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ESTIMATE FOR THE FIRST FOURTH STEKLOV EIGENVALUE OF A MINIMAL HYPERSURFACE WITH FREE BOUNDARY

RONDINELLE BATISTA, BARNABÉ LIMA, PAULO SOUSA AND BRUNO VIEIRA

We dedicate this paper to João Xavier da Cruz Neto on the occasion of his sixtieth birthday.

We explore the fourth-order Steklov problem of a compact embedded hypersurface Σ^n with free boundary in a $(n+1)$-dimensional compact manifold M^{n+1} which has nonnegative Ricci curvature and strictly convex boundary. If Σ is minimal we establish a lower bound for the first eigenvalue of this problem. When $M = B^{n+1}$ is the unit ball in $\mathbb{R}^{n+1}$, if Σ has constant mean curvature H^Σ we prove that the first eigenvalue satisfies $\sigma_1 \le n + |H^\Sigma|$. In the minimal case ($H^\Sigma = 0$), we prove that $\sigma_1 = n$.

1. Introduction

Let Σ^n be an n-dimensional compact Riemannian manifold with nonempty boundary $\partial\Sigma \neq \varnothing$. Consider the fourth-order Steklov eigenvalue problem

$$\begin{cases} \Delta^2\xi = 0 & \text{in } \Sigma, \\ \xi = 0 & \text{on } \partial\Sigma, \\ \Delta\xi = \sigma \frac{\partial\xi}{\partial\nu_\Sigma} & \text{on } \partial\Sigma, \end{cases} \tag{1}$$

where σ is a real number, Δ is the Laplacian operator on Σ and ν_Σ denotes the outward unit normal on $\partial\Sigma$. The first nonzero eigenvalue of the above problem will be denoted by $\sigma_1 = \sigma(\Sigma)$. The first eigenvalue of (1) has the following variational characterization:

$$\sigma_1 = \inf_{w_{|\partial\Sigma}=0} \frac{\int_\Sigma (\Delta w)^2}{\int_{\partial\Sigma}\left(\frac{\partial w}{\partial\nu_\Sigma}\right)^2}. \tag{2}$$

Wang and Xia [2009] proved that if Σ has nonnegative Ricci curvature and the mean curvature of $\partial\Sigma$ is bounded below by a positive constant c then $\sigma_1 \ge c \cdot n$. Furthermore, equality occurs if and only if Σ is isometric to an n-dimensional Euclidean ball of radius $\frac{1}{c}$.

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Since their first appearance in [Stekloff 1902], elliptic problems with parameters in the boundary conditions are called Steklov problems. Kuttler [1972] and Payne [1970] studied the isoperimetric properties of the first eigenvalue σ_1 of the fourth-order Steklov problem (1). Moreover, as already noticed in [Kuttler 1972; 1979; Kuttler and Sigillito 1985], σ_1 is the sharp constant for L^2 a priori estimates for solutions of the (second-order) Laplace equation under nonhomogeneous Dirichlet boundary conditions. In [Ferrero et al. 2005] the authors studied the spectrum of the biharmonic Steklov problem (1) and obtained a characterization of it, and presented a physical interpretation of σ_1. For comprehensive references on such Steklov problems, see [Berchio et al. 2006; Bucur et al. 2009; Wang and Xia 2009].

It should be pointed out that the problem

$$\begin{cases} \Delta^2\xi = 0 & \text{in } \Sigma, \\ \xi = 0 & \text{on } \partial\Sigma, \\ \frac{\partial^2\xi}{\partial\nu_\Sigma^2} = \lambda\frac{\partial\xi}{\partial\nu_\Sigma} & \text{on } \partial\Sigma, \end{cases} \tag{3}$$

is a natural Steklov problem and one can check that when the mean curvature of $\partial\Sigma$ is constant, it is equivalent to (1).

Let M be a compact Riemannian manifold with nonempty boundary ∂M and $\Sigma \subset M$ a compact hypersurface (with boundary $\partial\Sigma$) properly embedded into M, that is, $\Sigma \cap \partial M = \partial\Sigma$. We say that Σ is a minimal hypersurface with free boundary if Σ is a minimal hypersurface and Σ meets ∂M orthogonally along $\partial\Sigma$. In this setting, Fraser and Li [2014] obtained a lower bound for the first eigenvalue of the second-order Steklov problem.

If $M = B^n$ is the unit ball in $\mathbb{R}^n$, it is known [Fraser and Schoen 2013] that the coordinate functions are eigenfunctions of the second-order Steklov problem with eigenvalue 1. Taking that into consideration, Fraser and Li [2014] conjectured that the first eigenvalue of the second-order Steklov problem of a compact properly embedded minimal hypersurface in B^n is 1 and proved that this is limited from below by $\frac{1}{2}$.

On the one hand, we did not find in the literature an extrinsic approach to the fourth-order Steklov eigenvalue problem. Motivated by the work of Fraser and Li, in this paper we consider the fourth-order Steklov problem of a compact properly embedded minimal hypersurface Σ with free boundary in a compact manifold M.

On the other hand, Ferrero, Gazzola and Weth [Ferrero et al. 2005] explored the fourth-order Steklov eigenvalue problem in a bounded domain Ω of $\mathbb{R}^n$ and proved that the first eigenvalue of this problem is equal to n when $\Omega = B^n$. It is known that the unit ball B^n is a minimal hypersurface with free boundary in B^{n+1}. In this setting, we have established an upper estimate for the first eigenvalue of the fourth-order Steklov problem of a compact properly embedded CMC hypersurface in B^{n+1} with free boundary on ∂B^{n+1}:

Proposition 1. *Let Σ^n be a compact properly embedded hypersurface in the unit ball B^{n+1}, with free boundary on $\partial B^{n+1} = \mathbb{S}^n$. Assume that Σ has constant normalized mean curvature H^Σ. Then*

$$\sigma_1 \leq n + |H^\Sigma|.$$

It follows from Proposition 1 that if Σ is minimal ($H^\Sigma = 0$), then $\sigma_1 \leq n$. This, together with the result of Ferrero, Gazzola and Weth [Ferrero et al. 2005], naturally led us to formulate and prove the main result of this paper:

Theorem 2. *Let Σ^n be a compact properly embedded minimal hypersurface in the unit ball B^{n+1}, with free boundary on $\partial B^{n+1} = \mathbb{S}^n$. Then the first eigenvalue of the fourth-order Steklov problem of Σ is equal to n.*

Wang and Xia [2009] proved that any compact connected Riemannian manifold Σ with boundary $\partial\Sigma$ satisfies

$$|\Sigma|\sigma_1 \leq |\partial\Sigma|, \tag{4}$$

where $|\partial\Sigma|$ and $|\Sigma|$ denote the area of $\partial\Sigma$ and the volume of Σ, respectively. If in addition the Ricci curvature of Σ is nonnegative and the equality holds, then Σ is isometric to an n-dimensional Euclidean ball. In our context, the equality always holds even for codimension greater than 1 (see Proposition 2.4 in [Li 2020]), i.e.,

$$k|\Sigma| = |\partial\Sigma|$$

for every k-dimensional immersed free boundary minimal submanifold Σ^k in the unit ball B^{n+1}. As a consequence of this equality and from (4) we get that $\sigma_1 \leq k$ for free boundary minimal submanifolds $\Sigma^k \subset B^{n+1}$.

Taking that into consideration, it is natural to consider the following question.

Problem 3. *Under what additional assumption is it possible to ensure that a compact properly embedded minimal hypersurface in the unit ball B^{n+1}, with free boundary on $\partial B^{n+1} = \mathbb{S}^n$, such that $\sigma_1 = n$ is the unit ball B^n?*

In our next result, we prove a lower estimate for σ_1 when Σ^n is a compact properly embedded minimal hypersurface with free boundary in a compact manifold which has nonnegative Ricci curvature and strictly convex boundary. More precisely, we prove the following theorem.

Theorem 4. *Let M^{n+1} be an $(n+1)$-dimensional compact orientable Riemannian manifold with nonnegative Ricci curvature and nonempty boundary ∂M. Assume the second fundamental form of ∂M satisfies $A^{\partial M}(v, v) \geq k > 0$, for any unit vector v tangent to ∂M.*

Let Σ^n be a properly embedded minimal hypersurface in M with free boundary on ∂M. Assume $\partial\Sigma$ has constant mean curvature $H^{\partial\Sigma}$. If

(i) Σ *is orientable, or*

(ii) $\pi_1(M)$ *is finite,*

then we have the eigenvalue estimate $\sigma_1 \geq H^{\partial\Sigma} + \frac{k}{2}$, *where* σ_1 *is the first eigenvalue of the fourth-order Steklov problem on* Σ.

This estimate for σ_1 is analogous to the estimates of Fraser and Li [2014] for the first nonzero Steklov eigenvalue of the Dirichlet-to-Neumann map on Σ.

Remark 5. If $M = B^{n+1}$ is the unit ball in $\mathbb{R}^{n+1}$ and $\Sigma = B^n \subset B^{n+1}$ is the unit ball in $\mathbb{R}^n$ ("equatorial disk"), then $H^{\partial\Sigma} = n-1$ and $k = 1$, and we get that $\sigma_1 = H^{\partial\Sigma} + k$. For this reason, we believe that $\sigma_1 \geq H^{\partial\Sigma} + k$ is the sharp estimate. Consequently, the hypothesis in Theorem 4 that $\partial\Sigma$ has constant mean curvature becomes natural to assume.

Combining the inequality (4) with our Theorem 4 we deduce the following corollary.

Corollary 6. *Let* M^{n+1} *be an* $(n+1)$*-dimensional compact orientable Riemannian manifold with nonnegative Ricci curvature and nonempty boundary* ∂M. *Assume the second fundamental form of* ∂M *satisfies* $A^{\partial M}(v, v) \geq k > 0$, *for any unit vector* v *tangent to* ∂M.

Let Σ *be a properly embedded minimal hypersurface in* M *with free boundary on* ∂M. *Assume* $\partial\Sigma$ *has constant mean curvature* $H^{\partial\Sigma}$. *Then*

$$|\partial\Sigma| \geq \left(H^{\partial\Sigma} + \frac{k}{2}\right)|\Sigma|.$$

2. Preliminaries

In this section we will collect some basic results that are essential to deduce Theorem 4. Let M^{n+1} be a $(n+1)$-dimensional compact Riemannian manifold with nonempty boundary ∂M. Denote by $\langle \cdot, \cdot \rangle$ the metric on M and D the Riemannian connection on M. We define the second fundamental form of the boundary ∂M with respect to the outward unit normal μ by $A^{\partial M}(u, v) = \langle D_u \mu, v\rangle$, where u, v are tangent to ∂M. The mean curvature $H^{\partial M}$ of ∂M is then defined as the trace of $A^{\partial M}$, i.e.,

$$H^{\partial M} = \sum_{j=1}^{n} A^{\partial M}(e_j, e_j),$$

where $e_1, \ldots, e_n$ is any orthonormal basis for $T\partial M$.

The following, known as Reilly's formula, was settled in [Fraser and Li 2014, Lemma 2.6]; see also [Choi and Wang 1983].

Proposition 7 [Fraser and Li 2014]. *Let* Ω *be a compact* $(n+1)$*-manifold with piecewise smooth boundary* $\partial\Omega = \bigcup \sum_{i=1}^{k} \Sigma_i$. *Suppose* f *is a continuous function* *on* Ω *where* $f \in C^\infty(\Omega \setminus S)$, $S = \bigcup \sum_{i=1}^k \partial \Sigma_i$, *and there exists some* $C > 0$ *such that* $\|f\|_{C^3(\Omega')} \leq C$ *for all* $\Omega' \subset \Omega \setminus S$. *Then, Reilly's formula holds*:

$$
(5) \quad 0 = \int_\Omega \mathrm{Ric}^\Omega(Df, Df) - (\Delta_\Omega f)^2 + \|\mathrm{Hess}_\Omega f\|^2 \\
+ \sum_{i=1}^k \int_{\Sigma_i} \Big[\Big(\Delta_{\Sigma_i} f + H^{\Sigma_i} \frac{\partial f}{\partial \eta_i} \Big) \frac{\partial f}{\partial \eta_i} - \Big\langle \nabla^{\Sigma_i} f, \nabla^{\Sigma_i} \frac{\partial f}{\partial \eta_i} \Big\rangle + h^{\Sigma_i}(\nabla^{\Sigma_i} f, \nabla^{\Sigma_i} f) \Big].
$$

Here, Ric^Ω *is the Ricci tensor of* Ω; Δ_Ω, Hess_Ω *and* ∇_Ω *are the Laplacian*, *Hessian and gradient operators on* Ω, *respectively*; Δ_{Σ_i} *and* ∇^{Σ_i} *are the Laplacian and gradient operators on each* Σ_i, *respectively*; η_i *is the outward unit normal of* Σ_i; H^{Σ_i} *and* h^{Σ_i} *are the mean curvature and second fundamental form of* Σ_i *in* Ω *with respect to the outward unit normal*, *respectively.*

To prove our main result we need a few considerations. Let $\varphi : \Sigma \to M$ be a properly embedded minimal hypersurface with free boundary in a compact orientable manifold M. Assume that ∂M is strictly convex and M has nonnegative Ricci curvature. Under these assumptions, ∂M is connected [Fraser and Li 2014, Proposition 2.8], and any properly embedded minimal hypersurface in M with free boundary is connected [Fraser and Li 2014, Lemma 2.5]. Furthermore, if both Σ and M are orientable then $M \setminus \varphi(\Sigma)$ consists of two components Ω_1 and Ω_2 (see [Fraser and Li 2014, Corollary 2.10]). Take $\Omega = \Omega_1$. Let $\partial\Omega = \Sigma \cup \Gamma$ where $\Gamma \subset \partial M$. Thus, $\partial \Sigma = \partial \Gamma$. Note that Γ is not necessarily connected, but each component of Γ must intersect Σ along some component of $\partial\Sigma$. Otherwise, ∂M would have more than one component, a contradiction.

Remark 8. From a result due to M. C. Li [2011, Theorem 1.1.8], any compact Riemannian 3-manifold M with nonempty boundary ∂M admits a nontrivial compact embedded minimal surface Σ with free boundary. Some examples of free boundary submanifolds in the unit ball are given in [Fraser and Schoen 2013].

3. Proof of the results

3.1. *Proof of Proposition 1.*

Proof. Let $\xi : B^{n+1} \to \mathbb{R}$ be defined by $\xi(x) = 1 - \|x\|^2$. As can be easily seen

$$
\xi_{|\partial\Sigma} = 0 \quad \text{and} \quad \Delta_\Sigma \xi(x) = -2\big(n + H^\Sigma \langle x, N(x)\rangle\big),
$$

where N is a unit vector field normal to Σ^n in B^{n+1}. Thus,

$$
(\Delta_\Sigma \xi)^2 \leq 4n^2 \Big(1 + \frac{|H^\Sigma|}{n} \Big)^2.
$$

On the other hand, if ν_Σ is the outward unit conormal along $\partial\Sigma$ and x_i are the coordinate functions, the condition

$$\frac{\partial x_i}{\partial \nu_\Sigma} = x_i$$

is equivalent to $\nu_\Sigma = x$, which is equivalent to the condition that Σ meets ∂B^n orthogonally. Then, Σ meets ∂B^n orthogonally if and only if

$$\frac{\partial \xi}{\partial \nu_\Sigma} = -2.$$

Now, using the variational characterization of σ_1 we get

$$\sigma_1 \cdot |\partial\Sigma| \le n^2\left(1 + \frac{|H^\Sigma|}{n}\right)^2 \cdot |\Sigma|,$$

and applying inequality (4) we conclude that

$$\sigma_1 \le n + |H^\Sigma|. \qquad \square$$

3.2. *Proof of Theorem 2.*

Proof. Again let us consider the function $\xi : B^{n+1} \to \mathbb{R}$ defined by $\xi(x) = 1 - \|x\|^2$. Since Σ is minimal, it follows from the proof of Proposition 1 that $\Delta_\Sigma \xi = -2n$. Thus

$$\begin{cases} \Delta_\Sigma^2 \xi = 0 & \text{in } \Sigma, \\ \xi = 0 & \text{on } \partial\Sigma, \\ \Delta_\Sigma \xi = n \frac{\partial \xi}{\partial \nu_\Sigma} & \text{on } \partial\Sigma, \end{cases}$$

which implies that n is an eigenvalue. Now we will show that $\sigma_1 = n$.

It is known (see Theorem 1 in [Berchio et al. 2006]) that the infimum in (2) is achieved and that, up to a multiplicative constant, the minimizer is unique, smooth up to the boundary, positive in Σ, and the normal derivative relative to the outward unit normal is negative on $\partial\Sigma$. Arguing as in the proof of Lemma 2.2 in [Ferrero et al. 2005] we conclude that $\sigma_1 = n$. $\square$

3.3. *Proof of Theorem 4.*

Proof. Firstly suppose that Σ is orientable. Since M is orientable we have Σ is connected and $M \setminus \varphi(\Sigma)$ consists of two components Ω_1 and Ω_2 (see [Fraser and Li 2014, Corollaries 2.5 and 2.10]). Let $\Omega = \Omega_1$ and $\partial\Omega = \Sigma \cup \Gamma$, where $\Gamma \subset \partial M$, so that $\partial\Sigma = \partial\Gamma$.

Let $\xi \in C^\infty(\Sigma)$ be an eigenfunction corresponding to the first eigenvalue σ_1 of the fourth-order Steklov problem, that is,

$$\begin{cases} \Delta_\Sigma^2 \xi = 0 & \text{in } \Sigma, \\ \xi = 0 & \text{on } \partial\Sigma, \\ \Delta_\Sigma \xi = \sigma_1 \frac{\partial \xi}{\partial \nu_\Sigma} & \text{on } \partial\Sigma, \end{cases} \tag{6}$$

where ν_Σ is the outward conormal vector of $\partial\Sigma$ with respect to Σ. Next, we consider the Dirichlet–Neumann boundary value problem on the compact $(n+1)$-manifold Ω with piecewise smooth boundary $\partial\Omega = \Sigma \cup \Gamma$

$$\begin{cases} \Delta_\Omega f = 0 & \text{in } \Omega, \\ f = \Delta_\Sigma \xi & \text{on } \Sigma, \\ \frac{\partial f}{\partial \eta_\Gamma} = (\sigma_1 - H^{\partial\Sigma}) f & \text{on } \Gamma. \end{cases} \tag{7}$$

Analyzing the relationship between the first eigenvalues of problems (1) and (3) it is possible to conclude that $\sigma_1 > H^{\partial\Sigma}$. To ensure the existence of a solution for problem (7), we will consider the homogeneous problem

$$\begin{cases} \Delta_\Omega f = 0 & \text{in } \Omega, \\ f = 0 & \text{on } \Sigma, \\ \frac{\partial f}{\partial \eta_\Gamma} = \mu f & \text{on } \Gamma. \end{cases} \tag{8}$$

This mixed Steklov–Dirichlet problem has a discrete spectrum $\{\mu_i\}$ (see [Guo and Xia 2019, Section 2]) where

$$0 < \mu_1 \le \mu_2 \le \cdots \to +\infty.$$

Next, we will establish a lower bound for μ_1. Consider f_1 an eigenfunction associated with μ_1 and assume without loss of generality that $\int_\Sigma h^\Sigma(\nabla^\Sigma f_1, \nabla^\Sigma f_1) \ge 0$ (otherwise, we choose $\Omega = \Omega_2$ instead). We get by Reilly's formula (5) applied to f_1

$$0 \ge nk \int_\Gamma \left(\frac{\partial f_1}{\partial \eta_\Gamma}\right)^2 + \int_\Gamma \Delta_\Gamma f \frac{\partial f_1}{\partial \eta_\Gamma} - \int_\Gamma \left\langle \nabla^\Gamma f_1, \nabla^\Gamma \frac{\partial f_1}{\partial \eta_\Gamma} \right\rangle + k \int_\Gamma |\nabla^\Gamma f_1|^2,$$

where η_Σ and η_Γ are the outward unit normals of Σ and Γ, respectively, with respect to Ω. Integrating by parts we get

$$\int_\Gamma \Delta_\Gamma f_1 \frac{\partial f_1}{\partial \eta_\Gamma} = -\int_\Gamma \left\langle \nabla^\Gamma f_1, \nabla^\Gamma \frac{\partial f_1}{\partial \eta_\Gamma} \right\rangle + \int_{\partial\Gamma} \frac{\partial f_1}{\partial \nu_\Gamma} \frac{\partial f_1}{\partial \eta_\Gamma},$$

where ν_Σ and ν_Γ are the outward conormal vectors of $\partial\Sigma = \partial\Gamma$ with respect to Σ and Γ, respectively. Since Σ meets Γ orthogonally along $\partial\Sigma = \partial\Gamma$, we have $\nu_\Sigma = \eta_\Gamma$ and $\eta_\Sigma = \nu_\Gamma$ along the common boundary $\partial\Sigma$. Thereby

$$0 = \int_{\partial\Sigma} \frac{\partial f_1}{\partial \nu_\Sigma} \frac{\partial f_1}{\partial \eta_\Sigma} = \int_{\partial\Sigma} \frac{\partial f_1}{\partial \nu_\Sigma} \frac{\partial f_1}{\partial \nu_\Gamma} = \int_{\partial\Gamma} \frac{\partial f_1}{\partial \nu_\Gamma} \frac{\partial f_1}{\partial \eta_\Gamma},$$

which implies

$$2 \int_\Gamma \left\langle \nabla^\Gamma f_1, \nabla^\Gamma \frac{\partial f_1}{\partial \eta_\Gamma} \right\rangle \ge nk \int_\Gamma \left(\frac{\partial f_1}{\partial \eta_\Gamma}\right)^2 + k \int_\Gamma |\nabla^\Gamma f_1|^2.$$

We conclude that $\mu_1 \ge \frac{k}{2}$. Having proved this fact, we will make an analysis divided into two cases. Namely, if there is $i \in \mathbb{N}$ such that $\sigma_1 - H^{\partial\Sigma} = \mu_i \ge \mu_1$ we get $\sigma_1 \geq H^{\partial\Sigma} + \frac{k}{2}$. Otherwise, $\sigma_1 - H^{\partial\Sigma} \neq \mu_i$ for all $i \in \mathbb{N}$. So, the homogeneous problem (8) has only the trivial solution, and it follows from standard elliptic PDE theory, more specifically from the Fredholm alternative, that the problem (7) has a unique solution f. Note that $\Delta_\Sigma(f|_\Sigma) = \Delta_\Sigma^2 \xi = 0$ in Σ, and assuming without loss of generality that $\int_\Sigma h^\Sigma(\nabla^\Sigma f, \nabla^\Sigma f) \geq 0$, by substituting this function f in formula (5) we obtain

$$\begin{aligned} 0 \geq -\int_\Sigma \left\langle \nabla^\Sigma f, \nabla^\Sigma \frac{\partial f}{\partial \eta_\Sigma} \right\rangle + nk \int_\Gamma \left(\frac{\partial f}{\partial \eta_\Gamma} \right)^2 + \int_\Gamma \Delta_\Gamma f \frac{\partial f}{\partial \eta_\Gamma} \\ - \int_\Gamma \left\langle \nabla^\Gamma f, \nabla^\Gamma \frac{\partial f}{\partial \eta_\Gamma} \right\rangle + k \int_\Gamma |\nabla^\Gamma f|^2. \end{aligned}$$

Now, using that

$$\int_\Sigma \left\langle \nabla^\Sigma f, \nabla^\Sigma \frac{\partial f}{\partial \eta_\Sigma} \right\rangle = -\int_\Sigma \frac{\partial f}{\partial \eta_\Sigma} \Delta_\Sigma f + \int_{\partial\Sigma} \frac{\partial f}{\partial \nu_\Sigma} \frac{\partial f}{\partial \eta_\Sigma} = \int_{\partial\Sigma} \frac{\partial f}{\partial \nu_\Sigma} \frac{\partial f}{\partial \eta_\Sigma}$$

and

$$\int_\Gamma \Delta_\Gamma f \frac{\partial f}{\partial \eta_\Gamma} = -\int_\Gamma \left\langle \nabla^\Gamma f, \nabla^\Gamma \frac{\partial f}{\partial \eta_\Gamma} \right\rangle + \int_{\partial\Gamma} \frac{\partial f}{\partial \nu_\Gamma} \frac{\partial f}{\partial \eta_\Gamma},$$

we have

$$\begin{aligned} 0 \geq -\int_{\partial\Sigma} \frac{\partial f}{\partial \nu_\Sigma} \frac{\partial f}{\partial \eta_\Sigma} + \int_{\partial\Gamma} \frac{\partial f}{\partial \nu_\Gamma} \frac{\partial f}{\partial \eta_\Gamma} - 2 \int_\Gamma \left\langle \nabla^\Gamma f, \nabla^\Gamma \frac{\partial f}{\partial \eta_\Gamma} \right\rangle \\ + nk \int_\Gamma \left(\frac{\partial f}{\partial \eta_\Gamma} \right)^2 + k \int_\Gamma |\nabla^\Gamma f|^2. \end{aligned}$$

As we saw previously,

$$\int_{\partial\Sigma} \frac{\partial f}{\partial \nu_\Sigma} \frac{\partial f}{\partial \eta_\Sigma} = \int_{\partial\Sigma} \frac{\partial f}{\partial \nu_\Sigma} \frac{\partial f}{\partial \nu_\Gamma} = \int_{\partial\Gamma} \frac{\partial f}{\partial \nu_\Gamma} \frac{\partial f}{\partial \eta_\Gamma}.$$

Therefore,

$$2 \int_\Gamma \left\langle \nabla^\Gamma f, \nabla^\Gamma \frac{\partial f}{\partial \eta_\Gamma} \right\rangle \geq nk \int_\Gamma \left(\frac{\partial f}{\partial \eta_\Gamma} \right)^2 + k \int_\Gamma |\nabla^\Gamma f|^2.$$

Now, using the last equality in (7) we get

$$2(\sigma_1 - H^{\partial\Sigma}) \geq k \implies \sigma_1 \geq H^{\partial\Sigma} + \frac{k}{2}.$$

This proves the theorem when Σ is orientable. In the case when Σ nonorientable and $\pi_1(M)$ finite, we can argue as in [Fraser and Li 2014, Theorem 3.1]. □

3.4. *Proof of Corollary 6.*

Proof. The proof of Corollary 6 follows directly from (4) and Theorem 4. □

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