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THE MAXIMAL SYSTOLE OF HYPERBOLIC SURFACES WITH MAXIMAL S^3-EXTENDABLE ABELIAN SYMMETRY

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We study the maximal systole of hyperbolic surfaces with certain symmetries. We give the formula for the maximal systole of the surfaces that admit the largest S^3-extendable abelian group symmetry. The result is obtained by parametrizing such surfaces and enumerating all possible systoles.

1. Introduction

The systole is an important topic in the study of hyperbolic surfaces. The systole has applications in various areas on surfaces, e.g, the Mumford's classical compactness criterion [Mumford 1971] and the Weil–Peterson metric [Wolpert 2017; Wu 2019] in Teichmüller theory, and the spectrum of the Laplacian [Ballmann et al. 2016; 2018; Mondal 2014] and the optimal systolic ratio [Chen and Li 2015; Croke and Katz 2003; Gromov 1983] in differential geometry. For a survey on the study of the systole, see Parlier [2014].

We use the term "systole" to refer to either the minimal length of a closed geodesic on a hyperbolic surface, or a closed geodesic realizing this length, by abuse of notation. The systole can also be regarded as a real-valued function on the moduli space $\mathcal{M}_g$ of all closed hyperbolic surfaces of genus g or the Teichmüller space $\mathcal{T}_g$. The maximal value of the systole function on $\mathcal{M}_g$ is called the maximal systole in genus g. The maximal systole can be realized by Mumford's compactness criterion. It is quite difficult to compute the exact value of the maximal systole. The only known case is genus 2, for which the maximal systole is realized by the Bolza surface [Jenni 1984].

It is also interesting to study the maximal value of the systole function on certain subspaces of $\mathcal{M}_g$. Bavard [1992] obtained the maximal systole of genera 2 and 5 on hyperelliptic surfaces. Schmutz [1993] gave a necessary and sufficient condition for the local maxima of the systole function and he gave some examples of local maxima with polyhedral symmetry. Fortier Bourque and Rafi [2022] constructed surfaces with locally maximal systoles and trivial symmetry.

Buser and Sarnak [1994] constructed surfaces with systoles larger than $\frac{4}{3}\log g$ by arithmetic methods for infinitely many genera. Katz, Schaps, and Vishne [Katz

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et al. 2007] obtained more surfaces with this lower bound. Hurwitz surfaces are among their examples. Petri and Walker [2018] and Petri [2018] gave concrete examples with systoles larger than $\frac{4}{7}\log g - K$.

Inspired by [Katz et al. 2007; Schmutz 1993], we are interested in the maximal systole of hyperbolic surfaces with certain symmetries. We consider hyperbolic surfaces with the largest S^3-extendable abelian symmetry. The S^3-extendable symmetry on a topological surface was introduced in [Wang et al. 2013; 2015]. A group action G on the genus-g topological surface Σ_g is S^3*-extendable* if there exist an embedding $i : \Sigma_g \to S^3$ and an injective homomorphism $\phi : G \to \mathrm{SO}(4)$ such that for any $g \in G$, the following diagram commutes:

$$\begin{tikzcd} \Sigma_g \arrow[r, "g"] \arrow[d, "i"] & \Sigma_g \arrow[d, "i"] \\ S^3 \arrow[r, "\phi(g)"] & S^3 \end{tikzcd}$$

When restricted to finite abelian groups, the maximal order of an S^3-extendable group action on Σ_g is $2g+2$ [Wang et al. 2013]. Such a group action can be realized as an isometry group action on a hyperbolic surface, and we say such a hyperbolic surface has the *maximal* S^3*-extendable abelian symmetry* or call the surface a *hyperbolic* $\Gamma(2,n)$ *surface* [Wang et al. 2013] where $n = g+1$.

Hyperbolic surfaces that admit an isometric S^3-extendable abelian group action of maximal order form a 2-dimensional subset of $\mathcal{M}_g$, and we consider the systole function on this subspace. Our main result is the following:

Theorem 1. *The maximal value of the systole function on hyperbolic* $\Gamma(2,n)$ *surfaces is*

$$2\operatorname{arccosh} K,$$

where

$$K = \sqrt[3]{\tfrac{1}{216}L^3 + \tfrac{1}{8}L^2 + \tfrac{5}{8}L - \tfrac{1}{8} + \sqrt{\tfrac{1}{108}L(L^2+18L+27)}} + \sqrt[3]{\tfrac{1}{216}L^3 + \tfrac{1}{8}L^2 + \tfrac{5}{8}L - \tfrac{1}{8} - \sqrt{\tfrac{1}{108}L(L^2+18L+27)}} + \tfrac{1}{6}(L+3),$$

and $L = 4\cos^2\frac{\pi}{n}$. *The maximal value is obtained when*

$$(c,t) = \left(\operatorname{arccosh} K,\, 2\operatorname{arccosh}\frac{K+1}{2\cos\frac{\pi}{n}}\right).$$

(*The symbols* c *and* t *are defined in Section 2*).

The maximal value of systoles of the $\Gamma(2,n)$-surfaces for small genera are shown in Table 1. We remark that the $\Gamma(2,3)$ surface is exactly the Bolza surface that realizes the maximal systole in genus 2.

genus	$\Gamma(2, n)$ maximal systole
2	3.0571
3	3.6478
4	3.9078
5	4.0464
6	4.1291

Table 1. Maximal systole of surface with largest S^3-extendable abelian symmetry.

Compared with the work [Bai et al. 2021] on the systole of surfaces with large cyclic symmetry for which the surfaces with large cyclic symmetry have a unique geometric structure, the surfaces studied in this paper form a two-dimensional subspace of $\mathcal{M}_g$, and the methods in the papers are quite different. We obtain our result by classifying the family of curves that are potentially the shortest geodesics on surfaces admitting this symmetry, and then determine when the systole is maximal.

The paper is organized as follows. In Section 2 we describe how to construct all hyperbolic $\Gamma(2, n)$ surfaces and determine their symmetry. In Section 3, we give a useful lemma (Lemma 3) on the intersection properties of systoles. In Section 4, we prove that for any $\Gamma(2, n)$ surface, there are only four closed geodesics in the quotient orbifold of the surface by its symmetric group that can lift to systoles of the $\Gamma(2, n)$ surface (Proposition 7). In the last section, by calculating the length of these four curves and the differentials of these lengths, we get a condition for when the $\Gamma(2, n)$ surface has the maximal systole (Proposition 9) and calculate its length.

2. The symmetry of $\Gamma(2, n)$ surfaces

In this section, we construct the hyperbolic $\Gamma(2, n)$ surface and describe the geometry and topology of its quotient by its symmetry group.

Let $\Sigma_{0,n}$ be the surface of genus 0 with n boundaries, endowed with a hyperbolic structure so that its boundaries are geodesics and $\Sigma_{0,n}$ admits an isometric rotation of order n, as indicated in Figure 1. Each boundary circle is called a *cuff* of Σ. The shortest geodesic connecting two adjacent boundary circles or its image under the isometric rotation is called a *seam*. A seam is perpendicular to the two boundary circles it connects. The n seams cut $\Sigma_{0,n}$ into two isometric right-angled hyperbolic $2n$-gons with geodesic boundary edges. The hyperbolic structure of $\Sigma_{0,n}$ is parametrized by the length of its cuffs, called the *cuff length*. We denote the cuff length by $2c$, where $c \in \mathbb{R}_+$ is called the *half cuff length*.

Two copies of $\Sigma_{0,n}$ with the same cuff length can be glued together along the cuffs to form a closed surface, so that the two rotations on each copy can be extended

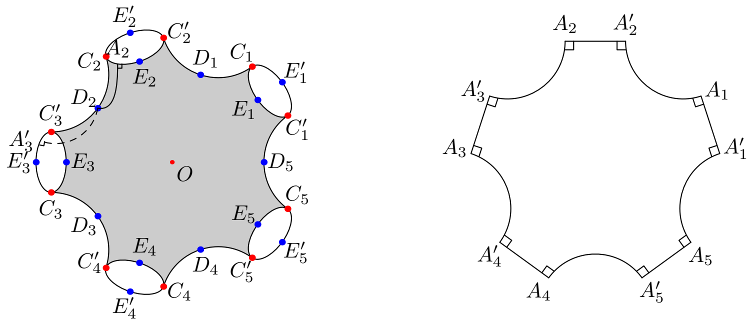

Figure 1. $\Sigma_{0,n}$ and the right-angled $2n$-gon.

to the glued surface. We call the resulting surface a *hyperbolic* $\Gamma(2, n)$*-surface*. Two seams on the surface are *paired* if they connect the same two cuffs. Similar to the Fenchel–Nielsen coordinates on the Teichmüller space, a hyperbolic $\Gamma(2, n)$-surface can be parametrized by (c, t), where c is the half cuff length and t is the "*twist parameter*". The twist parameter t equals 0 if (any) two paired seams form a closed geodesic. The hyperbolic $\Gamma(2, n)$ surface with parameter (c, t) is obtained from the hyperbolic $\Gamma(2, n)$ surface with parameter $(c, 0)$ by performing a Fenchel–Nielsen deformation of length t simultaneously along each cuff. Here the *Fenchel–Nielsen deformation* on a hyperbolic surface X along a simple closed geodesic $\alpha \subset X$ with length t is constructed by cutting X along α and then regluing the boundary curves with a left twist of length t. We may assume $0 \le t \le c$ since the surface with parameter (c, t) is isometric to the surface with parameter $(c, t + 2c)$ while the surface with parameter (c, t) is the reflection of the surface with parameter $(c, 2c - t)$ when $0 \le t \le 2c$.

The symmetry group of a hyperbolic $\Gamma(2, n)$ surface is $D_n \oplus (\mathbb{Z}/2\mathbb{Z})$, where D_n is the order n dihedral group. This symmetric group is generated by three rotations σ, τ, and ρ. As illustrated in Figure 2, σ is the order n rotation that maps each n-holed sphere to itself, τ is the order 2 rotation of each n-holed sphere, and ρ is the order 2 rotation exchanging the two n-holed spheres.

For a $\Gamma(2, n)$ surface X, the quotient $X/\langle\rho\rangle$ is a spherical orbifold with $2n$ singular points of order 2, denoted as $S^2(2, \dots, 2)_X$ (Figure 3, top left). The quotient $X/\langle\rho, \sigma\rangle$ is a spherical orbifold with four singular points of order 2, 2, n, n, respectively, denoted as $S^2(2, 2, n, n)_X$ (Figure 3, top right). The quotient $X/\langle\rho, \sigma, \tau\rangle$ is a spherical orbifold with four singular points of order 2, 2, 2, n, respectively, denoted as $S^2(2, 2, 2, n)_X$ (Figure 3, bottom). In Figure 3, top right, C and C' are the two order 2 singular points and O and O' are the order n singular points. In Figure 3, bottom, C, D, E are the order 2 singular points and O is the order n singular point. We will abbreviate the subscript X when there is no confusion. Denote the quotient

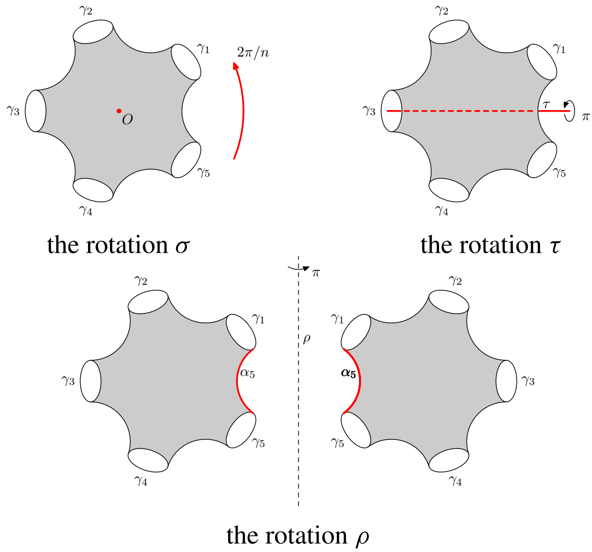

Figure 2. Generators for the isometry group of the hyperbolic $\Gamma(2, n)$ surface.

branched covering maps as

$$(2\text{-}1)\qquad \begin{tikzcd} X \arrow[r, "\pi"] \arrow[rr, bend left, "p"] \arrow[rrr, bend right, "q"'] & S^2(2,2,\dots,2)_X \arrow[r, "\pi'"] & S^2(2,2,n,n)_X \arrow[r, "\pi''"] & S^2(2,2,2,n)_X \end{tikzcd}$$

For a cuff γ_i in X, let A and $\rho(A)$ be the endpoints of two paired seams between γ_i and γ_{i-1} on the cuff γ_i. The Fenchel–Nielsen deformation gives a geodesic of length t between A and $\rho(A)$, and we let C be the midpoint. The other two paired

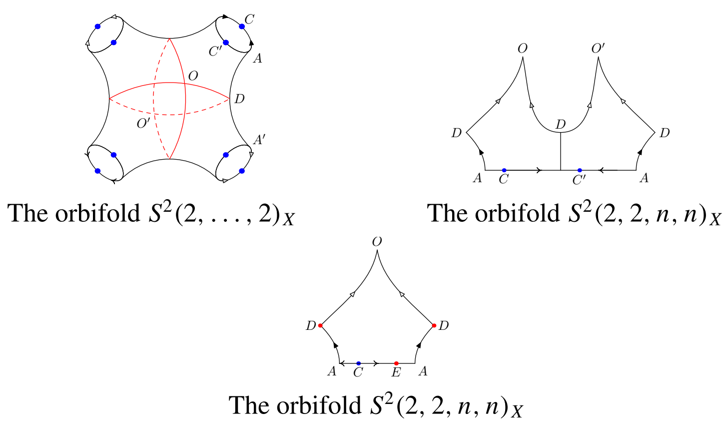

Figure 3. Orbifolds for the hyperbolic $\Gamma(2, n)$ surface.

seams give another midpoint C'. Then C and C' are fixed points of the rotation ρ on γ_i and $|AC|$ is half the twist parameter t. It follows that in $S^2(2, 2, n, n)_X$ (Figure 3, top right), we have

$$|CC'| = c, \quad |AC| = \tfrac{1}{2}t$$

and in $S^2(2, 2, 2, n)_X$ (Figure 3, bottom) we have

$$|CE| = \tfrac{1}{2}c, \quad |AC| = \tfrac{1}{2}t.$$

3. The intersection properties of the systoles

In this section, we rule out curves on $\Gamma(2, n)$ surface that cannot be the systole. The following lemma is classical and well known.

Lemma 2. *Any systole on a closed hyperbolic surface is simple and any two systoles intersect at most once. On the orbifold $S^2(2, 2, \dots, 2)$, $S^2(2, 2, n, n)$ or $S^2(2, 2, 2, n)$, any simple closed curve is separating, and any two simple closed curves are either disjoint or intersect at least twice.*

The following lemma is used to rule out curves on $\Gamma(2, n)$ that cannot be the systole.

Lemma 3. *Given a hyperbolic $\Gamma(2, n)$ surface X, let π, π', π'' denote the branched covering maps*

$$X \xrightarrow{\pi} S^2(2, 2, \dots, 2)_X \xrightarrow{\pi'} S^2(2, 2, n, n)_X \xrightarrow{\pi''} S^2(2, 2, 2, n)_X$$

Then under the maps π, $\pi' \circ \pi$ and $\pi'' \circ \pi' \circ \pi$, the image of a systole on X has no self-intersection at any regular point on the targeting orbifold, and the images of two systoles do not intersect at any regular point on the targeting orbifold.

Proof. (1) If α is a simple closed curve in X, $\pi(\alpha)$ has a self-intersection point p. Then $\pi^{-1}(p)$ consists of two points, both are the intersection points of $\pi^{-1}(\pi(\alpha))$. By the definition of double branched cover, $\pi^{-1}(\pi(\alpha))$ consists of either one curve or two curves with equal length. Since α is simple, $\pi^{-1}(\pi(\alpha))$ consists of two curves. These two curves intersect at least twice, therefore cannot be systole.

We assume α and β are two simple closed curves with equal length on X, Hence the shape of $\pi(\alpha)$ and $\pi(\beta)$ has two possibilities: S^1 or a segment whose endpoints

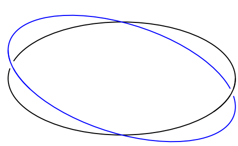

Figure 4. Case (a): $\pi(\alpha) \cup \pi(\beta)$.

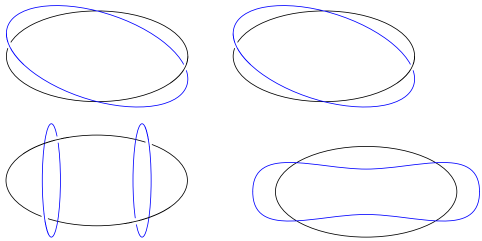

Figure 5. Case (a): The double covers of $\pi(\alpha)\cup\pi(\beta)$. The three types are shown in the top, the bottom left, and the bottom right, respectively.

are branched points of the branched cover π. We also assume p is the intersection point of $\pi(\alpha)$ and $\pi(\beta)$.

(a) If $\pi(\alpha)$ and $\pi(\beta)$ are simple closed curves, then $\pi(\alpha)$ intersects $\pi(\beta)$ at least twice by Lemma 2. Recall that there are two types of double covers of S^1, namely S^1 and $S^1 \coprod S^1$. Then there are three types of double covers of $\pi(\alpha)\cup\pi(\beta)$, shown in Figure 5.

In all the cases, α intersects β at least twice, which contradicts to Lemma 2.

(b) If $\pi(\alpha)$ is a segment while $\pi(\beta)$ is a simple closed curve (Figure 6, left), then there are two types of the double (branched) covers of $\pi(\alpha)\cup\pi(\beta)$ shown in Figure 6, middle and right.

If the double branched cover of $\pi(\alpha)\cup\pi(\beta)$ is the case shown in Figure 6, middle, then it is clear that the curve α and β have at least two intersections. Therefore, α and β cannot be systoles.

If the double branched cover of $\pi(\alpha)\cup\pi(\beta)$ is the case shown in Figure 6, right, we assume $\tilde{p}$ is one of the branched point of π in Figure 6, right. Therefore $\pi_*([\tilde{\beta}])=\pi_*([\tilde{\beta}'])$ in $\pi_1(\pi(X),\pi(\tilde{p}))$. Here $[\tilde{\beta}]$ and $[\tilde{\beta}']$ are elements of $\pi_1(X,\tilde{p})$ represented by $\tilde{\beta}$ and $\tilde{\beta}'$. It contradicts the injectivity of π_* (π is a covering map).

(c) If both $\pi(\alpha)$ and $\pi(\beta)$ are segments (Figure 7), then $|\pi^{-1}(\pi(\alpha))\cap\pi^{-1}(\pi(\beta))|\geq 2$ since the intersection point of $\pi(\alpha)$ and $\pi(\beta)$ is a regular point. However, both $\pi^{-1}(\pi(\alpha))$ and $\pi^{-1}(\pi(\beta))$ are connected. Therefore $|\alpha\cap\beta|\geq 2$, so that α and β cannot be systole.

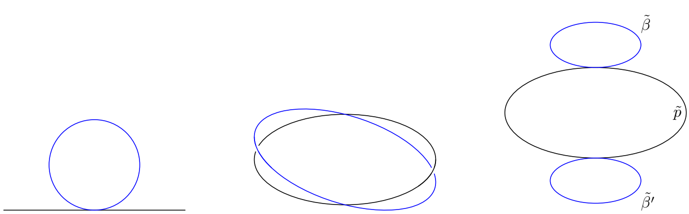

Figure 6. Case (b): $\pi(\alpha)\cup\pi(\beta)$ (left) and the double covers of $\pi(\alpha)\cup\pi(\beta)$ (middle and right).

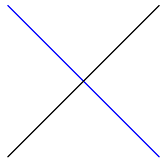

Figure 7. Case (c): $\pi(\alpha) \cup \pi(\beta)$.

(2) Let α be a systole of X; then by (1), $\pi(\alpha)$ has no self-intersection and won't intersect the image of another systole at regular points. Therefore, if $\pi'\pi(\alpha)$ has self-intersection at regular points, then it implies that either $\pi(\alpha)$ intersects itself or it intersects another lift of $\pi'\pi(\alpha)$. Therefore $\pi' \circ \pi(\alpha)$ has no self-intersections.

By exactly the same argument, we can prove that the images of two systoles of X on $S^2(2, 2, n, n)$ do not intersect at any regular point of the orbifold.

The case for $\pi'' \circ \pi' \circ \pi$ is similar to the case for $\pi' \circ \pi$. □

4. The image of systoles on $S^2(2, 2, 2, n)$

For a $\Gamma(2, n)$ surface X, we find geodesics in $S^2(2, 2, 2, n)_X$ that lift to the systoles in X in this section.

Lemma 4. *For a $\Gamma(2, n)$ surface X, a systole's image in the orbifold $S^2(2, 2, 2, n)_X$ (Figure 3, bottom) has only two possibilities*:

(1) *A geodesic segment joining two order-two singular points* (C *and* D, C *and* E *or* D *and* E).

(2) *A simple closed geodesic passing through* C.

Proof. By Lemma 3, the image of a systole of X is a simple closed geodesic or a geodesic segment joining two singular points.

(1) Image of a systole of X cannot be a simple closed curve not passing through any singular point of the orbifold. Such a curve separates $S^2(2, 2, 2, n)_X$ by Lemma 2. On each side of the curve, there are two singular points; otherwise, the curve lifts to null-homotopic curves in X. The order of both singular points on one side is two; hence the geodesic homotopic to this curve is the geodesic joining these two points.

(2) No systole's image passes through the order n singular point O. This point lifts to a regular point in $S^2(2, 2, \dots, 2)_X$, and a segment through O lifts to n segments intersecting at the preimage of O. Then by Lemma 3, this conclusion holds.

(3) The simple closed curve passing through D or E cannot lift to a systole of X, since D and E lift to regular points in $S^2(2, 2, n, n)_X$, and such curves lift to nonsimple curves (Figure 8).

By (1), (2), (3), this lemma holds. □

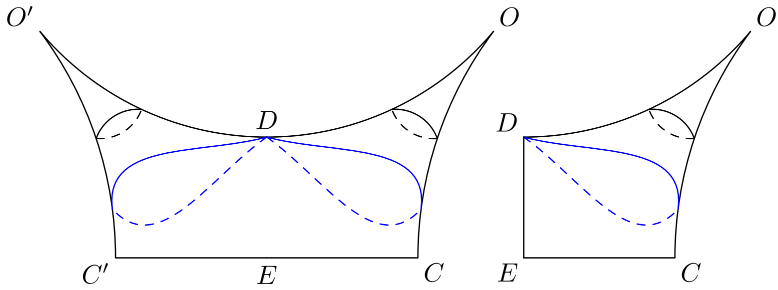

Figure 8. A simple closed curve passing through D in $S^2(2,2,2,n)$ and its lift.

In order to obtain the systole of a $\Gamma(2,n)$ surface, the next step is to find the geodesic in $S^2(2,2,2,n)_X$ joining C and D (or C and E or D and E or the simple closed geodesic through C), whose lift in X is the shortest one among all geodesics joining C and D (or C and E or D and E or the simple closed geodesic through C, respectively).

Lemma 5. *For a $\Gamma(2,n)$ surface X, let l and l' be two geodesics in $S^2(2,2,2,n)_X$ joining C and D (or joining C and E or D and E or the simple closed geodesics passing through C), and $\tilde{l}$ and $\tilde{l}'$ are their preimages in X, respectively. Then the covering $\tilde{l} \to l$ and $\tilde{l}' \to l'$ are topologically equivalent. More precisely, a homeomorphism $f : l \to l'$ can lift to $\tilde{f} : \tilde{l} \to \tilde{l}'$, letting this diagram commute*:

$$\begin{tikzcd} \tilde{l} \arrow[r, "\tilde{f}"] \arrow[d, "q"'] & \tilde{l}' \arrow[d, "q"] \\ l \arrow[r, "f"] & l' \end{tikzcd}$$

Proof. We provide the proof for the geodesics joining C and D only, since the proofs for other cases are exactly the same.

For any $l \subset S^2(2,2,2,n)_X$ joining C and D, there is a curve l'' joining D and E, intersecting l only at D. The double branched cover $\pi'' : S^2(2,2,n,n)_X \to S^2(2,2,2,n)_X$ can be constructed by gluing two copies of $S^2(2,2,2,n)_X \backslash l''$ along their boundaries. Hence the preimage of l in $S^2(2,2,n,n)_X$ is a segment (denoted as $\tilde{l}_1$) joining C and C' for any l. Therefore the coverings of any two segments joining C and D are equivalent (Figure 9).

Similarly, for any $\tilde{l}_1 \subset S^2(2,2,n,n)_X$ joining C and C', there is a segment l_1'' joining O and O', intersecting $\tilde{l}_1$ at exactly one point. Thus we can construct the n-fold cyclic branched cover of $S^2(2,2,n,n)_X$ by gluing n-copies of $S^2(2,2,n,n)_X \backslash l_1''$. Since for any $\tilde{l}_1$, we always choose a curve l_1'' intersecting $\tilde{l}_1$ once, the covering

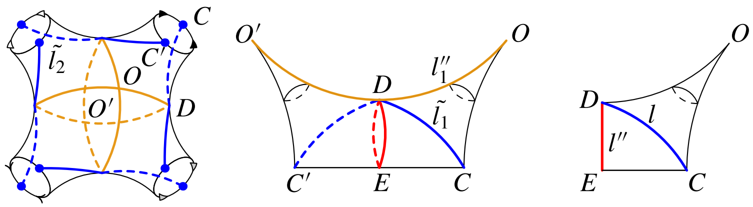

Figure 9. l and its lifts.

of $\tilde{l_1}$ by its preimage in $S^2(2, 2, \ldots, 2)_X$ (denoted by $\tilde{l_2}$) is topologically unique (Figure 9).

The multicurve $\tilde{l_2} \subset S^2(2, 2, \ldots, 2)_X$ consists of segments joining the singular points. Therefore its preimage in the $\Gamma(2, n)$ surface X (a manifold with no singular points) is topologically unique. □

Corollary 6. *Let $l, l' \subset S^2(2, 2, 2, n)_X$ be geodesic segments joining C and D, and $\alpha, \alpha' \subset X$ be simple closed geodesics lifted from l and l', respectively. If $|l| < |l'|$, then $|\alpha| < |\alpha'|$.*

This conclusion also holds for geodesics joining C and E, geodesics joining D and E, or simple closed geodesics passing through C.

Proof. By Lemma 5,

$$\frac{|\alpha|}{|l|} = \frac{|\alpha'|}{|l'|}.$$ □

Proposition 7. *For a $\Gamma(2, n)$ surface X, there are only four possible geodesics in $S^2(2, 2, 2, n)_X$ that lift to systoles in X. They are the shortest geodesics joining C and D, joining C and E and joining D and E, and the shortest simple closed geodesic passing through C, denoted as l_{CD}, l_{CE}, l_{DE} and l_C, respectively. Figure 10 describes the geometry of these four curves.*

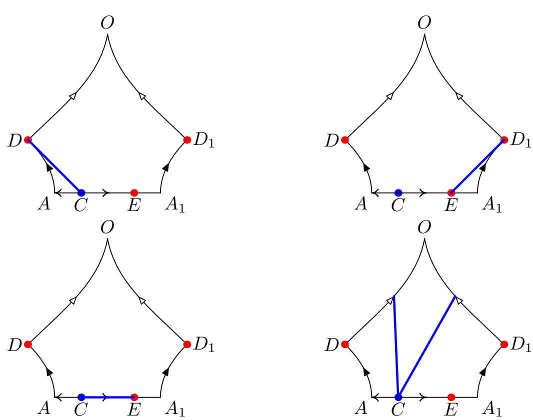

Figure 10. The four geodesics that possibly lift to systoles of X.

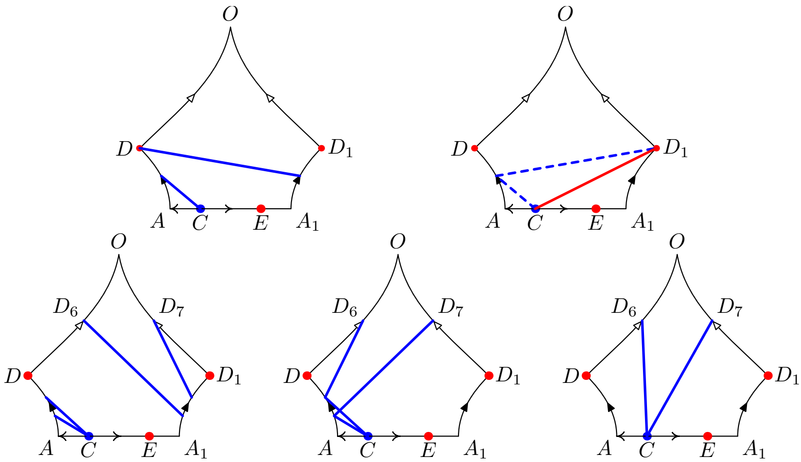

Figure 11. Looking for the shortest geodesic joining C and D (top) and looking for the shortest simple closed geodesic passing through C (bottom).

Proof. By Corollary 6, the goal of this proposition is to describe the shortest geodesics in $S^2(2, 2, 2, n)_X$ joining C and D, joining C and E, joining D and E and the simple closed geodesic passing through C.

(1) The shortest geodesic joining C and D: Let's consider the pentagon shown in Figure 11, a fundamental domain of $S^2(2, 2, 2, n)_X$. If a geodesic $l \subset S^2(2, 2, 2, n)_X$ joining C and D consists of more than one segment in the pentagon (Figure 11, top left), then by reflecting some of its segments, we obtain a bending geodesic segment joining C and D or C and D_1 with equal length to l and show that l is longer than the segment CD or CD_1.

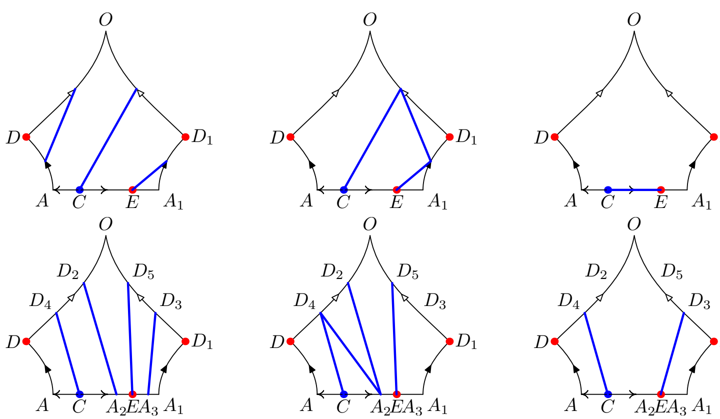

Figure 12. Looking for the shortest geodesic joining C and D (I) (top) and looking for the shortest geodesic joining C and D (II) (bottom).

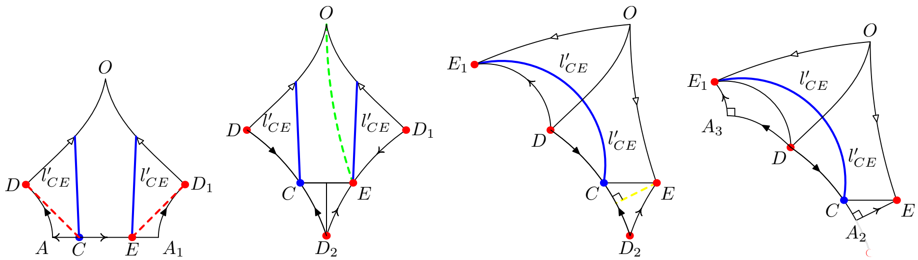

Figure 13. Cut and paste.

The segment CD is shorter than CD_1, because

$$|AD| = |AD_1| \quad \text{and} \quad |AC| < \frac{c}{2} < |A_1C|,$$

and it follows by the hyperbolic cosine law [Buser 2010, p. 454, 2.2.2 (i)]. Therefore, the shortest geodesic joining C and D is the segment l_{CD} shown in Figure 10, top left.

(2) The shortest geodesic joining D and E is the segment l_{DE} shown in Figure 10, top right, by the same argument.

(3) The shortest simple closed geodesic passing through C is the geodesic l_C shown in Figure 10, bottom right, by the same argument; see Figure 11.

(4) The shortest geodesic joining C and E: By reflecting some segments, we get the shortest geodesic is either the geodesic in Figure 12, top right, (denoted by l_{CE}) or the geodesic in Figure 12, bottom right (denoted by l'_{CE}). By the cut-and-paste shown in Figure 13, we see that l_{CE} is shorter than l'_{CE}, hence the shortest geodesic joining C and E is l_{CE} in Figure 10, bottom left. □

5. Calculations

In this section, we represent the length of the curves in Figure 10 by the parameters c and t. Then by these formulae, we give a condition of the $\Gamma(2, n)$ surface having the longest systole. For convenience, we call this surface a *maximal surface*. Finally, we calculate the systole length of this surface.

Recall that in the pentagons in Figure 10, $|CE| = c/2$, $|AC| = t/2$, $|A_1E| = (c-t)/2$ and $\angle DOD_1 = 2\pi/n$. We assume $AD = A_1D_1 = s/2$. Then in one of the two half pieces of the pentagon (Figure 14), by hyperbolic trigonometry [Buser 2010, p. 454, 2.3.1 (i)],

$$\sinh\frac{c}{2}\sinh\frac{s}{2} = \cos\frac{\pi}{n}. \tag{5-1}$$

Therefore, directly, for the lengths of the geodesics in Figure 10, we have

$$|l_{CE}| = \frac{c}{2}, \tag{5-2}$$

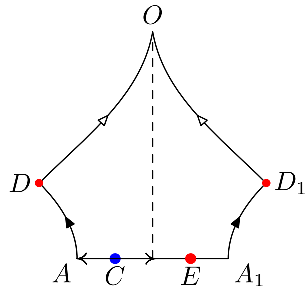

Figure 14. The pentagon.

and by the hyperbolic cosine law in right-angled triangles [Buser 2010, p. 454, 2.2.2 (i)]

$$\cosh|l_{CD}| = \cosh|CD| = \cosh\frac{t}{2}\cosh\frac{s}{2}; \tag{5-3}$$

$$\cosh|l_{DE}| = \cosh|D_1E| = \cosh\frac{c-t}{2}\cosh\frac{s}{2}. \tag{5-4}$$

To calculate the length of l_C (Figure 10, bottom right), we treat the pentagon as a fundamental domain for the orbifold $S^2(2,2,2,n)$ in $\mathbb{H}^2$. Then in the joining of two pentagons shown in Figure 15, left, l_C is realized by the segment CC'. Hence its length is

$$\begin{aligned}\cosh|l_C| &= \cosh|CC'| \\ &= \cosh|AA_1'|\cosh|AC|\cosh|A_1'C'| - \sinh|AC|\sinh|A_1'C'| \\ &= \cosh s\cosh\frac{t}{2}\cosh\Bigl(c-\frac{t}{2}\Bigr) - \sinh\frac{t}{2}\sinh\Bigl(c-\frac{t}{2}\Bigr)\end{aligned} \tag{5-5}$$

by a trigonometric formula [Buser 2010, p. 38, 2.3.2].

Now we are ready to prove the following:

Proposition 8. *In the $\Gamma(2,n)$ surface X_0 with maximal systole among all the $\Gamma(2,n)$ surfaces, l_{DE} in $S^2(2,2,2,n)_{X_0}$ cannot lift to a systole of this surface.*

Proof. Recall that by Proposition 7, in $S^2(2,2,2,n)_{X_0}$, there are only four geodesics that can lift to systoles of X_0, namely l_{CD}, l_{DE}, l_{CE} and l_C in Figure 10.

If l_{DE} lifts to a systole, then l_{CD} and l_{CE} cannot lift to systoles. This is because, when lifting to $S^2(2,2,n,n)$, l_{DE} intersects l_{CD} and l_{CE} at regular points D and E of the orbifold (Figure 15, right), therefore by Lemma 3, they cannot simultaneously become systoles.

Then we calculate the differentials of $|l_{DE}|$ and $|l_C|$, showing that there is vector $(A(c,t), B(c,t))$ such that $\mathrm{d}|l_C|(A,B) > 0$ and $\mathrm{d}|l_{DE}|(A,B) > 0$ simultaneously. Since only l_{DE} and l_C can lift to systoles, a surface with a systole lifted from l_{DE} cannot be a maximal surface.

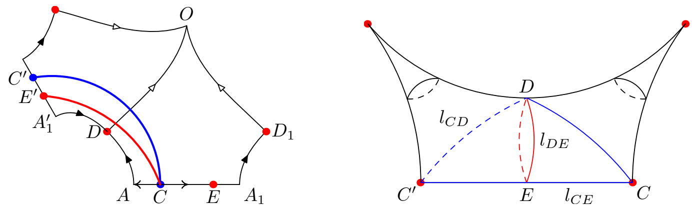

Figure 15. The geodesic l_C (left) and the lift of l_{DE}, l_{CD} and l_{CE} (right).

As a preparation, we differentiate both sides of (5-1) and get

$$\frac{\mathrm{d}s}{\mathrm{d}c} = -\frac{\cosh\frac{c}{2}\sinh\frac{s}{2}}{\cosh\frac{s}{2}\sinh\frac{c}{2}}. \tag{5-6}$$

Then for l_{DE}

$$\begin{aligned}
\frac{\partial|l_{DE}|}{\partial t} &= \frac{\partial}{\partial t}\Big(\cosh\frac{s}{2}\cosh\frac{c-t}{2}\Big) = -\frac{1}{2}\cosh\frac{s}{2}\sinh\frac{c-t}{2},\\
\frac{\partial|l_{DE}|}{\partial c} &= \frac{\partial}{\partial c}\Big(\cosh\frac{s}{2}\cosh\frac{c-t}{2}\Big)\\
&= \frac{1}{2}\Big(\sinh\frac{s}{2}\frac{\mathrm{d}s}{\mathrm{d}c}\cosh\frac{c-t}{2} + \cosh\frac{s}{2}\sinh\frac{c-t}{2}\Big),\\
\mathrm{d}|l_{DE}| &= \frac{\partial|l_{DE}|}{\partial t}\mathrm{d}t + \frac{\partial|l_{DE}|}{\partial c}\mathrm{d}c.
\end{aligned}$$

For l_C

$$\begin{aligned}
\frac{\partial|l_C|}{\partial t} &= \frac{\partial}{\partial t}\Big(\cosh s\cosh\frac{t}{2}\cosh\Big(c-\frac{t}{2}\Big) - \sinh\frac{t}{2}\sinh\Big(c-\frac{t}{2}\Big)\Big)\\
&= \frac{1}{2}(\cosh s + 1)\sinh(t-c),
\end{aligned} \tag{5-7}$$

and

$$\begin{aligned}
\frac{\partial|l_C|}{\partial c} &= \frac{\partial}{\partial c}\Big(\cosh s\cosh\frac{t}{2}\cosh\Big(c-\frac{t}{2}\Big) - \sinh\frac{t}{2}\sinh\Big(c-\frac{t}{2}\Big)\Big)\\
&= -\frac{\cosh\frac{c}{2}\sinh\frac{s}{2}}{\cosh\frac{s}{2}\sinh\frac{c}{2}}\sinh s\cosh\frac{t}{2}\cosh\Big(c-\frac{t}{2}\Big) + \cosh s\cosh\frac{t}{2}\sinh\Big(c-\frac{t}{2}\Big)\\
&\qquad - \sinh\frac{t}{2}\cosh\Big(c-\frac{t}{2}\Big),\\
\mathrm{d}|l_C| &= \frac{\partial|l_C|}{\partial t}\mathrm{d}t + \frac{\partial|l_C|}{\partial c}\mathrm{d}c.
\end{aligned}$$

The two tangent vectors $\mathrm{d}|l_{DE}|$, $\mathrm{d}|l_C|$ are nonzero vectors. When $c > 0$, $0 \le t \le c$,

$$\frac{\partial|l_{DE}|}{\partial t} < 0, \quad \frac{\partial|l_C|}{\partial t} < 0.$$

For any $k \le 0$, $\mathrm{d}|l_{DE}| \ne k\mathrm{d}|l_C|$. Then there is a vector $(A(c,t), B(c,t))$ such that

$$\mathrm{d}|l_{DE}|(A(c,t), B(c,t)) > 0, \quad \mathrm{d}|l_C|(A(c,t), B(c,t)) > 0.$$

By the assumption that l_{DE} lifts to a systole of X_0, only l_{DE} and l_C can lift to a systole of the surface. Then there is another surface with systole bigger than X_0. Therefore X_0 is not maximal. □

From Propositions 7 and 8, we know that only l_{CE}, l_{CD} and l_C in the orbifold $S^2(2, 2, 2, n)$ can lift to a systole of the maximal surface.

By the symmetry of $\Gamma(2, n)$ surfaces and Lemma 3, the preimage of the geodesic $l_{CE} \subset S^2(2, 2, 2, n)$ (l_{CD}, l_C, respectively) on the $\Gamma(2, n)$ surface consists of pairwise disjoint geodesics with equal length.

Proposition 9. *On the maximal $\Gamma(2, n)$ surface X_0, a simple closed geodesic is a systole if and only if it is lifted from l_{CE}, l_{CD} or l_C.*

Proof. It is sufficient to prove that in X_0, every geodesic lifted from l_{CE}, l_{CD} or l_C is a systole.

The proof is divided into two steps.

(1) If there is only one curve among l_{CE}, l_{CD} and l_C that lifts to the systoles of X_0, then X_0 is not maximal.

Without loss of generality, we assume that l_{CE} lifts to the systole of X_0, while l_{CD} and l_C do not lift to systoles of X_0. On the orbifold $S^2(2, 2, 2, n)_{X_0}$, there are deformations increasing or decreasing the length of the curve l_{CE}. A deformation increasing the length of l_{CE} increases the length of geodesics lifted from l_{CE} in X_0. If the deformation is small enough, then we get a new $\Gamma(2, n)$ surface, whose systoles are lifted from l_{CE} and the length of these curves are longer than the corresponding curves in X_0. Hence X_0 is not maximal.

(2) If there are exactly two curves among l_{CE}, l_{CD} and l_C lifting to the systole of the $\Gamma(2, n)$ surface, then the surface is not maximal.

(2a) We assume l_{CE} and l_{CD} lift to the systoles of X_0, while l_C does not lift to systoles of X_0. Then in the Fenchel–Nielsen coordinate (c, t), the length $|l_{CD}|(c, t)$ is monotonely increasing with respect to t by (5-3), while $l_{CE} = c/2$. We pick a sufficiently small $\varepsilon > 0$ and deform the Fenchel–Nielsen coordinate from (c, t) to $(c, t + \varepsilon)$; then we get a new surface X'. The systoles of X' are exactly the geodesics lifted from l_{CE}, and this surface has the same systole length to X_0. Then by (1), there exists a surface with longer systole than these two surfaces.

(2b) If l_{CE} and l_C lift to the systoles of X_0, while l_{CD} does not lift to systoles of X_0, the proof is similar. By (5-7), l_C is decreasing with respect to t when $t \leq c$. Thus using the deformation $(c, t) \mapsto (c, t - \varepsilon)$, we get a surface whose systoles are all lifted from l_{CE} and whose systole length is equal to X_0's. Thus by (1), we know X_0 is not maximal.

(2c) The last case is that l_{CD} and l_C lift to the systoles of X_0, while l_{CE} does not lift to systoles of X_0. Similarly, some of the Fenchel–Nielsen deformations along

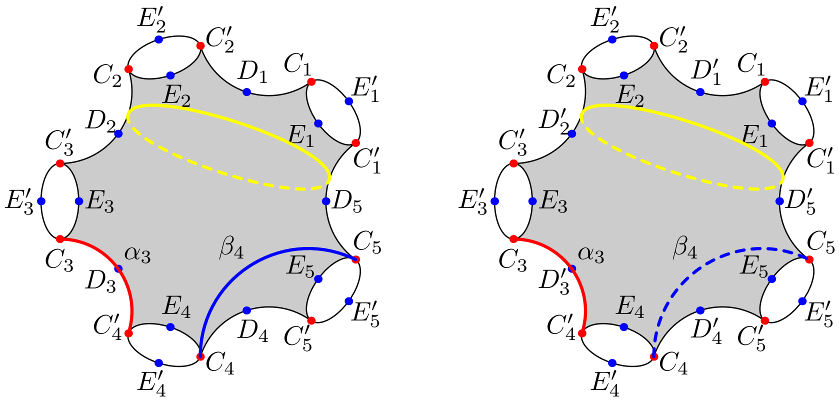

Figure 16. Cutting off a subsurface with signature (1, 2).

l_{CD} increase the length of l_C and let l_{CD} become the unique geodesic that can lift to systoles of X_0. Then by (1), this surface is not maximal.

By (1) and (2), all three curves l_{CD}, l_{CE} and l_C lift to the systoles of the maximal $\Gamma(2, n)$ surface. □

We are now ready to calculate the maximal systole on hyperbolic $\Gamma(2, n)$ surfaces.

Proof of Theorem 1. First we describe the lift of l_{CE}, l_{CD} and l_C in the $\Gamma(2, n)$ surface, respectively.

In the $\Gamma(2, n)$ surface shown in Figure 16, C_i and C_i' $(i = 1, \ldots, n)$ are the fixed points of the order-two involution τ (recall its definition in Section 2) and are the lifts of the singular point C in $S^2(2, 2, 2, n)$; D_i and D_i' are the mid-points of the seams, and the lifts of the point D in $S^2(2, 2, 2, n)$; E_i and E_i' are the mid-points of C_iC_i' and the lifts of the point E in $S^2(2, 2, 2, n)$.

The curve l_{CE} lifts to cuffs of the surface, denoted as γ_i $(i = 1, \ldots, 5$); l_{CD} lifts to geodesics passing through $C_iD_iC_{i+1}'D_i'$ denoted as α_i; l_C lifts to geodesics passing through C_iC_{i+1}, denoted as β_i.

To calculate the systole length, we cut off a subsurface with signature (1, 2) from the $\Gamma(2, n)$ surface containing γ_1, γ_2 and α_1 (Figures 16 and 17), the boundary length of this surface is given by [Buser 2010, p. 454, 2.4.1 (i)]. We take common perpendiculars between cuffs and boundary components as in Figure 17. Then in the hexagon $H_1H_2A_1A_2A_3A_4$, we have

$$\begin{aligned}\cosh|H_1H_2| &= \sinh|A_1A_2|\sinh|A_3A_4|\cosh|A_2A_3| - \cosh|A_1A_2|\cosh|A_3A_4| \\ &= \sinh^2 c\cosh s - \cosh^2 c.\end{aligned} \tag{5-8}$$

In this subsurface, C_1, C_1', C_2, C_2' are branched points of the hyperelliptic involution and α_1, β_1, γ_1, γ_2 are the systoles of the $\Gamma(2, n)$ surface (Figure 18). To calculate the systole length, we redraw the (1, 2)-subsurface as Figure 19. When the systole

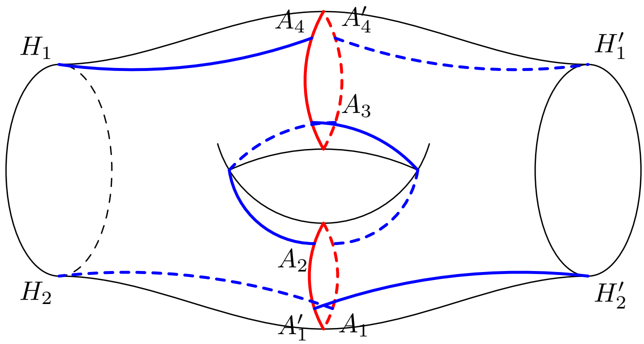

Figure 17. The red curves are cuffs. The blue curves are seams.

is maximal, the lengths of γ_1 and β_1 are the same, namely $|C_1C_1'| = |C_1C_2|$. In this case, the subsurface is shown in Figure 20.

When the surface is maximal, $|H_5H_6| = \frac{1}{2}|\alpha_1| = c$, $|C_1C_1'| = \frac{1}{2}\gamma_1 = c$. We assume $|H_3H_4| = l$ and $|H_5C_1'| = h$. Then in the hexagon $H_4H_3C_2'H_6H_5C_1$, by the symmetry of this hexagon, $|H_6C_2'| = |H_5C_1'| = h$, and by [Buser 2010, p. 454, 2.4.1(i)] we have

$$\cosh|H_3H_4| = \sinh|C_1H_5| \sinh|C_2'H_6| \cosh|H_5H_6| - \cosh|C_1H_5| \cosh|C_2'H_6|$$

and

$$\cosh l = \sinh^2 h \cosh c - \cosh^2 h. \tag{5-9}$$

In the triangle $\triangle C_1H_5C_1'$, by [Buser 2010, p. 454, 2.2.2 (i)], we have

$$\cosh|C_1C_1'| = \cosh|C_1H_5| \cosh|C_1'H_5| \cosh c = \cosh h \cosh \frac{c}{2}. \tag{5-10}$$

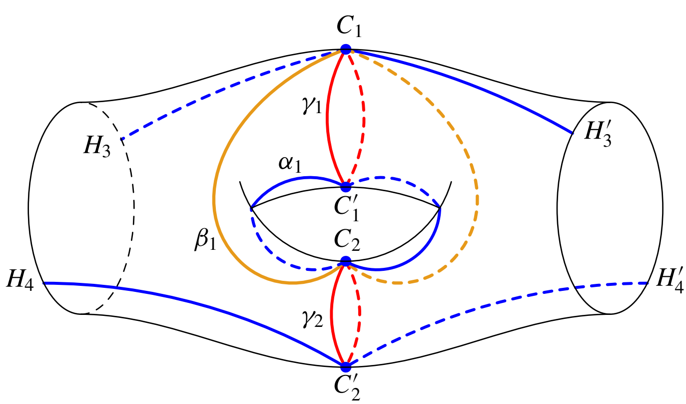

Figure 18. Systoles of X in the $(1, 2)$-subsurface.

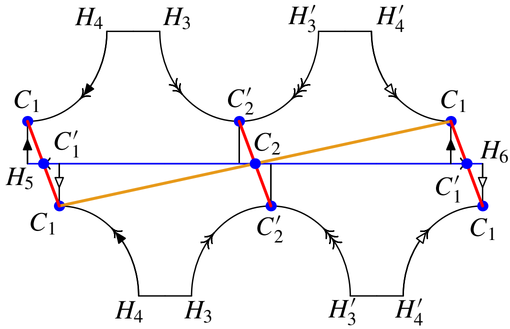

Figure 19. Redrawing the (1, 2)-subsurface.

For convenience, we denote $\cosh c$ by K. Then combining (5-9) and (5-10), we eliminate h and get

$$\frac{2K^2}{K+1} = \frac{K+\cosh l}{K-1}. \tag{5-11}$$

Recall that $l = |H_3H_4|$. Then combining (5-11), (5-8) and (5-1), we eliminate l and s, and get

$$2K^3 - 3K^2 + 1 - 4\cos^2\frac{\pi}{n}(K+1)^2 = 0.$$

The unique real solution of this equation is

$$K = \sqrt[3]{\tfrac{1}{216}L^3 + \tfrac{1}{8}L^2 + \tfrac{5}{8}L - \tfrac{1}{8} + \sqrt{\tfrac{1}{108}L(L^2+18L+27)}} \\ + \sqrt[3]{\tfrac{1}{216}L^3 + \tfrac{1}{8}L^2 + \tfrac{5}{8}L - \tfrac{1}{8} - \sqrt{\tfrac{1}{108}L(L^2+18L+27)}} + \tfrac{1}{6}(L+3),$$

where $L = 4\cos^2\frac{\pi}{n}$.

At last, we calculate the twist parameter t of the maximal surface, using (5-1) and (5-3):

$$\cosh\frac{t}{2} = \frac{\cosh\frac{c}{2}}{\cosh\frac{s}{2}} = \frac{\cosh^2\frac{c}{2}}{\cos\frac{\pi}{n}} = \frac{\cosh c+1}{2\cos\frac{\pi}{n}} = \frac{K+1}{2\cos\frac{\pi}{n}}.$$

Theorem 1 follows. □

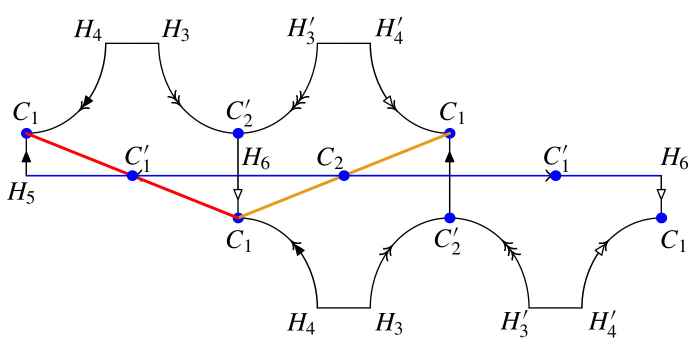

Figure 20. When $|C_1C'_1| = |C_1C_2|$.

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