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STABLE SYSTOLES OF HIGHER RANK IN RIEMANNIAN MANIFOLDS

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This paper introduces the stable systoles of higher rank of a Riemannian manifold as a generalization of the usual stable systoles. Several inequalities involving these higher rank systoles are proved.

1. Introduction

Let M be a smooth compact orientable manifold of dimension n. A Riemannian metric g on M induces an associated stable mass norm $\|\cdot\|$ on the real homology groups $H_p(M,\mathbb{R})$. Because the image of the p-th integral homology group $H_p(M,\mathbb{Z})$ in $H_p(M,\mathbb{R})$ is a lattice, denoted $H_p(M,\mathbb{Z})_{\mathbb{R}}$, in $H_p(M,\mathbb{R})$, we thereby obtain for each p a lattice in a normed vector space. Such structures are the central objects of study in the geometry of numbers [6], and their various numerical invariants thus give rise to a host of invariants of the Riemannian manifold (M, g).

For example, the p-dimensional stable systole $\mathrm{stsys}_p(M, g)$ is the minimum norm of the nonzero elements in the lattice $H_p(M,\mathbb{Z})_{\mathbb{R}}$. These have been studied extensively [8; 9; 12]. One can also consider the successive minimums of the lattice or its whole length spectrum [10]. The volume of the Jacobian variety $J_p = H_p(M,\mathbb{R})/H_p(M,\mathbb{Z})_{\mathbb{R}}$ with the (Finsler) metric induced from the stable norm gives an additional invariant. There are various natural ways to define the volume of the quotient tori [15]. In [8], the mass and mass* measures were used to define the volume of J_p. In this paper we will use the Busemann–Hausdorff measure to define the volume of J_p as well as the higher rank systoles of a Riemannian manifold.

Given a positive integer k less than or equal to the p-th Betti number b_p of M, we define $\mathrm{stsys}_{p,k}(M, g)$ to be the minimum Hausdorff–Busemann volume of the fundamental region of sublattices of $H_p(M,\mathbb{Z})_{\mathbb{R}}$ of rank k. This can be interpreted as the k-th systole of J_p. In particular $\mathrm{stsys}_{p,1}(M, g)$ is the ordinary stable p-th systole, and $\mathrm{stsys}_{p,b_p}(M, g)$ is the Hausdorff–Busemann volume of J_p.

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Among the results of this paper are sharp stable systolic inequalities. The first of these generalizes an inequality due to Bangert and Katz [1].

Theorem 5.1. *Let (M, g) be a compact oriented manifold of dimension n whose first Betti number is b. Then*

$$\operatorname{stsys}_{1,b}(M, g)\operatorname{stsys}_{n-1,b}(M, g) \le \operatorname{Vol}(M, g).$$

Equality holds if and only if there exists a Riemannian submersion of M onto a flat torus of dimension b with connected minimal fibers.

A further generalization is Theorem 5.2 which is stated and proved in Section 5. In addition, we prove a sharp inequality for conformally flat metrics on the 4-dimensional torus for the 2-dimensional stable systole of rank 6.

Theorem 6.7. *Let (M, g) be a conformally flat 4-dimensional torus. Then*

$$\operatorname{stsys}_{2,6}(M, g)^2 \le \left(\frac{3\pi}{4}\right)^{\frac{1}{3}} \operatorname{Vol}(M, g).$$

Equality holds if and only if (M, g) is flat.

This paper is organized as follows. In Section 2, we discuss lattices in normed vector spaces and their invariants, as well as the behavior of Hausdorff–Busemann volume under linear transformations. Section 3 reviews some properties of the mass, comass and L^2 norms on the (co)homology of a compact oriented manifold. The formal definition of the stable systoles of higher rank is given in Section 4. This section provides a number of inequalities involving them related to the properties of the cap product. In Section 5, we prove a sharp $(1, n-1)$-inequality that generalizes that of Bangert and Katz [1]. In Section 6, we calculate the 2-dimensional systole of rank 6 in flat 4-dimensional tori, and prove a sharp inequality for conformally flat metrics on the 4-dimensional torus. Finally, in the Appendix, we prove a result needed in Section 5 that the dual k-extreme lattices are dual k-perfect.

2. Normed vector spaces

Hausdorff measure. Let $(V, \|\cdot\|)$ be an n-dimensional normed vector space over the real numbers. Let $K \subset V$ be the unit ball in $(V, \|\cdot\|)$. According to a theorem of Busemann [5, (2.3)], the Hausdorff n-dimensional measure $\operatorname{Vol}_n(\,\cdot\,, \|\cdot\|)$ is the unique translation invariant (Haar) measure on V normalized such that

$$\operatorname{Vol}_n(K, \|\cdot\|) = \omega_n = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(1 + \frac{n}{2}\right)}.$$

Here ω_n denotes the Euclidean volume of the Euclidean unit ball, and Γ is the gamma function.

Proposition 2.1 (cf. [15]). *Suppose that $T : V \to W$ is a linear isomorphism between the n-dimensional normed spaces* $(V, \|\cdot\|)$ *and* $(W, \|\cdot\|')$ *such that*

$$\|T(x)\|' \le C\,\|x\|$$

for all $x \in V$ for some constant C. Then

$$\mathrm{Vol}_n(T(E), \|\cdot\|') \le C^n\, \mathrm{Vol}_n(E, \|\cdot\|) \tag{2-1}$$

for all Borel sets $E \subset V$. Moreover, if equality holds in (2-1) *for some E with nonzero volume, then equality holds in* (2-1) *for all E and*

$$\|T(x)\|' = C\,\|x\|$$

for all $x \in V$.

Proof. Let K and K' denote the unit balls of V and W, respectively. By assumption $T(K) \subset CK' = \{Cx : x \in K'\}$. We may choose inner products on V and W so that T is an isometry. Let $\mathcal{L}^n$ denote the n-dimensional Lebesgue measure on V and W for the Euclidean metrics induced from the inner products. Thus $\mathcal{L}^n(T(E)) = \mathcal{L}^n(E)$ for all Borel sets E in V. Hence $\mathcal{L}^n(K) = \mathcal{L}^n(T(K)) \le \mathcal{L}^n(CK') = C^n \mathcal{L}^n(K')$. Therefore

$$\begin{aligned} \mathrm{Vol}_n(T(E), \|\cdot\|') &= \frac{\omega_n}{\mathcal{L}^n(K')}\mathcal{L}^n(T(E)) \\ &\le C^n \frac{\omega_n}{\mathcal{L}^n(K)}\mathcal{L}^n(E) = C^n\, \mathrm{Vol}_n(E, \|\cdot\|) \end{aligned} \tag{2-2}$$

proving (2-1). If equality holds for some E with $\mathcal{L}^n(E) \ne 0$, then, by the proof of (2-2), $\mathcal{L}^n(TK) = \mathcal{L}^n(CK')$. Now $TK \subset CK'$ are both closed bounded convex sets of W. Thus, if $TK \ne CK'$, there would exist an open set contained in $CK' \backslash TK$ which would imply the contradiction $\mathcal{L}^n(CK' \backslash TK) > 0$. Hence $TK = CK'$, and therefore $\|T(x)\|' = C\,\|x\|$ for all $x \in V$. □

Lattices. Suppose that Λ is a lattice in $(V, \|\cdot\|)$. The Hausdorff measure in V passes down to the Hausdorff–Busemann measure on the n-dimensional torus V/Λ. Its volume is equal to the measure of a fundamental domain for Λ and will be denoted $\mathrm{Vol}_n(V/\Lambda, \|\cdot\|)$. The Hausdorff–Busemann volume has the following asymptotic interpretation. Let $N(R)$ equal the number of lattice points $x \in \Lambda$ with $\|x\| \le R$. Then

$$\lim_{R\to\infty} \frac{N(R)}{R^n} = \frac{\omega_n}{\mathrm{Vol}_n(V/\Lambda, \|\cdot\|)}.$$

Thus $\mathrm{Vol}_n(V/\Lambda, \|\cdot\|)$ depends only on the length spectrum of Λ. By the second Minkowski inequality [6, p. 218], the Hausdorff–Busemann volume also satisfies the inequality

$$\frac{2^n}{n!}\,\mathrm{Vol}_n(V/\Lambda, \|\cdot\|) \le \lambda_1 \cdots \lambda_n \omega_n \le 2^n\,\mathrm{Vol}_n(V/\Lambda, \|\cdot\|),$$

where the λ_i are the successive minimums of the lattice Λ.

Let V^* be the dual space of V with dual norm $\|\cdot\|^*$. The polar set $K^\circ \subset V^*$ is defined to be the unit ball in $(V^*, \|\cdot\|^*)$. Let $\Lambda^* \subset V^*$ denote the dual lattice of Λ.

Lemma 2.2. *Let Λ and Λ^* be dual lattices in the respective dual normed spaces $(V, \|\cdot\|)$ and $(V^*, \|\cdot\|^*)$. There exists a universal constant $c > 0$ such that*

$$1 \le \mathrm{Vol}_n(V/\Lambda, \|\cdot\|)\,\mathrm{Vol}_n(V^*/\Lambda^*, \|\cdot\|^*) \le \frac{1}{c^n}.$$

Proof. Let K° be the polar of K. Fix an inner product on V with its dual inner product, and let $\mathcal{L}^n$ denote the corresponding n-dimensional Lebesgue measures. Thus

$$\mathrm{Vol}_n(V/\Lambda, \|\cdot\|)\,\mathrm{Vol}_n(V^*/\Lambda^*, \|\cdot\|^*) = \omega_n^2\,\frac{\mathcal{L}^n(V/\Lambda)}{\mathcal{L}^n(K)}\,\frac{\mathcal{L}^n(V^*/\Lambda^*)}{\mathcal{L}^n(K^\circ)}.$$

But by the Santalo and the Bourgain–Milman [4] inequalities,

$$\omega_n^2 \ge \mathcal{L}^n(K)\mathcal{L}^n(K^\circ) \ge c^n \omega_n^2$$

for some universal constant c, and by Lemma 5 of [6, p. 24],

$$\mathcal{L}^n(V/\Lambda)\mathcal{L}^n(V^*/\Lambda^*) = 1.$$

The proof is completed by combining these three formulas. □

Remark 2.3. It is conjectured that $c = \frac{2}{\pi}$. Kuperberg [13] has shown $c \ge \frac{1}{2}$.

Sublattices. Let k be an integer with $1 \le k \le n$. By definition, a sublattice of Λ of rank k is a lattice Λ' in a k-dimensional vector subspace V' of V such that $\Lambda' = \Lambda \cap V'$. Let $\|\cdot\|'$ be the restriction of $\|\cdot\|$ to V'. Then the Hausdorff–Busemann volume of Λ' is $\mathrm{Vol}_k(V'/\Lambda', \|\cdot\|')$. Define

$$\Delta_k(\Lambda, \|\cdot\|) = \inf_{\Lambda'} \mathrm{Vol}_k(V'/\Lambda', \|\cdot\|'),$$

where Λ' runs over all sublattices of rank k in Λ. In particular $\Delta_1(\Lambda, \|\cdot\|)$ is just the length of the shortest nonzero element of Λ.

In the special case when $|\cdot|$ is a Euclidean norm obtained from an inner product on V,

$$\mathrm{Vol}_n(V/\Lambda, |\cdot|) = \det(\Lambda) \equiv |e_1 \wedge \cdots \wedge e_n|,$$

where $e_1, \ldots, e_n$ is a basis for Λ. Thus if k is an integer with $1 \le k \le n$, the numbers $\Delta_k(\Lambda, |\cdot|)$ are exactly the *carcans* of flat tori defined by Berger in [3, §7].

Following [2], define the Hermite–Rankin constants

$$\gamma_{n,k} = \sup_{\Lambda} \frac{\Delta_k(\Lambda, |\cdot|)^2}{\det(\Lambda)^{2k/n}}$$

and the Bergé–Martinet constants

$$\gamma'_{n,k} = \sup_{\Lambda} \Delta_k(\Lambda, |\cdot|)\Delta_k(\Lambda^*, |\cdot|^*),$$

where Λ runs over all lattices in the n-dimensional Euclidean space $(V, |\cdot|)$.

Proposition 2.4. *Let $(V, \|\cdot\|)$ be a normed vector space of dimension n. Then*

$$\frac{\Delta_k(\Lambda, \|\cdot\|)^2}{\mathrm{Vol}_n(V/\Lambda, \|\cdot\|)^{2k/n}} \leq n^k \gamma_{n,k}, \tag{2-3}$$

and

$$\Delta_k(\Lambda, \|\cdot\|)\Delta_k(\Lambda^*, \|\cdot\|^*) \leq n^{\frac{k}{2}} \gamma'_{n,k}. \tag{2-4}$$

Proof. Let E be the John ellipsoid for the unit ball in V [11]. Then E determines a Euclidean norm $|\cdot|_E$ on V that satisfies

$$\|\cdot\| \leq |\cdot|_E \leq \sqrt{n}\, \|\cdot\|. \tag{2-5}$$

Thus by Proposition 2.1

$$\mathrm{Vol}_n(V/\Lambda, |\cdot|_E) \leq n^{\frac{n}{2}}\, \mathrm{Vol}_n(V/\Lambda, \|\cdot\|),$$

and, for any sublattice Λ' of rank k in a k-dimensional subspace V',

$$\mathrm{Vol}_k(V'/\Lambda', \|\cdot\|) \leq \mathrm{Vol}_k(V'/\Lambda', |\cdot|_E).$$

Hence,

$$\Delta_k(\Lambda, \|\cdot\|) \leq \Delta_k(\Lambda, |\cdot|_E),$$

and therefore

$$\frac{\Delta_k(\Lambda, \|\cdot\|)^2}{\mathrm{Vol}_n(V/\Lambda, \|\cdot\|)^{2k/n}} \leq n^k \gamma_{n,k}.$$

Passing to the dual norms, inequality (2-5) implies

$$\|\cdot\|^* \geq |\cdot|_E^* \geq \frac{1}{\sqrt{n}}\, \|\cdot\|^*. \tag{2-6}$$

Thus

$$\Delta_k(\Lambda^*, \|\cdot\|^*) \leq n^{\frac{k}{2}}\, \Delta_k(\Lambda^*, |\cdot|_E^*),$$

and therefore

$$\Delta_k(\Lambda, \|\cdot\|)\Delta_k(\Lambda^*, \|\cdot\|^*) \leq n^{\frac{k}{2}} \gamma'_{n,k}.$$ □

Linear maps. If $T : V \to W$ is a linear isomorphism such that $T(\Lambda) \subset \Gamma$ where Λ and Γ are lattices in V and W, respectively, then T induces a covering map

$$\overline{T} : V/\Lambda \to W/\Gamma$$

between two n-dimensional tori. The degree of $\overline{T}$ is clearly equal to the index of $T(\Lambda)$ in Γ as free abelian groups. Let $\deg(T)$ denote this degree.

Proposition 2.5. *Suppose that $T : V \to W$ is a linear isomorphism between the n-dimensional normed spaces $(V, \|\cdot\|)$ and $(W, \|\cdot\|')$ such that*

$$\|T(x)\|' \le C\,\|x\|$$

for all $x \in V$. Suppose also that $T(\Lambda) \subset \Gamma$ where Λ and Γ are lattices in V and W, respectively. Then

$$\deg(T)\,\mathrm{Vol}_n(W/\Gamma, \|\cdot\|') \le C^n\,\mathrm{Vol}_n(V/\Lambda, \|\cdot\|). \tag{2-7}$$

Equality holds if and only if

$$\|T(x)\|' = C\,\|x\|$$

for all $x \in V$.

Proof. Let $E \subset V$ be the fundamental domain for the cover V over V/Λ and $F \subset W$ the fundamental domain for the cover W over W/Γ. Then $T(E)$ can be expressed as a union of the translates of $\deg(T)$ copies of F. By Proposition 2.1 and the translation invariance of the Hausdorff measure,

$$\deg(T)\,\mathrm{Vol}_n(F, \|\cdot\|') = \mathrm{Vol}_n(T(E), \|\cdot\|') \le C^n\,\mathrm{Vol}_n(E, \|\cdot\|).$$

The case of equality follows from Proposition 2.1 as well. □

Suppose now that $\dim(V) = n$, $\dim(W) = m$, and $T : V \to W$ is a linear transformation of rank k such that $T(\Lambda) \subset \Gamma$. Let $\overline{V} = V/\ker(T)$ be the cokernel of T, and $\widehat{W} = T(V) \subset W$ be the image of V under T. Also, let $\bar{\Lambda} \subset \overline{V}$ be the quotient lattice of Λ, and $\hat{\Gamma} = \Gamma \cap \widehat{W}$ the rank-k sublattice of Γ. Then T induces a linear isomorphism

$$\overline{T} : \overline{V} \to \widehat{W}$$

such that $\overline{T}(\bar{\Lambda}) \subset \hat{\Gamma}$. Let $\|\cdot\|_{\mathsf{q}}$ denote the quotient norm on $\overline{V}$, and let $\|\cdot\|'_{\mathsf{r}}$ denote the restriction of $\|\cdot\|'$ to $\widehat{W}$.

Corollary 2.6. *With this notation, if $\|T(x)\|' \le C\,\|x\|$ for all $x \in V$, then*

$$\deg(\overline{T})\,\mathrm{Vol}_k(\widehat{W}/\hat{\Gamma}, \|\cdot\|'_{\mathsf{r}}) \le C^k\,\mathrm{Vol}_k(\overline{V}/\bar{\Lambda}, \|\cdot\|_{\mathsf{q}}). \tag{2-8}$$

Equality holds if and only if

$$\|\overline{T}(x)\|'_{\mathsf{r}} = C\,\|x\|_{\mathsf{q}}$$

for all $x \in \overline{V}$ where $\|\cdot\|'_{\mathsf{r}}$ denotes the restriction of $\|\cdot\|'$ to $\widehat{W}$.

Proof. By hypothesis and the definition of the quotient norm, $\|\overline{T}(x)\|' \leq C\,\|x\|_{\mathsf{q}}$ for all $x \in \overline{V}$. The result follows immediately from Proposition 2.5. □

The next lemma identifies the dual norm of a quotient norm.

Lemma 2.7. *Let $(V, \|\cdot\|)$ be a normed vector space. Let $(\overline{V}, \|\cdot\|_{\mathsf{q}})$ be a quotient space of V with quotient norm. Let $(V^*, \|\cdot\|^*)$ be the dual space of V with the dual norm. Then the dual space $\overline{V}^*$ of $\overline{V}$ can be identified with a subspace of V^* and the dual norm $\|\cdot\|_{\mathsf{q}}^*$ is equal to the restriction of $\|\cdot\|^*$ to $\overline{V}^*$.*

Proof. Let V_0 be the kernel of the quotient map $\mathsf{q} : V \to \overline{V}$. Clearly

$$\overline{V}^* = \{\lambda \in V^* : \lambda(V_0) = 0\}.$$

By definition $\|\bar{v}\|_{\mathsf{q}} = \inf\{\|v\| : v \in V, \mathsf{q}(v) = \bar{v}\}$ for every $\bar{v} \in \overline{V}$. Let $\lambda \in \overline{V}^*$. Then

$$\|\lambda\|_{\mathsf{q}}^* = \sup\{\lambda(\bar{v}) : \bar{v} \in \overline{V}, \|\bar{v}\|_{\mathsf{q}} \leq 1\}$$

and

$$\|\lambda\|^* = \sup\{\lambda(v) : v \in V, \|v\| \leq 1\}.$$

Now, there exists $v_0 \in V$ such that $\|v_0\| \leq 1$ and $\lambda(v_0) = \|\lambda\|^*$. Thus $\|\mathsf{q}(v_0)\|_{\mathsf{q}} \leq \|v_0\| \leq 1$ and $\lambda(\mathsf{q}(v_0)) = \lambda(v_0) = \|\lambda\|^*$ which implies $\|\lambda_{\mathsf{q}}\|^* \geq \|\lambda\|^*$. But there also exists $\bar{v}_0 \in \overline{V}$ such that $\|\bar{v}_0\|_{\mathsf{q}} \leq 1$ and $\lambda(\bar{v}_o) = \|\lambda\|_{\mathsf{q}}^*$. Thus there exists $v_0 \in V$ with $\mathsf{q}(v_0) = \bar{v}_0$ and $\|v_0\| = \|\bar{v}_0\|_{\mathsf{q}} \leq 1$. Hence $\lambda(v_0) = \lambda(\mathsf{q}(v_0)) = \lambda(\bar{v}_0) = \|\lambda\|_{\mathsf{q}}^*$ which implies $\|\lambda\|^* \geq \|\lambda\|_{\mathsf{q}}^*$. Therefore $\|\lambda\|^* = \|\lambda\|_{\mathsf{q}}^*$. □

3. Norms on homology and cohomology

Throughout this section (M, g) is a compact oriented Riemannian manifold of dimension n.

Mass and comass. Recall that the comass norm $\|\cdot\|^*$ of a cohomology class $\alpha \in H^p(M, \mathbb{R})$ is defined by

$$\|\alpha\|^* = \inf\{\text{comass}(\omega) : \omega \text{ a closed } p\text{-form representing } \alpha\},$$

where $\text{comass}(\omega) = \max\{\omega_x(e_1, \ldots, e_p) : x \in M, e_i \in T_xM, |e_i| = 1\}$, and that the stable mass norm $\|\cdot\|$ of a homology class $h \in H_p(M, \mathbb{R})$ is defined by

$$\|h\| = \inf\left\{\sum_i |r_i|\text{vol}_p(\sigma_i) : \sum_i r_i\sigma_i \text{ is a Lipschitz cycle representing } h\right\}.$$

It is well known that comass and mass are dual norms relative to the Kronecker pairing

$$\langle\cdot,\cdot\rangle : H_p(M, \mathbb{R}) \times H^p(M, \mathbb{R}) \to \mathbb{R}.$$

Thus for all $h \in H_p(M,\mathbb{R})$ and $\alpha \in H^p(M,\mathbb{R})$

$$\langle h, \alpha\rangle \le \|h\|\,\|\alpha\|^*. \tag{3-1}$$

Moreover, for all $\alpha \in H^p(M,\mathbb{R})$ and $\beta \in H^q(M,\mathbb{R})$,

$$\|\alpha \smile \beta\|^* \le C(n;\, p, q)\|\alpha\|^*\|\beta\|^*, \tag{3-2}$$

where $\smile$ denotes the cup product; see [7, 1.8.1]. (Note that $C(n;\, p, q) \le \binom{p+q}{p}$ and $C(n;\, p, q) = 1$ if $p \in \{0, 1, n-1, n\}$. We will see in Section 6 that $C(4;\, 2, 2) = 2$.)

Lemma 3.1. *If $h \in H_{p+q}(M,\mathbb{R})$ and $\alpha \in H^p(M,\mathbb{R})$, then*

$$\|h \frown \alpha\| \le C(n;\, p, q)\|h\|\,\|\alpha\|^*,$$

where $\frown$ denotes the cap product.

Proof. If $\beta \in H^q(M,\mathbb{R})$, then using (3-1),

$$\langle h \frown \alpha, \beta\rangle = \langle h, \alpha \smile \beta\rangle \le \|h\|\,\|\alpha \smile \beta\|^*.$$

Applying inequality (3-2) gives

$$\langle h \frown \alpha, \beta\rangle \le \|h\|\, C(n;\, p, q)\,\|\alpha\|^*\,\|\beta\|^*,$$

and taking the supremum for all β with $\|\beta\|^* \le 1$ gives the result since $\|\cdot\|$ is dual to $\|\cdot\|^*$. ☐

The L^2 norm. According to Hodge theory the cohomology classes in $H^p(M,\mathbb{R})$ can be represented by the harmonic p-forms on (M, g). Moreover $H^p(M,\mathbb{R})$ is endowed with an inner product which for two harmonic p-forms φ and ψ is given by

$$\langle\langle \varphi, \psi\rangle\rangle = \int_M \varphi \wedge \star\psi,$$

where $\star$ denotes the Hodge star operator. We will denote the corresponding Euclidean norm as $|\cdot|_2^*$.

We have need of the following proposition proved in [9, Corollary 3].

Proposition 3.2. *Let $h \in H_p(M,\mathbb{R})$ be the Poincaré dual of the cohomology class $\alpha \in H^{n-p}(M,\mathbb{R})$. Then*

$$\|h\| \le \mathrm{Vol}(M, g)^{\frac{1}{2}}\, C(n, p)\,|\alpha|_2^*, \tag{3-3}$$

where $C(n, p)$ is a constant depending only on n and p. Moreover, if equality holds then α can be represented by a harmonic p-form of constant norm.

Remark 3.3. Conversely, the proof of [9, Corollary 3] also shows that equality holds in (3-3) for $p \in \{1, n-1\}$ when α is represented by a harmonic p-form of constant norm. Note that if p equals 0, 1, $n-1$, or n, then $C(n, p) = 1$, and that in any case $C(n, p) \le \binom{n}{p}^{\frac{1}{2}}$ always.

4. Higher rank systoles

Let (M, g) be a compact oriented Riemannian manifold of dimension n, and let $\|\cdot\|$ denote the stable mass norm on $H_p(M, \mathbb{R})$ induced from g. The image of the p-th integral homology group $H_p(M, \mathbb{Z})$ in $H_p(M, \mathbb{R})$ is a lattice in $H_p(M, \mathbb{R})$ which will be denoted $H_p(M, \mathbb{Z})_\mathbb{R}$. The following definition generalizes the well-known p-dimensional stable systole $\operatorname{stsys}_p(M, g)$.

Definition 4.1. For any integer k between 1 and the p-th Betti number b_p of M, the p-dimensional stable systole of rank k is defined to be

$$\operatorname{stsys}_{p,k}(M, g) = \Delta_k(H_p(M, \mathbb{Z})_\mathbb{R}, \|\cdot\|)^{\frac{1}{k}}.$$

Clearly $\operatorname{stsys}_{p,1}(M, g) = \operatorname{stsys}_p(M, g) = \inf\{\|x\| : 0 \neq x \in H_p(M, \mathbb{Z})_\mathbb{R}\}$.

The dual lattice $H^p(M, \mathbb{Z})_\mathbb{R}$ in $H^p(M, \mathbb{R})$ can be identified with the set of cohomology classes of degree p with integral periods [12, Lemma 15.4.2]. Thus

$$H^p(M, \mathbb{Z})_\mathbb{R} = \{\alpha \in H^p(M, \mathbb{R}) : \langle x, \alpha\rangle \in \mathbb{Z} \;\; \forall x \in H_p(M, \mathbb{Z})_\mathbb{R}\}.$$

As is Section 3, $\|\cdot\|^*$ is the comass norm and $|\cdot|_2^*$ is the L^2 norm on $H^p(M, \mathbb{R})$.

In general, the existence of nonzero cap products give rise to inequalities involving higher rank stable systoles. The results that follow give examples of such inequalities under various hypotheses. Proposition 4.2 is used in the proofs of Corollaries 4.3 and 4.4 and Theorems 4.5 and 4.6.

Proposition 4.2. *Suppose $h \in H_{p+q}(M, \mathbb{Z})_\mathbb{R}$, and $h \frown : H^p(M, \mathbb{R}) \to H_q(M, \mathbb{R})$ has rank k. Then*

$$\deg(\overline{h \frown})^{\frac{1}{k}} \operatorname{stsys}_{q,k}(M, g) \operatorname{stsys}_{p,k}(M, g) \leq \frac{C(n; p, q)}{c} \|h\|,$$

where $C(n; p, q)$ is the constant in (3-2) *and c is the constant in Lemma 2.2*

Proof. Using Lemma 3.1, apply Corollary 2.6 to obtain

$$\deg(\overline{h \frown}) \operatorname{Vol}_k(\widehat{W}/\hat{\Gamma}, \|\cdot\|'_{\mathsf{r}}) \leq C(n; p, q)^k \|h\|^k \operatorname{Vol}_k(\bar{V}/\bar{\Lambda}, \|\cdot\|_{\mathsf{q}}), \tag{4-1}$$

where $\widehat{W} \subset H_q(M, \mathbb{R})$ is the image of $H^p(M, \mathbb{R})$ under $h \frown$, $\hat{\Gamma} = H_q(M, \mathbb{Z})_\mathbb{R} \cap \widehat{W}$, and $\bar{V}$ and $\bar{\Lambda}$ are the quotients of $H^p(M, \mathbb{R})$ and $H^p(M, \mathbb{Z})_\mathbb{R}$ by the kernel of $h \frown$. Multiply both sides of (4-1) by $\operatorname{Vol}_k(\bar{V}^*/\bar{\Lambda}^*, \|\cdot\|^*_{\mathsf{q}})$ and use Lemma 2.2 to obtain

$$\deg(\overline{h \frown}) \operatorname{Vol}_k(\widehat{W}/\hat{\Gamma}, \|\cdot\|'_{\mathsf{r}}) \operatorname{Vol}_k(\bar{V}^*/\bar{\Lambda}^*, \|\cdot\|^*_{\mathsf{q}}) \leq C(n; p, q)^k \|h\|^k \frac{1}{c^k}. \tag{4-2}$$

But $\bar{V}^*$ is a k-dimensional subspace of $H_p(M, \mathrm{R})$ and $\bar{\Lambda}^* = H_p(M, \mathrm{Z})_\mathrm{R} \cap \bar{V}^*$. On taking k-th roots and using the Definition 4.1 we obtain the stated inequality. $\square$

Here is a simple application. Let T^4 be the 4-dimensional torus. It is easily checked that $h \frown : H^1(T^4, \mathbb{R}) \to H_2(T^4, \mathbb{R})$ has rank 3 for every nonzero $h \in H_3(T^4, \mathbb{R})$. Thus for nonzero $h \in H_3(T^4, \mathbb{Z})_{\mathbb{R}}$ and any Riemannian metric g on T^4 we have by Proposition 4.2,

$$\operatorname{stsys}_{1,3}(T^4, g)\operatorname{stsys}_{2,3}(T^4, g) \le 2\,\|h\|$$

because $C(4; 1, 2) = 1$, $c \ge \frac{1}{2}$ and $\deg(\overline{h \frown}) \ge 1$. This leads to the following intersystolic inequality.

Corollary 4.3. *For every Riemannian metric g on T^4,*

$$\operatorname{stsys}_{1,3}(T^4, g)\operatorname{stsys}_{2,3}(T^4, g) \le 2\operatorname{stsys}_{3,1}(T^4, g).$$

By taking $h = [M] \in H_n(M, \mathbb{Z})$, the fundamental class of M, in Proposition 4.2 we obtain:

Corollary 4.4. *Let $0 < p < n$, and let b_p the p-th Betti number of M. Then*

$$\operatorname{stsys}_{p,b_p}(M, g)\operatorname{stsys}_{n-p,b_p}(M, g) \le \frac{C(n; p, n-p)}{c}\operatorname{Vol}(M, g).$$

Proof. Capping by $[M]$ is the Poincaré duality map which is a linear isomorphism of rank b_p from $H^p(M, \mathbb{R})$ to $H_{n-p}(M, \mathbb{R})$ with degree 1. Also observe that $\|[M]\| = \operatorname{Vol}(M, g)$. □

Theorem 4.5. *Suppose for every nonzero $h \in H_{p+q}(M, \mathbb{R})$ there exists an $\alpha \in H^p(M, \mathbb{R})$ such that $h \frown \alpha \ne 0$. Then*

$$\min_{1\le k\le b}\operatorname{stsys}_{q,k}(M, g)\operatorname{stsys}_{p,k}(M, g) \le \frac{C(n; p, q)}{c}\operatorname{stsys}_{p+q,1}(M, g),$$

where $b = \min(b_p, b_q)$.

Proof. Take $h \in H_{p+q}(M, \mathbb{Z})_{\mathbb{R}}$ with $\|h\| = \operatorname{stsys}_{p+q,1}(M, g)$. By assumption,

$$h \frown : H^p(M, \mathbb{R}) \to H_q(M, \mathbb{R})$$

has rank k for some $1 \le k \le b = \min(b_p, b_q)$, and $\deg(\overline{h \frown}) \ge 1$. We apply Proposition 4.2 to obtain the stated inequality. □

The next result shows that the existence of just one nonzero cap product places a bound on the stable systoles in appropriate dimensions.

Theorem 4.6. *Suppose there exist $h \in H_{p+q}(M, \mathbb{R})$ and $\alpha \in H^p(M, \mathbb{R})$ such that $h \frown \alpha \ne 0$. Then*

$$\min_{1\le k\le b}\operatorname{stsys}_{q,k}(M, g)\operatorname{stsys}_{p,k}(M, g) \le \frac{C(n; p, q)}{c}\,\lambda_{b_{p+q}},$$

where $b = \min(b_p, b_q)$ and $\lambda_{b_{p+q}}$ is the b_{p+q}-th successive minimum of the lattice $(H_{p+q}(M, \mathbb{Z})_{\mathbb{R}}, \|\cdot\|)$.

Proof. There is a basis $h_1, \dots, h_{b_{p+q}}$ of $H_{p+q}(M, \mathbb{R})$ consisting of the successive minimums of $(H_{p+q}(M, \mathbb{Z})_{\mathbb{R}}, \|\cdot\|)$. Write $h = t_1 h_1 + \cdots + t_\beta h_\beta$. Since

$$h \frown \alpha = t_1 h_1 \frown \alpha + \cdots + t_\beta h_\beta \frown \alpha \neq 0,$$

there is at least one successive minimum $h_i \in H_{p+q}(M, \mathbb{Z})_{\mathbb{R}}$ with $h_i \frown \alpha \neq 0$. Thus capping by h_i has rank k for some $1 \leq k \leq b$ and $\deg(\overline{h_i \frown}) \geq 1$. By definition $\|h_i\| = \lambda_i \leq \lambda_{b_{p+q}}$. The result now follows by applying Proposition 4.2 □

Proposition 4.7. *Let $\alpha \in H^p(M, \mathbb{Z})_{\mathbb{R}}$. Suppose that $\frown \alpha : H_{p+q}(M, \mathbb{R}) \to H_q(M, \mathbb{R})$ is injective. Then*

$$\deg(\overline{\frown \alpha})^{\frac{1}{b_{p+q}}} \operatorname{stsys}_{q, b_{p+q}}(M, g) \leq C(n; p, q) \operatorname{stsys}_{p+q, b_{p+q}}(M, g) \|\alpha\|^*, \tag{4-3}$$

where b_{p+q} is the $(p+q)$-th Betti number.

Proof. By hypothesis the rank of $T = \frown \alpha$ is b_{p+q}. Using Lemma 3.1 and the injectivity of $T = \frown \alpha$, apply Corollary 2.6 to obtain

$$\deg(\overline{\frown \alpha}) \operatorname{Vol}_{b_{p+q}}(\widehat{W}/\hat{\Gamma}, \|\cdot\|') \leq (C(n; p, q)\|\alpha\|^*)^{b_{p+q}} \operatorname{Vol}_{b_{p+q}}(V/\Lambda, \|\cdot\|), \tag{4-4}$$

where $\widehat{W} \subset H_q(M, \mathbb{R})$ is the image of $H_{p+q}(M, \mathbb{R})$ under $\frown \alpha$, $\hat{\Gamma} = H_q(M, \mathbb{Z})_{\mathbb{R}} \cap \widehat{W}$, $V = H_{p+q}(M, \mathbb{R})$, and $\Lambda = H_{p+q}(M, \mathbb{Z})_{\mathbb{R}}$. Since $\widehat{W}$ has dimension b_{p+q} we have

$$\begin{aligned} &\deg(\overline{\frown \alpha}) \Delta_{b_{p+q}}(H_q(M, \mathbb{Z})_{\mathbb{R}}, \|\cdot\|) \\ &\qquad \leq (C(n; p, q)\|\alpha\|^*)^{b_{p+q}} \Delta_{b_{p+q}}(H_{p+q}(M, \mathbb{Z})_{\mathbb{R}}, \|\cdot\|). \end{aligned} \tag{4-5}$$

Finally (4-3) follows from (4-5) by taking b_{p+q}-th roots. □

The following lemma is needed to show, in Proposition 4.9, that the product of two stable systoles of the same rank in complementary dimensions is bounded from above in terms of the volume of the manifold.

Lemma 4.8. *Let $0 < p < n$, let b_p be the p-th Betti number of M, and let $1 \leq k \leq b_p$. Then*

$$\operatorname{stsys}_{p,k}(M, g) \leq C(n, p) \operatorname{Vol}(M, g)^{\frac{1}{2}} \Delta_k(H^{n-p}(M, \mathbb{Z})_{\mathrm{R}}, |\cdot|_2^*)^{\frac{1}{k}},$$

where $C(n, p)$ is the constant in Proposition 3.2.

Proof. Let Λ be a sublattice of $H^{n-p}(M, \mathbb{Z})_{\mathbb{R}}$ of rank k in a k-dimensional subspace $V \subset H^{n-p}(M, \mathbb{R})$. Then Poincaré duality maps Λ onto a sublattice Γ of $H_p(M, \mathbb{Z})_{\mathbb{R}}$ of rank k in a k-dimensional subspace $W \subset H_p(M, \mathbb{R})$ (because Poincaré duality is an isomorphism). Thus Poincaré duality is of rank k and degree 1. Applying Propositions 2.5 and 3.2, we obtain

$$\operatorname{Vol}_k(W/\Gamma, \|\cdot\|) \leq (\operatorname{Vol}(M, g))^{\frac{k}{2}} C(n, p)^k \operatorname{Vol}_k(V/\Lambda, |\cdot|_2^*). \tag{4-6}$$

The result follows by taking the infima over all rank-k sublattices and k-th roots. □

Proposition 4.9. *Let* $0 < p < n$, *and let* b_p *be the* p*-th Betti number of* M *and* $1 \le k \le b_p$. *Then*

$$\operatorname{stsys}_{p,k}(M, g)\operatorname{stsys}_{n-p,k}(M, g) \le C(n, p)^2 \operatorname{Vol}(M, g)(\gamma'_{b_p,k})^{\frac{1}{k}}.$$

Proof. Multiply the inequalities of Lemma 4.8 with p equal to p and to $n-p$. Observe that $C(n, p) = C(n, n-p)$, and that, because $H^p(M, \mathbb{Z})_{\mathbb{R}}$ and $H^{n-p}(M, \mathbb{Z})_{\mathbb{R}}$ are dual lattices,

$$\Delta_k(H^p(M, \mathbb{Z})_{\mathbb{R}}, |\cdot|_2^*)\,\Delta_k(H^{n-p}(M, \mathbb{Z})_{\mathbb{R}}, |\cdot|_2^*) \le \gamma'_{b_p,k}. \qquad \square$$

5. A sharp inequality in dimensions 1 and $n-1$

The two theorems in this section are analogs of the main theorem in [1].

Theorem 5.1. *Let* (M, g) *be a compact oriented manifold of dimension* n *whose first Betti number is* b. *Then*

$$\operatorname{stsys}_{1,b}(M, g)\operatorname{stsys}_{n-1,b}(M, g) \le \operatorname{Vol}(M, g). \tag{5-1}$$

Equality holds in (5-1) *if and only if there exists a Riemannian submersion of* M *onto a flat torus of dimension* b *with connected minimal fibers.*

Proof. The inequality (5-1) follows from Proposition 4.9 because $C(n, 1) = 1$ and $\gamma'_{b,b} = 1$.

Suppose now that equality holds in (5-1). Then by the proof of Proposition 4.9, equality holds in Lemma 4.8. Thus inequality (4-6) is an equality for $p \in \{1, n-1\}$ and $k = b$, that is,

$$\operatorname{Vol}_b(H_p(M, \mathbb{R})/H_p(M, \mathbb{Z})_{\mathbb{R}}, \|\cdot\|) = \operatorname{Vol}(M, g)^{\frac{b}{2}} \operatorname{Vol}_b(H^{n-p}(M, \mathbb{R})/H^{n-p}(M, \mathbb{Z})_{\mathbb{R}}, |\cdot|_2^*)$$

for $p \in \{1, n-1\}$. Consequently, by Propositions 2.5 and 3.2,

$$\|[M] \frown \alpha\| = \operatorname{Vol}(M, g)^{\frac{1}{2}} |\alpha|_2^*$$

for all $\alpha \in H^p(M, \mathbb{R})$ with $p \in \{1, n-1\}$. Hence by Proposition 3.2, the first (and $(n-1)$-st) degree cohomology classes of M with integral periods are represented by harmonic 1-forms (and $(n-1)$-forms) of constant norm. Applying [12, Proposition 16.7.3], there exists a Riemannian submersion of M onto a flat torus of dimension b with minimal fibers. In fact the submersion is the Abel–Jacobi map using a basis of harmonic 1-forms from $H^1(M, \mathbb{Z})_{\mathbb{R}}$ which induces an epimorphism on the fundamental groups. Thus the fibers are connected.

Conversely, if there exists a Riemannian submersion of M onto a flat torus of dimension b with connected minimal fibers, then each step in this argument is reversible with equality holding at every step. Therefore equality holds in (5-1). $\square$

This result can be generalized to stable systoles of rank k, $1 \leq k \leq b$.

Theorem 5.2. *Let (M, g) be a compact oriented manifold of dimension n whose first Betti number is b. Then, for each $1 \leq k \leq b$,*

$$\operatorname{stsys}_{1,k}(M, g) \operatorname{stsys}_{n-1,k}(M, g) \leq (\gamma'_{b,k})^{\frac{1}{k}} \operatorname{Vol}(M, g). \tag{5-2}$$

Equality holds if and only if there exists a Riemannian submersion with connected minimal fibers from M onto a flat b-dimensional torus $\mathbb{R}^b/\Lambda$ such that Λ is dual k-critical.

Proof. The inequality (5-2) follows from Proposition 4.9 because $C(n, 1) = 1$.

Suppose that equality holds in (5-2). Since $p = 1$, the proof of Proposition 4.9 implies

$$\Delta_k(H^1(M, \mathbb{Z})_{\mathbb{R}}, |\cdot|_2^*)\, \Delta_k(H^{n-1}(M, \mathbb{Z})_{\mathbb{R}}, |\cdot|_2^*) = \gamma'_{b,k}.$$

This means that the lattice $\Lambda = H^1(M, \mathbb{Z})_{\mathbb{R}}$ is dual k-critical (Definition A.1), and thus, by Lemma A.4, it is dual k-perfect (Definition A.3). Observe that the dual lattice $H^{n-1}(M, \mathbb{Z})_{\mathbb{R}}$ can be identified under the Hodge star operator with the lattice

$$\Lambda^* = \{\varphi \in H^1(M, \mathbb{R}) : \langle\langle \varphi, \psi \rangle\rangle \in \mathbb{Z}, \forall \psi \in \Lambda\}.$$

Let Q denote the vector space of quadratic forms on $H^1(M, \mathbb{R})$. That Λ is dual k-perfect implies that Q^* is generated by the linear functionals of the form $q \mapsto q(\alpha)$ where, in the notation of the Appendix, $\alpha \in W \in S(\Lambda) \cup S(\Lambda^*)$. We next need to prove that every $\alpha \in W \in S(\Lambda) \cup S(\Lambda^*)$ can be represented by a harmonic 1-form of constant norm. For then arguing as in [12, Remark 16.11.6], dual k-perfection implies that every harmonic 1-form on M has constant norm. This reduces us to the situation in Theorem 5.1, so that by [12, Proposition 16.7.3], the Abel–Jacobi map defines a Riemannian submersion with connected minimal fibers of M onto $\mathbb{R}^b/\Lambda$.

Let $V \in S(\Lambda)$ and set $\Lambda' = V \cap \Lambda$. Thus $\operatorname{Vol}_k(V/\Lambda', |\cdot|_2^*) = \Delta_k(\Lambda)$. The Poincaré duality map

$$T = [M] \frown : V \to H_{n-1}(M, \mathbb{R})$$

restricted to V has rank k and degree 1. Set $W = T(V)$ and $\Gamma = W \cap H_{n-1}(M, \mathbb{Z})_{\mathbb{R}}$. Since we are assuming equality in (5-2), equality holds in Proposition 4.9 which implies that equality holds in Lemma 4.8. Thus equality holds in (4-6) with $p = 1$, that is,

$$\operatorname{Vol}_k(W/\Gamma, \|\cdot\|) \leq (\operatorname{Vol}(M, g))^{\frac{k}{2}} \operatorname{Vol}_k(V/\Lambda', |\cdot|_2^*).$$

Hence by Propositions 2.5 and 3.2,

$$\|T(\alpha)\| = \operatorname{Vol}(M, g)^{\frac{1}{2}} |\alpha|_2^*$$

for all $\alpha \in V$, and thus by Proposition 3.2, every such α is represented by a harmonic 1-form of constant norm. A similar argument shows that if $V \in S(\Lambda^*)$, then every $\alpha \in V$ also can be represented by a harmonic $(n-1)$-form of constant norm, so that its Hodge star $\star\alpha \in \Lambda^*$ is represented by a harmonic 1-form of constant norm.

Conversely, if there exists a Riemannian submersion with connected minimal fibers from M onto a flat b-dimensional torus $\mathbb{R}^b/\Lambda$ such that Λ is dual k-critical, the steps of the above argument are reversible with equality holding at each step so that equality holds in (5-2). □

6. Example

Unless the manifold is nice enough, computing a stable systole of higher rank for a general Riemannian manifold is a difficult task. The purpose of this section is to illustrate the computation of a stable systole of higher rank in a case where the manifold is simple and nice enough to effect such a computation. In particular we compute the two dimensional stable systole of rank 6 in flat 4-dimensional tori. As a consequence we obtain a sharp stable systolic inequality for conformally flat 4-dimensional tori (Theorem 6.7).

Here we will consider a 4-dimensional flat torus $M = \mathbb{R}^4/\Lambda$ where Λ is a lattice in $\mathbb{R}^4$. According to [14] there are natural isomorphisms in cohomology

$$H^*(M, \mathbb{R}) \cong \Lambda^*(\mathbb{R}^4) \tag{6-1}$$

and

$$H^*(M, \mathbb{Z})_{\mathbb{R}} \cong \Lambda^*_{\mathbb{Z}}(\Lambda^*), \tag{6-2}$$

as well as in homology

$$H_*(M, \mathbb{R}) \cong \Lambda^*(\mathbb{R}^4) \tag{6-3}$$

and

$$H_*(M, \mathbb{Z})_{\mathbb{R}} \cong \Lambda^*_{\mathbb{Z}}(\Lambda). \tag{6-4}$$

Lemma 6.1. *The mass norm for $\xi \in \Lambda^2(\mathbb{R}^4)$ is given by*

$$\|\xi\| = (|\xi|^2 + |\xi \wedge \xi|)^{\frac{1}{2}}, \tag{6-5}$$

where $|\cdot|$ is the norm on the exterior algebra $\Lambda^*(\mathbb{R}^4)$ induced from the Euclidean norm of $\mathbb{R}^4$.

Proof. Under an orthogonal change of coordinates in $\mathbb{R}^4$ any given ξ can be put in the form

$$\xi = Ae_1 \wedge e_2 + Be_3 \wedge e_4.$$

Whitney [17, equation (13), p. 54] has proved that for such ξ, $\|\xi\| = |A| + |B|$. Thus

$$\|\xi\|^2 = |A|^2 + |B|^2 + 2|A||B| = |\xi|^2 + |\xi \wedge \xi|.$$

This completes the proof because the expression (6-5) is invariant under orthogonal changes of coordinates. □

Lemma 6.2. *The comass norm for* $\phi \in \Lambda^2(\mathbb{R}^4)$ *is given by*

$$\|\phi\|^* = \left(\frac{|\phi|^2 + \sqrt{|\phi|^4 - |\phi \wedge \phi|^2}}{2}\right)^{\frac{1}{2}}. \tag{6-6}$$

Thus

$$\|\phi\|^* \le |\phi| \le \sqrt{2}\, \|\phi\|^*. \tag{6-7}$$

In particular the constant $C(4, 2) = \sqrt{2}$.

Proof. By the invariance of the expression (6-6) under orthogonal changes of coordinates in $\mathbb{R}^4$, it suffices to consider the case $\phi = Ae_1 \wedge e_2 + Be_3 \wedge e_4$. In this case the left-hand side of (6-6) is $\|\phi\|^* = \max(|A|, |B|)$ according to [17, equation (12), p. 54]. The right-hand side of (6-6) becomes

$$\left(\frac{A^2 + B^2 + \sqrt{(A^2 + B^2)^2 - (2AB)^2}}{2}\right)^{\frac{1}{2}} = \left(\frac{A^2 + B^2 + |A^2 - B^2|}{2}\right)^{\frac{1}{2}},$$

which is equal to $\max(|A|, |B|)$. The inequality (6-7) follows easily from (6-6). □

Lemma 6.3. *Let* K *be the unit mass ball of* $\Lambda^2(\mathbb{R}^4)$. *Then*

$$\mathrm{Vol}_6(K, |\cdot|) = \frac{2\pi^2}{9}.$$

Proof. Setting

$$\xi = x_1 e_2 \wedge e_3 - x_2 e_1 \wedge e_3 + x_3 e_1 \wedge e_2 + y_1 e_1 \wedge e_4 + y_2 e_2 \wedge e_4 + y_3 e_3 \wedge e_4$$

gives an isomorphism between $\Lambda^2(\mathbb{R}^4)$ and $\mathbb{R}^3 \times \mathbb{R}^3 = \mathbb{R}^6$ with the Euclidean norms. If $dV_{\mathbb{R}^6}$ and dV_{S^5} are the volume elements in $\mathbb{R}^6$ and S^5, respectively, we have, on switching to spherical coordinates,

$$\mathrm{Vol}_6(K, |\cdot|) = \int_{\|\xi\| \le 1} dV_{\mathbb{R}^6} = \int_{\xi \in S^5} \int_0^{\frac{1}{\|\xi\|}} r^5\, dr\, dV_{S^5} = \frac{1}{6} \int_{\xi \in S^5} \frac{1}{\|\xi\|^6}\, dV_{S^5}.$$

The mapping $S^2 \times \left[0, \frac{\pi}{2}\right] \times S^2 \to S^5$ given by sending the ordered triplet (X, t, Y) to $\xi = (\cos(t)X, \sin(t)Y) \in \mathbb{R}^3 \times \mathbb{R}^3$ reparameterizes S^5 except on a set of measure 0. Making this change of variables in the integral gives us

$$\mathrm{Vol}_6(K, |\cdot|) = \frac{1}{6} \int_{Y \in S^2} \int_0^{\frac{\pi}{2}} \int_{X \in S^2} \frac{\cos^2(t) \sin^2(t)}{(1 + 2\cos(t)\sin(t)|X \cdot Y|)^3}\, dV_{S^2}\, dt\, dV_{S^2}.$$

By the invariance of the inner product under isometries of S^2, the value of the inner double integral is independent of $Y \in S^2$. Because the area of S^2 is 4π, integrating over Y gives the value

$$\mathrm{Vol}_6(K, |\cdot|) = \frac{4\pi}{6} \int_0^{\frac{\pi}{2}} \int_{X \in S^2} \frac{\cos^2(t)\sin^2(t)}{(1 + 2\cos(t)\sin(t)|X \cdot N|)^3}\, dV_{S^2}\, dt,$$

where

$$N = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

is the north pole of S^2. Now change to spherical coordinates (ϕ, θ) with $0 \le \theta \le 2\pi$, $0 \le \phi \le \pi$, on S^2. Since $X \cdot N = \cos(\phi)$ we obtain

$$\mathrm{Vol}_6(K, |\cdot|) = \frac{4\pi}{6} \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^{\pi} \frac{\cos^2(t)\sin^2(t)}{(1 + 2\cos(t)\sin(t)|\cos(\phi)|)^3} \sin(\phi)\, d\phi\, d\theta\, dt.$$

Using the double angle formula for $\sin(2t)$, the symmetry of the integrand in ϕ about $\frac{\pi}{2}$, and the independence of the integrand in θ we obtain

$$\mathrm{Vol}_6(K, |\cdot|) = \frac{4\pi^2}{6} \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{\sin^2(2t)}{(1 + \sin(2t)\cos(\phi))^3} \sin(\phi)\, d\phi\, dt.$$

This can be easily evaluated by iterated integration to obtain

$$\mathrm{Vol}_6(K, |\cdot|) = \frac{\pi^2}{3} \int_0^{\frac{\pi}{2}} \sin(2t) - \frac{\sin(2t)}{(1 + \sin(2t))^2}\, dt = \frac{2\pi^2}{9}.$$

□

Corollary 6.4. *The Hausdorff–Busemann measure in* $\Lambda^2(\mathrm{R}^4)$ *is given by*

$$\mathrm{Vol}_6(-, \|\cdot\|) = \frac{3\pi}{4} \mathrm{Vol}_6(-, |\cdot|).$$

Proof. Let K be the unit mass ball in $\Lambda^2(\mathbb{R}^4)$. By definition of the Hausdorff-Busemann measure and Lemma 6.3,

$$\mathrm{Vol}_6(-, \|\cdot\|) = \frac{\omega_6}{\mathrm{Vol}_6(K, \|\cdot\|)} \mathrm{Vol}_6(-, |\cdot|) = \frac{\pi^3/3!}{2\pi^2/9} \mathrm{Vol}_6(-, |\cdot|). \qquad \square$$

Lemma 6.5.

$$\mathrm{Vol}_6(\Lambda^2(\mathbb{R}^4)/\Lambda^2_{\mathbb{Z}}(\Lambda), |\cdot|) = \mathrm{Vol}_4(\mathbb{R}^4/\Lambda, |\cdot|)^3.$$

Proof. Let $v_1, v_2, v_3, v_4 \in \mathbb{R}^4$ be a set of generators for Λ over $\mathbb{Z}$. Then

$$\mathrm{Vol}_4(\mathbb{R}^4/\Lambda, |\cdot|) = |\det(v_1, v_2, v_3, v_4)|,$$

and

$$v_1 \wedge v_2, \quad v_1 \wedge v_3, \quad v_1 \wedge v_4, \quad v_2 \wedge v_3, \quad v_2 \wedge v_4, \quad v_3 \wedge v_4$$

is a set of generators for $\Lambda^2(\Lambda)$ over $\mathbb{Z}$. Recall that $\det(v_1, v_2, v_3, v_4)$ may be computed by keeping track of the sequence of elementary row operations that convert v_1, v_2, v_3, v_4 to the standard basis e_1, e_2, e_3, e_4. An elementary row operation either (i) adds one of the vectors to another, (ii) interchanges two vectors, or (iii) factors out a constant multiple c from one of the vectors. If v_1', v_2', v_3', v_4' is the result of applying an elementary row operation to v_1, v_2, v_3, v_4 we have $\det(v_1', v_2', v_3', v_4') = \det(v_1, v_2, v_3, v_4)$ in case (i), $-\det(v_1', v_2', v_3', v_4') = \det(v_1, v_2, v_3, v_4)$ in case (ii), and $c\det(v_1', v_2', v_3', v_4') = \det(v_1, v_2, v_3, v_4)$ in case (iii). Since $\det(e_1, e_2, e_3, e_4) = 1$, we see that $\det(v_1, v_2, v_3, v_4)$ is equal to the product of the constants c that we factored out in operations of type (ii) times ± 1 depending on whether there were an even or odd number of operations of type (ii). Now consider what happens to the generators of $\Lambda^2(\Lambda)$ under this sequence of operations. The operation that takes v_1, v_2, v_3, v_4 to v_1', v_2', v_3', v_4' will correspondingly take

$$v_1 \wedge v_2, \quad v_1 \wedge v_3, \quad v_1 \wedge v_4, \quad v_2 \wedge v_3, \quad v_2 \wedge v_4, \quad v_3 \wedge v_4$$

to

$$v_1' \wedge v_2', \quad v_1' \wedge v_3', \quad v_1' \wedge v_4', \quad v_2' \wedge v_3', \quad v_2' \wedge v_4', \quad v_3' \wedge v_4'.$$

If the operation is of type (i), the corresponding operation has the same effect as two operations of type (i). If the operation is of type (ii), the corresponding operation has the same effect as two operations of type (ii) and multiplying one vector by -1. If the operation is of type (iii), then the corresponding operation has the same effect as factoring out the same constant c from three of the vectors. As the result of the sequence of corresponding operations is the orthonormal basis

$$e_1 \wedge e_2, \quad e_1 \wedge e_3, \quad e_1 \wedge e_4, \quad e_2 \wedge e_3, \quad e_2 \wedge e_4, \quad e_3 \wedge e_4,$$

the result follows because the sequence of elementary row operations gives a power of 3 times those computing the determinant of v_1, v_2, v_3, v_4. □

Theorem 6.6. *Let (M, g) be a 4-dimensional flat torus $\mathbb{R}^4/\Lambda$. Then*

$$\operatorname{stsys}_{2,6}(M, g)^2 = \left(\frac{3\pi}{4}\right)^{\frac{1}{3}} \operatorname{Vol}(M, g)$$

Proof. By Corollary 6.4 and Lemma 6.5,

$$\operatorname{Vol}_6(\Lambda^2(\mathbb{R}^4)/\Lambda^2_{\mathbb{Z}}(\Lambda), \|\cdot\|) = \frac{3\pi}{4} \operatorname{Vol}_6(\Lambda^2(\mathbb{R}^4)/\Lambda^2_{\mathbb{Z}}(\Lambda), |\cdot|) = \frac{3\pi}{4} \operatorname{Vol}_4(\mathbb{R}^4/\Lambda, |\cdot|)^3. \quad \square$$

The paper [16] proved systolic inequalities for metrics on real projective spaces which are conformal to the constant curvature metric. Similar ideas combined with Theorem 6.6 lead to the following result about conformally flat metrics on 4-dimensional tori.

Theorem 6.7. *Let (M, g) be a conformally flat 4-dimensional torus. Then*

$$\operatorname{stsys}_{2,6}(M, g)^2 \le \left(\frac{3\pi}{4}\right)^{\frac{1}{3}} \operatorname{Vol}(M, g).$$

Equality holds if and only if (M, g) is flat.

Proof. Since (M, g) is conformally flat, we may assume that there exists a lattice Λ in $\mathbb{R}^4$ such that $M = \mathbb{R}^4/\Lambda$ and that $g = f^2 g_0$ for some positive real-valued function f on M where g_0 is the flat metric on M. Let G be the group of isometries of the flat metric g_0 with Haar measure da normalized so that $\int_{a\in G} da = 1$. Set $\bar f(x) = \left(\int_{a\in G} f(ax)^2\, da\right)^{\frac{1}{2}}$. Since G acts transitively on M, $\bar f$ would be a constant function. Set $\bar g = \bar f^2 g_0$. Then $(M, \bar g)$ is flat. Thus if dx is the volume form for (M, g_0),

$$\begin{aligned}
\text{(6-8)}\qquad \operatorname{Vol}(M, \bar g) &= \int_{x\in M} \bar f(x)^4\, dx = \int_{x\in M}\left(\int_{a\in G} f(ax)^2\, da\right)^2 dx \\
&\le \int_{x\in M}\int_{a\in G} f(ax)^4\, da\, dx \\
&= \int_{a\in G}\int_{x\in M} f(ax)^4\, dx\, da = \int_{a\in G}\int_{x\in M} f(ax)^4\, dx\, da \\
&= \int_{a\in G} \operatorname{Vol}(M, g)\, da = \operatorname{Vol}(M, g),
\end{aligned}$$

where we have used successively Jensen's inequality, Fubini's theorem, the change of variables formula, that a is an isometry of g_0, and that G has unit measure. Note on account of Jensen's inequality, if equality holds then f is a constant function, and thus (M, g) would be flat.

On the other hand, given a homology class $h \in H_2(M, R)$ taking a 2-chain S in M representing h that gives the least mass (area) $\|h\|_{\bar g}$ in the homology class, one has

$$\|h\|_{\bar g} = \operatorname{Area}(S, \bar g) = \int_{a\in G} \operatorname{Area}(aS, g) \ge \|h\|_g.$$

As an explanation, suppose that S is a surface and $j : S \to M$ is the inclusion mapping. Then $j^* g_0$ induces an area form ds on S. Thus $j^* g$ induces the area form $(f \circ j)^2 ds$ and $j^*\bar g$ induces $(\bar f \circ j)^2 ds$. Thus

$$\begin{aligned}
\text{(6-9)}\qquad \operatorname{Area}(S, \bar g) &= \int_{s\in S} (\bar f \circ j)^2\, ds = \int_{s\in S}\int_{a\in G} f(aj(s))^2\, da\, ds \\
&= \int_{a\in G}\int_{s\in S} f(aj(s))^2\, ds\, da = \int_{a\in G} \operatorname{Area}(aS, g)\, da.
\end{aligned}$$

Therefore by Proposition 2.1

$$\text{(6-10)}\quad \operatorname{Vol}_6(H_2(M, \mathbb{R})/H_2(M, \mathbb{Z}), \|\cdot\|_g) \le \operatorname{Vol}_6(H_2(M, \mathbb{R})/H_2(M, \mathbb{Z}), \|\cdot\|_{\bar g}).$$

Thus, since $(M, \bar g)$ is flat,

$$\operatorname{stsys}_{2,6}(M, g) \le \operatorname{stsys}_{2,6}(M, \bar g) = \left(\frac{3\pi}{4}\right)^{\frac{1}{3}} \operatorname{Vol}(M, \bar g) \le \left(\frac{3\pi}{4}\right)^{\frac{1}{3}} \operatorname{Vol}(M, g).$$

If equality holds, then f must be constant and thus (M, g) would be flat. ☐

It is an open question whether this inequality also holds for metrics on 4-dimensional tori which are not conformally flat.

Theorem 6.6 gives information about the conformal volume norm of flat 4-dimensional tori. The conformal volume norm is an invariant of a conformal class of Riemannian metrics; see [8, 7.4; 12, 15.8]. When M is a 4-dimensional manifold, $h \in H_2(M, \mathbb{R})$, and $\mathcal{G}$ is a conformal class of Riemannian metrics on M, the conformal volume norm satisfies

$$\|h\|_{L^2} = \sup\left\{ \frac{\|h\|_g}{\sqrt{\mathrm{Vol}(M, g)}} : g \in \mathcal{G} \right\},$$

where $\|h\|_g$ is the stable mass norm for the Riemannian metric g. Thus for any $g \in \mathcal{G}$ and $h \in H_2(M, \mathbb{R})$, one has

$$\|h\|_g \le \sqrt{\mathrm{Vol}(M, g)}\, \|h\|_{L^2}.$$

Corollary 6.8. *Let (M, g) be a 4-dimensional flat torus $\mathbb{R}^4/\Lambda$. Then*

$$\left(\frac{3\pi}{4}\right)^{\frac{1}{6}} \le \mathrm{confsys}_{2,6}(M, g).$$

Proof. Applying Proposition 2.5, the Hausdorff–Busemann volumes satisfy

$$\mathrm{Vol}_6(\Lambda^2(\mathbb{R}^4)/\Lambda^2_{\mathbb{Z}}(\Lambda), \|\cdot\|) \le \mathrm{Vol}(\mathbb{R}^4/\Lambda, g)^3\, \mathrm{Vol}_6(\Lambda^2(\mathbb{R}^4)/\Lambda^2_{\mathbb{Z}}(\Lambda), \|\cdot\|_{L^2}).$$

Dividing by $\mathrm{Vol}(\mathbb{R}^4/\Lambda, g)^3$, extracting 6-th roots, and using Theorem 6.6 gives

$$\left(\frac{3\pi}{4}\right)^{\frac{1}{6}} = \frac{\mathrm{stsys}_{2,6}(M, g)}{\mathrm{Vol}(M, g)^{\frac{1}{2}}} \le \mathrm{Vol}_6(\Lambda^2(\mathbb{R}^4)/\Lambda^2_{\mathbb{Z}}(\Lambda), \|\cdot\|_{L^2})^{\frac{1}{6}},$$

where the right side of the inequality is by the definition of $\mathrm{confsys}_{2,6}(M, g)$. ☐

Appendix

The following proof that a dual k-extreme lattice is dual k-perfect is a modification of the argument in [2] that a dual extreme lattice is dual perfect.

Let V be a Euclidean space of dimension n with Euclidean norm $|\cdot|$, and let Λ be a lattice in V. Define $S_k(\Lambda)$ to be the collection of all k-dimensional subspaces W of V for which $\Gamma = \Lambda \cap W$ is a rank-k sublattice of Λ such that $\mathrm{Vol}_k(W/\Gamma, |\cdot|) = \Delta_k(\Lambda)$.

Definition A.1. A lattice Λ of rank k is dual k-extreme if it is a local maximum of the function

$$\Lambda \mapsto \Delta_k(\Lambda)\, \Delta_k(\Lambda^*)$$

and is dual k-critical if it is an absolute maximum, that is, if

$$\Delta_k(\Lambda)\, \Delta_k(\Lambda^*) = \gamma'_{n,k}.$$

Lemma A.2. *Suppose there exists a hyperspace H of V such that $W \subset H$ for all $W \in S_k(\Lambda)$. Then there exists a lattice M near Λ such that $\Delta_k(\Lambda)\,\Delta_k(\Lambda^*) < \Delta_k(\mathsf{M})\,\Delta_k(\mathsf{M}^*)$. In other words, Λ is not dual k-extreme.*

Proof. Let $0 < r < 1$, and consider the linear transformation $\tau : V \to V$ such that τ is the identity on H and contracts by a factor of r on $H^\perp$. As $W \subset H$ for all $W \in S(\Lambda)$, by continuity, if r is chosen close enough to 1, then $S_k(\tau(\Lambda)) = S_k(\Lambda)$ and $\Delta_k(\Lambda) = \Delta_k(\tau(\Lambda))$. Note that the adjoint map $(\tau^\dagger)^{-1}$ is the identity on H and expands by a factor of $\frac{1}{r} > 1$ on $H^\perp$ and that $(\tau\Lambda)^* = (\tau^\dagger)^{-1}\Lambda^*$. Thus if W is a k-dimensional subspace such that $\Gamma^* = \Lambda^* \cap W$ is a rank-k sublattice of Λ^* which is not contained in H, then

$$\mathrm{Vol}_k((\tau^\dagger)^{-1}(W)/(\tau^\dagger)^{-1}(\Gamma^*), |\cdot|) > \mathrm{Vol}_k(W/\Gamma^*, |\cdot|). \tag{A-1}$$

Thus if no $W \in S_k(\Lambda^*)$ is contained in H, we have $\Delta_k((\tau\Lambda)^*) > \Delta_k(\Lambda^*)$ and we may take $\mathsf{M} = \tau(\Lambda)$. However if $W \subset H$ for some $W \in S_k(\Lambda^*)$, then on account of our hypothesis and (A-1), $W \subset H$ for all $W \in S_k(\tau\Lambda) \cup S_k((\tau\Lambda)^*)$ and

$$\Delta_k(\tau(\Lambda))\,\Delta_k((\tau\Lambda)^*) = \Delta_k(\Lambda)\,\Delta_k(\Lambda^*).$$

Now proceed by taking a hyperplane F in H which contains no $W \in S_k((\tau\Lambda)^*)$. We have the orthogonal decomposition

$$V = H^\perp \oplus H = H^\perp \oplus F^\perp \oplus F,$$

where $F^\perp$ is the orthogonal complement to F in H. Consider a linear transformation $\sigma : V \to V$ whose matrix, relative to an orthonormal basis which respects this orthogonal decomposition, can be written in the block form

$$\sigma = \begin{pmatrix} q & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & \mathrm{Id} \end{pmatrix},$$

where $q > 1$ and $a \neq 0$. Then

$$(\sigma^\dagger)^{-1} = \begin{pmatrix} \frac{1}{q} & -\frac{a}{q} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mathrm{Id} \end{pmatrix}.$$

Observe that σ is the identity on H, so that if q is sufficiently close to 1 and a to 0, $S_k(\sigma\tau\Lambda) = S_k(\tau\Lambda)$ and $\Delta_k(\sigma\tau\Lambda) = \Delta_k(\tau\Lambda) = \Delta_k(\Lambda)$. However, σ increases lengths of vectors in H which are not in F. Thus $\Delta_k((\sigma\tau\Lambda)^*) > \Delta_k((\tau\Lambda)^*) = \Delta_k(\Lambda^*)$. Therefore $\Delta_k(\Lambda)\,\Delta_k(\Lambda^*) < \Delta_k(\mathsf{M})\,\Delta_k(\mathsf{M}^*)$ where $\mathsf{M} = \sigma\tau\Lambda$. □

Definition A.3. Let Q be the vector space of quadratic forms on V. We say that a lattice Λ is dual k-perfect if the linear functionals $q \mapsto q(x)$ for $x \in W \in S_k(\Lambda) \cup S_k(\Lambda^*)$ generate Q^*.

Clearly Λ is k-perfect if $q(x) = 0$ for all $x \in W \in S_k(\Lambda) \cup S_k(\Lambda^*)$ implies $q = 0$. Note that the elements q of Q correspond to symmetric endomorphisms υ of V such that $q(x) = \upsilon(x) \cdot x$ for $x \in V$ where $\cdot$ denotes the inner product on V.

Lemma A.4. *If Λ is dual k-extreme, then Λ is dual k-perfect.*

Proof. If Λ is not dual k-perfect, then there exists a nonzero symmetric endomorphism υ of V such that $\upsilon(x) \cdot x = 0$ for all $x \in W \in S_k(\Lambda) \cup S_k(\Lambda^*)$. Consider the linear isomorphism $\tau = \mathrm{id} + \epsilon\upsilon$. Since υ is symmetric, $\tau^\dagger = \tau$. If $\epsilon > 0$ is sufficiently small, $S_k(\tau\Lambda) \subset S_k(\Lambda)$ and $S_k((\tau^\dagger)^{-1}\Lambda^*) \subset S_k(\Lambda^*)$. Thus if $x \in W \in S_k(\Lambda)$, then $\tau(x) = x + \epsilon\upsilon(x)$ and thus, using $\upsilon(x) \cdot x = 0$,

$$|\tau(x)|^2 = |x|^2 + \epsilon^2 |\upsilon(x)|^2.$$

Hence τ increases lengths of vectors $x \in W$ for $W \in S_k(\Lambda)$. Therefore $\Delta_k(\Lambda^*) \geq \Delta_k(\Lambda)$, with equality if and only if $\upsilon(x) = 0$ for all $x \in W \in S_k(\tau\Lambda)$.

Choosing $\epsilon > 0$ sufficiently small so that $\epsilon|\upsilon| < 1$, where $|\upsilon|$ is the operator norm, one has the series expansion

$$(\tau^\dagger)^{-1} = \tau^{-1} = \mathrm{id} - \epsilon\upsilon + \epsilon^2\upsilon^2 - \epsilon^3\upsilon^3 + \cdots .$$

Thus if $y \in W \in S_k(\Lambda^*)$, then

$$(\tau^\dagger)^{-1}(y) = y - \epsilon\upsilon(y) + \epsilon^2\upsilon^2(y) - \epsilon^3\upsilon^3(y) + \cdots .$$

Using the symmetry of υ, it follows that

$$|(\tau^\dagger)^{-1}(y)|^2 = |y|^2 - 2\epsilon\upsilon(y) \cdot y + 3\epsilon^2\upsilon(y) \cdot \upsilon(y) - 4\epsilon^3\upsilon^2(y) \cdot \upsilon(y) + \cdots .$$

But by assumption, $\upsilon(y) \cdot y = 0$. Hence, using the symmetry of υ,

$$\begin{aligned} \text{(A-2)} \quad |(t^\dagger)^{-1}(y)|^2 &= |y|^2 + 3\epsilon^2\upsilon(y) \cdot \upsilon(y) - 4\epsilon^3\upsilon^2(y) \cdot \upsilon(y) + 5\epsilon^4\upsilon^3(y) \cdot \upsilon(y) - \cdots \\ &= |y|^2 + 3\epsilon^2|\upsilon(y)|^2 - 4\epsilon^3\upsilon^2(y) \cdot \upsilon(y) + 5\epsilon^4|\upsilon^2(y)|^2 - \cdots \\ &\geq |y|^2 + \epsilon^2|\upsilon(y)|^2(3 - 4\epsilon|\upsilon| - 6\epsilon^3|\upsilon|^3 - \cdots), \end{aligned}$$

since the terms with odd coefficients are nonnegative. If ϵ is sufficiently small, inequality (A-2) shows that $(\tau^\dagger)^{-1}$ increases lengths of vectors $y \in W \in S_k(\Lambda^*)$. Thus $\Delta_k((\tau\Lambda)^*) \geq \Delta_k(\Lambda^*)$ with equality if and only if $\upsilon(y) = 0$ for all $y \in W \in S_k((\tau\Lambda)^*)$.

Since Λ is assumed to be dual k-extreme, these inequalities show

$$\Delta_k(\tau\Lambda)\,\Delta_k((\tau\Lambda)^*) \leq \Delta_k(\Lambda)\,\Delta_k(\Lambda^*) \leq \Delta_k(\tau\Lambda)\,\Delta_k((\tau\Lambda)^*).$$

Thus $\Delta_k((\tau\Lambda)) = \Delta_k(\Lambda)$ and $\Delta_k((\tau\Lambda)^*) = \Delta_k(\Lambda^*)$. Consequently, as υ is nonzero, every $W \in S_k(\Lambda) \cup S_k(\Lambda^*)$ is contained in the hyperplane $\upsilon(x) = 0$ contradicting the dual k-extremality of Λ by Lemma A.2. Therefore Λ is dual k-perfect. $\square$

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