\times[0,T],S) \to t^{\frac{\alpha}{2}}\,\mathcal{H}^{k+2,0}_{-2+\gamma}(M\times[0,T],S),$$

where the first mapping property is due to (7-15) and the second is obtained by similar arguments, converting the lesser target regularity into a time weight. Hence

$LH_p[\varphi f] \in \mathcal{H}^{k,0}_{-4+\gamma}$ vanishes identically at $t=0$ and hence

$$\|LH_p[\varphi f]\|_{\mathcal{H}^{k,0}_{-4+\gamma}} = \|LH_p[\varphi f] - LH_p[\varphi f](t=0)\|_{\mathcal{H}^{k,0}_{-4+\gamma}} \tag{7-23}$$

$$\leq CT^{\alpha/2}\|[\varphi f]\|_{\mathcal{H}^{k,\alpha}_{-4+\gamma}} \leq C(\varepsilon^\alpha + T^{\alpha/2})\|[\varphi f]\|_{\mathcal{H}^{k,\alpha}_{-4+\gamma}}. \tag{7-24}$$

Analogous estimates hold for Hölder differences, if we take supremums over $\operatorname{supp}\psi$. Hence overall we arrive at the estimate

$$\|LH_p[\varphi f]\|_{\mathcal{H}^{k,\alpha}_{-4+\gamma}(\operatorname{supp}\psi\times[0,T],S)} \leq C(\varepsilon^\alpha + T^{\alpha/2})\|[\varphi f]\|_{\mathcal{H}^{k,\alpha}_{-4+\gamma}}. \tag{7-25}$$

Plugging the estimates (7-22) and (7-25) into (7-21), we conclude for $T/\varepsilon^2 < 1$

$$\begin{aligned}
\|E^0 f\|_{\mathcal{H}^{k,\alpha}_{-4+\gamma}} &\leq C(\varepsilon^\gamma + T^{\alpha/2})\varepsilon^{-\alpha}(\varepsilon^\alpha + T^{\alpha/2})\|[\varphi f]\|_{\mathcal{H}^{k,\alpha}_{-4+\gamma}} \\
&\leq C(\varepsilon^\gamma + T^{\alpha/2})(1 + (T/\varepsilon^2)^{\alpha/2})\|[\varphi f]\|_{\mathcal{H}^{k,\alpha}_{-4+\gamma}} \\
&\leq C(\varepsilon^\gamma + T^{\alpha/2})\|[\varphi f]\|_{\mathcal{H}^{k,\alpha}_{-4+\gamma}}.
\end{aligned}$$

This finishes the proof. □

**Proposition 7.8.** *Let $\alpha \in (0, \min\{\gamma, 1\})$. For every $\delta > 0$, there exists $\varepsilon > 0$ and $T_0 > 0$ sufficiently small, such that the heat parametrix defined in (7-20)*

$$Q_b : \mathcal{H}^{k,\alpha}_{-4+\gamma}(M \times [0, T_0], S) \to \mathcal{H}^{k+2,\alpha}_{-2+\gamma}(M \times [0, T_0], S),$$

*is a bounded linear map, solving*

$$(\partial_t + a^{-1}L)(Q_b f) = \Phi f + E^0 f + E^1 f,$$

*where $E^0$ and $E^1$ are bounded linear maps on $\mathcal{H}^{k,\alpha}_{-4+\gamma}(M\times[0,T_0],S)$ with operator norms $\|E^0\| < \delta$ and $\|E^1\| \to 0$ as $T_0 \to 0^+$.*

*Proof.* By Lemma 7.7 we compute for any $f \in \mathcal{H}^{k,\alpha}_{-4+\gamma}(M \times [0, T_0], S)$

$$(\partial_t + a^{-1}L)(Q_b f) = \sum_{j=1}^{N}\sum_{p\in L_j}(\partial_t + a^{-1}L)\psi_{j,p}H_p[\varphi_{j,p} f] = \Phi f + E^0 f + E^1 f,$$

where we have defined for $i = 0, 1$

$$E^i f := \sum_{j=1}^{N}\sum_{p\in L_j} E^i_{j,p} f.$$

The operators $E^i_{j,p}$, and hence also both $E^0$ and $E^1$, are bounded operators on $\mathcal{H}^{k,\alpha}_{-4+\gamma}(M \times [0, T_0], S)$ by Lemma 7.7. It remains to estimate their operator norms as $T_0 \to 0$. The fact that $\|E^1\| \to 0$ as $T_0 \to 0^+$ is a direct consequence of the second statement in Lemma 7.7. For the estimate of $\|E^0\|$ we argue as follows. We choose $\varepsilon > 0$ and $T_0 > 0$ with $T_0 < \varepsilon^2$. Then by Lemma 7.7 and (7-19) we have

$$\begin{aligned}\|E^0_{j,p} f\|_{\mathcal{H}^{k,\alpha}_{-4+\gamma}} &\le C(\varepsilon^\gamma + T_0^{\alpha/2})\|\varphi_{j,p} f\|_{\mathcal{H}^{k,\alpha}_{-4+\gamma}} \\ &\le C\varepsilon^{-\alpha}(\varepsilon^\gamma + T_0^{\alpha/2})\|f\|_{\mathcal{H}^{k,\alpha}_{-4+\gamma}} \\ &= C(\varepsilon^{\gamma-\alpha} + \varepsilon^{-\alpha} T_0^{\alpha/2})\|f\|_{\mathcal{H}^{k,\alpha}_{-4+\gamma}}\end{aligned}$$

Now set $\delta' := \delta / \big(\sum_j |L_j| N\big) > 0$ for a given $\delta > 0$. Then fix $\varepsilon > 0$ sufficiently small $C\varepsilon^{\gamma-\alpha} < \delta'/2$ (note that $\gamma - \alpha > 0$). For the fixed $\varepsilon > 0$ choose $T_0 > 0$ sufficiently small, such that $C\varepsilon^{-\alpha} T_0^{\alpha/2} < \delta'/2$. These choices yield $\|E^0\| < \delta$, and the proof is finished. $\square$

*Construction of an interior parametrix.* Next we construct an approximate interior parametrix. This construction is analogous to the one in [2], but for the convenience of the reader we repeat it here.

Recall that the radial function of the cone $x : \mathcal{C}(F) \to (0, 1)$ is extended smoothly to a nowhere vanishing function $x \in C^\infty(M)$. We assume that $x \ge 1$ outside of the singular neighborhood $\mathcal{C}(F)$. For $\varepsilon > 0$ small enough, $Y_\varepsilon := \{x \ge \varepsilon/2\} \subset M$ is a manifold with smooth boundary $\{\varepsilon/2\} \times F$. Let $\overline{Y}$ denote the double of $Y_\varepsilon$. Since the Riemannian metric on $\overline{Y}$ need not be smooth, we smoothen the metric in a small collar neighborhood of $\{\varepsilon/2\} \times F$, such that the metrics on $\overline{Y}$ and $M$ coincide over $Y_{2\varepsilon}$.

Since $\Phi \in C^\infty(\overline{M})$ is by construction identically 1 in $\{x < \varepsilon\}$, the function $1 - \Phi$ defines a smooth cutoff function on the closed double $\overline{Y}$, which is again denoted by $1 - \Phi$. Let $\overline{P}$ denote the extension of $P = \partial_t + a^{-1} L$ to a uniformly parabolic operator on $\overline{Y}$. Note that $\mathcal{H}^{k,\alpha}_{-4+\gamma} \restriction Y_\varepsilon \equiv \mathcal{C}^{k,\alpha}_{\mathrm{ie}}(Y_\varepsilon \times [0, T_0], S)$. Let $\widetilde{Q}_i : \mathcal{C}^{k,\alpha}_{\mathrm{ie}}(Y_\varepsilon \times [0, T_0], S) \to \mathcal{C}^{k+2,\alpha}_{\mathrm{ie}}(Y_\varepsilon \times [0, T_0], S)$ be the solution operator of the inhomogeneous Cauchy problem

$$\overline{P} u = (1 - \Phi) f, \quad u(\,\cdot\,, 0) = 0.$$

Also let $\chi$ be any smooth cutoff function which is identically 1 on $\operatorname{supp}(1 - \Phi)$. Then we define the interior parametrix for any $f \in \mathcal{H}^{k,\alpha}_{-4+\gamma}$ by

$$Q_i f = \chi \widetilde{Q}_i[(1 - \Phi) f].$$

*Construction of the parametrix.* From the approximate boundary and interior parametrices we obtain an approximate parametrix by

$$Qf := Q_b f + Q_i f$$

for $f \in \mathcal{H}^{k,\alpha}_{-4+\gamma}(M \times [0, T_0], S)$. From there we get a parametrix as follows.

*Proof of Theorem 7.5.* By Proposition 7.8 we conclude

$$(\partial_t + a^{-1}L)Qf = \Phi f + E^0 f + E^1 f + (1-\Phi)f + E^2 f =: f + Ef,$$

where $E^2 f := [\chi, a^{-1}L]\widetilde{Q}_i[(1-\Phi)f]$. Now similarly as in Lemma 7.3 and Corollary 7.4 we find that $\|E^2\| \to 0$ as $T_0 \to 0$. Hence we can choose $T_0$ sufficiently small, such that the error term $Ef := E^0 f + E^1 f + E^2 f$ has operator norm less than 1. Then $I + E$, as an operator on $\mathcal{H}^{k,\alpha}_{-4+\gamma}(M \times [0, T_0], S)$, is invertible, and we set

$$\mathcal{Q} := Q(I+E)^{-1}. \qquad \square$$

***Extension of parametrices to the full time interval.*** In this subsection we extend the existence results of Theorems 7.5 and 7.6 from the shorter time interval $[0, T_0]$ to the full time interval $[0, T]$. By Theorem 7.5, for any $f \in \mathcal{H}^{k,\alpha}_{-4+\gamma}(M \times [0, T], S)$ there exists a $T_0 \in (0, T]$ and a solution $u = \mathcal{Q}f \in \mathcal{H}^{k,\alpha}_{-2+\gamma}(M \times [0, T_0], S)$ to the parabolic initial value problem

$$(\partial_t + a^{-1}L)u = f, \quad u(t=0) = 0.$$

If $T_0 < T$, we consider the homogeneous Cauchy problem

$$(\partial_t + a^{-1}L)v_1 = 0, \quad v_1(t=0) = u(t=T_0),$$

where the initial data $u(t = T_0) \in \mathcal{H}^{k,\alpha}_{-2+\gamma}(M, S)$. By Theorem 7.6, the solution to this problem, $v_1 = \mathcal{R}v_1(t=0)$, exists on the time interval $[0, T_0]$ independent of the initial value $u(t = T_0)$. We may use Theorem 7.5 to solve

$$(\partial_t + a^{-1}L)u_1 = f(p, t+T_0), u_1(0) = 0,$$

on the interval $[T_0, 2T_0]$, and then the function

$$\widetilde{u}(p,t) = \begin{cases} u(p,t), & \text{for } 0 \le t \le T_0, \\ u_1(p, t-T_0) + v_1(p, t-T_0), & \text{for } T_0 < t \le 2T_0, \end{cases}$$

extends $u$ past $T_0$. This process continues until $nT_0 > T$, and produces a solution $u$ in $\mathcal{H}^{k,\alpha}_{-2+\gamma}(M \times [0, T], S)$. Thus Theorem 7.5 and hence also Theorem 7.6 hold on the full time interval $[0, T]$.

**Remark 7.9.** To see that the solution produced in this way indeed has the claimed regularity (in particular including Hölder regularity in time), we observe that instead of piecing together solutions on the time intervals $[0, T_0]$, $[T_0, 2T_0]$ etc. we could have instead chosen overlapping intervals $[0, T_0]$, $[T_0 - \varepsilon, 2T_0 - \varepsilon]$ etc. with a small $\varepsilon > 0$. Then these solutions agree on the overlaps by a uniqueness argument analogous to the proof of Theorem 7.10, based on the fact that the Friedrichs self adjoint extension $L$ is bounded from below.

***Regularity of Ricci curvature along the flow.*** We continue under the previously fixed notation where the superscript "$\sim$" indicates that the quantity is taken with respect to the initial metric. The following result proves our main Theorem 1.1.

**Theorem 7.10.** *Let $(M, g)$ be a tangentially stable conical manifold of dimension at least* 4, *with $(\alpha, \gamma, k+1)$ Hölder regular geometry. Let $g(t)$, $t \in [0, T]$ be the solution of the Ricci de Turck flow with initial metric $g$ and reference metric as in Theorem 5.1. Then, assuming $\gamma \in (0, 2+\min\{\mu_0, \mu_1\})$, if $\widetilde{\mathrm{Ric}} \in \mathcal{H}^{k,\alpha}_{-2+\gamma}(M \times [0, T], S)$, then* $\mathrm{Ric} \in \mathcal{H}^{k,\alpha}_{-2+\gamma}(M \times [0, T], S)$.

*Proof.* By Corollary 7.2 we know

$$(\partial_t + a^{-1} L)\,\mathrm{Ric} =: X(\mathrm{Ric}) \in \mathcal{C}^{k-1,\alpha}_{\mathrm{ie}}(M \times [0, T], S)_{-4+2\bar{\gamma}} \subseteq \mathcal{H}^{k,\alpha}_{-4+2\bar{\gamma}}(M \times [0, T], S).$$

Consider the parametrix $\mathcal{Q}$ of Theorem 7.5 and the parametrix $\mathcal{R}$ of Theorem 7.6. We set $\gamma' := \min\{\gamma, 2\bar{\gamma}\}$ and define

$$\mathrm{Ric}' := \mathcal{Q}(X(\mathrm{Ric})) + \mathcal{R}(\widetilde{\mathrm{Ric}}) \in \mathcal{H}^{k,\alpha}_{-2+\gamma'}(M \times [0, T], S), \tag{7-26}$$

which is a solution of the parabolic initial value problem

$$(\partial_t + a^{-1} L)\,\mathrm{Ric}' = X(\mathrm{Ric}), \quad \mathrm{Ric}(t = 0) = \widetilde{\mathrm{Ric}}. \tag{7-27}$$

We define $u := \mathrm{Ric}' - \mathrm{Ric}$. For $n \geq 3$ (recall that $\dim M \geq 4$) we can integrate by parts without boundary terms and conclude (recall $L = a^{1/2} \widetilde{\Delta} a^{-1/2}$)

$$\partial_t \|u\|^2_{L^2(M,g_0)} = -(a^{-1/2} \widetilde{\Delta} a^{-1/2} u, u)_{L^2(M,g_0)} = -\|\widetilde{\nabla}(a^{-1/2} u)\|^2_{L^2(M,g_0)} \leq 0.$$

Since $u(t = 0) \equiv 0$, we find that $u \equiv 0$. Hence we still conclude, despite having no maximum principle at hand

$$\mathrm{Ric} \equiv \mathrm{Ric}' \in \mathcal{H}^{k,\alpha}_{-2+\gamma'}(M \times [0, T], S).$$

We iterate the argument, improving the weight as long as $\gamma' < \gamma$. This proves the theorem. □

## 8. Positivity of scalar curvature along the Ricci de Turck flow

We can now prove our second main result on positivity of scalar curvature along the singular Ricci de Turck flow. At this final step, we will need a stronger tangential stability hypothesis. We impose, see Remark 5.2, the following additional

**Assumption 8.1.** We assume strong tangential stability: $u_0, u_1 > n$, i.e., $\Box_L > n$, so that we may choose $\gamma_0, \gamma_1 \geq 1$ satisfying (5-1). This stronger condition is studied in the Appendix, where a list of examples is provided. Amongst the symmetric spaces of compact type, only

$$E_8, \quad \mathrm{E}_7/[\mathrm{SU}(8)/\{\pm I\}], \quad \mathrm{E}_8/\mathrm{SO}(16), \quad \mathrm{E}_8/\mathrm{E}_7 \cdot \mathrm{SU}(2)$$

satisfy the strong tangential stability condition.[5]

The following theorem proves our second main result, Theorem 1.2.

**Theorem 8.2.** *Let $(M, g)$ be a strong tangentially stable conical manifold of dimension at least 4, with $(\alpha, \gamma, k+1)$ Hölder regular geometry, where we assume $\gamma > 3$. Let $g(t)$, $t \in [0, T]$ be the solution of the Ricci de Turck flow with initial metric $g$ and reference metric as in Theorem 5.1. Assume that $R_g \geq 0$. Then $R_{g(t)} \geq 0$ for all $t \in [0, T]$. Furthermore, if $R_g$ is positive at some point in the interior $M$, then $R_{g(t)}$ is positive in the interior $M$ for all $t \in (0, T]$.*

*Proof.* By Theorem 7.10, $\mathrm{Ric} \in \mathcal{H}^{k,\alpha}_{-2+\gamma'}(M \times [0, T], S)$, where $\gamma'$ is any weight smaller than $\{\gamma, 2 + \mu_0, 2 + \mu_1\}$. Due to Assumption 8.1 of strong tangential stability, we find in the evolution equation (6-18) for the scalar curvature along the flow

$$\partial_t R + \Delta R = \langle W, \nabla R \rangle + 2|\mathrm{Ric}|^2 \in \mathcal{C}^{\alpha}_{\mathrm{ie}}(M \times [0, T]). \tag{8-1}$$

We want to express the Laplace Beltrami operator $\Delta$ in terms of $a^{-1}\widetilde{\Delta}$, where $\widetilde{\Delta}$ denotes the Laplace Beltrami operator with respect to $\tilde{g} \equiv g(0)$. Consider first the Laplace Beltrami operator for $\hat{g} = a\tilde{g}$,

$$\begin{aligned}\Delta_{\hat{g}} &= -\frac{1}{\det \hat{g}} \partial_i (\sqrt{\det \hat{g}} \cdot \hat{g}^{ij} \partial_j) \\ &= a^{-1}\widetilde{\Delta} - \left(\frac{m}{2} - 1\right) \cdot \partial_i a \cdot a^{-2} \cdot \tilde{g}^{ij} \partial_j \\ &= a^{-1}\widetilde{\Delta} - \{x^{-1}\mathcal{V}_b a, a\} x^{-1}\mathcal{V}_b,\end{aligned}$$

where $\{x^{-1}\mathcal{V}_b a, a\}$ refers to a linear combination of monomials consisting of the terms in the brackets. If we take into account higher order terms with $g = a\tilde{g} \oplus \omega$, we obtain in the same notation and higher regularity of $R$

$$\begin{aligned}\Delta R - a^{-1}\widetilde{\Delta} R &= \{x^{-1}\mathcal{V}_b a, a\} x^{-1}\mathcal{V}_b R + \{x^{-1}\mathcal{V}_b\, \omega, x^{-1}\, \omega, \omega\} x^{-1}\mathcal{V}_b R + \{a, \omega\} x^{-1}\mathcal{V}_b^2 R \\ &\in \mathcal{C}^{\alpha}_{\mathrm{ie}}(M \times [0, T]).\end{aligned} \tag{8-2}$$

The relation above is similar to Theorem 7.1, where we note that the Lichnerowicz Laplacian acting on the pure trace component $S_1$ coincides with the Laplace Beltrami operator. Combining (8-1) and (8-2), we conclude

$$\partial_t R + a^{-1}\widetilde{\Delta} R = P \in \mathcal{C}^{\alpha}_{\mathrm{ie}}(M \times [0, T]). \tag{8-3}$$

By [2, Propositions 4.1 and 4.6] there exists a solution $R' \in \mathcal{C}^{\alpha}_{\mathrm{ie}}(M \times [0, T])$ of (8-3), for a given initial value $R(0) \in \mathcal{C}^{\alpha}_{\mathrm{ie}}(M)$, such that $\Delta R' \in \mathcal{C}^{\alpha}_{\mathrm{ie}}(M \times [0, T])$

[5] We hope to lift that restriction in the forthcoming work.

again. Exactly as in Theorem 7.10, we find $R = R'$. Hence the maximum principle obtained in [2, Theorem 3.1] applies to $R$ and, denoting by $R_{\min}(t)$ the minimum of the scalar curvature at time $t$, which is attained at $p_{\min}(t) \in \overline{M}$, we conclude

$$\begin{aligned}\partial_t R_{\min}(t) &\geq \langle W(p_{\min}(t)), \nabla R_{\min}(t)\rangle + 2|\mathrm{Ric}(p_{\min}(t))|^2 \\ &\geq \langle W(p_{\min}(t)), \nabla R_{\min}(t)\rangle\end{aligned}$$

Now, if $p_{\min}(t)$ lies in the open interior of $M$, then $\nabla R_{\min}(t) = 0$. If $p_{\min}(t)$ lies at the conical singularity, we argue as follows: note that $\langle W, \nabla R\rangle \in \mathcal{C}^{\alpha}_{\mathrm{ie}}(M \times [0, T])_{\gamma'-3}$. Since by strong tangential stability, we can choose $\gamma' > 3$, we find that $\langle W, \nabla R\rangle$ is vanishing at the conical singularity and hence

$$\partial_t R_{\min}(t) \geq \langle W(p_{\min}(t)), \nabla R_{\min}(t)\rangle = 0.$$

This implies that $R \geq 0$ for all $t \in [0, T]$. Now the last statement follows from the strong maximum principle [11, Theorem A.5]. □

## Appendix: Characterizing the spectral conditions

In this appendix, we aim to characterize strong tangential stability in terms of eigenvalues of geometric operators on the cross-section of a cone. Note that this condition is only used in the last part of our argument, proving Theorem 1.1. It is not used in our first main result Theorem 1.2 on higher regularity of Ricci curvature.

Note that the operator $\square_L$ can be entirely be described as an operator on the cross section $(F^n, g_F)$ of the cone. Therefore in this section, all scalar products and geometric operators are taken with respect to $(F^n, g_F)$. The operator $\Delta_E$ denotes the Einstein operator on symmetric two-tensors over $F$. It is given by

$$\Delta_E\, \omega_{ij} = \Delta\, \omega_{ij} - 2g^{pq}\, \mathrm{Rm}^r_{qij}\, \omega_{rp},$$

where $\Delta$ is the rough Laplacian. We write $\Delta$ for the Laplace Beltrami operator on $F$. Moreover, $TT$ denotes the space of symmetric two-tensors which are trace-free and divergence-free at each point.

**Theorem A.1.** *Let $(F^n, g_F)$, $n \geq 3$ be a compact Einstein manifold with constant $n-1$. Then $(F, g_F)$ is strongly tangentially stable if and only if we have the conditions*

$$\mathrm{Spec}(\Delta_E|_{TT}) > n, \quad \mathrm{Spec}(\Delta_{\Omega^1(F)\cap\ker(\mathrm{div})}) > n + \sqrt{n^2 + 2n + 1},$$

*and if for all positive eigenvalues $\lambda$ of the Laplace–Beltrami operator on functions satisfies*

$$\begin{aligned} &n(\lambda-3n+2)(\lambda+4-n)n(\lambda+n+2) \\ &\qquad -8n(n-1)(\lambda-n)(\lambda+n+2)-8\lambda n(n+1)(\lambda-3n+2) > 0.\end{aligned}$$

*Proof.* We first recall from the discussion before Remark 5.2 that strong tangential stability is equivalent to the two estimates

$$u_0 = \min(\operatorname{Spec} \square'_L \backslash \{0\}) > n, \quad u_1 = \min(\operatorname{Spec} \Delta_F \backslash \{0\}) > n.$$

However, the condition $u_1 > n$ holds for any Einstein metric except the sphere, where we have equality [17]. For the rest of the proof, it thus suffices to consider the bundle $S_0 \restriction F$. We use the same methodology as in [13] which builds up on a decomposition of symmetric 2-tensors established in [12]. We use the notation in [12, Section 2] and the calculations in Section 3 of the same paper where we remove all terms containing radial derivatives in order to obtain expressions for the tangential operator. More precisely, we write:

- $\{h_i\}$, a basis of $L^2(TT)$, $\Delta_E h_i = \kappa_i h_i$, $V_{1,i} := \langle r^2 h_i \rangle$.
- $\{\omega_i\}$, a basis of coclosed sections of $L^2(T^*F)$, $\Delta\omega_i = \mu_i \omega_i$, $V_{3,i} := \langle r^2 \delta^* \omega_i \rangle \oplus \langle dr \odot r\omega_i \rangle$.
- $\{v_i\}$, a basis of $L^2(F)$, $\Delta v_i = \lambda_i v_i$, $V_{4,i} := \langle r^2(n\nabla^2 v_i + \Delta v_i g)\rangle \oplus \langle dr \odot r \nabla v_i \rangle \oplus \langle v_i(r^2 g - n dr \otimes dr)\rangle$.

Here, $\Delta$ in $\Delta\omega_i$ denotes the connection Laplacian, while $\Delta$ in $\Delta v_i$ denotes the Laplace Beltrami operator. The spaces $V_{1,i}$, $V_{3,i}$, $V_{4,i}$, with $L^2(0, 1)$ coefficients, span all trace-free sections $L^2(S_0 \restriction F)$ over $F$, and are invariant under the action of the Lichnerowicz Laplacian. In [12, Section 2], there is also a notion for the spaces $V_{2,i} := \langle v_i(r^2 g + n dr \otimes dr)\rangle$. But these spaces span the full trace sections $L^2(S_1 \restriction F)$, whose discussion is not relevant here. At first, if $\tilde{h} = r^2 h_i \in V_{1,i}$,

$$(\square_L \tilde{h}, \tilde{h})_{L^2} = \kappa_i \|\tilde{h}\|_{L^2}$$

such that $\square_L > n$ on $V_{1,i}$ for all $i$ if and only if $\kappa_i > n$ for all eigenvalues of the Einstein operator on $TT$-tensors are positive (nonnegative).

Let now $\tilde{h} \in V_{3,i}$ so that it is of the form $\tilde{h} = \tilde{h}_1 + \tilde{h}_2 = \varphi r^2 \delta^* \omega_i + \psi dr \odot r\omega_i$. In this case, we have the scalar products

$$\begin{aligned}
(\square_L \tilde{h}_1, \tilde{h}_1)_{L^2} &= \tfrac{1}{2}\varphi^2(\mu_i - (n-1))^2, \\
(\square_L \tilde{h}_2, \tilde{h}_2)_{L^2} &= \psi^2[2\mu_i + (2n+6)], \\
(\square_L \tilde{h}_1, \tilde{h}_2)_{L^2} &= -2(\mu_i - (n-1))\psi\varphi.
\end{aligned}$$

Taking $r^2\delta^*\omega_i$ and $dr \odot r\omega_i$ as a basis, $\square_L$ respects the subspace and acts as $2 \times 2$-matrix

$$\begin{pmatrix} \frac{1}{2}(\mu_i - (n-1))^2 & -2(\mu_i - (n-1)) \\ -2(\mu_i - (n-1)) & 2\mu_i + (2n+6) \end{pmatrix}.$$

Because

$$\|\tilde{h}_1\|^2_{L^2} = \tfrac{1}{2}(\mu_i - (n-1)) \cdot |\varphi|^2, \quad \|\tilde{h}_2\|^2_{L^2} = 2|\psi|^2,$$

the operator $\Box_L - n\cdot\mathrm{id} > 0$ is represented by

$$A := \begin{pmatrix} \frac{1}{2}(\mu_i-(n-1))^2-\frac{n}{2}(\mu_i-(n-1)) & -2(\mu_i-(n-1)) \\ -2(\mu_i-(n-1)) & 2\mu_i+6 \end{pmatrix}.$$

The matrix $A$ is positive definite if and only if the matrix

$$B := \begin{pmatrix} \frac{1}{2}(\mu_i-(2n-1)) & -2(\mu_i-(n-1)) \\ -2 & 2\mu_i+6 \end{pmatrix}$$

is positive definite because $A$ is obtained from $B$ by multiplying the first column by $\mu_i-(n-1)$. By computing principal minors, this holds if

$$\det(B) = \tfrac{1}{2}(\mu_i-(2n-1))(2\mu_i+6)-4(\mu_i-(n-1)) > 0.$$

This in turn holds if

$$\mu_i > n+\sqrt{n^2+2n+1}.$$

Therefore, $\Box_L > n$ on the spaces $V_{3,i}$ if and only if $\Delta > n+\sqrt{n^2+2n+1}$ on coclosed sections $L^2(T^*F)$. It remains to consider the case $\tilde{h}\in V_{4,i}$, so that it is of the form

$$\tilde{h} = \tilde{h}_1+\tilde{h}_2+\tilde{h}_3 = \varphi r^2(n\nabla^2 v_i+\Delta v_i g)+\psi dr\odot r\nabla v_i+\mathcal{X} v_i(r^2 g-n dr\otimes dr).$$

This case is the most delicate one. We have the scalar products

$$\begin{aligned}
(\Box_L\tilde{h}_1,\tilde{h}_1) &= n(n-1)\lambda_i(\lambda_i-n)(\lambda_i-2(n-1))\varphi^2,\\
(\Box_L\tilde{h}_2,\tilde{h}_2) &= [2\lambda_i(\lambda_i-(n-1))+(2n+6)\lambda_i]\psi^2,\\
(\Box_L\tilde{h}_3,\tilde{h}_3) &= [n\{(n+1)\lambda_i-2(n+1)\}+2n^2(n+3)]\mathcal{X}^2,\\
(\Box_L\tilde{h}_1,\tilde{h}_2) &= -4(n-1)\lambda_i(\lambda_i-n)\psi\varphi,\\
(\Box_L\tilde{h}_2,\tilde{h}_3) &= 4(n+1)\lambda_i\psi\mathcal{X},\\
(\Box_L\tilde{h}_1,\tilde{h}_3) &= 0
\end{aligned}$$

and the norms

$$\begin{aligned}
\|\tilde{h}_1\|_{L^2}^2 &= n(n-1)\lambda_i(\lambda_i-n)\varphi^2,\\
\|\tilde{h}_2\|_{L^2}^2 &= 2\varphi^2\lambda_i,\\
\|\tilde{h}_3\|_{L^2}^2 &= (n+1)n.
\end{aligned}$$

Consider $(\Box_L - n\cdot \mathrm{id})$. It acts as a matrix $A=(a_{ij})_{1\le n\le 3}$, whose coefficients are given by

$$\begin{aligned}
a_{11} &= n(n-1)\lambda_i(\lambda_i-n)[\lambda_i-2(n-1)-n],\\
a_{22} &= 2\lambda_i[\lambda_i-(n-1)+3],\\
a_{33} &= n\{(n+1)\lambda_i-2(n-1)-n(n+1)+2n(n+3)\},\\
a_{12} &= a_{21} = -4(n-1)\lambda_i(\lambda_i-n),\\
a_{23} &= a_{32} = 4(n+1)\lambda_i,\\
a_{13} &= a_{31} = 0.
\end{aligned}$$

In order to prove positivity of this matrix, we consider its principal minors $A_{33}$ (which is the lower right entry), $A_{23}$ (the lower right $2\times 2$-matrix) and $A$ (the whole matrix). At first,

$$A_{33} = n\{(n+1)\lambda_i + n^2+3n+2\} > 0.$$

Observe that in the case $\lambda_i=0$, $\tilde{h}_1\equiv 0$ and $\tilde{h}_2\equiv 0$, so that $V_{4i}=\mathrm{span}\{\tilde{h}_3\}$ and hence, $(\Box_L - n\cdot\mathrm{id})$ acts as $A_{33}>0$. Therefore, we may from now on assume that $\lambda_i>0$, which means that actually $\lambda_i\ge n$ (due to eigenvalue estimates for Einstein manifolds, see, e.g., [17]) with $\lambda_i=n$ only for $\mathbb{S}^n$. By considering the matrix

$$\begin{pmatrix} 2(\lambda_i+4-n) & 4(n+1)\lambda_i \\ 4(n+1) & n\{(n+1)\lambda_i+n^2+3n+2\}\end{pmatrix},$$

from which one recovers $A_{23}$ by multiplying the first column by $\lambda_i$, we see that

$$\begin{aligned}
\frac{\det A_{23}}{\lambda_i} &= 2(\lambda_i+4-\epsilon)n\cdot[(n+1)\lambda_i+n^2+3n+2]-16(n+1)^2\lambda_i\\
&= 2n(n+1)\lambda_i^2-4(n+1)(n+4)\lambda_i-2n(n-4)(n^2+3n+2),
\end{aligned}$$

which is positive if

$$\lambda_i > \frac{n+4}{n}+\sqrt{\frac{n+4}{n}+(n-4)(n+2)}.$$

Here, the right hand side is smaller than $n$ if $n\ge 4$ such that this condition holds anyway. Before we compute the full determinant of $A$, we remark that in the case $\lambda_i=n$, the tensor $\tilde{h}_1$ is vanishing so that in this case, the matrix $A$ describing $\Box_L$ on $V_{4,i}$ reduces to the matrix $A_{23}$ which just has been considered. Therefore, there is nothing more to prove in this case and we may assume $\lambda=\lambda_i>n$ from now on. To compute the full determinant of $A$, we first consider the matrix

$$\begin{pmatrix} n[\lambda-2(n-1)-n] & -2(n-1)(\lambda-n) & 0\\ -4 & \lambda-(n-1)+3 & 4\lambda \\ 0 & 2(n+1) & n\{\lambda+2(n+1)-n\}\end{pmatrix}$$

| type | G | dim(G) | $\Lambda$ | $\Theta$ | STS |
|---|---|---|---|---|---|
| $\mathrm{A}_p$ | $\mathrm{SU}(p+1)$, $p \geq 2$ | $p^2-1$ | $\frac{2p(p+2)}{(p+1)^2}$ | $\frac{2p(p+2)}{(p+1)^2}$ | no |
| $\mathrm{B}_n$ | $\mathrm{Spin}(5)$ | 10 | $\frac{5}{3}$ | $\frac{4}{3}$ | no |
| | $\mathrm{Spin}(7)$ | 21 | $\frac{21}{10}$ | $\frac{12}{5}$ | no |
| | $\mathrm{Spin}(2p+1), n \geq 4$ | $2p(p+1)$ | $\frac{4p}{2p-1}$ | $\frac{4p}{2p-1}$ | no |
| $\mathrm{C}_p$ | $\mathrm{Sp}(p)$, $p \geq 3$ | $p(2p+1)$ | $\frac{2p+1}{p+1}$ | $\frac{4p-1}{2(p+1)}$ | no |
| $\mathrm{D}_p$ | $\mathrm{Spin}(2p)$, $p \geq 3$ | $p(2p+1)$ | $\frac{2p-1}{p-1}$ | $\frac{2p-1}{p-1}$ | no |
| $\mathrm{E}_6$ | $\mathrm{E}_6$ | 156 | $\frac{26}{9}$ | $\frac{17}{6}$ | no |
| $\mathrm{E}_7$ | $\mathrm{E}_7$ | 266 | $\frac{19}{6}$ | 3 | no |
| $\mathrm{E}_8$ | $\mathrm{E}_8$ | 496 | 4 | $\frac{47}{15}$ | yes |
| $\mathrm{F}_4$ | $\mathrm{F}_4$ | 52 | $\frac{8}{3}$ | $\frac{8}{3}$ | no |
| $\mathrm{G}_2$ | $\mathrm{G}_2$ | 14 | 2 | 2 | no |

**Table 1.** Conditions of Theorem A.1 for simple Lie groups.

from which we recover $A$ by multiplying the three columns by $(n-1)\lambda(\lambda-n)$ and $2\lambda$, $(n+1)$, respectively. We get

$$\begin{aligned}
&[(n-1)\lambda(\lambda-n)2\cdot\lambda\cdot(n+1)]^{-1}\det A \\
&\qquad = n(\lambda-3n+2)(\lambda+4-n)n(\lambda+n+2) \\
&\qquad\qquad - 8n(n-1)(\lambda-n)(\lambda+n+2) - 8\lambda n(n+1)(\lambda-3n+2)
\end{aligned}$$

which finishes the proof. □

**Theorem A.2.** *Amongst the symmetric spaces of compact type, only*

$$E_8, \quad \mathrm{E}_7/[\mathrm{SU}(8)/\{\pm I\}], \quad \mathrm{E}_8/\mathrm{SO}(16), \quad \mathrm{E}_8/\mathrm{E}_7\cdot\mathrm{SU}(2)$$

*are strongly tangentially stable.*

*Proof.* We merge and analyze Tables 2 and 3 in [12] and Tables 1 and 2 in [3]. In Tables 1–3, $\Lambda$ denotes the smallest nonzero eigenvalue of the Laplace–Beltrami operator divided by the Einstein constant $n-1$ and $\Theta$ denotes the smallest eigenvalue of the Lichnerowicz Laplacian $\Delta_L$ on symmetric 2-tensors with $\int_F \operatorname{tr} h\, dV = 0$ divided by the same constant $n-1$.

To check that the above mentioned spaces satisfy the condition of Theorem A.1, we check on one hand that the estimate for $\lambda$ in this theorem holds for $(n-1)\cdot\Lambda$. On the other hand, we have the relations $\Delta_L = \Delta_E + 2(n-1)\,\mathrm{id}$, $\delta^* \circ \Delta_H = \Delta_L \circ \delta^*$

| type | $G/K$ | $\dim(G/K)$ | $\Lambda$ | $\Theta$ | STS |
|---|---|---|---|---|---|
| A I | $\mathrm{SU}(p)/\mathrm{SO}(p)$, $5 \geq p \geq 3$ | $\frac{(p-1)(p+2)}{2}$ | $\frac{2(p-1)(p+2)}{p^2}$ | 2 | no |
| | $\mathrm{SU}(p)/\mathrm{SO}(p)$, $p \geq 6$ | $\frac{(p-1)(p+2)}{2}$ | $\frac{2(p-1)(p+2)}{p^2}$ | 2 | no |
| A II | $\mathrm{SU}(4)/\mathrm{Sp}(2) = S^5$ | 5 | $\frac{5}{4}$ | 3 | no |
| | $\mathrm{SU}(2p)/\mathrm{Sp}(p)$, $p \geq 3$ | $2p^2 - p - 1$ | $\frac{(2p+1)(p-1)}{p^2}$ | 2 | no |
| A III | $\frac{\mathrm{U}(p+1)}{\mathrm{U}(p)\times\mathrm{U}(1)} = \mathbb{CP}^p$ | $2p$ | 2 | 2 | no |
| | $\frac{\mathrm{U}(p+q)}{\mathrm{U}(q)\times\mathrm{U}(p)}$, $q \geq p \geq 2$ | $2pq$ | 2 | 2 | no |
| B I | $\frac{\mathrm{SO}(5)}{\mathrm{SO}(3)\times SO(2)}$ | 6 | 2 | $\frac{4}{3}$ | no |
| | $\frac{\mathrm{SO}(2p+3)}{\mathrm{SO}(2p+1)\times\mathrm{SO}(2)}$, $p \geq 2$ | $4p+2$ | 2 | $\frac{8}{2p+1}$ | no |
| | $\frac{\mathrm{SO}(7)}{\mathrm{SO}(4)\times\mathrm{SO}(3)}$ | 12 | $\frac{12}{5}$ | $\frac{8}{5}$ | no |
| | $\frac{\mathrm{SO}(2p+3)}{\mathrm{SO}(3)\times\mathrm{SO}(2p)}$, $p \geq 3$ | $6p$ | $\frac{4p+6}{2p+1}$ | $\frac{8}{2q+1}$ | no |
| | $\frac{\mathrm{SO}(2q+2p+1)}{\mathrm{SO}(2q+1)\times\mathrm{SO}(2p)}$, $p, q \geq 2$ | $2n(2m+1)$ | $\frac{4m+4n+2}{2m+2n-1}$ | $\frac{8}{2p+2q-1}$ | no |
| B II | $\frac{\mathrm{SO}(2p+1)}{\mathrm{SO}(2p)} = \mathbb{S}^{2p}$, $p \geq 1$ | $2p$ | $\frac{2p}{2p-1}$ | $\frac{4p+2}{2p-1}$ | no |
| C I | $\mathrm{Sp}(p)/\mathrm{U}(p)$, $p \geq 3$ | $p(p+1)$ | 2 | $\frac{2p}{p+1}$ | no |
| C II | $\frac{\mathrm{Sp}(2)}{\mathrm{Sp}(1)\times\mathrm{Sp}(1)} = \mathbb{S}^4$ | 4 | $\frac{4}{3}$ | $\frac{10}{3}$ | no |
| | $\frac{\mathrm{Sp}(p+1)}{\mathrm{Sp}(p)\times\mathrm{Sp}(1)} = \mathbb{HP}^p$, $p \geq 2$ | $4p$ | $\frac{2(p+1)}{p+2}$ | $\frac{2(p+1)}{p+2}$ | no |
| | $\frac{\mathrm{Sp}(p+q)}{\mathrm{Sp}(q)\times\mathrm{Sp}(p)}$, $q \geq p \geq 2$ | $4pq$ | $\frac{2(p+q)}{p+q+1}$ | $\frac{2(p+q)}{p+q+1}$ | no |
| D I | $\frac{\mathrm{SO}(8)}{\mathrm{SO}(5)\times\mathrm{SO}(3)}$ | 15 | $\frac{5}{2}$ | $\frac{5}{2}$ | no |
| | $\frac{\mathrm{SO}(2p+2)}{\mathrm{SO}(2p)\times\mathrm{SO}(2)}$, $p \geq 3$ | $4p$ | 2 | 2 | no |
| | $\frac{\mathrm{SO}(2p)}{\mathrm{SO}(p)\times\mathrm{SO}(p)}$, $p \geq 4$ | $p^2$ | $\frac{2p}{p-1}$ | $\frac{2p}{p-1}$ | no |
| | $\frac{\mathrm{SO}(2p+2)}{\mathrm{SO}(p+2)\times\mathrm{SO}(p)}$, $p \geq 4$ | $p(p+2)$ | $\frac{2p+2}{p}$ | $\frac{2p+2}{p}$ | no |
| | $\frac{\mathrm{SO}(2p)}{\mathrm{SO}(2p-q)\times\mathrm{SO}(q)}$, $p-2 \geq q \geq 3$ | $(2p-q)q$ | $\frac{2p}{p-1}$ | $\frac{2p}{p-1}$ | no |
| D II | $\frac{\mathrm{SO}(2p+2)}{\mathrm{SO}(2p+1)} = S^{2p+1}$, $p \geq 3$ | $2p+1$ | $\frac{2p+1}{2p}$ | $\frac{2(p+1)}{p}$ | no |
| D III | $\mathrm{SO}(2p)/\mathrm{U}(p)$, $p \geq 5$ | $p(p-1)$ | 2 | 2 | no |
| E I | $\mathrm{E}_6/[\mathrm{Sp}(4)/\{\pm I\}]$ | 42 | $\frac{28}{9}$ | 3 | no |
| E II | $\mathrm{E}_6/\mathrm{SU}(2)\cdot\mathrm{SU}(6)$ | 40 | 3 | 3 | no |
| E III | $\mathrm{E}_6/\mathrm{SO}(10)\cdot\mathrm{SO}(2)$ | 32 | 2 | 2 | no |

**Table 2.** *Continued in Table 3.*

| type | $G/K$ | $\dim(G/K)$ | $\Lambda$ | $\Theta$ | STS |
|---|---|---|---|---|---|
| E IV | $\mathrm{E}_6/\mathrm{F}_4$ | 26 | $\frac{13}{9}$ | $\frac{13}{9}$ | no |
| E V | $\mathrm{E}_7/[\mathrm{SU}(8)/\{\pm I\}]$ | 70 | $\frac{10}{3}$ | $\frac{28}{9}$ | yes |
| E VI | $\mathrm{E}_7/\mathrm{SO}(12)\cdot\mathrm{SU}(2)$ | 64 | $\frac{28}{9}$ | $\frac{28}{9}$ | no |
| E VII | $\mathrm{E}_7/\mathrm{E}_6\cdot\mathrm{SO}(2)$ | 54 | 2 | 2 | no |
| E VIII | $\mathrm{E}_8/\mathrm{SO}(16)$ | 128 | $\frac{62}{15}$ | $\frac{16}{5}$ | yes |
| E IX | $\mathrm{E}_8/\mathrm{E}_7\cdot\mathrm{SU}(2)$ | 112 | $\frac{16}{5}$ | $\frac{16}{5}$ | yes |
| F I | $\mathrm{F}_4/Sp(3)\cdot\mathrm{SU}(2)$ | 28 | $\frac{26}{9}$ | $\frac{26}{9}$ | no |
| F II | $\mathrm{F}_4/\mathrm{Spin}(9)$ | 16 | $\frac{4}{3}$ | $\frac{4}{3}$ | no |
| G | $\mathrm{G}_2/\mathrm{SO}(4)$ | 8 | $\frac{7}{3}$ | $\frac{7}{3}$ | no |

**Table 3.** Conditions of Theorem A.1 for symmetric spaces of nongroup type.

and $\Delta_H = \Delta + (n-1)\,\mathrm{id}$. Therefore, the condition $\mathrm{Spec}(\Delta_E|_{TT}) > n$ holds, if $(n-1)\Theta - 2(n-1) > n$ and the condition $\mathrm{Spec}(\Delta_{\Omega^1(F)\cap\ker(\mathrm{div})}) > n+\sqrt{n^2+2n+1}$ holds if $(n-1)\Theta > 2n-1+\sqrt{n^2+2n+1}$.

To show that all the other examples do not satisfy $\Box_L > n$, we proceed as follows: We check that $(n-1)\Lambda$ satisfies

$$\Lambda \le \frac{n+4}{n} + \sqrt{\frac{n+4}{n} + (n-4)(n+2)}$$

or

$$\begin{aligned} &n(\Lambda-3n+2)(\Lambda+4-n)n(\Lambda+n+2)\\ &\qquad - 8n(n-1)(\Lambda-n)(\Lambda+n+2) - 8\Lambda n(n+1)(\Lambda-3n+2) \le 0. \end{aligned}$$

This holds for example, if $\Lambda \le 3$. Because $\Delta_L(f\cdot g_F) = \Delta f\cdot g_F$ and $\Delta_L\circ\nabla^2 = \nabla^2\circ\Delta$, we clearly have $\Theta \le \Lambda$. Because of the decomposition

$$C^\infty(\mathrm{Sym}^2(T^*F)) = C^\infty(F)\cdot g_F \oplus \nabla^2(C^\infty(F)) \oplus \delta^*(\Omega^1(F)\cap\ker(\mathrm{div})) \oplus TT,$$

(see [12]) $\Theta$ is attained on

$$\delta^*(\Omega^1(F)\cap\ker(\mathrm{div})) \oplus TT$$

if $\Theta < \Lambda$. In this case, we check if $\Theta$ satisfies $(n-1)\Theta - 2(n-1) \le n$ and $(n-1)\Theta < 2n-1+\sqrt{n^2+2n+1}$ (both conditions are satisfied if $\Theta \le 3$). Due to the relations above, this implies that either $\mathrm{Spec}(\Delta_E|_{TT}) > n$ or $\mathrm{Spec}(\Delta_{\Omega^1(F)\cap\ker(\mathrm{div})}) > n + \sqrt{n^2+2n+1}$ fails so that $\square_L > n$ fails.

Therefore, the condition $\mathrm{Spec}(\Delta_E|_{TT}) > n$ holds, if $(n-1)\Theta - 2(n-1) > n$ and the condition $\mathrm{Spec}(\Delta_{\Omega^1(F)\cap\ker(\mathrm{div})}) > n + \sqrt{n^2+2n+1}$ holds if $(n-1)\Theta > 2n-1+\sqrt{n^2+2n+1}$.

To finish the proof, one just has to go through all the values of $\Theta$ and $\Lambda$ in Table 1. Strong tangential stability is abbreviated by STS. As was already said, for the spaces mentioned in the statement of the theorem, one manually checks the estimates above to verify that the conditions of Theorem A.1 are satisfied. In all the other cases, one finds $\Theta \leq 3$ except in the case $\mathrm{E}_7/\,\mathrm{SO}(12)\cdot\mathrm{SU}(2)$ where one manually checks the condition on $\Lambda$ mentioned above. This finishes the proof of the theorem.

In the case of irreducible rank-1 symmetric spaces of compact type, an analogous argumentation yields Tables 2 and 3. □

## References


[1] E. Bahuaud and B. Vertman, "Yamabe flow on manifolds with edges", *Math. Nachr.* **287**:2-3 (2014), 127–159. MR Zbl

[2] E. Bahuaud and B. Vertman, "Long-time existence of the edge Yamabe flow", *J. Math. Soc. Japan* **71**:2 (2019), 651–688. MR Zbl

[3] H.-D. Cao and C. He, "Linear stability of Perelman's $\nu$-entropy on symmetric spaces of compact type", *J. Reine Angew. Math.* **709** (2015), 229–246. MR Zbl

[4] X. Chen and Y. Wang, "Bessel functions, heat kernel and the conical Kähler–Ricci flow", *J. Funct. Anal.* **269**:2 (2015), 551–632. MR Zbl

[5] B. Chow and D. Knopf, *The Ricci flow: an introduction*, Mathematical Surveys and Monographs **110**, American Mathematical Society, Providence, RI, 2004. MR Zbl

[6] S. K. Donaldson, "Kähler metrics with cone singularities along a divisor", pp. 49–79 in *Essays in mathematics and its applications*, edited by P. M. Pardalos and T. M. Rassias, Springer, 2012. MR Zbl

[7] G. Giesen and P. M. Topping, "Ricci flow of negatively curved incomplete surfaces", *Calc. Var. Partial Differential Equations* **38**:3-4 (2010), 357–367. MR Zbl

[8] G. Giesen and P. M. Topping, "Existence of Ricci flows of incomplete surfaces", *Comm. Partial Differential Equations* **36**:10 (2011), 1860–1880. MR Zbl

[9] T. Jeffres, R. Mazzeo, and Y. A. Rubinstein, "Kähler–Einstein metrics with edge singularities", *Ann. of Math.* (2) **183**:1 (2016), 95–176. MR Zbl

[10] K. Kirsten, P. Loya, and J. Park, "Functional determinants for general self-adjoint extensions of Laplace-type operators resulting from the generalized cone", *Manuscripta Math.* **125**:1 (2008), 95–126. MR Zbl

[11] B. Kleiner and J. Lott, "Notes on Perelman's papers", *Geom. Topol.* **12**:5 (2008), 2587–2855. MR Zbl

[12] K. Kröncke, "Stable and unstable Einstein warped products", *Trans. Amer. Math. Soc.* **369**:9 (2017), 6537–6563. MR Zbl

[13] K. Kröncke and B. Vertman, "Stability of Ricci de Turck flow on singular spaces", *Calc. Var. Partial Differential Equations* **58**:2 (2019), Paper No. 74, 40. MR Zbl

[14] K. Kröncke and B. Vertman, "Perelman's entropies for manifolds with conical singularities", *Trans. Amer. Math. Soc.* **374**:4 (2021), 2873–2908. MR Zbl

[15] J. Liu and X. Zhang, "Conical Kähler–Ricci flows on Fano manifolds", *Adv. Math.* **307** (2017), 1324–1371. MR Zbl

[16] R. Mazzeo, Y. A. Rubinstein, and N. Sesum, "Ricci flow on surfaces with conic singularities", *Anal. PDE* **8**:4 (2015), 839–882. MR Zbl

[17] M. Obata, "Certain conditions for a Riemannian manifold to be isometric with a sphere", *J. Math. Soc. Japan* **14** (1962), 333–340. MR Zbl

[18] F. Riesz and B. Sz.-Nagy, *Functional analysis*, Dover Publications, New York, 1990. MR

[19] M. Simon, "Local smoothing results for the Ricci flow in dimensions two and three", *Geom. Topol.* **17**:4 (2013), 2263–2287. MR Zbl

[20] G. Tian, "K-stability and Kähler–Einstein metrics", *Comm. Pure Appl. Math.* **68**:7 (2015), 1085–1156. MR Zbl

[21] B. Vertman, "Ricci de Turck flow on singular manifolds", *J. Geom. Anal.* **31**:4 (2021), 3351–3404. MR Zbl

[22] Y. Wang, "Smooth approximations of the conical Kähler–Ricci flows", *Math. Ann.* **365**:1-2 (2016), 835–856. MR Zbl

[23] H. Yin, "Ricci flow on surfaces with conical singularities", *J. Geom. Anal.* **20**:4 (2010), 970–995. MR Zbl

Received May 4, 2022. Revised May 6, 2023.

KLAUS KRÖNCKE
DEPARTMENT OF MATHEMATICS
KTH STOCKHOLM
STOCKHOLM
SWEDEN
kroncke@kth.se

TOBIAS MARXEN
DEPARTMENT OF MATHEMATICS
UNIVERSITY OLDENBURG
OLDENBURG
GERMANY
tobias.marxen@uni-oldenburg.de

BORIS VERTMAN
INSTITUT FUR MATHEMATIK
UNIVERSITÄT OLDENBURG
OLDENBURG
GERMANY
boris.vertman@uni-oldenburg.de

PACIFIC JOURNAL OF MATHEMATICS
Vol. 324, No. 2, 2023
https://doi.org/10.2140/pjm.2023.324.333

# POLYNOMIAL DEDEKIND DOMAINS WITH FINITE RESIDUE FIELDS OF PRIME CHARACTERISTIC

GIULIO PERUGINELLI

*To the everlasting memory of Robert Gilmer*

**We show that every Dedekind domain $R$ lying between the polynomial rings $\mathbb{Z}[X]$ and $\mathbb{Q}[X]$ with the property that its residue fields of prime characteristic are finite fields is equal to a generalized ring of integer-valued polynomials; that is, for each prime $p \in \mathbb{Z}$ there exists a finite subset $E_p$ of transcendental elements over $\mathbb{Q}$ in the absolute integral closure $\overline{\mathbb{Z}_p}$ of the ring of $p$-adic integers such that $R = \{f \in \mathbb{Q}[X] \mid f(E_p) \subseteq \overline{\mathbb{Z}_p},$ for each prime $p \in \mathbb{Z}\}$. Moreover, we prove that the class group of $R$ is isomorphic to a direct sum of a countable family of finitely generated abelian groups. Conversely, any group of this kind is the class group of a Dedekind domain $R$ between $\mathbb{Z}[X]$ and $\mathbb{Q}[X]$.**

## 1. Introduction

Given a Dedekind domain $D$, the class group of $D$ measures how far $D$ is from being a UFD and it is therefore an important object in the study of factorization problems in the ring $D$. It is well-known that the class group of the ring of integers of a number field is a finite abelian group. In contrast with this result, Claborn [1966] proved the groundbreaking result that every abelian group occurs as the class group of a suitable Dedekind domain.

Eakin and Heinzer [1973] showed that every finitely generated abelian group is the class group of a Dedekind domain between $\mathbb{Z}[X]$ and $\mathbb{Q}[X]$. More generally, they proved that if $V_1, \ldots, V_n$ are distinct DVRs with same quotient field $K$ and, for each $i = 1, \ldots, n$, $\{V_{i,j}\}_{j=1}^{g_i}$ is a finite collection of DVRs extending $V_i$ to $K(X)$, each of which is residually algebraic over $V_i$ (i.e., the extension of the residue fields is algebraic), then

$$R = \bigcap_{i,j} V_{i,j} \cap K[X]$$

is a Dedekind domain. They also give an explicit description of the class group of such a domain $R$, thanks to which they showed the quoted result by considering

*MSC2020:* 13B25, 13F05, 13F20, 20K99.
*Keywords:* Dedekind domain, class group, integer-valued polynomials.

suitable residually algebraic extensions of a finite set of DVRs of $\mathbb{Q}$ to $\mathbb{Q}(X)$.

Actually, if we suppose that each residue field extension of $V_{i,j}$ over $V_i$ is finite, a ring $R$ constructed as above can be represented as a ring of integer-valued polynomials in the following way. For each $i, j$, by [Peruginelli 2017, Theorem 2.5 and Proposition 2.2], there exists an element $\alpha_{i,j}$ in the algebraic closure $\overline{\widehat{K}_i}$ of the $V_i$-adic completion $\widehat{K}_i$ of $K$, $\alpha_{i,j}$ transcendental over $K$, such that

$$V_{i,j} = V_{i,\alpha_{i,j}} = \{\varphi \in K(X) \mid \varphi(\alpha_{i,j}) \in \overline{\widehat{V}_i}\},$$

where $\overline{\widehat{V}_i}$ is the absolute integral closure of $\widehat{V}_i$, the completion of $V_i$. Hence, the above ring $R$ can be represented as $R = \{f \in K[X] \mid f(\alpha_{i,j}) \in \overline{\widehat{V}_i}, \forall i, j\}$ (for more details, see [Peruginelli 2017, Remark 2.8]).

More recently, Glivický and Šaroch [2013] investigated a family of quasieuclidean subrings of $\mathbb{Q}[X]$ depending on a parameter $\alpha \in \widehat{\mathbb{Z}}$, the profinite completion of $\mathbb{Z}$. A ring of this family is always a Bézout domain (i.e., finitely generated ideals are principal) and might be a PID or not, according to the finiteness of some set of primes depending on $\alpha$ and the set of polynomials in $\mathbb{Z}[X]$. Glivická et al. [2023] observed that these rings can be realized as overrings of the classical ring of integer-valued polynomials $\mathrm{Int}(\mathbb{Z}) = \{f \in \mathbb{Q}[X] \mid f(\mathbb{Z}) \subseteq \mathbb{Z}\}$, which is a two-dimensional nonnoetherian Prüfer domain; such overrings have been completely characterized in [Chabert and Peruginelli 2016]. We will review this representation in Section 2.

In the same area, Chang [2022] generalized Eakin and Heinzer's result, proving that there exists an almost Dedekind domain $R$ (i.e., $R_M$ is a DVR for each maximal ideal $M$ of $R$) which is not noetherian, lies between $\mathbb{Z}[X]$ and $\mathbb{Q}[X]$ and has class group isomorphic to a direct sum of a prescribed countable family of finitely generated abelian groups. As before, assuming the finiteness of the residue field extensions of the involved DVRs, Chang's construction falls in the class of integer-valued polynomial rings that we consider in this paper.

Here, we provide a complete description of the class of Dedekind domains $R$ lying between $\mathbb{Z}[X]$ and $\mathbb{Q}[X]$ such that their residue fields of prime characteristic are finite fields. Throughout the paper, for short we denote the last property by saying that $R$ has finite residue fields of prime characteristic. We remark that the residue fields of such a domain $R$ cannot be all finite fields. In fact, since $R \subseteq \mathbb{Q}[X]_{(q)}$ for every irreducible $q \in \mathbb{Q}[X]$, the residue field of the center of the DVR $\mathbb{Q}[X]_{(q)}$ on $R$ is a finite extension of $\mathbb{Q}$, hence an infinite field. However, since $R$ is supposed to be Dedekind (in particular, a Prüfer domain) the residue fields of prime characteristic are algebraic extensions of the corresponding prime field (see, for example, [Peruginelli 2018, Theorem 3.14]). Infinite algebraic extensions of the prime fields of prime characteristic are also allowed, and that is the content of another work on this subject [Peruginelli 2023].

The paper is organized as follows. We first set the notation we will use throughout the paper and introduce the class of *generalized rings of integer-valued polynomials*, which are subrings of $\mathbb{Q}[X]$ formed by polynomials which are simultaneously integer-valued over different subsets of integral elements over $\mathbb{Z}_p$, the ring of $p$-adic integers, for $p$ running over the set of integer primes. In Section 2, we review Loper and Werner's construction [2012] of Prüfer domains and recall that it falls into the class of generalized rings of integer-valued polynomials, as already observed in [Peruginelli 2017, Remark 2.8]. We then characterize when a ring of their construction is a Dedekind domain in Theorem 2.15. In order to accomplish this objective, we introduce the definition of *polynomially factorizable* subsets $\underline{E}$ of $\widehat{\overline{\mathbb{Z}}} = \prod_p \overline{\mathbb{Z}_p}$ (we refer to Section 1 for unexplained notation), which turns out to be the key assumption for such a ring to be of finite character (hence, a noetherian Prüfer domain, thus Dedekind). Furthermore, we show in Theorem 2.17 that every Dedekind domain $R$ with finite residue fields of prime characteristic lying between $\mathbb{Z}[X]$ and $\mathbb{Q}[X]$ is equal to a generalized ring of integer-valued polynomials with class group equal to a direct sum of a countable family of finitely generated abelian groups (Recall that the Picard group of $\operatorname{Int}(\mathbb{Z})$ is a free abelian group of countably infinite rank [Gilmer et al. 1990]). Among other things, we will also characterize the PIDs among these class of domains, generalizing the aforementioned work of Glivický and Šaroch [2013] (see also [Glivická et al. 2023]). We will also give a criteria for when two such generalized rings of integer-valued polynomials are equal. Finally, in Section 3, by means of a suitable modification of Chang's construction, given a group $G$ which is the direct sum of a countable family of finitely generated abelian groups, we prove that there exists a Dedekind domain $R$ with finite residue fields of prime characteristic, $\mathbb{Z}[X] \subset R \subseteq \mathbb{Q}[X]$, with class group $G$, thus giving a positive answer to a question raised by Chang [2022]. By the previous results, such a domain is a generalized ring of integer-valued polynomials.

It has come to our attention that Theorem 7 of [Chang and Geroldinger 2024] shows the existence of a Dedekind domain with class group equal to a direct sum of a countable family of prescribed finitely generated abelian groups. However, that construction is based on a polynomial ring with an infinite set of indeterminates with the additional property that each ideal class contains infinitely many height-one prime ideals.

***Notation.*** The generalized rings of integer-valued polynomials considered in this paper fall into the class of integer-valued polynomials on algebras (see for example [Frisch 2013; 2014; Peruginelli and Werner 2017]), which encompasses also the classical definition of ring of integer-valued polynomials. We now recall the latter definition. Let $D$ be an integral domain with quotient field $K$ and $A$ a torsion-free $D$-algebra such that $A \cap K = D$. We may evaluate polynomials $f \in K[X]$ at any element $a \in A$ inside the extended algebra $A \otimes_D K$. The $D$-algebra $A$ clearly embeds into $A \otimes_D K$ and if $f(a) \in A$ we say that $f$ is integer-valued at $a$. In general, given a subset $S$ of $A$, we define the ring of integer-valued polynomials over $S$ as

$$\mathrm{Int}_K(S, A) = \{f \in K[X] \mid f(s) \in A, \forall s \in S\}.$$

Note that when $A = D$ we get the usual definition of ring of integer-valued polynomials on a subset $S$ of $D$, and in that case we omit the subscript $K$. If $S = D = A$, then we set $\mathrm{Int}(D, D) = \mathrm{Int}(D)$.

For an integral domain $D$, we define the Picard group of $D$, denoted by $\mathrm{Pic}(D)$, as the quotient of the abelian group of the invertible fractional ideals of $D$ by the subgroup generated by the nonzero principal fractional ideals, where the operation is the ideal multiplication (see [Cahen and Chabert 1997, §VIII.1]). If $D$ is a Dedekind domain, then $\mathrm{Pic}(D)$ is the usual ideal class group of $D$.

Let $\mathbb{P}$ be the set of all prime numbers. For a fixed $p \in \mathbb{P}$, we adopt the following notation:

- $\mathbb{Z}_{(p)}$ denotes the localization of $\mathbb{Z}$ at $p\mathbb{Z}$.
- $\mathbb{Z}_p$ and $\mathbb{Q}_p$ denote the ring of $p$-adic integers and the field of $p$-adic numbers, respectively.
- $\overline{\mathbb{Q}_p}$ and $\overline{\mathbb{Z}_p}$ denote a fixed algebraic closure of $\mathbb{Q}_p$ and the absolute integral closure of $\mathbb{Z}_p$, respectively.
- For a finite extension $K$ of $\mathbb{Q}_p$, we denote by $O_K$ the ring of integers of $K$.
- $v_p$ denotes the unique extension of the $p$-adic valuation on $\mathbb{Q}_p$ to $\overline{\mathbb{Q}_p}$.
- If $\alpha \in \overline{\mathbb{Q}_p}$, we denote the ramification index $e(\mathbb{Q}_p(\alpha) \mid \mathbb{Q}_p)$ by $e_\alpha$.
- $\widehat{\mathbb{Z}} = \prod_{p \in \mathbb{P}} \mathbb{Z}_p$, the profinite completion of $\mathbb{Z}$.
- $\overline{\widehat{\mathbb{Z}}} = \prod_{p \in \mathbb{P}} \overline{\mathbb{Z}_p}$.
- For $\alpha \in \overline{\mathbb{Q}_p}$, we set

$$V_{p,\alpha} = \{\varphi \in \mathbb{Q}(X) \mid \varphi(\alpha) \in \overline{\mathbb{Z}_p}\}.$$

Clearly, $V_{p,\alpha}$ is a valuation domain of $\mathbb{Q}(X)$ extending $\mathbb{Z}_{(p)}$ with maximal ideal equal to $M_{p,\alpha} = \{\varphi \in V_{p,\alpha} \mid v_p(\varphi(\alpha)) > 0\}$. Moreover, $V_{p,\alpha}$ is a DVR if $\alpha$ is transcendental over $\mathbb{Q}$ and it has rank 2 otherwise. In the former case, the ramification index $e(V_{p,\alpha} \mid \mathbb{Z}_{(p)})$ is equal to $e_\alpha$. In either case, let $O_\alpha$ and $M_\alpha$ be the valuation domain and maximal ideal of $\mathbb{Q}_p(\alpha)$, respectively. Then, the residue field of $V_{p,\alpha}$ is equal to $O_\alpha/M_\alpha$ and $pO_\alpha = M_\alpha^e$, for some integer $e$, which is equal to $e_\alpha$ (for all these results, see [Peruginelli 2017, Proposition 2.2 and Theorem 2.5]).

The following result, mentioned in the introduction, characterizes residually algebraic extensions of $\mathbb{Z}_{(p)}$ to $\mathbb{Q}(X)$ of a certain kind; the valuation overrings of the Dedekind domains we are dealing with belong to this class.

**Theorem 1.1** [Peruginelli 2017, Theorems 2.5 and 3.2]. *Let $W \subset \mathbb{Q}(X)$ be a valuation domain with maximal ideal $M$ extending $\mathbb{Z}_{(p)}$ for some $p \in \mathbb{P}$. If $pW = M^e$ for some $e \geq 1$ and $W/M \supseteq \mathbb{Z}/p\mathbb{Z}$ is a finite extension, then there exists $\alpha \in \overline{\mathbb{Q}_p}$ such that $W = V_{p,\alpha}$. Moreover, for $\alpha, \beta \in \overline{\mathbb{Q}_p}$, we have $V_{p,\alpha} = V_{p,\beta}$ if and only if $\alpha, \beta$ are conjugate over $\mathbb{Q}_p$.*

Clearly, if $W$ is as in the assumptions of Theorem 1.1 and $\mathbb{Z}[X] \subset W$, then $\alpha \in \overline{\mathbb{Z}_p}$.

Given $f \in \mathbb{Q}[X]$, the evaluation of $f(X)$ at an element $\alpha = (\alpha_p) \in \overline{\widehat{\mathbb{Z}}}$ is done componentwise:

$$f(\alpha) = (f(\alpha_p)) \in \prod_{p \in \mathbb{P}} \overline{\mathbb{Q}_p}.$$

We say that $f$ is *integer-valued* at $\alpha$ if $f(\alpha) \in \overline{\widehat{\mathbb{Z}}}$, which is equivalent to $f \in V_{p,\alpha_p}$ for all $p \in \mathbb{P}$.

**Definition 1.2.** Given a subset $\underline{E}$ of $\overline{\widehat{\mathbb{Z}}}$, we define the *generalized ring of integer-valued polynomials on* $\underline{E}$ as:

$$\mathrm{Int}_{\mathbb{Q}}(\underline{E}, \overline{\widehat{\mathbb{Z}}}) = \{f \in \mathbb{Q}[X] \mid f(\alpha) \in \overline{\widehat{\mathbb{Z}}}, \forall \alpha \in \underline{E}\}.$$

If $\underline{E} = \widehat{\mathbb{Z}}$, then $\mathrm{Int}_{\mathbb{Q}}(\widehat{\mathbb{Z}}, \overline{\widehat{\mathbb{Z}}}) = \mathrm{Int}_{\mathbb{Q}}(\widehat{\mathbb{Z}}, \widehat{\mathbb{Z}}) = \mathrm{Int}(\mathbb{Z})$; in fact, the first equality follows easily from the fact that the polynomials have rational coefficients; for the last equality, see [Chabert and Peruginelli 2016, Remark 6.4] (essentially, $\mathbb{Z}$ is dense in $\widehat{\mathbb{Z}}$). We recall that the family of overrings of $\mathrm{Int}(\mathbb{Z})$ which are contained in $\mathbb{Q}[X]$ is formed exactly by the rings $\mathrm{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\mathbb{Z}})$, as $\underline{E}$ ranges through the subsets of $\widehat{\mathbb{Z}}$ of the form $\prod_{p \in \mathbb{P}} E_p$, where for each prime $p$, $E_p$ is a closed (possibly empty) subset of $\mathbb{Z}_p$ [Theorem 6.2]. In the study of a generalized ring of integer-valued polynomials $\mathrm{Int}_{\mathbb{Q}}(\underline{E}, \overline{\widehat{\mathbb{Z}}})$, without loss of generality we may suppose that the subset $\underline{E}$ of $\overline{\widehat{\mathbb{Z}}}$ is of the form $\underline{E} = \prod_{p \in \mathbb{P}} E_p$ (see the arguments given in [Remark 6.3]). Note that we allow each component $E_p$ of $\underline{E}$ to be equal to the empty set.

## 2. Polynomial Dedekind domains

Loper and Werner [2012] exhibited a construction of Prüfer domains between $\mathbb{Z}[X]$ and $\mathbb{Q}[X]$ in order to show the existence of a Prüfer domain strictly contained in $\mathrm{Int}(\mathbb{Z})$. As earlier in [Eakin and Heinzer 1973], their construction is obtained by intersecting a suitable family of valuation domains of $\mathbb{Q}(X)$ indexed by $\mathbb{P}$ with $\mathbb{Q}[X]$. A valuation domain of this family is equal to $V_{p,\alpha}$, for some $\alpha \in \overline{\mathbb{Z}_p}$, by Theorem 1.1 and the fact that $X$ is in every valuation domain of this family. By [Peruginelli 2017, Remark 2.8], a ring in Loper and Werner's construction can be represented as a generalized ring of integer-valued polynomials $\mathrm{Int}_{\mathbb{Q}}(\underline{E}, \overline{\widehat{\mathbb{Z}}})$, for a suitable subset $\underline{E}$ of $\overline{\widehat{\mathbb{Z}}}$ which satisfies the following definition.

**Definition 2.1.** Let $\underline{E}=\prod_{p\in\mathbb{P}} E_p\subset\overline{\widehat{\mathbb{Z}}}$. We say that $\underline{E}$ is locally bounded, if, for each prime $p$, $E_p$ is a subset of $\overline{\mathbb{Z}_p}$ of bounded degree, that is, $\{[\mathbb{Q}_p(\alpha):\mathbb{Q}_p]\mid\alpha\in E_p\}$ is bounded.

As we have already said above, some of the components $E_p$ of $\underline{E}$ may be equal to the empty set. Since $\mathbb{Q}_p$ has at most finitely many extensions of degree bounded by some fixed positive integer, if $E_p\subset\overline{\mathbb{Z}_p}$ has bounded degree then $E_p$ is contained in a finite extension of $\mathbb{Q}_p$.

By Theorem 1.1, a Prüfer domain constructed in [Loper and Werner 2012] can be represented as an intersection of valuation domains (see also [Chabert and Peruginelli 2016]):

$$\mathrm{Int}_{\mathbb{Q}}(\underline{E},\overline{\widehat{\mathbb{Z}}})=\bigcap_{p\in\mathbb{P}}\bigcap_{\alpha_p\in E_p}V_{p,\alpha_p}\cap\bigcap_{q\in\mathcal{P}^{\mathrm{irr}}}\mathbb{Q}[X]_{(q)}. \tag{2.2}$$

Here $\underline{E}=\prod_{p\in\mathbb{P}} E_p\subset\overline{\widehat{\mathbb{Z}}}$ is locally bounded and $\mathcal{P}^{\mathrm{irr}}$ denotes the set of irreducible polynomials in $\mathbb{Q}[X]$; note that the intersection on the right in this display equals $\mathbb{Q}[X]$. Similarly, for the ring $\mathrm{Int}_{\mathbb{Q}}(E_p,\overline{\mathbb{Z}_p})=\{f\in\mathbb{Q}[X]\mid f(E_p)\subseteq\overline{\mathbb{Z}_p}\}$ we have

$$\mathrm{Int}_{\mathbb{Q}}(E_p,\overline{\mathbb{Z}_p})=\bigcap_{\alpha_p\in E_p}V_{p,\alpha_p}\cap\bigcap_{q\in\mathcal{P}^{\mathrm{irr}}}\mathbb{Q}[X]_{(q)}. \tag{2.3}$$

In particular, $\mathrm{Int}_{\mathbb{Q}}(\underline{E},\overline{\widehat{\mathbb{Z}}})=\bigcap_{p\in\mathbb{P}}(\mathbb{Z}\setminus p\mathbb{Z})^{-1}\,\mathrm{Int}_{\mathbb{Q}}(\underline{E},\overline{\widehat{\mathbb{Z}}})=\bigcap_{p\in\mathbb{P}}\mathrm{Int}_{\mathbb{Q}}(E_p,\overline{\mathbb{Z}_p})$ by Lemma 2.5.

By means of the representation (2.2), the main result of [Loper and Werner 2012, Corollary 2.12] can now be restated as follows:

**Theorem 2.4.** *Let $\underline{E}\subset\overline{\widehat{\mathbb{Z}}}$ be locally bounded. Then the ring* $\mathrm{Int}_{\mathbb{Q}}(\underline{E},\overline{\widehat{\mathbb{Z}}})$ *is a Prüfer domain.*

We want to characterize when a ring of the form $\mathrm{Int}_{\mathbb{Q}}(\underline{E},\overline{\widehat{\mathbb{Z}}})$, $\underline{E}\subseteq\overline{\widehat{\mathbb{Z}}}$, is a Dedekind domain. In order to accomplish this objective, we need to describe the prime spectrum of this ring when $E$ is locally bounded. It is customary for rings of integer-valued polynomials to distinguish the prime ideals into two different kinds, and we do the same here in our setting: given a prime ideal $P$ of $R=\mathrm{Int}_{\mathbb{Q}}(\underline{E},\overline{\widehat{\mathbb{Z}}})$, we say that $P$ is *nonunitary* if $P\cap\mathbb{Z}=(0)$ and that $P$ is *unitary* if $P\cap\mathbb{Z}=p\mathbb{Z}$ for some $p\in\mathbb{P}$.

It is a classical result that each nonunitary prime ideal of $R$ is equal to

$$\mathfrak{P}_q=q(X)\mathbb{Q}[X]\cap R$$

for some $q\in\mathcal{P}^{\mathrm{irr}}$ (see for example [Cahen and Chabert 1997, Corollary V.1.2]).

If $P\cap\mathbb{Z}=p\mathbb{Z}$, $p\in\mathbb{P}$, and $\alpha\in E_p$, the following is a unitary prime ideal of $R$:

$$\mathfrak{M}_{p,\alpha}=\{f\in R\mid v_p(f(\alpha))>0\}.$$

If $E_p$ is a closed subset of $\overline{\mathbb{Z}_p}$ for each prime $p$, and $\underline{E} = \prod_p E_p$ is locally bounded, we are going to show that each unitary prime ideal of $R$ is equal to $\mathfrak{M}_{p,\alpha}$, for some $p \in \mathbb{P}$ and $\alpha \in E_p$.

**Lemma 2.5.** *Let $\underline{E} \subseteq \widehat{\overline{\mathbb{Z}}}$ be any subset, $P$ be a finite subset of $\mathbb{P}$ and $S$ the multiplicative subset of $\mathbb{Z}$ generated by $\mathbb{P} \setminus P$. Then $S^{-1}\operatorname{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\overline{\mathbb{Z}}}) = \bigcap_{p\in P}\operatorname{Int}_{\mathbb{Q}}(E_p, \overline{\mathbb{Z}_p})$.*

*In particular, for each $p \in \mathbb{P}$, $(\mathbb{Z} \setminus p\mathbb{Z})^{-1}\operatorname{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\overline{\mathbb{Z}}}) = \operatorname{Int}_{\mathbb{Q}}(E_p, \overline{\mathbb{Z}_p})$.*

*Proof.* The proof follows by an argument similar to the one of [Chabert and Peruginelli 2018, Proposition 4.2]. Let $R = \operatorname{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\overline{\mathbb{Z}}})$ and $R_p = \operatorname{Int}_{\mathbb{Q}}(E_p, \overline{\mathbb{Z}_p})$, for each $p \in P$. The containment $S^{-1}R \subseteq \bigcap_{p\in P} R_p$ is clear, since $R \subseteq R_p$ and for every $d \in S$, $d$ is a unit in $R_p$, for each $p \in P$. Conversely, let $f \in \bigcap_{p\in P} R_p$. Let $d \in \mathbb{Z}$, $d \neq 0$, be such that $df \in \mathbb{Z}[X]$ and let $d = t\prod_{p\in P} p^{a_p}$, $a_p \geq 0$ and $t \in \mathbb{Z}$ not divisible by any $p \in P$. Then, letting $g = tf$, we have that $g$ is in $\mathbb{Z}_{(q)}[X] \subset R_q$ for each $q \notin P$ and $g$ is in $R_p$ for each $p \in P$ because $t$ is a unit in $\mathbb{Z}_{(p)}$, for all $p \in P$. Hence, $f = \frac{g}{t} \in S^{-1}R$, as desired. $\square$

**Proposition 2.6.** *Let $\underline{E} = \prod_p E_p \subset \widehat{\overline{\mathbb{Z}}}$ be locally bounded and closed. If $M$ is a unitary prime ideal of $\operatorname{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\overline{\mathbb{Z}}})$ such that $M \cap \mathbb{Z} = p\mathbb{Z}$ for some $p \in \mathbb{P}$, then $M$ is maximal and there exists $\alpha \in E_p$ such that $M = \mathfrak{M}_{p,\alpha}$.*

*Proof.* Let $R = \operatorname{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\overline{\mathbb{Z}}})$. We use the fact that $R$ is a Prüfer domain by Theorem 2.4.

Let $M$ be a unitary prime ideal of $R$ and let $V = R_M$. Then, by Lemma 2.5, we have $R_p = \operatorname{Int}_{\mathbb{Q}}(E_p, \overline{\mathbb{Z}_p}) \subset V$, since $(\mathbb{Z} \setminus p\mathbb{Z})^{-1}V = V$. Let $M'$ be the center of $V$ on $R_p$. Since $M' \cap R = M$, it is sufficient to show that

$$M' = \mathfrak{M}_{p,\alpha} = \{f \in R_p \mid v_p(f(\alpha)) > 0\},$$

for some $\alpha \in E_p$ (with a slight abuse of notation, we denote the unitary prime ideals of $R$ and $R_p$ in the same way). Let $f \in R_p$. Let $K$ be a finite extension of $\mathbb{Q}_p$ such that $O_K$ contains $E_p$ and let $i_0, \ldots, i_{q-1} \in O_K$ be a set of representatives for $O_K/\pi O_K \cong \mathbb{F}_q$, where $\pi$ is a uniformizer of $O_K$ (i.e., a generator of the maximal ideal of $O_K$). For each $\alpha \in E_p$, there exists some $j \in \{0, \ldots, q-1\}$ such that $f(\alpha) - i_j \in \pi O_K$. In particular, $\prod_{j=0}^{q-1}(f(\alpha) - i_j) \in \pi O_K$ for each $\alpha \in E_p$. Observe that the polynomials $X^q - X$ and $\prod_{j=0}^{q-1}(X - i_j)$ coincide modulo $\pi$, so in particular $f(\alpha)^q - f(\alpha) \in \pi O_K$. If $e = e(O_K \mid \mathbb{Q}_p)$, we have $(f(\alpha)^q - f(\alpha))^e \in pO_K$. Equivalently, $(f^q - f)^e \in pR_p$, which is contained in $M'$. Since $M'$ is a prime ideal, it follows that $f^q - f \in M'$, so modulo $M'$, $f$ satisfies the equation $X^q - X = 0$. This shows that $R_p/M'$ is contained in the finite field $\mathbb{F}_q$, so it is a finite domain, hence a field. This proves that $M'$ is maximal. Note that, since $R/M \subseteq R_p/M'$ and the latter is a finite field, it follows also that $M$ is a maximal ideal of $R$.

Since $R_p$ is countable, $M'$ is countably generated, say $M' = \bigcup_{n\in\mathbb{N}} I_n$, where $I_n = (p, f_1, \dots, f_n)$ for each $n \in \mathbb{N}$. By [Gilmer and Heinzer 1968, Proposition 1.4], for each $n \in \mathbb{N}$, there exists $\alpha_n \in E_p$ such that $I_n \subset \mathfrak{M}_{p,\alpha_n}$ (we may exclude the nonunitary prime ideals of $R_p$ because they do not contain $p$, hence neither $I_n$ for every $n$). Suppose first that $E_p$ is finite. Then there exists $\alpha \in E_p$ such that the set $J = \{n \in \mathbb{N} \mid I_n \subset \mathfrak{M}_{p,\alpha}\}$ is a cofinal subset of $\mathbb{N}$. Hence, for each $f \in M'$, there exists $n \in J$ such that $f \in I_n \subset \mathfrak{M}_{p,\alpha}$, so that $M' \subseteq \mathfrak{M}_{p,\alpha}$ and therefore equality holds since $M'$ is maximal. If $E_p$ is infinite, since it is a closed subset (because $\underline{E}$ is closed) contained in a finite extension of $\mathbb{Q}_p$, by compactness we may extract a sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ from $E_p$ converging to some element $\alpha \in E_p$. Without loss of generality we suppose that $\alpha_n \to \alpha$. Now, for each $f \in M'$, $f \in I_n \subset \mathfrak{M}_{p,\alpha_n}$ for some $n$. Since $I_n \subseteq I_{n+1}$ for each $n \in \mathbb{N}$, $f \in \mathfrak{M}_{p,\alpha_m}$ for each $m \geq n$, that is, $v_p(f(\alpha_m)) > 0$. By continuity we get that $v_p(f(\alpha)) > 0$, that is, $f \in \mathfrak{M}_{p,\alpha}$. Therefore as before we conclude that $M' = \mathfrak{M}_{p,\alpha}$. □

Thus, if $\mathrm{Int}_{\mathbb{Q}}(\underline{E}, \overline{\widehat{\mathbb{Z}}})$ is a Prüfer domain, given a maximal unitary ideal $\mathfrak{M}_{p,\alpha}$, $p \in \mathbb{P}$ and $\alpha \in E_p$, we have

$$\mathrm{Int}_{\mathbb{Q}}(\underline{E}, \overline{\widehat{\mathbb{Z}}})_{\mathfrak{M}_{p,\alpha}} = V_{p,\alpha}. \tag{2.7}$$

Similarly, for $q \in \mathcal{P}^{\mathrm{irr}}$, we have

$$\mathrm{Int}_{\mathbb{Q}}(\underline{E}, \overline{\widehat{\mathbb{Z}}})_{\mathfrak{P}_q} = \mathbb{Q}[X]_{(q)}. \tag{2.8}$$

We call the valuation domains $V_{p,\alpha}$ unitary, and the others $\mathbb{Q}[X]_{(q)}$ nonunitary. Similar equalities hold for the Prüfer domain $\mathrm{Int}_{\mathbb{Q}}(E_p, \overline{\mathbb{Z}_p})$. Note that the residue field of $\mathrm{Int}_{\mathbb{Q}}(\underline{E}, \overline{\widehat{\mathbb{Z}}})$ at a unitary prime ideal is a finite field (by the property of the unitary valuation overrings we discussed about in Section 1), while the residue field of a nonunitary prime ideal of $\mathrm{Int}_{\mathbb{Q}}(\underline{E}, \overline{\widehat{\mathbb{Z}}})$ is a finite extension of the rationals, hence an infinite field.

We finish this section with the following remark.

**Remark 2.9.** By Theorem 1.1, given a ring $\mathrm{Int}_{\mathbb{Q}}(E_p, \overline{\mathbb{Z}_p})$, without loss of generality we may assume that the elements of $E_p$ are pairwise nonconjugate over $\mathbb{Q}_p$. Under this further assumption and if $E_p$ is bounded (i.e., contained in a finite extension of $\mathbb{Q}_p$), Theorem 2.4, (2.7) and Proposition 2.6 imply that there is a one-to-one correspondence between the elements of $E_p$ and the unitary valuation overrings $V_{p,\alpha_p}$, $\alpha_p \in E_p$, of $\mathrm{Int}_{\mathbb{Q}}(E_p, \overline{\mathbb{Z}_p})$.

**2A.** ***The local case.*** For a fixed $p \in \mathbb{P}$, we characterize in this section the subsets $E_p$ of $\overline{\mathbb{Z}_p}$ for which the corresponding ring of integer-valued polynomials $\mathrm{Int}_{\mathbb{Q}}(E_p, \overline{\mathbb{Z}_p})$ is a Dedekind domain. The following proposition is a generalization of [Chang 2022, Theorem 4.3 (2)].

**Proposition 2.10.** *Let $E_p$ be a subset of $\overline{\mathbb{Z}_p}$. Then $\mathrm{Int}_{\mathbb{Q}}(E_p, \overline{\mathbb{Z}_p})$ is a Dedekind domain with finite residue fields of prime characteristic if and only if $E_p$ is a finite subset of transcendental elements over $\mathbb{Q}$.*

*Suppose that $E_p = \{\alpha_1, \dots, \alpha_n\}$ and the $\alpha_i$'s are pairwise nonconjugate over $\mathbb{Q}_p$. Then, then the class group of $\mathrm{Int}_{\mathbb{Q}}(E_p, \overline{\mathbb{Z}_p})$ is isomorphic to $\mathbb{Z}/e\mathbb{Z} \oplus \mathbb{Z}^{n-1}$, where $e = \gcd\{e_{\alpha_i} \mid i = 1, \dots, n\}$. Thus $\mathrm{Int}_{\mathbb{Q}}(E_p, \overline{\mathbb{Z}_p})$ is a PID if and only if $E_p$ contains at most one element $\alpha_p \in \overline{\mathbb{Z}_p}$, such that $\alpha_p$ is transcendental over $\mathbb{Q}$ and unramified over $\mathbb{Q}_p$.*

*Proof.* Let $R_p = \mathrm{Int}_{\mathbb{Q}}(E_p, \overline{\mathbb{Z}_p})$. Note that, if $E_p$ is the empty set, then $R_p = \mathbb{Q}[X]$. We assume henceforth that $E_p \neq \varnothing$.

Suppose $R_p$ is a Dedekind domain with finite residue fields of prime characteristic. We show first that each maximal unitary ideal $M$ of $R_p$ is equal to $\mathfrak{M}_{p,\alpha_p}$, for some $\alpha_p \in E_p$. Let $V$ be a unitary valuation overring of $R_p$ which is centered on $M$. By Theorem 1.1, there exists $\alpha_0 \in \overline{\mathbb{Z}_p}$ such that $V = V_{p,\alpha_0}$. Then, $M = \mathfrak{M}_{p,\alpha_0}$. Since $M$ is finitely generated and $R_p$ is Prüfer, by [Gilmer and Heinzer 1968, Proposition 1.4] $M \subseteq \mathfrak{M}_{p,\alpha_p}$ for some $\alpha_p \in E_p$ (we may exclude the nonunitary prime ideals of $R_p$ because they do not contain $p$, hence neither $M$). Since $M$ is maximal, it follows that $M = \mathfrak{M}_{p,\alpha_p}$, which means that $\alpha_0$ and $\alpha_p$ are conjugate over $\mathbb{Q}_p$ by [Peruginelli 2017, Theorem 3.2]. Hence, without loss of generality, we may suppose that $\alpha_0 \in E_p$. Note that each $\alpha_p \in E_p$ is transcendental over $\mathbb{Q}$, otherwise the valuation overring $V_{p,\alpha_p}$ of $R_p$ would have rank 2. Since $R_p$ is Dedekind, $p$ is contained in only finitely many maximal ideals of this ring; necessarily, such ideals are unitary. By the previous argument, such ideals are equal to $\mathfrak{M}_{p,\alpha_p}$, for $\alpha_p \in E_p$. Since by Theorem 1.1 and (2.7), $\mathfrak{M}_{p,\alpha_p} = \mathfrak{M}_{p,\beta_p}$ if and only if $\alpha_p, \beta_p \in E_p$ are conjugate over $\mathbb{Q}_p$, it follows that $E_p$ is a finite subset of $\overline{\mathbb{Z}_p}$.

Conversely, suppose now that $E_p \subset \overline{\mathbb{Z}_p}$ is a finite subset of transcendental elements over $\mathbb{Q}$. The fact that $\mathrm{Int}_{\mathbb{Q}}(E_p, \overline{\mathbb{Z}_p})$ is a Dedekind domain follows from [Eakin and Heinzer 1973, Theorem], but we give a different self-contained argument based on the previous results. We know that $E_p$ has bounded degree, so $R_p$ is Prüfer, by Theorem 2.4. By (2.3), $R_p$ is equal to an intersection of DVRs which are essential over it. Moreover, each nonzero $f \in R_p$ belongs to finitely many maximal ideals, since $E_p$ is finite and $f$ has finitely many irreducible factors in $\mathbb{Q}[X]$. Hence, $R_p$ is a Krull domain, so, by [Gilmer 1992, Theorem 43.16], $R_p$ is a Dedekind domain. Finally, $R_p$ has finite residue fields of prime characteristic, because each of the unitary valuation overrings of $R_p$ (namely, $V_{p,\alpha_p}$, $\alpha_p \in E_p$) have finite residue field.

Assuming that the elements of $E_p$ are pairwise nonconjugate over $\mathbb{Q}_p$, the claim regarding the class group follows easily from [Eakin and Heinzer 1973, Theorem], taking into account the representation (2.3). If $E_p = \{\alpha_1, \dots, \alpha_n\}$, let

$\boldsymbol{e} = (e_{\alpha_1}, \dots, e_{\alpha_n}) \in \mathbb{Z}^n$ and $e = \gcd(e_{\alpha_1}, \dots, e_{\alpha_n})$. Then, the class group of $R_p$ is isomorphic to

$$\mathbb{Z}^n/\langle \boldsymbol{e} \rangle \cong \mathbb{Z}/e\mathbb{Z} \oplus \mathbb{Z}^{n-1}.$$

The last claim follows at once from the description of the class group. □

**2B.** ***The global case.*** If, for each $p \in \mathbb{P}$, $E_p \subset \overline{\mathbb{Z}_p}$ is a finite subset of transcendental elements over $\mathbb{Q}$ and $\underline{E} = \prod_p E_p$, then, by [Chang 2022, Corollary 2.6], $\mathrm{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\overline{\mathbb{Z}}})$ is an almost Dedekind domain. However, this ring might not be noetherian, that is, a Dedekind domain. See for example the construction of [Chang 2022, Theorem 3.1], in which the polynomial $X$ is divisible by infinitely many primes $p \in \mathbb{P}$. In general, an almost Dedekind domain $R$ is Dedekind if and only if it has finite character, that is, each nonzero $f \in R$ belongs to finitely many maximal ideals of $R$ [Gilmer 1992, Theorem 37.2], or, equivalently, $v(f) \neq 0$ only for finitely many valuation overrings $V$ of $R$ (which are only DVRs). We aim to characterize the subsets $\underline{E} = \prod_p E_p$ of $\widehat{\overline{\mathbb{Z}}}$ such that $\mathrm{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\overline{\mathbb{Z}}})$ is Dedekind.

**Definition 2.11.** We say that $\underline{E}$ is *polynomially factorizable* if, for each $g \in \mathbb{Z}[X]$ and $\alpha = (\alpha_p) \in \underline{E}$, there exist $n, d \in \mathbb{Z}$, $n, d \geq 1$ such that $g(\alpha)^n/d$ is a unit of $\widehat{\overline{\mathbb{Z}}}$, that is, $v_p(g(\alpha_p)^n/d) = 0$, for all $p \in \mathbb{P}$.

Note that $g(\alpha)^n = (g(\alpha_p)^n) \in \widehat{\overline{\mathbb{Z}}}$. Loosely speaking, a subset $\underline{E}$ of $\widehat{\overline{\mathbb{Z}}}$ is polynomially factorizable if, for every $g \in \mathbb{Z}[X]$ and $\alpha \in \underline{E}$, $g(\alpha) \in \widehat{\overline{\mathbb{Z}}}$ is divisible only by finitely many primes $p \in \mathbb{P}$ (up to some exponent $n \geq 1$), or, equivalently, all but finitely many components of $g(\alpha)$ are units. Note that, if the above condition of the definition holds, then $g(\alpha)^n$ and $d$ generate the same principal ideal of $\widehat{\overline{\mathbb{Z}}}$.

The next lemma gives a simple characterization of polynomially factorizable subsets $\underline{E}$ of $\widehat{\overline{\mathbb{Z}}}$ in terms of the finiteness of some sets of primes associated to every polynomial in $\mathbb{Z}[X]$. For every $g \in \mathbb{Z}[X]$ and subset $\underline{E} = \prod_p E_p \subseteq \widehat{\overline{\mathbb{Z}}}$, we set

$$\mathbb{P}_{g,\underline{E}} = \{p \in \mathbb{P} \mid \exists \alpha_p \in E_p \text{ such that } v_p(g(\alpha_p)) > 0\}.$$

The next result shows that $\underline{E}$ is polynomially factorizable if and only if $\mathbb{P}_{g,\underline{E}}$ is finite for every $g \in \mathbb{Z}[X]$.

**Lemma 2.12.** *Let $g \in \mathbb{Z}[X]$ and $\underline{E} = \prod_p E_p \subset \widehat{\overline{\mathbb{Z}}}$, where each $E_p \subset \overline{\mathbb{Z}_p}$ is a closed set of transcendental elements over $\mathbb{Q}$. Then the following conditions are equivalent*:

i) *The set $\mathbb{P}_{g,\underline{E}}$ is finite.*

ii) *For each $\alpha \in \underline{E}$, there exist $n, d \in \mathbb{Z}$, $n, d \geq 1$ such that $g(\alpha)^n/d$ is a unit of $\widehat{\overline{\mathbb{Z}}}$.*

*Proof.* We use the following easy remark: for $\alpha = (\alpha_p) \in \widehat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$, the set $\{p \in \mathbb{P} \mid v_p(\alpha_p) > 0\}$ is finite if and only if there exists $d \in \mathbb{Z}$, $d \geq 1$, such that $\alpha\widehat{\mathbb{Z}} = d\widehat{\mathbb{Z}}$.

Suppose i) holds and let $\alpha = (\alpha_p) \in \underline{E}$. By assumption, there are only finitely many $p \in \mathbb{P}$ such that $v_p(g(\alpha_p)) > 0$, for some $\alpha_p \in E_p$, say, $p_1, \dots, p_k$. Let $\alpha \in \underline{E}$ be fixed; in particular, there exists $n \in \mathbb{N}$ such that $nv_p(g(\alpha_p)) = a_p \in \mathbb{Z}$ for each prime $p$ (where $a_p = 0$ for all $p \notin \{p_1, \dots, p_k\}$). Hence, if we let $d = \prod_{i=1}^k p_i^{a_{p_i}}$ we get $v_p(g(\alpha_p)^n) = v_p(d)$ for all $p \in \mathbb{P}$, thus ii) holds.

Assume now that ii) holds and suppose that $\mathbb{P}_{g,\underline{E}}$ is infinite. For each $p \in \mathbb{P}_{g,\underline{E}}$, let $\alpha_p \in E_p$ be such that $v_p(g(\alpha_p)) > 0$ and consider the element $\alpha = (\alpha_p) \in \underline{E}$, where $\alpha_p$ is any element of $E_p$ for $p \notin \mathbb{P}_{g,\underline{E}}$. If there is no $n \geq 1$ such that $nv_p(g(\alpha_p)) = a_p \in \mathbb{Z}$ for all $p \in \mathbb{P}$ we immediately get a contradiction. Suppose instead that such an $n$ exists. Since $a_p$ is nonzero for infinitely many $p \in \mathbb{P}$, there is no $d \in \mathbb{Z}$ such that $v_p(g(\alpha_p)^n/d) = 0$ for each $p \in \mathbb{P}$, which again is a contradiction. $\square$

**Remark 2.13.** By Lemma 2.12, it follows easily that a subset $\underline{E} \subseteq \widehat{\overline{\mathbb{Z}}}$ is polynomially factorizable if and only if $\mathbb{P}_{g,\underline{E}}$ is finite for each irreducible $g \in \mathbb{Z}[X]$. In fact, if $g = \prod_i g_i$, where $g_i \in \mathbb{Z}[X]$ are irreducible, then $\mathbb{P}_{g,\underline{E}} = \bigcup_i \mathbb{P}_{g_i,\underline{E}}$.

It is well-known that, given a nonconstant $q \in \mathbb{Z}[X]$, there exist infinitely many $p \in \mathbb{P}$ for which there exists $n \in \mathbb{Z}$ such that $q(n)$ is divisible by $p$ (see for example the proof of [Cahen and Chabert 1997, Proposition V.2.8]). In particular, $\widehat{\mathbb{Z}}$ is not polynomially factorizable by Lemma 2.12.

The next lemma describes the Picard group of $\mathrm{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\overline{\mathbb{Z}}})$ in terms of the Picard groups of the localizations $\mathrm{Int}_{\mathbb{Q}}(E_p, \overline{\mathbb{Z}_p})$, $p \in \mathbb{P}$ (see Lemma 2.5).

**Lemma 2.14.** *Let $\underline{E} = \prod_p E_p \subset \widehat{\overline{\mathbb{Z}}}$ be a subset. Then*

$$\mathrm{Pic}(\mathrm{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\overline{\mathbb{Z}}})) \cong \bigoplus_{p \in \mathbb{P}} \mathrm{Pic}(\mathrm{Int}_{\mathbb{Q}}(E_p, \overline{\mathbb{Z}_p})).$$

*Proof.* Let $R = \mathrm{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\overline{\mathbb{Z}}})$ and $R_p = (\mathbb{Z} \setminus p\mathbb{Z})^{-1}R$, for $p \in \mathbb{P}$; by Lemma 2.5, $R_p = \mathrm{Int}_{\mathbb{Q}}(E_p, \overline{\mathbb{Z}_p})$. Since the proof follows by the same arguments of [Gilmer et al. 1990, Theorem 1], we just sketch it and refer to the cited paper for the details. By a classical argument (see for example [McQuillan 1985, Lemma 1]), every finitely generated ideal $J$ of $R$ (in particular, every invertible ideal of $R$) is isomorphic to a finitely generated unitary ideal $I$, that is, $I \cap \mathbb{Z} = d\mathbb{Z} \neq (0)$. For such an ideal, $(I \cap \mathbb{Z})_{(p)} = \mathbb{Z}_{(p)}$ for all $p \in \mathbb{P}$ not dividing $d$, so $IR_p = R_p$. This argument shows that we have a well-defined map from $\mathrm{Pic}(R)$ to $\bigoplus_{p \in \mathbb{P}} \mathrm{Pic}(R_p)$.

If $I$ is a unitary ideal of $R$, say $I \cap \mathbb{Z} = d\mathbb{Z}$, such that $IR_p$ is principal, it is generated by $d$. Hence, $I$ and $dR$ have the same localizations at each prime $p \in \mathbb{P}$, so they are equal. This shows that the previous map is injective.

For the surjectivity, it is sufficient to show that, if $J_p$ is an invertible unitary ideal of $R_p$, for some $p \in \mathbb{P}$, then there exists an invertible ideal $J$ of $R$ such that

$JR_p = J_p$ and $JR_q = R_q$ for each $q \in \mathbb{P} \setminus \{p\}$. The ideal $J = J_p \cap R$ has the required properties. □

Now we may characterize when a generalized ring of integer-valued polynomials $\mathrm{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\overline{\mathbb{Z}}})$ is Dedekind and describe its class group.

**Theorem 2.15.** *Let* $\underline{E} = \prod_p E_p \subset \widehat{\overline{\mathbb{Z}}}$ *be a subset. Then* $\mathrm{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\overline{\mathbb{Z}}})$ *is a Dedekind domain with finite residue fields of prime characteristic if and only if* $E_p$ *is a finite set of transcendental elements over* $\mathbb{Q}$ *for each* $p \in \mathbb{P}$ *and* $\underline{E}$ *is polynomially factorizable.*

*In this case, the class group of* $\mathrm{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\overline{\mathbb{Z}}})$ *is equal to a direct sum of a countable family of finitely generated abelian groups.*

*Proof.* Let $R = \mathrm{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\overline{\mathbb{Z}}})$ and suppose the conditions for $\underline{E}$ in the statement are satisfied. Then $\underline{E}$ is locally bounded and closed so, by Theorem 2.4, $R$ is Prüfer. For $R$ to be Dedekind, it is sufficient to show that it is a Krull domain [Gilmer 1992, Theorem 43.16]. By assumption, each of the unitary valuation overrings of $R$ in the representation (2.2) is a DVR with finite residue field, so $R$ has finite residue fields of prime characteristic by Proposition 2.6. We have to show that $R$ has finite character, that is, for each nonzero $f = \frac{g}{n} \in R$, $g \in \mathbb{Z}[X]$ and $n \in \mathbb{Z} \setminus \{0\}$, $f$ is contained in only finitely many maximal ideals of $R$. As in the proof of Proposition 2.10, $f$ is contained in only finitely many nonunitary prime ideals of $R$. We now check the maximal unitary ideals of $R$, described in the Proposition 2.6, which contain $f$. Since the denominator $n$ of $f$ is divisible by only finitely many $p \in \mathbb{P}$, $f$ is contained in only finitely many maximal unitary ideals if and only if the same condition holds for $g$. Since $E_p$ is finite for each $p \in \mathbb{P}$, this is equivalent to the finiteness of the set $\mathbb{P}_{g,\underline{E}}$. Since $\underline{E}$ is polynomially factorizable, by Lemma 2.12, $\mathbb{P}_{g,\underline{E}}$ is finite.

Conversely, if $\mathrm{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\overline{\mathbb{Z}}})$ is a Dedekind domain with finite residue fields of prime characteristic, then, for each prime $p$, the overring $\mathrm{Int}_{\mathbb{Q}}(E_p, \overline{\mathbb{Z}_p})$ is a Dedekind domain with finite residue fields of prime characteristic [Gilmer 1992, Theorem 40.1]. By Proposition 2.10, $E_p$ is a finite subset of $\overline{\mathbb{Z}_p}$ formed by transcendental elements over $\mathbb{Q}$ (so, in particular, $\underline{E}$ is locally bounded). If there exists some $g \in \mathbb{Z}[X]$ such that the set $\mathbb{P}_{g,\underline{E}}$ is infinite, then $g(X)$ would be contained in infinitely many unitary prime ideals of $\mathrm{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\overline{\mathbb{Z}}})$, a contradiction with [Gilmer 1992, Theorem 37.2]. Therefore, $\underline{E}$ is polynomially factorizable by Lemma 2.12.

The final claim follows from Lemma 2.14 and Proposition 2.10. □

The next corollary is a generalization of [Glivický and Šaroch 2013, Lemma 3.3]: it characterizes the elements $\alpha$ in $\widehat{\overline{\mathbb{Z}}}$ for which the ring $\mathrm{Int}_{\mathbb{Q}}(\{\alpha\}, \widehat{\overline{\mathbb{Z}}})$ is a PID.

**Corollary 2.16.** *Let* $\underline{E} = \prod_p E_p \subset \widehat{\overline{\mathbb{Z}}}$ *be a subset such that, for each* $p \in \mathbb{P}$, *the elements of* $E_p$ *are pairwise nonconjugate over* $\mathbb{Q}_p$. *Then* $\mathrm{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\overline{\mathbb{Z}}})$ *is a PID with* *finite residue fields of prime characteristic if and only if, for each prime $p$, $E_p$ contains at most one element of $\overline{\mathbb{Z}_p}$, unramified over $\mathbb{Q}_p$ and transcendental over $\mathbb{Q}$, and $\underline{E}$ is polynomially factorizable.*

Note that if the conditions of Corollary 2.16 occur, namely, $E_p = \{\alpha_p\}$ for each $p \in \mathbb{P}$, then $\underline{E}$ is the singleton $\{\alpha\}$, where $\alpha = (\alpha_p) \in \widehat{\overline{\mathbb{Z}}}$. The condition that $\underline{E}$ is polynomially factorizable appears in other equivalent forms in [Glivický and Šaroch 2013, Lemma 3.3] and [Glivická et al. 2023, Proposition 1.1], in the case $\alpha \in \widehat{\mathbb{Z}}$.

*Proof.* The proof follows from Theorem 2.15, Lemma 2.14 and Proposition 2.10. □

An argument similar to the one in the proof of [Eakin and Heinzer 1973, Theorem] shows that a PID $\mathrm{Int}_{\mathbb{Q}}(\{\alpha\}, \widehat{\overline{\mathbb{Z}}})$ as in the statement of Corollary 2.16 is never a Euclidean domain.

We now show that each Dedekind domain with finite residue fields of prime characteristic between $\mathbb{Z}[X]$ and $\mathbb{Q}[X]$ is indeed a generalized ring of integer-valued polynomials.

**Theorem 2.17.** *Let $R$ be a Dedekind domain with finite residue fields of prime characteristic such that $\mathbb{Z}[X] \subset R \subseteq \mathbb{Q}[X]$. Then $R$ is equal to $\mathrm{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\overline{\mathbb{Z}}})$, for some subset $\underline{E} = \prod_p E_p \subset \widehat{\overline{\mathbb{Z}}}$ such that $E_p$ is a finite set of transcendental elements over $\mathbb{Q}$ for each prime $p$ and $\underline{E}$ is polynomially factorizable.*

*In particular, the class group of $R$ is isomorphic to a direct sum of a countable family of finitely generated abelian groups.*

*Proof.* Let $\mathbb{P}_R = \{p \in \mathbb{P} \mid \exists P \in \mathrm{Spec}(R) \text{ such that } P \cap \mathbb{Z} = p\mathbb{Z}\}$. Clearly, $\mathbb{P}_R$ is empty if and only if $R = \mathbb{Q}[X]$; in this case for $\underline{E}$ equal to the empty set we have the claim. Suppose $\mathbb{P}_R$ is not empty. For each $p \in \mathbb{P}_R$, we denote by $\mathbb{P}_{R,p}$ the set of unitary prime ideals of $R$ lying above $p$. By assumption, for each $P \in \mathbb{P}_{R,p}$, $p \in \mathbb{P}$, $R_P$ is a DVR of $\mathbb{Q}(X)$ with finite residue field extending $\mathbb{Z}_{(p)}$. By Theorem 1.1, there exists $\alpha_p \in \overline{\mathbb{Z}_p}$, transcendental over $\mathbb{Q}$, such that $R_P = V_{p,\alpha_p}$. Let $E_p$ be the subset of $\overline{\mathbb{Z}_p}$ formed by such $\alpha_p$'s, for each $P \in \mathbb{P}_{R,p}$. Since $R$ is Dedekind and by (2.2) and (2.3), we have the equalities

$$\begin{aligned} R = \bigcap_{p \in \mathbb{P}_R} \bigcap_{P \in \mathbb{P}_{R,p}} R_P \cap \mathbb{Q}[X] &= \bigcap_{p \in \mathbb{P}_R} \bigcap_{\alpha_p \in E_p} V_{p,\alpha_p} \cap \mathbb{Q}[X] \\ &= \bigcap_{p \in \mathbb{P}_R} \mathrm{Int}_{\mathbb{Q}}(E_p, \overline{\mathbb{Z}_p}) = \mathrm{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\overline{\mathbb{Z}}}), \end{aligned}$$

where $\underline{E} = \prod_{p \in \mathbb{P}_R} E_p \subset \widehat{\overline{\mathbb{Z}}}$. By Theorem 2.15, for each $p \in \mathbb{P}$, $E_p$ is a finite subset of $\overline{\mathbb{Z}_p}$ of transcendental elements over $\mathbb{Q}$, $\underline{E}$ is polynomially factorizable and the class group of $R$ is isomorphic to a direct sum of a countable family of finitely generated abelian groups. □

It was shown in [Glivický and Šaroch 2013, Proposition 3.4] that the cardinality of the set of $\alpha \in \widehat{\mathbb{Z}}$ such that $\mathrm{Int}_{\mathbb{Q}}(\{\alpha\}, \widehat{\mathbb{Z}})$ is a PID is $2^{\aleph_0}$. The next corollary describes all the PIDs with finite residue fields of prime characteristic between $\mathbb{Z}[X]$ and $\mathbb{Q}[X]$.

**Corollary 2.18.** *Let $R$ be a PID with finite residue fields of prime characteristic such that $\mathbb{Z}[X] \subset R \subset \mathbb{Q}[X]$. Then $R$ is equal to $\mathrm{Int}_{\mathbb{Q}}(\{\alpha\}, \overline{\widehat{\mathbb{Z}}})$, for some $\alpha = (\alpha_p) \in \overline{\widehat{\mathbb{Z}}}$ such that, for each $p \in \mathbb{P}$, $\alpha_p$ is transcendental over $\mathbb{Q}$, $\alpha_p$ is unramified over $\mathbb{Q}_p$ and $\{\alpha\}$ is polynomially factorizable.*

*Proof.* The proof follows from Theorem 2.17 and Corollary 2.16. □

**2C.** ***Equality of generalized rings of integer-valued polynomials.*** Given two locally bounded closed subsets $\underline{E}, \underline{F}$ of $\overline{\widehat{\mathbb{Z}}}$, we characterize when the associated generalized ring of integer-valued polynomials $\mathrm{Int}_{\mathbb{Q}}(\underline{E}, \overline{\widehat{\mathbb{Z}}})$, $\mathrm{Int}_{\mathbb{Q}}(\underline{F}, \overline{\widehat{\mathbb{Z}}})$ are the same.

The following is a general result about integral extensions of rings of integer-valued polynomials. For an integral domain $D$ with quotient field $K$, let $\overline{K}$ and $\overline{D}$ be the algebraic closure of $K$ and the absolute integral closure of $D$, respectively. We let $G_K = \mathrm{Gal}(\overline{K}/K)$ be the absolute Galois group of $K$. For a subset $\Omega$ of $\overline{K}$ we set $G_K(\Omega) = \{\sigma(a) \mid \sigma \in G_K, a \in \Omega\} = \bigcup_{\sigma \in G_K} \sigma(\Omega)$. We say that $\Omega$ is *$G_K$-invariant* if $G_K(\Omega) = \Omega$. Note that in general we have

$$\mathrm{Int}_K(\Omega, \overline{D}) = \mathrm{Int}_K(G_K(\Omega), \overline{D}) \tag{2.19}$$

because if $f(\alpha) \in \overline{D}$ for some $f \in K[X]$ and $\alpha \in \Omega$, then, for every $\sigma \in G_K$, we have $f(\sigma(\alpha)) = \sigma(f(\alpha)) \in \overline{D}$ because $\sigma(\overline{D}) \subseteq \overline{D}$.

**Lemma 2.20.** *Let $D$ be an integrally closed domain with quotient field $K$. Let $\Omega \subset \overline{D}$ be $G_K$-invariant. Let $F$ be an algebraic extension of $K$ containing $\Omega$. Then $\mathrm{Int}_F(\Omega, \overline{D})$ is the integral closure of $\mathrm{Int}_K(\Omega, \overline{D})$ in $F(X)$.*

*Proof.* By [Cahen and Chabert 1997, Proposition IV.4.1], $\mathrm{Int}_{\overline{K}}(\Omega, \overline{D})$ is integrally closed. In particular, $\mathrm{Int}_F(\Omega, \overline{D}) = \mathrm{Int}_{\overline{K}}(\Omega, \overline{D}) \cap F(X)$ is integrally closed, too. Hence, we just need to show that $\mathrm{Int}_K(\Omega, \overline{D}) \subseteq \mathrm{Int}_F(\Omega, \overline{D})$ is an integral ring extension.

Without loss of generality, we may enlarge $F$ and suppose that $F$ is normal over $K$ (e.g., we may take $F = \overline{K}$). Let $f \in \mathrm{Int}_F(\Omega, \overline{D}) \subset F[X]$. In particular, $f$ is integral over $K[X]$, that is, it satisfies a monic equation of the form

$$f^n + g_{n-1} f^{n-1} + \cdots + g_1 f + g_0 = 0,$$

where $g_i \in K[X]$, for $i = 0, \ldots, n-1$. We claim that $g_i \in \mathrm{Int}_K(\Omega, \overline{D})$, for $i = 0, \ldots, n-1$, which will prove the claim. In fact, let

$$\Phi(T) = T^n + g_{n-1} T^{n-1} + \cdots + g_0 \in K[X][T],$$

and suppose that $\Phi(T)$ is irreducible over $K(X)$. The roots of $\Phi(T)$ are the conjugates of $f$ under the action of the Galois group $\mathrm{Gal}(F(X)/K(X)) \cong \mathrm{Gal}(F/K)$, which acts on the coefficients of the polynomial $f$. If $\sigma \in \mathrm{Gal}(F/K)$, then $\sigma(f) \in \mathrm{Int}_F(\Omega, \overline{D})$. In fact, for each $\alpha \in \Omega$, since $\Omega$ is $\mathrm{Gal}(F/K)$-invariant, we have $\alpha = \sigma(\alpha')$ for some $\alpha' \in \Omega$, therefore $\sigma(f)(\alpha) = \sigma(f(\alpha'))$ which still is an element of $\overline{D}$ (which likewise is left invariant under the action of $\mathrm{Gal}(F/K)$). Now, since each coefficient $g_i$ of $\Phi(T)$ is an elementary symmetric function of the elements $\sigma(f), \sigma \in \mathrm{Gal}(F/K)$, we have $g_i(\alpha) \in \overline{D}$, for each $\alpha \in \Omega$; thus $g_i \in \mathrm{Int}_K(\Omega, \overline{D})$, as claimed. □

To ease notation, we denote the absolute Galois group of $\mathbb{Q}_p$ ($p$ prime) by $G_p$.

**Theorem 2.21.** *Suppose* $\underline{E} = \prod_p E_p$ *and* $\underline{F} = \prod_p F_p$ *are locally bounded closed subsets of* $\widehat{\overline{\mathbb{Z}}}$. *Then the rings* $\mathrm{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\overline{\mathbb{Z}}})$ *and* $\mathrm{Int}_{\mathbb{Q}}(\underline{F}, \widehat{\overline{\mathbb{Z}}})$ *are equal if and only if* $G_p(E_p) = G_p(F_p)$, *for each* $p \in \mathbb{P}$.

*Proof.* Clearly, $\mathrm{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\overline{\mathbb{Z}}}) = \mathrm{Int}_{\mathbb{Q}}(\underline{F}, \widehat{\overline{\mathbb{Z}}})$ if and only if the two rings have the same localization at each $p \in \mathbb{P}$, that is, by Lemma 2.5, $\mathrm{Int}_{\mathbb{Q}}(E_p, \overline{\mathbb{Z}_p}) = \mathrm{Int}_{\mathbb{Q}}(F_p, \overline{\mathbb{Z}_p})$. Such a condition is equivalent to $\mathrm{Int}_{\mathbb{Q}_p}(E_p, \overline{\mathbb{Z}_p}) = \mathrm{Int}_{\mathbb{Q}_p}(F_p, \overline{\mathbb{Z}_p})$. In fact, one implication is obvious because $\mathrm{Int}_{\mathbb{Q}}(E_p, \overline{\mathbb{Z}_p})$ is the contraction to $\mathbb{Q}[X]$ of $\mathrm{Int}_{\mathbb{Q}_p}(E_p, \overline{\mathbb{Z}_p})$. Conversely, suppose that $\mathrm{Int}_{\mathbb{Q}}(E_p, \overline{\mathbb{Z}_p}) = \mathrm{Int}_{\mathbb{Q}}(F_p, \overline{\mathbb{Z}_p})$ and let $f \in \mathrm{Int}_{\mathbb{Q}_p}(E_p, \overline{\mathbb{Z}_p})$, say $f(X) = \sum_i \alpha_i X^i$. We can choose $g \in \mathbb{Q}[X]$ sufficiently $v_p$-adically close to $f(X)$, that is, $g(X) = \sum_i a_i X^i$, where $v_p(\alpha_i - a_i) \geq n$ for each $i \geq 0$, where $n \in \mathbb{N}$ is arbitrary large. Then $h = f - g \in p^n \mathbb{Z}_p[X]$, so, if $\alpha_p \in E_p$, it follows that $g(\alpha_p) = f(\alpha_p) + h(\alpha_p) \in \overline{\mathbb{Z}_p}$. Hence, $g \in \mathrm{Int}_{\mathbb{Q}}(E_p, \overline{\mathbb{Z}_p}) = \mathrm{Int}_{\mathbb{Q}}(F_p, \overline{\mathbb{Z}_p})$. If now $\beta_p \in F_p$, we have $f(\beta_p) = g(\beta_p) + h(\beta_p) \in \overline{\mathbb{Z}_p}$, which proves that $f \in \mathrm{Int}_{\mathbb{Q}_p}(F_p, \overline{\mathbb{Z}_p})$. The other containment $\mathrm{Int}_{\mathbb{Q}_p}(F_p, \overline{\mathbb{Z}_p}) \subseteq \mathrm{Int}_{\mathbb{Q}_p}(E_p, \overline{\mathbb{Z}_p})$ follows in the same way.

Let $p \in \mathbb{P}$ be a fixed prime and set $\widehat{R}_{p,E_p} = \mathrm{Int}_{\mathbb{Q}_p}(E_p, \overline{\mathbb{Z}_p})$ and $\widehat{R}_{p,F_p} = \mathrm{Int}_{\mathbb{Q}_p}(F_p, \overline{\mathbb{Z}_p})$. Since $E_p$, $F_p$ are subsets of $\overline{\mathbb{Z}_p}$ of bounded degree, there exists a finite Galois extension $K$ of $\mathbb{Q}_p$ containing both of them. By (2.19), $\widehat{R}_{p,E_p} = \mathrm{Int}_{\mathbb{Q}_p}(G_p(E_p), \overline{\mathbb{Z}_p})$ and $\widehat{R}_{p,F_p} = \mathrm{Int}_{\mathbb{Q}_p}(G_p(F_p), \overline{\mathbb{Z}_p})$. Clearly, $\widehat{R}_{p,E_p}$ and $\widehat{R}_{p,F_p}$ are equal if and only if they have the same integral closure in $K(X)$. By Lemma 2.20, this amounts to say that

$$\mathrm{Int}_K(G_p(E_p), \overline{\mathbb{Z}_p}) = \mathrm{Int}_K(G_p(F_p), \overline{\mathbb{Z}_p}). \tag{2.22}$$

Note that the rings of (2.22) are equal to $\mathrm{Int}_K(G_p(E_p), O_K)$, $\mathrm{Int}_K(G_p(F_p), O_K)$, respectively, where $O_K$ is the ring of integers of $K$. Moreover, $G_p(E_p)$ is a closed subset of $O_K$, being a finite union of closed sets $\sigma(E_p)$, $\sigma \in \mathrm{Gal}(K/\mathbb{Q}_p)$. Similarly, $G_p(F_p)$ is closed.

Finally, by [McQuillan 1991, Lemma 2], (2.22) holds if and only if $G_p(E_p) = G_p(F_p)$. □

Theorem 2.21 implies that the rings $\mathrm{Int}_{\mathbb{Q}}(\{\alpha\}, \widehat{\mathbb{Z}})$, $\alpha \in \widehat{\mathbb{Z}}$, are in one-to-one correspondence with the elements of $\widehat{\mathbb{Z}}$.

## 3. Construction of a Dedekind domain with prescribed class group

We review Chang's construction [2022] mentioned in the introduction and modify it in order to show that, given a group $G$ which is the direct sum of a countable family of finitely generated abelian groups, there exists a Dedekind domain $R$ with finite residue fields of prime characteristic, $\mathbb{Z}[X] \subset R \subseteq \mathbb{Q}[X]$, such that the class group of $R$ is $G$. As in [Eakin and Heinzer 1973], we show first that the ring constructed by Chang can also be represented as a generalized ring of integer-valued polynomials. In [Chang 2022, Lemma 3.4] it is proved that for each $n \in \mathbb{N}$ and $p \in \mathbb{P}$, there exists a DVR of $\mathbb{Q}(X)$ which is a residually algebraic extension of $\mathbb{Z}_{(p)}$ with ramification index equal to $n$; by means of Theorem 1.1, we can give an explicit representation of such an extension in terms of a valuation domain $V_{p,\alpha}$ associated to some $\alpha \in \overline{\mathbb{Z}_p}$ which generates a totally ramified extension of $\mathbb{Q}_p$ of degree $n$.

Let $I$ be a countable set and $G = \bigoplus_{i \in I} G_i$ be a direct sum of finitely generated abelian groups $G_i$. Suppose that for each $i \in I$ we have

$$G_i \cong \mathbb{Z}^{m_i} \oplus \mathbb{Z}/n_{i,1}\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/n_{i,k_i}\mathbb{Z}$$

for some uniquely determined nonnegative integers $m_i, n_{i,1}, \ldots, n_{i,k_i}$ satisfying $n_{i,j} \mid n_{i,j+1}$. We partition $\mathbb{P}$ into a family of finite subsets $\{\mathbb{P}_i\}_{i \in I}$ each of which contains arbitrary chosen $1 + k_i$ primes, namely $\mathbb{P}_i = \{p_i, q_{i,1}, \ldots, q_{i,k_i}\}$ and correspondingly for each $i \in I$ we fix the following $1 + k_i$ sets:

i) $E_{p_i}$ is a subset of $\mathbb{Z}_{p_i}$ of $m_i + 1$ elements $\{\alpha_{p_i,1}, \ldots, \alpha_{p_i,m_i+1}\}$ which are transcendental over $\mathbb{Q}$.

ii) For $j = 1, \ldots, k_i$, $E_{q_{i,j}} = \{\alpha_{q_{i,j}}\}$ a singleton of $\overline{\mathbb{Z}_{q_{i,j}}}$ such that $\alpha_{q_{i,j}}$ is transcendental over $\mathbb{Q}$ and $n_{i,j} = e_{\alpha_{q_{i,j}}}$, the ramification index of $\mathbb{Q}_p(\alpha_{q_{i,j}})$ over $\mathbb{Q}_p$.

We set $\underline{E}_i = E_{p_i} \times \prod_{j=1}^{k_i} E_{q_{i,j}}$ and also

$$R_i = \mathrm{Int}_{\mathbb{Q}}(E_{p_i}, \mathbb{Z}_{p_i}) \cap \bigcap_{j=1}^{k_i} \mathrm{Int}_{\mathbb{Q}}(E_{q_{i,j}}, \overline{\mathbb{Z}}_{q_{i,j}}) = \mathrm{Int}_{\mathbb{Q}}(\underline{E}_i, \overline{\widehat{\mathbb{Z}}}).$$

Since each of the unitary valuation overrings of $R_i$, namely $V_{p,\alpha_p}$, $p \in \mathbb{P}_i$ and $\alpha_p \in E_p$, is a DVR which is residually algebraic over $\mathbb{F}_p$ [Peruginelli 2017, Proposition 2.2], by [Eakin and Heinzer 1973, Theorem and Corollary] $R_i$ is a Dedekind domain with class group isomorphic to $G_i$.

We also set

$$R = \bigcap_{i \in I} R_i = \mathrm{Int}_{\mathbb{Q}}(\underline{E}, \overline{\widehat{\mathbb{Z}}}),$$

where $\underline{E} = \prod_i \underline{E}_i$. By [Chang 2022, Corollary 2.6], $R$ is an almost Dedekind domain with class group isomorphic to $G$.

As we already mentioned at the beginning of Section 2B, the ring $R = \mathrm{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\overline{\mathbb{Z}}})$ is not Dedekind in general. By Theorem 2.15, this happens precisely when $\underline{E}$ is polynomially factorizable. By a suitable modification of the above construction, we are going to show that there exists a polynomially factorizable subset $\underline{E}$ of $\widehat{\overline{\mathbb{Z}}}$ such that $R$ is Dedekind with class group isomorphic to $G$, thus giving a positive answer to [Chang 2022, Question 3.7].

**Theorem 3.1.** *Let $G$ be a direct sum of a countable family $\{G_i\}_{i\in I}$ of finitely generated abelian groups (which are not necessarily distinct). Then there exists a Dedekind domain $R$ between $\mathbb{Z}[X]$ and $\mathbb{Q}[X]$ with class group isomorphic to $G$. Moreover, for each $i \in I$, there exists a multiplicative subset $S_i$ of $\mathbb{Z}$ such that $S_i^{-1}R$ is a Dedekind domain with class group $G_i$.*

*Proof.* We keep the notation used in the above construction. Let $\mathbb{P}_r = \bigcup_{i\in I}(\mathbb{P}_i \setminus \{p_i\})$. For each $q = q_{i,j} \in \mathbb{P}_r$, for some $i \in I$ and $j \in \{1, \dots, k_i\}$, we set $n_q = n_{i,j}$. We choose a uniformizer $\tilde{q}$ of $\mathbb{Z}_q$ which is transcendental over $\mathbb{Q}$. Let $\tilde{\alpha}_q \in \overline{\mathbb{Z}_q}$ be a root of the Eisenstein polynomial $X^{n_q} - \tilde{q}$. Clearly, $\tilde{\alpha}_q$ is still transcendental over $\mathbb{Q}$ and it is well-known that $\mathbb{Q}_q(\tilde{\alpha}_q)$ is a totally ramified extension of $\mathbb{Q}_q$ of degree $n_q$. We now let $\alpha_q = \tilde{\alpha}_q + \lfloor \log q \rfloor$: this is another generator of $\mathbb{Q}_q(\tilde{\alpha}_q)$ over $\mathbb{Q}_q$ which still is transcendental over $\mathbb{Q}$ and has $v_q$-adic valuation zero. We then set $E_q = \{\alpha_q\}$ in the above construction.

Similarly, for each $p = p_i \in \mathbb{P} \setminus \mathbb{P}_r$, for some $i \in I$, let $m_p = m_{p_i}$. We choose distinct elements $\alpha_{p,i} \in \lfloor \log p \rfloor + p\mathbb{Z}_p$, for $i = 1, \dots, m_p + 1$, which are transcendental over $\mathbb{Q}$ and set $E_p = \{\alpha_{p,1}, \dots, \alpha_{p,m_p+1}\}$.

We show now that with these choices the subset $\underline{E} = \prod_p E_p \subset \widehat{\overline{\mathbb{Z}}}$ is polynomially factorizable, and therefore the corresponding domain $R = \mathrm{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\overline{\mathbb{Z}}})$ is a Dedekind domain by Theorem 2.15. By Lemma 2.12, we need to show that for each $g \in \mathbb{Z}[X]$, $\mathbb{P}_{g,\underline{E}}$ is finite. Let $g \in \mathbb{Z}[X]$ be a fixed polynomial. For $\alpha = (\alpha_p) \in \underline{E}$, we have:

- $\alpha_p = pa + \lfloor \log p \rfloor$, for some $a \in \mathbb{Z}_p$, if $p \in \mathbb{P} \setminus \mathbb{P}_r$.
- $\alpha_p = \tilde{\alpha}_p + \lfloor \log p \rfloor$, if $p \in \mathbb{P}_r$, where $\tilde{\alpha}_p$ is a root of an Eisenstein polynomial, so, in particular, $v_p(\tilde{\alpha}_p) > 0$.

For each $p \in \mathbb{P}$, let $\pi_p$ be a uniformizer of $\mathbb{Q}_p(\alpha_p)$ (which is just $p$ if $p \notin \mathbb{P}_r$). We then have

$$g(\alpha_p) \equiv g(\lfloor \log p \rfloor) \pmod{\pi_p}.$$

Now, for all $p$ sufficiently large, $g(\lfloor \log p \rfloor)$ is not divisible by $p$, since

$$\lim_{x\to\infty} \frac{g(\log x)}{x} = 0.$$

Hence, $\mathbb{P}_{g,\underline{E}}$ is finite.

The fact that $\mathrm{Int}_{\mathbb{Q}}(\underline{E}, \overline{\widehat{\mathbb{Z}}})$ has class group equal to $G$ follows either by [Chang 2022, Corollary 2.6] or by applying Lemma 2.14 and Proposition 2.10, by noting that $\mathrm{Pic}(\mathrm{Int}_{\mathbb{Q}}(E_p, \overline{\mathbb{Z}_p})) = \mathbb{Z}^{m_p}$ for each $p \in \mathbb{P} \setminus \mathbb{P}_r$ and $\mathrm{Pic}(\mathrm{Int}_{\mathbb{Q}}(E_q, \overline{\mathbb{Z}_q})) = \mathbb{Z}/n_q\mathbb{Z}$ for each $q \in \mathbb{P}_r$.

For the last claim, if $i \in I$, we let $S_i$ be the multiplicative subset of $\mathbb{Z}$ generated by $\mathbb{P} \setminus \mathbb{P}_i$. Then, by Lemma 2.5, $S_i^{-1}\,\mathrm{Int}_{\mathbb{Q}}(\underline{E}, \overline{\widehat{\mathbb{Z}}}) = \mathrm{Int}_{\mathbb{Q}}(\underline{E}_i, \overline{\widehat{\mathbb{Z}}})$ which has class group isomorphic to $G_i$ by Lemma 2.14 and Proposition 2.10. □

## Acknowledgments

The author wishes to thank Jean-Luc Chabert for reading a preliminary version of this paper. The author also wishes also to thank the anonymous referees for helpful remarks.

## References

[Cahen and Chabert 1997] P.-J. Cahen and J.-L. Chabert, *Integer-valued polynomials*, Mathematical Surveys and Monographs **48**, American Mathematical Society, Providence, RI, 1997. MR Zbl

[Chabert and Peruginelli 2016] J.-L. Chabert and G. Peruginelli, "Polynomial overrings of Int($\mathbb{Z}$)", *J. Commut. Algebra* **8**:1 (2016), 1–28. MR Zbl

[Chabert and Peruginelli 2018] J.-L. Chabert and G. Peruginelli, "Adelic versions of the Weierstrass approximation theorem", *J. Pure Appl. Algebra* **222**:3 (2018), 568–584. MR Zbl

[Chang 2022] G. W. Chang, "The ideal class group of polynomial overrings of the ring of integers", *J. Korean Math. Soc.* **59**:3 (2022), 571–594. MR Zbl

[Chang and Geroldinger 2024] G. W. Chang and A. Geroldinger, "On Dedekind domains whose class groups are direct sums of cyclic groups", *J. Pure Appl. Algebra* **228**:1 (2024), art. id. 107470. MR

[Claborn 1966] L. Claborn, "Every abelian group is a class group", *Pacific J. Math.* **18** (1966), 219–222. MR Zbl

[Eakin and Heinzer 1973] P. Eakin and W. Heinzer, "More noneuclidian PID's and Dedekind domains with prescribed class group", *Proc. Amer. Math. Soc.* **40** (1973), 66–68. MR Zbl

[Frisch 2013] S. Frisch, "Integer-valued polynomials on algebras", *J. Algebra* **373** (2013), 414–425. MR Zbl

[Frisch 2014] S. Frisch, "Corrigendum to "Integer-valued polynomials on algebras" [J. Algebra 373 (2013) 414–425]", *J. Algebra* **412** (2014), 282. MR Zbl

[Gilmer 1992] R. Gilmer, *Multiplicative ideal theory*, Queen's Papers in Pure and Applied Mathematics **90**, Queen's University, Kingston, ON, 1992. MR

[Gilmer and Heinzer 1968] R. Gilmer and W. Heinzer, "Irredundant intersections of valuation rings", *Math. Z.* **103** (1968), 306–317. MR Zbl

[Gilmer et al. 1990] R. Gilmer, W. Heinzer, D. Lantz, and W. Smith, "The ring of integer-valued polynomials of a Dedekind domain", *Proc. Amer. Math. Soc.* **108**:3 (1990), 673–681. MR Zbl

[Glivická et al. 2023] J. Glivická, E. Sgallová, and J. Šaroch, "Controlling distribution of prime sequences in discretely ordered principal ideal subrings of $\mathbb{Q}[x]$", *Proc. Amer. Math. Soc.* **151**:8 (2023), 3281–3290. MR Zbl

[Glivický and Šaroch 2013] P. Glivický and J. Šaroch, "Quasi-Euclidean subrings of $\mathbb{Q}[X]$", *Comm. Algebra* **41**:11 (2013), 4267–4277. MR Zbl

[Loper and Werner 2012] K. A. Loper and N. J. Werner, "Generalized rings of integer-valued polynomials", *J. Number Theory* **132**:11 (2012), 2481–2490. MR Zbl

[McQuillan 1985] D. L. McQuillan, "Rings of integer-valued polynomials determined by finite sets", *Proc. Roy. Irish Acad. Sect. A* **85**:2 (1985), 177–184. MR Zbl

[McQuillan 1991] D. L. McQuillan, "On a theorem of R. Gilmer", *J. Number Theory* **39**:3 (1991), 245–250. MR Zbl

[Peruginelli 2017] G. Peruginelli, "Transcendental extensions of a valuation domain of rank one", *Proc. Amer. Math. Soc.* **145**:10 (2017), 4211–4226. MR

[Peruginelli 2018] G. Peruginelli, "Prüfer intersection of valuation domains of a field of rational functions", *J. Algebra* **509** (2018), 240–262. MR Zbl

[Peruginelli 2023] G. Peruginelli, "Stacked pseudo-convergent sequences and polynomial Dedekind domains", preprint, 2023. arXiv 2303.11740

[Peruginelli and Werner 2017] G. Peruginelli and N. J. Werner, "Non-triviality conditions for integer-valued polynomial rings on algebras", *Monatsh. Math.* **183**:1 (2017), 177–189. MR Zbl

Received July 9, 2022. Revised April 19, 2023.

GIULIO PERUGINELLI
DIPARTIMENTO DI MATEMATICA "TULLIO LEVI-CIVITA"
UNIVERSITÀ DI PADOVA
PADOVA
ITALY
gperugin@math.unipd.it

PACIFIC JOURNAL OF MATHEMATICS
Vol. 324, No. 2, 2023
https://doi.org/10.2140/pjm.2023.324.353

# THE COHOMOLOGICAL BRAUER GROUP OF WEIGHTED PROJECTIVE SPACES AND STACKS

MINSEON SHIN

**We compute the cohomological Brauer groups of twists of weighted projective spaces and weighted projective stacks, generalizing Gabber's computation of the Brauer group of Brauer–Severi varieties. A key ingredient in our proof is a description of the Brauer group of toric varieties due to DeMeyer, Ford, Miranda (1993).**

## 1. Introduction

Weighted projective spaces and stacks are a natural generalization of projective space that often arise in the construction of certain moduli spaces. For example, the moduli space of cubic surfaces is isomorphic to $\mathbb{P}(1, 2, 3, 4, 5)$, see, e.g., [13, Section 9.4.5]. Over a field $k$ of characteristic not 2 or 3, the moduli stack of elliptic curves $\mathcal{M}_{1,1,k}$ is isomorphic to an open substack of $\mathcal{P}_k(4, 6)$.

To recall the construction of weighted projective spaces and stacks, let $n \geq 1$ and let $\rho = (\rho_0, \dots, \rho_n)$ be an $(n+1)$-tuple of positive integers. For any field $k$, we may define an equivalence relation on $k^{n+1} \setminus \{(0, \dots, 0)\}$ by

$$(x_0, \dots, x_n) \sim (u^{\rho_0} x_0, \dots, u^{\rho_n} x_n)$$

for all units $u \in k^\times$. The *weighted projective space* $\mathbb{P}_{\mathbb{Z}}(\rho)$ is the scheme-theoretic quotient of this action; it is the scheme whose $k$-rational points correspond to the equivalence classes of this equivalence relation. Taking the stack-theoretic quotient of the above action gives the *weighted projective stack* $\mathcal{P}_{\mathbb{Z}}(\rho)$ associated to $\rho$. If every $\rho_i$ is equal to 1, then $\mathbb{P}_{\mathbb{Z}}(\rho)$ and $\mathcal{P}_{\mathbb{Z}}(\rho)$ are isomorphic to the (unweighted) projective space $\mathbb{P}^n$.

In this paper, we are interested in the cohomological Brauer groups of étale twists of weighted projective spaces and weighted projective stacks. For any scheme $S$, we denote $\mathrm{Br}'(S) := \mathrm{H}^2_{\text{ét}}(S, \mathbb{G}_m)_{\text{tors}}$ the *cohomological Brauer group* of $S$. In the unweighted case, an étale twist of projective space is called a *Brauer–Severi scheme*; it is well known that to every Brauer–Severi scheme $f : X \to S$ there is an associated class $[X] \in \mathrm{Br}'(S)$ and that the pullback of $[X]$ to $\mathrm{Br}'(X)$ is trivial.

*MSC2020:* 14F22.
*Keywords:* Brauer group, weighted projective spaces.

A theorem of Gabber states that the induced map $\mathrm{Br}'(S)/\langle[X]\rangle \to \mathrm{Br}'(X)$ is in fact an isomorphism.

**Theorem 1.1** (Gabber [18, Chapter II, Theorem 2]). *Let $S$ be a scheme and let $f : X \to S$ be a Brauer–Severi scheme. Then the sequence*

$$\Gamma(S, \underline{\mathbb{Z}}) \to \mathrm{Br}'(S) \xrightarrow{f^*} \mathrm{Br}'(X) \to 0 \tag{1.1.1}$$

*is exact, where the first map sends $1 \mapsto [X]$.*

The purpose of this paper is to extend the above theorem to include weighted projective spaces and stacks.

**Theorem 1.2.** *Let $f_X : X \to S$ be a morphism of schemes such that there exists an étale surjection $S' \to S$ such that $X \times_S S' \simeq \mathbb{P}_{S'}(\rho)$. Then there is a natural Brauer class $[X] \in \mathrm{Br}'(S)$ associated to $X$, and the sequence*

$$\Gamma(S, \underline{\mathbb{Z}}) \to \mathrm{Br}'(S) \xrightarrow{f_X^*} \mathrm{Br}'(X) \to 0 \tag{1.2.1}$$

*is exact, where the first map sends $1 \mapsto [X]$.*

**Theorem 1.3.** *Let $S$ be a scheme, let $f_{\mathcal{X}} : \mathcal{X} \to S$ be a morphism of algebraic stacks such that there exists an étale surjection $S' \to S$ such that $\mathcal{X} \times_S S' \simeq \mathcal{P}_{S'}(\rho)$. Then there is a natural Brauer class $[\mathcal{X}] \in \mathrm{Br}'(S)$ associated to $\mathcal{X}$, and the sequence*

$$\Gamma(S, \underline{\mathbb{Z}}) \to \mathrm{Br}'(S) \xrightarrow{f_{\mathcal{X}}^*} \mathrm{Br}'(\mathcal{X}) \to 0 \tag{1.3.1}$$

*is exact, where the first map sends $1 \mapsto [\mathcal{X}]$. If $\pi : \mathcal{X} \to X$ denotes the coarse moduli space of $\mathcal{X}$, then $X$ satisfies the hypothesis of Theorem 1.2, and pullback by $\pi$ induces a commutative diagram*

$$\begin{tikzcd}
\Gamma(S, \underline{\mathbb{Z}}) \arrow[r] & \mathrm{Br}'(S) \arrow[r, "f_{\mathcal{X}}^*"] & \mathrm{Br}'(\mathcal{X}) \arrow[r] & 0 \\
\Gamma(S, \underline{\mathbb{Z}}) \arrow[r] \arrow[u, "\times\,\mathrm{lcm}(\rho)"] & \mathrm{Br}'(S) \arrow[r, "f_X^*"'] \arrow[u, equal] & \mathrm{Br}'(X) \arrow[r] \arrow[u, "\pi^*"'] & 0
\end{tikzcd} \tag{1.3.2}$$

*where the rows are (1.2.1) and (1.3.1) and the leftmost vertical map denotes multiplication-by-*$\mathrm{lcm}(\rho)$.

**1.4.** ***Outline of the paper.*** To prove Theorems 1.2 and 1.3, we show that $\boldsymbol{R}^1 f_* \mathbb{G}_m \simeq \underline{\mathbb{Z}}$ and $\boldsymbol{R}^2 f_* \mathbb{G}_m = 0$ and apply the Leray spectral sequence to the morphism $f$. For the claim that $\boldsymbol{R}^2 f_* \mathbb{G}_m = 0$, a deformation theory argument of Mathur (personal communication, 2019) which uses a Tannaka duality result of Hall and Rydh [22], reduces us to the case where $S$ is the spectrum of a field. Here, the proofs of Theorems 1.2 and 1.3 require different approaches (indeed, a Deligne–Mumford stack $\mathcal{X}$ and its coarse moduli space $X$ may have nonisomorphic (Picard groups and) Brauer groups in general).

As we recall in Section 3.2, a weighted projective space $\mathbb{P}(\rho)$ is a toric variety. In Section 3, we use the results of DeMeyer, Ford and Miranda [11] on the Brauer group of toric varieties to compute the Brauer group of $\mathbb{P}(\rho)$ over an algebraically closed field; taking the prime-to-$p$ limit of dilations of the toric variety reduces us to computing the $p$-torsion when each weight $\rho_i$ is a power of $p$. In Section 5 we prove Theorem 1.3, for which the key observation turns out to be that the $\mathbb{G}_m$-action on $\mathbb{A}^{n+1}$ extends to an action of the multiplicative monoid $\mathbb{A}^1$ on $\mathbb{A}^{n+1}$.

## 2. Weighted projective spaces

In this section, we recall basic facts about weighted projective spaces (in particular, regarding their Picard group Lemma 2.8 and cohomology of line bundles Lemma 2.9) which will be used in the proof of Theorem 1.2. For general background on weighted projective spaces, we refer to [12; 29].

**2.1.** For a weight vector $\rho = (\rho_0, \ldots, \rho_n)$, the *weighted projective space* associated to $\rho$ is

$$\mathbb{P}_{\mathbb{Z}}(\rho) := \operatorname{Proj} \mathbb{Z}[t_0, \ldots, t_n],$$

where $\mathbb{Z}[t_0, \ldots, t_n]$ has the $\mathbb{Z}$-grading defined by $\deg(t_i) = \rho_i$. We set

$$\mathbb{P}_S(\rho) := \mathbb{P}_{\mathbb{Z}}(\rho) \times_{\operatorname{Spec} \mathbb{Z}} S$$

for any scheme $S$. By [14, Lemme (2.1.6), Proposition (2.4.7)], the weighted projective space $\mathbb{P}_{\mathbb{Z}}(\rho)$ is projective. Thus, if $S$ is quasicompact and admits an ample line bundle, then the same is true for $\mathbb{P}_S(\rho)$; hence in this case $\mathrm{Br} = \mathrm{Br}'$ for $\mathbb{P}_S(\rho)$ by a theorem of Gabber [9] (i.e., the Azumaya Brauer group coincides with the cohomological Brauer group).

**2.2.** Suppose a positive integer $d$ divides all $\rho_i$ and set $\rho/d := (\rho_0/d, \ldots, \rho_n/d)$. Then there is a natural isomorphism $\mathbb{P}_{\mathbb{Z}}(\rho) \simeq \mathbb{P}_{\mathbb{Z}}(\rho/d)$ by [14, Proposition (2.4.7)(i)], and under this isomorphism $\mathcal{O}_{\mathbb{P}_{\mathbb{Z}}(\rho/d)}(\ell)$ corresponds to $\mathcal{O}_{\mathbb{P}_{\mathbb{Z}}(\rho)}(d\ell)$ for all $\ell \in \mathbb{Z}$.

If $\rho, \sigma$ are two weight vectors such that one is a permutation of the other, then the corresponding weighted projective spaces $\mathbb{P}_{\mathbb{Z}}(\rho)$, $\mathbb{P}_{\mathbb{Z}}(\sigma)$ are isomorphic. The converse is not true in general, but is true if $\rho, \sigma$ satisfy a certain "normalization" condition.

**Definition 2.3** (normalized weight vectors [2, Section 2]). We say that $\rho = (\rho_0, \ldots, \rho_n)$ satisfies (N) if, for all $0 \le i \le n$, we have $\gcd(\{\rho_j\}_{j \ne i}) = 1$.

**Lemma 2.4** [2, Section 8]. *Let $\rho, \sigma$ be two weight vectors satisfying* (*N*). *We have $\mathbb{P}_{\mathbb{Z}}(\rho) \simeq \mathbb{P}_{\mathbb{Z}}(\sigma)$ if and only if $\rho$ is a permutation of $\sigma$.*

By Lemma 2.5 below, every weight vector $\rho$ has an associated normalized weight vector $\rho'$ such that $\rho'$ satisfies (N) and $\mathbb{P}_{\mathbb{Z}}(\rho) \simeq \mathbb{P}_{\mathbb{Z}}(\rho')$. Thus in Theorem 1.2 we may always assume that our weight vector $\rho$ satisfies (N).

**Lemma 2.5** (reduction of weights [10, Proposition 1.3; 12, Section 1.3.1; 2, Sections 1.3, 1.4]). *Suppose* $\gcd(\rho) = 1$. *Define the constants*

$$d_i := \gcd(\{\rho_j\}_{j \neq i}), \quad s_i := \operatorname{lcm}(\{d_j\}_{j \neq i}), \quad s := \operatorname{lcm}(s_0, \dots, s_n),$$
$$\rho'_i := \rho_i / s_i, \quad \rho' := (\rho'_0, \dots, \rho'_n)$$

*and let* $R' := \mathbb{Z}[t'_0, \dots, t'_n]$ *be the ring with the* $\mathbb{Z}$*-grading determined by* $\deg(t'_i) = \rho'_i$. *The ring homomorphism* $R' \to R$ *sending* $t'_i \mapsto t_i^{d_i}$ (*which multiplies the degree by* $s$) *induces an isomorphism*

$$\varphi : \mathbb{P}_{\mathbb{Z}}(\rho) \to \mathbb{P}_{\mathbb{Z}}(\rho')$$

*of schemes. We have*

$$\operatorname{lcm}(\rho) = s \cdot \operatorname{lcm}(\rho') \tag{2.5.1}$$

*since* $v_p(\operatorname{lcm}(\rho)) = \alpha_{i_0}$ *and* $v_p(\operatorname{lcm}(\rho')) = \alpha_{i_0} - \alpha_{i_{n-1}}$ *for any prime* $p$, *in the notation of* [2, Section 1.2].

*For any integer* $\ell$, *there exists a unique pair* $(b_i(\ell), c_i(\ell)) \in \mathbb{Z}^2$ *satisfying* $0 \leq b_i(\ell) < d_i$ *and* $\ell = b_i(\ell)\rho_i + c_i(\ell)\, d_i$; *set* $\ell' := \ell - \sum_{i=0}^n b_i(\ell)\rho_i$. *The multiplication-by-*$(t_0^{b_0(\ell)} \cdots t_n^{b_n(\ell)})$ *map* $R(\ell') \to R(\ell)$ *induces an isomorphism* $\mathcal{O}_{\mathbb{P}_{\mathbb{Z}}(\rho)}(\ell') \simeq \mathcal{O}_{\mathbb{P}_{\mathbb{Z}}(\rho)}(\ell)$ *of* $\mathcal{O}_{\mathbb{P}_{\mathbb{Z}}(\rho)}$*-modules. Furthermore* $\ell'$ *is divisible by* $s$ *and we obtain an isomorphism*

$$\mathcal{O}_{\mathbb{P}_{\mathbb{Z}}(\rho')}(\ell'/s) \simeq \varphi_*(\mathcal{O}_{\mathbb{P}_{\mathbb{Z}}(\rho)}(\ell))$$

*of* $\mathcal{O}_{\mathbb{P}_{\mathbb{Z}}(\rho)}$*-modules. In particular, we have*

$$\varphi^*(\mathcal{O}_{\mathbb{P}_{\mathbb{Z}}(\rho')}(\ell)) \simeq \mathcal{O}_{\mathbb{P}_{\mathbb{Z}}(\rho)}(s\ell) \tag{2.5.2}$$

*for all* $\ell \in \mathbb{Z}$ *since* $b_i(s\ell) = 0$.

**Remark 2.6.** By Lemma 2.5, all weighted projective lines $\mathbb{P}_{\mathbb{Z}}(q_0, q_1)$ are isomorphic to $\mathbb{P}^1_{\mathbb{Z}}$; thus, for Theorem 1.2, we may assume $n \geq 2$.

**Lemma 2.7.** *The sheaf* $\mathcal{O}_{\mathbb{P}_{\mathbb{Z}}(\rho)}(r)$ *is reflexive for any* $r \in \mathbb{Z}$. *If* $\rho$ *satisfies* (*N*), *the sheaf* $\mathcal{O}_{\mathbb{P}_{\mathbb{Z}}(\rho)}(r)$ *is invertible if and only if* $\operatorname{lcm}(\rho)$ *divides* $r$.

**Lemma 2.8** (Picard group of $\mathbb{P}(\rho)$ [2, Section 6.1]). *For any connected locally Noetherian scheme* $S$, *the map*

$$\mathbb{Z} \oplus \operatorname{Pic}(S) \to \operatorname{Pic}(\mathbb{P}_S(\rho))$$

*sending*

$$(\ell, \mathcal{L}) \mapsto \mathcal{O}_{\mathbb{P}_S(\rho)}(\ell \cdot \operatorname{lcm}(\rho)) \otimes f_S^* \mathcal{L}$$

*is an isomorphism.* (*See also* [26, Section 6].)

*Proof.* By Section 2.2, we may assume $\gcd(\rho) = 1$. In [2] the desired claim is proved assuming that $\rho$ satisfies (N). If $\rho$ does not satisfy (N), then we conclude using (2.5.1) and (2.5.2). □

**Lemma 2.9** (cohomology of $\mathcal{O}_{\mathbb{P}(\rho)}(\ell)$ [10, Section 3]). *Let $A$ be a ring and set $X := \mathbb{P}_A(\rho)$.*

(1) *For $\ell \geq 0$, the $A$-module $\mathrm{H}^0(X, \mathcal{O}_X(\ell))$ is free with basis consisting of monomials $t_0^{e_0} \cdots t_n^{e_n}$ such that $e_0, \ldots, e_n \in \mathbb{Z}_{\geq 0}$ and $\rho_0 e_0 + \cdots + \rho_n e_n = \ell$.*

(2) *For $\ell < 0$, the $A$-module $\mathrm{H}^n(X, \mathcal{O}_X(\ell))$ is free with basis consisting of monomials $t_0^{e_0} \cdots t_n^{e_n}$ such that $e_0, \ldots, e_n \in \mathbb{Z}_{<0}$ and $\rho_0 e_0 + \cdots + \rho_n e_n = \ell$.*

(3) *If $(i, \ell) \notin (\{0\} \times \mathbb{Z}_{\geq 0}) \cup (\{n\} \times \mathbb{Z}_{<0})$, then $\mathrm{H}^i(X, \mathcal{O}_X(\ell)) = 0$.*

(4) *For any $A$-module $M$ and any $(i, \ell)$, the canonical map*

$$\mathrm{H}^i(X, \mathcal{O}_X(\ell)) \otimes_A M \to \mathrm{H}^i(X, \mathcal{O}_X(\ell) \otimes_A M)$$

*is an isomorphism.*

**Remark 2.10.** The projection $\mathbb{P}_{\mathbb{Z}}(\rho) \to \operatorname{Spec} \mathbb{Z}$ is a flat morphism of relative dimension $n$, and its geometric fibers are normal. By [12, Section 1.3.3(iii)], we have that $\mathbb{P}_S(\rho) \to S$ is smooth if and only if $\mathbb{P}_S(\rho) \simeq \mathbb{P}^n_S$. If $\rho$ satisfies (N), then Lemma 2.4 implies that $\mathbb{P}_S(\rho) \simeq \mathbb{P}^n_S$ if and only if $\rho = (1, \ldots, 1)$.

## 3. Over an algebraically closed field

In this section, we prove Lemma 3.1 (i.e., Theorem 1.2 when $S = \operatorname{Spec} k$ for an algebraically closed field $k$). We will consider arbitrary fields in Lemma 4.1, and generalize from fields to (strictly henselian) local rings in Lemma 4.3.

**Lemma 3.1.** *If $k$ is an algebraically closed field, then $\mathrm{H}^2_{ét}(\mathbb{P}_k(\rho), \mathbb{G}_m) = 0$.*

*Proof (outline of argument).* In Section 3.2, we recall how to construct a fan $\Delta$ such that $\mathbb{P}_k(\rho)$ is isomorphic to the toric variety $X = X(\Delta)$. In Section 3.3, we recall a result of DeMeyer, Ford and Miranda giving an isomorphism

$$\mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_m) \simeq \check{\mathrm{H}}^2(\mathfrak{U}, \mathbb{G}_m),$$

where $\mathfrak{U}$ denotes the standard affine open cover of the toric variety $X$ (corresponding to the maximal cones of the fan $\Delta$). We show (in Section 3.4 and Lemma 3.5) that it suffices to show that the $p$-torsion vanishes, when each weight in $\rho = (\rho_0, \ldots, \rho_n)$ is a power of $p$. In Sections 3.7–3.9, we define a double complex $A^{\bullet,\bullet}$ such that the spectral sequence $\{\mathrm{E}_\bullet^{\bullet,\bullet}\}$ corresponding to the horizontal filtration on $A^{\bullet,\bullet}$ satisfies $\check{\mathrm{H}}^p(\mathfrak{U}, \mathbb{G}_m) \simeq \mathrm{E}_2^{p,0}$ for all $p$. We compute $\mathrm{E}_2^{2,0}$ in Sections 3.10 and 3.11. □

**3.2.** ***Presentation as a toric variety.*** We recall from [17, Section 2.2; 8, Example 3.1.17] how to view a weighted projective space as a toric variety (i.e., what the fan is).

Let $\mathrm{U} \in \mathrm{GL}_{n+1}(\mathbb{Z})$ be an invertible matrix which has $\rho$ as its first row (using the Euclidean algorithm, do column operations on $\rho$ to reduce to $(1, 0, \dots, 0)$, then apply the inverse column operations in the reverse order on the identity matrix $\mathrm{id}_{n+1}$); let $\mathrm{Y} \in \mathrm{Mat}_{(n+1)\times n}(\mathbb{Z})$ be the matrix obtained by removing the leftmost column of $\mathrm{U}^{-1}$; let $\boldsymbol{v}_0, \dots, \boldsymbol{v}_n \in \mathbb{Z}^n$ be the rows of Y; then $\mathbb{P}(\rho)$ is isomorphic to the toric variety associated to the fan $\Delta$ whose maximal cones are generated by the $n$-element subsets of $\{\boldsymbol{v}_0, \dots, \boldsymbol{v}_n\}$.

**3.3.** ***Reduce to computing the subgroup of Zariski-locally trivial Brauer classes.*** Let $\Delta'$ be a nonsingular subdivision of $\Delta$, and let $X'$ be the toric variety associated to $\Delta'$. The morphism of fans $\Delta' \to \Delta$ gives rise to a morphism of toric varieties $X' \to X$ which is a resolution of singularities for $X$. As in [11], we set

$$\mathrm{H}^2(K/X_{\text{ét}}, \mathbb{G}_m) := \ker(\mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_m) \to \mathrm{H}^2_{\text{ét}}(K, \mathbb{G}_m)),$$

since $X'$ is regular, the restriction $\mathrm{H}^2_{\text{ét}}(X', \mathbb{G}_m) \to \mathrm{H}^2_{\text{ét}}(K, \mathbb{G}_m)$ is injective; hence there is an exact sequence

$$0 \to \mathrm{H}^2(K/X_{\text{ét}}, \mathbb{G}_m) \to \mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_m) \to \mathrm{H}^2_{\text{ét}}(X', \mathbb{G}_m)$$

of abelian groups. Here $X'$ is a smooth, proper, geometrically connected, rational $k$-scheme; hence $\mathrm{H}^2_{\text{ét}}(X', \mathbb{G}_m) = 0$ by birational invariance of the Brauer group (see [21, corollaire 7.3] in characteristic 0 and [6, Corollary 5.2.6] in general); thus it remains to compute $\mathrm{H}^2(K/X_{\text{ét}}, \mathbb{G}_m)$. By [11, 4.3, 5.1], there are natural isomorphisms

$$\check{\mathrm{H}}^2(\mathfrak{U}, \mathbb{G}_m) \simeq \mathrm{H}^2_{\text{zar}}(X, \mathbb{G}_m) \simeq \mathrm{H}^2(K/X_{\text{ét}}, \mathbb{G}_m), \tag{3.3.1}$$

where $\mathfrak{U} = \{U_{\sigma_0}, \dots, U_{\sigma_n}\}$ is the Zariski cover of $X$ corresponding to the set of maximal cones of $\Delta$.

**3.4.** ***Limit of dilations.*** Let $A$ be a ring and let $X$ be the toric variety (over $A$) associated to a fan $\Delta$ of cones in $\mathsf{N}_{\mathbb{Q}}$. For any positive integer $d$, the multiplication-by-$d$ map $\times d : \mathsf{N} \to \mathsf{N}$ induces a finite $A$-morphism

$$\theta_d : X \to X,$$

which is equivariant for the $d$-th power map on tori. This is called a *dilation* [7, Section 6] (or *toric Frobenius* [23, Remark 4.14]). For a cone $\sigma$ of $\Delta$, this is the $A$-algebra endomorphism of $\Gamma(U_\sigma, \mathcal{O}_{U_\sigma}) = A[\sigma^\vee \cap \mathsf{M}]$ sending $\chi^{\mathsf{m}} \mapsto \chi^{d\mathsf{m}}$ for $\mathsf{m} \in \sigma^\vee \cap \mathsf{M}$. If $\sigma$ is a smooth cone, then $\theta_d : U_\sigma \to U_\sigma$ is flat for any $d$.

We view $\mathbb{N}$ as a category whose objects correspond to positive integers $m \in \mathbb{N}$ and there is a morphism $m_1 \to m_2$ if $m_1$ divides $m_2$. Let $S \subset \mathbb{N}$ be a multiplicatively closed subset; there is a functor $S^{\mathrm{op}} \to (\mathrm{Sch})$ sending $m \mapsto X$ and $\{m_1 \to m_2\} \mapsto \theta_{m_2/m_1}$; the limit

$$X^{1/S} := \varprojlim(\theta_{m_2/m_1} : X \to X)$$

of the resulting projective system is representable by a scheme since all the transition maps are affine. The scheme $X^{1/S}$ is isomorphic to the monoid scheme obtained by the usual construction with the finite free $\mathbb{Z}$-module $\mathsf{N}$ and its dual $\mathsf{M}$ replaced by the $S^{-1}\mathbb{Z}$-module $S^{-1}\mathsf{N}$ and its dual $S^{-1}\mathsf{M} = \mathrm{Hom}_{S^{-1}\mathbb{Z}}(S^{-1}\mathsf{N}, S^{-1}\mathbb{Z})$. More precisely, set

$$U_\sigma^{1/S} := \mathrm{Spec}\, A[\sigma^\vee \cap S^{-1}\mathsf{M}];$$

for any face $\tau$ of $\sigma$, the canonical map $U_\tau^{1/S} \to U_\sigma^{1/S}$ is an open immersion; then $U_{\sigma_1}^{1/S}$ and $U_{\sigma_2}^{1/S}$ are glued along the common open subscheme $U_{\sigma_1 \cap \sigma_2}^{1/S}$.

If $A$ is reduced, then we have

$$\Gamma(U_\sigma, \mathbb{G}_m) = (A[\sigma^\vee \cap \mathsf{M}])^\times = A^\times \cdot (\sigma^\perp \cap \mathsf{M}) \tag{3.4.1}$$

for any cone $\sigma \in \Delta$; hence, by (3.3.1), the pullback

$$\theta_d^* : \mathrm{H}^p_{\mathrm{zar}}(X, \mathbb{G}_m) \to \mathrm{H}^p_{\mathrm{zar}}(X, \mathbb{G}_m)$$

is multiplication-by-$d$. In the limit, we obtain a natural isomorphism

$$S^{-1}(\mathrm{H}^p_{\mathrm{zar}}(X, \mathbb{G}_m)) \simeq \mathrm{H}^p_{\mathrm{zar}}(X^{1/S}, \mathbb{G}_m) \tag{3.4.2}$$

of $S^{-1}\mathbb{Z}$-modules.

**Lemma 3.5.** *Let $d$ be a positive integer dividing $\rho_i$, and set $\rho' := (\rho'_0, \dots, \rho'_n)$ where $\rho'_i := \rho_i/d$ and $\rho'_j := \rho_j$ for $j \neq i$. If $d \in S$, then $\mathbb{P}_{\mathbb{Z}}(\rho)^{1/S} \simeq \mathbb{P}_{\mathbb{Z}}(\rho')^{1/S}$.*

*Proof.* As in Section 3.2, let $\mathrm{U}, \mathrm{U}' \in \mathrm{GL}_{n+1}(\mathbb{Z})$ be invertible matrices whose first rows are $\rho$ and $\rho'$, respectively. Let $\mathrm{U}^\circ \in \mathrm{GL}_{n+1}(S^{-1}\mathbb{Z})$ be the matrix obtained by dividing the $i$-th column of $\mathrm{U}$ by $d$; then $(\mathrm{U}^\circ)^{-1}$ is obtained by multiplying the $i$-th row of $\mathrm{U}^{-1}$ by $d$; this does not change the cones since we are just replacing $\mathrm{v}'_i$ by $\frac{1}{d}\mathrm{v}'_i$. Set $\mathrm{V} := \mathrm{U}' \cdot (\mathrm{U}^\circ)^{-1} \in \mathrm{GL}_{n+1}(S^{-1}\mathbb{Z})$; since the first rows of $\mathrm{U}^\circ, \mathrm{U}'$ are the same, the matrix $\mathrm{V}$ has the form

$$\mathrm{V} = \begin{bmatrix} 1 & \mathbf{0} \\ \mathrm{V}' & \mathrm{V}'' \end{bmatrix}$$

for some $\mathrm{V}' \in \mathrm{Mat}_{n\times 1}(S^{-1}\mathbb{Z})$ and $\mathrm{V}'' \in \mathrm{GL}_n(S^{-1}\mathbb{Z})$. Let $\mathrm{Y}^\circ, \mathrm{Y}' \in \mathrm{Mat}_{(n+1)\times n}(S^{-1}\mathbb{Z})$ be the matrices obtained by removing the leftmost column of $(\mathrm{U}^\circ)^{-1}$, $(\mathrm{U}')^{-1}$ respectively; then $(\mathrm{U}')^{-1} \cdot \mathrm{V} = (\mathrm{U}^\circ)^{-1}$ implies $\mathrm{Y}' \cdot \mathrm{V}'' = \mathrm{Y}^\circ$; then $\mathrm{V}'' : S^{-1}\mathsf{N} \to S^{-1}\mathsf{N}$ induces the desired isomorphism $\mathbb{P}_{\mathbb{Z}}(\rho)^{1/S} \to \mathbb{P}_{\mathbb{Z}}(\rho')^{1/S}$. ☐

**3.6.** We show that $\mathrm{H}^2_{\mathrm{zar}}(X, \mathbb{G}_m) = 0$ by showing that the localization

$$\mathrm{H}^2_{\mathrm{zar}}(X, \mathbb{G}_m) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$$

is 0 for every prime $p$. By (3.4.2) and Lemma 3.5, we may thus assume that

$$\rho = (1, p^{e_1}, \dots, p^{e_n})$$

for some nonnegative integers $e_1 \leq \cdots \leq e_n$. In this case, in Section 3.2 we may take the first row of U to be $\rho$ and the other rows to coincide with the identity $\mathrm{id}_{n+1}$, so that

$$\mathrm{Y} = \begin{bmatrix} -p^{e_1} & \cdots & -p^{e_n} \\ 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} \tag{3.6.1}$$

and thus $\boldsymbol{v}_0 = (-p^{e_1}, \dots, -p^{e_n})$ and $\boldsymbol{v}_i$ is the $i$-th standard basis vector of $\mathbb{Z}^n$.

**3.7.** ***Definition of*** $\mathsf{A}^{\bullet,\bullet}$**.** For convenience, we set $[n] := \{0, 1, \dots, n\}$; we will use $I$ to denote a subset of $[n]$. We construct a double complex

$$\big(\{\mathsf{A}^{p,q}\}, \{\mathsf{d}_{\mathrm{v}}^{p,q} : \mathsf{A}^{p,q} \to \mathsf{A}^{p,q+1}\}, \{\mathsf{d}_{\mathrm{h}}^{p,q} : \mathsf{A}^{p,q} \to \mathsf{A}^{p+1,q}\}\big)$$

as follows: for $-1 \leq p \leq n$, we set

$$\mathsf{A}^{p,1} = \bigoplus_{|I|=n-p} \mathbb{Z}^{n-p}, \qquad \mathsf{A}^{p,0} = \bigoplus_{|I|=n-p} \mathbb{Z}^{n} \tag{3.7.1}$$

and $\mathsf{A}^{p,q} = 0$ if $(p, q) \notin \{-1, \dots, n\} \times \{0, 1\}$.

For the vertical differential $\mathsf{d}_{\mathrm{v}}^{p,0} : \mathsf{A}^{p,0} \to \mathsf{A}^{p,1}$, the $I$-th component (with $|I| = n - p$) of this map is the group homomorphism $\mathbb{Z}^n \to \mathbb{Z}^{n-p}$ whose corresponding matrix has rows $\boldsymbol{v}_i$ for $i \in I$.

The horizontal differentials $\mathsf{d}_{\mathrm{h}}^{p,q}$ are defined with the sign conventions as follows: if $I = \{i_0, \dots, i_{n-p-1}\} \subset [n]$ is a subset of size $|I| = n - p$ and $I'$ is obtained by removing the $i$-th element of $I$ (where $0 \leq i \leq n - p - 1$), then the restriction from the $I$-th to $I'$-th components has sign $(-1)^i$.

The subcomplex of $A^{\bullet,\bullet}$ obtained by restricting to $p \geq 1$ is isomorphic to the morphism of Čech complexes

$$\check{\mathrm{C}}^\bullet(\Delta, \mathcal{M}) \to \check{\mathrm{C}}^\bullet(\Delta, \mathcal{SF}),$$

in the notation of [11, (5.0.1)].

**3.8.** ***Diagram of*** $\mathsf{A}^{\bullet,\bullet}$**.** Here is a diagram of the double complex $\mathsf{A}^{\bullet,\bullet}$:

$$\begin{array}{ccccccccccccc}
\mathsf{A}^{-1,1} & \xrightarrow{\mathsf{d}_{\mathrm{h}}^{-1,1}} & \mathsf{A}^{0,1} & \xrightarrow{\mathsf{d}_{\mathrm{h}}^{0,1}} & \mathsf{A}^{1,1} & \xrightarrow{\mathsf{d}_{\mathrm{h}}^{1,1}} & \mathsf{A}^{2,1} & \longrightarrow & \cdots & \longrightarrow & \mathsf{A}^{n-1,1} & \xrightarrow{\mathsf{d}_{\mathrm{h}}^{n-1,1}} & \mathsf{A}^{n,1} \\
{\scriptstyle \mathsf{d}_{\mathrm{v}}^{-1,0}}\uparrow & & {\scriptstyle \mathsf{d}_{\mathrm{v}}^{0,0}}\uparrow & & {\scriptstyle \mathsf{d}_{\mathrm{v}}^{1,0}}\uparrow & & {\scriptstyle \mathsf{d}_{\mathrm{v}}^{2,0}}\uparrow & & & & {\scriptstyle \mathsf{d}_{\mathrm{v}}^{n-1,0}}\uparrow & & {\scriptstyle \mathsf{d}_{\mathrm{v}}^{n,0}}\uparrow \\
\mathsf{A}^{-1,0} & \xrightarrow[\mathsf{d}_{\mathrm{h}}^{-1,1}]{} & \mathsf{A}^{0,0} & \xrightarrow[\mathsf{d}_{\mathrm{h}}^{0,1}]{} & \mathsf{A}^{1,0} & \xrightarrow[\mathsf{d}_{\mathrm{h}}^{1,1}]{} & \mathsf{A}^{2,0} & \longrightarrow & \cdots & \longrightarrow & \mathsf{A}^{n-1,0} & \xrightarrow[\mathsf{d}_{\mathrm{h}}^{n-1,0}]{} & \mathsf{A}^{n,0}
\end{array}$$

For a weighted projective surface (i.e., $n = 2$), this looks like

$$\begin{array}{ccccccc}
\mathbb{Z}^3 & \longrightarrow & \mathbb{Z}^2 \oplus \mathbb{Z}^2 \oplus \mathbb{Z}^2 & \longrightarrow & \mathbb{Z}^1 \oplus \mathbb{Z}^1 \oplus \mathbb{Z}^1 & \longrightarrow & 0 \\
\uparrow {\scriptstyle \begin{bmatrix} v_0 \\ v_1 \\ v_2 \end{bmatrix}} & & \uparrow {\scriptstyle \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} \begin{bmatrix} v_0 \\ v_2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}} & & \uparrow {\scriptstyle [v_0]\,[v_1]\,[v_2]} & & \uparrow \\
\mathbb{Z}^2 & \longrightarrow & \mathbb{Z}^2 \oplus \mathbb{Z}^2 \oplus \mathbb{Z}^2 & \longrightarrow & \mathbb{Z}^2 \oplus \mathbb{Z}^2 \oplus \mathbb{Z}^2 & \longrightarrow & \mathbb{Z}^2
\end{array}$$

**3.9.** Let $\boldsymbol{C}_n^\bullet$ be the complex with $\boldsymbol{C}_n^k = \mathbb{Z}^{\binom{n}{k}}$ and such that the differentials $\boldsymbol{C}_n^k \to \boldsymbol{C}_n^{k+1}$ have sign conventions as above. Then $\boldsymbol{C}_n^\bullet$ is isomorphic to a direct sum of shifts of $\mathrm{id} : \mathbb{Z} \to \mathbb{Z}$, and hence is exact. The complex $\mathsf{A}^{\bullet,0}$ is isomorphic to the direct sum $(\boldsymbol{C}_{n+1}^\bullet)^n$, and hence is exact. The complex $\mathsf{A}^{\bullet,1}$ is isomorphic to the direct sum $(\boldsymbol{C}_{n-1}^\bullet)^{n+1}$, and hence is exact. Let

$$\left(\{\mathrm{E}_r^{p,q}\}, \{\mathrm{d}_r^{p,q} : \mathrm{E}_r^{p,q} \to \mathrm{E}_r^{p+r,q-r+1}\}\right) \tag{3.9.1}$$

denote the spectral sequence corresponding to the horizontal filtration on $\mathsf{A}^{\bullet,\bullet}$, so that $\mathrm{E}_0^{p,q} = \mathsf{A}^{p,q}$ and $\mathrm{d}_0^{p,q} = \mathsf{d}_{\mathrm{v}}^{p,q}$. Then there is a natural isomorphism

$$\mathrm{E}_2^{p,0} \simeq \check{\mathrm{H}}^p(\mathfrak{U}, \mathbb{G}_m),$$

where $\mathfrak{U}$ is the Zariski open cover of $X$ corresponding to the maximal cones of $\Delta$. Since there are only two nonzero rows, the differentials

$$\mathrm{d}_2^{p,1} : \mathrm{E}_2^{p,1} \to \mathrm{E}_2^{p+2,0}$$

are isomorphisms for all $p$. We are interested in $\check{\mathrm{H}}^2(\mathfrak{U}, \mathbb{G}_m) \simeq \mathrm{E}_2^{2,0} \simeq \mathrm{E}_2^{0,1}$.

**3.10.** For the differential $\mathsf{d}_{\mathrm{v}}^{0,0} : \mathsf{A}^{0,0} \to \mathsf{A}^{0,1}$, the $I$-th component (with $|I| = n$) of this map is the group homomorphism $\mathbb{Z}^n \to \mathbb{Z}^n$ whose corresponding matrix is obtained by removing the $i$-th rows from Y (3.6.1) for $i \notin I$; hence

$$\mathrm{E}_1^{0,1} \simeq \bigoplus_{i \in [n]} \mathbb{Z}/(p^{e_i}), \tag{3.10.1}$$

where a generator of the $i$-th component $\mathbb{Z}/(p^{e_i})$ is given by the image of the first standard basis vector of $\mathbb{Z}^n$ (see (3.7.1)).

For the differential $\mathrm{d}_{\mathrm{v}}^{1,0} : \mathsf{A}^{1,0} \to \mathsf{A}^{1,1}$, the $I$-th component (with $|I| = n-1$) of this map is the group homomorphism $\mathbb{Z}^n \to \mathbb{Z}^{n-1}$ whose corresponding matrix is obtained by removing the $i$-th rows from Y (3.6.1) for $i \notin I$; hence

$$\mathrm{E}_1^{1,1} \simeq \bigoplus_{i_1<i_2} \mathbb{Z}/(p^{\min\{e_{i_1}, e_{i_2}\}}) \tag{3.10.2}$$

where a generator of the $i$-th component $\mathbb{Z}/(p^{\min\{e_{i_1}, e_{i_2}\}})$ is given by the image of the first standard basis vector of $\mathbb{Z}^{n-1}$ (see (3.7.1)).

**3.11.** We compute $\mathrm{E}_2^{0,1} = \ker \mathrm{d}_1^{0,1} / \operatorname{im} \mathrm{d}_1^{-1,1}$ in (3.9.1). With identifications as in (3.10.1) and (3.10.2), the image of $(x_0, x_1, \dots, x_n) \in \mathrm{E}_1^{0,1}$ under the differential $\mathrm{d}_1^{0,1} : \mathrm{E}_1^{0,1} \to \mathrm{E}_1^{1,1}$ has $(i_1, i_2)$-th coordinate $(-1)^{i_1} x_{i_1} + (-1)^{i_2 - 1} x_{i_2}$. Suppose $(x_0, x_1, \dots, x_n) \in \ker \mathrm{d}_1^{0,1}$; using the differential $\mathrm{d}_1^{-1,1} : \mathrm{E}_1^{-1,1} \to \mathrm{E}_1^{0,1}$, we may assume that $x_n = 0$ in $\mathbb{Z}/(p^{e_n})$. Since $e_{n-1} \le e_n$, the condition

$$(-1)^{n-1} x_{n-1} + (-1)^{n-1} x_n = 0$$

in $\mathbb{Z}/(p^{e_{n-1}})$ forces $x_{n-1} = 0$ in $\mathbb{Z}/(p^{e_{n-1}})$. Using downward induction on $i$, we conclude that $x_i = 0$ in $\mathbb{Z}/(p^{e_i})$ for all $i$. Thus we have $\mathrm{E}_2^{0,1} = 0$.

**Remark 3.12** (assumptions on the base field). In [11], there are two implicit assumptions regarding the base field $k$:

(1) It is assumed that $k$ is algebraically closed, as we do throughout Section 3 (this assumption will be removed in Lemma 4.2). This is used to conclude that all closed points are $k$-points and to identify the henselization and the strict henselization at a closed point of a variety. In the proof of Lemma 4.1, the reference to [31, Chapter VI, Section 14, Theorem 32, p. 92] (in showing that an affine toric variety is analytically normal) requires $k$ to be perfect (here we may also use [24, (33.I) Theorem 79]).

(2) It is assumed that $k$ has characteristic 0. This is used to conclude that (5.1.1) is a split surjection; we only use their Lemmas 4.3 and 5.1, which do not depend on the characteristic of $k$. (There are potential subtleties when considering the Brauer group of (affine) toric varieties in positive characteristic; for example, if $k$ is an algebraically closed field of characteristic $p$, the Brauer group of $\mathbb{A}_k^2$ has nontrivial $p$-torsion by [4, Theorem 7.5]. These classes are not cup products since $\mathrm{H}^1_{\mathrm{fppf}}(\mathbb{A}_k^2, \mu_p) = 0$.)

## 4. Over a general base scheme

In this section, we prove Theorem 1.2 for an arbitrary scheme $S$ (see Section 4.4). This is a Leray spectral sequence argument for the structure morphism $f : X \to S$. For this, we show that $\boldsymbol{R}^1 f_* \mathbb{G}_m = \underline{\mathbb{Z}}$ and $\boldsymbol{R}^2 f_* \mathbb{G}_m = 0$ (see Lemma 4.3).

We first generalize Lemma 3.1 to arbitrary fields:

**Lemma 4.1.** *For any field $k$, the pullback map*

$$\mathrm{H}^2_{ét}(\operatorname{Spec} k, \mathbb{G}_m) \to \mathrm{H}^2_{ét}(\mathbb{P}_k(\rho), \mathbb{G}_m)$$

*is an isomorphism.*

*Proof.* Let $P \in (\mathbb{P}_k(\rho))(k)$ be a $k$-point, and let $\alpha \in \mathrm{H}^2_{\text{ét}}(\mathbb{P}_k(\rho), \mathbb{G}_m)$ be a Brauer class such that $\alpha_P = 0$ in $\mathrm{H}^2_{\text{ét}}(\operatorname{Spec} k, \mathbb{G}_m)$. It suffices to show that $\alpha = 0$; this follows from Lemma 4.2, whose hypotheses are satisfied by Lemma 3.1. □

**Lemma 4.2.** *Let $A$ be a local ring, set $X := \mathbb{P}_A(\rho)$, let $P \in X(A)$ be an $A$-rational point and let $\alpha \in \mathrm{H}^2_{ét}(X, \mathbb{G}_m)$ be a class such that $\alpha_P = 0$. If there exists a finite faithfully flat $A$-algebra $A'$ such that $\alpha_{A'} = 0$, then $\alpha = 0$.*

*Proof.* Let $\mathcal{G} \to X$ be the $\mathbb{G}_m$-gerbe corresponding to $\alpha$. Since $\mathcal{G}_{A'}$ is trivial, there is a 1-twisted line bundle $\mathcal{L}'$ on $\mathcal{G}_{A'}$; set $A'' := A' \otimes_A A'$ and $A''' := A' \otimes_A A' \otimes_A A'$; then there exists a line bundle $L''$ on $X_{A''}$ such that $L''|_{\mathcal{G}_{A''}} \simeq (p_1^* \mathcal{L}')^{-1} \otimes p_2^* \mathcal{L}'$; this line bundle $L''$ satisfies $p_{13}^* L'' \simeq p_{23}^* L'' \otimes p_{12}^* L''$; hence $L''$ is trivial since $p_{12}^*, p_{13}^*, p_{23}^* : \operatorname{Pic}(X_{A''}) \to \operatorname{Pic}(X_{A'''})$ are the same maps $\mathbb{Z} \to \mathbb{Z}$ (see Lemma 2.8). Choose an isomorphism $\varphi : p_1^* \mathcal{L}' \to p_2^* \mathcal{L}'$ of $\mathcal{O}_{\mathcal{G}_{A''}}$-modules; the isomorphisms $p_{13}^* \varphi$ and $p_{23}^* \varphi \circ p_{12}^* \varphi$ differ by an element $u_\alpha \in \Gamma(X_{A'''}, \mathbb{G}_m) \simeq \Gamma(A''', \mathbb{G}_m)$. Since $\mathcal{G}|_P$ is trivial, we may refine the finite flat cover $A \to A'$ if necessary so that $u_\alpha$ is the coboundary of some $u_\beta \in \Gamma(X_{A''}, \mathbb{G}_m)$. After modifying $\varphi$ by this $u_\beta$, we have that the descent datum $(\mathcal{L}', \varphi)$ gives a 1-twisted line bundle on $\mathcal{G}$. □

We use deformation theory of twisted sheaves to deduce Theorem 1.2 over strictly henselian local rings:

**Lemma 4.3.** *Let $A$ be a strictly henselian local ring. Then $\mathrm{H}^2_{ét}(\mathbb{P}_A(\rho), \mathbb{G}_m) = 0$.*

*Proof.* This proof is an argument of Siddharth Mathur (personal communication, 2019). By standard limit techniques, we may assume that $A$ is the strict henselization of a localization of a finite type $\mathbb{Z}$-algebra; in particular, $A$ is excellent [20, Corollary 5.6(iii)]. Let $\mathfrak{m}$ be the maximal ideal of $A$ and let $k := A/\mathfrak{m}$ be the residue field.

We first consider the case when $A$ is complete. Set $X := \mathbb{P}_A(\rho)$ and let $\pi : \mathcal{G} \to X$ be a $\mathbb{G}_m$-gerbe corresponding to a class $[\mathcal{G}] \in \mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_m)$. The class $[\mathcal{G}]$ is trivial if and only if $\pi$ admits a section. We have that $\mathcal{G}_0$ is a $\mathbb{G}_m$-gerbe over $X_0 = \mathbb{P}_k(\rho)$, which is a trivial gerbe by Lemma 4.1 since $k$ is separably closed. For $\ell \in \mathbb{N}$, set $X_\ell := X \times_{\operatorname{Spec} A} \operatorname{Spec} A/\mathfrak{m}^{\ell+1}$ and $\mathcal{G}_\ell := \mathcal{G} \times_X X_\ell$. We have equivalences of categories

$$\begin{aligned}\operatorname{Mor}(X, \mathcal{G}) &\overset{1}{\simeq} \operatorname{Hom}_{r\otimes,\simeq}(\mathsf{Coh}(\mathcal{G}), \mathsf{Coh}(X)) \\ &\overset{2}{\simeq} \operatorname{Hom}_{r\otimes,\simeq}(\mathsf{Coh}(\mathcal{G}), \varprojlim \mathsf{Coh}(X_\ell)) \\ &\overset{3}{\simeq} \varprojlim \operatorname{Hom}_{r\otimes,\simeq}(\mathsf{Coh}(\mathcal{G}), \mathsf{Coh}(X_\ell)) \\ &\overset{1}{\simeq} \varprojlim \operatorname{Mor}(X_\ell, \mathcal{G}),\end{aligned}$$

where the equivalences marked 1 are by [22, Theorem 1.1] (here we use that $A$ is excellent), the equivalence marked 2 is Grothendieck existence [15, Scholie 5.1.4], the equivalence marked 3 is [22, Lemma 3.8].

It remains now to construct a compatible system of morphisms $X_\ell \to \mathcal{G}$. A morphism $X_\ell \to \mathcal{G}$ over $\mathbb{P}_A(\rho)$ corresponds to a 1-twisted line bundle on $\mathcal{G}_\ell$; the obstruction to lifting a line bundle via $\mathcal{G}_\ell \to \mathcal{G}_{\ell+1}$ lies in $\mathrm{H}^2_{\text{ét}}(\mathcal{G}_\ell, \mathfrak{m}^\ell \mathcal{O}_{\mathcal{G}_\ell})$. Since $\mathcal{G} \to X$ is a cohomologically affine morphism and the diagonal of $\mathcal{G}$ is affine, the pullback $\mathrm{H}^2_{\text{ét}}(X_\ell, \mathfrak{m}^\ell \mathcal{O}_{X_\ell}) \to \mathrm{H}^2_{\text{ét}}(\mathcal{G}_\ell, \mathfrak{m}^\ell \mathcal{O}_{\mathcal{G}_\ell})$ is an isomorphism by [3, Remark 3.5] and a Leray spectral sequence argument; by Lemma 2.9, we have $\mathrm{H}^2_{\text{ét}}(X_\ell, \mathfrak{m}^\ell \mathcal{O}_{X_\ell}) = 0$.

In general, if $A$ is not complete, we use Artin approximation to descend a 1-twisted line bundle from $\mathcal{G}^\wedge$ to $\mathcal{G}$. □

**4.4.** ***Proof of Theorem 1.2.*** Set $f := f_X$. The Leray spectral sequence associated to the map $f$ and sheaf $\mathbb{G}_m$ is of the form

$$\mathrm{E}_2^{p,q} = \mathrm{H}^p_{\text{ét}}(S, \boldsymbol{R}^q f_* \mathbb{G}_m) \Rightarrow \mathrm{H}^{p+q}_{\text{ét}}(X, \mathbb{G}_m) \tag{4.4.1}$$

with differentials $\mathrm{d}_2^{p,q} : \mathrm{E}_2^{p,q} \to \mathrm{E}_2^{p+2,q-1}$. For any strictly henselian local ring $A$, we have $\mathrm{H}^2_{\text{ét}}(\mathbb{P}_A(\rho), \mathbb{G}_m) = 0$ by Lemma 4.3, and hence $\boldsymbol{R}^2 f_* \mathbb{G}_m = 0$ since its stalks vanish. The sheaf $\boldsymbol{R}^1 f_* \mathbb{G}_m$ is the sheaf associated to $T \mapsto \operatorname{Pic}(X_T)$; by Lemma 2.8, every line bundle on $\mathbb{P}_T(\rho)$ is, locally on $T$, isomorphic to one pulled back from $\mathbb{P}_{\mathbb{Z}}(\rho)$; hence $\boldsymbol{R}^1 f_* \mathbb{G}_m$ is isomorphic to the constant sheaf $\underline{\mathbb{Z}}$. Hence we have an exact sequence

$$\mathrm{H}^0_{\text{ét}}(S, \underline{\mathbb{Z}}) \xrightarrow{\dagger} \mathrm{H}^2_{\text{ét}}(S, \mathbb{G}_m) \xrightarrow{f^*} \mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_m) \to \mathrm{H}^1_{\text{ét}}(S, \underline{\mathbb{Z}}) \tag{4.4.2}$$

and we may argue as in [30, Section 4.3] to show that $f^*$ restricts to a surjection on the torsion subgroups, inducing an exact sequence (1.2.1) as desired. □

**Remark 4.5** (the Brauer class of a twist of weighted projective space). By the argument in Section 4.4, the map $\Gamma(S, \underline{\mathbb{Z}}) \to \operatorname{Br}'(S)$ in (1.2.1) corresponds to the differential $\mathrm{d}_2^{0,1}$ in the Leray spectral sequence for $f : X \to S$. The Brauer class $[X] \in \operatorname{Br}'(S)$ is defined to be the image of $1 \in \Gamma(S, \underline{\mathbb{Z}})$ under † in (4.4.2). We have the following alternative description of $[X]$. Let $R := \mathbb{Z}[t_0, \ldots, t_n]$ be the

$\mathbb{Z}$-graded ring with $\deg(t_i) = \rho_i$, and let $\mathrm{Aut}_{\mathrm{gr.alg.}}(R)$ denote the group sheaf sending a scheme $T$ to the set of $\mathbb{Z}$-graded $\mathcal{O}_T$-algebra automorphisms of $R \otimes_{\mathbb{Z}} \mathcal{O}_T$. By [2, Section 8], we have an exact sequence

$$1 \to \mathbb{G}_m \to \mathrm{Aut}_{\mathrm{gr.alg.}}(R) \to \mathrm{Aut}_{\mathrm{sch}}(\mathbb{P}_{\mathbb{Z}}(\rho)) \to 1$$

of sheaves of groups for the étale topology on the category of schemes, where the image of $\mathbb{G}_m$ is contained in the center of $\mathrm{Aut}_{\mathrm{gr.alg.}}(R)$. By definition, $X$ is an $\mathrm{Aut}_{\mathrm{sch}}(\mathbb{P}_{\mathbb{Z}}(\rho))$-torsor over $S$, and the class of $[X]$ under the coboundary map

$$\mathrm{H}^1_{\text{ét}}\big(S, \mathrm{Aut}_{\mathrm{sch}}(\mathbb{P}_{\mathbb{Z}}(\rho))\big) \to \mathrm{H}^2_{\text{ét}}(S, \mathbb{G}_m)$$

is the desired Brauer class.

Alternatively, fix an étale surjection $S' \to S$ and set $S'' := S' \times_S S'$ and $S''' := S' \times_S S' \times_S S'$; the choice of an isomorphism $X \times_S S' \simeq \mathbb{P}_{S'}(\rho)$ yields an automorphism $\varphi : \mathbb{P}_{S''}(\rho) \to \mathbb{P}_{S''}(\rho)$ satisfying the cocycle condition $p_{13}^*\varphi = p_{23}^*\varphi \circ p_{12}^*\varphi$ over $S'''$. Choose $\ell \gg 0$ so that $\mathcal{O}_{\mathbb{P}(\rho)}(\ell)$ is very ample; fixing a $\mathbb{Z}$-basis of $\Gamma(\mathbb{P}_{\mathbb{Z}}(\rho), \mathcal{O}_{\mathbb{P}_{\mathbb{Z}}(\rho)}(\ell))$ gives an invertible matrix $\varphi^\sharp \in \mathrm{GL}_r(\Gamma(S'', \mathcal{O}_{S''}))$, where $r = \mathrm{rank}_{\mathbb{Z}}\, \Gamma(\mathbb{P}_{\mathbb{Z}}(\rho), \mathcal{O}_{\mathbb{P}_{\mathbb{Z}}(\rho)}(\ell))$; here the invertible matrices

$$p_{13}^*\varphi^\sharp,\ p_{12}^*\varphi^\sharp \cdot p_{23}^*\varphi^\sharp \in \mathrm{GL}_r(\Gamma(S''', \mathcal{O}_{S'''}))$$

differ by a unit $u \in \Gamma(S''', \mathbb{G}_m)$, which is the desired class in $\mathrm{H}^2_{\text{ét}}(S, \mathbb{G}_m)$. In other words, given a $\mathbb{Z}$-graded algebra automorphism of $R$, it restricts to a $\mathbb{Z}$-graded algebra automorphism of its $\ell$-th Veronese subring $R^{(\ell)} := \bigoplus_{i \geq 0} R_{i\ell}$, which restricts to an abelian group automorphism of $R_\ell$ and thus a $\mathbb{Z}$-graded algebra automorphism of the standard graded algebra $\mathrm{Sym}^\bullet_{\mathbb{Z}} R_\ell \simeq \mathbb{Z}[t'_1, \ldots, t'_r]$; the induced group homomorphism $\mathrm{Aut}_{\mathrm{gr.alg.}}(R) \to \mathrm{Aut}_{\mathrm{gr.alg.}}(\mathrm{Sym}^\bullet_{\mathbb{Z}} R_\ell)$ induces a commutative diagram of exact sequences which we may use to compare the two constructions above.

**Remark 4.6** (comparison to the argument of Gabber). Gabber [18] computes the Brauer group of Brauer–Severi schemes over an arbitrary base scheme by combining the following two facts to reduce to the $\mathbb{P}^1$ case:

(1) Suppose $Y \to X$ is a closed immersion locally defined by a regular sequence, and let $B \to X$ be the blowup of $X$ at $Y$; then $\mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_m) \to \mathrm{H}^2_{\text{ét}}(B, \mathbb{G}_m)$ is injective.

(2) The blowup of $\mathbb{P}^n$ at a point is a $\mathbb{P}^1$-bundle over $\mathbb{P}^{n-1}$.

In our case, we may ask whether the analogous statement to (2) holds — namely, whether a (weighted) blowup of $\mathbb{P}(\rho)$ at a (torus-invariant) local complete intersection subscheme is isomorphic to a $\mathbb{P}(\rho')$-bundle over $\mathbb{P}(\rho'')$ for some $\rho'$, $\rho''$ such that $|\rho| - 1 = |\rho'| - 1 + |\rho''| - 1$. Indeed, the blowup of the weighted projective surface $\mathbb{P}(1, 1, q_2)$ at its unique singular point gives the $q_2$-th Hirzebruch surface $\mathbb{F}_{q_2}$ (see [12, Section 1.2.3; 19]). Such a result for arbitrary $\rho$ would give an alternative proof of Theorem 1.2. This seems unlikely, however, as it (with Remark 2.6) would imply that every weighted projective surface $\mathbb{P}(\rho_0, \rho_1, \rho_2)$ is a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$, which has Picard group $\mathbb{Z}^2$, but $\mathbb{P}(2, 3, 5)$ has three isolated singular points and blowing up these points increases the Picard rank by 3.

## 5. Weighted projective stacks

In this section we prove Theorem 1.3 (see Section 5.9).

**5.1.** Let $\rho = (\rho_0, \dots, \rho_n)$ be a weight vector, and consider the $\mathbb{G}_m$-action on $\mathbb{A}^{n+1}$ sending $u \cdot (t_0, \dots, t_n) \mapsto (u^{\rho_0} t_0, \dots, u^{\rho_n} t_n)$. The *weighted projective stack* associated to $\rho$ is the quotient stack

$$\mathcal{P}_{\mathbb{Z}}(\rho) := [(\mathbb{A}^{n+1}_{\mathbb{Z}} \setminus \{0\})/\mathbb{G}_m]$$

for this action. For any scheme $S$, we denote the base change of $\mathcal{P}_{\mathbb{Z}}(\rho)$ to $S$ by $\mathcal{P}_S(\rho) := \mathcal{P}_{\mathbb{Z}}(\rho) \times_{\operatorname{Spec} \mathbb{Z}} S$.

The weighted projective stack $\mathcal{P}_{\mathbb{Z}}(\rho)$ admits a natural morphism

$$\pi_\rho : \mathcal{P}_{\mathbb{Z}}(\rho) \to \mathbb{P}_{\mathbb{Z}}(\rho)$$

to the weighted projective space $\mathbb{P}_{\mathbb{Z}}(\rho)$, which is a coarse moduli space morphism [1, Section 2.1]. Since $\mathcal{P}_{\mathbb{Z}}(\rho)$ is smooth for any $\rho$, the morphism $\pi_\rho$ is not an isomorphism if $\rho \neq (1, \dots, 1)$.

**Lemma 5.2.** *For any field $k$, the pullback map*

$$\mathrm{H}^2_{\acute{e}t}(\operatorname{Spec} k, \mathbb{G}_m) \to \mathrm{H}^2_{\acute{e}t}(\mathcal{P}_k(\rho), \mathbb{G}_m)$$

*is an isomorphism.*

*Proof.* We have a descent spectral sequence

$$\mathrm{E}_1^{p,q} = \mathrm{H}^q_{\text{ét}}(\mathbb{G}_{m,k}^{\times p} \times_k (\mathbb{A}^{n+1}_k \setminus \{0\}), \mathbb{G}_m) \Longrightarrow \mathrm{H}^{p+q}_{\text{ét}}(\mathcal{P}_k(\rho), \mathbb{G}_m) \tag{5.2.1}$$

with differentials $\mathrm{d}_1^{p,q} : \mathrm{E}_1^{p,q} \to \mathrm{E}_1^{p+1,q}$. Each $\mathbb{G}_{m,k}^{\times p} \times_k (\mathbb{A}^{n+1}_k \setminus \{0\})$ is an open subscheme of $\mathbb{A}^{n+p+1}_k$, hence has trivial Picard group; hence $\mathrm{E}_1^{p,1} = 0$ for all $p$. The pullback $\mathrm{B}\mathbb{G}_{m,k} \to \mathcal{P}_k(\rho)$ induces an isomorphism of complexes $\mathrm{H}^0_{\text{ét}}(\mathbb{G}_{m,k}^{\times \bullet}, \mathbb{G}_m) \to \mathrm{E}_1^{\bullet,0}$; hence, by the proof of [30, Lemma 4.2], we have $\mathrm{E}_2^{2,0} = 0$.

It remains to compute $\mathrm{E}_2^{0,2}$, which is isomorphic to the equalizer of the two pullback maps

$$a^*, p_2^* : \mathrm{H}^2_{\text{ét}}(\mathbb{A}^{n+1}_k \setminus \{0\}, \mathbb{G}_m) \rightrightarrows \mathrm{H}^2_{\text{ét}}(\mathbb{G}_m \times_k (\mathbb{A}^{n+1}_k \setminus \{0\}), \mathbb{G}_m)$$

corresponding to the action map and second projection, respectively; by purity for the Brauer group (see Gabber [16] and Česnavičius [5]), this is isomorphic to the equalizer of

$$a^*, p_2^* : \mathrm{H}^2_{\text{ét}}(\mathbb{A}^{n+1}_k, \mathbb{G}_m) \rightrightarrows \mathrm{H}^2_{\text{ét}}(\mathbb{G}_m \times_k \mathbb{A}^{n+1}_k, \mathbb{G}_m),$$

and also to the equalizer of

$$a^*,\, p_2^* : \mathrm{H}^2_{\text{ét}}(\mathbb{A}^{n+1}_k, \mathbb{G}_m) \rightrightarrows \mathrm{H}^2_{\text{ét}}(\mathbb{A}^1_k \times_k \mathbb{A}^{n+1}_k, \mathbb{G}_m)$$

since the restriction

$$\mathrm{H}^2_{\text{ét}}(\mathbb{A}^1_k \times_k \mathbb{A}^{n+1}_k, \mathbb{G}_m) \to \mathrm{H}^2_{\text{ét}}(\mathbb{G}_m \times_k \mathbb{A}^{n+1}_k, \mathbb{G}_m)$$

is injective. With coordinates $\mathbb{A}^1_k = \operatorname{Spec} k[u]$, let $f : \mathbb{A}^{n+1}_k \to \mathbb{A}^1_k \times_k \mathbb{A}^{n+1}_k$ be the morphism of $k$-schemes obtained by setting $u = 0$; note that $p_2 f = \mathrm{id}$ and $af$ factors through $\operatorname{Spec} k$. Let $\alpha \in \mathrm{H}^2_{\text{ét}}(\mathbb{A}^{n+1}_k, \mathbb{G}_m)$ be a Brauer class such that $a^*\alpha = p_2^*\alpha$ in $\mathrm{H}^2_{\text{ét}}(\mathbb{A}^1_k \times_k \mathbb{A}^{n+1}_k, \mathbb{G}_m)$; then $f^*a^*\alpha = f^*p_2^*\alpha = \alpha$; hence $\alpha$ is in the image of $\mathrm{H}^2_{\text{ét}}(\operatorname{Spec} k, \mathbb{G}_m)$. □

**Lemma 5.3** [26, Corollary 4.3]. *For any connected scheme $S$, the map*

$$\mathbb{Z} \oplus \operatorname{Pic}(S) \to \operatorname{Pic}(\mathcal{P}_S(\rho))$$

*sending*

$$(\ell, \mathcal{L}) \mapsto \mathcal{O}_{\mathcal{P}_S(\rho)}(\ell) \otimes \pi_S^*\mathcal{L}$$

*is an isomorphism.*

**Lemma 5.4** (cohomology of $\mathcal{O}_{\mathcal{P}(\rho)}(\ell)$ [25, Proposition 2.5]). *Let $A$ be a ring and set $X := \mathcal{P}_A(\rho)$.*

(1) *For $\ell \geq 0$, the $A$-module $\mathrm{H}^0(X, \mathcal{O}_X(\ell))$ is free with basis consisting of monomials $t_0^{e_0} \cdots t_n^{e_n}$ such that $e_0, \ldots, e_n \in \mathbb{Z}_{\geq 0}$ and $\rho_0 e_0 + \cdots + \rho_n e_n = \ell$.*

(2) *For $\ell < 0$, the $A$-module $\mathrm{H}^n(X, \mathcal{O}_X(\ell))$ is free with basis consisting of monomials $t_0^{e_0} \cdots t_n^{e_n}$ such that $e_0, \ldots, e_n \in \mathbb{Z}_{<0}$ and $\rho_0 e_0 + \cdots + \rho_n e_n = \ell$.*

(3) *If $(i, \ell) \notin (\{0\} \times \mathbb{Z}_{\geq 0}) \cup (\{n\} \times \mathbb{Z}_{<0})$, then $\mathrm{H}^i(X, \mathcal{O}_X(\ell)) = 0$.*

(4) *For any $A$-module $M$ and any $(i, \ell)$, the canonical map*

$$\mathrm{H}^i(X, \mathcal{O}_X(\ell)) \otimes_A M \to \mathrm{H}^i(X, \mathcal{O}_X(\ell) \otimes_A M)$$

*is an isomorphism.*

**Lemma 5.5.** *Let $A$ be a strictly henselian local ring. Then $\mathrm{H}^2_{ét}(\mathcal{P}_A(\rho), \mathbb{G}_m) = 0$.*

*Proof.* The proof is the same as that of Lemma 4.3 with the following modifications: for the triviality of the gerbe $\mathcal{G}_0$ over the special fiber, we use Lemma 5.2; to obtain the equivalence marked 2, we use Grothendieck existence for stacks [28, Theorem 1.4] (using that $\mathcal{P}(\rho)$ is proper [25, Proposition 2.1]); to conclude that $\mathrm{H}^2_{\text{ét}}(X_\ell, \mathfrak{m}^\ell \mathcal{O}_{X_\ell}) = 0$, we use Lemma 5.4. □

**Lemma 5.6.** *Let*

$$\pi_\rho : \mathcal{P}_{\mathbb{Z}}(\rho) \to \mathbb{P}_{\mathbb{Z}}(\rho)$$

*denote the coarse moduli space morphism. For any $\ell \in \mathbb{Z}$, there is a canonical $\mathcal{O}_{\mathcal{P}_{\mathbb{Z}}(\rho)}$-linear map*

$$\pi_\rho^*(\mathcal{O}_{\mathbb{P}_{\mathbb{Z}}(\rho)}(\ell)) \to \mathcal{O}_{\mathcal{P}_{\mathbb{Z}}(\rho)}(\ell), \tag{5.6.1}$$

*which is an isomorphism if $\ell$ is divisible by* $\operatorname{lcm}(\rho)$.

*Proof.* Set $R := \mathbb{Z}[t_0, \dots, t_n]$ with the $\mathbb{Z}$-grading determined by $\deg(t_i) = \rho_i$. The restriction of (5.6.1) to the open substack $[(\operatorname{Spec} R[t_i^{-1}])/\mathbb{G}_m]$ corresponds to the graded homomorphism

$$R(\ell)[t_i^{-1}]_0 \otimes_{R[t_i^{-1}]_0} R[t_i^{-1}] \to R(\ell)[t_i^{-1}] \tag{5.6.2}$$

of $\mathbb{Z}$-graded $R[t_i^{-1}]$-modules; the $m$-th component of (5.6.2) is isomorphic to the $R[t_i^{-1}]_0$-linear map

$$R[t_i^{-1}]_\ell \otimes_{R[t_i^{-1}]_0} R[t_i^{-1}]_m \to R[t_i^{-1}]_{\ell+m} \tag{5.6.3}$$

induced by multiplication.

If $\ell$ is divisible by $\rho_i$, then the multiplication-by-$t_i^{\ell/\rho_i}$ map $R[t_i^{-1}] \to R[t_i^{-1}](\ell)$ is an isomorphism of $\mathbb{Z}$-graded $R[t_i^{-1}]$-modules, thus (5.6.3) is an isomorphism for all $m \in \mathbb{Z}$, in other words the restriction of (5.6.1) to $[(\operatorname{Spec} R[t_i^{-1}])/\mathbb{G}_m]$ is an isomorphism. □

**Lemma 5.7.** *The pullback*

$$\pi_\rho^* : \operatorname{Pic}(\mathbb{P}_{\mathbb{Z}}(\rho)) \to \operatorname{Pic}(\mathcal{P}_{\mathbb{Z}}(\rho))$$

*is multiplication by* $\operatorname{lcm}(\rho)$.

*Proof.* We have that $\operatorname{Pic}(\mathbb{P}_{\mathbb{Z}}(\rho)) \simeq \mathbb{Z}$ is generated by the class of $\mathcal{O}_{\mathbb{P}_{\mathbb{Z}}(\rho)}(\operatorname{lcm}(\rho))$ and that $\operatorname{Pic}(\mathcal{P}_{\mathbb{Z}}(\rho)) \simeq \mathbb{Z}$ is generated by the class of $\mathcal{O}_{\mathcal{P}_{\mathbb{Z}}(\rho)}(\operatorname{lcm}(1))$ by Lemma 2.8 and Lemma 5.3, respectively. We have the desired claim by Lemma 5.6. □

**Remark 5.8.** There exist $\rho, \ell$ for which the natural map (5.6.1) is not an isomorphism. For example, in case $\rho = (1, 2)$ and $\ell = 1$, the element $t_0 \in R[t_0^{-1}]_2$ is not in the image of the map (5.6.3) for $m = 1$ and $i = 0$. We have $\mathcal{O}_{\mathbb{P}(\rho)}(1) \simeq \mathcal{O}_{\mathbb{P}(\rho)}$, and the pullback (5.6.1) is multiplication by $t_1 \in \Gamma(\mathcal{P}(\rho), \mathcal{O}_{\mathcal{P}(\rho)}(1))$; see Lemma 2.5 for details. Furthermore, the natural map $\mathcal{O}_{\mathbb{P}(\rho)}(1) \otimes \mathcal{O}_{\mathbb{P}(\rho)}(1) \to \mathcal{O}_{\mathbb{P}(\rho)}(2)$ is not an isomorphism; here [14, Proposition 2.5.13] does not apply since $R$ is not generated in degree 1. (See also [10, Exemple 4.8; 12, Section 1.5.3].)

**5.9. *Proof of Theorem 1.3.*** The proof of the exactness of (1.3.1) is the same as in Section 4.4 with the following modifications: to show $\boldsymbol{R}^2 f_* \mathbb{G}_m = 0$, we use Lemma 5.5; to show $\boldsymbol{R}^1 f_* \mathbb{G}_m \simeq \underline{\mathbb{Z}}$, we use Lemma 5.3.

For any faithfully flat morphism $S' \to S$, the pullback $\pi_{S'} : \mathcal{X} \times_S S' \to X \times_S S'$ is a coarse moduli space morphism. Since $\mathcal{X} \times_S S' \simeq \mathcal{P}_{S'}(\rho)$, we have $X \times_S S' \simeq \mathcal{P}_{S'}(\rho)$.

We have a morphism between Leray spectral sequences for $\mathcal{X}$ and $X$ induced by pullback via $\pi$, from which we obtain the vertical maps in (1.3.2). The description of the left vertical arrow in (1.3.2) follows from Lemma 5.7. □

**Remark 5.10.** It should be possible to describe the Brauer class $[\mathcal{X}] \in \mathrm{Br}'(S)$ in a similar way to Remark 4.6, using Noohi's description of the automorphism 2-group of weighted projective stacks in [27].

## Acknowledgements

I am grateful to Tim Ford, Daniel Huybrechts, Max Lieblich, and Siddharth Mathur for helpful conversations. I am also grateful to the referees for helpful comments and suggestions that improved the exposition of the paper. Part of this project was completed during my visit to the Max Planck Institute for Mathematics in Bonn, and I would like to thank the institute for their hospitality and financial support.

## References

[1] D. Abramovich and B. Hassett, "Stable varieties with a twist", pp. 1–38 in *Classification of algebraic varieties*, Eur. Math. Soc., Zürich, 2011. MR Zbl

[2] A. Al Amrani, "Classes d'idéaux et groupe de Picard des fibrés projectifs tordus", *K-Theory* **2**:5 (1989), 559–578. MR Zbl

[3] J. Alper, "Good moduli spaces for Artin stacks", *Ann. Inst. Fourier* (*Grenoble*) **63**:6 (2013), 2349–2402. MR Zbl

[4] M. Auslander and O. Goldman, "The Brauer group of a commutative ring", *Trans. Amer. Math. Soc.* **97** (1960), 367–409. MR

[5] K. Česnavičius, "Purity for the Brauer group", *Duke Math. J.* **168**:8 (2019), 1461–1486. MR

[6] J.-L. Colliot-Thélène and A. N. Skorobogatov, *The Brauer–Grothendieck group*, A Series of Modern Surveys in Mathematics **71**, Springer, Cham, 2021. MR Zbl

[7] G. Cortiñas, C. Haesemeyer, M. E. Walker, and C. Weibel, "The K-theory of toric varieties", *Trans. Amer. Math. Soc.* **361**:6 (2009), 3325–3341. MR

[8] D. A. Cox, J. B. Little, and H. K. Schenck, *Toric varieties*, Graduate Studies in Mathematics **124**, American Mathematical Society, Providence, RI, 2011. MR Zbl

[9] A. J. de Jong, "A result of Gabber", preprint, 2003, available at http://www.math.columbia.edu/~dejong/papers/2-gabber.pdf.

[10] C. Delorme, "Espaces projectifs anisotropes", *Bull. Soc. Math. France* **103**:2 (1975), 203–223. MR Zbl

[11] F. R. DeMeyer, T. J. Ford, and R. Miranda, "The cohomological Brauer group of a toric variety", *J. Algebraic Geom.* **2**:1 (1993), 137–154. MR

[12] I. Dolgachev, "Weighted projective varieties", pp. 34–71 in *Group actions and vector fields* (Vancouver, B. C., 1981), Lecture Notes in Math. **956**, Springer, Berlin, 1982. MR Zbl

[13] I. V. Dolgachev, *Classical algebraic geometry: a modern view*, Cambridge University Press, Cambridge, 2012. MR Zbl

[14] A. Grothendieck, "Éléments de géométrie algébrique, II: Étude globale élémentaire de quelques classes de morphismes", *Inst. Hautes Études Sci. Publ. Math.* **8** (1961), 5–222. MR Zbl

[15] A. Grothendieck, "Éléments de géométrie algébrique, III: Étude cohomologique des faisceaux cohérents, I", *Inst. Hautes Études Sci. Publ. Math.* **11** (1961), 5–167. MR Zbl

[16] K. Fujiwara, "A proof of the absolute purity conjecture (after Gabber)", pp. 153–183 in *Algebraic geometry* 2000, *Azumino* (Hotaka), Adv. Stud. Pure Math. **36**, Math. Soc. Japan, Tokyo, 2002. MR Zbl

[17] W. Fulton, *Introduction to toric varieties*, Annals of Mathematics Studies **131**, Princeton University Press, Princeton, NJ, 1993. MR Zbl

[18] O. Gabber, *Some theorems on Azumaya algebras*, Ph.D. thesis, Harvard University, 1978. Published in *The Brauer group*, Lecture Notes in Math. **844**, Springer-Verlag, Berlin, 1981, pp. 129–209. MR Zbl

[19] P. Gauduchon, "Hirzebruch surfaces and weighted projective planes", pp. 25–48 in *Riemannian topology and geometric structures on manifolds*, Progr. Math. **271**, Springer, 2009. MR Zbl

[20] S. Greco, "Two theorems on excellent rings", *Nagoya Math. J.* **60** (1976), 139–149. MR Zbl

[21] A. Grothendieck, "Le groupe de Brauer, III: exemples et compléments", pp. 88–188 in *Dix exposés sur la cohomologie des schémas*, Adv. Stud. Pure Math. **3**, North-Holland, Amsterdam, 1968. MR Zbl

[22] J. Hall and D. Rydh, "Coherent Tannaka duality and algebraicity of Hom-stacks", *Algebra Number Theory* **13**:7 (2019), 1633–1675. MR Zbl

[23] M. Hering, M. Mustaţă, and S. Payne, "Positivity properties of toric vector bundles", *Ann. Inst. Fourier* (*Grenoble*) **60**:2 (2010), 607–640. MR Zbl

[24] H. Matsumura, *Commutative algebra*, W. A. Benjamin, New York, 1970. MR Zbl

[25] L. Meier, "Vector bundles on the moduli stack of elliptic curves", *J. Algebra* **428** (2015), 425–456. MR Zbl

[26] B. Noohi, "Picard stack of a weighted projective stack", preprint, 2011, available at http://www.maths.qmul.ac.uk/~noohi/papers/PicardSt.pdf.

[27] B. Noohi, "Group actions on algebraic stacks via butterflies", *J. Algebra* **486** (2017), 36–63. MR Zbl

[28] M. C. Olsson, "On proper coverings of Artin stacks", *Adv. Math.* **198**:1 (2005), 93–106. MR Zbl

[29] M. Rossi and L. Terracini, "Weighted Projective Spaces from the toric point of view with computational applications", preprint, 2011. arXiv 1112.1677

[30] M. Shin, "The cohomological Brauer group of a torsion $\mathbb{G}_m$-gerbe", *Int. Math. Res. Not.* **2021**:19 (2021), 14480–14507. MR Zbl

[31] O. Zariski and P. Samuel, *Commutative algebra*, *II*, Graduate Texts in Mathematics **29**, Springer, 1960. MR Zbl

Received August 23, 2022. Revised May 3, 2023.

MINSEON SHIN
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF WASHINGTON
SEATTLE, WA
UNITED STATES
shinms@uw.edu

PACIFIC JOURNAL OF MATHEMATICS
Vol. 324, No. 2, 2023
https://doi.org/10.2140/pjm.2023.324.371

# POCHETTE SURGERY OF 4-SPHERE

TATSUMASA SUZUKI AND MOTOO TANGE

**Iwase and Matsumoto (2004) defined "pochette surgery" as a cut-and-paste operation on 4-manifolds along a 4-manifold homotopy equivalent to $S^2 \vee S^1$. Suzuki (2022) studied infinitely many homotopy 4-spheres obtained by pochette surgery. We compute the homology of pochette surgery of any homology 4-sphere by using "linking number" of a pochette embedding. We prove that pochette surgery with the trivial cord does not change the diffeomorphism type or gives a Gluck surgery. We also show that there exist pochette surgeries on the 4-sphere with a nontrivial core sphere and a nontrivial cord such that the surgeries give the 4-sphere.**

## 1. Introduction

**1A.** ***Pochette surgery.*** Let $D^n$ be an $n$-dimensional disk and $S^n$ an $n$-dimensional sphere. Let $P$ denote the boundary-sum $S^1 \times D^3 \natural D^2 \times S^2$. It is called a *pochette*. Throughout this paper, all manifolds are assumed smooth, and connected, and all maps are smooth. For a manifold $M$, the open tubular neighborhood for a submanifold $A \subset M$ is denoted by $N(A)$. Let $E(X)$ denote the exterior $M - N(X)$ of a submanifold $X$ in $M$.

Here we define pochette surgery, which was initially defined by Iwase and Matsumoto in [7]. Let $e$ be an embedding $P \hookrightarrow M$ in a 4-manifold $M$. Let $Q_e$ denote the image $e(Q)$ of a submanifold $Q$ in $P$.

**Definition 1.1.** Let $g$ be a diffeomorphism $g : \partial P \to \partial E(P_e)$. Gluing $E(P_e)$ and $P$ via $g$, we construct a manifold $M(e, g) := E(P_e) \cup_g P$. We call this operation a *pochette surgery*. We say that the diffeomorphism $g$ is a *gluing map* for the pochette surgery.

We call the curves $l := S^1 \times \{\text{pt}\}$ and $m := \partial D^2 \times \{\text{pt}\}$ on $\partial P$ a *longitude* and a *meridian* of $P$, respectively. According to [7, Theorem 2], the diffeomorphism type of $M(e, g)$ is uniquely determined by the following data:

Suzuki is supported by JST SPRING, grant number JPMJSP2106.
Tange is supported by JSPS KAKENHI, grant number 21K03216.
*MSC2020:* primary 57K45, 57R65; secondary 57K40.
*Keywords:* 4-manifolds, pochette surgery, handle calculus.

(i) An embedding $e : P \hookrightarrow M$.

(ii) A homology class $g_*([m]) \in H_1(\partial E(P_e)) = \mathbb{Z}[m_e] \oplus \mathbb{Z}[l_e]$.

(iii) A mod 2 framing $\epsilon$ around $g(m)$.

The mod 2 framing will be defined in Section 2C. The induced map $g_*$ maps the primitive element $[m]$ in $H_1(\partial P)$ to $p[m_e] + q[l_e]$ in $H_1(\partial E(P_e))$, where $p, q$ are relatively prime integers. Then, we call the element $p/q \in \mathbb{Q} \cup \{\infty\}$ a *slope* of the pochette surgery. Any slope $p/q$ gives an unoriented image $g(m)$ of $m$. Hence, for some embedding $e$, the pochette surgery with the slope $p/q$ and the mod 2 framing $\epsilon$ is called $(p/q, \epsilon)$*-pochette surgery* of $M$ and this denotes $M(e, p/q, \epsilon)$. We call the 2-sphere $S := \{\mathrm{pt}\} \times S^2 \subset P$ a *core sphere* of $P$ and the meridian 2-sphere $B := \{\mathrm{pt}\} \times \partial D^3 \subset P$ a *belt sphere* of $P$.

Consider $P$ as $D^2 \times S^2 \cup h^1$, where $h^1$ is a 1-handle. In order to embed $P$ into a 4-manifold $M$, we have only to determine an embedding of $D^2 \times S^2$ and the 1-handle $h^1$. First we take an embedding $e : D^2 \times S^2 \hookrightarrow M$.

**Definition 1.2** (cord). The 1-handle gives a properly embedded, simple arc in $E(S_e^2)$ by taking the core of $h^1$. We call this arc a *cord* here. If a cord is boundary parallel, then the cord is called *trivial*.

**1B.** ***Gluck surgery and circle surgery.*** Let $S'$ be an embedded sphere with a product neighborhood in a 4-manifold $M$. *Gluck surgery* along $S'$ is an operation $\mathrm{Gl}(S') := E(S') \cup_\varphi (D^2 \times S^2)$, where $\varphi$ is a diffeomorphism $\partial D^2 \times S^2 \to \partial N(S') \cong S^1 \times S^2$ which is not homotopy equivalent to the identity. From the construction of pochette surgery, for an embedding $e : P \hookrightarrow M$, any $(\infty, 0)$-pochette surgery is the trivial surgery and any $(\infty, 1)$-pochette surgery yields $\mathrm{Gl}(S_e)$. In the case of $(0, \epsilon)$-pochette surgery, it is an operation $E(l_e) \cup (D^2 \times S^2)$ along the curve $l_e \subset M$. This surgery means that the result is one side of the manifold obtained by attaching 5-dimensional 2-handle on $M \times I$ along $l_e$. We call the result an $S^1$*-surgery* (*circle surgery*). Thus, any pochette surgery with the slope $p/q$ can be regarded as an intermediate between a Gluck surgery and an $S^1$-surgery.

Pochette surgery is a generalization of Gluck surgery as mentioned above. Gluck surgery gave exotic nonorientable 4-manifolds in [1]. It is natural to think pochette surgery may give interesting orientable 4-manifolds, possibly exotic 4-spheres and so on. In this article, we focus on pochette surgeries yielding homotopy 4-spheres.

**1C.** ***Other results.*** Since the definition of pochette surgery was done, some people have studied pochette surgery. Murase [9] studied pochette surgeries of the double of $P$. Let $D(P)$ be the double of $P$ which means $P \cup_{\mathrm{id}} (-P)$. In fact, $D(P)$ is diffeomorphic to $S^1 \times S^3 \# S^2 \times S^2$. Let $i_P$ be the inclusion map $i_P : P \to D(P)$. He shows the resulting manifold $D(P)(i_P, p/q, \epsilon)$ is diffeomorphic to a rational homology 4-sphere with type $L$, which is defined in [13].

In the next section, we will share Okawa's result with readers. He investigates pochette surgeries yielding homotopy 4-spheres with the core sphere ribbon and with the cord trivial. We generalize this in Theorem 1.4.

Suzuki [14] computed the homology of some types of pochette surgeries. These results are generalized in this paper (Proposition 2.5).

Pochette surgery can easily extend to a surgery along $\natural^a S^1 \times D^3 \natural^b D^2 \times S^2$ for some positive integers $a, b$. This is called *outer surgery* defined in [10]. In the future, we expect to find many exotic 4-manifolds by pochette surgery or outer surgery. See Section 5 for questions for pochette surgery or outer surgery.

**1D.** ***Pochette surgery with trivial cord or trivial core sphere.*** After the definition of pochette surgery by Iwase and Matsumoto, pochette surgeries for embedding of $P$ with trivial cord or trivial core sphere in $S^4$ have been considered to construct a new type of homotopy 4-spheres.

*The case of trivial cord.* In this paper, we clarify diffeomorphism types of pochette surgeries of closed 4-manifolds with the trivial cord. Okawa proved the following.

**Theorem 1.3** (Okawa [12]). *Let $e$ be an embedding of $P$ into $S^4$ with the cord trivial. If the core sphere $S_e$ is a ribbon 2-knot, then any pochette surgery $S^4(e, 1/q, \epsilon)$ is diffeomorphic to $S^4$ for any integer $q$.*

Here we state the first main theorem.

**Theorem 1.4.** *Let $e$ be an embedding of $P$ into a closed 4-manifold $M$ with the trivial cord. Then for any integer $q$, the following holds*:

$$M(e, 1/q, \epsilon) \cong \begin{cases} M, & \epsilon = 0, \\ \mathrm{Gl}(S_e), & \epsilon = 1. \end{cases}$$

The Gluck surgery along any ribbon 2-knot is diffeomorphic to the standard 4-sphere; see, for example, [5]. Hence, Theorem 1.4 implies Theorem 1.3. It is also known that Gluck surgeries of some nonribbon 2-knots give the standard $S^4$; see, for example, [6; 8; 11]. Pochette surgeries for such examples give the standard $S^4$.

Theorem 1.4 determines diffeomorphism types of $(1/q, \epsilon)$-pochette surgeries with the trivial cord. As a corollary, we clarify the diffeomorphism type of any pochette surgery on a homology 4-sphere with the complement of the core sphere homotopically trivial.

Gluck surgery can produce nonorientable exotic 4-manifolds due to Akbulut [1]. Hence, Theorem 1.4 implies that pochette surgery also produces nonorientable exotic 4-manifolds. As in the case of Gluck surgery, it remains uncertain whether pochette surgery has the potential to produce orientable exotic 4-manifolds (Question 5.6).

*The case of trivial core sphere.* Suzuki [14] proved that several examples of infinitely many homotopy 4-spheres with the trivial core sphere are all diffeomorphic to the standard 4-sphere.

Theorem 1.4 immediately leads to the following theorem. This is a generalization of first author's result.

**Theorem 1.5.** *Let $M$ be a homology* 4*-sphere. Let $e$ be an embedding $P \hookrightarrow M$ with $\pi_1(E(S_e)) = \mathbb{Z}$. If a pochette surgery produces a homology* 4*-sphere, then the result is diffeomorphic to $M$ or* $\mathrm{Gl}(S_e)$*. In particular suppose $M$ is $S^4$ and $e : P \hookrightarrow S^4$ is an embedding that the core sphere $S_e$ is the unknot. Then if a pochette surgery by $e$ yields a homology* 4*-sphere $M'$, then $M'$ is diffeomorphic to $S^4$.*

**1E.** ***Pochette surgeries with nontrivial core sphere and cord.*** Next, we consider several examples of pochette surgeries with nontrivial core sphere and cord.

First, we prove the existence of such an example.

**Theorem 1.6.** *There exists a pochette embedding $e : P \hookrightarrow S^4$ with a nontrivial core sphere and a nontrivial cord such that the pochette surgery $S^4(e, g)$ is diffeomorphic to $S^4$.*

Further, the following theorem gives a sufficient condition for the existence of nontrivial cords whose surgery yielding homotopy 4-sphere is trivialized.

**Theorem 1.7.** *Let $S \subset S^4$ be any ribbon* 2*-knot of* 1*-fusion with $\pi_1(E(S)) \ncong \mathbb{Z}$. Then there exists a nontrivial cord $c$ in $E(S)$ and an embedding*

$$e : P \to P_e = N(S) \cup N(c) \subset S^4$$

*such that the pochette surgery $S^4(e, p/(p+1), \epsilon)$ is diffeomorphic to $S^4$.*

Actually, as proven in Theorem 1.7, the core sphere of $e$ is any nontrivial ribbon 2-knot of 1-fusion. Furthermore, there exist infinitely many cords for such a ribbon 2-knot such that the results all obtain the standard $S^4$.

**Theorem 1.8.** *Let $S \subset S^4$ be any ribbon* 2*-knot with $\pi_1(E(S)) \ncong \mathbb{Z}$. Then there exists a nontrivial cord $C$ in $E(S)$ satisfying the following conditions*:

(1) *The embedding $e : P \hookrightarrow S^4$ has the core sphere $S$ and the cord $C$.*

(2) *If for a gluing map $g$, $S^4(e, g)$ is a homology* 4*-sphere then it is diffeomorphic to the double of a homology* 4*-ball $H$ without* 3*-handles.*

For a general ribbon 2-knot, it is uncertain whether the homology 4-ball $H$ is contractible or not. In Theorem 1.8 we show that for any nontrivial ribbon 2-knot there exists a nontrivial cord such that any pochette surgery yielding a homology 4-sphere gives the double of a homology 4-ball without 3-handles.

Furthermore, when $S^4(e, g)$ is a homotopy 4-sphere, for $S^4(e, g)$ to be the standard $S^4$, we have only to assume the AC-triviality of the presentation of $\pi_1$. As a result, we obtain the following theorem.

**Theorem 1.9.** *If the homology* 4*-ball* $H$ *obtained in Theorem 1.8 is contractible and the presentation of* $\pi_1(H)$ *for a handle decomposition of* $H$ *without* 3*-handles is AC-trivial*, *then* $S^4(e, g)$ *is standard* $S^4$.

In Lemma 4.5, we actually give infinitely many presentations for $\pi_1(H)$ satisfying this condition. This means that such a type of ribbon 2-knots has a nontrivial cord satisfying $S^4(e, g) = S^4$.

It is unknown whether a pochette surgery with nontrivial $S_e$ gives an exotic manifold or not. In general, even if $S_e$ is trivial in a 4-manifold $M$, then it is unclear whether the pochette surgery is trivial or not. We expect that some pochette surgery creates a new exotic 4-manifold.

**1F.** ***Aims of this paper.*** The first aim of this paper is to investigate pochette surgeries $M(e, g)$ yielding homotopy 4-spheres and to determine the diffeomorphism types. What occurs in the case of nontrivial core sphere? The second aim is what even in this case, we clarify the existence of nontrivial cords that pochette surgeries give the standard $S^4$.

**1G.** ***Organization of this paper.*** In Section 2, we give a review for pochette surgery. We define several definitions and lemmas. To carry out the second aim above, we compute the homology of $M(e, g)$ for any homology 4-sphere $M$. In order to compute the homology, we need to introduce the notion of a linking number for an embedding of a pochette as well as the slope which was defined by Iwase and Matsumoto [7]. The linking number of an embedded pochette is the usual linking number of the embedded core sphere $S_e$ and the longitude $l_e$ in $M$. It depends on the choice of a meridian $m$, a longitude $l$ and an embedding $e : P \hookrightarrow M$. Actually, we show that the homology of a pochette surgery is uniquely determined by the slope and the linking number (Proposition 2.5).

In Section 3, first, we prove Theorem 1.4 and clarify that pochette surgeries $M(e, g)$ of the case where the cord is trivial is diffeomorphic to $M$ or some Gluck surgery. Second, we prove Theorem 1.5, by using this result, and we give a sufficient condition that any pochette surgery of $M$ for some core sphere gives the same manifold $M$ or the Gluck surgery. As a particular condition, any $(1/q, \epsilon)$-pochette surgery of 4-sphere whose core sphere is the unknot is diffeomorphic to $S^4$.

In Section 4, we investigate cases where the core sphere $S_e$ is a nontrivial 2-knot and the cord is a nontrivial (Theorem 1.6). These surgeries give the standard 4-sphere. Actually, we use a ribbon 2-knot of 1-fusion as $S_e$. The proof is essentially proven in Theorem 1.7. We generalize this situation to some cases where the core spheres are any general nontrivial ribbon 2-knots $S$ with $\pi_1(E(S)) \ncong \mathbb{Z}$ (Theorem 1.8). However, we did not see whether the resulting manifold is a homotopy 4-sphere or not. In Theorem 1.9, we give a sufficient condition of ribbon 2-knots for the existence of a nontrivial cord such that any surgery yielding homotopy 4-sphere gives the standard $S^4$.

## 2. Preliminaries

**2A. *Embedding of $P$.*** To consider an embedding of $P$ in a 4-manifold $M$, as mentioned in the previous section, we embed a 2-sphere $S$ in $M$ with product neighborhood and embed a cord in the exterior $E(S)$. In 4-dimension, the isotopy class of any 1-manifold coincides with the homotopy class. Thus, the isotopy class of any embedding of $P$ is determined by a 2-knot with product neighborhood and the homotopy class of a cord as a proper embedding in $E(S)$.

Let $S$ be a 2-knot in a homology 4-sphere $M$. Here we clarify the isotopy classes of embedding $e$ of $P$ with $S_e = S$. We put $G(S) = \pi_1(E(S))$. $G(S)$ includes a subgroup $\langle m \rangle$ that is isomorphic to $\mathbb{Z}$. In this section, $m$ is regarded as the class represented by the meridian circle. Here we call $\langle m \rangle$ a *boundary-subgroup*.

In fact, the abelianization map induces the surjection $G(S) \to H_1(E(S)) \cong \mathbb{Z}$ and the meridian is mapped to a generator in $\mathbb{Z} \subset H_1(E(S))$. Thus $m$ is nontorsion in $G(S)$. We define the set of isotopy classes of cords in $E(S)$ to be

$$\Pi_1(E(S), \partial E(S)) := [(I, \partial I), (E(S), \partial E(S))],$$

and the double coset space $G(S)//\langle m \rangle := \langle m \rangle \backslash G(S)/\langle m \rangle$. Let

$$\varphi : \pi_1(E(S), \partial E(S)) \to \Pi_1(E(S), \partial E(S))$$

be the natural map.

**Lemma 2.1.** *Let $S$ be a 2-knot in a homology 4-sphere $M$. The set of properly embedded cords up to isotopy with the end points included in $\partial E(S)$ has a bijection to the double coset space $G(S)//\langle m \rangle$.*

*Proof.* By the short exact sequence

$$1 \to \pi_1(\partial E(S)) \to \pi_1(E(S)) = G(S) \to \pi_1(E(S), \partial E(S)) \to 1$$

induced from the homotopy long exact sequence of the pair $(E(S), \partial E(S))$, we have the bijection

$$\pi_1(E(S), \partial E(S)) \cong \langle m \rangle \backslash G(S).$$

Here $\pi_1(E(S), \partial E(S))$ is the relative homotopy set.

Any element in $\Pi_1(E(S), \partial E(S))$ can be realized as one in $\pi_1(E(S), \partial E(S))$ by homotoping a starting point of the path to the base point $x_0$ of $\pi_1(E(S), \partial E(S))$. If $\varphi(\gamma_0) = \varphi(\gamma_1)$ for some $\gamma_0, \gamma_1 \in \pi_1(E(S), \partial E(S))$, then

$$\gamma_0(0) = \gamma_1(0) = x_0, \gamma_0(1), \gamma_1(1) \in \partial E(S).$$

There is a homotopy $H : I \times I \to E(S)$ such that $H(i, \cdot\,) = \gamma_i$ and $H(t, i) \in \partial E(S)$ $(i = 0, 1)$. Then $c(t) := H(t, 0)$ is a loop in $\partial E(S)$ with a base point $x_0$, we have $\gamma_0 = \gamma_1 \cdot c \in \pi_1(E(S), \partial E(S))$. Therefore, $\varphi$ is surjective. If

$$\gamma_0 = \gamma_1 \cdot c \in \pi_1(E(S), \partial E(S))$$

for some $c \in \pi_1(\partial E(S))$, then $\gamma_0 = \gamma_1$ in $\Pi_1(E(S), \partial E(S))$. Thus

$$\pi_1(E(S), \partial E(S))/\langle m\rangle \to \Pi_1(E(S), \partial E(S))$$

is bijective.

Then we obtain the bijection

$$\Pi_1(E(S), \partial E(S)) \to \pi_1(E(S), \partial E(S))/\langle m\rangle \to G(S)/\!/\langle m\rangle. \qquad \square$$

Let $[\![\mathrm{id}]\!]$ be the element in $G(S)/\!/\langle m\rangle$ represented by the trivial cord. Here the class in the double coset is represented by $[\![\,\cdot\,]\!]$ and id stands for the identity element in $G(S)$. Hence, if the boundary-subgroup $\langle m\rangle$ is a proper subgroup in $G(S)$, then $G(S)/\!/\langle m\rangle \neq \{[\![\mathrm{id}]\!]\}$. If $S$ is the trivial 2-knot in the 4-sphere, then $G(S) = \langle m\rangle$ and it has a unique isotopy class of a cord. If $G(S)$ is not isomorphic to $\mathbb{Z}$, then there exists a nontrivial cord.

**2B.** ***Fundamental group of pochette surgery.*** In general, to find a homotopy 4-sphere obtained by applying pochette surgery, we need to compute the fundamental group. Let $M$ be a 4-manifold and $e$ an embedding $e : P \hookrightarrow M$. According to [7], we see that a free isotopy class of an unoriented curve with slope $p/q$ is uniquely determined as an image of $m$. We call the class a *natural lift*. Let $c_{p,q}$ be the natural lift of $p[m_e] + q[l_e]$ to $\pi_1(\partial E(P_e))$, which is defined in [7]. Let $l'$, and $m'$ be the images on $\pi_1(\partial E(P_e))$ of the based, oriented, longitude and meridian in $\partial P$ via $e$ respectively. Let $c'_{p,q}$ be an element in $\pi_1(\partial E(P_e))$ presenting $c_{p,q}$. Concretely, the element is given by

$$c'_{p,q} = l'^{\lfloor q/p\rfloor} m' l'^{\lfloor 2q/p\rfloor - \lfloor q/p\rfloor} m' l'^{\lfloor 3q/p\rfloor - \lfloor 2q/p\rfloor} \cdots\, m' l'^{\lfloor pq/p\rfloor - \lfloor ((p-1)q)/p\rfloor} m'.$$

See Theorem 6 in [7].

We assume that the group presentation of $\pi_1(E(S))$ is $\pi_1(E(S)) = \langle \mathcal{S} \mid \mathcal{R}\rangle$, where $\mathcal{S}$ is a set of generators and $\mathcal{R}$ is a set of relators. For the inclusion maps $i : \partial P_e \to E(P_e)$ and $j : \partial P \to P$, the following maps are induced:

$$i_\# : \pi_1(\partial P_e) \to \pi_1(E(P_e)), \quad j_\# : \pi_1(\partial P) \to \pi_1(P).$$

From the Seifert–Van Kampen theorem, we have

$$\pi_1(M(e, p/q, \epsilon)) = \langle \mathcal{S} \mid \mathcal{R}, c'_{p,q} \rangle. \tag{1}$$

**2C.** ***Mod 2 framing.*** For a gluing map $g : \partial P \to \partial E(P_e)$ we define mod 2 framing of $g(m)$ as explained in [7, first paragraph in p. 162]. Let us consider a pochette surgery on $M$. After attaching $D^2 \times S^2$ in $P$ along $g(m)$, we can uniquely attach the remaining $S^1 \times D^3$. Hence, we have only to consider an identification between neighborhoods of $m$ and $g(m)$ via $g$ to attach $P$.

We fix an identification

$$\partial P = S^1 \times \partial D^3 \# \partial D^2 \times S^2 = S^1 \times S^2 \# S^1 \times S^2.$$

The meridian $m = \partial D^2 \times \{\mathrm{pt}\} \subset \partial P$ has the natural product framing. We obtain an identification $\iota : \partial E(P_e) \to S^1 \times S^2 \# S^1 \times S^2$ through the embedding $e$. Then, $S^1 \times S^2 \# S^1 \times S^2$ can be presented by the 2-component unlink with 0-framings. We map the natural framing on $m \subset \partial P$ to a framing on $g(m)$. The framing is presented by an integer by the identification $\iota$. As far as we consider the diffeomorphism type of the result of the pochette surgery, we have only to consider an integer modulo 2 as the framing on $g(m)$. In fact, consider $P$ as $S^1 \times D^3$ attaching a 2-handle with the cocore $m$. For two gluing maps $g_1, g_2 : \partial P \to \partial E(P_e)$ with $g_1(m) = g_2(m)$ but with framings whose difference is divisible by 2, the map $g_1^{-1} \circ g_2|_{N(m)}$ can be extended to the inside of the 2-handle. Namely, two 4-manifolds attached by such gluing maps are diffeomorphic each other. Such a framing on $g(m)$ is called a *mod* 2 *framing* and written by $\epsilon$.

**2D.** ***Linking number.*** Let $l$ and $S$ be the longitude and the core sphere of a pochette $P$ respectively. Let $M$ be an oriented homology 4-sphere and $e : P \hookrightarrow M$ an embedding. The images $l_e$, and $S_e$ in $M$ give submanifolds of $M$. Then they can give the linking number

$$\ell = L(S_e, l_e)$$

according to [3]. In fact, we extend an embedding $e|_S : S \to M$ to a map $\mathcal{B}^3 \to M$, where $\mathcal{B}^3$ is a homology 3-ball. The orientation of $\mathcal{B}^3$ is induced by the one of $S_e$. We count the intersection points between the image of $\mathcal{B}^3$ and $l_e$ with sign. Here we deform $l_e$ in $E(S_e)$ so that $l_e$ can meet with $\mathcal{B}^3$ transversely. For each intersection point if the concatenation of orientations on $\mathcal{B}^3$ and $l_e$ at the point coincides with the orientation of $M$, then the sign is $+1$, otherwise $-1$. We call the sign a *local intersection number* at the intersection point. In the end, we sum up the local intersection numbers through all the intersection points. In the same way, we can compute $L(l_e, S_e)$ by changing the order of $l_e$ and $S_e$.

In the general theory of linking number, the absolute values of $L(S_e, l_e)$ and $L(l_e, S_e)$ are the same. Actually, by the careful consideration of orientation we can easily obtain $L(S_e, l_e) = -L(l_e, S_e) =: \ell$. We call this number $\ell$ *linking number* of the embedding $e$. We must notice that the linking number is *not* an invariant of the embedding of $P$. If we fix the coordinate $m$ and $l$, then the linking number can be determined. This is due to what the 3-disk separating $S^1 \times D^3$ and $D^2 \times S^2$ is not unique.

Here let us reinterpret the linking number $L(S_e, l_e)$ in terms of the homology. We use the intersection pairing:

$$\langle\,\cdot\,,\,\cdot\,\rangle_3^4 : H_3\big(E(S_e), \partial(E(S_e))\big) \times H_1(E(S_e)) \to \mathbb{Z}.$$

Let $\mathcal{M}^3$ be a Seifert hypersurface of $S_e$ in $E(S_e)$, namely $\mathcal{M}^3$ is a properly embedded 3-manifold in $E(S_e)$ satisfying $\partial\mathcal{M}^3 = S_e$. $H_3\big(E(S_e), \partial(E(S_e))\big)$ is isomorphic to $\mathbb{Z}[\mathcal{M}^3]$. Here $\mathcal{M}^3 \cap E(S_e)$ and $\mathcal{M}^3$ are identified. $H_3(E(P_e), \partial E(P_e))$ is isomorphic to $\mathbb{Z}[\mathcal{M}^3]$.

The intersection point between $\mathcal{M}^3$ and $m_e$ is one point. Here we give an orientation on $\mathcal{M}^3$ satisfying $\langle[\mathcal{M}^3], [m_e]\rangle_3^4 = +1$.

By the definition of linking number, it follows that $\langle[\mathcal{M}^3], [l_e]\rangle_3^4 = \ell$. Since $H_1(E(S_e))$ is also isomorphic to $\mathbb{Z}$ generated by $[m_e]$, we have $[l_e] = \ell[m_e]$.

In the similar way we consider the next intersection pairing:

$$\langle\,\cdot\,,\,\cdot\,\rangle_2^4 : H_2\big(E(l_e), \partial(E(l_e))\big) \times H_2(E(l_e)) \to \mathbb{Z}.$$

Here we take a proper embedded surface $\Sigma$ satisfying $\partial\Sigma = l_e$ in $E(l_e)$. We take the usual orientation of the meridian $B_e$ of $l_e$ and the orientation on $\Sigma$ by using $\langle[\Sigma], [B_e]\rangle_2^4 = +1$. From the computation $L(l_e, S_e) = -\ell$ of the linking number, we obtain $\langle[\Sigma], [S_e]\rangle_2^4 = -\ell$. Since $H_2(E(l_e))$ is isomorphic to $\mathbb{Z}$ generated by the belt sphere $[B_e]$, $[S_e] = -\ell[B_e]$ holds.

**2E.** ***The homology of a pochette surgery.*** Let $M$ be a homology 4-sphere. Here we compute the homology of the result by pochette surgery. Let $g : \partial P \to \partial E(P_e)$ be a gluing map with the slope $p/q$ and the mod 2 framing $\epsilon$. Let $i$ be the inclusion map $\partial E(P_e) \to E(P_e)$.

To compute the homology group of any pochette surgery of a homology 4-sphere, we prove lemmas needed later. First, we compute the homology of $E(P_e)$ here. Since $E(P_e)$ is connected, we have $H_0(E(P_e)) \cong \mathbb{Z}$.

**Lemma 2.2.** *$E(P_e)$ has the following homology groups*:

$$H_n(E(P_e)) = \begin{cases} \mathbb{Z}\cdot[m_e], & n = 1, \\ \mathbb{Z}\cdot[B_e], & n = 2, \\ 0, & n \geq 3. \end{cases}$$

*Proof.* Let $h^3$ be a 4-dimensional 3-handle. Attaching $h^3$ on the belt sphere of $P_e$, we obtain $E(P_e) \cup h^3 = E(S_e)$ and $E(P_e) \cap h^3 = \partial D^3 \times D^1 = S^2 \times D^1$. The homology of $E(S_e)$ is the same as the homology of $S^1$ and the first homology group is generated by the meridian $m_e$. Since $H_1$ is independent of attaching any 3-handle, we have $H_1(E(P_e)) = H_1(E(P_e) \cup h^3) = H_1(E(S_e)) = \mathbb{Z}[m_e] \cong \mathbb{Z}$. Then we obtain the Mayer–Vietoris sequence:

$$\cdots \to H_n(S^2 \times D^1) \to H_n(E(P_e)) \oplus H_n(h^3) \to H_n(E(S_e)) \to \cdots .$$

Thus, we can easily check

$$H_n(E(P_e)) = \begin{cases} \mathbb{Z}, & n = 2, \\ 0, & n = 3, 4. \end{cases}$$

The generator of $H_1$ clearly corresponds to the meridian $m_e$ of $E(S_e)$ and the one of $H_2$ corresponds to the generator, the belt sphere $B_e$ which is the image of $H_2(S^2 \times D^1)$. □

From this lemma, we obtain natural isomorphisms $H_1(E(P_e)) \cong H_1(E(S_e))$ and $H_2(E(P_e)) \cong H_2(E(l_e))$. The isomorphisms are induced by the inclusions and connect the corresponding elements $[m_e]$ and $[B_e]$.

Let $g$ be a gluing map from $\partial P$ to $\partial E(P_e)$. Suppose that $g_*([m]) = p[m_e] + q[l_e]$ is satisfied on the first homology group.

**Lemma 2.3.** *If $g_*([m]) = p[m_e] + q[l_e]$, then we have $g_*([B]) = p[B_e] - q[S_e]$.*

*Proof.* We put $g_*([l]) = r[m_e] + s[l_e]$, $g_*([B]) = x[B_e] + y[S_e]$. Then, we can define the nondegenerate bilinear form $\langle \cdot, \cdot \rangle_3 : H_1(\partial P) \times H_2(\partial P) \to \mathbb{Z}$ from the cup product $H^2(\partial P) \times H^1(\partial P) \to H^3(\partial P)$.

By defining

$$\langle [m], [B] \rangle_3 = 0, \quad \langle [l], [B] \rangle_3 = 1, \quad \langle [m], [S] \rangle_3 = 1 \quad \text{and} \quad \langle [l], [S] \rangle_3 = 0,$$

we determine the orientations on $m$ and $B$. These orientations coincide with the ones determined Section 2D via the map $H_n(\partial P_e) \to H_n(E(P_e))$. Since $g : \partial P \to \partial E(P_e)$ is a diffeomorphism, we can define the nondegenerate bilinear form $\langle \cdot, \cdot \rangle_3^e : H_1(\partial E(P_e)) \times H_2(\partial E(P_e)) \to \mathbb{Z}$ from the nondegenerate bilinear form $\langle \cdot, \cdot \rangle_3 : H_1(\partial P) \times H_2(\partial P) \to \mathbb{Z}$. Since $g : \partial P \to \partial E(P_e)$ is an orientation preserving diffeomorphism, the determinant of the matrix given by

$$\begin{pmatrix} g_*([m]) & g_*([l]) \end{pmatrix} = \begin{pmatrix} [m_e] & [l_e] \end{pmatrix} \begin{pmatrix} p & r \\ q & s \end{pmatrix}$$

is 1. Hence we obtain $ps - qr = 1$. Thus the inverse is as

$$\begin{pmatrix} g_*^{-1}([m_e]) & g_*^{-1}([l_e]) \end{pmatrix} = \begin{pmatrix} [m] & [l] \end{pmatrix} \begin{pmatrix} s & -r \\ -q & p \end{pmatrix}.$$

Since

$$\langle g_*(\alpha), g_*(\beta) \rangle_3^e = \langle \alpha, \beta \rangle_3 \quad \text{for all } \alpha \in H_1(\partial P),\ \beta \in H_2(\partial P),$$

we have

$$x = \langle [l_e], x[B_e] + y[S_e] \rangle_3^e = \langle [l_e], g_*([B]) \rangle_3^e = \langle g_*^{-1}([l_e]), [B] \rangle_3$$
$$= \langle -r[m] + p[l], [B] \rangle_3 = p$$

and

$$y = \langle [m_e], x[B_e] + y[S_e] \rangle_3^e = \langle [m_e], g_*([B]) \rangle_3^e = \langle g_*^{-1}([m_e]), [B] \rangle_3$$
$$= \langle s[m] - q[l], [B] \rangle_3 = -q.$$

Therefore, we obtain the desired result above. □

**Lemma 2.4.** *Let $e$ be an embedding $P \hookrightarrow M$ with linking number $\ell$. Let $i$ be an inclusion $i : \partial E(P_e) \to E(P_e)$. Then $i_*([l_e]) = \ell[m_e]$ and $i_*([S_e]) = -\ell[B_e]$ are satisfied.*

*Proof.* The image of $[l_e] \in H_1(\partial E(P_e))$ by $i_*$ is also $[l_e]$ in $H_1(E(P_e))$. Since $H_1(E(P_e))$ and $H_1(E(S_e))$ are identified with each other by the natural isomorphism by the inclusion, the elements $[m_e]$ having in these homology groups are mapped. Hence, from Section 2D, $[l_e] = \ell[m_e]$ also holds in $H_1(E(P_e))$. In the same way, we have $i_*([S_e]) = -\ell[B_e]$. □

Here, we compute the homology groups of the pochette surgery $M(e, p/q, \epsilon)$. Since $M$ is connected and oriented, $H_0(M(e, p/q, \epsilon)) \cong H_4(M(e, p/q, \epsilon)) \cong \mathbb{Z}$ is satisfied. We compute $H_n$ of $M$ for $n = 1, 2, 3$.

**Proposition 2.5.** *Let $M$ be a homology 4-sphere. Let $e$ be an embedding with linking number $\ell$. Then, $M(e, p/q, \epsilon)$ has the following homology groups*:

(i) *If $p + q\ell \neq 0$, then*

$$H_n(M(e, p/q, \epsilon)) \cong \begin{cases} \mathbb{Z}/(p+q\ell)\mathbb{Z}, & n = 1, 2, \\ 0, & n = 3. \end{cases}$$

(ii) *If $p + q\ell = 0$, then*

$$H_n(M(e, p/q, \epsilon)) \cong \begin{cases} \mathbb{Z}, & n = 1, 3, \\ \mathbb{Z}^2, & n = 2. \end{cases}$$

Note that the case of $p + q\ell = 0$ means $(p, q) = (\ell, -1), (-\ell, 1)$ because $p, q$ are relatively prime.

*Proof.* The embedding map $e : P \hookrightarrow M$ induces the map

$$H_n(\partial P) \xrightarrow{g_*} H_n(\partial E(P_e)) \xrightarrow{i_*} H_n(E(P_e)).$$

Then we have $H_1(\partial E(P_e)) = \mathbb{Z} \cdot [m_e] \oplus \mathbb{Z} \cdot [l_e]$, $H_2(\partial E(P_e)) = \mathbb{Z} \cdot [B_e] \oplus \mathbb{Z} \cdot [S_e]$ and obtain $g_*([m]) = p[m_e] + q[l_e]$, $i_*([m_e]) = [m_e]$ and $i_*([B_e]) = [B_e]$. By Lemma 2.3, we obtain $g_*([B]) = p[B_e] - q[S_e]$. By Lemma 2.4, we have $i_*([l_e]) = \ell[m_e]$ and $i_*([S_e]) = -\ell[B_e]$. By Lemma 2.2 and the Mayer–Vietoris sequence

$$\cdots \to H_*(E(P_e)) \oplus H_*(P) \to H_*(M(e, p/q, \epsilon)) \to H_{*-1}(\partial P) \to \cdots,$$

we obtain

$$\begin{aligned}
&\longrightarrow 0 \longrightarrow H_3(M(e, p/q, \epsilon)) \xrightarrow{\partial_3} \mathbb{Z}\cdot[B] \oplus \mathbb{Z}\cdot[S] \\
&\xrightarrow{j_{21}\oplus j_{22}} \mathbb{Z}\cdot[B_e] \oplus \mathbb{Z}\cdot[S] \xrightarrow{i_2} H_2(M(e, p/q, \epsilon)) \xrightarrow{\partial_2} \mathbb{Z}\cdot[m] \oplus \mathbb{Z}\cdot[l] \\
&\xrightarrow{j_{11}\oplus j_{12}} \mathbb{Z}\cdot[m_e] \oplus \mathbb{Z}\cdot[l] \xrightarrow{i_1} H_1(M(e, p/q, \epsilon)) \xrightarrow{\partial_1 = 0} H_0(\partial P)
\end{aligned}$$

We put $j_n = j_{n1} \oplus j_{n2}$ for any $n \in \mathbb{Z}$. Since we have $\partial_1 = 0$, $i_1$ is a surjection. Since we have $j_1([m]) = (p + q\ell)[m_e]$ and $j_1([l]) = (r + s\ell)[m_e] + [l]$, we obtain

$$\begin{aligned}
H_1(M(e, p/q, \epsilon)) = \operatorname{Im} i_1 &\cong \mathbb{Z}\cdot[m_e] \oplus \mathbb{Z}\cdot[l] / \langle (p+q\ell)[m_e], (r+s\ell)[m_e] + [l] \rangle \\
&\cong \mathbb{Z}\cdot[m_e]/\langle (p+q\ell)[m_e] \rangle \cong \mathbb{Z}/(p+q\ell)\mathbb{Z}.
\end{aligned}$$

Here $r$, $s$ are the same coefficients as the ones used in the proof of Lemma 2.3.

Next, we compute $H_2$ and $H_3$ of the result of the pochette surgery.

If $p + q\ell \neq 0$, then $j_1$ is an injection. Since $i_2$ is a surjection, we obtain the following isomorphism:

$$\begin{aligned}
H_2(M(e, p/q, \epsilon)) &= \operatorname{Im} i_2 \\
&\cong \mathbb{Z}\cdot[B_e] \oplus \mathbb{Z}\cdot[S] / \langle (p+q\ell)[B_e], (r'+s'\ell)[B_e] + [S] \rangle \\
&\cong \mathbb{Z}\cdot[B_e]/\langle (p+q\ell)[B_e] \rangle \cong \mathbb{Z}/(p+q\ell)\mathbb{Z}.
\end{aligned}$$

Here, $r'$, $s'$ are some integers satisfying $ps' + qr' = 1$. In this case, $\operatorname{Im} \partial_3 = \operatorname{Ker} j_2 = 0$. Thus we have

$$H_3(M(e, p/q, \epsilon)) = \operatorname{Ker} \partial_3 = 0.$$

If $p + q\ell = 0$, then $\operatorname{Im} \partial_2 = \operatorname{Ker} j_1 = \mathbb{Z}\cdot[m]$. Thus we have

$$H_2(M(e, p/q, \epsilon)) \cong \operatorname{Im} i_2 \oplus \mathbb{Z}\cdot[m] \cong \mathbb{Z}\cdot[B_e] \oplus \mathbb{Z}\cdot[m].$$

In this case, $\operatorname{Im} \partial_3 = \operatorname{Ker} j_2 = \mathbb{Z}\cdot[B]$. Thus we have

$$H_3(M(e, p/q, \epsilon)) \cong \mathbb{Z}\cdot[B].$$

Therefore, we obtain the desired result above. □

The theorems by Whitehead [15], Freedman [4] and Proposition 2.5 imply the next corollary.

**Corollary 2.6.** *Let $M$ be a homology 4-sphere. $M(e, p/q, \epsilon)$ is homeomorphic to $S^4$ if and only if $M(e, p/q, \epsilon)$ is a simply connected 4-manifold and $|p + q\ell|$ is equal to* 1.

*Proof.* By Freedman's theorem, $M(e, p/q, \epsilon)$ is homeomorphic to $S^4$ if and only if $M(e, p/q, \epsilon)$ is homotopy equivalent to $S^4$. We will only show that $M(e, p/q, \epsilon)$ is homotopy equivalent to $S^4$ if and only if $M(e, p/q, \epsilon)$ is a simply connected 4-manifold and $|p+q\ell| = 1$. By the Whitehead theorem, the necessary and sufficient condition for a manifold to be homotopy equivalent to $S^4$ is $\pi_1 = \{\mathrm{id}\}$ and $H_n = 0$ for $n = 1, 2, 3$. From Proposition 2.5, we can easily check this corollary follows. □

**2F.** ***Images of the meridian by diffeomorphism.*** In this section we describe images of $m$ via some gluing maps $g : \partial P \to \partial E(P_e)$ with slope $1/p$ and $p/(p+1)$. In the first diagram in Figure 1 we describe $m, l \subset \#^2 S^2 \times S^1$. By sliding along the dashed arrow in the first picture, $m$ is moved to a curve represented by $[m] + [l]$ in the second picture. Furthermore, sliding the diagram along the dashed arrow, we obtain the third picture. Then $[m]+[l]$ is moved to a curve by represented by $[m]+2[l]$. By the same diffeomorphism, $[m] + 2[l]$ is moved to a curve represented by $[m] + 3[l]$ in the fourth picture.

Thus, by the diffeomorphism $h : \#^2 S^2 \times S^1 \to \#^2 S^2 \times S^1$ with slope $1/p$, meridian $m$ is moved to a curve represented in $[m] + p[l]$ as in the bottom picture in Figure 1. This position will be used when we describe the handle diagram of $M(e, 1/p, \epsilon)$.

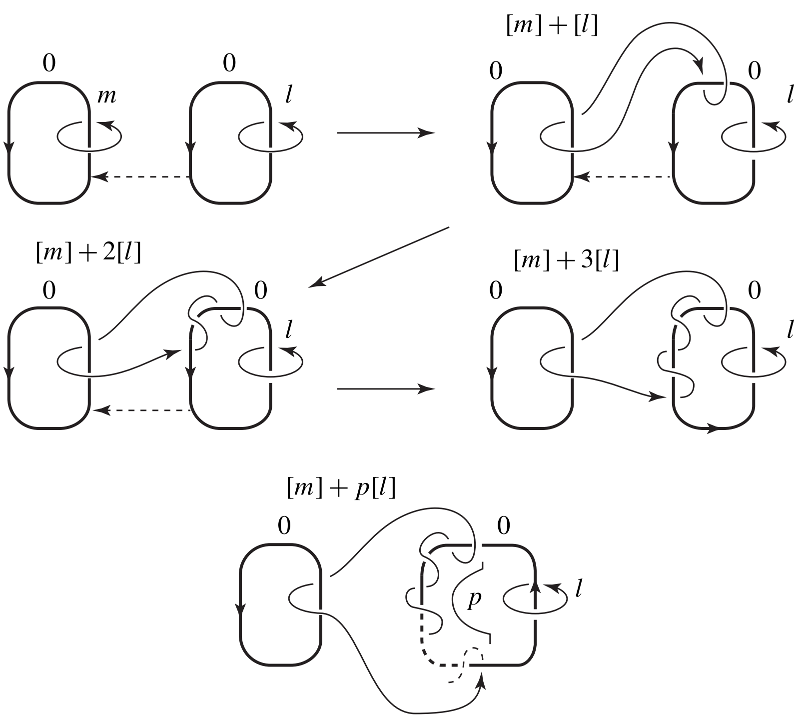



**Figure 1.** Images of $m$ and $l$ via a gluing map $\#^2 S^2 \times S^1 \to \#^2 S^2 \times S^1$.

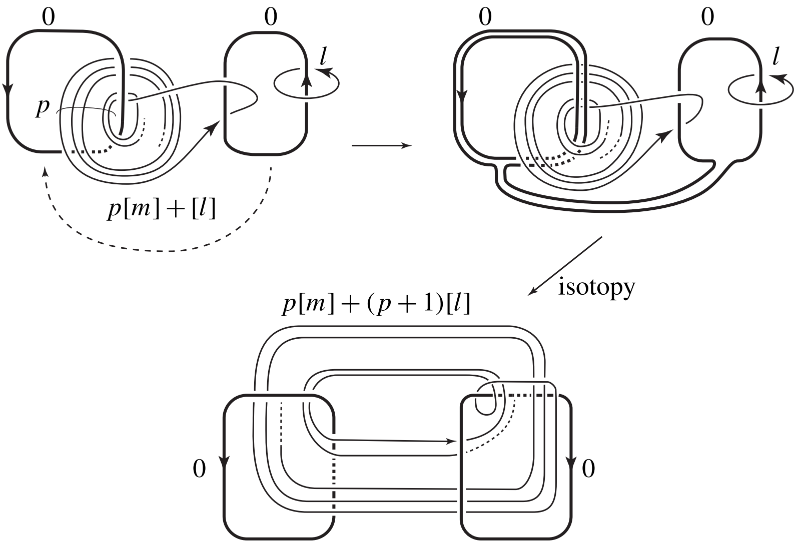



**Figure 2.** Case (I).

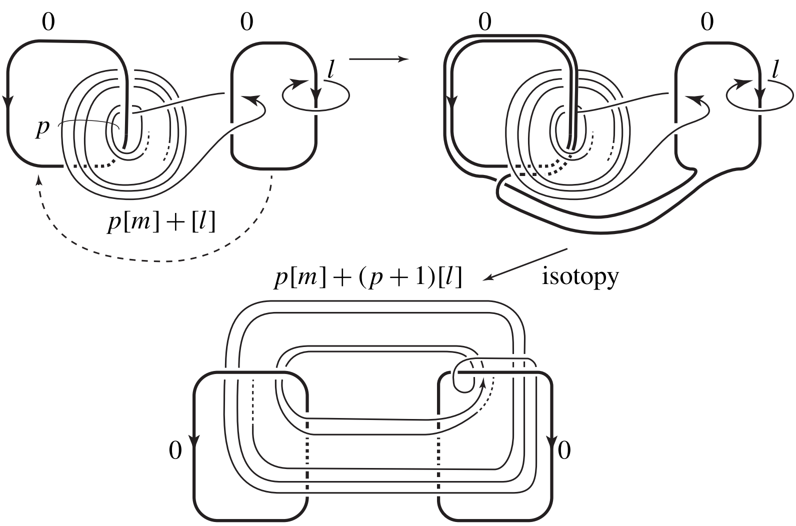



**Figure 3.** Case (II).

Furthermore, exchanging $m$ and $l$ in the last picture in Figure 1 and doing an isotopy, we obtain a curve represented by $p[m] + [l]$ as in the first pictures in Figures 2 and 3. We call these cases Case (I) and Case (II) respectively. Sliding a 0-framed 2-handle, we obtain the second picture. The thin curves in the figures are represented by $p[m] + (p + 1)[l]$. By an isotopy we obtain the last pictures in Figures 2 and 3.

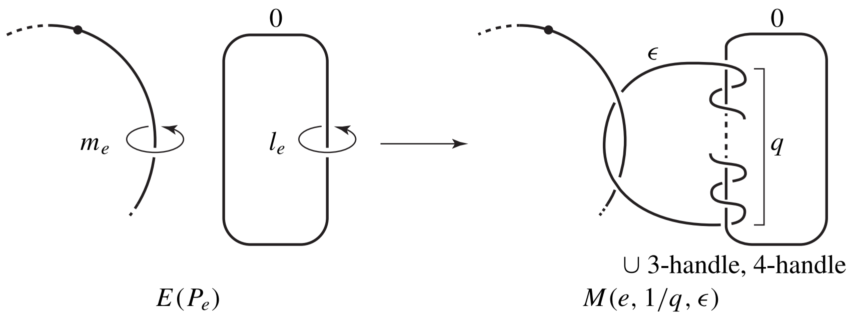



**Figure 4.** Attaching $P$ on $E(P_e)$ with the trivial cord.

## 3. Proofs of Main theorems

In this section we prove Theorem 1.4.

*Proof of Theorem 1.4.* Let $e$ be an embedding $P \hookrightarrow M$ with a trivial cord. The exterior $E(P_e)$ is obtained by attaching a 0-framed 2-handle on $E(S_e)$ in a separated position from the diagram of $E(S_e)$ as in the left picture of Figure 4. The circle $m_e$ in the figure is the image of meridian of $P$. For example, when we describe $E(S_e)$ along the motion picture as in [5, Section 6.2], it is a meridian of a 1-handle corresponding to a 0-handle of the embedded sphere. Hence, the pochette surgery on $M$ can be obtained by attaching an $\epsilon$-framed 2-handle on $E(P_e)$ plus a 3-handle and a 4-handle. The position of the $\epsilon$-framed 2-handle is understood from the argument in Section 2F. The right picture in Figure 4 is the local picture of the handle diagram of $M(e, 1/q, \epsilon)$.

Here, we prove that the rightmost 0-framed knot in Figure 4 is isotopic to the unknot in $\partial(E(S_e) \cup h^2(\epsilon)) = S^3$, where $h^2(\epsilon)$ is the $\epsilon$-framed 2-handle. We remove the previous 3- and 4-handle in $M(e, 1/q, \epsilon)$. Since the boundary of obtained manifold is diffeomorphic to the $\epsilon$-Dehn surgery of $\partial E(S_e)$. By several handle moves, we obtain the Hopf link surgery that the framing coefficients of the two components are $\langle 0 \rangle$ and $\langle \epsilon \rangle$. Then we get the second picture in Figure 5. From this point, doing slides by $q$-times, we obtain the fifth picture. Canceling the Hopf link component, we obtain 0-framed knot as in the last picture in Figure 5. Hence, this 0-framed unknot is isotopic to the unknot.

Since we can move the 0-framed unknot in the last picture in Figure 4 to the unlink position in the same picture, we cancel this component with a 3-handle. The remaining diagram is obtained by attaching an $\epsilon$-framed 2-handle and a 4-handle on $E(S_e)$. Therefore, the resulting manifold is the trivial surgery or the Gluck surgery along $S_e$ depending on $\epsilon = 0$ or 1 respectively. □

Using this theorem, we can prove Theorem 1.5.

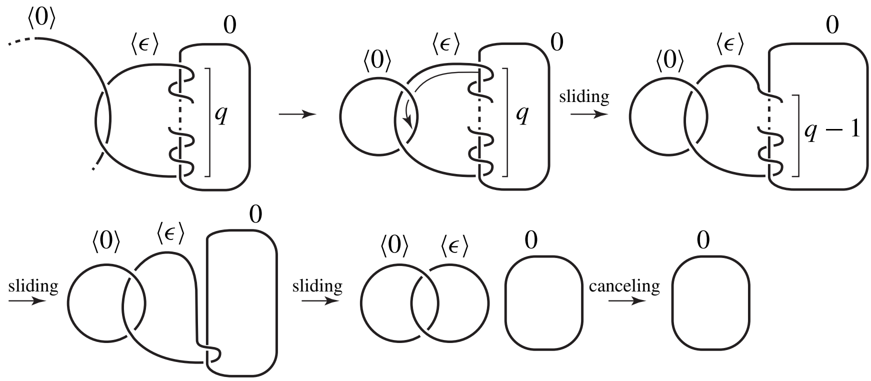



**Figure 5.** The isotopy type of the rightmost component.

*Proof of Theorem 1.5.* Let $e$ be an embedding $P \hookrightarrow M$. If $G(S_e) \cong \mathbb{Z}$ holds, then $\pi_1(E(S_e), \partial E(S))$ consists of one element. This means that any cord in $E(S_e)$ is isotopic to the trivial cord. Moving the embedded 1-handle in $P$ around the meridian $\partial D^2 \times \{\mathrm{pt}\}$ as an isotopy of $e$, we can make the linking number zero. Hence, if the pochette surgery produces a homology 4-sphere, then the slope is $1/q$ for some meridian and longitude in $P$. From Theorem 1.4, the result is $M$ (when $\epsilon = 0$) or $\mathrm{Gl}(S_e)$ (when $\epsilon = 1$).

If $M$ is diffeomorphic to $S^4$ and $S_e$ is the unknot, then any cord is isotopic to the trivial one. In the same way as above, any pochette surgery yielding a homology 4-sphere gives $S^4$. □

## 4. Examples

**4A.** ***Pochette surgeries along ribbon 2-knots of 1-fusion.*** In this section, we consider diffeomorphism types of pochette surgeries on the 4-sphere with nontrivial core spheres and nontrivial cords.

Now we define *ribbon* 2*-knot* and *fusion*.

**Definition 4.1** (ribbon 2-knot). Let $\{D_1^3, \dots, D_m^3\}$ be $m$ pairwise disjoint 3-disks in $S^4$. We take $m-1$ pairwise disjoint embeddings $f_1, \dots, f_{m-1} : D^2 \times [0, 1] \to S^4$. We assume that the embeddings satisfy the following conditions:

- $f_k(D^2 \times [0, 1]) \cap \bigcup_{u=1}^{m} \partial D_u^3 = f_k(D^2 \times \{0, 1\})$ for any $1 \le k \le m-1$.
- $\bigcup_{k=1}^{m-1} f_k(D^2 \times [0, 1]) \cup \bigcup_{u=1}^{m} \partial D_u^3$ is connected.

Then the boundary of union of these $m$ 3-disks and $m-1$ $D^2 \times [0, 1]$

$$\bigcup_{u=1}^{m} \partial D_u^3 \cup \bigcup_{k=1}^{m-1} f_k(\partial D^2 \times [0, 1])$$

is a 2-knot and called a *ribbon* 2*-knot* of $(m-1)$*-fusion*.

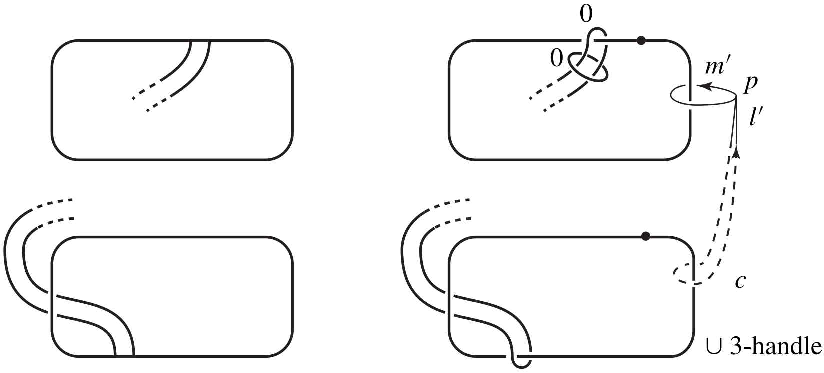



**Figure 6.** A ribbon 2-knot of 1-fusion (left) and the diagram of the complement of the 2-knot (right).

We take any ribbon 2-knot of 1-fusion as core spheres. Let $S$ denote a ribbon 2-knot of 1-fusion in the 4-sphere. The sphere $S$ is the double of a disk obtained by attaching one band over two 2-disks as presented by the left picture in Figure 6. The right diagram is the handle diagram of the complement of $S$. Let $m' \subset \partial E(S_e)$ be the oriented meridian of a dotted 1-handle indicated in Figure 6 with a base point $p$. Let $l'$ be an oriented meridian of the other dotted 1-handle passing $p$. Pushing the complement (the dashed line in the right picture in Figure 6) of the neighborhood of $l'$ in the interior of $E(S_e)$, we obtain a cord $c$. Then the following holds.

**Lemma 4.2.** *If $G(S_e)$ is not isomorphic to $\mathbb{Z}$, then this cord $c$ is nontrivial.*

Recall the triviality of a cord was defined in Definition 1.2.

*Proof.* The fundamental group $G(S_e)$ is presented by

$$\langle x, y \mid wxw^{-1}y^{\pm 1}\rangle,$$

where $x$ and $y$ are the elements presented by the meridian $m'$ and the longitude $l'$ respectively, and $w$ is a word obtained by reading $x$, $y$ along the 2-handle corresponding to the band. Here the boundary-subgroup in $G(S_e)$ is $\langle x\rangle$.

Let $p : G(S_e) \to G(S_e)//\langle x\rangle$ be the projection for the double coset. Let $[\![\mathrm{id}]\!]$ be the trivial coset in $G(S_e)//\langle x\rangle$, which is the coset including the identity element $\mathrm{id} \in G(S_e)$. The inverse image $p^{-1}([\![\mathrm{id}]\!])$ is equal to $\langle x\rangle$. In fact $\langle x\rangle \subset p^{-1}([\![\mathrm{id}]\!])$ is clear. For any $z \in p^{-1}([\![\mathrm{id}]\!])$, there exist some integers $r, s$ such that $x^r z x^s = \mathrm{id}$ is satisfied. Then $z = x^{-r-s} \in \langle x\rangle$.

The homotopy class of the cord $c$ corresponds to $[\![y]\!] \in G(S_e)//\mathbb{Z}$. If the cord $c$ is trivial, then $y \in p^{-1}([\![\mathrm{id}]\!]) = \langle x\rangle$ holds. Hence we have $y = x^n$ for some integer $n$. This means $G(S_e)$ is an abelian group. Since the abelianization of $G(S_e)$ is $\mathbb{Z}$, we have $G(S_e) \cong \mathbb{Z}$. □

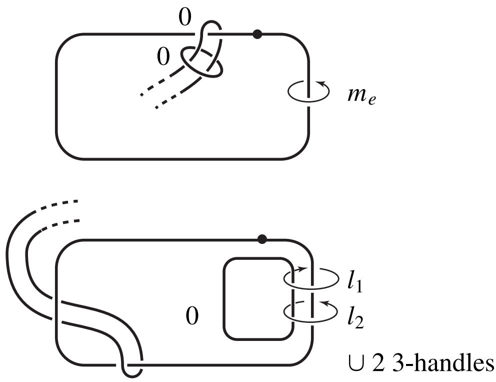



**Figure 7.** The pochette complement whose core sphere is a ribbon 2-knot of 1-fusion.

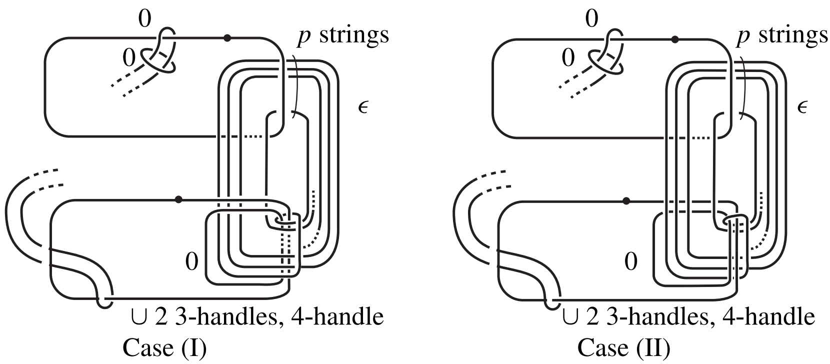



**Figure 8.** (I): $m_e$ is isotopic to $l_1$. (II): $m_e$ is isotopic to $l_2$.

In general, it is well-known that $G(S_e) \ncong \mathbb{Z}$ is satisfied for many nontrivial 2-knot $S_e$. Then the cord $c$ is nontrivial.

By using this cord $c$, we obtain an embedding $e : P \hookrightarrow S^4$ whose core sphere is $S$. Then the handle diagram of the complement $E(P_e)$ of $P$ is Figure 7. The meridian $m_e$ is isotopic to $l_1$ or $l_2$ in $E(S_e)$. Here we assume that $m_e$ is isotopic to $l_i$. Then, we put the orientation of the longitude as $[l_e] = -[l_i]$ in $E(P_e)$. Then $[m_e] = -[l_e]$ in $H_1(E(S_e))$ is satisfied. In this situation, the linking number of $P_e$ is $-1$. Consider the $(p/(p+1), \epsilon)$-pochette surgery by using the embedding $e$ and these oriented meridian and longitude in $P$. The element $y \in \pi_1(E(P_e))$ is a lift of $-[l_1]$ and $y^{-1}$ is a lift of $-[l_2]$, and hence $y^{\pm 1}$ is a lift of the longitude $l_e$.

According to the last pictures in Figures 2 and 3, the cases (I) and (II) in Figure 8 are obtained as results of attaching $P$ along $p[m_e] + (p+1)[l_e]$ with the mod 2 framing $\epsilon$. The case (I) is the one which $m_e$ is isotopic to $l_1$ (as an oriented loop), while (II) is the case where $m_e$ is isotopic to $l_2$ in the same way.

To prove Theorem 1.7, we first prove the following:

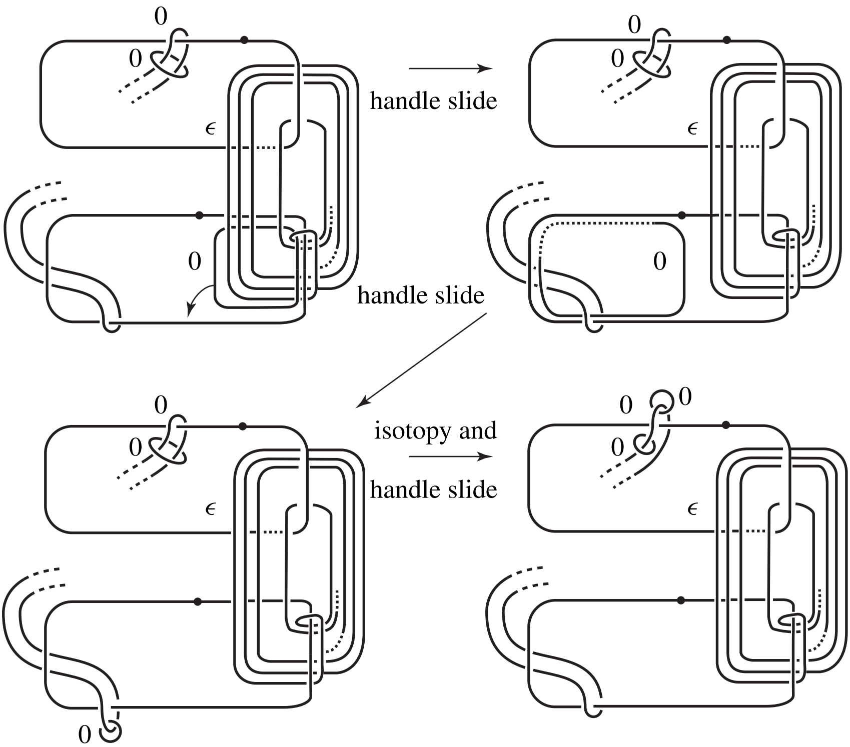



**Figure 9.** Handle moves.

**Proposition 4.3.** *$S^4(e, p/(p+1), \epsilon)$ is diffeomorphic to the double of a contractible 4-manifold without no 3-handles.*

*Proof.* Here we will consider the case where $m_e$ is isotopic to $l_2$. The case where $m_e$ is isotopic to $l_1$ can be proved in the same way.

We deform the handle diagram of (II) as in Figure 9. Continuously, we deform the handle diagram according to Figure 10. We show that the last picture presents that $S^4(e, p/(p+1), \epsilon)$ is diffeomorphic to the double of a contractible 4-manifold $C$. The fundamental group $\pi_1(C)$ of $C$ has the following presentation

$$\langle x, y \mid wxw^{-1}y^{\pm 1}, y^{\pm 1}(xy^{\pm 1})^p \rangle, \tag{2}$$

according to the last picture in Figure 10. The proof of the triviality of this group is postponed in Lemma 4.4. The homology group of $C$ is easily found out to be trivial from the handle decomposition. □

As mentioned in [2, second paragraph in p. 36], the following result holds. Let $\mathcal{C}$ be a contractible 4-manifold with $n$ 1-handles, $n$ 2-handles and no 3-handles. If the presentation $\pi_1(\mathcal{C})$ with respect to the handle decomposition is AC-trivial, which is defined in the next section, then the double satisfies $D(\mathcal{C}) := \mathcal{C} \cup_{\mathrm{id}} (-\mathcal{C}) = \partial(\mathcal{C} \times I)$.

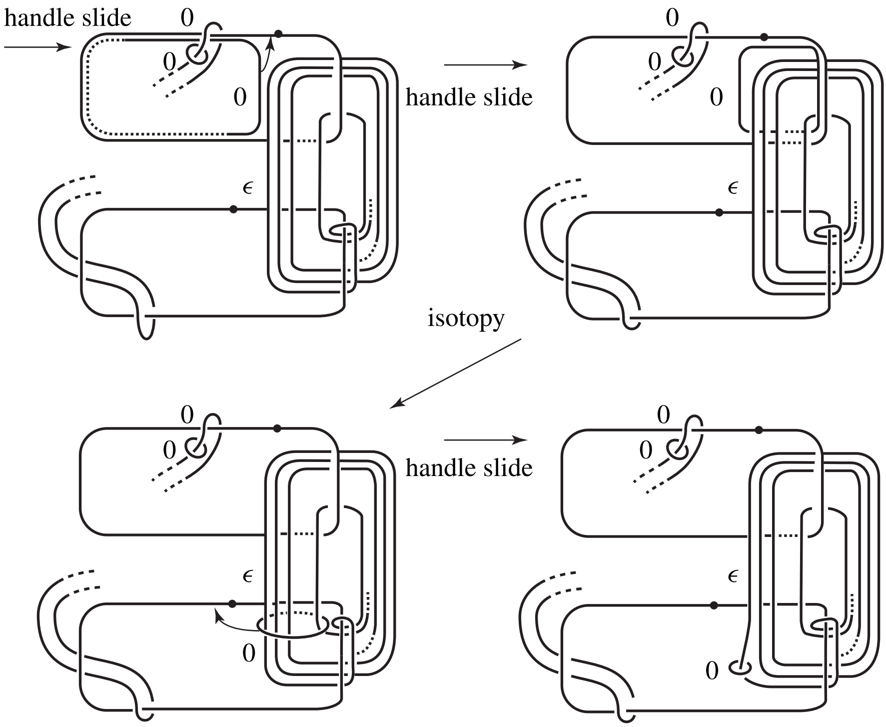



**Figure 10.** Handle moves.

Since the handle decomposition of $\mathcal{C} \times I$ depends only on the homotopy classes of the 2-handles, $\mathcal{C} \times I$ is diffeomorphic to the standard $D^5$. In the next section, we give a brief review of Andrews–Curtis moves and Andrews–Curtis trivial.

**4B.** ***AC-triviality.*** Let $F = F(X)$ be a free group of rank $n \geq 2$ with a basis $X = \{x_1, \ldots, x_n\}$ and $W = (w_1, \ldots, w_n)$ an $n$-tuple of words of $X$. Consider the following three types of transformations of $W$:

(AC1): Replace $w_i$ by $w_i w_j$ if $j \neq i$.

(AC2): Replace $w_i$ by $w_i^{-1}$.

(AC3): Replace $w_i$ by $v w_i v^{-1}$ for some $v \in F$, and leave $w_k$ fixed for all $k \neq i$.

Let $R = \langle x_1, \ldots, x_n \mid w_1, \ldots, w_n \rangle$ be a presentation of the trivial group. We call base transformations (inversion and permutation of generators and relators) of $X$, the transformations (AC1)–(AC3) for relators $w_1, \ldots, w_n$, and adding or deleting a generator $g$ and a relator $g$ as the same element *Andrews–Curtis moves* (or *AC-moves*). If $R$ can be reduced to the empty presentation $\langle \varnothing \mid \varnothing \rangle$ by a finite sequence of AC-moves for the basis and relators, then $R$ is called an *AC-trivial* presentation.

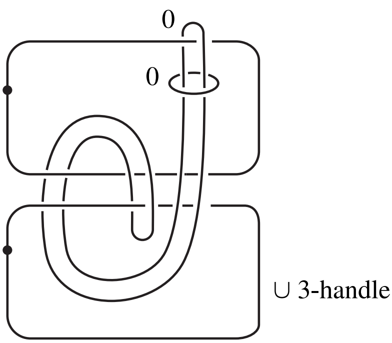



**Figure 11.** A handle diagram of a ribbon 2-knot exterior.

**Lemma 4.4.** *The presentation (2) is an AC-trivial presentation of the trivial group.*

*Proof.* We give the following sequence of AC-moves:

$$\begin{aligned}\langle x, y \mid wxw^{-1}y^{\pm 1}, y^{\pm 1}(xy^{\pm 1})^p\rangle &= \langle x, xy^{\pm 1} \mid wxw^{-1}x^{-1}(xy^{\pm 1}), x^{-1}(xy^{\pm 1})^{p+1}\rangle \\ &= \langle x, z \mid wxw^{-1}x^{-1}z, x^{-1}z^{p+1}\rangle \\ &= \langle x^{-1}z^{p+1}, z \mid wxw^{-1}x^{-1}z, x^{-1}z^{p+1}\rangle \\ &= \langle u, z \mid w(z^{p+1}u^{-1})w^{-1}uz^{-p}, u\rangle = \langle z \mid z^m\rangle.\end{aligned}$$

Here since this group is trivial, $m = \pm 1$. Thus the presentation is AC-trivial. □

We left the proof of the triviality of $\pi_1(C)$ in Proposition 4.3. Lemma 4.4 implies the proof of Proposition 4.3 completes.

*Proof of Theorem 1.7.* Let $e : S^2 \hookrightarrow S^4$ be a ribbon 2-knot of 1-fusion. We take the same cord $c$ as the one chosen in Section 4A, which is used in Figure 6. By using Proposition 4.3, the pochette surgery $S^4(e, p/(p+1), \epsilon)$ is diffeomorphic to the double of a contractible 4-manifold $C$. The $C$ has an AC-trivial presentation of $\pi_1$ coming from a handle decomposition of $C$ with no 3-handles. By applying the method in [2], $S^4(e, p/(p+1), \epsilon) = D(C)$ is diffeomorphic to the standard 4-sphere. □

*Proof of Theorem 1.6.* Let $S$ and $c$ be the ribbon 2-knot and the cord that we dealt with in Theorem 1.7. Then $S$ is nontrivial and $c$ is nontrivial. The pochette surgery gives the standard $S^4$. □

**4C.** ***A case of spun trefoil knot.*** As an example, we give a concrete diagram for the spun trefoil knot as a ribbon 2-knot of 1-fusion. Figure 11 is the handle diagram of the complement.

We choose $m_e$ and $l_e$ as in Figure 12 (left), then the embedding $i : \partial P_e \hookrightarrow E(P_e)$ gives $i_*([l_e]) = -[m_e]$. Namely the linking number is $\ell = -1$. Let $x, y$ be lifts

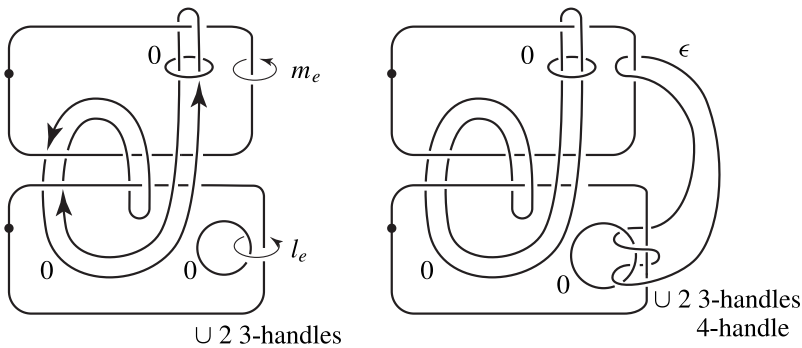



**Figure 12.** A pochette surgery $S^4\left(e, \frac{1}{2}, \epsilon\right)$ with a nontrivial 2-knot $S_e$ and a nontrivial cord.

in $\pi_1\left(S^4\left(e, \frac{1}{2}, \epsilon\right)\right)$ of generators $m_e$ and $l_e$ respectively. Then the presentation of $\pi_1\left(S^4\left(e, \frac{1}{2}, \epsilon\right)\right)$ is the following:

$$\langle x, y \mid yx^{-1}yxy^{-1}x,\, y^2x\rangle \cong \{\mathrm{id}\}.$$

The diagram of this homotopy 4-sphere becomes the right picture in Figure 12. In this case, we can deform this diagram into the double of a contractible 4-manifold with no 3-handles as in Figure 13.

**4D.** ***Pochette surgeries along ribbon 2-knots of n-fusion.*** The method to prove Theorem 1.7 can be easily extended to the case of the surgery that the core sphere is any ribbon 2-knot of $n$-fusion.

*Proof of Theorem 1.8.* Let $S$ be any ribbon 2-knot of $n$-fusion. We fix the handle decomposition of $E(S)$ corresponding to the fusion. That is, the decomposition has one 0-handle, $n+1$ dotted 1-handles, $n$ 2-handles and $n$ dual 2-handles and $n+1$ 3-handles and one 4-handle. See [5, Section 6.2] for the description of ribbon 2-knot complement. We take two based meridians $m'$ and $l'$ of the dotted 1-handles with a base point $p_0 \in \partial E(S)$. We suppose that $m'$ lies in $\partial E(S)$ and is a meridian of $\partial E(S)$. Let $x$, $y$ be elements in $\pi_1(E(S))$ corresponding to $m'$ and $l'$ respectively. Here we can assume that $y^{\pm 1}$ is conjugate to $x$ but $y^{\pm 1} \notin \langle x\rangle$. Actually, if any based meridian of each dotted 1-handle of $E(S)$ is in an element in $\langle x\rangle$, then $\pi_1(E(S))$ is a quotient of $\mathbb{Z}$, because the set of the meridians of the dotted 1-handles is a generator of $\pi_1(E(S))$. Actually using the abelianization map $\pi_1(E(S)) \xrightarrow{ab} H_1(E(S)) = \mathbb{Z}$, we conclude that $\pi_1(E(S))$ is isomorphic to $\mathbb{Z}$. Now this case is ruled out. Thus, there exists a based meridian $l' \subset E(S)$ such that $y := [l']$ is conjugate to $x$ but $y \notin \langle x\rangle$.

In the same way as the proof of Theorem 1.7, from $l'$ we produce a cord in $E(S)$. Thus, by taking such a cord, we obtain a pochette embedding $e : P \hookrightarrow S^4$. By moving the 0-framed 2-handle by the process in Figures 9 and 10, we can take the 0-framed 2-handle in the position of the meridian of the $\epsilon$-framed 2-handle.

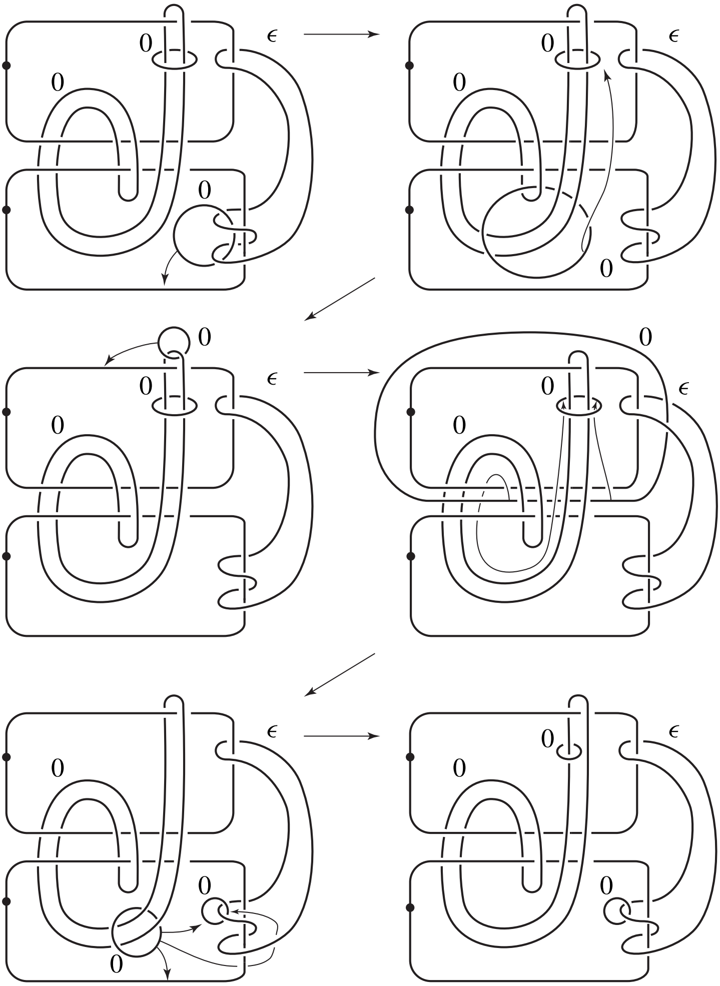



**Figure 13.** A diffeomorphism to the double of a contractible 4-manifold.

If the graph for the $n$-fusion is as in Figure 14. This is just a schematic picture for the fusion, and the edges stand for connecting 0-framed 2-handles coming from the bands of the ribbon disk. Actually, in the true picture, the edges should be drawn as some bands and might be linking to several dotted 1-handles. For our proof, we may omit these data because sliding the 0-framed 2-handle to dual 2-handles, we can ignore the linking.

We take the two based oriented meridians $m'$ and $l'$ in the positions in the figure. We suppose that the below 0-framed 2-handle in the first picture in Figure 9 is attached in the dashed circle in Figure 14 in our situation. From the 1-handle $k$ linking to $l'$ to the 1-handle $k'$ linking to $m'$, the 0-framed 2-handle can be moved by doing several handle slides and some isotopy. See Figure 15 for the handle

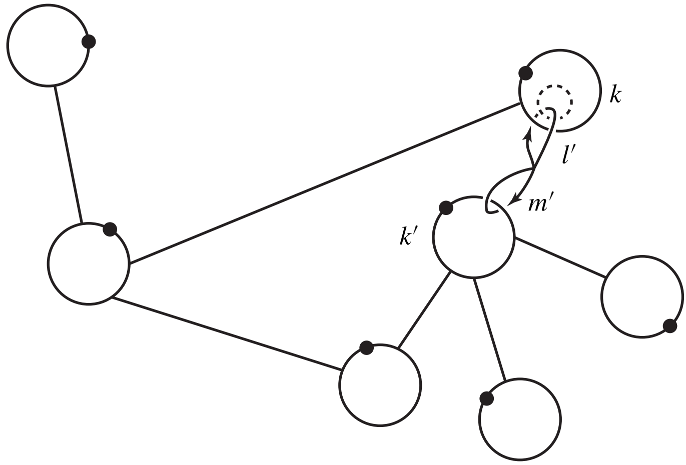



**Figure 14.** A graph for the fusion of a ribbon 2-knot.

moves. This also generalizes the moves from the first picture in Figure 9 to the second picture in Figure 10. Hence, we can freely move the 0-framed 2-handle from a dotted 1-handle to another dotted 1-handle.

By these handle slides, all 0-framed 2-handles corresponding to the dual bands can be moved in the meridians of all 2-handles. This means that $S^4(e, p/(p+1), \epsilon)$ is the double of a homology 4-ball $H$ without 3-handles. $\square$

As mentioned in Section 1 as well, it is unclear whether any homology 4-sphere obtained by this pochette surgery is simply connected or not.

If the 2-knot is $n$-fusion ribbon knot, the fundamental group of $S^4(e, p/(p+1), \epsilon)$ has the form

$$\langle x_1, \dots, x_{n+1} \mid w_1 x_{i_1} w_1^{-1} x_{j_1}^{-1}, \dots, w_n x_{i_n} w_n^{-1} x_{j_n}^{-1}, x_s^{-1}(x_r x_s^{-1})^p \rangle, \tag{3}$$

where for $k = 1, 2, \dots, n$, $w_k$ is a word in $x_1, \dots, x_{n+1}$, the set

$$\{\{i_k, j_k\} \mid k = 1, \dots, n\}$$

is the set of edges of the graph, and $r, s$ are some integers in $\{1, \dots, n\}$. Even if $H$ in the proof of Theorem 1.8 is contractible, that is, the fundamental group is trivial then it is unclear whether $S^4(e, p/(p+1), \epsilon)$ is diffeomorphic to $S^4$ or not.

*Proof of Theorem 1.9.* If the homology 4-ball $H$ in the proof above is contractible, then $S^4(e, g) = H \cup (-H)$ is a homotopy 4-sphere. Furthermore, if the presentation of $\pi_1$ coming from the handle decomposition is AC-trivial, then from the method mentioned right after the proof of Theorem 1.7, therefore, $S^4(e, g)$ is diffeomorphic to the standard $S^4$. $\square$

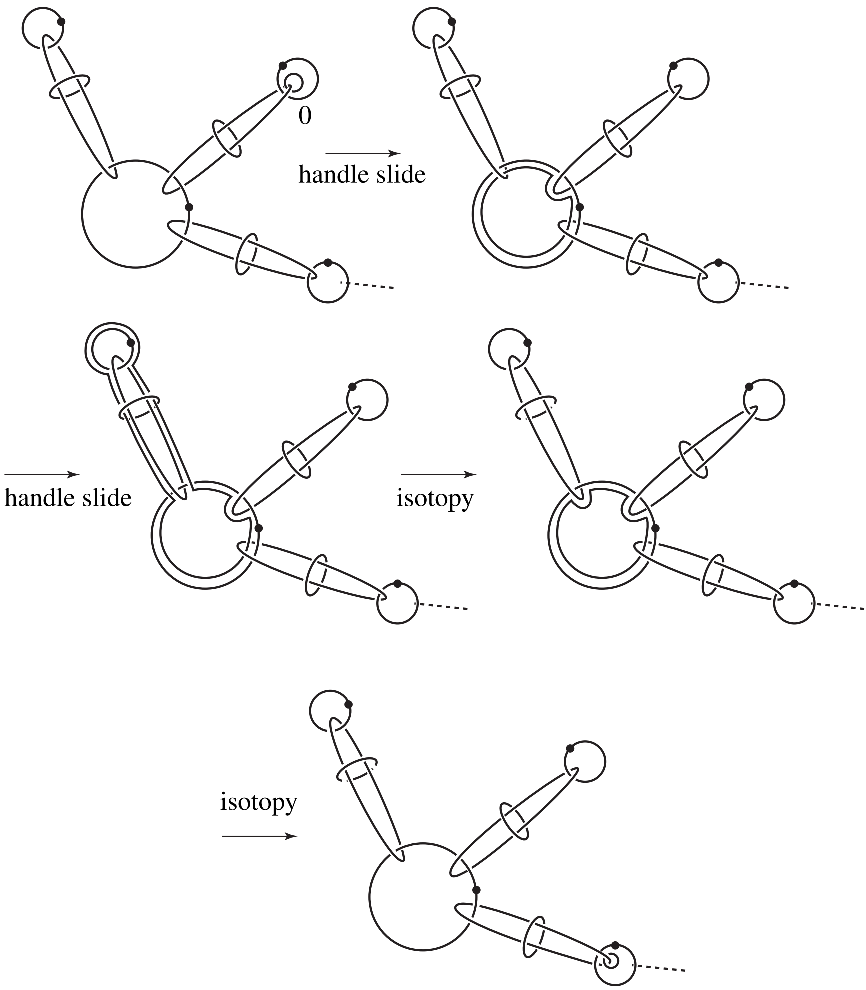



**Figure 15.** Deformations to move the 0-framed meridian in the position.

We give a sufficient condition that the presentation (3) is AC-trivial. Let $\{x_1, \ldots, x_{n+1}\}$ be a generator of the free group $F_{n+1}$. For any word $w$ of $x_1, \ldots, x_{n+1}$, we put $r_{2i-1} = wx_{2i}w^{-1}x_{2i+1}^{-1}$ for $2i-1 < n$, $r_{2i} = wx_{2i+2}w^{-1}x_{2i+1}^{-1}$ for $2i < n$ and

$$r_n = \begin{cases} wx_{n+1}w^{-1}x_1^{-1}, & n \text{ is odd,} \\ wx_1w^{-1}x_{n+1}^{-1}, & n \text{ is even.} \end{cases}$$

Then we consider the presentation

$$\langle x_1, \ldots, x_{n+1} \mid r_1, \ldots, r_n \rangle.$$

This presentation gives the fundamental group of the complement of a ribbon 2-knot of $n$-fusion.

**Lemma 4.5.** *Let $n$ be a positive integer. For any word $w$ of $x_1, \dots, x_{n+1}$, the relators $r_1, \dots, r_n$ are the same as above. For $r_{n+1} = x_1^{-1}(x_2 x_1^{-1})^p$, the presentation*

$$\langle x_1, \dots, x_{n+1} \mid r_1, \dots, r_n, r_{n+1} \rangle \tag{4}$$

*is the trivial group presentation with AC-trivial.*

*Proof.* We obtain

$$r_{2i-1}\, r_{2i}^{-1} = w x_{2i}\, x_{2i+2}^{-1} w^{-1} \sim x_{2i}\, x_{2i+2}^{-1} \quad \text{and} \quad r_{2i+1}^{-1}\, r_{2i} = x_{2i+3}^{-1} x_{2i+2},$$

$r_{n-1}^{-1}\, r_n = x_n x_1^{-1}$ if $n$ is odd, $r_{n-1}\, r_n^{-1} = w x_n\, x_1^{-1} w^{-1} \sim x_n\, x_1^{-1}$ if $n$ is even, where $\sim$ presents the relation between conjugate elements. Then we have

$$\begin{aligned}
&\langle x_1, \dots, x_{n+1} \mid r_1, r_2, r_3, \dots, r_n, r_{n+1} \rangle \\
&\qquad \cong \langle x_1, \dots, x_{n+1} \mid r_1 r_2^{-1}, r_2, r_3, \dots, r_n, r_{n+1} \rangle \\
&\qquad \cong \langle x_1, \dots, x_{n+1} \mid r_1 r_2^{-1}, r_2^{-1}\, r_3, r_3, \dots, r_n, r_{n+1} \rangle \\
&\qquad \cong \langle x_1, \dots, x_{n+1} \mid r_1 r_2^{-1}, r_2^{-1}\, r_3, r_3 r_4^{-1}, \dots, r_n, r_{n+1} \rangle \\
&\qquad \cong \begin{cases} \langle x_1, \dots, x_{n+1} \mid r_1 r_2^{-1}, r_2^{-1}\, r_3, r_3 r_4^{-1}, \dots, r_{n-1}^{-1}\, r_n, r_n, r_{n+1} \rangle, & n \text{ is odd,} \\ \langle x_1, \dots, x_{n+1} \mid r_1 r_2^{-1}, r_2^{-1}\, r_3, r_3 r_4^{-1}, \dots, r_{n-1}\, r_n^{-1}, r_n, r_{n+1} \rangle, & n \text{ is even} \end{cases} \\
&\qquad \cong \langle x_1, \dots, x_{n+1} \mid x_2 x_4^{-1}, x_3 x_5^{-1}, x_4 x_6^{-1}, \dots, x_{n-1}\, x_{n+1}^{-1}, x_n x_1^{-1}, r_n, r_{n+1} \rangle.
\end{aligned}$$

Replacing $x_i\, x_{i+2}^{-1}$ with $x_i'$ for $i = 2, \dots, n-1$ and $x_n\, x_1^{-1}$ with $x_n'$, we give

$$x_2 = \begin{cases} x_2' x_4' x_6' \cdots x_{n-1}' x_{n+1}, & n \text{ is odd,} \\ x_2' x_4' x_6' \cdots x_n' x_1, & n \text{ is even,} \end{cases}$$

$$r_n = \begin{cases} w' x_{n+1} (w')^{-1} x_1^{-1}, & n \text{ is odd,} \\ w' x_1 (w')^{-1} x_{n+1}^{-1}, & n \text{ is even,} \end{cases}$$

where $w'$ is a word of $x_1$, $x_i'$ and $x_{n+1}$ and we have

$$\begin{aligned}
&\langle x_1, \dots, x_{n+1} \mid r_1, r_2, r_3, r_4, \dots, r_n, r_{n+1} \rangle \\
&\qquad\qquad \cong \langle x_1, x_2', \dots, x_n', x_{n+1} \mid x_2', x_3', x_4', \dots, x_{n-1}', x_n', r_n, r_{n+1} \rangle \\
&\qquad\qquad \cong \begin{cases} \langle x_1, x_{n+1} \mid w' x_{n+1} (w')^{-1} x_1^{-1}, x_1^{-1} (x_{n+1} x_1^{-1})^p \rangle, & n \text{ is odd,} \\ \langle x_1, x_{n+1} \mid w' x_1 (w')^{-1} x_{n+1}^{-1}, x_1^{-1} \rangle, & n \text{ is even.} \end{cases}
\end{aligned}$$

By applying Lemma 4.4, we see that this presentation is AC-trivial. Therefore, we obtain the desired result above. □

## 5. Questions

In this section we raise several questions. We leave the following problem about Theorem 1.8.

**Question 5.1.** *Let $S$ be any ribbon 2-knot with $G(S) \not\cong \mathbb{Z}$. Does there exist a nontrivial cord $c$ in $E(S)$ such that any nontrivial surgery with respect to the embedding $e : P \hookrightarrow S^4$ with the cord $c$ and the core sphere $S$ yielding a homology 4-sphere gives the standard 4-sphere?*

Since pochette surgery is a generalization of Gluck surgery, the triviality of Gluck surgery on any ribbon 2-knot might also hold in the pochette surgery situation.

**Question 5.2.** *Let $S$ be any ribbon 2-knot with $G(S) \not\cong \mathbb{Z}$. Suppose that $e : P \hookrightarrow S^4$ is any embedding with $S_e = S$. Does any pochette surgery $S^4(e, g)$ yielding a homology 4-sphere for some gluing map $g$ give the 4-sphere?*

It might be possible that we answer the following question affirmatively.

**Question 5.3.** *Let $S$ be any ribbon 2-knot in $S^4$ with $G(S) \not\cong \mathbb{Z}$. If a pochette surgery with the core sphere $S$ yields a homology 4-sphere, is the pochette surgery the standard 4-sphere?*

Can the diffeomorphisms in the previous section be generalized to cases of any nontrivial core sphere?

**Question 5.4.** *Let $S$ be any 2-knot with $G(S) \not\cong \mathbb{Z}$. Then, does there exist a nontrivial cord in $E(S)$ such that any pochette surgery for a pochette embedding $e : P \hookrightarrow S^4$ with the core sphere $S$ is $S^4$ or $\mathrm{Gl}(S)$?*

Can we construct a homotopy 4-sphere other than $\mathrm{Gl}(S)$ by pochette surgery? Furthermore, we raise two questions in more generalized settings.

**Question 5.5.** *Can a pochette surgery of $S^4$ construct an exotic $S^4$?*

More generally, we ask the following question.

**Question 5.6.** *Can a pochette surgery of an oriented 4-manifold $M$ construct an exotic structure on $M$?*

Pochette surgery can be generalized to a surgery on a *generalized pochette* $P_{a,b} = \natural^a S^1 \times D^3 \natural^b D^2 \times S^2$. Such a surgery is called an *outer surgery* and it is studied by Nakamura in [10]. Would studying outer surgery lead to the construction of interesting 4-manifolds? Investigating outer surgery is a potential avenue for future research about exotic 4-manifolds.

## Acknowledgements

Hisaaki Endo, an adviser of Suzuki, has constantly encouraged and advised our work and we thank him. The authors appreciate a referee for giving us helpful comments and suggestions.

## References


[1] S. Akbulut, "Constructing a fake 4-manifold by Gluck construction to a standard 4-manifold", *Topology* **27**:2 (1988), 239–243. MR Zbl

[2] S. Akbulut, "Cork twisting Schoenflies problem", *J. Gökova Geom. Topol. GGT* **8** (2014), 35–43. MR Zbl

[3] G. E. Bredon, *Topology and geometry*, Graduate Texts in Mathematics **139**, Springer, New York, 1997. MR Zbl

[4] M. H. Freedman, "The topology of four-dimensional manifolds", *J. Differential Geometry* **17**:3 (1982), 357–453. MR Zbl

[5] R. E. Gompf and A. I. Stipsicz, *4-manifolds and Kirby calculus*, Graduate Studies in Mathematics **20**, American Mathematical Society, Providence, RI, 1999. MR Zbl

[6] C. M. Gordon, "Knots in the 4-sphere", *Comment. Math. Helv.* **51**:4 (1976), 585–596. MR Zbl

[7] Z. Iwase and Y. Matsumoto, "4-dimensional surgery on a "pochette"", pp. 161–166 in *Proceedings of the East Asian School of knots, links and related topics*, 2004.

[8] P. M. Melvin, *Blowing up and down in 4-manifolds*, Ph.D. thesis, University of California, Berkeley, 1977, available at https://www.proquest.com/docview/302865311. MR

[9] Y. Murase, *Pochette surgery and Kirby diagrams*, master's thesis, Tokyo Institute of Technology, 2015. In Japanese.

[10] S. Nakamura, *On a generalization of Iwase–Matsumoto's pochette surgery and its applications*, master's thesis, The University of Tokyo, 2018. In Japanese.

[11] P. Naylor and H. R. Schwartz, "Gluck twisting roll spun knots", *Algebr. Geom. Topol.* **22**:2 (2022), 973–990. MR Zbl

[12] T. Okawa, *On pochette surgery on the* 4-*sphere*, master's thesis, Tokyo Institute of Technology, 2020. In Japanese.

[13] P. S. Pao, "The topological structure of 4-manifolds with effective torus actions, I", *Trans. Amer. Math. Soc.* **227** (1977), 279–317. MR Zbl

[14] T. Suzuki, "Constructions of homotopy 4-spheres by pochette surgery", preprint, 2022. arXiv 2205.05239

[15] J. H. C. Whitehead, "On simply connected, 4-dimensional polyhedra", *Comment. Math. Helv.* **22** (1949), 48–92. MR Zbl


Received September 11, 2022. Revised March 27, 2023.

TATSUMASA SUZUKI
DEPARTMENT OF MATHEMATICS
TOKYO INSTITUTE OF TECHNOLOGY
TOKYO
JAPAN
suzuki.t.do@m.titech.ac.jp

MOTOO TANGE
INSTITUTE OF MATHEMATICS
UNIVERSITY OF TSUKUBA
IBARAKI
JAPAN
tange@math.tsukuba.ac.jp

## Guidelines for Authors

Authors may submit articles at msp.org/pjm/about/journal/submissions.html and choose an editor at that time. Exceptionally, a paper may be submitted in hard copy to one of the editors; authors should keep a copy.

By submitting a manuscript you assert that it is original and is not under consideration for publication elsewhere. Instructions on manuscript preparation are provided below. For further information, visit the web address above or write to pacific@math.berkeley.edu or to Pacific Journal of Mathematics, University of California, Los Angeles, CA 90095–1555. Correspondence by email is requested for convenience and speed.

Manuscripts must be in English, French or German. A brief abstract of about 150 words or less in English must be included. The abstract should be self-contained and not make any reference to the bibliography. Also required are keywords and subject classification for the article, and, for each author, postal address, affiliation (if appropriate) and email address if available. A home-page URL is optional.

Authors are encouraged to use LaTeX, but papers in other varieties of TeX, and exceptionally in other formats, are acceptable. At submission time only a PDF file is required; follow the instructions at the web address above. Carefully preserve all relevant files, such as LaTeX sources and individual files for each figure; you will be asked to submit them upon acceptance of the paper.

Bibliographical references should be listed alphabetically at the end of the paper. All references in the bibliography should be cited in the text. Use of BibTeX is preferred but not required. Any bibliographical citation style may be used but tags will be converted to the house format (see a current issue for examples).

Figures, whether prepared electronically or hand-drawn, must be of publication quality. Figures prepared electronically should be submitted in Encapsulated PostScript (EPS) or in a form that can be converted to EPS, such as GnuPlot, Maple or Mathematica. Many drawing tools such as Adobe Illustrator and Aldus FreeHand can produce EPS output. Figures containing bitmaps should be generated at the highest possible resolution. If there is doubt whether a particular figure is in an acceptable format, the authors should check with production by sending an email to pacific@math.berkeley.edu.

Each figure should be captioned and numbered, so that it can float. Small figures occupying no more than three lines of vertical space can be kept in the text (“the curve looks like this:”). It is acceptable to submit a manuscript will all figures at the end, if their placement is specified in the text by means of comments such as “Place Figure 1 here”. The same considerations apply to tables, which should be used sparingly.

Forced line breaks or page breaks should not be inserted in the document. There is no point in your trying to optimize line and page breaks in the original manuscript. The manuscript will be reformatted to use the journal’s preferred fonts and layout.

Page proofs will be made available to authors (or to the designated corresponding author) at a website in PDF format. Failure to acknowledge the receipt of proofs or to return corrections within the requested deadline may cause publication to be postponed.

PACIFIC JOURNAL OF MATHEMATICS

Volume 324 No. 2 June 2023