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NAIHUAN JING AND NING LIU

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# SPIN KOSTKA POLYNOMIALS AND VERTEX OPERATORS

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**An algebraic iterative formula for the spin Kostka–Foulkes polynomial $K^-_{\xi\mu}(t)$ is given using vertex operator realizations of Hall–Littlewood symmetric functions and Schur $Q$-functions. Based on the operational formula, more favorable properties are obtained parallel to the Kostka polynomial. In particular, we obtain some formulae for the number of (unshifted) marked tableaux. As an application, we confirmed a conjecture of Aokage on the expansion of the Schur $P$-function in terms of Schur functions. Tables of $K^-_{\xi\mu}(t)$ for $|\xi| \leq 6$ are listed.**

## 1. Introduction

The Hall–Littlewood symmetric functions $P_\mu(x;t)$ and the Kostka–Foulkes polynomials $K_{\lambda\mu}(t)$ both have played an active role in algebraic combinatorics and representation theory. On one hand, the Hall–Littlewood symmetric functions $P_\mu(x;t)$ are certain deformations of the Schur functions $s_\lambda(x)$, and the Kostka–Foulkes polynomials $K_{\lambda\mu}(t)$ are the transition coefficients between the two bases. On the other hand, $K_{\lambda\mu}(t)$ have the following representation theoretic interpretation. Let $\mathfrak{B}_\mu$ be the variety of flags preserved by a nilpotent matrix with Jordan block of shape $\mu$. The cohomology group $H^\bullet(\mathfrak{B}_\mu)$ affords a graded $\mathfrak{S}_n$-module structure. Set

$$C_{\lambda\mu}(t) = \sum_{i\geq 0} t^i (\dim \mathrm{Hom}_{\mathfrak{S}_n}(S^\lambda, H^{2i}(\mathfrak{B}_\mu))),$$

where $S^\lambda$ denotes the Specht module of $\mathfrak{S}_n$ associated with $\lambda$. Garsia and Procesi [1992] proved that

$$K_{\lambda\mu}(t) = C_{\lambda\mu}(t^{-1})t^{n(\mu)}, \tag{1-1}$$

which confirms geometrically the positivity of the Kostka–Foulkes polynomials [Lascoux and Schützenberger 1978].

Recently, Wan and Wang [2013] have introduced the spin Kostka–Foulkes polynomials $K^-_{\xi\mu}(t)$ as the transition coefficients between the Hall–Littlewood functions

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$P_\mu(x;t)$ and Schur $Q$-functions $Q_\xi$ with interesting representation theoretic interpretations. As is well-known, the Schur $Q$-functions are indexed by strict partitions and were used by Schur [Stembridge 1989] in generalizing the Frobenius character formula for projective irreducible characters of the symmetric group $\mathfrak{S}_n$. Schur $Q$-functions form a distinguished basis in the subring of symmetric functions generated by $p_1, p_3, \ldots$. Yamaguchi [1999] has shown that the category of irreducible $\mathfrak{S}_n$-supermodules is equivalent to that of supermodules of the Hecke–Clifford algebra $\mathcal{H}_n = \mathcal{C}_n \rtimes \mathbb{C}\mathfrak{S}_n$ and the irreducible objects $D^\xi$ are parametrized by strict partitions $\xi \in \mathcal{SP}_n$. Wan and Wang [2013] have shown that the spin Kostka polynomials admit the interpretation

$$K^-_{\lambda\mu}(t) = 2^{[l(\xi)/2]} C_{\xi\mu}(t^{-1}) t^{n(\mu)}, \tag{1-2}$$

and

$$C^-_{\xi\mu}(t) = \sum_{i\geq 0} t^i (\dim \mathrm{Hom}_{\mathcal{H}_n}(D^\xi, \mathcal{C}_n \otimes H^{2i}(\mathfrak{B}_\mu)).$$

Let $\mathfrak{q}(n)$ be the queer Lie superalgebra containing the general linear Lie algebra $\mathfrak{gl}(n)$ as its even subalgebra. Sergeev [1984] has shown that the irreducible $\mathfrak{q}(n)$-modules $V(\xi)$ are also parametrized by strict partitions $\xi \in \mathcal{SP}_n$. It turns out that the $q$-weight multiplicity $\gamma^-_{\xi\mu}(t)$ associated with the weight space $V(\xi)_\mu$ also appears as the spin Kostka polynomial [Wan and Wang 2013]:

$$K^-_{\lambda\mu}(t) = 2^{[l(\xi)/2]} \gamma^-_{\xi\mu}(t). \tag{1-3}$$

The purpose of this paper is to give an operational algebraic formula for the spin Kostka–Foulkes polynomials $K^-_{\xi\mu}(t)$. The method we adopt is similar to that of [Bryan and Jing 2021], in which the vertex operator realizations of the Hall–Littlewood polynomials and Schur functions were employed. However, there is some subtlety in the spin situation.

In the usual vertex realization of Schur $Q$-functions [Jing 1991b], only the modes of odd indices (of the twisted Heisenberg algebra) were used in the definition. Should this vertex operator be employed, the commutation relations of its components with those of the vertex operator for the Hall–Littlewood symmetric functions would have infinitely many terms in the quadratic relations. To salvage the situation, we introduce a new vertex operator realization of Schur $Q$-functions using a larger Heisenberg algebra graded by all integers (see (2-8) and (2-9)). The new vertex operator realization enables us to get a finite quadratic relation between the operators realizing both the Hall–Littlewood and Schur $Q$-functions and then the matrix coefficients express the spin Kostka polynomials.

As matrix coefficients, the spin Kostka–Foulkes polynomials can be computed in general, and exact formulas are given in some special cases. We also prove a stability formula for the spin Kostka polynomials. We have clarified some questions regarding them (in Example 3.11, we disproved the symmetric property) and obtained counting formulas for the Stembridge coefficients [Stembridge 1989] between the Schur $P$-functions and Schur functions. As applications, we answer a recent conjecture of Aokage and are able to derive a tensor decomposition in the general situation.

The paper is organized as follows. In Section 2 we recall the vertex operator realization of the Hall–Littlewood functions and give a new vertex operator construction of the Schur $Q$-functions, which is specifically tailored for taming the commutation relation between the two vertex operators. In Section 3 we express the spin Hall–Littlewood polynomials as matrix coefficients of vertex operators and derive an iterative formula (see Theorem 3.5). Finally in Section 4 we use the iterative formulas to verify Aokage's conjecture on multiplicities of tensor products of spin modules, and a formula is also obtained for the general case.

## 2. Vertex operator realization of Hall–Littlewood and Schur $Q$-functions

A partition (resp. strict partition) $\lambda = (\lambda_1, \lambda_2, \ldots)$, denoted $\lambda \vdash n$, is a weakly (resp. strictly) decreasing sequence of positive integers such that $\sum_i \lambda_i = n$. The sum $|\lambda| = \sum_i \lambda_i$ is called the weight and the number $l(\lambda)$ of nonzero parts is called the length. We also define $\lambda \models n$ if $\lambda$ is a composition of $n$ when the past $\lambda_i$ are not necessarily ordered. The set of partitions (resp. strict partitions) of weight $n$ will be denoted by $\mathcal{P}_n$ (resp. $\mathcal{SP}_n$). The dominance order $\lambda \geq \mu$ is defined by $|\lambda| = |\mu|$ and $\lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i$ for each $i$.

Let $m_i$ be the multiplicity of $i$ in $\lambda$ and set $z_\lambda = \prod_{i\geq 1} i^{m_i(\lambda)} m_i(\lambda)!$; we define the parity $\varepsilon_\lambda = (-1)^{|\lambda| - l(\lambda)}$ and

$$z_\lambda(t) = \frac{z_\lambda}{\prod_{i\geq 1}(1 - t^{\lambda_i})}, \quad n(\lambda) = \sum_{i\geq 1} (i-1)\lambda_i. \tag{2-1}$$

A partition $\lambda$ can be visualized by its Young diagram when $\lambda$ is identified with $\{(i, j) \in \mathbb{Z}^2 \mid 1 \leq i \leq l(\lambda), 1 \leq j \leq \lambda_i\}$. To each cell $(i, j) \in \lambda$, we define its content $c_{ij} = j - i$ and hook length $h_{ij} = \lambda_i + \lambda'_j - i - j + 1$, where the partition $\lambda' = (\lambda'_1, \ldots, \lambda'_{\lambda_1})$ is the dual partition of $\lambda$ obtained by reflecting the Young diagram of $\lambda$ along the diagonal.

In this paper, we use the $t$-integer $[n] = t^{n-1} + t^{n-2} + \cdots + t + 1$. Similarly $[n]! = [n] \cdots [1]$, and the Gauss $t$-binomial symbol $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right] = \frac{[n]!}{[k]![n-k]!}$.

Let $\Lambda_F$ be the ring of symmetric functions over $F = \mathbb{Q}(t)$, the field of rational functions in $t$. We also consider $\Lambda$ over the ring of integers. The space $\Lambda_F$ is graded and decomposes into a direct sum:

$$\Lambda_F = \bigoplus_{n=0}^{\infty} \Lambda_F^n, \tag{2-2}$$

where $\Lambda_F^n$ is the subspace of degree $n$, spanned by the elements $p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_l}$ with $|\lambda| = n$. Here $p_r$ is the degree $r$ power sum symmetric function.

Let $\Gamma_{\mathbb{Q}}$ be the subring of $\Lambda_{\mathbb{Q}}$ generated by the $p_{2r-1}$, $r \in \mathbb{N}$. Then

$$\Gamma_{\mathbb{Q}} = \mathbb{Q}[p_r : r \text{ odd}]. \tag{2-3}$$

The Schur $Q$-functions $Q_\xi$, $\xi$ strict, form a $\mathbb{Q}$-basis of $\Gamma_{\mathbb{Q}}$ [Macdonald 1979]. Also, $\Gamma$ is a graded ring $\Gamma = \oplus_{n\geq 0}\Gamma^n$, where $\Gamma^n = \Gamma \cap \Lambda^n$.

The space $\Lambda_F$ is equipped with the bilinear form $\langle \cdot, \cdot \rangle$ defined by

$$\langle p_\lambda, p_\mu \rangle = \delta_{\lambda\mu} z_\lambda(t). \tag{2-4}$$

As $\{z_\lambda(t)^{-1} p_\lambda\}$ is the dual basis of the power sum basis, the adjoint operator of the multiplication operator $p_n$ is the differential operator $p_n^* = (n/(1-t^n))\,\partial/\partial p_n$ of degree $-n$.

We recall the vertex operator realization of the Hall–Littlewood symmetric functions [Jing 1991a] and construct a variant vertex operator for the Schur $Q$-function on the space $\Lambda_F$. The *vertex operators* $H(z)$ and its adjoint $H^*(z)$ are $t$-parametrized linear maps, $\Lambda_F \longrightarrow \Lambda_F[\![z, z^{-1}]\!] = \Lambda_F \otimes F[z, z^{-1}]$, defined by

$$H(z) = \exp\Big(\sum_{n\geq 1} \frac{1-t^n}{n} p_n z^n\Big) \exp\Big(-\sum_{n\geq 1} \frac{\partial}{\partial p_n} z^{-n}\Big) = \sum_{n\in\mathbb{Z}} H_n z^n, \tag{2-5}$$

and

$$H^*(z) = \exp\Big(-\sum_{n\geq 1} \frac{1-t^n}{n} p_n z^n\Big) \exp\Big(\sum_{n\geq 1} \frac{\partial}{\partial p_n} z^{-n}\Big) = \sum_{n\in\mathbb{Z}} H_n^* z^{-n}. \tag{2-6}$$

Note that * is $\mathbb{Q}(t)$-linear and anti-involutive satisfying

$$\langle H_n u, v \rangle = \langle u, H_n^* v \rangle \tag{2-7}$$

for $u, v \in \Lambda_F$.

We now introduce the *vertex operators* $Q(z)$ and its *adjoint* $Q^*(z)$ as the linear maps, $\Lambda_F \longrightarrow \Lambda_F[\![z, z^{-1}]\!]$, defined by

$$Q(z) = \exp\Big(\sum_{n\geq 1,\text{odd}} \frac{2}{n} p_n z^n\Big) \exp\Big(-\sum_{n\geq 1} \frac{\partial}{\partial p_n} z^{-n}\Big) = \sum_{n\in\mathbb{Z}} Q_n z^n, \tag{2-8}$$

and

$$Q^*(z) = \exp\Big(-\sum_{n\geq 1} \frac{1-t^n}{n} p_n z^n\Big) \exp\Big(\sum_{n\geq 1,\text{odd}} \frac{2}{1-t^n} \frac{\partial}{\partial p_n} z^{-n}\Big) = \sum_{n\in\mathbb{Z}} Q_n^* z^{-n}. \tag{2-9}$$

The components $H_n, H_{-n}^* \in \mathrm{End}_F(\Lambda)$ are of degree $n$, and so are the annihilation operators for $n > 0$. Similarly $Q_n, Q_{-n}^* \in \mathrm{End}_{\mathbb{Q}}(\Lambda)$. We remark that the second exponential factor of $Q(z)$ is different from the usual construction in [Jing 1991b], and this will be crucial for our later discussion. In particular, note that $Q(-z) \neq Q^*(z)$ in the current situation due to different inner product.

We collect the relations of the vertex operators as follows.

**Proposition 2.1** [Jing 1991a; 1991b]. (1) *The operators $H_n$ and $H_n^*$ satisfy the relations*

$$H_m H_n - t H_n H_m = t H_{m+1} H_{n-1} - H_{n-1} H_{m+1}, \tag{2-10}$$

$$H_m^* H_n^* - t H_n^* H_m^* = t H_{m-1}^* H_{n+1}^* - H_{n+1}^* H_{m-1}^*, \tag{2-11}$$

$$H_m H_n^* - t H_n^* H_m = t H_{m-1} H_{n-1}^* - H_{n-1}^* H_{m-1} + (1-t)^2 \delta_{m,n}, \tag{2-12}$$

$$H_{-n}.1 = Q_{-n}.1 = \delta_{n,0}, \quad H_n^*.1 = Q_n^*.1 = \delta_{n,0}, \tag{2-13}$$

*where $\delta_{m,n}$ is the Kronecker delta function.*

(2) *The operators $Q_n$ satisfy the Clifford algebra relations*

$$\{Q_m, Q_n\} = (-1)^n 2\delta_{m,-n}, \tag{2-14}$$

*where $\{A, B\} = AB + BA$.*

*Proof.* Commutation relations (2-10)–(2-13) were from [Jing 1991a]. We focus on (2). Define the normal ordering product by

$$:Q(z)Q(w): = \exp\biggl(\sum_{n\geq 1,\text{odd}} \frac{2}{n} p_n (z^n + w^n)\biggr) \exp\biggl(-\sum_{n\geq 1} \frac{\partial}{\partial p_n} (z^{-n} + w^{-n})\biggr).$$

Then we have for $|z| < |w|$

$$Q(z)Q(w) = :Q(z)Q(w): \exp\biggl(-\sum_{n\geq 1,\text{odd}} \frac{2}{n} \Bigl(\frac{w}{z}\Bigr)^n\biggr) = :Q(z)Q(w): \frac{z-w}{z+w}.$$

The rest of the argument is similar to Proposition 4.15 in [Jing 1991b]. □

Note that the vacuum vector 1 is annihilated by $p_n^*$, so

$$H(z).1 = \exp\biggl(\sum_{n=1}^{\infty} \frac{1-t^n}{n} p_n z^n\biggr) = \sum_{n=0}^{\infty} q_n z^n = q(z), \tag{2-15}$$

where $q_n$ is the Hall–Littlewood polynomial of one-row partition $(n)$, and clearly

$$q_n = H_n.1 = \sum_{\lambda\vdash n} \frac{1}{z_\lambda(t)} p_\lambda. \tag{2-16}$$

We also introduce a spin analogue $h(z)$ by

$$\tilde{h}(z) = \exp\biggl(\sum_{n=1}^{\infty} \frac{t^n - (-1)^n}{n} p_n z^n\biggr) = \sum_{n\geq 0} \tilde{h}_n z^n; \tag{2-17}$$

then

$$\tilde{h}_n = \sum_{\lambda \vdash n} \frac{\varepsilon_\lambda}{z_\lambda(-t)} p_\lambda. \tag{2-18}$$

Moreover,

$$\tilde{h}_n(-t) = \sum_{\lambda \vdash n} \varepsilon_\lambda u_\lambda q_\lambda, \tag{2-19}$$

where $\varepsilon_\lambda = (-1)^{|\lambda|-l(\lambda)}$ and $u_\lambda = l(\lambda)!/\prod_{i\geq 1} m_i(\lambda)!$.

As consequences of the proposition, one also has that

$$H_n H_{n+1} = t H_{n+1} H_n, \tag{2-20}$$

$$H_n^* H_{n-1}^* = t H_{n-1}^* H_n^*, \tag{2-21}$$

$$\langle H_n.1, H_n.1\rangle = \sum_{\lambda \vdash n} \frac{1}{z_\lambda(t)} = 1 - t, \quad n > 0, \tag{2-22}$$

$$\langle H_n.1, H_{-n}^*.1\rangle = \sum_{\lambda \vdash n} \frac{(-1)^{l(\lambda)}}{z_\lambda(t)} = t^n - t^{n-1}, \quad n > 0, \tag{2-23}$$

where the last two identities follow from (2-12) and (2-10) by induction.

In general, expressing $H_\mu$ for any composition $\mu$ in terms of the basis elements $H_\lambda$, $\lambda \in \mathcal{P}$, can be formulated as follows. Let $S_{i,a}$ be the transformation $(\lambda_1, \dots, \lambda_i, \lambda_{i+1}, \dots) \mapsto (\lambda_1, \dots, \lambda_{i+1} - a, \lambda_i + a, \dots)$, where $\lambda_{i+1} > \lambda_i$. Define

$$C(S_{i,a}) = \begin{cases} t, & a = 0, \\ t^{a+1} - t^{a-1}, & 1 \leq a < \left[\frac{\lambda_{i+1}-\lambda_i}{2}\right], \\ t^{a+\epsilon} - t^{a-1}, & 1 \leq a = \left[\frac{\lambda_{i+1}-\lambda_i}{2}\right], \end{cases} \tag{2-24}$$

where $\epsilon \equiv \lambda_{i+1} - \lambda_i \pmod 2$. For $\underline{i} = (i_1, \dots, i_r)$ and $\underline{a} = (a_1, \dots, a_r)$ let

$$C(S_{\underline{i},\underline{a}}) = C(S_{i_1,a_1}) C(S_{i_2,a_2}) \cdots C(S_{i_r,a_r}), \tag{2-25}$$

where the product order follows that of $S_{i_1,a_1} S_{i_2,a_2} \cdots S_{i_r,a_r} \lambda$, i.e., from the right to the left. In particular, when $t = 0$, $C(S_{\underline{i},\underline{a}}) = 0$ unless all $a_i = 1$; in that case, $C(S_{\underline{i},\underline{1}}) = (-1)^r$ which is possible only when $\lambda_{i+1} - \lambda_i \geq 2$. When $t = -1$, $C(S_{\underline{i},\underline{a}}) = 0$ unless all $a_i = 0$ and $C(S_{\underline{i},\underline{0}}) = (-1)^r$.

Let $\mu$ be a composition and $\lambda$ be a partition. Define

$$B(\lambda, \mu) \triangleq \sum_{\underline{i},\underline{a}} C(S_{\underline{i},\underline{a}}) \tag{2-26}$$

summed over $\underline{i} = (i_1, i_2, \dots, i_r)$, $\underline{a} = (a_1, a_2, \dots, a_r)$ such that $S_{\underline{i},\underline{a}}\mu = \lambda$.

**Proposition 2.2** [Jing and Liu 2022]. *Suppose $\mu$ is a composition. Then*

$$H_\mu = \sum_{\lambda \vdash |\mu|} B(\lambda, \mu) H_\lambda. \tag{2-27}$$

We remark that $\lambda$ appears only when $\lambda \geq \mu$ in (2-27). Let $\mu$ be a composition and $\lambda$ be a partition. If there exists $\underline{i} = (i_1, i_2, \ldots, i_r)$, $\underline{a} = (a_1, a_2, \ldots, a_r)$ such that $S_{\underline{i},\underline{a}}\mu = \lambda$, then $\sum_{i=1}^{k} \lambda_i \geq \sum_{i=1}^{k} \mu_i$, $k = 1, 2, \ldots$.

**Proposition 2.3** [Jing 1991a; 1991b]. (1) *Let $\lambda = (\lambda_1, \ldots, \lambda_l)$ be a partition. The vertex operator products $H_{\lambda_1} \cdots H_{\lambda_l}.1$ is the Hall–Littlewood function $Q_\lambda(t)$:*

$$H_{\lambda_1} \cdots H_{\lambda_l}.1 = Q_\lambda(t) = \prod_{i<j} \frac{1 - R_{ij}}{1 - tR_{ij}} q_{\lambda_1} \cdots q_{\lambda_l}, \tag{2-28}$$

*where the raising operator is given by $R_{ij} q_\lambda = q_{(\lambda_1, \ldots, \lambda_i + 1, \ldots, \lambda_j - 1, \ldots, \lambda_l)}$.*

(2) *Let $\xi = (\xi_1, \xi_2, \ldots, \xi_l)$ be a strict partition. Then*

$$Q_\xi = Q_{\xi_1} Q_{\xi_2} \cdots Q_{\xi_l}.1 \tag{2-29}$$

*is the Schur $Q$-function indexed by $\xi$. Moreover, $Q_\xi.1$, where $\xi$ ranges over strict partitions, form an orthogonal $\mathbb{Z}$-base of $\Gamma$ under the specialized inner product $\langle \cdot, \cdot \rangle_{t=-1}$, explicitly*

$$\langle Q_\lambda.1, Q_\xi.1 \rangle|_{t=-1} = 2^{l(\lambda)} \delta_{\lambda\xi}, \quad \lambda, \xi \in \mathcal{SP}. \tag{2-30}$$

*Proof.* Part (1) is from [Jing 1991a]. Since our vertex operator $Q(z)$ is different from that of [Jing 1991b], we explain why the new vertex operator also realizes the Schur $Q$-functions. From the argument in proving (2-14) in Proposition 2.1 it follows that

$$\begin{aligned} Q(z_1)Q(z_2) \cdots Q(z_l).1 &= \prod_{i<j} \frac{z_i - z_j}{z_i + z_j} :Q(z_1)Q(z_2) \cdots Q(z_l):.1 \\ &= \prod_{i<j} \frac{z_i - z_j}{z_i + z_j} \exp\biggl( \sum_{n \geq 1, \text{odd}} \frac{2p_n}{n} (z_1^n + \cdots + z_l^n) \biggr). \end{aligned}$$

Taking coefficients of $z_1^{\xi_1} \cdots z_l^{\xi_l}$, we obtain that $Q_\xi$ is exactly the Schur $Q$-function indexed by $\xi$ (cf. [Jing 1991b]). □

## 3. Spin Hall–Littlewood polynomials and vertex operators

Wan and Wang [2013] have introduced an extremely interesting spin analogue of Kostka(–Foulkes) polynomials and shown that these polynomials enjoy favorable properties parallel to those of the Kostka polynomials.

**Definition 3.1** [Wan and Wang 2013]. The spin Kostka polynomials $K^-_{\xi\mu}(t)$ for $\xi \in \mathcal{SP}$ and $\mu \in \mathcal{P}$ are defined by

$$Q_\xi(x) = \sum_\mu K^-_{\xi\mu}(t) P_\mu(x;t), \tag{3-1}$$

where $Q_\xi(x)$ (resp. $P_\mu(x;t)$) are Schur $Q$-functions (resp. Hall–Littlewood functions).

From the above discussion and Proposition 2.3, it is clear that the spin Kostka polynomials can be expressed as matrix coefficients:

$$\begin{aligned} K^-_{\xi\mu}(t) &= \langle Q_\mu(x;t), Q_\xi(x)\rangle \\ &= \langle H_{\mu_1} H_{\mu_2} \cdots H_{\mu_l}.1, Q_{\xi_1} Q_{\xi_2} \cdots Q_{\xi_k}.1\rangle. \end{aligned}$$

To compute the matrix coefficients, we first get the commutation relations by usual techniques of vertex operators:

$$H^*(z)Q(w)(w-tz) + Q(w)H^*(z)(z+w) = 2(1-t)z\delta\Big(\frac{w}{z}\Big)\tilde{h}(z), \tag{3-2}$$

$$\tilde{h}^*(z)H(w) = H(w)\tilde{h}^*(z)\frac{w+z}{w-tz}, \tag{3-3}$$

$$Q(z)\tilde{h}(w) = \tilde{h}(w)Q(z)\frac{z-tw}{z+w}. \tag{3-4}$$

We remark that if the old vertex operator $\tilde{Q}(w)$ from [Jing 1991b] were used, then the commutation relations between $H^*(z)$ and $\tilde{Q}(w)$ would have been an infinite quadratic relation.

Taking coefficients we obtain the following commutation relations.

**Proposition 3.2.** *The commutation relations between the Hall–Littlewood vertex operators and Schur $Q$-function operators are*

$$H^*_n Q_m = t^{-1} H^*_{n-1} Q_{m-1} + t^{-1} Q_m H^*_n + t^{-1} Q_{m-1} H^*_{n-1} + 2(1-t^{-1})\tilde{h}_{m-n}, \tag{3-5}$$

$$\tilde{h}^*_m H_n = H_n \tilde{h}^*_m + (1+t)\sum_{k=0}^{m-1} t^{m-k-1} H_{n-m+k}\tilde{h}^*_k, \tag{3-6}$$

$$Q_n \tilde{h}_m = \tilde{h}_m Q_n + (1+t)\sum_{k=0}^{m-1} (-1)^{m-k}\tilde{h}_k Q_{n-k+m}. \tag{3-7}$$

Now we can state our formulas to compute the spin Kostka polynomials. To this end, we prepare some necessary notation. Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$ and $\mu = (\mu_1, \mu_2, \dots, \mu_m)$ be (strict) partitions. We define $\lambda^{[i]} = (\lambda_{i+1}, \dots, \lambda_l)$, $\lambda^{\hat{i}} = (\lambda_1, \dots, \lambda_{i-1}, \lambda_{i+1}, \dots, \lambda_l)$, and $\lambda - \mu = (\lambda_1 - \mu_1, \lambda_2 - \mu_2, \dots)$.

**Theorem 3.3.** *For an integer $k$, strict partition $\xi = (\xi_1, \xi_2, \dots, \xi_l)$ and partition $\mu = (\mu_1, \mu_2, \dots, \mu_m)$,*

$$H_k^* Q_\xi = \sum_{i=1}^{l} (-1)^{i-1} 2\tilde{h}_{\xi_i - k} Q_{\xi^{\hat{i}}}, \tag{3-8}$$

$$\tilde{h}_k^* H_\mu = \sum_{\tau \models k} t^{k-l(\tau)} (1+t)^{l(\tau)} H_{\mu-\tau}. \tag{3-9}$$

*Proof.* We show the first relation by induction on $k + |\xi|$. The case of $k + |\xi| = 1$ is clear. Assume that (3-8) holds for $k + |\xi| = n - 1$. Using the induction hypothesis and (3-5) we have that

$$\begin{aligned}
&H_k^* Q_{\xi_1} Q_{\xi_2} \cdots Q_{\xi_l} \\
&\quad = t^{-1} H_{k-1}^* Q_{\xi_1 - 1} Q_{\xi_2} \cdots Q_{\xi_l} + t^{-1} Q_{\xi_1} H_k^* Q_{\xi_2} \cdots Q_{\xi_l} + t^{-1} Q_{\xi_1 - 1} H_{k-1}^* Q_{\xi_2} \cdots Q_{\xi_l} \\
&\qquad\qquad + 2(1 - t^{-1}) \tilde{h}_{\xi_1 - k} Q_{\xi_2} \cdots Q_{\xi_l} \\
&\quad = t^{-1} \big( 2\tilde{h}_{\xi_1 - k} Q_{\xi_2} \cdots Q_{\xi_l} - 2\tilde{h}_{\xi_2 - k + 1} Q_{\xi_1 - 1} Q_{\xi_3} \cdots Q_{\xi_l} \\
&\qquad + 2\tilde{h}_{\xi_3 - k + 1} Q_{\xi_1 - 1} Q_{\xi_2} Q_{\xi_4} \cdots Q_{\xi_l} + \cdots + (-1)^{l+1} 2\tilde{h}_{\xi_l - k + 1} Q_{\xi_1 - 1} Q_{\xi_2} \cdots Q_{\xi_{l-1}} \big) \\
&\qquad + t^{-1} Q_{\xi_1} \big( 2\tilde{h}_{\xi_2 - k} Q_{\xi_3} \cdots Q_{\xi_l} - 2\tilde{h}_{\xi_3 - k} Q_{\xi_2} Q_{\xi_4} \cdots Q_{\xi_l} \\
&\qquad\quad + 2\tilde{h}_{\xi_4 - k} Q_{\xi_2} Q_{\xi_3} Q_{\xi_5} \cdots Q_{\xi_l} + \cdots + (-1)^l 2\tilde{h}_{\xi_l - k} Q_{\xi_2} Q_{\xi_3} \cdots Q_{\xi_{l-1}} \big) \\
&\qquad + t^{-1} Q_{\xi_1 - 1} \big( 2\tilde{h}_{\xi_2 - k + 1} Q_{\xi_3} \cdots Q_{\xi_l} - 2\tilde{h}_{\xi_3 - k + 1} Q_{\xi_2} Q_{\xi_4} \cdots Q_{\xi_l} \\
&\qquad\quad + 2\tilde{h}_{\xi_4 - k + 1} Q_{\xi_2} Q_{\xi_3} Q_{\xi_5} \cdots Q_{\xi_l} \\
&\qquad\quad + \cdots + (-1)^l 2\tilde{h}_{\xi_l - k + 1} Q_{\xi_2} Q_{\xi_3} \cdots Q_{\xi_{l-1}} \big) \\
&\qquad + 2(1 - t^{-1}) \tilde{h}_{\xi_1 - k} Q_{\xi_2} Q_{\xi_3} \cdots Q_{\xi_l}.
\end{aligned}$$

Simplifying the expression, we see the above is

$$\begin{aligned}
&t^{-1} \big( 2\tilde{h}_{\xi_1 - k} Q_{\xi_2} \cdots Q_{\xi_l} - 2\tilde{h}_{\xi_2 - k + 1} Q_{\xi_1 - 1} Q_{\xi_3} \cdots Q_{\xi_l} \\
&\qquad + 2\tilde{h}_{\xi_3 - k + 1} Q_{\xi_1 - 1} Q_{\xi_2} Q_{\xi_4} \cdots Q_{\xi_l} \\
&\qquad + \cdots + (-1)^{l+1} 2\tilde{h}_{\xi_l - k + 1} Q_{\xi_1 - 1} Q_{\xi_2} \cdots Q_{\xi_{l-1}} \big) \\
&\qquad + 2t^{-1} \big( \tilde{h}_{\xi_2 - k + 1} Q_{\xi_1 - 1} Q_{\xi_3} \cdots Q_{\xi_l} - \tilde{h}_{\xi_3 - k + 1} Q_{\xi_1 - 1} Q_{\xi_2} Q_{\xi_4} \cdots Q_{\xi_l} \\
&\qquad\quad + \tilde{h}_{\xi_4 - k + 1} Q_{\xi_1 - 1} \cdots Q_{\xi_l} \\
&\qquad\quad + \cdots + (-1)^l \tilde{h}_{\xi_l - k + 1} Q_{\xi_1 - 1} Q_{\xi_2} Q_{\xi_3} \cdots Q_{\xi_{l-1}} \big) \\
&\qquad - 2 \big( \tilde{h}_{\xi_2 - k} Q_{\xi_1} Q_{\xi_3} \cdots Q_{\xi_l} - \tilde{h}_{\xi_3 - k} Q_{\xi_1} Q_{\xi_2} Q_{\xi_4} \cdots Q_{\xi_l} \\
&\qquad\quad + \tilde{h}_{\xi_4 - k} Q_{\xi_1} Q_{\xi_2} Q_{\xi_3} Q_{\xi_5} \cdots Q_{\xi_l} \\
&\qquad\quad + \cdots + (-1)^l \tilde{h}_{\xi_l - k} Q_{\xi_1} Q_{\xi_2} Q_{\xi_3} \cdots Q_{\xi_{l-1}} \big) \\
&\qquad + 2(1 - t^{-1}) \tilde{h}_{\xi_1 - k} Q_{\xi_2} Q_{\xi_3} \cdots Q_{\xi_l} \quad \text{(by (3-7))} \\
&= 2\tilde{h}_{\xi_1 - k} Q_{\xi_2} \cdots Q_{\xi_l} - 2\tilde{h}_{\xi_2 - k} Q_{\xi_1} Q_{\xi_3} \cdots Q_{\xi_l} + \cdots + 2(-1)^{l-1} \tilde{h}_{\xi_l - k} Q_{\xi_1} \cdots Q_{\xi_{l-1}},
\end{aligned}$$

which has proved (3-8). The second relation is similarly shown by (3-6) and induction on $l(\mu)$. □

**Example 3.4.** Let $\mu = (2, 2)$ and $\xi = (3, 1)$. Then by Theorem 3.3

$$\begin{aligned} K^-_{\xi\mu}(t) &= \langle H_2 H_2.1,\ Q_3 Q_1.1\rangle \\ &= \langle H_2.1, 2h_1 Q_1.1\rangle \\ &= 2\langle t(1+t^{-1})H_1.1,\ Q_1.1\rangle \\ &= 4t+4. \end{aligned}$$

By Theorem 3.3, we now obtain an algebraic formula for $K^-_{\xi\mu}(t)$.

**Theorem 3.5.** *For $\xi = (\xi_1, \dots, \xi_l) \in \mathcal{SP}_n$ and $\mu = (\mu_1, \dots, \mu_m) \in \mathcal{P}_n$, $K^-_{\xi\mu}(t)$ is given by the iterative formula*

$$K^-_{\xi\mu}(t) = \sum_{i=1}^{l} \sum_{\tau \models \xi_i - \mu_1} \sum_{\lambda \vdash n-\xi_i} (-1)^{i-1} 2t^{\xi_i-\mu_1}(1+t^{-1})^{l(\tau)} B(\lambda, \mu^{[1]}-\tau) K^-_{\xi^{\hat{i}}\lambda}(t). \tag{3-10}$$

*Proof.* It follows readily from (3-8), (3-9) and (2-27). □

Equation (3-10) shows that all spin Kostka polynomials are integral polynomials, and it also gives an effective recurrence of $K^-_{\xi\mu}(t)$ as shown by the following example.

**Example 3.6.** Let $\xi = (4, 3, 1)$ and $\mu = (3, 3, 2)$. Then

$$\begin{aligned} K^-_{\xi\mu}(t) &= \langle H_3 H_3 H_2.1,\ Q_4 Q_3 Q_1.1\rangle \\ &= \langle H_3 H_2.1, 2\tilde{h}_1 Q_3 Q_1.1\rangle - \langle H_3 H_2.1, 2\tilde{h}_0 Q_4 Q_1.1\rangle \\ &= 2\langle t(1+t^{-1})(H_2H_2.1 + H_3H_1.1),\ Q_3Q_1.1\rangle - 2\langle H_3H_2.1,\ Q_4Q_1.1\rangle \\ &= 2(t+1)(K^-_{(3,1)(2,2)}(t) + K^-_{(3,1)(3,1)}(t)) - 2K^-_{(4,1)(3,2)}(t). \end{aligned}$$

The spin Kostka polynomials have quite a few remarkable properties resembling those of the Kostka–Foulkes polynomials. As a consequence of the recurrence we have the following.

**Corollary 3.7.** *Let $\xi$ be a strict partition and $\mu$ be a partition. We have*:

(1) *If there exists $k \in \mathbb{N}$, such that $\xi_i = \mu_i$, $i = 1, 2, \dots, k$, then*

$$K^-_{\xi\mu}(t) = 2^k K^-_{\xi^{[k]}\mu^{[k]}}(t). \tag{3-11}$$

*In particular*, $K^-_{\xi\xi}(t) = 2^{l(\xi)}$.

(2) $2^{l(\xi)} \mid K^-_{\xi\mu}(t)$.

(3) $K^-_{\xi\mu}(-1) = 2^{l(\xi)}\delta_{\xi\mu}$.

*Proof.* They are immediate consequences of Theorem 3.5. □

Some special cases of Theorem 3.5 are listed as follows.

**Example 3.8.** Suppose $\xi \in \mathcal{SP}_n$, $\mu \in \mathcal{P}_n$. We have

$$K^-_{\xi(n)}(t) = 2\delta_{\xi,(n)}, \tag{3-12}$$

$$K^-_{(n)\mu}(t) = 2t^{n-\mu_1} \sum_{\tau \models n-\mu_1} (1+t^{-1})^{l(\tau)} B(\varnothing, \mu^{(1)} - \tau), \tag{3-13}$$

$$K^-_{\xi(\mu_1,\mu_2)}(t) = \begin{cases} 2^{2-\delta_{0,\xi_2}} t^{\xi_1-\mu_1}(1+t^{-1}) & \text{if } \xi > (\mu_1, \mu_2), \\ 4 & \text{if } \xi = (\mu_1, \mu_2), \\ 0 & \text{otherwise.} \end{cases} \tag{3-14}$$

There is a compact formula of $K^-_{(n)\mu}(t)$ [Wan and Wang 2013] by using a result of [Macdonald 1979]. We will come back to the Wan–Wang formula using the iteration in the next section.

The following result was first proved in [Wan and Wang 2013] using the similar property of the Kostka–Foulkes polynomials. Using our iterative formula, one can give an independent proof from that of the Kostka–Foulkes polynomials. We remark that the method can also be used to show this property for the Kostka–Foulkes polynomial by the iterative formula in [Bryan and Jing 2021].

**Corollary 3.9.** *Let $\xi = (\xi_1, \xi_2, \dots) \in \mathcal{SP}_n$, $\mu = (\mu_1, \mu_2, \dots) \in \mathcal{P}_n$. Then $K^-_{\xi\mu}(t) = 0$, unless $\xi \geq \mu$.*

*Proof.* It is equivalent to prove $K^-_{\xi\mu}(t) = 0$, if $\xi \ngeq \mu$. We argue it by induction on $n$. The initial step is obvious. Suppose it holds for weight $< n$. There exists a smallest $k \geq 1$, such that $\xi_1 + \xi_2 + \cdots + \xi_k < \mu_1 + \mu_2 + \cdots + \mu_k$.

If $k = 1$, then it's evident that $K^-_{\xi\mu}(t) = 0$ by the iterative formula (3-10).

If $k > 1$, then there exists $k > j \geq 1$, such that $\xi_{j+1} < \mu_1 \leq \xi_j$. We have

$$\begin{aligned} K^-_{\xi\mu}(t) &= \sum_{i=1}^{j} (-1)^{i-1} \langle H_{\mu_2} H_{\mu_3} \cdots, 2\tilde{h}_{\xi_i-\mu_1} Q_{\xi_1} \cdots \hat{Q}_{\xi_i} \cdots \rangle \\ &= \sum_{i=1}^{j} (-1)^{i-1} \sum_{\tau \models \xi_i - \mu_1} 2t^{\xi_i-\mu_1} (1+t^{-1})^{l(\tau)} \langle H_{\mu^{[1]}-\tau}, Q_{\xi^{\hat{i}}} \rangle \\ &= \sum_{i=1}^{j} (-1)^{i-1} \sum_{\tau \models \xi_i - \mu_1} 2t^{\xi_i-\mu_1} (1+t^{-1})^{l(\tau)} \sum_{\nu \vdash n-\xi_i} B(\nu, \mu^{[1]} - \tau) \langle H_\nu, Q_{\xi^{\hat{i}}} \rangle \\ &= \sum_{i=1}^{j} (-1)^{i-1} \sum_{\tau \models \xi_i - \mu_1} 2t^{\xi_i-\mu_1} (1+t^{-1})^{l(\tau)} \sum_{\nu \vdash n-\xi_i} B(\nu, \mu^{[1]} - \tau) K^-_{\xi^{\hat{i}}\nu}(t). \end{aligned}$$

By the remark below Proposition 2.2, for each $1 \le i \le j$, we have $\nu_1 + \cdots + \nu_{k-1} \ge \mu_2 + \cdots + \mu_k - \tau_1 - \cdots - \tau_{k-1} \ge \mu_2 + \cdots + \mu_k + \mu_1 - \xi_i > \xi_1 + \cdots + \xi_{i-1} + \xi_{i+1} + \cdots \xi_k$. By induction, we have $K^-_{\xi\mu}(t) = 0$. □

The Kostka–Foulkes polynomials have the stability property [Bryan and Jing 2021], which says that if $\mu_1 \ge \lambda_2$, then $K_{\lambda+(r),\mu+(r)}(t) = K_{\lambda\mu}(t)$ for all $r \ge 1$. Here, $\lambda+(r) = (\lambda_1+r, \lambda_2, \ldots)$. The spin Kostka polynomials also enjoy the same stability.

**Proposition 3.10.** *Let* $\xi = (\xi_1, \ldots, \xi_l) \in \mathcal{SP}$, $\mu = (\mu_1, \ldots, \mu_m) \in \mathcal{P}$, *and* $\mu_1 > \xi_2$. *Then for any* $r \ge 1$, *we have*

$$K^-_{\xi+(r)\mu+(r)}(t) = K^-_{\xi\mu}(t). \tag{3-15}$$

*Proof.* By Theorem 3.3, it follows that

$$K^-_{\xi+(r)\mu+(r)}(t) = \langle H_{\mu_2} H_{\mu_3} \cdots H_{\mu_m}.1, 2\tilde{h}_{\xi_1-\mu_1} Q_{\xi_2} \cdots Q_{\xi_l}.1\rangle = K^-_{\xi\mu}(t). \qquad \square$$

The spin Kostka–Foulkes polynomials $K_{\lambda\mu}(t)$ were conjecturally symmetric [Wan and Wang 2013, Question 4.10] in the sense that

$$K^-_{\lambda\mu}(t) = t^{m_{\lambda\mu}} K^-_{\lambda\mu}(t^{-1})$$

for some $m_{\lambda\mu} \in \mathbb{Z}$. However, the following is a counterexample.

**Example 3.11.** Given $\xi = (3, 2)$ and $\mu = (2, 1^3)$, we have

$$\begin{aligned} K^-_{\xi\mu}(t) &= \langle H_2 H_1 H_1 H_1.1,\ Q_3 Q_2.1\rangle \\ &= \langle H_1 H_1 H_1.1, 2\tilde{h}_1 Q_2.1\rangle - \langle H_1 H_1 H_1.1, 2\tilde{h}_0 Q_3.1\rangle \\ &= 2\langle t(1+t^{-1})[3] H_1 H_1.1,\ Q_2.1\rangle - 2K^-_{(3)(1^3)}(t) \\ &= 4t(t^3 + 2t^2 + 3t + 2). \end{aligned}$$

## 4. Marked tableaux

To study projective representations of the symmetric group, Stembridge [1989] introduced the number $g_{\xi\lambda}$ as follows:

$$Q_\xi(x) = \sum_\lambda b_{\xi\lambda} s_\lambda(x), \quad g_{\xi\lambda} = 2^{-l(\xi)} b_{\xi\lambda}. \tag{4-1}$$

Note that $b_{\xi\lambda} = K^-_{\xi\lambda}(0)$, but we will see that $g_{\xi\lambda}$ can be extended to any partition $\xi$, so we reserve this notation in this section.

Let $\xi, \lambda$ be partitions with $\xi$ strict. The coefficient $g_{\xi\lambda}$ of $s_\lambda$ in the expansion of the Schur $Q$-function $2^{-l(\xi)} Q_\xi$ counts the number of (unshifted) marked tableaux $T$ of shape $\lambda$ and weight $\xi$ such that

(a) $w(T)$ has the lattice property;

(b) for each $k \geq 1$, the last occurrence of $k'$ in $w(T)$ precedes the last occurrence of $k$.

Here $w(T)$ is the word of $T$ by reading the symbols in $T$ from right to left in successive rows, starting with the top row.

The combinatorial interpretation and the representation-theoretic interpretation of $g_{\xi\lambda}$ are known [Sagan 1987; Stembridge 1989; Wan and Wang 2013; Worley 1984]. However, no effective formula for $g_{\xi\mu}$ is available. As an application of the preceding section, we give an algebraic formula for $g_{\xi\lambda}$.

The ring $\Lambda_{\mathbb{Q}}$ of symmetric functions has the canonical bilinear form $\langle\cdot,\cdot\rangle_0 = \langle\cdot,\cdot\rangle_{t=0}$ under which Schur functions are orthonormal:

$$\langle p_\lambda, p_\mu\rangle_0 = \delta_{\lambda,\mu} z_\lambda. \tag{4-2}$$

Thus the adjoint operator of the multiplication operator $p_n$ is the differential operator $p_n^- = n(\partial/(\partial p_n))$.

With respect to $\langle\cdot,\cdot\rangle_0$, the *vertex operators* and their adjoint operators for Schur functions and Schur $Q$-functions are given by [Jing 1991b; 2000]

$$S^{\pm}(z) = \exp\Big(\pm\sum_{n\geq 1}\frac{1}{n}p_n z^n\Big)\exp\Big(\mp\sum_{n\geq 1}\frac{\partial}{\partial p_n}z^{-n}\Big) = \sum_{n\in\mathbb{Z}} S_n^{\pm} z^{\pm n}, \tag{4-3}$$

$$Q^{+}(z) = Q(z) = \sum_{n\in\mathbb{Z}} Q_n^{+} z^n, \tag{4-4}$$

$$Q^{-}(z) = \exp\Big(-\sum_{n\geq 1}\frac{1}{n}p_n z^n\Big)\exp\Big(\sum_{n\geq 1,\mathrm{odd}} 2\frac{\partial}{\partial p_n}z^{-n}\Big) = \sum_{n\in\mathbb{Z}} Q_n^{-} z^{-n}. \tag{4-5}$$

Note that $Q^-(z)$ is the specialized vertex operator $Q^*(z)|_{t=0}$. Here we denote the adjoint operators by $S_n^+$ and $Q_n^+$, respectively, to distinguish from the preceding section.

Therefore $g_{\xi\lambda}$ can be expressed in terms of this inner product:

$$g_{\xi\lambda} = 2^{-l(\xi)} b_{\xi\lambda} = 2^{-l(\xi)}\langle s_\lambda, Q_\xi\rangle_0 = 2^{-l(\xi)}\langle S_\lambda.1, Q_\xi.1\rangle_0. \tag{4-6}$$

Recall that the involution $\omega : \Lambda \to \Lambda$ defined by $\omega(p_\lambda) = \varepsilon_\lambda p_\lambda$ [Macdonald 1979] is an isometry with respect to the canonical inner product $\langle\cdot,\cdot\rangle_0$ such that

$$\omega(s_\lambda) = s_{\lambda'}, \quad \omega(Q_\xi) = Q_\xi.$$

**Proposition 4.1.** *If $\lambda \in \mathcal{P}_n$, $\xi \in \mathcal{SP}_n$, then $g_{\xi\lambda}$ or $b_{\xi\lambda}$ has the property*

$$g_{\xi\lambda} = g_{\xi\lambda'}. \tag{4-7}$$

We introduce the operators for the elementary symmetric functions $e_n$

$$e^{\pm}(z) = \exp\left(\sum_{n\geq 1} \frac{(-1)^{n+1}}{n} p_n^{\pm} z^{\pm n}\right) = \sum_{n\geq 0} e_n^{\pm} z^{\pm n}, \tag{4-8}$$

where $p_n^+ = p_n$, $p_n^- = n(\partial/(\partial p_n))$, and $e^+(z) = h(z)|_{t=0}$.

Then by Theorem 3.3 we have:

**Proposition 4.2.** *For any strict partition $\xi = (\xi_1, \xi_2, \dots, \xi_l)$, any partition $\lambda = (\lambda_1, \lambda_2, \dots)$ and integer $k$,*

$$S_k^- Q_\xi = \sum_{i=1}^{l} (-1)^{i-1} 2 e_{\xi_i - k} Q_{\xi_1} Q_{\xi_2} \cdots \hat{Q}_{\xi_i} \cdots Q_{\xi_l}, \tag{4-9}$$

$$e_k^- S_\lambda = \sum_\rho S_\rho, \tag{4-10}$$

*where $\rho$ runs through the partitions such that $\lambda/\rho$ are vertical $k$-strips.*

The algebraic iterative formula for $b_{\xi\lambda}$ is then natural:

**Theorem 4.3.** *Let $\xi \in \mathcal{SP}_n$, $\lambda \in \mathcal{P}_n$. Then*

$$b_{\xi\lambda} = \sum_{i=1}^{l(\xi)} 2(-1)^{i-1} \sum_{\rho^i} b_{\xi^{(i)}\rho^i}, \tag{4-11}$$

*where $\rho^i$ runs through the partitions such that $\lambda^{[1]}/\rho^i$ are vertical $\xi_i - \lambda_1$-strips.*

**Example 4.4.** Let $\lambda \in \mathcal{P}_n$. We have

$$b_{(n)\lambda} = \begin{cases} 2 & \text{if } \lambda \text{ is a hook,} \\ 0 & \text{if } \lambda \text{ is not a hook.} \end{cases} \tag{4-12}$$

Combining (3-1) and (4-1), we have

$$K_{\xi\mu}^-(t) = \sum_\lambda b_{\xi\lambda} K_{\lambda\mu}(t), \tag{4-13}$$

where $K_{\lambda\mu}(t)$ are the Kostka–Foulkes polynomials.

By (4-12), we have

$$K_{(n)\mu}^-(t) = \sum_{\lambda \text{ hook}} 2K_{\lambda\mu}(t). \tag{4-14}$$

Recall that a compact formula for the Kostka–Foulkes polynomials $K_{\lambda\mu}(t)$ is known for $\lambda$ being hook-shaped [Kirillov 2001; Bryan and Jing 2021]:

$$K_{(n-k,1^k)\mu}(t) = t^{n(\mu) + \frac{k(k+1-2l)}{2}} \begin{bmatrix} l-1 \\ k \end{bmatrix}, \tag{4-15}$$

where $n = |\mu|, l = l(\mu)$. Therefore, we have that for any partition $\mu \vdash n$

$$K^{-}_{(n)\mu}(t) = \sum_{k=0}^{l(\mu)-1} 2t^{n(\mu)+\frac{k(k+1-2l(\mu))}{2}} \begin{bmatrix} l(\mu)-1 \\ k \end{bmatrix}, \tag{4-16}$$

$$= t^{n(\mu)} \prod_{i=1}^{l(\mu)} (1 + t^{1-i}). \tag{4-17}$$

Here the second equation follows from the $t$-binomial expansion [Andrews 1986, (2.9)] or an easy induction on $l(\mu)$ from (4-16). We remark that (4-17) was first given by Wan and Wang [2013] using identities of Hall–Littlewood polynomials.

For a given partition $\lambda$, we define

$$\{\lambda\}_s \doteq \{\rho \subset \lambda^{[1]} \mid \rho \text{ is a hook and } \lambda^{[1]}/\rho \text{ is a vertical } s\text{-strip}\}.$$

Set $N^{(s)}(\lambda) = \mathrm{Card}\{\lambda\}_s$. It is clear that $N^{(s)}(\lambda) = 0$ when $s < 0$ or $s > |\lambda^{[1]}|$. Now we can give a two-row formula by the iterative formula for $b_{\xi\lambda}$.

**Theorem 4.5.** *Let* $1 \le m < \frac{n}{2}$, $\lambda \in \mathcal{P}_n$. *We have*

$$b_{(n-m,m)\lambda} = 4(N^{(n-m-\lambda_1)}(\lambda) - N^{(m-\lambda_1)}(\lambda)). \tag{4-18}$$

To compute $N^{(s)}(\lambda)$, we denote all hook (resp. double hook) partitions of $n$ by $\mathrm{HP}(n)$ (resp. $\mathrm{DHP}(n)$). That is, $\mathrm{HP}(n) \doteq \{(\lambda_1, 1^{m_1}) \mid \lambda_1 + m_1 = n\}$, $\mathrm{DHP}(n) \doteq \{(\lambda_1, \lambda_2, 2^{m_2}, 1^{m_1}) \mid \lambda_1+\lambda_2+2m_2+m_1 = n\}$. Clearly, $\mathrm{HP}(n) \subset \mathrm{DHP}(n)$. We remark that $N^{(s)}(\lambda) = 0$ unless $\lambda \in \mathrm{DHP}(n)$. Now let's consider $N^{(s)}(\lambda)$, for $0 \le s \le |\lambda^{[1]}|$ and $\lambda \in \mathrm{DHP}(n)$, case by case.

**Case 1:** If $\lambda \in \mathrm{HP}(n)$, then $N^{(s)}(\lambda) = 1$.

Before considering the case $\lambda \in \mathrm{DHP}(n) \setminus \mathrm{HP}(n)$, we look at the following special case.

**Case 2:** If $\lambda = (\lambda_1, \lambda_2, 1^{m_1})$ and $\lambda \notin \mathrm{HP}(n)$, then we have

$$N^{(s)}(\lambda) = \begin{cases} 0 & \text{if } s \ge m_1 + 2, \\ 1 & \text{if } s = 0 \text{ or } s = m_1 + 1, \\ 2 & \text{if } 1 \le s \le m_1. \end{cases} \tag{4-19}$$

**Case 3:** If $\lambda = (\lambda_1, \lambda_2, 2^{m_2}, 1^{m_1}) \in \mathrm{DHP}(n) \setminus \mathrm{HP}(n)$, then it follows from case 2 that

$$\begin{aligned} N^{(s)}(\lambda) &= N^{(s-m_2)}((\lambda_1, \lambda_2, 1^{m_1})) \\ &= \begin{cases} 0 & \text{if } 0 \le s \le m_2 - 1 \text{ or } s \ge m_1 + m_2 + 2, \\ 1 & \text{if } s = m_2 \text{ or } s = m_1 + m_2 + 1, \\ 2 & \text{if } 1 + m_2 \le s \le m_1 + m_2. \end{cases} \end{aligned} \tag{4-20}$$

**Example 4.6.** Given $\xi = (4, 3)$, $\lambda = (2, 2, 2, 1)$, we have $\lambda_1 = \lambda_2 = 2$, $m_1 = m_2 = 1$, and

$$b_{(4,3)(2,2,2,1)} = 4(N^{(2)}(\lambda) - N^{(1)}(\lambda)) = 4 \times (2 - 1) = 4.$$

The symmetric group $\mathfrak{S}_n$ has a two-valued representation, known as the spin representation studied by Schur, and this is actually a representation of the double covering group $\widetilde{\mathfrak{S}}_n$ of $\mathfrak{S}_n$ [Schur 1911]. It is known that the irreducible spin representations of $\mathfrak{S}_n$ are parametrized by strict partitions of $n$. Let $\zeta^\lambda$ be the irreducible spin character of the Schur double covering group $\widetilde{\mathfrak{S}}_n$ afforded by the module $V^\lambda$, $\lambda \in \mathcal{SP}_n$. Stembridge [1989] obtained the irreducible decomposition for the twisted tensor product of $\widetilde{\mathfrak{S}}_n$ [Kleshchev 2005]

$$\boldsymbol{ch}(\zeta^{(n)} \otimes \zeta^\lambda) = P_\lambda(x; -1),$$

where $\boldsymbol{ch}$ is the characteristic map (cf. [Jing 1991b]).

**Corollary 4.7.** *Let $S^\lambda$ be the Specht module corresponding to partition $\lambda \vdash n$ and $1 \le m < \frac{n}{2}$. Then we have the irreducible decomposition as $\mathfrak{S}_n$-modules*

$$V^{(n)} \otimes V^{(n-m,m)} \simeq \bigoplus_{\lambda \in \mathrm{DHP}(n)} (N^{(n-m-\lambda_1)}(\lambda) - N^{(m-\lambda_1)}(\lambda)) S^\lambda. \tag{4-21}$$

Aokage [2021b] obtained the explicit irreducible decomposition of $(V^{(n)})^{\otimes 2}$ when $n$ is even, so (4-21) offers the formula for a general tensor product. Recall that the symmetric functions $P_\mu(x; -1)$ are well defined for all partitions $\mu$, so $g_{\mu\lambda}$ are defined similarly as (4-1) for any partitions $\lambda$, $\mu$:

$$P_\mu(x; -1) = \sum_\lambda g_{\mu\lambda} s_\lambda(x). \tag{4-22}$$

Note that the following identities between the Schur $P$-functions and the Schur functions hold by using the tensor product of the spin representations of the symmetric group [Aokage 2021a]:

$$\begin{aligned} \sum_{\lambda \in \mathrm{HP}(n) \setminus \mathrm{HOP}(n)} s_\lambda(x) &= \sum_{l(\mu) \le 2} (-1)^{\mu_2} P_\mu(x; -1), \\ \sum_{\lambda \in \mathrm{HOP}(n)} s_\lambda(x) &= \sum_{l(\mu) = 2} (-1)^{\mu_2 + 1} P_\mu(x; -1), \end{aligned} \tag{4-23}$$

where $\mathrm{HOP}(n) \doteq \{\lambda \in \mathrm{HP}(n) \mid \lambda_1 \text{ is odd}\}$ and $n = 2r$ is even.

Aokage [2021a] has this conjecture at the end of his paper:

**Theorem 4.8.** *For $\lambda = (n - j, 1^j) \in \mathrm{HP}(n)$,*

$$g_{(r^2)\lambda} = \begin{cases} 0 & \text{if } j < r, \\ (-1)^{r+j} & \text{if } j \ge r. \end{cases} \tag{4-24}$$

As an application of our two-row formula for $b_{\xi\lambda}$, we will present a proof of Aokage's conjecture.

Combining with the above two identities in (4-23), we have

$$P_n(x;-1)+2\sum_{i\geq 1}^{r}(-1)^i P_{(n-i,i)}(x;-1)=\sum_{j=0}^{n}(-1)^j s_{(n-j,1^j)}(x).$$

Thus,

$$P_{(r^2)}(x;-1)=\frac{1}{4}\sum_{i\geq 0}^{r-1}(-1)^{i+r+1}Q_{(n-i,i)}(x;-1)+\frac{1}{2}\sum_{j=0}^{n}(-1)^{r+j}s_{(n-j,1^j)}(x).$$

By the orthonormality of $s_\lambda$,

$$g_{(r^2)\lambda}=\frac{1}{4}\sum_{i\geq 0}^{r-1}(-1)^{i+r+1}b_{(n-i,i)\lambda}+\frac{1}{2}(-1)^{r+j}\delta_{(n-j,1^j)\lambda}.$$

It follows from the remark below Theorem 4.5, we have $g_{(r^2)\lambda}=0$ unless $\lambda\in\mathrm{DHP}(n)$. Now let's show Theorem 4.8.

*Proof.* Let $\lambda=(n-j,1^j)\in\mathrm{HP}(n)$. We have

$$\begin{aligned}
g_{(r^2)\lambda}&=\frac{1}{2}(-1)^{r+1}+\sum_{i=1}^{r-1}(-1)^{i+r+1}(N^{(j-i)}(\lambda)-N^{(i+j-n)}(\lambda))+\frac{1}{2}(-1)^{r+j}\\
&=\frac{1}{2}(-1)^{r+1}+(-1)^{r+1}\left(\sum_{i=1}^{\min\{r-1,j\}}(-1)^i-\sum_{i=n-j}^{r-1}(-1)^i\right)+\frac{1}{2}(-1)^{r+j}.
\end{aligned}$$

Then the result follows immediately by a careful analysis of $j$ and direct computation. □

We remark that there exists a quadratic expression of the $P$-function in terms of Schur functions [Lascoux et al. 1993]. Explicit and direct linear expansion (4-22) in general is thus needed. Indeed, we can give a compact formula of $g_{(r^2)\lambda}$ for any partition $\lambda$.

**Theorem 4.9.** *For* $\lambda=(\lambda_1,\lambda_2,2^{m_2},1^{m_1})\in\mathrm{DHP}(n)\setminus\mathrm{HP}(n)$, *we have that*

$$g_{(r^2)\lambda}=\sum_{i=1}^{r-1}(-1)^{i+r+1}(N^{(n-i-\lambda_1)}(\lambda)-N^{(i-\lambda_1)}(\lambda)). \tag{4-25}$$

*By considering* $\lambda$ *case by case, we have that*

$$g_{(r^2)\lambda}=\begin{cases}1 & \text{if } \lambda_2+m_1-1\leq\lambda_1\leq\lambda_2+m_1+1,\\ 0 & \text{otherwise.}\end{cases}$$

## Tables for $K^-_{\xi\mu}(t)$, $2 \le n \le 6$

Here

$$[n] = t^{n-1} + \cdots + t + 1, \quad [n]!! = [n][n-2] \cdots .$$

For completeness, we include $n = 2, 3, 4$ from [Wan and Wang 2013].

| $\mu$ | $\xi =$ (2) |
|---|---|
| (2) | 2 |
| $(1^2)$ | 2[2] |

**Table 1.** $n = 2$.

| $\mu$ | $\xi =$ (3) | (2, 1) |
|---|---|---|
| (3) | 2 | 0 |
| (2, 1) | 2[2] | 4 |
| $(1^3)$ | 2[4] | $4t[2]$ |

**Table 2.** $n = 3$.

| $\mu$ | $\xi =$ (4) | (3, 1) |
|---|---|---|
| (4) | 2 | 0 |
| (3, 1) | 2[2] | 4 |
| $(2^2)$ | $2t[2]$ | 4[2] |
| $(2, 1^2)$ | 2[4] | $4[2]^2$ |
| $(1^4)$ | 2[6]!!/[3]! | $4t[4]!!$ |

**Table 3.** $n = 4$.

| $\mu$ | $\xi =$ (5) | (4, 1) | (3, 2) |
|---|---|---|---|
| (5) | 2 | 0 | 0 |
| (4, 1) | 2[2] | 4 | 0 |
| (3, 2) | $2t[2]$ | 4[2] | 4 |
| $(3, 1^2)$ | 2[4] | $4[2]^2$ | 4[2] |
| $(2^2, 1)$ | $2t[4]$ | 4[2][3] | $4[2]^2$ |
| $(2, 1^3)$ | 2[6]!!/[3]! | 4[4][3] | $4t[2]([3]+1)$ |
| $(1^5)$ | 2[8]!!/[4]! | $4t[6]!!/[2]$ | $4t^2[4]^2$ |

**Table 4.** $n = 5$.

| $\mu$ | $\xi =$ (6) | (5, 1) | (4, 2) | (3, 2, 1) |
|---|---|---|---|---|
| (6) | 2 | 0 | 0 | 0 |
| (5, 1) | 2[2] | 4 | 0 | 0 |
| (4, 2) | $2t[2]$ | 4[2] | 4 | 0 |
| $(4, 1^2)$ | 2[4] | $4[2]^2$ | 4[2] | 0 |
| (3, 3) | $2t^2[2]$ | $4t[2]$ | 4[2] | 0 |
| (3, 2, 1) | $2t[4]$ | 4[2][3] | $4[2](t+2)$ | 8 |
| $(3, 1^3)$ | 2[6]!!/[3]! | 4[4][3] | $4[2]^2[3]$ | $8t[2]$ |
| $(2^3)$ | $2t^3[4]$ | $4t[4]!!$ | $4[2]([4]+t^2)$ | $8t[2]$ |
| $(2^2, 1^2)$ | $2t[6]!!/[3]!$ | $4[4]^2$ | $4[2]^2([4]+t)$ | $8t[2]^2$ |
| $(2, 1^4)$ | 2[8]!!/[4]! | 4[4][6]!! | $4t[4]!!([4]+1)$ | $8t^2[4]!!$ |
| $(1^6)$ | 2[10]!!/[5]! | $4t[8]!!/[3]!$ | $4t^2[5][6]!!/[3]$ | $8t^4[6]!!/[3]$ |

**Table 5.** $n = 6$.

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NAIHUAN JING
DEPARTMENT OF MATHEMATICS
NORTH CAROLINA STATE UNIVERSITY
RALEIGH, NC
UNITED STATES

jing@ncsu.edu

NING LIU
SCHOOL OF MATHEMATICS
SOUTH CHINA UNIVERSITY OF TECHNOLOGY
GUANGZHOU
CHINA

mathliu123@outlook.com

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