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WADE HINDES AND JOSEPH H. SILVERMAN

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Let V be a projective variety defined over a number field K, let S be a polarized set of endomorphisms of V all defined over K, and let $P \in V(K)$. For each prime $\mathfrak{p}$ of K, let $m_{\mathfrak{p}}(S, P)$ denote the number of points in the orbit of P mod $\mathfrak{p}$ for the semigroup of maps generated by S. Under suitable hypotheses on S and P, we prove an analytic estimate for $m_{\mathfrak{p}}(S, P)$ and use it to show that the set of primes for which $m_{\mathfrak{p}}(S, P)$ grows subexponentially as a function of $\mathsf{N}_{K/\mathbb{Q}}\,\mathfrak{p}$ is a set of density zero. For $V = \mathbb{P}^1$ we show that this holds for a generic set of maps S provided that at least two of the maps in S have degree at least four.

1. Introduction

A general expectation in arithmetic dynamics over number fields is that the dynamical systems generated by "unrelated" self-maps $f_1, f_2 : V \to V$ should not be too similar. For example, they should not have identical canonical heights [16], they should not have infinitely many common preperiodic points [2; 8; 11], their orbits should not have infinite intersection [10], and arithmetically their orbits should not have unexpectedly large common divisors [15]. It is not always clear what "unrelated" should mean, but in any case it includes the assumption that f_1 and f_2 do not share a common iterate.

Similarly, we expect that the points in semigroup orbits generated by all finite compositions of "unrelated" maps f_1 and f_2 should be asymptotically large [4; 13] when ordered by height, where now unrelated means that the semigroup is not unexpectedly small. For example, the semigroup is small if it contains no free

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subsemigroups requiring at least 2 generators; cf. [3].

In this note, we study the size of semigroup orbits over finite fields. In particular, we show that, under suitable hypotheses, a free semigroup of maps defined over a number field generates many large orbits when reduced modulo primes. See [1; 6; 7; 21] for additional results in this vein.

Definition 1. We set notation that will remain in effect throughout this note.

$K/\mathbb{Q}$	a number field
V/K	a smooth projective variety defined over K
$r \geq 1$	an integer
$S = \{f_1, \ldots, f_r\}$	a set of morphisms $f_i : V \to V$ defined over K
$d_1, \ldots, d_r$	real numbers satisfying $d_i > 1$ for $i = 1, \ldots, r$
$\mathcal{L} \in \operatorname{Pic}_K(V) \otimes \mathbb{R}$	line bundles satisfying $f_i^*\mathcal{L} \cong \mathcal{L}^{\otimes d_i}$
M_S	the semigroup generated by S under composition
$\operatorname{Orb}_S(P)$	the orbit $\{f(P) : f \in M_S\}$ of a point $P \in V$

The following property will play a crucial role in some of our results.

Definition 2. A point $P \in V$ is called *strongly S-wandering* if the evaluation map

$$M_S \to V, \quad f \mapsto f(P), \tag{1}$$

is injective.

Remark 3. If $V = \mathbb{P}^1$ and S is any sufficiently generic set of maps as described in Section 3, then the set of points that fail to be strongly S-wandering is a set of bounded height. In particular, it follows in this case that all infinite orbits contain strongly wandering points, and this weaker condition is sufficient for our orbit bounds.

Our goal is to study the number of points in the reduction of $\operatorname{Orb}_S(P)$ modulo primes. We set some additional notation, briefly recall a standard definition, and then define our principal object of study.

R_K	the ring of integers of K
$\operatorname{Spec}(R_K)$	the set of prime ideals of R_K
$\mathsf{N}\mathfrak{p}$	the norm of $\mathfrak{p} \in \operatorname{Spec}(R_K)$, that is, $\mathsf{N}\mathfrak{p} := \#R_K/\mathfrak{p}$.

Definition 4. Let $\mathfrak{p} \in \operatorname{Spec}(R_K)$, and let $R_\mathfrak{p}$ denote the localization of R_K at $\mathfrak{p}$, and let $k_\mathfrak{p} = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ denote the residue field. A finite K-morphism $f : V \to V$ has *good reduction at* $\mathfrak{p}$ if there is a scheme $\mathcal{V}_\mathfrak{p}/R_\mathfrak{p}$ that is proper and smooth over $R_\mathfrak{p}$, and there is an $R_\mathfrak{p}$-morphism $F_\mathfrak{p} : \mathcal{V}_\mathfrak{p} \to \mathcal{V}_\mathfrak{p}$ whose generic fiber is $f : V \to V$.[1]

[1]Intuitively, this means that we can find equations for V and for f that have coefficients in R_K, and so that when we reduce the equations modulo $\mathfrak{p}$, the reduced variety $\tilde{V}$ mod $\mathfrak{p}$ is non-singular and the reduced map $\tilde{f} : \tilde{V} \to \tilde{V}$ is a morphism having the same degree as f. Of course, when we say "find equations", this needs to be done locally on an appropriately fine cover by affine neighborhoods.

We write $\tilde{\mathcal{V}}_\mathfrak{p} = \mathcal{V}_\mathfrak{p} \times_{R_\mathfrak{p}} k_\mathfrak{p}$ for the special fiber of $\mathcal{V}_\mathfrak{p}$. Properness implies that each point $Q \in V(K)$ extends to a section $\mathcal{Q}_\mathfrak{p} \in \mathcal{V}_\mathfrak{p}(R_\mathfrak{p})$, and the *reduction* $\tilde{Q}_\mathfrak{p} \in \tilde{\mathcal{V}}_\mathfrak{p}(k_\mathfrak{p})$ *of* Q *modulo* $\mathfrak{p}$ is the intersection of the image of $\mathcal{Q}_\mathfrak{p}$ with the fiber $\tilde{\mathcal{V}}_\mathfrak{p}$, i.e.,

$$\{\tilde{Q}_\mathfrak{p}\} = \mathcal{Q}_\mathfrak{p}(\operatorname{Spec} R_\mathfrak{p}) \cap \mathcal{V}_\mathfrak{p}.$$

Similarly, the *reduction* $\tilde{f}_\mathfrak{p}$ *of* f *modulo* $\mathfrak{p}$ is the restriction of $F_\mathfrak{p}$ to the special fiber $\tilde{\mathcal{V}}_\mathfrak{p}$.

Remark 5. Continuing with notation from Definition 4, we note that if f has good reduction at $\mathfrak{p}$, then reduction modulo $\mathfrak{p}$ commutes with evaluation,

$$\widetilde{f(Q)}_\mathfrak{p} = \tilde{f}_\mathfrak{p}(\tilde{Q}_\mathfrak{p}).$$

Further, composition commutes with reduction for good reduction maps. In other words, if f and g have good reduction at $\mathfrak{p}$, then

$$\big(\widetilde{f \circ g}\big)_\mathfrak{p} = \tilde{f}_\mathfrak{p} \circ \tilde{g}_\mathfrak{p}.$$

Definition 6. Let $\mathfrak{p} \in \operatorname{Spec}(R_K)$. Continuing with notation from Definition 4, let $f_1, \dots, f_r : V \to V$ be maps that have good reduction modulo $\mathfrak{p}$, and let $P \in V(K)$. Then the reduction of the S-orbit of P modulo $\mathfrak{p}$ is the set

$$\operatorname{Orb}_{\tilde{S}}(\tilde{P} \bmod \mathfrak{p}) := \big\{\tilde{f}_\mathfrak{p}(\tilde{P}_\mathfrak{p}) : f \in M_S\big\}.$$

We define

$$m_\mathfrak{p} := m_\mathfrak{p}(S, P) = \#\operatorname{Orb}_{\tilde{S}}(\tilde{P} \bmod \mathfrak{p})$$

to be the size of the mod $\mathfrak{p}$ reduction of $\operatorname{Orb}_S(P)$. (If any of the maps $f_1, \dots, f_r$ has bad reduction at $\mathfrak{p}$, then we formally set $m_\mathfrak{p} = \infty$.)

Our main result is an analytic formula that implies that $m_\mathfrak{p}$ is not too small on average.

Theorem 7. *Assume that M_S is a free semigroup, that $P \in V(K)$ is a strongly S-wandering point, and that $r = \#S \geq 2$. Then there exists a constant $C_1 = C_1(K, V, S, P)$ such that, for all $\epsilon > 0$,*

$$\sum_{\mathfrak{p} \in \operatorname{Spec}(R_K)} \frac{\log \mathsf{N}\mathfrak{p}}{\mathsf{N}\mathfrak{p} \cdot m_\mathfrak{p}(S, P)^\epsilon} \leq C_1 \epsilon^{-1}. \tag{2}$$

Remark 8. The principal result of the paper [21] is an estimate exponentially weaker than (2) in the case that $r = \#S = 1$, while a principal result of the paper [17] is an estimate that exactly mirrors (2) with $m_\mathfrak{p}$ equal to the number of points on the mod $\mathfrak{p}$ reduction of the multiples of a point on an abelian variety. Thus the present paper, as well as the papers [4; 13], suggest that the analogy

$$\begin{pmatrix}\text{arithmetic of points}\\ \text{of an abelian variety}\end{pmatrix} \iff \begin{pmatrix}\text{arithmetic of points in orbits}\\ \text{of a dynamical system}\end{pmatrix}$$

described in [5] and [22, §6.5] may be more accurate when the dynamical system on the right-hand side is generated by at least two non-commuting maps, rather than using orbits coming from iteration of a single map.

Estimate (2) can be used to show that there are few primes $\mathfrak{p}$ for which $m_{\mathfrak{p}}(S, P)$ is subexponential compared to $\mathsf{N}\mathfrak{p}$. We quantify this assertion in the following corollary.

Corollary 9. *Let S, M_S and P be as in Theorem 7, and let $\overline{\delta}$ and δ denote the (upper) logarithmic analytic densities on sets of primes as described in Definition 12.*

(a) *There is a constant $C_2 = C_2(K, V, S, P)$ such that*

$$\overline{\delta}\big(\{\mathfrak{p} \in \operatorname{Spec}(R_K) : m_{\mathfrak{p}}(S, P) \le \mathsf{N}\mathfrak{p}^{\gamma}\}\big) \le C_2\gamma$$

holds for all $0 < \gamma < 1$.

(b) *Let $L(t)$ be a subexponential function, i.e., a function with the property that*

$$\lim_{t\to\infty} \frac{L(t)}{t^{\mu}} = 0 \quad \textit{for all } \mu > 0.$$

Then

$$\delta\big(\{\mathfrak{p} \in \operatorname{Spec}(R_K) : m_{\mathfrak{p}}(S, P) \le L(\mathsf{N}\mathfrak{p})\}\big) = 0.$$

In the special case that $V = \mathbb{P}^1$, we show that the conclusions of Theorem 7 and Corollary 9 are true for generic sets of maps. In the statement of the next result, we write Rat_d for the space of rational maps of $\mathbb{P}^1$ of degree $d \ge 2$, so in particular Rat_d is an affine variety of dimension $2d + 1$; see [20, §4.3] for details.

Theorem 10. *Let $r \ge 2$, and let $d_1, \dots, d_r$ be integers satisfying*

$$d_1, d_2 \ge 4 \quad \textit{and} \quad d_3, \dots, d_r \ge 2.$$

Then there is a Zariski dense subset

$$\mathcal{U} = \mathcal{U}(d_1, \dots, d_r) \subseteq \operatorname{Rat}_{d_1} \times \cdots \times \operatorname{Rat}_{d_r}$$

such that the inequality (2) in Theorem 7 and the density estimates in Corollary 9 are true for all number fields $K/\mathbb{Q}$, all $S \in \mathcal{U}(K)$, and all $P \in \mathbb{P}^1(K)$ for which $\operatorname{Orb}_S(P)$ is infinite.

The contents of this paper are as follows. In Section 2 we build upon prior work [17; 21] of the second author to prove Theorem 7 and Corollary 9. Then in Section 3 we use results from [10; 13; 23] to construct many sets of maps on $\mathbb{P}^1$

for which the bounds in Section 2 apply. The key step is to construct a point in every infinite orbit that is strongly wandering. The construction is explicit, and in particular, Theorem 15 describes an explicit set $\mathcal{U}$ for which Theorem 10 is true.

2. The size of orbits modulo $\mathfrak{p}$

We start with a key estimate.

Proposition 11. *Let S, M_S and P be as in Theorem 7. For each $m \geq 2$, we define an integral ideal*

$$\mathfrak{D}(m) = \mathfrak{D}(m; K, V, S, P) := \prod_{\substack{\mathfrak{p}\in \operatorname{Spec}(R_K) \\ m_{\mathfrak{p}}(S,P)\leq m}} \mathfrak{p}. \tag{3}$$

There are constants $C_i = C_i(K, V, S, P)$ for $i = 3, 4$ such that the following hold:

(a) *If $r = \#S = 1$, then*

$$\log\log \mathsf{N}\mathfrak{D}(m) \leq C_3 m \quad \textit{for all } m \geq 2.$$

(b) *Assume that S generates a free semigroup, that $P \in V(K)$ is strongly S-wandering, and that $r = \#S \geq 2$. Then*

$$\log\log \mathsf{N}\mathfrak{D}(m) \leq C_4 \log m \quad \textit{for all } m \geq 2.$$

Proof. (a) This is [21, Proposition 10].

(b) Next, since V and S are polarized with respect to some line bundle $\mathcal{L}$, we may choose $N \geq 1$ and an embedding $V \subseteq \mathbb{P}^N$ such that the f_i extend to self-morphisms of $\mathbb{P}^N$. Next, for notational convenience, we write $m_{\mathfrak{p}}$ for $m_{\mathfrak{p}}(S, P)$ and use the standard combinatorics notation $[r] = \{1, 2, \dots, r\}$. Also to ease notation, we write

$$f_{\boldsymbol{i}} := f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_k} \quad \text{for} \quad \boldsymbol{i} = (i_1, \dots, i_k) \in [r]^k.$$

Let

$$m \geq 1 \quad \text{and} \quad k = k(m) := \left\lceil \frac{\log(m+1)}{\log r} \right\rceil.$$

For each good reduction prime $\mathfrak{p}$, we consider the map that sends a function $f_{\boldsymbol{i}}$ to the image of P under reduction modulo $\mathfrak{p}$,

$$[r]^k \to \mathcal{O}_{\tilde{S}}(\tilde{P} \bmod \mathfrak{p}), \quad \boldsymbol{i} \mapsto f_{\boldsymbol{i}}(P) \pmod{\mathfrak{p}}. \tag{4}$$

If

$$m_{\mathfrak{p}} \leq m, \quad \text{then} \quad r^k > m_{\mathfrak{p}} \quad \text{by our choice of } k,$$

so the map (4) cannot be injective (pigeonhole principle) and there exist

$$\boldsymbol{i} \neq \boldsymbol{j} \text{ in } [r]^k \quad \text{satisfying} \quad f_{\boldsymbol{i}}(P) \equiv f_{\boldsymbol{j}}(P) \pmod{\mathfrak{p}}.$$

Sine we have assumed that P is strongly wandering, i.e., that the map

$$M_S \to V(K), \quad f \mapsto f(P),$$

is injective, it follows that the global points are distinct,

$$f_{\boldsymbol{i}}(P) \neq f_{\boldsymbol{j}}(P),$$

so the ideals generated by their "differences" are non-zero.

More formally, [21, Lemma 9] says that there is an integral ideal $\mathfrak{C} = \mathfrak{C}_K \subseteq R_K$ with the property that every point $Q \in \mathbb{P}^N(K)$ can be written with homogeneous coordinates

$$Q = [\alpha_0, \dots, \alpha_N]$$

satisfying[2]

$$\alpha_0, \dots, \alpha_N \in R_K \quad \text{and} \quad (\alpha_0 R_K + \cdots \alpha_N R_K) \bigm| \mathfrak{C}_K.$$

Applying [21, Lemma 9] to our situation, for the given $P \in V(K)$, we can write

$$f_{\boldsymbol{i}}(P) = \bigl[A_0(\boldsymbol{i}), \dots, A_N(\boldsymbol{i})\bigr]$$

with $A_0(\boldsymbol{i}), \dots, A_N(\boldsymbol{i}) \in R_K$ and such that the ideal

$$\mathfrak{A}(\boldsymbol{i}) := A_0(\boldsymbol{i})R_K + \cdots + A_N(\boldsymbol{i})R_K \quad \text{divides the ideal } \mathfrak{C}. \tag{5}$$

Then for $\mathfrak{p} \nmid \mathfrak{C}$ we have

$$\begin{aligned} &f_{\boldsymbol{i}}(P) \equiv f_{\boldsymbol{j}}(P) \pmod{\mathfrak{p}} \\ &\qquad \iff \quad A_u(\boldsymbol{i})A_v(\boldsymbol{j}) \equiv A_v(\boldsymbol{i})A_u(\boldsymbol{j}) \pmod{\mathfrak{p}} \quad \text{for all } 0 \le u, v \le N. \end{aligned}$$

We define a difference ideal

$$\mathfrak{B}(\boldsymbol{i}, \boldsymbol{j}) := \sum_{0 \le u, v \le N} \bigl(A_u(\boldsymbol{i})A_v(\boldsymbol{j}) - A_v(\boldsymbol{i})A_u(\boldsymbol{j})\bigr)R_K,$$

and the product of the difference ideals

$$\mathfrak{D}'(m) := \prod_{\boldsymbol{i}, \boldsymbol{j} \in [r]^k,\ \boldsymbol{i} \neq \boldsymbol{j}} \mathfrak{B}(\boldsymbol{i}, \boldsymbol{j}).$$

Then

$$\mathfrak{p} \nmid \mathfrak{C} \text{ and } m_{\mathfrak{p}} \le m \quad \Longrightarrow \quad \mathfrak{p} \mid \mathfrak{D}'(m),$$

and hence

$$\mathfrak{D}(m) \mid \mathfrak{C} \cdot \mathfrak{D}'(m).$$

[2] If R_K is a PID, then we can take $\mathfrak{C}_K = R_K$, so $\gcd(\alpha_0, \dots, \alpha_N) = 1$. Thus [21, Lemma 9] provides a weaker version of this gcd result that holds for all R_K.

Since $\mathfrak{C}$ depends only on K, it remains to estimate the norm of $\mathfrak{D}'(m)$.

Let $h(\cdot)$ denote the logarithmic Weil height on $\mathbb{P}^N$, and let $\mathfrak{A}(\boldsymbol{i})$ for $\boldsymbol{i} \in [r]^k$ be the ideals defined by (5). Then, using [21, Proposition 7], we find for all $\boldsymbol{i}$ and $\boldsymbol{j}$ that

$$\frac{1}{[K:\mathbb{Q}]} \log \frac{\mathsf{N}\mathfrak{B}(\boldsymbol{i}, \boldsymbol{j})}{\mathsf{N}\mathfrak{A}(\boldsymbol{i}) \cdot \mathsf{N}\mathfrak{A}(\boldsymbol{j})} \le h(f_{\boldsymbol{i}}(P)) + h(f_{\boldsymbol{j}}(P)) + C_5,$$

where C_5 is an absolute constant. Since $\mathsf{N}\mathfrak{A}(\boldsymbol{i})$ and $\mathsf{N}\mathfrak{A}(\boldsymbol{j})$ are smaller than $\mathsf{N}\mathfrak{C}$, this implies that

$$\frac{1}{[K:\mathbb{Q}]} \log \mathsf{N}\mathfrak{B}(\boldsymbol{i}, \boldsymbol{j}) \le h(f_{\boldsymbol{i}}(P)) + h(f_{\boldsymbol{j}}(P)) + C_6.$$

Next we apply the height estimate

$$h(f_{\boldsymbol{i}}(P)) \le C_7 \cdot \prod_{u=1}^{k} d_{i_u},$$

which is a weak form of [12, Lemma 2.1]. This yields

$$\frac{1}{[K:\mathbb{Q}]} \log \mathsf{N}\mathfrak{B}(\boldsymbol{i}, \boldsymbol{j}) \le C_7 \cdot \prod_{u=1}^{k} d_{i_u} + C_7 \cdot \prod_{u=1}^{k} d_{j_u} + C_6.$$

This gives

$$\begin{aligned}
\log \mathfrak{D}'(m) &= \sum_{\boldsymbol{i},\boldsymbol{j} \in [r]^k,\, \boldsymbol{i} \ne \boldsymbol{j}} \log \mathfrak{B}(\boldsymbol{i}, \boldsymbol{j}) \\
&\le \sum_{\boldsymbol{i},\boldsymbol{j} \in [r]^k,\, \boldsymbol{i} \ne \boldsymbol{j}} \left(C_7 \cdot \prod_{u=1}^{k} d_{i_u} + C_7 \cdot \prod_{u=1}^{k} d_{j_u} + C_6 \right) \\
&\le C_8 \cdot r^k \cdot \sum_{\boldsymbol{i} \in [r]^k} \prod_{u=1}^{k} d_{i_u} \\
&= C_8 \cdot \left(r \cdot \sum_{i \in [r]} d_i \right)^k \\
&\le C_9 \cdot \left(r \cdot \sum_{i \in [r]} d_i \right)^{1 + \frac{\log(m+1)}{\log r}}.
\end{aligned}$$

Hence

$$\log \log \mathfrak{D}'(m) \le C_{10} \cdot \log(m+1) + C_{11}.$$

Since $m \ge 2$, we can absorb C_{11} into C_{10}, although we remark that if we leave in $C_{11}(K, V, S, P)$, then we can take C_{10} to depend on only the degrees of the maps in S,

$$C_{10} = C_{10}(d_1, \dots, d_r) = 1 + \frac{\log(d_1 + \cdots + d_r)}{\log r}.$$

This completes the proof of Proposition 11. $\square$

Proof of Theorem 7. To ease notation, we let

$$g(t) = \frac{\log t}{t} \quad \text{and} \quad G(t) = \frac{1}{t^{\epsilon}}.$$

We start with two elementary estimates. First, the mean value theorem gives

$$G(m) - G(m+1) \le \sup_{m \le t \le m+1} -G'(t) = \sup_{m \le t \le m+1} \frac{\epsilon}{t^{1+\epsilon}} = \frac{\epsilon}{m^{1+\epsilon}}. \tag{6}$$

Second, an easy integral calculation gives

$$\sum_{m \ge 1} g(m)G(m) \le \int_1^{\infty} \frac{\log x}{x^{1+\epsilon}}\, \mathrm{dx} = \frac{1}{\epsilon^2}. \tag{7}$$

We use these and our other calculations to estimate

$$\begin{aligned}
&\sum_{\mathfrak{p} \in \operatorname{Spec}(R_K)} \frac{\log \mathsf{N}\mathfrak{p}}{\mathsf{N}\mathfrak{p} \cdot m_{\mathfrak{p}}^{\epsilon}} \\
&\qquad = \sum_{\mathfrak{p} \in \operatorname{Spec}(R_K)} g(\mathsf{N}\mathfrak{p}) \cdot G(m_{\mathfrak{p}}) && \text{(by definition of } g \text{ and } G) \\
&\qquad = \sum_{m \ge 1} G(m) \sum_{\substack{\mathfrak{p} \in \operatorname{Spec}(R_K) \\ m_{\mathfrak{p}} = m}} g(\mathsf{N}\mathfrak{p}) \\
&\qquad = \sum_{m \ge 1} \bigl(G(m) - G(m+1)\bigr) \sum_{\substack{\mathfrak{p} \in \operatorname{Spec}(R_K) \\ m_{\mathfrak{p}} \le m}} g(\mathsf{N}\mathfrak{p}) && \text{(Abel summation)} \\
&\qquad \le \sum_{m \ge 1} \frac{\epsilon}{m^{1+\epsilon}} \sum_{\substack{\mathfrak{p} \in \operatorname{Spec}(R_K) \\ m_{\mathfrak{p}} \le m}} g(\mathsf{N}\mathfrak{p}) && \text{(from (6))} \\
&\qquad = \sum_{m \ge 1} \frac{\epsilon}{m^{1+\epsilon}} \sum_{\substack{\mathfrak{p} \in \operatorname{Spec}(R_K) \\ \mathfrak{p} \mid \mathfrak{D}(m)}} g(\mathsf{N}\mathfrak{p}) && \text{(by definition (3) of } \mathfrak{D}(m)) \\
&\qquad \le \sum_{m \ge 1} \frac{\epsilon}{m^{1+\epsilon}} \cdot \bigl(C_{12} \log\log \mathfrak{D}(m) + C_{13}\bigr) && \text{(from [17, Corollary 2.3])} \\
&\qquad \le C_{14} \sum_{m \ge 1} \frac{\epsilon}{m^{1+\epsilon}} \cdot \log m && \text{(from Proposition 11(b))} \\
&\qquad = C_{14} \cdot \epsilon \cdot \sum_{m \ge 1} g(m) \cdot G(m) && \text{(by definition of } g \text{ and } G) \\
&\qquad \le C_{15} \epsilon^{-1} && \text{(from (7)).}
\end{aligned}$$

$\square$

Definition 12. Let $\mathcal{P} \subset \operatorname{Spec}(R_K)$ be a set of primes. The *upper logarithmic analytic density of* $\mathcal{P}$ is

$$\overline{\delta}(\mathcal{P}) := \limsup_{s\to 1^+}(s-1)\sum_{\mathfrak{p}\in\mathcal{P}} \frac{\log \mathsf{N}\mathfrak{p}}{\mathsf{N}\mathfrak{p}^s}.$$

Similarly, the *logarithmic analytic density of* $\mathcal{P}$, denoted $\delta(P)$, is given by the same formula with a limit, instead of a lim sup.

Proof of Corollary 9. (a) For any $0 < \gamma < 1$, we let

$$\mathcal{P}_\gamma := \left\{\mathfrak{p} \in \operatorname{Spec}(R_K) : m_\mathfrak{p} \le \mathsf{N}\mathfrak{p}^\gamma\right\}.$$

Then

$$\begin{aligned} \frac{C_1}{\epsilon} &\ge \sum_{\mathfrak{p}\in\operatorname{Spec}(R_K)} \frac{\log \mathsf{N}\mathfrak{p}}{\mathsf{N}\mathfrak{p}\cdot m_\mathfrak{p}^\epsilon} && \text{(from Theorem 7)} \\ &\ge \sum_{\mathfrak{p}\in\mathcal{P}_\gamma} \frac{\log \mathsf{N}\mathfrak{p}}{\mathsf{N}\mathfrak{p}\cdot m_\mathfrak{p}^\epsilon} && \text{(summing over a smaller set)} \\ &\ge \sum_{\mathfrak{p}\in\mathcal{P}_\gamma} \frac{\log \mathsf{N}\mathfrak{p}}{\mathsf{N}\mathfrak{p}^{1+\gamma\epsilon}} && \text{(by definition of } \mathcal{P}_\gamma). \end{aligned} \tag{8}$$

This allows us to estimate the upper logarithmic density of $\mathcal{P}_\gamma$ by

$$\begin{aligned} \overline{\delta}(\mathcal{P}_\gamma) &= \limsup_{s\to 1^+}(s-1)\sum_{\mathfrak{p}\in\mathcal{P}_\gamma} \frac{\log \mathsf{N}\mathfrak{p}}{\mathsf{N}\mathfrak{p}^s} \\ &= \limsup_{\epsilon\to 0^+} \gamma\epsilon \sum_{\mathfrak{p}\in\mathcal{P}_\gamma} \frac{\log \mathsf{N}\mathfrak{p}}{\mathsf{N}\mathfrak{p}^{1+\gamma\epsilon}} && \text{(setting } s = 1+\gamma\epsilon) \\ &\le \limsup_{\epsilon\to 0^+} \gamma\epsilon \cdot \frac{C_1}{\epsilon} && \text{(from (8))} \\ &= C_1\gamma. \end{aligned}$$

This completes the proof of Corollary 9(a).

(b) We let

$$\mathcal{P}_L := \left\{\mathfrak{p} \in \operatorname{Spec}(R_K) : m_\mathfrak{p} \le L(\mathsf{N}\mathfrak{p})\right\}.$$

The assumption that L is subexponential means that for all $\mu > 0$ there exists a constant $C_{16}(L,\mu)$ depending only on L and μ such that

$$L(t) \le t^\mu \quad \text{for all } t > C_{16}(L,\mu).$$

We also note that

$$
\begin{aligned}
\mathfrak{p} \in \mathcal{P}_L \quad &\iff \quad m_{\mathfrak{p}} \le L(\mathsf{N}\mathfrak{p}) \\
&\implies \quad m_{\mathfrak{p}} \le (\mathsf{N}\mathfrak{p})^{\mu} \quad \text{for all } \mathsf{N}\mathfrak{p} > C_{16}(L,\mu).
\end{aligned} \tag{9}
$$

We now fix a $\mu > 0$ and estimate

$$
\begin{aligned}
\overline{\delta}(\mathcal{P}_L) &= \limsup_{\lambda\to 0^+} \lambda \sum_{\mathfrak{p}\in\mathcal{P}_L} \frac{\log \mathsf{N}\mathfrak{p}}{\mathsf{N}\mathfrak{p}^{1+\lambda}} \\
&= \limsup_{\lambda\to 0^+} \lambda \sum_{\substack{\mathfrak{p}\in\mathcal{P}_L \\ \mathsf{N}\mathfrak{p}\ge C_{16}(L,\mu)}} \frac{\log \mathsf{N}\mathfrak{p}}{\mathsf{N}\mathfrak{p}^{1+\lambda}} && \text{(since μ is fixed, so we can discard finitely many terms)} \\
&\le \limsup_{\lambda\to 0^+} \lambda \sum_{\substack{\mathfrak{p}\in\mathcal{P}_L \\ \mathsf{N}\mathfrak{p}\ge C_{16}(L,\mu)}} \frac{\log \mathsf{N}\mathfrak{p}}{\mathsf{N}\mathfrak{p}} \cdot \frac{1}{m_{\mathfrak{p}}^{\lambda/\mu}} && \text{(from (9))} \\
&\le \limsup_{\lambda\to 0^+} \lambda \sum_{\mathfrak{p}\in\operatorname{Spec} R_K} \frac{\log \mathsf{N}\mathfrak{p}}{\mathsf{N}\mathfrak{p}} \cdot \frac{1}{m_{\mathfrak{p}}^{\lambda/\mu}} \\
&\le \limsup_{\lambda\to 0^+} \lambda \cdot C_1 \cdot \left(\frac{\lambda}{\mu}\right)^{-1} && \text{(from Theorem 7)} \\
&= C_1\mu.
\end{aligned}
$$

This estimate holds for all $\mu > 0$, so we find that

$$\overline{\delta}(\mathcal{P}_L) \le \inf_{\mu>0} C_1 \cdot \mu = 0,$$

which completes the proof that $\delta(\mathcal{P}_L) = 0$. □

3. Orbits of generic families of maps of $\mathbb{P}^1$

In this section, we show that there are many sets of endomorphisms of $\mathbb{P}^1$ for which Theorem 7 holds. To make this statement precise, we need some definitions.

Definition 13. Let f be a non-constant rational map of $\mathbb{P}^1$ defined over $\overline{\mathbb{Q}}$. A point $w \in \mathbb{P}^1(\overline{\mathbb{Q}})$ is a *critical value* of f if $f^{-1}(w)$ contains fewer than $\deg(f)$ elements. It is a *simple critical value* if

$$\# f^{-1}(w) = \deg(f) - 1.$$

The map f is *critically simple* if all of its critical values are simple.

Definition 14. Let f and g be non-constant rational maps of $\mathbb{P}^1$ with respective critical value sets $\mathrm{CritVal}_f$ and $\mathrm{CritVal}_g$. We say that f and g are *critically separated* if

$$\mathrm{CritVal}_f \cap \mathrm{CritVal}_g = \varnothing.$$

Our first result says that the conclusions of Theorem 7 and Corollary 9 hold for certain sets S that contain a pair of critically simple and critically separated maps and initial points P with infinite orbit.

Theorem 15. *Let $K/\mathbb{Q}$ be a number field, let S be a set of endomorphisms of $\mathbb{P}^1$ defined over K containing a pair of critically simple and critically separated maps of degree at least* 4, *and let $P \in \mathbb{P}^1(K)$ be a point with infinite S-orbit. Then there is a constant $C_{17} = C_{17}(K, S, P)$ such that for all $\epsilon > 0$,*

$$\sum_{\mathfrak{p}\in\operatorname{Spec}(R_K)} \frac{\log \mathsf{N}\mathfrak{p}}{\mathsf{N}\mathfrak{p} \cdot m_{\mathfrak{p}}(S, P)^{\epsilon}} \le C_{17} \cdot \epsilon^{-1}.$$

Remark 16. In particular, there is a constant $C_{18}(S)$ such that Theorem 15 holds for all $P \in \mathbb{P}^1(K)$ satisfying $h(P) > C_{18}(S)$; see Lemma 18.

We start with a definition and some basic height estimates. In what follows, we fix an embedding $V \subseteq \mathbb{P}^N$ and extend the maps f_i to self-morphisms of $\mathbb{P}^N$; here we use our assumption that S is polarizable with respect to some line bundle $\mathcal{L}$. Moreover, $h(\,\cdot\,)$ denotes the logarithmic Weil height on $\mathbb{P}^N$.

Definition 17. A point $P \in V$ is *moderately S-preperiodic* if

$$g \circ f(P) = f(P) \quad \text{for some } f, g \in M_S \text{ with } g \neq 1.$$

Lemma 18. *Let $V/\overline{\mathbb{Q}}$ be a variety, and let $S = \{f_1, \dots, f_r\}$ be a set of polarized endomorphisms as described in Definition 1. Then there exists a constant $C_{19} = C_{19}(S, V, \mathcal{L})$ such that the following statements hold for all $Q \in V(\overline{\mathbb{Q}})$:*

(a) *If Q is moderately S-preperiodic as described in Definition 17, then $h(Q) \le C_{19}$. In particular, this is true if* $\operatorname{Orb}_S(Q)$ *is finite.*

(b) *If $h(Q) > C_{19}$, then*

$$h(f(Q)) \ge h(Q) \quad \textit{for all } f \in M_S.$$

Proof. These estimates are proven in [4, Lemma 2.11]. □

We combine Lemma 18 with the techniques in [13; 19] to obtain the following result for pairs of maps that are critically simple and critically separated.

Proposition 19. *Let f_1 and f_2 be endomorphisms of $\mathbb{P}^1$ of degree at least* 4, *let $S = \{f_1, f_2\}$, and suppose that f_1 and f_2 are critically simple and critically separated. Then the following statements hold*:

(a) *The semigroup M_S is free.*

(b) *Let $P \in \mathbb{P}^1(\overline{\mathbb{Q}})$ be a point whose S-orbit* $\operatorname{Orb}_S(P)$ *is infinite. Then there exists a point $Q \in$* $\operatorname{Orb}_S(P)$ *such that Q is strongly S-wandering as described in Definition 2.*

Proof. (a) See [13, Proposition 4.1].

(b) We fix a number field K over which P, f_1, and f_2 are defined. Letting $\Delta \subset \mathbb{P}^1 \times \mathbb{P}^1$ be the diagonal, we define three curves

$$\begin{aligned} \Gamma_i &:= (f_i \times f_i)^{-1}(\Delta) \quad \text{for } i = 1, 2, \\ \Gamma_{1,2} &:= (f_1 \times f_2)^{-1}(\Delta). \end{aligned}$$

Then the main results in [19] (see also [13, Proposition 4.6]) imply that the curves Γ_1 and Γ_2 are each the union of Δ and an irreducible curve of geometric genus ≥ 2, while $\Gamma_{1,2}$ is itself an irreducible curve of geometric genus ≥ 2.

More specifically, the assumption that f_1 and f_2 are critically simple implies from [19, Corollary 3.6] that $\mathcal{C}_1 \smallsetminus \Delta$ and $\mathcal{C}_2 \smallsetminus \Delta$ are irreducible, while the assumption that f_1 and f_2 are critically separated implies from [19, Proposition 3.1] that $\mathcal{C}_{1,2}$ is irreducible. It then follows from [19, pages 208 and 210] that the geometric genera of these curves are given by the formulas

$$\begin{aligned} \operatorname{genus}(\Gamma_i \smallsetminus \Delta) &= \bigl(\deg(f_i) - 2\bigr)^2 \quad \text{for } i = 1, 2, \\ \operatorname{genus}(\Gamma_{1,2} \smallsetminus \Delta) &= (\deg(f_1) - 1)(\deg(f_2) - 1). \end{aligned}$$

In particular, the assumption that f_1 and f_2 have degree at least 4 ensures that these genera are at least 2.

We now invoke Faltings's theorem [9], [14, Theorem E.0.1] to deduce that the set

$$\Sigma := \Gamma_{1,2}(K) \cup (\Gamma_1 \smallsetminus \Delta)(K) \cup (\Gamma_2 \smallsetminus \Delta)(K)$$

is finite. We note that the definition of Σ says that for all $P, Q \in \mathbb{P}^1(K)$, we have

$$\begin{bmatrix} P \neq Q & \text{and} & f_1(P) = f_1(Q) & \Longrightarrow & (P, Q) \in \Sigma \\ P \neq Q & \text{and} & f_2(P) = f_2(Q) & \Longrightarrow & (P, Q) \in \Sigma \\ & & f_1(P) = f_2(Q) & \Longrightarrow & (P, Q) \in \Sigma \end{bmatrix}. \tag{10}$$

Let $\pi_1, \pi_2 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ be the two projection maps, and let

$$C_{20} := \max\bigl(\{h(P) : P \in \pi_1(\Sigma)\} \cup \{h(P) : P \in \pi_2(\Sigma)\}\bigr)$$

be the maximum of the heights of the coordinates of the finitely many points in Σ. We then set

$$C_{21} := \max\{C_{19}, C_{20}\},$$

where C_{19} is the constant that appears in Lemma 18.

The fact that $\operatorname{Orb}_S(P) \subseteq \mathbb{P}^1(K)$ is infinite, combined with Northcott's theorem [18] saying that $\mathbb{P}^1(K)$ has only finitely many points of bounded height, implies that there exists a point $Q \in \mathrm{Orb}_S(P)$ satisfying

$$h(Q) > C_{21}. \tag{11}$$

We claim that Q is strongly wandering for S. To see this, suppose that

$$f_{i_1} \circ \cdots \circ f_{i_n}(Q) = f_{j_1} \circ \cdots \circ f_{j_m}(Q), \tag{12}$$

where without loss of generality we may assume that $n \geq m$. Our goal is to prove that $m = n$ and $i_k = j_k$ for all $1 \leq i \leq n$.

To ease notation, we let

$$F = f_{i_2} \circ \cdots \circ f_{i_n} \quad \text{and} \quad G = f_{j_2} \circ \cdots \circ f_{j_m} \tag{13}$$

be the compositions with the initial map omitted. Thus (12) and (13) say that

$$f_{i_1}\big(F(Q)\big) = f_{j_1}\big(G(Q)\big). \tag{14}$$

It follows from (14) and (10) that one of the following is true:

(i) $i_1 = j_1$ and $F(Q) = G(Q)$.

(ii) $i_1 = j_1$ and $F(Q) \neq G(Q)$ and $\big(F(Q), G(Q)\big) \in \Sigma$.

(iii) $i_1 \neq j_1$ and $\big(F(Q), G(Q)\big) \in \Sigma$.

On the other hand, we know that

$$\big(F(Q), G(Q)\big) \in \Sigma \quad \Longrightarrow \quad h\big(F(Q)\big) \leq C_{21} < h(Q)$$

by (11). But this contradicts Lemma 18. Hence (ii) and (iii) are false, so (i) is true.

We recall that $m \leq n$, so repeating this argument, we conclude that

$$i_k = j_k \quad \text{for all } 1 \leq k \leq m.$$

If $m < n$ is a strict inequality, then we see that

$$f_{i_{m+1}} \circ \cdots \circ f_{i_n}(Q) = Q.$$

But then Lemma 18 implies that $h(Q) \leq C_{19} \leq C_{21}$, and we obtain a contradiction of (11). Thus $m = n$ and $i_k = j_k$ for all $1 \leq k \leq n$, which completes the proof that Q is a strongly S-wandering point. $\square$

We now have the tools in place to prove Theorem 15.

Proof of Theorem 15. Let S be the given set of endomorphisms of $\mathbb{P}^1$, and let f_1 and f_2 be the given maps in S that have degree at least 4 and that are critically simple and critically separated. We let

$$S' = \{f_1, f_2\}.$$

We are given that the point $P \in \mathbb{P}^1(K)$ has infinite S-orbit, and hence by Northcott's theorem [18], there are points of arbitrarily large height in $\mathrm{Orb}_S(P)$. We choose a point

$$Q' \in \mathrm{Orb}_S(P) \quad \text{satisfying} \quad h(Q') > C_{19}(S'),$$

where $C_{19}(S')$ is the constant associated to the set S' appearing in Lemma 18. In particular, it follows from Lemma 18(b) and Northcott's theorem that $\mathrm{Orb}_{S'}(Q')$ must be infinite. Then Proposition 19 implies that $M_{S'}$ is free and that there is a point

$$Q \in \mathrm{Orb}_{S'}(Q') \subseteq \mathrm{Orb}_S(P)$$

that is strongly S'-wandering. Applying Theorem 7 to the set S' and the point Q, we deduce that

$$\sum_{\mathfrak{p} \in \mathrm{Spec}(R_K)} \frac{\log \mathsf{N}\mathfrak{p}}{\mathsf{N}\mathfrak{p} \cdot m_{\mathfrak{p}}(S, P)^{\epsilon}} \le \sum_{\mathfrak{p} \in \mathrm{Spec}(R_K)} \frac{\log \mathsf{N}\mathfrak{p}}{\mathsf{N}\mathfrak{p} \cdot m_{\mathfrak{p}}(S', Q)^{\epsilon}} \le C_{17}\epsilon^{-1}$$

for some constant C_{17} depending on S, Q (and so P) and K. For this last conclusion, we have also used the fact that

$$m_{\mathfrak{p}}(S', Q) \le m_{\mathfrak{p}}(S, P),$$

which is immediate from the inclusion $\mathrm{Orb}_{S'}(Q) \subseteq \mathrm{Orb}_S(P)$. □

Proof of Theorem 10. We recall that Rat_d denotes the space of rational maps of degree d. Then it follows from Theorems 1.1–1.4 in [19] that if $d_1, d_2 \ge 4$, then the set

$$\mathcal{V}_{d_1,d_2} := \left\{ (f_1, f_2) \in \mathrm{Rat}_{d_1} \times \mathrm{Rat}_{d_2} : \begin{array}{l} f_1 \text{ and } f_2 \text{ are critically simple} \\ \text{and critically separated} \end{array} \right\}$$

is Zariski dense in $\mathrm{Rat}_{d_1} \times \mathrm{Rat}_{d_2}$. Then for any $d_3, \dots, d_r \ge 2$, the set

$$\mathcal{U}(d_1, \dots, d_r) := \mathcal{V}_{d_1,d_2} \times \mathrm{Rat}_{d_3} \times \cdots \times \mathrm{Rat}_{d_r}$$

is Zariski dense in $\mathrm{Rat}_{d_1} \times \cdots \times \mathrm{Rat}_{d_r}$, and Theorem 15 gives us that the desired inequality (2) for every S generated by a set of maps

$$(f_1, \dots, f_r) \in \mathcal{U}(d_1, \dots, d_r). \qquad \square$$

We conclude with a variant of Theorem 15 in which the maps are polynomials. We start with a definition.

Definition 20. A polynomial $f(x) \in \overline{\mathbb{Q}}[x]$ is *power-like* if there exist polynomials $R(x), C(x), L(x) \in \overline{\mathbb{Q}}[x]$ such that

$$f = R \circ C \circ L, \deg(L) = 1, \deg(C) \ge 2,$$
$$C(x) = \text{a power map or a Chebyshev polynomial.}$$

Theorem 21. *Let $K/\mathbb{Q}$ be a number field, let S be a set of endomorphisms of $\mathbb{P}^1$ defined over K, and let $P \in \mathbb{P}^1(K)$ be a point such that $\operatorname{Orb}_S(P)$ is infinite. Suppose further that S contains polynomials $f_1(x), f_2(x) \in K[x]$ having the following properties*:

(1) *Neither f_1 nor f_2 is power-like*; *see Definition 20.*

(2) *For all $g \in \overline{\mathbb{Q}}[x]$ satisfying $\deg(g) \geq 2$, we have*

$$f_1 \neq f_2 \circ g \quad \textit{and} \quad f_2 \neq f_1 \circ g.$$

Then there is a constant $C_{22} = C_{22}(K, S, P)$ such that for all $\epsilon > 0$,

$$\sum_{\mathfrak{p} \in \operatorname{Spec}(R_K)} \frac{\log \mathsf{N}\mathfrak{p}}{\mathsf{N}\mathfrak{p} \cdot m_{\mathfrak{p}}(S, P)^{\epsilon}} \leq C_{22} \cdot \epsilon^{-1}.$$

The proof of Theorem 21 is similar to the proof of Theorem 15, except that we use [10; 23] instead of [13; 19]. As a first step, we need the following result, which is a polynomial analogue of Proposition 19.

Proposition 22. *Let f_1 and f_2 be polynomials satisfying the hypotheses of Theorem 21, and let $S = \{f_1, f_2\}$. Then the following statements hold*:

(a) *The semigroup M_S is free.*

(b) *Let $P \in \mathbb{P}^1(\overline{\mathbb{Q}})$ be a point whose S-orbit $\operatorname{Orb}_S(P)$ is infinite. Then there exists a strongly S-wandering point $Q \in \operatorname{Orb}_S(P)$.*

Proof. (a) See [13, Proposition 4.5].

(b) The proof is very similar to the proof of Proposition 19, so we just give a brief sketch, highlighting the differences. We note that we have picked a coordinate function x on $\mathbb{P}^1$. We let $\infty \in \mathbb{P}^1$ be the pole of x and let $\mathbb{A}^1 = \mathbb{P}^1 \smallsetminus \{\infty\}$. Replacing P with another point in $\operatorname{Orb}_S(P)$ if necessary, we may assume that $P \neq \infty$ is not the point at infinity. We choose a set $\mathfrak{S}$ of primes of K so that the ring of $\mathfrak{S}$-integers $R_{K,\mathfrak{S}}$ satisfies

$$P \in \mathbb{A}^1(R_{K,\mathfrak{S}}) \quad \text{and} \quad f_1(x), f_2(x) \in R_{K,\mathfrak{S}}[x].$$

We use the map $f_1 \times f_2 : \mathbb{A}^2 \to \mathbb{A}^2$ to define three affine curves,

$$\Gamma_1 := (f_1 \times f_1)^{-1}(\Delta), \quad \Gamma_2 := (f_2 \times f_2)^{-1}(\Delta), \quad \Gamma_{1,2} := (f_1 \times f_2)^{-1}(\Delta).$$

Then [13, Proposition 4.5], itself a consequence of the main results of [10; 23], tells us that these are geometrically irreducible curves of geometric genus at least 1. (This is where we use the assumptions (1) and (2) of Theorem 21 on f_1 and f_2.)

The Siegel–Mahler theorem for integral points on affine curves [14, Theorem D.9.1] then implies that

$$\Gamma_1(R_{K,\mathfrak{S}}), \quad \Gamma_2(R_{K,\mathfrak{S}}), \quad \text{and} \quad \Gamma_{1,2}(R_{K,\mathfrak{S}}) \quad \text{are finite sets,}$$

and hence that

$$\Sigma := \Gamma_{1,2}(R_{K,\mathfrak{S}}) \cup (\Gamma_1 \smallsetminus \Delta)(R_{K,\mathfrak{S}}) \cup (\Gamma_2 \smallsetminus \Delta)(R_{K,\mathfrak{S}})$$

is finite.

The remainder of the proof of Proposition 22 follows the proof of Proposition 19, starting with the three possibilities described in (10). □

Proof of Theorem 21. The proof of Theorem 21 is identical to that of Theorem 15. We first use Lemma 18, Proposition 22, and the fact that $\mathrm{Orb}_S(P)$ is infinite to find a point $Q \in \mathrm{Orb}_S(P)$ that is strongly wandering for $S' = \{f_1, f_2\}$. We then apply Theorem 7 to the point Q and the set $S' = \{f_1, f_2\}$ to deduce the desired result for P and S. □

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WADE HINDES
DEPARTMENT OF MATHEMATICS
TEXAS STATE UNIVERSITY
SAN MARCOS, TX
UNITED STATES

wmh33@txstate.edu

JOSEPH H. SILVERMAN
DEPARTMENT OF MATHEMATICS
BROWN UNIVERSITY
PROVIDENCE, RI
UNITED STATES

joseph_silverman@brown.edu

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