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SPIN LEFSCHETZ FIBRATIONS ARE ABUNDANT

MIHAIL ARABADJI AND R. İNANÇ BAYKUR

We prove that any finitely presented group can be realized as the fundamental group of a spin Lefschetz fibration over the 2-sphere. We also show that any admissible lattice point in the symplectic geography plane below the Noether line can be realized by a simply connected spin Lefschetz fibration.

1. Introduction

Explicit constructions of Lefschetz fibrations with prescribed fundamental groups were given by Amorós, Bogomolov, Katzarkov and Pantev [1] and by Korkmaz [14]; also see [12]. We show that the same result holds for a much smaller family of Lefschetz fibrations:

Theorem A. *Given any finitely presented group G, there exists a spin symplectic Lefschetz fibration $X \to S^2$ with $\pi_1(X) \cong G$.*

These results were inspired by the pioneering work of Gompf, who proved that any finitely presented group G is the fundamental group of a closed symplectic 4-manifold [9], which can be assumed to be spin. By the existence of Lefschetz pencils on any symplectic 4-manifold due to Donaldson [6], it follows a priori that, after blowing up the base points of the pencil, one can realize G as the fundamental group of a symplectic Lefschetz fibration; however, these are never spin.

On the other hand, unlike Kähler surfaces, there are minimal symplectic 4-manifolds of general type violating the Noether inequality, which was shown again by Gompf [9]. More recently, Korkmaz, Simone, and Baykur showed that all the lattice points in the symplectic geography plane below the Noether line can be further realized by simply connected symplectic Lefschetz fibrations [4]. We prove that a similar result holds in the spin case:

Theorem B. *For any pair of nonnegative integers (m, n) satisfying the inequalities $n \geq 0$, $n \equiv 8m \pmod{16}$, $n \leq 8(m-6)$ and $n \leq \frac{16}{3}m$, there exists a simply connected spin symplectic Lefschetz fibration $X \to S^2$ such that $\chi_h(X) = m$ and $c_1^2(X) = n$. In particular, any admissible point in the symplectic geography plane below the Noether line is realized by a simply connected spin Lefschetz fibration.*

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Among the hypotheses in the theorem, the first inequality is due to a theorem of Taubes, who showed that $c_1^2(X) \geq 0$ for any nonruled minimal symplectic 4-manifold X, whereas the second equality follows from Rokhlin's theorem. A pair $(m, n) \in \mathbb{N}^2$ satisfying this condition is called *admissible*. The first systematic production of spin symplectic 4-manifolds realizing the above admissible lattice points, but without the Lefschetz fibration structure we get, was first obtained by J. Park in [15].

Our examples are produced explicitly via positive Dehn twist factorizations in the mapping class group. The spin Lefschetz fibrations for Theorems A and B are obtained by adapting the strategies of [14] and [4], respectively, together with a subtle use of the breeding technique [2; 3] for the latter. The main challenge in producing the examples in either theorem is due to the fact that the monodromy of a spin Lefschetz fibration lies in a proper subgroup of the mapping class group (fixing a spin structure on the fiber), so throughout our work, we restrain ourselves to algebraic manipulations in this smaller mapping class group.

2. Preliminaries

We begin with a review of the concepts and background results underlying the rest of our article, along with our conventions. We refer the reader to [10] for more details and comprehensive references on Lefschetz fibrations, symplectic 4-manifolds, and monodromy factorizations, and to [3] for their interplay with spin structures.

2.1. ***Lefschetz fibrations and positive factorizations.*** A *Lefschetz fibration* on a closed smooth oriented 4-manifold X is a smooth surjective map $f : X \to S^2$, a submersion on the complement of finitely many points $\{p_i\} \neq \varnothing$ all in distinct fibers, around which f conforms (compatibly with fixed global orientations on X and S^2) to the local complex model of a nodal singularity $(z_1, z_2) \mapsto z_1 z_2$. We assume that there are no exceptional spheres contained in the fibers. Each nodal fiber of the Lefschetz fibration (X, f) is obtained by crashing a simple closed curve, called a *vanishing cycle*, on a reference regular fiber F.

We denote by Σ_g^b a compact connected oriented surface of genus g with b boundary components. Let $\mathrm{Diff}^+(\Sigma_g^b)$ denote the group of orientation-preserving diffeomorphisms of Σ_g^b compactly supported away from the boundary. The *mapping class group* of Σ_g^b is defined as $\mathrm{Mod}(\Sigma_g^b) := \pi_0(\mathrm{Diff}^+(\Sigma_g^b))$. When $b = 0$, we simply drop b from the above notation. Unless mentioned otherwise, by a *curve* c on Σ_g^b we mean a smooth simple closed curve.

We denote by $t_c \in \mathrm{Mod}(\Sigma_g^b)$ the positive (right-handed) Dehn twist along the curve $c \subset \Sigma_g^b$. For any $\psi, \phi \in \mathrm{Mod}(\Sigma_g^b)$ we write the conjugate of ψ by ϕ as $\psi^\phi = \phi\psi\phi^{-1}$. We act on any curve c in the order $(\varphi\phi)(c) = \varphi(\phi(c))$. An elementary but crucial point is that $t_c^{\phi} = t_{\phi(c)}$. For any product of Dehn twists $W = \prod_{i=1}^{\ell} t_{c_i}^{k_i}$ and ϕ in $\mathrm{Mod}(\Sigma_g^b)$, we denote the conjugated product by $W^{\phi} = \prod_{i=1}^{\ell} t_{\phi(c_i)}^{k_i}$.

Let $\{c_i\}$ be a nonempty collection of curves on Σ_g^b which do not become null-homotopic after an embedding $\Sigma_g^b \hookrightarrow \Sigma_g$. Let $\{\delta_j\}$ be a collection of b curves parallel to distinct boundary components of Σ_g^b. A relation of the form

$$t_{c_1} t_{c_2} \cdots t_{c_l} = t_{\delta_1}^{k_1} \cdots t_{\delta_b}^{k_b} \quad \text{in } \mathrm{Mod}(\Sigma_g^b) \tag{1}$$

corresponds to a genus-g Lefschetz fibration (X, f) with a reference regular fiber F identified with Σ_g, with vanishing cycles $\{c_i\}$ and b disjoint sections $\{S_j\}$ of self-intersections $S_j \cdot S_j = -k_j$.

The product on the left-hand side of the equality (1), the word P in positive Dehn twists, is called a *positive factorization* of the mapping class on the right-hand side that maps to the trivial word under the homomorphism induced by an embedding $\Sigma_g^b \hookrightarrow \Sigma_g$. We will often denote the corresponding Lefschetz fibration as X_P.

As shown by Gompf, every Lefschetz fibration (X, f) admits a Thurston-type symplectic form with respect to which the fibers are symplectic.

2.2. ***Fiber sums and fundamental groups.*** A Lefschetz fibration X_P corresponding to a positive factorization $P := t_{c_1} t_{c_2} \cdots t_{c_l}$ in $\mathrm{Mod}(\Sigma_g^1)$ (of some power of the boundary twist) has $\pi_1(X_P) \cong \pi_1(\Sigma_g)/N(\{c_i\})$, where $N(\{c_i\})$ is the subgroup of $\pi_1(\Sigma_g)$ generated normally by collection of the vanishing cycles c_i.

Given $P_1 := t_{c_1} t_{c_2} \cdots t_{c_l} = t_{\delta}^{k_1}$ and $P_2 := t_{d_1} t_{d_2} \cdots t_{d_l} = t_{\delta}^{k_2}$, and any $\phi \in \mathrm{Mod}(\Sigma_g^1)$, we can always derive another positive factorization $P_1 P_2^{\phi} = t_{\delta}^{k_1+k_2}$ in $\mathrm{Mod}(\Sigma_g^1)$, prescribing a new Lefschetz fibration $X_{P_1 P_2^{\phi}}$ with a section of self-intersection $-(k_1 + k_2)$. This coincides with the well-known *twisted fiber sum* operation applied to the Lefschetz fibrations X_{P_1} and X_{P_2}. We have

$$\pi_1(X_{P_1 P_2^{\phi}}) \cong \pi_1(\Sigma_g)/N\big(\{c_i\} \cup \{\phi(d_j)\}\big).$$

A neat trick of Korkmaz, applicable in the more special setting described in the next proposition, will come in handy for our arguments to follow:

Proposition 1 (Korkmaz [14]). Let $P = t_{c_1} t_{c_2} \cdots t_{c_\ell}$ be a positive factorization of (some power of) a boundary twist in $\mathrm{Mod}(\Sigma_g^1)$. Let d be a curve on Σ_g intersecting at least one c_i transversally at one point. Then $\pi_1(X_{P P^{t_d}}) \cong \pi_1(\Sigma_g)/N(\{c_i\} \cup \{d\})$.

2.3. ***Spin monodromies and fibrations.*** A *spin structure* s on Σ_g is a cohomology class $s \in H^1(UT(\Sigma_g); \mathbb{Z}_2)$ evaluating to 1 on a fiber of the unit tangent bundle $UT(\Sigma_g)$. There is a bijection between the set of spin structures on Σ_g, which we denote by $\mathrm{Spin}(\Sigma_g)$, and the set of quadratic forms on $H_1(\Sigma_g; \mathbb{Z}_2)$ with respect to the intersection pairing. Recall that $q : H_1(\Sigma_g; \mathbb{Z}_2) \to \mathbb{Z}_2$ is such a quadratic form if $q(a+b) = q(a) + q(b) + a \cdot b$ for every $a, b \in H_1(\Sigma_g; \mathbb{Z}_2)$.

For a fixed spin structure s on Σ_g, the *spin mapping class group* $\mathrm{Mod}(\Sigma_g, s)$ is the stabilizer group of s, or equivalently that of the corresponding quadratic form q, in $\mathrm{Mod}(\Sigma_g)$. For any nonseparating curve $c \subset \Sigma_g$, we have $t_c \in \mathrm{Mod}(\Sigma_g, s)$ if and only if $q(c) = 1$.

The following, which is a reformulation of a theorem of Stipsicz, provides us with a criterion for the existence of a spin structure on a Lefschetz fibration:

Theorem 2 (Stipsicz [16]). *Let X_P be the Lefschetz fibration prescribed by a positive factorization*

$$P := t_{c_1} t_{c_2} \cdots t_{c_l} = t_\delta^k \quad \text{in} \;\; \mathrm{Mod}(\Sigma_g^1),$$

and let us denote the images of the twist curves under the embedding $\Sigma_g^1 \hookrightarrow \Sigma_g$ also by $\{c_i\}$. Then, X_P admits a spin structure with a quadratic form q if and only if k is even and $q(c_i) = 1$ for all i.

3. Spin Lefschetz fibrations with prescribed fundamental group

We prove Theorem A, adapting the strategy in [14], where Korkmaz takes twisted fiber sums of many copies of the same Lefschetz fibration (the building block) to obtain a new Lefschetz fibration whose fundamental group is the prescribed finitely presented group. To accomplish the same with spin fibrations, there are two essential refinements we will need to make. First is to identify a building block X_P where the monodromy curves in the positive factorization P will satisfy the spin condition for some quadratic form we will describe. That is, we will show that $P := t_{c_1} t_{c_2} \cdots t_{c_\ell}$ in $\mathrm{Mod}(\Sigma_g, s)$ for a carefully chosen spin structure s. Second is to make sure that when taking the twisted fiber sums to land on the desired fundamental group, in the corresponding positive factorization $P P^{\phi_1} \cdots P^{\phi_m}$, we only use conjugations $\phi_i \in \mathrm{Mod}(\Sigma_g, s)$.

3.1. ***The building block.*** A generalization of the monodromy factorization of the well-known genus-1 Lefschetz fibration on $\mathbb{CP}^2 \# 9\, \overline{\mathbb{CP}}^2 \cong S^2 \times S^2 \# 8\, \overline{\mathbb{CP}}^2$ to any odd genus $g = 2n+1$ Lefschetz fibration on $S^2 \times \Sigma_n \# 8\, \overline{\mathbb{CP}}^2$ was given by Korkmaz in [13], and by Cadavid in [5]. It has the monodromy factorization

$$(t_{B_0} \cdots t_{B_g} t_a^2 t_b^2)^2 = t_\delta \quad \text{in} \;\; \mathrm{Mod}(\Sigma_g^1),$$

where the curves B_i, a, b are shown in the Figure 1. Capping off the boundary component of Σ_g^1, we will regard the same curves also in Σ_g. Let us denote the above positive factorization by $P_g := (t_{B_0} \cdots t_{B_g} t_a^2 t_b^2)^2$.

Clearly, $\pi_1(X_{P_g}) \cong \pi_1(\Sigma_n)$ will have larger number of generators we can work with as we increase $g = 2n+1$. Let us first review the presentation for $\pi_1(X_{P_g})$. Consider the geometric basis $\{a_i, b_i\}_{i=1}^g$ for $\pi_1(\Sigma_g)$, where the based oriented curves

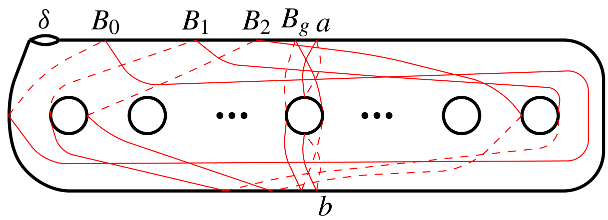

Figure 1. The vanishing cycles B_i, a, b of X_{P_g} in $\Sigma_g^1 \subset \Sigma_g$.

a_i, b_i are as shown in Figure 2. We have

$$\pi_1(X) \cong \langle a_1, \dots, a_g, b_1, \dots, b_g \mid C_g\, a, b, B_0, \dots, B_g \rangle,$$

where[1]

$$(2) \qquad \begin{cases} B_0 = b_1 \cdots b_g, & \\ B_{2k-1} = a_k b_k \cdots b_{g+1-k} C_{g+1-k} a_{g+1-k}, & 1 \le k, \le n+1, \\ B_{2k} = a_k b_{k+1} \cdots b_{g-k} C_{g-k} a_{g+1-k}, & 1 \le k, \le n, \\ a = a_{n+1}, & \\ b = C_n a_{n+1}, & \\ C_1 = b_1^{-1} a_1 b_1 a_1^{-1}, & \\ C_i = b_i^{-1} C_{i-1} a_i b_i a_i^{-1}, & 2 \le i \le g. \end{cases}$$

Next, we will describe a spin structure for which the vanishing cycles of this Lefschetz fibration satisfy the monodromy condition.[2] Forgetting the base point, the geometric basis $\{a_j, b_j\}$ for $\pi_1(\Sigma_g)$ in Figure 2 becomes freely homotopic to a standard symplectic basis on Σ_g. We can then describe a quadratic form with respect to this basis and evaluate it on the mod-2 homology classes of the vanishing cycles described in this basis. The latter is easily derived from (2):

$$\begin{cases} B_0 = b_1 + \cdots + b_g, & \\ B_{2k-1} = a_k + (b_k + \cdots + b_{g+1-k}) + a_{g+1-k}, & 1 \le k \le n+1, \\ B_{2k} = a_k + (b_{k+1} + \cdots + b_{g-k}) + a_{g+1-k}, & 1 \le k \le n, \\ a = a_{n+1}, & \\ b = a_{n+1}. & \end{cases}$$

[1] Here we adopted Korkmaz's generating set to make our calculations comparable to his work in [14], which yields a nonstandard expression for the surface relator as iterated conjugates, resulting in $C_g = 1$.

[2] It may be worth noting that this is not a trial and error process. By the heuristic arguments of [3], we are proceeding with an educated guess, since these fibrations are known to come from pencils on the spin manifolds $S^2 \times \Sigma_n$; see [11].

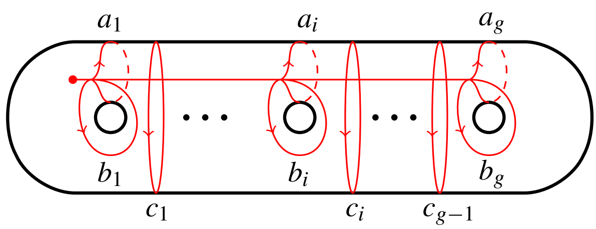

Figure 2. The generators for $\pi_1(\Sigma_g)$ and the C_i curves.

Set $q : H_1(\Sigma_g, \mathbb{Z}/2) \to \mathbb{Z}/2$ as $q(a_i) = q(b_i) = 1$. (There are in fact 2^n different spin structures that would work here; we are picking the one that will serve our needs the most in the next stages of the proof.) Note that for any ordered set of curves $\{d_j\}$ we have $q\left(\sum_{i=1}^n d_i\right) = \sum_{i=1}^n q(d_i) + \sum_{i<j} d_i \cdot d_j$. Thus, for each k as above,

$$
\begin{aligned}
q(B_0) &= \sum_{i=1}^{g} q(b_i) = g = 1, \\
q(B_{2k-1}) &= q(a_k) + \sum_{i=k}^{g+1-k} q(b_i) + q(a_{g+1-k}) + a_k \cdot b_k + b_{g+1-k} \cdot a_{g+1-k} \\
&= 1 + (g+1-k-k+1) + 1 + 1 + 1 = 1, \\
q(B_{2k}) &= q(a_k) + \sum_{i=k+1}^{g-k} q(b_i) + q(a_{g+1-k}) \\
&= 1 + (g-k-k) + 1 = 1, \\
q(a) &= q(a_{n+1}) = 1, \\
q(b) &= q(a_{n+1}) = 1.
\end{aligned}
$$

Hence all the monodromy curves of X_{P_g} satisfy the spin condition, which is all we needed at this point.[3] To sum up, we have the following:

Lemma 3. *Let $s \in \mathrm{Spin}(\Sigma_g)$ correspond to the quadratic form q that satisfies $q(a_i) = q(b_i) = 1$, for $i = 1, \ldots, g$, on the symplectic basis $\{a_i, b_i\}$ above. We have*

$$(t_{B_0} \cdots t_{B_g} t_a^2 t_b^2)^2 = 1 \quad \textit{in} \ \ \mathrm{Mod}(\Sigma_g, s),$$

where B_i, a, b are the curves on $\Sigma_g^1 \subset \Sigma_g$ in Figure 1.

3.2. ***The construction.*** In anticipation of a forthcoming issue, here we deviate a bit from Korkmaz's steps. In order to guarantee that we can represent the relators by embedded curves on Σ_g, we change the given presentation. Instead of reinventing the wheel here, we invoke the following result (cf. [8, Lemma 6.2]):

[3] Recall that t_δ has odd power in Section 3.1, so X_{P_g} is not a spin Lefschetz fibration, as it shouldn't be, remembering that $X_{P_g} \cong S^2 \times \Sigma_n \# 8\overline{\mathbb{CP}}^2$.

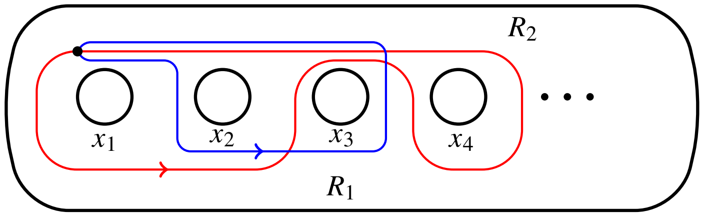

Figure 3. Relator curves on Σ_g.

Lemma 4 (Ghiggini, Golla and Plamanevskaya [8]). *For any finitely presented group G, there exists a presentation $G \cong \langle x_1, \dots, x_n \mid r_1, \dots, r_m \rangle$ such that*

(i) *each r_j is a positive (no inverses) word in $x_1, \dots, x_n$;*

(ii) *each generator x_i appears at most once in each r_j;*

(iii) *the cyclic order (by index) of the generators $x_1, \dots, x_n$ is preserved in each r_j.*

This means that if our generating set consists of only the curves $\{b_i\}$, we can assume that all the relators in the generating set can be nicely represented by the embedded curves as in Figure 3, where R_1 represents x_2x_3, R_2 represents $x_1x_2x_4$, and so on.

We are now ready to present our construction.

Proof of Theorem A. Given a finitely presented group G, take a (new) presentation of $G \cong \langle x_1, \dots, x_n \mid r_1, \dots, r_m \rangle$ as in Lemma 4. Set $g = 2n + 1$.

Let $P_g := (t_{B_0} \cdots t_{B_g} t_a^2 t_b^2)^2$ be the positive factorization in $\mathrm{Mod}(\Sigma_g, s)$ given in Lemma 3. Because $q(a_i) = 1$, we have $t_{a_i} \in \mathrm{Mod}(\Sigma_g, s)$ for all i. So we get a new spin factorization

$$P_g P_g^{t_{a_1}} P_g^{t_{a_2}} \cdots P_g^{t_{a_g}} = 1 \quad \text{in } \mathrm{Mod}(\Sigma_g, s)$$

for each odd $g \in \mathbb{Z}^+$, which lifts to a positive factorization of t_δ^{g+1} in $\mathrm{Mod}(\Sigma_g^1)$.

From the expression of the monodromy curves of P_g in the $\pi_1(\Sigma_g)$ basis $\{a_i, b_i\}$ given in (2), one easily deduces that

$$\begin{aligned}
&\langle a_1, \dots, a_g, b_1, \dots, b_g \mid C_g, a, b, B_0, \dots, B_g, a_1, \dots, a_g \rangle, \\
&\qquad \cong \langle b_1, \dots, b_{2n+1} \mid b_1 \cdots b_{2n+1}, b_2 \cdots b_{2n}, \dots, b_n b_{n+1} b_{n+2}, b_{n+1} \rangle \\
&\qquad \cong \langle b_1, \dots, b_n \rangle,
\end{aligned}$$

that is, we get a free group on n generators. For the first step, simply note that all a_j and C_j we had in (2) are trivial in this group.

Now, identifying each generator x_i with b_i, for $i = 1, \dots, n$, we can represent each relator r_j by an embedded curve on R_j on Σ_g. (This is why we switched to this special presentation.) All $\{R_j\}$ can be contained on $\Sigma_n^1 \subset \Sigma_g$ bounded by c_n. It is possible that some $q(R_j) = 0$. If that is the case, we replace this R_j with an embedded curve R'_j representing $R_j a_{n+1}$ in $\pi_1(\Sigma_g)$. Such an embedded curve always exists; R_j can be isotoped to meet a_{n+1} only at the base point and one can then resolve the intersection point compatibly with the orientations. So now $q(R'_j) = 1$. Otherwise we just take $R'_j := R_j$. We have $t_{R'_j} \in \mathrm{Mod}(\Sigma_g, s)$ for all $j = 1, \dots, m$.

It follows that we have a spin positive factorization

$$P_g P_g^{t_{a_1}} P_g^{t_{a_2}} \cdots P_g^{t_{a_g}} P_g^{R'_1} P_g^{R'_2} \cdots P_g^{R'_m} = 1 \quad \text{in} \ \ \mathrm{Mod}(\Sigma_g, s),$$

which now lifts to a positive factorization of t_δ^{g+m+1} in $\mathrm{Mod}(\Sigma_g^1)$. If m is odd, we add one more P_g factor to the positive factorization above, so then its lift is a positive factorization of t_δ^{g+m+2}. If m is even, leave it as it is. In either case let us denote this final positive factorization in $\mathrm{Mod}(\Sigma_g, s)$ simply by P. Let X_P denote the corresponding Lefschetz fibration. By Theorem 2, X_P is spin. By Proposition 1, and the above discussion, we have

$$\begin{aligned} \pi_1(X_P) &\cong \langle a_1, \dots, a_g, b_1, \dots, b_g \mid C_g, a, b, B_0, \dots, B_g, a_1, \dots, a_g, R'_1, \dots, R'_m \rangle \\ &\cong \langle b_1, b_2, \dots, b_n \mid R_1, \dots, R_m \rangle, \end{aligned}$$

which is the presentation we had for G. □

4. Geography of spin Lefschetz fibrations

We prove Theorem B by a direct construction of a family of spin Lefschetz fibrations $Z_{g,k}$ populating the region below the Noether line in the geography plane. We prescribe these fibrations via new positive factorizations via algebraic manipulations in the mapping class group corresponding to twisted fiber sums and breedings [2; 3]. We then verify how our careful choice of building blocks out of monodromy factorizations for Lefschetz pencils and fibrations indeed yields positive factorizations in spin mapping class groups. A somewhat longer calculation will show that our choices also guarantee that $Z_{g,k}$ are simply connected. We will then conclude by describing the portion of the geography plane spanned by our spin fibrations.

While some of the particular choices we will make in the construction of $Z_{g,k}$ may look arbitrary at first, they are to achieve two somewhat competing properties simultaneously: the existence of a spin structure on $Z_{g,k}$ and the simple-connectivity of $Z_{g,k}$. The latter calculation implies that the spin structure we describe on $Z_{g,k}$ is in fact unique.

4.1. ***The construction.*** Our first building block is a positive factorization for a Lefschetz fibration on $\mathbb{CP}^2 \# (4g+5)\overline{\mathbb{CP}}^2$ given in [4]. Taking $p = q = 2g + 2$ in

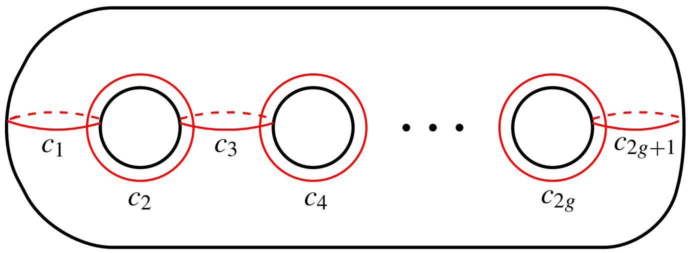

Figure 4. Dehn twist curves c_i on Σ_g.

Lemma 4 of [4], we obtain

$$(3)\quad U := t_1^{2g+2} t_3^{2g+2} (t_1^{t_2} t_2^{t_3} \cdots t_{2g}^{t_{2g+1}})(t_{2g+1}^{t_{2g}} \cdots t_4^{t_3})(t_3^{t_3^q t_2} t_2^{t_3^q t_1}) = 1 \quad \text{in } \operatorname{Mod}(\Sigma_g),$$

which is in fact Hurwitz equivalent to the square of the positive factorization of the hyperelliptic involution $h := (t_1 \cdots t_{2g} t_{2g+1}^2 t_{2g} \cdots t_1)$ in $\operatorname{Mod}(\Sigma_g)$. Here t_i denotes a Dehn twist along the curve c_i shown in Figure 4. We also assume that $g \geq 5$ and is odd. Let also V be the following conjugate of U:

$$(4)\quad V := (t_1^{t_2} t_2^{t_3} \cdots t_{2g}^{t_{2g+1}})(t_{2g+1}^{t_{2g}} \cdots t_4^{t_3})(t_3^{t_3^{2g+2} t_2} t_2^{t_3^{2g+2} t_1}) t_1^{2g+2} t_3^{2g+2} = 1 \quad \text{in } \operatorname{Mod}(\Sigma_g).$$

Consider the two mapping classes

$$\phi := (t_8 t_7 t_6 t_a)(t_5 t_6 t_7 t_8)(t_4 t_5 t_6 t_7)(t_3 t_4 t_5 t_6)(t_2 t_3 t_4 t_5)(t_1 t_2 t_3 t_4),$$
$$\psi := (t_8 t_9 t_{10} t_d)(t_7 t_8 t_9 t_{10})(t_6 t_7 t_8 t_9) \cdots (t_1 t_2 t_3 t_4).$$

We claim that $\phi(c_1) = a$, $\phi(c_3) = b$ and $\psi(c_1) = c$, $\psi(c_3) = d$; see Figure 6. This can be easily verified because of the following elementary observation: whenever we have a k-chain of curves $u_1, \ldots u_k$,

$$t_{u_1} t_{u_2} \cdots t_{u_k}(u_i) = u_{i+1} \quad \text{for every } 1 \leq i \leq k-1.$$

Let us denote by Z_g the Lefschetz fibration corresponding to the positive factorization $P := V^\phi U^\psi$ in $\operatorname{Mod}(\Sigma_g)$, a twisted fiber sum of the Lefschetz fibration on $\mathbb{CP}^2 \# (4g+5)\overline{\mathbb{CP}}^2$ with itself. Note that we have

$$(5)\qquad \begin{aligned} P = V^\phi U^\psi &= V_1\, t_a^{2g+2} t_b^{2g+2} \cdot t_c^{2g+2} t_d^{2g+2}\, U_1 \\ &= V_1\, (t_a t_b t_c t_d)^{2g+2} U_1 \\ &= 1 \quad \text{in } \operatorname{Mod}(\Sigma_g), \end{aligned}$$

where V_1, U_1 are the products of positive Dehn twists

$$U_1 := ((t_1^{t_2} t_2^{t_3} \cdots t_{2g}^{t_{2g+1}})(t_{2g+1}^{t_{2g}} \cdots t_4^{t_3})(t_3^{t_3^q t_2} t_2^{t_3^q t_1}))^\psi,$$
$$V_1 := ((t_1^{t_2} t_2^{t_3} \cdots t_{2g}^{t_{2g+1}})(t_{2g+1}^{t_{2g}} \cdots t_4^{t_3})(t_3^{t_3^{2g+2} t_2} t_2^{t_3^{2g+2} t_1}))^\phi.$$

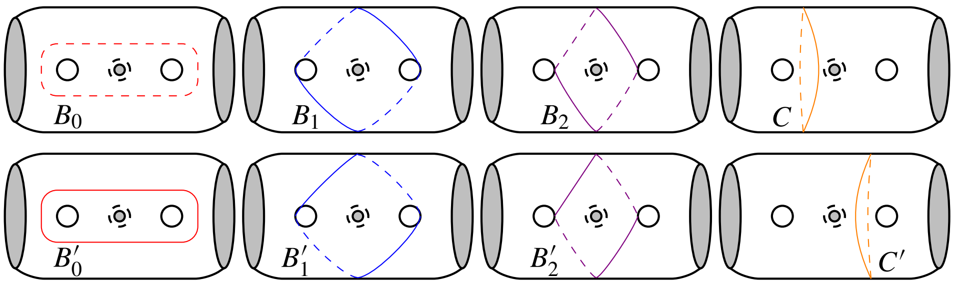

Figure 5. Vanishing cycles of the genus-2 pencil.

Our second building block is the following positive factorization by Hamada:

$$Q := t_{B_0} t_{B_1} t_{B_2} t_C t_{C'} t_{B_2'} t_{B_1'} t_{B_0'} = t_{\delta_1} t_{\delta_2} t_{\delta_3} t_{\delta_4} \quad \text{in } \operatorname{Mod}(\Sigma_2^4) \tag{6}$$

for a genus-2 Lefschetz pencil on $S^2 \times T^2$, where the twist curves are as shown in Figure 5; see [3; 11].

Since the curves $\{a, b, c, d\}$ cobound a subsurface Σ_2^4 of Σ_g, we can *breed* (see [2; 3]) the genus-2 pencil prescribed by (6) into the Lefschetz fibration prescribed by (5) for k times, for any $k \leq 2g+2$, and get a new positive factorization

$$P_{g,k} := V_1 (t_a t_b t_c t_d)^{2g+2-k} R^k U_1 = 1 \quad \text{in } \operatorname{Mod}(\Sigma_g), \tag{7}$$

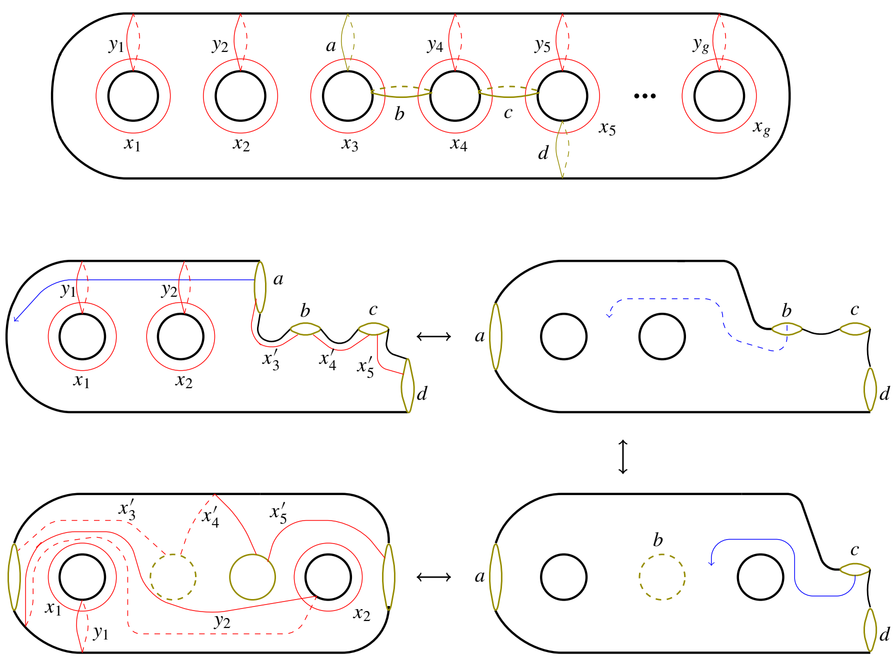

Figure 6. Embedding of Σ_2^4

where R is the image of the positive factorization Q under the homomorphism induced by a specific embedding $\Sigma_2^4 \hookrightarrow \Sigma_g$ we describe below. We let $Z_{g,k}$ denote the Lefschetz fibration corresponding to the positive factorization $P_{g,k}$.

The embedding $\Sigma_2^4 \hookrightarrow \Sigma_g$ is described in Figure 6. Brown curves indicate where the boundary curves $\{\delta_i\}$ of Σ_2^4 in the positive factorization (6) are mapped to. Blue arrows illustrate how we isotope the boundaries of $\Sigma_2^4 \subset \mathbb{R}^3$ before embedding it into $\Sigma_g \subset \mathbb{R}^3$. Red curves constitute a geometric generating set for $H_1(\Sigma_g; \mathbb{Z}_2)$. Red arcs are the parts of these curves contained in the image of the embedding $\Sigma_2^4 \hookrightarrow \Sigma_g$.

4.2. ***The spin structure on $Z_{g,k}$.*** We are going to invoke Theorem 2 to confirm that $Z_{g,k}$ admits a spin structure. The curves $\{x_i, y_i\}$ in Figure 6 constitute a symplectic basis for $H_1(\Sigma_g; \mathbb{Z}_2)$. Consider the quadratic form q for a spin structure $s \in \mathrm{Spin}(\Sigma_g)$, where for any $1 \le i \le g$,

$$\begin{aligned} q(x_i) &= 1 \quad \text{for all } i,\\ q(y_i) &= 1 \quad \text{for } i \text{ odd},\\ q(y_i) &= 0 \quad \text{for } i \text{ even}. \end{aligned}$$

First of all, $c_{2i} = x_i$, $c_1 = y_1$, $c_{2g+1} = y_g$ and $c_{2i+1} = y_i - y_{i+1}$. This means that $q(c_i) = 1$ for each i. Therefore, $t_i := t_{c_i} \in \mathrm{Mod}(\Sigma_g, s)$ for all i and the positive factorizations given in (3) and (4) are in fact factorizations in $\mathrm{Mod}(\Sigma_g, s)$.

Secondly, $a = y_3$ and $d = y_5$ in $H_1(\Sigma_g; \mathbb{Z}_2)$, so $q(a) = 1 = q(d)$, in addition to $t_i \in \mathrm{Mod}(\Sigma_g, s)$, so $\phi, \psi \in \mathrm{Mod}(\Sigma_g, s)$. It follows that $P := V^{\phi} U^{\psi} = 1$ is a positive factorization in $\mathrm{Mod}(\Sigma_g, s)$.

Thirdly, to check the spin condition for the new monodromy curves in R, we would like to express these curves in terms of the generators $\{x_i, y_i\}$.[4] We get in $H_1(\Sigma_g, \mathbb{Z}/2)$ the expressions

$$\begin{aligned} B_0 &= x_1 + x_2 + y_3 + y_4, & B_0' &= x_1 + x_2 + y_4 + y_5,\\ B_1 &= x_1 + x_2 + y_1 + y_2 + y_3 + y_4 + y_5, & B_1' &= x_1 + x_2 + y_1 + y_2 + y_4,\\ B_2 &= y_1 + y_2 + y_3 + y_4 + y_5, & B_2' &= y_1 + y_2 + y_4,\\ C &= y_3, & C' &= y_5. \end{aligned}$$

So we have

$$\begin{aligned} q(B_0) &= q(x_1 + x_2 + y_3 + y_4) = q(x_1) + q(x_2) + q(y_3) + q(y_4) = 1+1+1+0 = 1,\\ q(B_0') &= q(x_1 + x_2 + y_4 + y_5) = q(x_1) + q(x_2) + q(y_4) + q(y_5) = 1+1+0+1 = 1, \end{aligned}$$

[4]Let F denote the embedding $\Sigma_2^4 \hookrightarrow \Sigma_g$ and let v_j be a Dehn twist curve in R. Instead of $F(v_j) \cdot x_i$ and $F(v_j) \cdot y_i$ we can look at $v_j \cdot F^{-1}(x_i)$ and $v_j \cdot F^{-1}(y_i)$ to run the calculation here. Note that if x_i or y_i is only partially contained in the image of F, then we denote the arc in its preimage by x_i' or y_i'.

$$\begin{aligned}
q(B_1) &= q(x_1)+q(x_2)+q(y_1)+q(y_2)+q(y_3)+q(y_4)+q(y_5)+2\\
&= 1+1+1+0+1+0+1=1,\\
q(B_1') &= q(x_1)+q(x_2)+q(y_1)+q(y_2)+q(y_4)+2 = 1+1+1+0+0=1,\\
q(B_2) &= q(y_1)+q(y_2)+q(y_3)+q(y_4)+q(y_5) = 1+0+1+0+1=1,\\
q(B_2') &= q(y_1)+q(y_2)+q(y_4) = 1+0+0=1,\\
q(C) &= q(y_3)=1,\\
q(C') &= q(y_5)=1.
\end{aligned}$$

Hence, all the vanishing cycles of the Lefschetz fibration $Z_{g,k}$ satisfy the spin condition.

It is well known that the Lefschetz fibration with positive factorization U admits a (-1)-section; in fact this fibration is Hurwitz equivalent to a Lefschetz fibration obtained by blowing up all $4g+4$ base points of a genus-g pencil on $S^2\times S^2$ [17]. Therefore U, V, and in turn U^{ψ}, V^{ϕ}, all lift to a positive factorization of t_δ in $\mathrm{Mod}(\Sigma_g^1)$, where δ is a boundary parallel curve on Σ_g^1. We can pick a (-1)-section so that in the lift of U (and V), the lifts of t_{c_1}, t_{c_3} are still along disjoint curves in Σ_g^1.[5] The same goes for t_a, t_b of V^{ϕ} and t_c, t_d of U^{ψ}. Let us continue denoting the twist curves in their lifts by a, b and c, d. After an isotopy, we can assume that $P = V^{\phi}U^{\psi} = 1$ lifts to a positive factorization of t_δ^2 in $\mathrm{Mod}(\Sigma_g^1)$ so that the boundary component is not contained in the subsurface $\Sigma_2^4 \subset \Sigma_g$ cobounded by $\{a,b,c,d\}$.

Therefore, for any $k \le 2g+2$ we have a spin positive factorization

$$P_{g,k} = V_1(t_at_bt_ct_d)^{2g+2-k}R^kU_1 = 1 \quad \text{in } \mathrm{Mod}(\Sigma_g, s),$$

which lifts to a positive factorization

$$\tilde{P}_{g,k} = \tilde{V}_1(t_at_bt_ct_d)^{2g+2-k}\tilde{R}^k\tilde{U}_1 = t_\delta^2 \quad \text{in } \mathrm{Mod}(\Sigma_g^1).$$

Hence every $Z_{g,k}$ admits a spin structure by Theorem 2.

4.3. ***The fundamental group.*** Let $\{x_i, y_i\}$ be a geometric basis for $\pi_1(\Sigma_g)$ as shown in Figure 7. Since $Z_{g,k}$ has a section, we have $G := \pi_1(Z_{g,k}) \cong \pi_1(\Sigma_g)/N(\{v_j\})$, where v_j are the Dehn twist curves in the positive factorization $P_{g,k}$ of $Z_{g,k}$.

Set

$$S := (t_1^{t_2}t_2^{t_3}\cdots t_{2g}^{t_{2g+1}})(t_{2g+1}^{t_{2g}}\cdots t_4^{t_3})(t_3^{t_3^{2g+2}t_2}t_2^{t_3^{2g+2}t_1}).$$

So $U^{\psi} = t_c^{2g+2}t_d^{2g+2}U_1$ with $U_1 = S^{\psi}$, and $V^{\phi} = V_1t_a^{2g+2}t_b^{2g+2}$ with $V_1 = S^{\phi}$. While the fundamental group of $Z_{g,k}$ can be calculated from the factorization

[5] The Dehn twist curves may get entangled when we take lifts, but for just one section we are after, this is not a problem for our positive factorization; see, e.g., [17] for many possible choices.

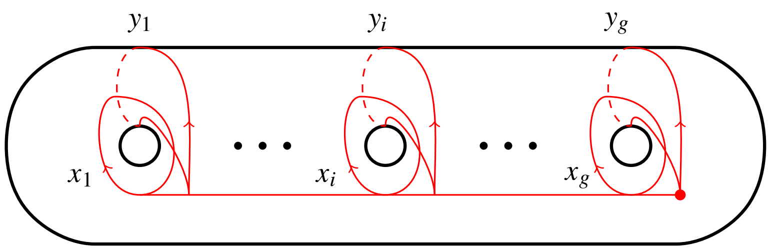

Figure 7. The generators x_i, y_i for $\pi_1(\Sigma_g)$.

$P_{g,k} = S^{\phi}(t_a t_b t_c t_d)^{2g+2-k} R^k S^{\psi}$, it can also be calculated from the factorization $P^{\phi^{-1}} = S(t_1 t_3 t_{\phi^{-1}(c)} t_{\phi^{-1}(d)})^{2g+2-k}(R^{\phi^{-1}})^k S^{\phi^{-1}\psi}$. We will run our calculations for the latter.

For $k < 2g+2$ the Dehn twist curves of the latter factorization contain all the vanishing cycles $\{c_i\}$ in U, which we know kill all the generators of $\pi_1(\Sigma_g)$ to yield trivial the fundamental group, as P_U has total space $\mathbb{CP}^2\#(4g+5)\overline{\mathbb{CP}}^2$, a simply connected space. Thus, $\pi_1(Z_{g,k}) = 1$ for any $k < 2g+2$.

For $k = 2g+2$, first note that we can connect the vanishing cycles $\{c_i\}$ of U or V to the basepoint (where any two different paths connecting them to the base point will yield the same normal generating set) so that in $\pi_1(\Sigma_g)$ we have $c_{2i} = x_i$, $c_1 = y_1$, $c_{2g+1} = y_g$ and $c_{2i+1} = x_i x_{i+1}^{-1}$ for each i. It follows that $\pi_1(\Sigma_g)$ is generated by $\{c_i\}$. To get G we quotient $\pi_1(\Sigma_g)$ by normally generated subgroup by relators coming from the Dehn twist curves in S (and not U), which are of the form $t_i(c_{i-1})$ with $2 \leq i \leq 2g+1$ and $t_3(c_4)$, along with several other relations. We may assume that c_i are oriented so that $c_{i-1} \cdot c_i = +1$ for all i. Then $t_i(c_{i-1}) = c_{i-1}c_i = 1$ and $t_3(c_4) = c_4 c_3^{-1}$. These relations imply that

$$c_1 = c_2^{-1} = c_3 = c_4^{-1} = \cdots = c_{2g}^{-1} = c_{2g+1}$$

and

$$c_3 = c_4.$$

We thus see that

$$G \cong \langle c_1 \mid c_1^2, \text{rest of the relators coming from other vanishing cycles} \rangle$$

for our positive factorization $S(R^{\phi^{-1}})^{2g+2} S^{\phi^{-1}\psi}$.

At this point G is a quotient of the abelian group $\mathbb{Z}_2$ generated by c_1, so it is certainly an abelian group, and it suffices to show that $H_1(Z_{g,2g+2}) = 0$.

We will argue this by observing that the vanishing cycle coming from $t_{\phi^{-1}(B_2)}$ induces a relator killing the homology class of c_1. This is because it is homologous to an odd factor of c_1. For this reason, it is in fact enough to consider $t_{\phi^{-1}(B_2)}$ in

$H_1(Z_{g,2g+2};\mathbb{Z}_2)$. By the previous computations, we have

$$B_2 = (c_1) + (c_1 + c_3) + \cdots + (c_1 + c_3 + \cdots c_9) = c_1 + c_5 + c_9.$$

Let's apply ϕ^{-1}. Then

$$\begin{aligned}
c_1+c_5+c_9 &\xmapsto{t_8^{-1}} c_1+c_5+c_8+c_9 \xmapsto{t_7^{-1}} c_1+c_5+c_7+c_8+c_9 \xmapsto{t_6^{-1}} c_1+c_5+c_7+c_8+c_9\\
&\xmapsto{t_a^{-1}} c_1+c_5+c_7+c_8+c_9 \xmapsto{t_5^{-1}} c_1+c_5+c_7+c_8+c_9\\
&\xmapsto{t_6^{-1}} c_1+c_5+c_7+c_8+c_9 \xmapsto{t_7^{-1}} c_1+c_5+c_8+c_9 \xmapsto{t_8^{-1}} c_1+c_5+c_9\\
&\xmapsto{t_4^{-1}} c_1+c_4+c_5+c_9 \xmapsto{t_5^{-1}} c_1+c_4+c_9 \xmapsto{t_6^{-1}} c_1+c_4+c_9 \xmapsto{t_7^{-1}} c_1+c_4+c_9\\
&\xmapsto{t_3^{-1}} c_1+c_3+c_4+c_9 \xmapsto{t_4^{-1}} c_1+c_3+c_9 \xmapsto{t_5^{-1}} c_1+c_3+c_9 \xmapsto{t_6^{-1}} c_1+c_3+c_9\\
&\xmapsto{t_2^{-1}} c_1+c_3+c_9 \xmapsto{t_3^{-1}} c_1+c_3+c_9 \xmapsto{t_4^{-1}} c_1+c_3+c_4+c_9\\
&\xmapsto{t_5^{-1}} c_1+c_3+c_4+c_5+c_9 \xmapsto{t_1^{-1}} c_1+c_3+c_4+c_5+c_9\\
&\xmapsto{t_2^{-1}} c_1+c_3+c_4+c_5+c_9 \xmapsto{t_3^{-1}} c_1+c_4+c_5+c_9 \xmapsto{t_4^{-1}} c_1+c_5+c_9,
\end{aligned}$$

which gives c_1 in G (after killing all other c_i). Hence $G \cong 1$.

4.4. ***The geography.*** We are left with determining the portion of the geography plane populated by our simply connected spin Lefschetz fibrations

$$\{Z_{g,k} \mid g \geq 5 \text{ and odd, } k \leq 2g+2 \text{ and nonnegative}\}.$$

The Euler characteristic of $Z_{g,k}$ is given by the formula

$$\mathrm{e}(Z_{g,k}) = 4 - 4g + \ell = 4 - 4g + (16g + 8 + 4k) = 12(g+1) + 4k,$$

where ℓ is the number of Dehn twist curves in $P_{g,k}$.

Since the positive factorization U commutes with a hyperelliptic involution on Σ_g (after all, it is Hurwitz equivalent to the positive factorization of a hyperelliptic involution itself), by Endo's signature formula for hyperelliptic fibrations [7], it has signature $-4g-4$ (as expected, since the total space is $\mathbb{CP}^2\#(4g+5)\overline{\mathbb{CP}^2}$). By the Novikov additivity, we then get $\sigma(Z_g) = -8g-8$. Breeding the signature zero genus-2 Lefschetz pencil into this fibration (any number of times) does not change the signature [2] and we get

$$\sigma(Z_{g,k}) = -8(g+1).$$

We thus have

$$\chi_h(Z_{g,k}) = \tfrac{1}{4}(\mathrm{e}(Z_{g,k}) + \sigma(Z_{g,k})) = g + 1 + k$$

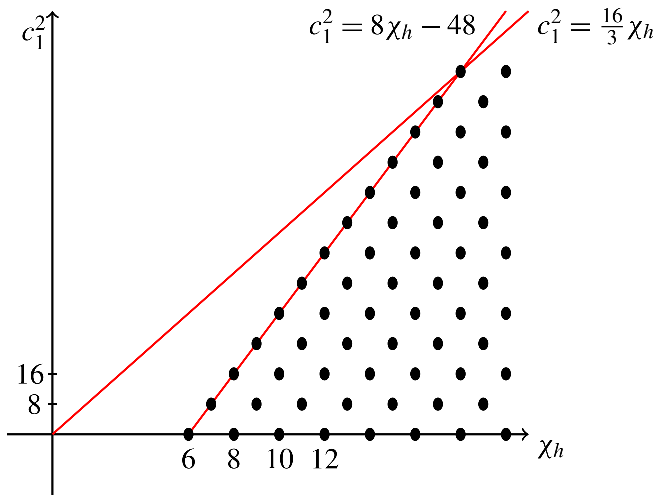

Figure 8. The region populated by spin $Z_{g,k}$.

and

$$c_1^2(Z_{g,k}) = 2\mathrm{e}(Z_{g,k}) + 3\sigma(Z_{g,k}) = 8k.$$

Thus, setting $g = 2r+5$, we see that $\{(\chi_h, c_1^2)(Z_{g,k})\}$ populate the region

$$\mathcal{R} = \{(6,0) + r(2,0) + k(1,8) \mid (r,k) \in \mathbb{N} \times \mathbb{N} \text{ with } k \leq 4(r+3)\}$$

of the geography plane, or equivalently,

$$\mathcal{R} = \left\{(m,n) \in \mathbb{N}^2 \mid n \geq 0,\ n \leq 8(m-6),\ n \leq \tfrac{16}{3}m \text{ and } n \equiv 8m \pmod{16}\right\};$$

see Figure 8. In particular, one can easily see from the first description of $\mathcal{R}$ above that we cover all of the admissible lattice points in $\mathbb{N}^2$ under the Noether line.

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SOME ARITHMETICAL PROPERTIES OF CONVERGENTS TO ALGEBRAIC NUMBERS

YANN BUGEAUD AND KHOA D. NGUYEN

Let ξ be an irrational algebraic real number and let $(p_k/q_k)_{k\geq 1}$ denote the sequence of its convergents. Let $(u_n)_{n\geq 1}$ be a nondegenerate linear recurrence sequence of integers, which is not a polynomial sequence. We show that if the intersection of the sequences $(q_k)_{k\geq 1}$ and $(u_n)_{n\geq 1}$ is infinite, then ξ is a quadratic number. This extends an earlier work of Lenstra and Shallit (1993). We also discuss several arithmetical properties of the base-b representation of the integers q_k, $k \geq 1$, where $b \geq 2$ is an integer. Finally, when ξ is a (possibly transcendental) non-Liouville number, we prove a result implying the existence of a large prime factor of $q_{k-1}\, q_k\, q_{k+1}$ for large k. This is related to earlier results of Erdős and Mahler (1939), Shorey and Stewart (1983), and Shparlinskii (1987).

1. introduction

Let θ be an arbitrary irrational real number and $(p_k(\theta)/q_k(\theta))_{k\geq 1}$ (we will use the shorter notation p_k/q_k when no confusion is possible and ξ instead of θ if the number is known to be algebraic) denote the sequence of its convergents.

Let $\mathcal{N}$ be an infinite set of positive integers. It follows from a result of Borosh and Fraenkel [6] that the set

$$\mathcal{K}(\mathcal{N}) = \{\theta \in \mathbb{R} : q_k(\theta) \text{ is in } \mathcal{N} \text{ for arbitrarily large } k\}$$

has always Hausdorff dimension at least $\frac{1}{2}$ and its Lebesgue measure is zero if there is some positive δ such that the series $\sum_{q\in\mathcal{N}} q^{-1+\delta}$ converges. Examples of sets $\mathcal{N}$ (or integer sequences $(u_n)_{n\geq 1}$) with the latter property include nondegenerate linear recurrence sequences, the set of integers having a bounded number of nonzero digits in their base-10 representation, sets of positive values taken at integer values by a given integer polynomial of degree at least 2, and sets of positive integers divisible only by prime numbers from a given, finite set.

MSC2020: primary 11J68; secondary 11J87.
Keywords: approximation to algebraic numbers, Schmidt subspace theorem, recurrence sequence, continued fraction.

Our main purpose is to discuss whether $\mathcal{K}(\mathcal{N})$ contains algebraic numbers for some special sets $\mathcal{N}$ for which $\mathcal{K}(\mathcal{N})$ has zero Lebesgue measure. Said differently, for an arbitrary irrational real algebraic number ξ, we investigate various arithmetical properties of the sequence $(q_k(\xi))_{k\geq 1}$. We consider the following questions:

A. Does the greatest prime factor of $q_k(\xi)$ tends to infinity with k? If yes, how rapidly?

B. Does the number of nonzero digits in the base-10 representation of $q_k(\xi)$ tends to infinity with k? If yes, how rapidly?

C. Are there infinitely many squares (cubes, perfect powers) in $(q_k(\xi))_{k\geq 1}$?

D. Is the intersection of $(q_k(\xi))_{k\geq 1}$ with a given linear recurrence sequence of integers finite or infinite?

First, let us recall that very few is known on the continued fraction expansion of an algebraic number of degree at least 3, while the continued fraction expansion of a quadratic real number ξ is ultimately periodic and takes the form

$$\xi = [a_0; a_1, \dots, a_r, \overline{a_{r+1}, \dots, a_{r+s}}].$$

Consequently, we have $q_{k+2s} = tq_{k+s} - (-1)^s\, q_k$ for $k > r$, where t is the trace of

$$\begin{pmatrix} a_{r+1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{r+2} & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_{r+s} & 1 \\ 1 & 0 \end{pmatrix};$$

see [18; 19]. This shows that $(q_k(\xi))_{k\geq 1}$ is the union of s binary recurrences whose roots are the roots of the polynomial $X^2 - tX + (-1)^s$, that is, the real numbers $\frac{1}{2}(t \pm \sqrt{t^2 - 4(-1)^s})$. Thus, for a quadratic real number ξ, we immediately derive Diophantine results on $(q_k(\xi))_{k\geq 1}$ from results on binary recurrences of the above form.

Question A has already been discussed in [7] and earlier works. Let us mention that it easily follows from Ridout's theorem [23] that the greatest prime factor of $q_k(\xi)$ tends to infinity with n, but we have no estimate of the rate of growth, except when ξ is quadratic (by known effective results on binary recurrences, see [28]). Furthermore, the theory of linear forms in logarithms gives a lower bound for the greatest prime factor of the product $p_k(\xi)\, q_k(\xi)$, which tends to infinity at least as fast as some constant times $\log_2 q_k(\xi) \log_3 q_k(\xi)/\log_4 q_k(\xi)$, where $\log_j$ denotes the j-th iterated logarithm function. Although we have no new contribution to Question A as stated for algebraic numbers ξ, we obtain new results on prime factors of $q_k(\theta)$ for a transcendental number θ. In 1939, Erdős and Mahler [16] proved that the greatest prime factor of $q_{k-1}(\theta)\, q_k(\theta)\, q_{k+1}(\theta)$ tends to infinity as k tends to infinity. In this paper, we obtain a more explicit result involving the irrationality exponent of θ.

We give a partial answer to Question B, which has not been investigated up to now. Question C is solved when ξ is quadratic: there are only finitely many perfect powers in the sequence $(q_k(\xi))_{k\geq 1}$ thanks to results of Pethő [22] and Shorey and Stewart [25] stating that there are only finitely many perfect powers in binary recurrence sequences of integers. This result is effective. When ξ has degree at least 3, Question C appears to be very difficult. Since $(n^d)_{n\geq 1}$ is a linear recurrence sequence for any given positive integer d, a large part of Question C is contained in Question D.

Question D is interesting for several reasons. First, some assumption on the linear recurrence must be added, since the linear recurrence $(n)_{n\geq 1}$ has infinite intersection with the sequence $(q_k(\xi))_{k\geq 1}$. Second, as already mentioned, when ξ is quadratic, its continued fraction expansion is ultimately periodic and the sequence $(q_k(\xi))_{k\geq 1}$ is the union of a finite set of binary recurrences. Among our results, we show that if a "nonsingular" linear recurrence has an infinite intersection with $(q_k(\xi))_{k\geq 1}$, then ξ must be quadratic. Unfortunately we must exclude linear recurrences of the form $(n^d)_{n\geq 1}$, and hence we do not have any contribution to Question C.

Recall that any nonzero linear recurrence sequence $(u_n)_{n\geq 1}$ of complex numbers can be expressed as

$$u_n = P_1(n)\,\alpha_1^n + \cdots + P_r(n)\,\alpha_r^n \quad \text{for } n \geq 1,$$

where $r \geq 1$, $\alpha_1, \ldots, \alpha_r$ are distinct nonzero complex numbers (called the roots of the recurrence), and $P_1, \ldots, P_r$ are nonzero polynomials with complex coefficients. This expression is unique up to rearranging the terms. The sequence $(u_n)_{n\geq 1}$ is called nondegenerate if α_i/α_j is not a root of unity for $1 \leq i \neq j \leq r$. For most problems about linear recurrence sequences, it is harmless to assume that $(u_n)_{n\geq 1}$ is nondegenerate. Indeed, if $(u_n)_{n\geq 1}$ is degenerate and L denotes the lcm of the orders of the roots of unity of the form α_i/α_j, then each of the subsequences $(u_{nL+m})_{n\geq 1}$ with $m \in \{0, \ldots, L-1\}$ is either identically zero or nondegenerate.

The proofs of our results rest on the p-adic Schmidt subspace theorem. This powerful tool was first applied to the study of continued fraction expansions of algebraic numbers by Corvaja and Zannier in [13; 14]. They proved in [13] that, for any positive real quadratic irrational α which is neither the square root of a rational number, nor a unit in the ring of integers of $\mathbb{Q}(\alpha)$, the period length of the continued fraction for α^n tends to infinity with n. They established in [14] that if $\alpha(n)$ and $\beta(n)$ are power sums over the rationals satisfying suitable necessary assumptions, then the length of the continued fraction for $\alpha(n)/\beta(n)$ tends to infinity with n; see also [12; 17; 24] for related questions. The Schmidt subspace theorem has also been used by Adamczewski and Bugeaud in [1; 2; 3; 9] to prove that the continued fraction expansion of an algebraic number of degree at least 3 cannot have arbitrary long repetitions nor quasipalindromes close to its start.

Let $(u_n)_{n\geq 1}$ be a nonconstant linear recurrence sequence with integral roots greater than 1 and rational coefficients. It follows from [15, Theorem 4.16] that the intersection of the sequences $(u_n)_{n\geq 1}$ and $(q_k)_{k\geq 1}$ is finite. This gives a first partial result toward Question D. For a real number θ, we let $\|\theta\|$ denote the distance from θ to the nearest integer. Our first main result gives a full answer to Question D. Its proof uses results of Kulkarni, Mavraki, and Nguyen [20], which extend a seminal work of Corvaja and Zannier [13], who showed that, if a real algebraic number $\alpha > 1$ and ℓ in (0, 1) are such that $\|\alpha^n\| < \ell^n$ for infinitely many positive integers n, then there is a positive integer d such that α^d is a Pisot number (observe that this conclusion is best possible).

Theorem 1.1. *Let $(p_k/q_k)_{k\geq 1}$ be the sequence of convergents to an irrational real algebraic number ξ of degree d. Let $\varepsilon > 0$. Let $(u_n)_{n\geq 1}$ be a nondegenerate linear recurrence sequence of integers, which is not a polynomial sequence. Then the set*

$$\left\{ n \in \mathbb{N} : u_n \neq 0 \ \textit{and} \ \|u_n \xi\| < \frac{1}{|u_n|^{(1/(d-1))+\varepsilon}} \right\}$$

is finite. In particular, if $d \geq 3$, then there are only finitely many pairs (n, k) such that $u_n = q_k$.

The case $d = 2$ of Theorem 1.1 is immediate, since quadratic real numbers have bounded partial quotients in their continued fraction expansion. Consequently, we restrict our attention to the case $d \geq 3$. Theorem 1.1 is a special case of Theorem 3.6, which deals with a larger class of integer sequences than that of recurrence sequences.

When $d = 3$, the exponent $\frac{1}{d-1} = \frac{1}{2}$ is best possible, as can be seen with the following example. Let $K \subset \mathbb{R}$ be a cubic field with a pair of complex-conjugate embeddings. Let $\xi \in K$ with $|\xi| > 1$ be a unit of the ring of integers. Let α and $\bar{\alpha}$ denote the remaining Galois conjugates of ξ. We have $|\alpha| = |\xi|^{-1/2}$ and, setting $u_n = \xi^n + \alpha^n + \bar{\alpha}^n$ for $n \geq 1$, we check that

$$|u_n \xi - u_{n+1}| \ll_\xi |\alpha^n| \ll_\xi |u_n|^{-1/2} \quad \text{for } n \geq 1,$$

where $\ll_\xi$ means that the implicit constant is positive and depends only on ξ. When $d \geq 4$, we do not know if Theorem 1.1 remains valid with a smaller exponent than $\frac{1}{d-1}$.

Theorem 1.1 allows us to complement the result of Lenstra and Shallit [21]:

Theorem 1.2 (Lenstra and Shallit [21]). *Let θ be an irrational real number, whose continued fraction expansion is given by $\theta = [a_0; a_1, a_2, \dots]$, and let $(p_k)_{k\geq 1}$ and $(q_k)_{k\geq 1}$ be the sequence of numerators and denominators of the convergents to θ. Then the following four conditions are equivalent:*

(i) *The sequence* $(p_k)_{k\geq 1}$ *satisfies a linear recurrence with constant complex coefficients.*

(ii) *The sequence* $(q_k)_{k\geq 1}$ *satisfies a linear recurrence with constant complex coefficients.*

(iii) *The sequence* $(a_n)_{n\geq 0}$ *is ultimately periodic.*

(iv) θ *is a quadratic irrational.*

The proof of Theorem 1.2 rests on the Hadamard quotient theorem. A simpler proof of a more general statement has been given by Bézivin [4], who instead of (ii) only assumes that $(q_k)_{k\geq 1}$ satisfies a linear recurrence with coefficients being polynomials in k and that the series $\sum_{k\geq 1} q_k\, z^k$ has a nonzero convergence radius.

We strengthen Theorem 1.2 for convergents of algebraic numbers as follows.

Corollary 1.3. *Let* $\xi = [a_0; a_1, a_2, \dots]$ *be an irrational real algebraic number, and let* $(p_k)_{k\geq 1}$ *and* $(q_k)_{k\geq 1}$ *be the sequence of numerators and denominators of the convergents to* ξ. *Then the following four conditions are equivalent*:

(i) *The sequence* $(p_k)_{k\geq 1}$ *has an infinite intersection with some nondegenerate linear recurrence sequence that is not a polynomial sequence.*

(ii) *The sequence* $(q_k)_{k\geq 1}$ *has an infinite intersection with some nondegenerate linear recurrence sequence that is not a polynomial sequence.*

(iii) *The sequence* $(a_n)_{n\geq 0}$ *is ultimately periodic.*

(iv) ξ *is a quadratic irrational.*

Now we present our results concerning Question B. Let $b \geq 2$ be an integer. Every positive integer N can be written uniquely as

$$N = d_k\, b^k + \cdots + d_1\, b + b_0,$$

where

$$d_0, d_1, \dots, d_k \in \{0, 1, \dots, b-1\}, \quad d_k \neq 0.$$

We define the length

$$\mathcal{L}(N, b) = \mathrm{Card}\{0 \leq j \leq k : d_j \neq 0\}$$

of the b-ary representation of N. We also define the number of digit changes by

$$\mathcal{DC}(N, b) = \mathrm{Card}\{2 \leq j \leq k : d_j \neq d_{j-1}\}.$$

Theorem 1.4. *Let* ξ *be an irrational real algebraic number and let* $b \geq 2$ *be an integer. Let* $(u_n)_{n\geq 1}$ *be a strictly increasing sequence of positive integers and* $\lambda \in (0, 1]$ *such that for every* $\varepsilon > 0$, *the inequality*

$$\|u_n\, \xi\| < u_n^{-\lambda+\varepsilon}$$

holds for all but finitely many n. *We have*:

(i) *Let k be a positive integer and let $\varepsilon > 0$. For all sufficiently large n, if δ is a divisor of u_n with $\mathcal{L}(\delta, b) \le k$ then $\delta < u_n^{(k-\lambda)/k+\varepsilon}$.*

(ii) *Let k be a nonnegative integer and let $\varepsilon > 0$. For all sufficiently large n, if δ is a divisor of u_n with $\mathcal{DC}(\delta, b) \le k$ then $\delta < u_n^{(k+2-\lambda)/(k+2)+\varepsilon}$.*

Consequently, let $(p_k/q_k)_{k\ge1}$ denote the sequence of convergents to ξ then each one of the limits $\lim_{k\to+\infty} \mathcal{L}(q_k, b)$, $\lim_{k\to+\infty} \mathcal{DC}(q_k, b)$, $\lim_{k\to+\infty} \mathcal{L}(p_k, b)$, *and* $\lim_{k\to+\infty} \mathcal{DC}(p_k, b)$ *is infinite.*

Except for certain quadratic numbers, it seems to be a very difficult problem to get an effective version of the last assertion of Theorem 1.4. Stewart [27, Theorem 2] established that if $(u_n)_{n\ge1}$ is a binary sequence of integers, whose roots ξ, ξ' are quadratic numbers with $|\xi| > \max\{1, |\xi'|\}$, then there exists a positive real number C such that

$$\mathcal{L}(u_n, b) > \frac{\log n}{\log\log n + C} - 1, \quad n \ge 5.$$

Consequently, if $(p_k/q_k)_{k\ge1}$ denote the sequence of convergents to a quadratic real algebraic number, then for $k \ge 4$ we have

$$\mathcal{L}(q_k, b) > \frac{\log k}{\log\log k + C} - 1 \quad \text{and} \quad \mathcal{DC}(q_k, b) > \frac{\log k}{\log\log k + C} - 1.$$

A similar question can be asked for the Zeckendorf representation [30] of q_k. Let $(F_n)_{n\ge0}$ denote the Fibonacci sequence defined by

$$F_0 = 0, \quad F_1 = 1, \quad \text{and} \quad F_{n+2} = F_{n+1} + F_n \qquad \text{for } n \ge 0.$$

Every positive integer N can be written uniquely as a sum:

$$N = \varepsilon_\ell F_\ell + \varepsilon_{\ell-1} F_{\ell-1} + \cdots + \varepsilon_2 F_2 + \varepsilon_1 F_1,$$

with $\varepsilon_\ell = 1$, ε_j in $\{0, 1\}$, and $\varepsilon_j \varepsilon_{j+1} = 0$ for $j = 1, \dots, \ell - 1$. This representation of N is called its Zeckendorf representation. The number of digits of N in its Zeckendorf representation is the number of positive integers j for which ε_j is equal to 1. By using the Schmidt subspace theorem we can in a similar way prove that the number of digits of $q_k(\xi)$ in its Zeckendorf representation tends to infinity with k, we omit the details (but see [10]).

Our last result is motivated by a theorem of Erdős and Mahler [16] on convergents to real numbers. Let S be a set of prime numbers. For a nonzero integer N, let $[N]_S$ denote the largest divisor of N composed solely of primes from S. Set $[0]_S = 0$. Recall that the irrationality exponent $\mu(\theta)$ of an irrational real number θ is the supremum of the real numbers μ such that there exist infinitely many rational numbers r/s with $s \ge 1$ and $|\theta - r/s| < 1/s^\mu$. It is always at least equal to 2 and, by definition, θ is called a Liouville number when $\mu(\theta)$ is infinite. Erdős and Mahler [16] established that, when θ is irrational and not a Liouville number, then the greatest prime factor of $q_{k-1}\, q_k\, q_{k+1}$ tends to infinity with k. We obtain the following more precise version of their result.

Theorem 1.5. *Let θ be an irrational real number and μ its irrationality exponent. Let $(p_k/q_k)_{k\geq 1}$ denote the sequence of convergents to θ. Let S be a finite set of prime numbers. If μ is finite, then, for every $\varepsilon > 0$ and every k sufficiently large (depending on ε), we have*

$$[q_{k-1}\, q_k\, q_{k+1}]_S < (q_{k-1}\, q_k\, q_{k+1})^{\mu/(\mu+1)+\varepsilon}. \tag{1-1}$$

The same conclusion holds when the sequence $(q_k)_{k\geq 1}$ is replaced by $(|p_k|)_{k\geq 1}$.

When θ is algebraic irrational and $\varepsilon > 0$, we have $[q_k]_S < q_k^{\varepsilon}$ for all large k by Ridout's theorem. The interesting feature of Theorem 1.5 is that it holds for all transcendental non-Liouville numbers.

Theorem 1.5 is ineffective. Under its assumption, it is proved in [11] that there exists a (large) positive, effectively computable $c = c(S)$ such that

$$[q_{k-1}\, q_k\, q_{k+1}]_S < (q_{k-1}\, q_k\, q_{k+1})^{1-1/(c\mu \log \mu)}, \quad k \geq 2.$$

For $\mu = 2$ (that is, for almost all θ), the exponent in (1-1) becomes $\frac{2}{3} + \varepsilon$. It is an interesting question to determine whether it is best possible. It cannot be smaller than $\frac{1}{3}$. Indeed, the Folding lemma (see, e.g., [8, Section 7.6]) allows one, for any given integer $b \geq 2$, to construct explicitly real numbers θ with $\mu(\theta) = 2$ and having infinitely many convergents whose denominator is a power of b.

Furthermore, there exist irrational real numbers $\theta = [a_0; a_1, a_2, \dots]$ with convergents p_k/q_k such that the q_k's are alternating among powers of 2 and 3. Indeed, let $k \geq 2$ and assume that $q_{k-1} = 2^c$ and $q_k = 3^d$ for positive integers c, d. Then, we have to find a positive integer a_{k+1} such that $2^c + a_{k+1}\, 3^d$ is a power of 2. To do this, it is sufficient to take for m the smallest integer greater than c such that 2^{m-c} is congruent to 1 modulo 3^d and then define $a_{k+1} = (2^{m-c} - 1)/3^d$. The sequence $(a_k)_{k\geq 1}$ increases very fast and θ is a Liouville number.

We are grateful to Professor Igor Shparlinskii for bringing our attention to [26]. Suppose the irrational number θ has the property that $\log q_n \ll n$ for every n; the set of all such θ's is strictly smaller than the set of all non-Liouville numbers. Then [26, Theorem 5] implies $P[q_1 \cdots q_n] \gg n$ for all sufficiently large n where $P[\,\cdot\,]$ denotes the largest prime factor. It seems possible to relax the condition $\log q_n \ll n$ at the expense of a weaker lower bound for $P[q_1 \cdots q_n]$ in order to allow θ to be certain Liouville numbers. On the other hand, it seems possible to extend the proof of Theorem 1.5 to get a lower bound for $P[q_1 \cdots q_n]$ in terms of n and the irrationality exponent of θ. We leave this further discussion for future work.

The outline of this paper is as follows. The proof of Theorem 1.5 and additional remarks on [16] are given in Section 4. Theorem 1.4 is established in Section 2 and the other results are proved in Section 3.

2. Proof of Theorem 1.4

For a prime number ℓ, we let $v_\ell : \mathbb{Q} \to \mathbb{Z} \cup \{\infty\}$ be the additive ℓ-adic valuation and let $|\cdot|_\ell = \ell^{-v_\ell(\,\cdot\,)}$ be the ℓ-adic absolute value.

Proof of Theorem 1.4. First, we prove part (i). Let $\mathcal{N}_1$ be the set of tuples $(m, n_1, \ldots, n_a)$ such that:

- $1 \le a \le k$ and $n_1 < n_2 < \cdots < n_a$ are nonnegative integers.
- There exist $d_1, \ldots, d_a$ in $\{1, \ldots, b-1\}$ such that $\delta := d_a\, b^{n_a} + \cdots + d_1\, b^{n_1}$ is a divisor of u_m and $\delta \ge u_m^{(k-\lambda)/k+\varepsilon}$.

Assume that $\mathcal{N}_1$ is infinite. Then, there exist an integer h with $1 \le h \le k$, positive integers $D_1, \ldots, D_h$, an infinite set $\mathcal{N}_2$ of $(h+1)$-tuples $(m_i, n_{1,i}, \ldots, n_{h,i})$ for $i \ge 1$ such that:

- $n_{1,i} < \cdots < n_{h,i}$ are nonnegative integers.
- For $i \ge 1$, we have a divisor of u_{m_i}:
$$\delta_{m_i} := D_h\, b^{n_{h,i}} + \cdots + D_1\, b^{n_{1,i}},$$
with $\delta_{m_i} \ge u_{m_i}^{(k-\lambda)/k+\varepsilon}$.
- We have

$$\lim_{i \to +\infty} (n_{j,i} - n_{j-1,i}) = +\infty, \quad j = 2, \ldots, h. \tag{2-1}$$

For $i \ge 1$, let w_{m_i} denote the nearest integer to $u_{m_i}\xi$ and let

$$v_{m_i} := u_{m_i}/\delta_{m_i} \le u_m^{\lambda/k-\varepsilon}. \tag{2-2}$$

When m_i is sufficiently large, we have

$$|\xi D_h\, v_{m_i}\, b^{n_{h,i}} + \cdots + \xi D_1\, v_{m_i}\, b^{n_{1,i}} - w_{m_i}| = \|\xi u_{m_i}\| < |u_{m_i}|^{-\lambda+\varepsilon/2}, \tag{2-3}$$

thanks to the given properties of $(u_m)_{m \ge 1}$ and λ. We are in position to apply the Schmidt subspace theorem.

Let S denote the set of prime divisors of b. Consider the linear forms in

$$\boldsymbol{X} = (X_0, X_1, \ldots, X_h)$$

given by

$$L_{j,\infty}(\boldsymbol{X}) := X_j, \quad j = 1, \ldots, h,$$
$$L_{0,\infty}(\boldsymbol{X}) := \xi D_h\, X_h + \cdots + \xi D_1 X_1 - X_0,$$

and, for every prime number ℓ in S,

$$L_{j,\ell}(\boldsymbol{X}) := X_j, \quad j = 0, \dots, h.$$

For the tuple

$$\boldsymbol{b}_i = (w_{m_i}, v_{m_i} b^{n_{h,i}}, \dots, v_{m_i} b^{n_{2,i}}, v_{m_i} b^{n_{1,i}}),$$

with a sufficiently large m_i, we use (2-2) and (2-3) to obtain

$$\prod_{j=0}^{h} |L_{j,\infty}(\boldsymbol{b}_i)| \times \prod_{\ell \in S} \prod_{j=0}^{h} |L_{j,\ell}(\boldsymbol{b}_i)|_\ell \le \|\xi u_{m_i}\| \cdot |v_{m_i}|^h < |u_{m_i}|^{-(h-1/2)\,\varepsilon} \ll H(\boldsymbol{b}_i)^{-(h-1/2)\,\varepsilon},$$

where the implied constant is independent of i and $H(\boldsymbol{b}_i)$ is the Weil height of the projective point $[w_{m_i} : v_{m_i} b^{n_{h,i}} : \cdots : v_{m_i} b^{n_{1,i}}]$.

The subspace theorem [5, Corollary 7.2.5] implies that there exist integers $t_0, t_1, \dots, t_h$, not all zero, and an infinite subset $\mathcal{N}_3$ of $\mathcal{N}_2$ such that

$$v_{m_i}(t_h\, b^{n_{h,i}} + \cdots + t_1\, b^{n_{1,i}}) + t_0\, w_{m_i} = 0 \quad \text{for } (w_{m_i}, n_{h,i}, \dots, n_{1,i}) \in \mathcal{N}_3. \tag{2-4}$$

Dividing the above equation by u_{m_i} and letting i tend to infinity, we deduce that

$$\frac{t_h}{D_h} + t_0\, \xi = 0.$$

Since ξ is irrational, we must have $t_0 = t_h = 0$. Then, we use (2-1) and (2-4) to derive that $t_1 = \cdots = t_{h-1} = 0$, a contradiction. This finishes the proof of (i).

We now prove part (ii) using a similar method. Let $s \ge 0$ and let x be a positive integer such that $\mathcal{DC}(x, b) = s$. If $s = 0$, we can write

$$x = d + db + \cdots + db^n = \frac{db^{n+1} - d}{b-1},$$

with $n \ge 0$ and $d \in \{1, \dots, b-1\}$. If $s > 0$, let $0 < c_1 < c_2 < \cdots < c_s$ denote the exponents of b where digit changes take place:

$$\begin{aligned} x &= d_0(1 + \cdots + b^{c_1 - 1}) + d_1(b^{c_1} + \cdots + b^{c_2 - 1}) + \cdots + d_s(b^{c_s} + \cdots + b^n) \\ &= \frac{-d_0 + (d_0 - d_1)\, b^{c_1} + (d_1 - d_2)\, b^{c_2} + \cdots + (d_{s-1} - d_s)\, b^{c_s} + d_s\, b^{n+1}}{b-1}, \end{aligned}$$

with $n \ge c_s$, $d_0, \dots, d_s \in \{0, \dots, b-1\}$, and $d_{i+1} \ne d_i$ for $0 \le i \le s-1$.

Let $\mathcal{N}_4$ be the set of tuples $(m, n_0, n_1, \dots, n_a)$ such that:

- $0 \le a \le k+1$ and $n_0 < \ldots < n_a$ are nonnegative integers.
- There exist integers $e_0, \dots, e_a$ in $[-(b-1), b-1]$ such that

$$\delta := \frac{e_0\, b^{n_0} + \cdots + e_{k+1}\, b^{n_{k+1}}}{b-1}$$

  is a divisor of u_m and $\delta \ge u_m^{(k+2-\lambda)/(k+2)+\varepsilon}$.

Assume that $\mathcal{N}_4$ is infinite. Then, there exist an integer h with $0 \le h \le k+1$, nonzero integers $E_0, \dots, E_h$, an infinite set $\mathcal{N}_5$ of $(h+2)$-tuples $(m_i, n_{h,i}, \dots, n_{0,i})$ for $i \ge 1$ such that:

- $n_{0,i} < \dots < n_{h,i}$ are nonnegative integers.
- For $i \ge 1$,
$$\delta_{m_i} := \frac{E_h\, b^{n_{h,i}} + \cdots + E_0\, b^{n_{0,i}}}{b-1}$$
is a divisor of u_{m_i} with $\delta_{m_i} \ge u_{m_i}^{(k+2-\lambda)/(k+2)+\varepsilon}$.
- We have
$$\lim_{i\to+\infty} (n_{j,i} - n_{j-1,i}) = +\infty, \quad j = 1, \dots, h.$$

We can now apply the subspace theorem in essentially the same way as before to finish the proof. □

3. Proof of Theorem 1.1 and Corollary 1.3

In Corollary 1.3, the equivalence (iii) $\Leftrightarrow$ (iv) and the implications (iv) $\Rightarrow$ (i) and (iv) $\Rightarrow$ (ii) are well known and have already appeared in Theorem 1.2. The implication (ii) $\Rightarrow$ (iv) is essentially the last assertion of Theorem 1.1 while the remaining implication (i) $\Rightarrow$ (iv) follows from the inequality $\|p_k/\xi\| \ll_\xi |p_k|^{-1}$ and Theorem 1.1 again. We spend the rest of this section to discuss Theorem 1.1.

From now on $\mathbb{N}$ is the set of positive integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\boldsymbol{\mu}$ is the group of roots of unity, and $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Let h denote the absolute logarithmic Weil height on $\overline{\mathbb{Q}}$. Let $k \in \mathbb{N}$, a tuple $(\alpha_1, \dots, \alpha_k)$ of nonzero complex numbers is called nondegenerate if $\alpha_i/\alpha_j \notin \boldsymbol{\mu}$ for $1 \le i \ne j \le k$. We consider the following more general family of sequences than (nondegenerate) linear recurrence sequences:

Definition 3.1. Let K be a number field. Let $\mathscr{S}(K)$ be the set of all sequences $(u_n)_{n\ge1}$ of complex numbers with the following property. There exist $k \in \mathbb{N}_0$ together with a nondegenerate tuple $(\alpha_1, \dots, \alpha_k) \in (K^*)^k$ such that, when n is sufficiently large, we can express

$$u_n = q_{n,1}\, \alpha_1^n + \cdots + q_{n,k}\, \alpha_k^n \tag{3-1}$$

for $q_{n,1}, \dots, q_{n,k} \in K^*$ and $\max_{1\le i\le k} h(q_{n,i}) = o(n)$.

In Definition 3.1, we allow $k = 0$ for which the empty sum in the right-hand side of (3-1) means 0. Any sequence $(u_n)_{n\ge1}$ that is eventually 0 is in $\mathscr{S}(K)$.

Example 3.2. Consider a linear recurrence sequence $(v_n)_{n\ge1}$ of the form

$$v_n = P_1(n)\, r_1^n + \cdots + P_k(n)\, r_k^n,$$

with $k \in \mathbb{N}$, distinct $r_1, \ldots, r_k \in K^*$, and nonzero $P_1, \ldots, P_k \in K[X]$. Let L be the lcm of the order of the roots of unity that appear among the r_i/r_j for $1 \le i, j \le k$. Then each one of the L sequences $(v_{nL+r})_{n\ge 1}$ for $r = 0, \ldots, L-1$ is a member of $\mathscr{S}(K)$.

As an explicit example, consider $v_n = 2^n + (-2)^n + n$ for $n \in \mathbb{N}$. The sequence $(v_{2n} = 2\cdot 4^n + 2n)_{n\ge 1}$ is in $\mathscr{S}(\mathbb{Q})$ and a tuple $(\alpha_1, \ldots, \alpha_k)$ satisfying the requirement in Definition 3.1 is $(\alpha_1 = 4,\ \alpha_2 = 1)$. The sequence $(v_{2n+1} = 2n+1)_{n\ge 1}$ is in $\mathscr{S}(\mathbb{Q})$ and a tuple $(\alpha_1, \ldots, \alpha_k)$ satisfying the requirement in Definition 3.1 is $(\alpha_1 = 1)$.

Lemma 3.3. *Let K be a number field and let $(u_n)_{n\ge 1}$ be an element of $\mathscr{S}(K)$. Let $k, \ell \in \mathbb{N}_0$ and let $(\alpha_1, \ldots, \alpha_k)$ and $(\beta_1, \ldots, \beta_\ell)$ be nondegenerate tuples of nonzero elements of K. Suppose that when n is sufficiently large, we can express*

$$u_n = q_{n,1}\,\alpha_1^n + \cdots + q_{n,k}\,\alpha_k^n = r_{n,1}\,\beta_1^n + \cdots + r_{n,\ell}\,\beta_\ell^n$$

for $q_{n,1}, \ldots, r_{n,\ell} \in K^*$ such that

$$\max\{h(q_{n,i}), h(r_{n,j}) : 1 \le i \le k,\ 1 \le j \le \ell\} = o(n)$$

as n tends to infinity. Then $k = \ell$ and there exist a permutation σ of $\{1, \ldots, k\}$ together with roots of unity $\zeta_1, \ldots, \zeta_k$ in K such that $\alpha_i = \zeta_i\,\beta_{\sigma(i)}$ for $1 \le i \le k$ and $q_{n,i}\,\zeta_i^n = r_{n,\sigma(i)}$ for every sufficiently large n and for every $1 \le i \le k$.

Proof. This follows from [20, Proposition 2.2]. □

Definition 3.4. Let K be a number field and let $(u_n)_{n\ge 1}$ be in $\mathscr{S}(K)$. Let $(\alpha_1, \ldots, \alpha_k)$ satisfy the requirement in Definition 3.1. We call k the number of $\mathscr{S}(K)$-roots of $(u_n)_{n\ge 1}$; this is well defined, thanks to Lemma 3.3. We call $(\alpha_1, \ldots, \alpha_k)$ a tuple of $\mathscr{S}(K)$-roots of $(u_n)_{n\ge 1}$; this is well defined up to permuting the α_i's and multiplying each α_i by a root of unity in K.

Here is the reason why we use the strange terminology "$\mathscr{S}(K)$-roots" instead of the usual "characteristic roots". In the theory of linear recurrence sequences, we have the well defined notion of characteristic roots. For example, *the* characteristic roots of $(u_n = 2^n + 1)_{n\ge 1}$ are 2 and 1. When regarding $(u_n)_{n\ge 1}$ as an element of $\mathscr{S}(K)$, we may say that any tuple $(2\zeta, \zeta')$ where ζ and ζ' are roots of unity in K is a tuple of $\mathscr{S}(K)$-roots of $(u_n)_{n\ge 1}$.

Definition 3.5. Let K be a number field. Let $(u_n)_{n\ge 1}$ be an element of $\mathscr{S}(K)$ and let $k \in \mathbb{N}_0$ be its number of $\mathscr{S}(K)$-roots. We say that $(u_n)_{n\ge 1}$ is admissible if

- either $k = 0$, i.e., $(u_n)_{n\ge 1}$ is eventually 0,
- or $k > 0$ and at least one entry in a tuple of $\mathscr{S}(K)$-roots of $(u_n)_{n\ge 1}$ is not a root of unity.

Since every nondegenerate linear recurrence sequence of algebraic numbers that is not a polynomial sequence is an admissible element of $\mathscr{S}(K)$ for some number field K, Theorem 1.1 follows from the below theorem.

Theorem 3.6. *Let ξ be an algebraic number of degree $d \geq 3$. Let $\varepsilon > 0$ and let K be a number field. Let $(u_n)_{n\geq 1}$ be a sequence of integers that is also an admissible element of $\mathscr{S}(K)$. Then the set*

$$\left\{ n \in \mathbb{N} : u_n \neq 0 \ \text{ and } \ \|u_n \xi\| < \frac{1}{|u_n|^{(1/(d-1))+\varepsilon}} \right\}$$

is finite.

The proof of Theorem 3.6 relies on a result of Kulkarni et al. [20], which extends a seminal work of Corvaja and Zannier [13]. By a sublinear function, we mean a function $f : \mathbb{N} \to (0, \infty)$ such that $\lim_{n\to\infty} f(n)/n = 0$, that is, $f(n) = o(n)$. We need the following slightly more flexible version of [20, Theorem 1.4]:

Theorem 3.7. *Let $C \in (0, 1]$. Let K be a number field, let $k \in \mathbb{N}$, let $(\alpha_1, \ldots, \alpha_k)$ be a nondegenerate tuple of algebraic numbers satisfying $|\alpha_i| \geq C$ for $1 \leq i \leq k$, and let f be a sublinear function. Assume that for some $\theta \in (0, C)$, the set $\mathcal{M}$ of $(n, q_1, \ldots, q_k) \in \mathbb{N} \times (K^*)^k$ satisfying*

$$\left\| \sum_{i=1}^{k} q_i \alpha_i^n \right\| < \theta^n \text{ and } \max_{1\leq i\leq k} h(q_i) < f(n)$$

is infinite. Then:

(i) *α_i is an algebraic integer for $i = 1, \ldots, k$.*

(ii) *For each $\sigma \in G_{\mathbb{Q}}$ and $i = 1, \ldots, k$ such that $\frac{\sigma(\alpha_i)}{\alpha_j} \notin \boldsymbol{\mu}$ for $j = 1, \ldots, k$, we have $|\sigma(\alpha_i)| < C$.*

Moreover, for all but finitely many $(n, q_1, \ldots, q_k) \in \mathcal{M}$ we have

$$\text{for } (\sigma, i, j) \in G_{\mathbb{Q}} \times \{1, \ldots, k\}^2, \quad \sigma(q_i \alpha_i^n) = q_j \alpha_j^n \text{ if and only if } \frac{\sigma(\alpha_i)}{\alpha_j} \in \boldsymbol{\mu}.$$

Remark 3.8. Theorem 3.7 in the case $C = 1$ is exactly [20, Theorem 1.4].

Proof of Theorem 3.7. When n is fixed, there are only finitely many tuples $(n, q_1, \ldots, q_k)$ in $\mathcal{M}$, thanks to the upper bound on $\max h(q_i)$ and Northcott's property. In the following, for $(n, q_1, \ldots, q_k)$ in $\mathcal{M}$, we tacitly assume that n is sufficiently large.

For N large enough, we have $1/\theta^N > 3/C^N$ and the interval $[1/C^N, 1/\theta^N)$ contains a prime number D which does not divide the denominator of α_i for $i = 1, \ldots, k$. We have

$$D\theta^N < 1 \leq DC^N.$$

Fix $\theta' \in (D\theta^N, 1)$. Let $\beta_i = D\alpha_i^N$ for $1 \le i \le k$. We now define the set $\mathcal{M}'$ as follows. Consider $(n, q_1, \dots, q_k) \in \mathcal{M}$ with $n \equiv r \bmod N$, write $n = mN + r$ with $r \in \{0, \dots, N-1\}$, then we have

$$\Big\| \sum_{i=1}^{k} q_i\, \alpha_i^r \beta_i^m \Big\| = \Big\| \sum_{i=1}^{k} q_i\, \alpha_i^r (D\alpha_i^N)^m \Big\| < D^m \theta^n = \theta^r (D\theta^N)^m < \theta'^m,$$

assuming n and hence m are sufficiently large so that the last inequality holds, thanks to the choice $\theta' \in (D\theta^n, 1)$. We include the tuple $(m, q_1\, \alpha_1^r, \dots, q_k\, \alpha_k^r)$ in $\mathcal{M}'$. Finally, consider the sublinear function

$$g(n) = f(n) + (N-1) \max_{1 \le i \le k} h(\alpha_i),$$

so that $\max_{1 \le i \le k} h(q_i\, \alpha_i^r) < g(n)$.

We apply [20, Theorem 1.4] for the tuple $(\beta_1, \dots, \beta_k)$, the function g, the number θ', and the set $\mathcal{M}'$ to conclude that:

- $D\alpha_i^N$ is an algebraic integer for $1 \le i \le k$. Our choice of D implies that α_i is an algebraic integer for $1 \le i \le k$.
- For each $\sigma \in G_{\mathbb{Q}}$ and $i \in \{1, \dots, k\}$ such that $\frac{\sigma(\alpha_i)}{\sigma(\alpha_j)} \notin \boldsymbol{\mu}$ for every $j \in \{1, \dots, k\}$, we have $\sigma(D\alpha_i^N) < 1$ consequently $\sigma(\alpha_i) < 1/D^{1/N} \le C$.
- The last assertion of Theorem 3.7 holds.

This finishes the proof. □

Proof of Theorem 3.6. Let k denote the number of $\mathscr{S}(K)$-roots of $(u_n)_{n\ge1}$. The case $k = 0$ (i.e., $(u_n)_{n\ge1}$ is eventually 0) is obvious. Assume $k > 0$ and let $(\alpha_1, \dots, \alpha_k)$ be a tuple of $\mathscr{S}(K)$-roots of $(u_n)_{n\ge1}$. For $L \in \mathbb{N}$ and $r \in \{0, \dots, L-1\}$, each sequence $(u_{nL+r})_{n\ge1}$ is an admissible element of $\mathscr{S}(K)$ and admits $(\alpha_1^L, \dots, \alpha_k^L)$ as a tuple of $\mathscr{S}(K)$-roots. Let L be the lcm of the order of roots of unity among the $\sigma(\alpha_i)/\tau(\alpha_j)$ for $\sigma, \tau \in G_{\mathbb{Q}}$ and $1 \le i, j \le k$ and replace $(u_n)_{n\ge1}$ by each $(u_{nL+r})_{n\ge1}$, we may assume

(3-2) for $\sigma, \tau \in G_{\mathbb{Q}}$ and $1 \le i, j \le k$, $\quad \dfrac{\sigma(\alpha_i)}{\tau(\alpha_j)} \in \boldsymbol{\mu}$ if and only if $\sigma(\alpha_i) = \tau(\alpha_j)$.

We first prove that the set $\{\alpha_1, \dots, \alpha_k\}$ is Galois invariant.

For sufficiently large n, express

$$u_n = q_{n,1}\, \alpha_1^n + \dots + q_{n,k}\, \alpha_k^n$$

as in Definition 3.1. Let $\sigma \in G_{\mathbb{Q}}$, since $u_n \in \mathbb{Z}$ we have

$$q_{n,1}\, \alpha_1^n + \dots + q_{n,k}\, \alpha_k^n = \sigma(q_{n,1})\, \sigma(\alpha_1)^n + \dots + \sigma(q_{n,k})\, \sigma(\alpha_k)^n \quad \text{for all large } n.$$

From [20, Proposition 2.2], we have that for every $i \in \{1, \dots, k\}$ there exists $j \in \{1, \dots, k\}$ such that $\sigma(\alpha_i)/\alpha_j \in \boldsymbol{\mu}$ and this gives $\sigma(\alpha_i) = \alpha_j$, thanks to (3-2). Theorem 3.7 implies that the α_i's are algebraic integers and for every sufficiently large n, for $(\sigma, i, j) \in G_{\mathbb{Q}} \times \{1, \dots, k\}^2$ we have

$$\sigma(q_{n,i}) = q_{n,j}, \quad \text{whenever } \sigma(\alpha_i) = \alpha_j. \tag{3-3}$$

Since $(u_n)_{n\geq 1}$ is admissible, at least one of the α_i's is not a root of unity and hence

$$M := \max_{1\leq i\leq k} |\alpha_i| > 1. \tag{3-4}$$

Suppose the set

$$T := \left\{ n \in \mathbb{N} : u_n \neq 0 \text{ and } \|u_n \xi\| < \frac{1}{|u_n|^{(1/(d-1))+\varepsilon}} \right\}$$

is infinite, then we will arrive at a contradiction. Let δ denote a sufficiently small positive real number that will be specified later. By [20, Section 2], we have

$$|u_n| > M^{(1-\delta)\, n} \tag{3-5}$$

for all large n. Therefore

$$\|\xi\, q_{n,1}\, \alpha_1^n + \cdots + \xi\, q_{n,k}\, \alpha_k^n\| < \frac{1}{M^{(1-\delta)(1/(d-1)+\varepsilon)\, n}} \tag{3-6}$$

for all large n in T.

We relabel the α_i's and let $m \leq \ell \leq k$ such that

(i) $|\alpha_1| = M$.

(ii) $|\alpha_i| \geq \frac{1}{M^{1/(d-1)+\delta}}$ for $1 \leq i \leq \ell$ while $|\alpha_i| < \frac{1}{M^{1/(d-1)+\delta}}$ for $\ell+1 \leq i \leq k$.

(iii) Among the $\alpha_1, \dots, \alpha_\ell$, we have that $\alpha_1, \dots, \alpha_m$ are exactly the Galois conjugates of α_1. When combining with (ii), this means that $\alpha_1, \dots, \alpha_m$ are precisely the Galois conjugates of α_1 with modulus at least $M^{-(1/(d-1)+\delta)}$.

We require δ small enough so that

$$\frac{1}{d-1} + \delta < (1-\delta)\Big(\frac{1}{d-1} + \varepsilon\Big). \tag{3-7}$$

Choose the real number c such that:

$$\frac{1}{d-1} + \delta < c < (1-\delta)\Big(\frac{1}{d-1} + \varepsilon\Big) \quad \text{and} \quad |\alpha_i| < \frac{1}{M^c} \text{ for } \ell+1 \leq i \leq k. \tag{3-8}$$

Thanks to this choice of c and the assumption that $h(q_{n,i}) = o(n)$ for $1 \leq i \leq k$, we have

$$|\xi\, q_{n,\ell+1}\, \alpha_{\ell+1}^n + \cdots + \xi\, q_{n,k}\, \alpha_k^n| < \frac{1}{2M^{cn}} \tag{3-9}$$

for all sufficiently large n. From (3-6) and (3-8), we have

$$\|\xi\, q_{n,1}\, \alpha_1^n + \cdots + \xi\, q_{n,k}\, \alpha_k^n\| < \frac{1}{2M^{cn}} \tag{3-10}$$

for all large n in T. Combining the above inequalities, we have

$$\|\xi\, q_{n,1}\, \alpha_1^n + \cdots + \xi\, q_{n,\ell}\, \alpha_\ell^n\| < \frac{1}{M^{cn}} \tag{3-11}$$

for all large n in T.

Let F be the Galois closure of $K(\xi)$. We apply Theorem 3.7 for the tuple $(\alpha_1, \dots, \alpha_\ell)$, $C = M^{-(1/(d-1)+\delta)}$, $\theta = M^{-c}$, and the inequality (3-11) then use (3-2) and (3-3) to have that for every large n in T, $\sigma \in \mathrm{Gal}(F/\mathbb{Q})$, and $1 \le i, j \le \ell$,

$$\text{if } \sigma(\alpha_i) = \alpha_j, \text{ then } \sigma(\xi\, q_{n,i}\, \alpha_i^n) = \xi\, q_{n,j}\, \alpha_j^n \text{ and hence } \sigma(\xi) = \xi. \tag{3-12}$$

Since $\alpha_1, \dots, \alpha_m$ are exactly the Galois conjugates of α_1 among the $\alpha_1, \dots, \alpha_\ell$, equation (3-12) implies that ξ is fixed by at least $m|\,\mathrm{Gal}(F/\mathbb{Q}(\alpha_1))| = m[F : \mathbb{Q}(\alpha_1)]$ many automorphisms in $\mathrm{Gal}(F/\mathbb{Q})$. Put $d' = [\mathbb{Q}(\alpha_1) : \mathbb{Q}]$, we have

$$[F : \mathbb{Q}(\xi)] = |\,\mathrm{Gal}(F/\mathbb{Q}(\xi))| \ge m[F : \mathbb{Q}(\alpha_1)] = \frac{m}{d'}[F : \mathbb{Q}]. \tag{3-13}$$

Since $[\mathbb{Q}(\xi) : \mathbb{Q}] = d$, equation (3-13) implies $m \le d'/d$. This means α_1 has at least $d'(d-1)/d$ many Galois conjugates with modulus less than $M^{-(1/(d-1)+\delta)}$. Combining with the fact that all Galois conjugates of α_1 have modulus at most M, we have

$$|\,\mathrm{N}_{\mathbb{Q}(\alpha_1)/\mathbb{Q}}(\alpha_1)| \le M^{d'/d} M^{-(1/(d-1)+\delta)\, d'(d-1)/d} < 1,$$

since $M > 1$ and $\delta > 0$. This contradicts the fact that α_1 is a nonzero algebraic integer and we finish the proof. □

4. Proof of Theorem 1.5 and further discussion on Erdős and Mahler [16]

Proof of Theorem 1.5. We assume that θ is not a Liouville number, that is, we assume that μ is finite. Define

$$Q_k = q_{k-1}\, q_k\, q_{k+1}, \quad k \ge 2.$$

Let S be a finite set of prime numbers. Write $\theta = [a_0; a_1, a_2, \dots]$ and recall that

$$q_{k+1} = a_{k+1}\, q_k + q_{k-1}, \quad k \ge 2.$$

Let $k \ge 2$ and set $d_k = \gcd(q_{k-1}, q_{k+1})$. Since q_{k-1} and q_k are coprime, we see that d_k divides a_{k+1}. Define

$$q_{k-1}^* = q_{k-1}/d_k, \quad q_{k+1}^* = q_{k+1}/d_k, \quad a_{k+1}^* = a_{k+1}/d_k.$$

Then, we have

$$q_{k+1}^* = a_{k+1}^* \, q_k + q_{k-1}^*, \quad k \geq 2.$$

Let $\varepsilon > 0$. By the Schmidt subspace theorem, the set of points (q_{k-1}^*, q_{k+1}^*) such that

$$q_{k-1}^* \, q_{k+1}^* \prod_{p \in S} |q_{k-1}^* \, q_{k+1}^* (q_{k+1}^* - q_{k-1}^*)|_p < (q_{k+1}^*)^{-\varepsilon}$$

is contained in a union of finitely many proper subspaces. Since q_{k-1}^* and q_{k+1}^* are coprime, this set is finite. We deduce that, for k large enough, we get

$$\prod_{p \in S} |q_{k-1}^* \, q_{k+1}^* (q_{k+1}^* - q_{k-1}^*)|_p > (q_{k-1}^* \, q_{k+1}^*)^{-1} (q_{k+1}^*)^{-\varepsilon},$$

thus

$$\prod_{p \in S} |q_{k-1} \, q_{k+1} (a_{k+1}^* \, q_k)|_p > (q_{k-1} \, q_{k+1})^{-1} (q_{k+1}^*)^{-\varepsilon},$$

and hence

$$\prod_{p \in S} |q_{k-1} \, q_{k+1} \, q_k|_p > (q_{k-1} \, q_{k+1})^{-1} (q_{k+1})^{-\varepsilon}.$$

Recalling that $q_{k-1} < q_k$ and $q_{k+1} < q_k^{\mu - 1 + \varepsilon}$ for k large enough, we get

$$[Q_k]_S < q_{k-1} \, q_{k+1}^{1+\varepsilon} < q_k^{\mu+\varepsilon} \, Q_k^{\varepsilon}.$$

Since

$$Q_k < q_k^2 \, q_{k+1} < q_k^{\mu+1+\varepsilon},$$

we get

$$[Q_k]_S < Q_k^{(\mu+\varepsilon)/(\mu+1+\varepsilon)} \, Q_k^{\varepsilon}.$$

This proves (1-1). The last assertion can be proved in the same manner, thanks to the identity $p_{k+1} = a_{k+1} \, p_k + p_{k-1}$ and the inequalities

$$|p_{k-1}| < |p_k| \quad \text{and} \quad |p_{k+1}| < |p_k|^{\mu-1+\varepsilon}$$

for large k. □

The following was suggested at the end of [16]:

Question 4.1 (Erdős and Mahler [16]). *Let θ be an irrational real number such that the largest prime factor of $p_n(\theta) \, q_n(\theta)$ is bounded for infinitely many n. Is it true that θ is a Liouville number?*

Without further details, Erdős and Mahler stated the existence of θ with the given properties in Question 4.1. We provide a construction here for the sake of completeness.

Let S and T be disjoint nonempty sets of prime numbers such that S has at least two elements. We construct uncountably many θ such that for infinitely many n the prime factors of $p_n(\theta)$ belong to S while the prime factors of $q_n(\theta)$ belong to T. To simplify the notation, we consider the case $S = \{2, 3\}$ and $T = \{5\}$. The construction for general S and T follows the same method. The constructed numbers θ have the form

$$\theta = \sum_{i=1}^{+\infty} \frac{a_i}{5^{3^i}}.$$

Let Γ be the set of positive integers with only prime factors in $\{2, 3\}$. For every positive integer m, let $\gamma(m)$ denote the smallest element of Γ that is greater than m. Let $f(m) := \frac{\gamma(m)-m}{m}$. By [29], we have

$$\lim_{m\to+\infty} f(m) = 0. \tag{4-1}$$

First, we construct the sequence of positive integers $s(1) < s(2) < \dots$ recursively:

- $s(1) = 1$.
- After having $s(1), \dots, s(k)$, let N_k be a positive integer depending on $s(k)$ such that

$$f(m) < \frac{1}{5^{3^{s(k)+1}}} \quad \text{for } m \geq N_k. \tag{4-2}$$

Then we choose $s(k+1)$ so that

$$5^{2\cdot 3^{s(k+1)-1}} \geq N_k \quad \text{and} \quad s(k+1) > s(k)+1. \tag{4-3}$$

Now we construct the a_i's:

- $a_1 = 1$.
- Choose arbitrary $a_i \in \{1, 2\}$ for $i \notin \{s(1), s(2), \dots\}$. Since $s(k+1) > s(k)+1$ for every k, the set $\mathbb{N} \setminus \{s(1), s(2), \dots\}$ is infinite. Hence there are uncountably many choices here.
- Since $s(1) = 1$, we already had $a_{s(1)}$. Suppose we have $a_{s(1)}, \dots, a_{s(k)}$ positive integers with the following properties:

  (i) For $1 \leq j \leq k$, we have $\sum_{i=1}^{s(j)} \frac{a_i}{5^{3^i}} = \frac{u_{s(j)}}{5^{3^{s(j)}}}$ with $u_{s(j)} \in \Gamma$.

  (ii) For $2 \leq j \leq k$, we have $\frac{a_{s(j)}}{5^{3^{s(j)}}} < \frac{1}{5^{3^{s(j-1)+1}}}$.

We now define $a_{s(k+1)}$ so that the above two properties continue to hold with $j = k+1$ as well. Thanks to property (ii) and the fact that $a_i \leq 2$ for $i \notin \{s(1), s(2), \dots\}$, we have the rough estimate

$$\frac{u}{5^{3^{s(k+1)-1}}} := \sum_{i=1}^{s(k+1)-1} \frac{a_i}{5^{3^i}} \leq \sum_{i=1}^{s(k+1)-1} \frac{2}{5^{3^i}} + \sum_{j=1}^{k-1} \frac{1}{5^{3^{s(j)+1}}} < 1,$$

and hence, $u < 5^{3^{s(k+1)-1}}$. From

$$\sum_{i=1}^{s(k+1)} \frac{a_i}{5^{3^i}} = \frac{u}{5^{3^{s(k+1)-1}}} + \frac{a_{s(k+1)}}{5^{3^{s(k+1)}}} = \frac{u \cdot 5^{2\cdot 3^{s(k+1)-1}} + a_{s(k+1)}}{5^{3^{s(k+1)}}},$$

we now define

$$a_{s(k+1)} = \gamma(u \cdot 5^{2\cdot 3^{s(k+1)-1}}) - u \cdot 5^{2\cdot 3^{s(k+1)-1}}.$$

Recall that $\gamma(u \cdot 5^{2\cdot 3^{s(k+1)-1}})$ is the smallest element of Γ that is greater than $u \cdot 5^{2\cdot 3^{s(k+1)-1}}$. This verifies property (i) for $j = k+1$. To verify (ii) for $j = k+1$, we have

$$\begin{aligned} \frac{a_{s(k+1)}}{5^{3^{s(k+1)}}} &= \frac{\gamma(u \cdot 5^{2\cdot 3^{s(k+1)-1}}) - u \cdot 5^{2\cdot 3^{s(k+1)-1}}}{u \cdot 5^{2\cdot 3^{s(k+1)-1}}} \cdot \frac{u \cdot 5^{2\cdot 3^{s(k+1)-1}}}{5^{3^{s(k+1)}}} \\ &< \frac{\gamma(u \cdot 5^{2\cdot 3^{s(k+1)-1}}) - u \cdot 5^{2\cdot 3^{s(k+1)-1}}}{u \cdot 5^{2\cdot 3^{s(k+1)-1}}} \quad [\text{since } u < 5^{3^{s(k+1)-1}}] \\ &= f(u \cdot 5^{2\cdot 3^{s(k+1)-1}}) \\ &< \frac{1}{5^{3^{s(k)+1}}} \quad \text{by (4-2) and (4-3).} \end{aligned}$$

By the principle of recursive definition, we have a_i for $i \in \{s(1), s(2), \dots\}$ such that property (i) holds for every $j \geq 1$ and property (ii) holds for every $j \geq 2$.

Write $u_n/v_n = \sum_{i \leq n} \frac{a_i}{5^{3^i}}$ with $v_n = 5^{3^n}$. We have

$$|\theta - u_{s(k)}/v_{s(k)}| = \sum_{i=s(k)+1}^{\infty} \frac{a_i}{5^{3^i}} < \sum_{i=s(k)+1}^{\infty} \frac{2}{5^{3^i}} + \sum_{j=k}^{\infty} \frac{1}{5^{3^{s(j)+1}}} < \frac{4}{5^{3^{s(k)+1}}} = \frac{4}{v_{s(k)}^3}.$$

Therefore the $u_{s(k)}/v_{s(k)}$ are among the convergents to θ.

It is not clear to us whether the above numbers θ are always Liouville numbers. However, we suspect that this is the case. In order to construct Liouville numbers, we can use a similar method for numbers of the form $\sum_{i \geq 1} \frac{b_i}{5^{i!}}$.

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LOCAL GALOIS REPRESENTATIONS OF SWAN CONDUCTOR ONE

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We construct the local Galois representations over the complex field whose Swan conductors are one by using étale cohomology of Artin–Schreier sheaves on affine lines over finite fields. Then, we study the Galois representations, and give an explicit description of the local Langlands correspondences for simple supercuspidal representations. We discuss also a more natural realization of the Galois representations in the étale cohomology of Artin–Schreier varieties.

Introduction

Let K be a nonarchimedean local field. Let n be a positive integer. The existence of the local Langlands correspondence for $\mathrm{GL}_n(K)$, proved in [Laumon et al. 1993] and [Harris and Taylor 2001], is one of the fundamental results in the Langlands program. However, even in this fundamental case, an explicit construction of the local Langlands correspondence has not yet been obtained. One of the most striking results in this direction is the result of Bushnell and Henniart [2005a; 2005b; 2010] for essentially tame representations. On the other hand, we don't know much about the explicit construction outside essentially tame representations.

We discuss this problem for representations of Swan conductor 1. The irreducible supercuspidal representations of $\mathrm{GL}_n(K)$ of Swan conductor 1 are equivalent to the simple supercuspidal representations in the sense of Adrian and Liu [2016] (see [Gross and Reeder 2010; Reeder and Yu 2014]). Such representations are called "epipelagic" in [Bushnell and Henniart 2014].

Let p be the characteristic of the residue field k of K. If n is prime to p, the simple supercuspidal representations of $\mathrm{GL}_n(K)$ are essentially tame. Hence, this case is covered by the work of Bushnell and Henniart. See also [Adrian and Liu 2016]. It is discussed in [Kaletha 2015] to generalize the construction of the local Langlands correspondence for essentially tame epipelagic representations to other reductive groups.

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In this paper, we consider the case where p divides n. In this case, the simple supercuspidal representations of $\mathrm{GL}_n(K)$ are not essentially tame. Moreover, if n is a power of p, the irreducible representations of the Weil group W_K of Swan conductor 1, which correspond to the simple supercuspidal representations via the local Langlands correspondence, cannot be induced from any proper subgroup. Such representations are called primitive (see [Koch 1977]). For simple supercuspidal representations, we have a straightforward characterization of the local Langlands correspondence given in [Bushnell and Henniart 2014]. Further, Bushnell and Henniart study the restriction to the wild inertia subgroup of the Langlands parameters for the simple supercuspidal representations explicitly. Actually, the restriction to the wild inertia subgroup already determines the original Langlands parameters up to character twists, but we need additional data, which appear in Bushnell and Henniart's characterization, to pin down the correct Langlands parameters. On the other hand, the construction of the irreducible representations of W_K of Swan conductor 1 is a nontrivial problem. What we will do in this paper is

- to construct the irreducible representations of W_K of Swan conductor 1 without appealing to the existence of the local Langlands correspondence, and
- to give a description of the Langlands parameters themselves for the simple supercuspidal representations.

Let ℓ be a prime number different from p. For the construction of the irreducible representations of W_K of Swan conductor 1, we use étale cohomology of an Artin–Schreier ℓ-adic sheaf on $\mathbb{A}^1_{k^{\mathrm{ac}}}$, where k^{ac} is an algebraic closure of k. It will be possible to avoid usage of geometry in the construction of the irreducible representations of W_K of Swan conductor 1. However, we prefer this approach, because

- we can use geometric tools such as the Lefschetz trace formula and the product formula of Deligne–Laumon to study the constructed representations, and
- the construction works also for ℓ-adic integral coefficients and mod ℓ coefficients.

A description of the local Langlands correspondence for the simple supercuspidal representations is discussed in [Imai and Tsushima 2022] in the special case where $n = p = 2$. Even in the special case, our method in this paper is totally different from that in [Imai and Tsushima 2022].

We explain the main result. We write $n = p^e n'$, where n' is prime to p. We fix a uniformizer ϖ of K and an isomorphism $\iota : \overline{\mathbb{Q}}_\ell \simeq \mathbb{C}$.

Let $\mathcal{L}_\psi$ be the Artin–Schreier $\overline{\mathbb{Q}}_\ell$-sheaf on $\mathbb{A}^1_{k^{\mathrm{ac}}}$ associated to a nontrivial character ψ of $\mathbb{F}_p$. Let $\pi : \mathbb{A}^1_{k^{\mathrm{ac}}} \to \mathbb{A}^1_{k^{\mathrm{ac}}}$ be the morphism defined by $\pi(y) = y^{p^e+1}$. Let $\zeta \in \mu_{q-1}(K)$, where $q = |k|$. We put $E_\zeta = K[X]/(X^{n'} - \zeta\varpi)$. Then we can define a natural action of W_{E_ζ} on $H^1_{\mathrm{c}}(\mathbb{A}^1_{k^{\mathrm{ac}}}, \pi^*\mathcal{L}_\psi)$. Using this action, we can associate a primitive representation $\tau_{n,\zeta,\chi,c}$ of W_{E_ζ} to $\zeta \in \mu_{q-1}(K)$, a character χ of $k^\times$ and $c \in \mathbb{C}^\times$. We construct an irreducible representation $\tau_{\zeta,\chi,c}$ of Swan conductor 1 as the induction of $\tau_{n,\zeta,\chi,c}$ to W_K.

We can associate a simple supercuspidal representation $\pi_{\zeta,\chi,c}$ of $\mathrm{GL}_n(K)$ to the same triple (ζ, χ, c) by type theory. Any simple supercuspidal representation can be written in this form uniquely (see [Imai and Tsushima 2018, Proposition 1.3]).

Theorem. *The representations $\tau_{\zeta,\chi,c}$ and $\pi_{\zeta,\chi,c}$ correspond via the local Langlands correspondence.*

In Section 1, we recall a general fact on representations of a semidirect product of a Heisenberg group with a cyclic group. In Section 2, we give a construction of the irreducible representations of W_K of Swan conductor 1. To construct a representation of W_K which naturally fits a description of the local Langlands correspondence, we need a subtle character twist. Such a twist appears also in the essentially tame case in [Bushnell and Henniart 2010], where it is called a rectifier. Our twist can be considered as an analogue of the rectifier. We construct the representations of W_K using geometry, but we give also a representation theoretic characterization of the constructed representations. In Section 3, we give a construction of the simple supercuspidal representations of $\mathrm{GL}_n(K)$ using the type theory.

In Section 4, we state the main theorem and recall a characterization of the local Langlands correspondence for simple supercuspidal representations given in [Bushnell and Henniart 2014]. The characterization consists of the three equalities of (i) the determinant and the central character, (ii) the refined Swan conductors, and (iii) the epsilon factors.

In Section 5, we recall some general facts on epsilon factors. In Section 6, we recall facts on Stiefel–Whitney classes, multiplicative discriminants and additive discriminants. We use these facts to calculate Langlands constants of wildly ramified extensions. In Section 7, we recall the product formula of Deligne–Laumon. In Section 8, we show the equality of the determinant and the central character using the product formula of Deligne–Laumon.

In Section 9, we construct a field extension T^{u}_ζ of E_ζ such that the restriction of $\tau_{n,\zeta,\chi,c}$ to $W_{T^{\mathrm{u}}_\zeta}$ is an induction of a character and $p \nmid [T^{\mathrm{u}}_\zeta : E_\zeta]$, which we call an imprimitive field. In Section 10, we show the equality of the refined Swan conductors. We see also that the constructed representations of W_K are actually of Swan conductor 1.

In Section 11, we show the equality of the epsilon factors. It is difficult to calculate the epsilon factors of irreducible representations of W_K of Swan conductor 1 directly, because primitive representations are involved. However, we know the equality of the epsilon factors up to p^e-th roots of unity if $n = p^e$, since we have already checked the conditions (i) and (ii) in the characterization. Using this fact and $p \nmid [T_\zeta^{\mathrm{u}} : E_\zeta]$, the problem is reduced to study an epsilon factor of a character. Next we reduce the problem to the case where the characteristic of K is p and $k = \mathbb{F}_p$. At this stage, it is possible to calculate the epsilon factor if $p \neq 2$. However, it is still difficult if $p = 2$, because the direct calculation of the epsilon factor involves an explicit study of the Artin reciprocity map for a wildly ramified extension with a nontrivial ramification filtration. This is a special phenomenon in the case where $p = 2$. We will avoid this difficulty by reducing the problem to the case where $e = 1$. In this case, we have already known the equality up to sign. Hence, it suffices to show the equality of nonzero real parts. This is easy, because the difficult study of the Artin reciprocity map involves only the imaginary part of the equality.

In Appendix, we discuss a construction of irreducible representations of W_K of Swan conductor 1 in the cohomology of Artin–Schreier varieties. This geometric construction incorporates a twist by a “rectifier”. We see that the “rectifier” parts come from the cohomology of Artin–Schreier varieties associated to quadratic forms. The Artin–Schreier varieties which we use have origins in studies of Lubin–Tate spaces in [Imai and Tsushima 2017; 2021].

Notation. Let $A^\vee$ denote the character group $\mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{C}^\times)$ for a finite abelian group A. For a nonarchimedean local field K, let

- $\mathcal{O}_K$ denote the ring of integers of K,
- $\mathfrak{p}_K$ denote the maximal ideal of $\mathcal{O}_K$,
- v_K denote the normalized valuation of K which sends a uniformizer of K to 1,
- $\mathrm{ch}\, K$ denote the characteristic of K,
- G_K denote the absolute Galois group of K,
- W_K denote the Weil group of K,
- I_K denote the inertia subgroup of W_K,
- P_K denote the wild inertia subgroup of W_K,

and we put $U_K^m = 1 + \mathfrak{p}_K^m$ for any positive integer m.

1. Representations of finite groups

First, we recall a fact on representations of Heisenberg groups. Let G be a finite group with center Z. We assume:

(i) The group G/Z is an elementary abelian p-group.

(ii) For any $g \in G \setminus Z$, the map $c_g : G \to Z,\ g' \mapsto [g, g']$ is surjective.

Remark 1.1. The map c_g in (ii) is a group homomorphism. Hence, Z is automatically an elementary abelian p-group.

Let $\psi \in Z^\vee$ be a nontrivial character.

Proposition 1.2. *There is a unique irreducible representation ρ_ψ of G such that $\rho_\psi|_Z$ contains ψ. Moreover, we have $(\dim \rho_\psi)^2 = [G : Z]$ and we can construct ρ_ψ as follows: Take an abelian subgroup G_1 of G such that $Z \subset G_1$ and $2\dim_{\mathbb{F}_p}(G_1/Z) = \dim_{\mathbb{F}_p}(G/Z)$. Extend ψ to a character ψ_1 of G_1. Then $\rho_\psi = \mathrm{Ind}_{G_1}^G \psi_1$.*

Proof. The claims other than the construction of ρ_ψ is Proposition 8.3.3 in [Bushnell and Fröhlich 1983]. Note that if an abelian subgroup G_1 of G satisfies the conditions in the claim, then G_1/Z is a maximal totally isotropic subspace of G/Z under the pairing

$$(G/Z) \times (G/Z) \to \mathbb{C}^\times, \quad (gZ, g'Z) \mapsto \psi([g, g']).$$

Hence the construction follows from the proof of [Bushnell and Fröhlich 1983, Proposition 8.3.3]. □

Next, we consider representations of a semidirect product of a Heisenberg group with a cyclic group. Let $A \subset \mathrm{Aut}(G)$ be a cyclic subgroup of order $p^e + 1$ where $e = \frac{1}{2}(\log_p[G : Z])$. We assume:

(3) The group A acts on Z trivially.

(4) For any nontrivial element $a \in A$, the action of a on G/Z fixes only the unit element.

We consider the semidirect product $A \ltimes G$ by the action of A on G.

Lemma 1.3. *There is a unique irreducible representation ρ'_ψ of $A \ltimes G$ such that $\rho'_\psi|_G \simeq \rho_\psi$ and $\mathrm{tr}\, \rho'_\psi(a) = -1$ for every nontrivial element $a \in A$.*

Proof. The claim is proved in the proof of Lemma 22.2 in [Bushnell and Henniart 2006] if Z is cyclic and ψ is a faithful character. In fact, the same proof works also in our case. □

Corollary 1.4. *There exists a unique representation ρ'_ψ of $A \ltimes G$ such that*

$$\rho'_\psi|_Z \simeq \psi^{\oplus p^e} \quad \text{and} \quad \mathrm{tr}\, \rho'_\psi(a) = -1$$

for every nontrivial element $a \in A$. Further, the representation $\rho'_\psi|_G$ is irreducible.

Proof. First we show the existence. We take the representation ρ'_ψ in Lemma 1.3. Then ρ'_ψ has a central character equal to ψ by Proposition 1.2. This shows the existence.

We show the uniqueness and the irreducibility of $\rho'_\psi|_G$. Assume that ρ'_ψ satisfies the condition in the claim. Take an irreducible subrepresentation ρ_ψ of $\rho'_\psi|_G$. Then ρ_ψ satisfies the condition of Proposition 1.2. Hence, $\dim \rho_\psi = p^e$. Then we have $\rho_\psi = \rho'_\psi|_G$ and $\rho'_\psi|_G$ is irreducible. Such ρ_ψ is unique by Lemma 1.3. □

2. Galois representations

2A. ***Swan conductor.*** Let K be a nonarchimedean local field with residue field k. Let p be the characteristic of k. Let f be the extension degree of k over $\mathbb{F}_p$. We put $q = p^f$.

Let
$$\mathrm{Art}_K : K^\times \xrightarrow{\sim} W_K^{\mathrm{ab}}$$
be the Artin reciprocity map, which sends a uniformizer to a lift of the geometric Frobenius element.

Let τ be a finite dimensional irreducible continuous representation of W_K over $\mathbb{C}$. Let $\Psi : K \to \mathbb{C}^\times$ be a nontrivial additive character. Let $\varepsilon(\tau, s, \Psi)$ denote the Deligne–Langlands local constant of τ with respect to Ψ. We simply write $\varepsilon(\tau, \Psi)$ for $\varepsilon\bigl(\tau, \frac{1}{2}, \Psi\bigr)$.

We define an unramified character $\omega_s : K^\times \to \mathbb{C}^\times$ by $\omega_s(\varpi) = q^{-s}$ for $s \in \mathbb{R}$, where ϖ is a uniformizer of K. We recall that

$$\varepsilon(\tau, s, \Psi) = \varepsilon(\tau \otimes \omega_s, 0, \Psi) \tag{2-1}$$

(see [Tate 1979, (3.6.4)]).

Let $\psi_0 \in \mathbb{F}_p^\vee$ by $\psi_0(1) = e^{2\pi\sqrt{-1}/p}$. We take an additive character $\psi_K : K \to \mathbb{C}^\times$ such that $\psi_K(x) = \psi_0(\mathrm{Tr}_{k/\mathbb{F}_p}(\bar{x}))$ for $x \in \mathcal{O}_K$. By [Bushnell and Henniart 2006, Proposition 29.4], there exists an integer $\mathrm{sw}(\tau)$ such that

$$\varepsilon(\tau, s, \psi_K) = q^{-\mathrm{sw}(\tau)s}\varepsilon(\tau, 0, \psi_K).$$

We put $\mathrm{Sw}(\tau) = \max\{\mathrm{sw}(\tau), 0\}$, which we call the Swan conductor of τ.

2B. ***Construction.*** We construct a group Q which acts on a curve C over an algebraic closure of k. By using this action of Q and Frobenius action, we construct a representation of a semidirect product $Q \rtimes \mathbb{Z}$ in étale cohomology of C. Then we use the representation of $Q \rtimes \mathbb{Z}$ to construct a representation of a Weil group.

We fix an algebraic closure K^{ac} of K. Let k^{ac} be the residue field of K^{ac}. Let n be a positive integer. We write $n = p^e n'$ with $(p, n') = 1$. Throughout this paper, we assume that $e \geq 1$. Let

$$Q = \{(a, b, c) \mid a \in \mu_{p^e+1}(k^{\mathrm{ac}}),\ b, c \in k^{\mathrm{ac}},\ b^{p^{2e}} + b = 0,\ c^p - c + b^{p^e+1} = 0\}$$

be the group whose multiplication is given by

$$(a_1, b_1, c_1) \cdot (a_2, b_2, c_2) = \left(a_1 a_2, b_1 + a_1 b_2, c_1 + c_2 + \sum_{i=0}^{e-1} (a_1 b_1^{p^e} b_2)^{p^i}\right).$$

Remark 2.1. The construction of the group Q has its origin in a study of the automorphism of a curve C defined below. We can check that the above multiplication gives a group structure on Q directly, but it's also possible to show this by checking that the inclusion from Q to the automorphism group of C defined below is compatible with the multiplications.

Note that $|Q| = p^{2e+1}(p^e+1)$. Let $Q \rtimes \mathbb{Z}$ be a semidirect product, where $m \in \mathbb{Z}$ acts on Q by $(a, b, c) \mapsto (a^{p^{-m}}, b^{p^{-m}}, c^{p^{-m}})$. We put

$$\mathrm{Fr}(m) = ((1,0,0), m) \in Q \rtimes \mathbb{Z} \quad \text{for } m \in \mathbb{Z}. \tag{2-2}$$

Let C be the smooth affine curve over k^{ac} defined by

$$x^p - x = y^{p^e+1} \quad \text{in } \mathbb{A}^2_{k^{\mathrm{ac}}}.$$

We define a right action of $Q \rtimes \mathbb{Z}$ on C by

$$(x, y)((a, b, c), 0) = \left(x + \sum_{i=0}^{e-1} (by)^{p^i} + c,\, a(y + b^{p^e})\right), \quad (x, y)\,\mathrm{Fr}(1) = (x^p, y^p).$$

We consider the morphisms

$$h : \mathbb{A}^1_{k^{\mathrm{ac}}} \to \mathbb{A}^1_{k^{\mathrm{ac}}}, \quad x \mapsto x^p - x, \qquad \pi : \mathbb{A}^1_{k^{\mathrm{ac}}} \to \mathbb{A}^1_{k^{\mathrm{ac}}}, \quad y \mapsto y^{p^e+1}.$$

Then we have the fiber product

$$\begin{tikzcd} C \arrow[r, "h'"] \arrow[d, "\pi'"'] \arrow[dr, phantom, "\square"] & \mathbb{A}^1_{k^{\mathrm{ac}}} \arrow[d, "\pi"] \\ \mathbb{A}^1_{k^{\mathrm{ac}}} \arrow[r, "h"'] & \mathbb{A}^1_{k^{\mathrm{ac}}} \end{tikzcd}$$

where π' and h' are the natural projections to the first and second coordinates respectively. Let $g = ((a, b, c), m) \in Q \rtimes \mathbb{Z}$. We consider the morphism

$$g_0 : \mathbb{A}^1_{k^{\mathrm{ac}}} \to \mathbb{A}^1_{k^{\mathrm{ac}}}, \quad y \mapsto (a(y + b^{p^e}))^{p^m}.$$

Let ℓ be a prime number different from p. Then we have a natural isomorphism

$$c_g : g_0^* h'_* \overline{\mathbb{Q}}_\ell \xrightarrow{\sim} h'_* g^* \overline{\mathbb{Q}}_\ell \xrightarrow{\sim} h'_* \overline{\mathbb{Q}}_\ell.$$

We take an isomorphism $\iota : \overline{\mathbb{Q}}_\ell \simeq \mathbb{C}$. We sometimes view a character over $\mathbb{C}$ as a character over $\overline{\mathbb{Q}}_\ell$ by ι. Let $\psi \in \mathbb{F}_p^\vee$. We write $\mathcal{L}_\psi$ for the Artin–Schreier $\overline{\mathbb{Q}}_\ell$-sheaf on $\mathbb{A}^1_{k^{\mathrm{ac}}}$ associated to ψ, which is equal to $\mathfrak{F}(\psi)$ in the notation of [Deligne 1977, Sommes trig. 1.8(i)]. Then we have a decomposition $h_* \overline{\mathbb{Q}}_\ell = \bigoplus_{\psi \in \mathbb{F}_p^\vee} \mathcal{L}_\psi$. This decomposition gives canonical isomorphisms

$$h'_* \overline{\mathbb{Q}}_\ell \cong \pi^* h_* \overline{\mathbb{Q}}_\ell \cong \bigoplus_{\psi \in \mathbb{F}_p^\vee} \pi^* \mathcal{L}_\psi. \tag{2-3}$$

The isomorphisms c_g and (2-3) induce $c_{g,\psi} : g_0^* \pi^* \mathcal{L}_\psi \to \pi^* \mathcal{L}_\psi$. We define a left action of $Q \rtimes \mathbb{Z}$ on $H_c^1(\mathbb{A}^1_{k^{\mathrm{ac}}}, \pi^* \mathcal{L}_\psi)$ by

$$H_c^1(\mathbb{A}^1_{k^{\mathrm{ac}}}, \pi^* \mathcal{L}_\psi) \xrightarrow{g_0^*} H_c^1(\mathbb{A}^1_{k^{\mathrm{ac}}}, g_0^* \pi^* \mathcal{L}_\psi) \xrightarrow{c_{g,\psi}} H_c^1(\mathbb{A}^1_{k^{\mathrm{ac}}}, \pi^* \mathcal{L}_\psi) \quad \text{for } g \in Q \rtimes \mathbb{Z}.$$

Let τ_ψ be the representation of $Q \rtimes \mathbb{Z}$ over $\mathbb{C}$ defined by $H_c^1(\mathbb{A}^1_{k^{\mathrm{ac}}}, \pi^* \mathcal{L}_\psi)$ and ι. For $\theta \in \mu_{p^e+1}(k^{\mathrm{ac}})^\vee$, let $\mathcal{K}_\theta$ be the smooth Kummer $\overline{\mathbb{Q}}_\ell$-sheaf on $\mathbb{G}_{\mathrm{m},k^{\mathrm{ac}}}$ associated to θ. We view $\mu_{p^e+1}(k^{\mathrm{ac}}) \times \mathbb{F}_p$ as a subgroup of Q by $(a, c) \mapsto (a, 0, c)$.

Lemma 2.2. *We have a natural isomorphism*

$$H_c^1(\mathbb{A}^1_{k^{\mathrm{ac}}}, \pi^* \mathcal{L}_\psi) \simeq \bigoplus_{\theta \in \mu_{p^e+1}(k^{\mathrm{ac}})^\vee \setminus \{1\}} H_c^1(\mathbb{G}_{\mathrm{m},k^{\mathrm{ac}}}, \mathcal{L}_\psi \otimes \mathcal{K}_\theta),$$

which is compatible with the actions of $\mu_{p^e+1}(k^{\mathrm{ac}}) \times \mathbb{F}_p$ *where*

$$(a, c) \in \mu_{p^e+1}(k^{\mathrm{ac}}) \times \mathbb{F}_p$$

acts on $H_c^1(\mathbb{G}_{\mathrm{m},k^{\mathrm{ac}}}, \mathcal{L}_\psi \otimes \mathcal{K}_\theta)$ *by* $\theta(a)\psi(c)$. *Further, we have*

$$\dim H_c^1(\mathbb{G}_{\mathrm{m},k^{\mathrm{ac}}}, \mathcal{L}_\psi \otimes \mathcal{K}_\theta) = 1$$

for any $\theta \in \mu_{p^e+1}(k^{\mathrm{ac}})^\vee \setminus \{1\}$.

Proof. By the projection formula, we have natural isomorphisms

$$\pi_* \pi^* \mathcal{L}_\psi \simeq \pi_*(\pi^* \mathcal{L}_\psi \otimes \overline{\mathbb{Q}}_\ell) \simeq \mathcal{L}_\psi \otimes \pi_* \overline{\mathbb{Q}}_\ell \quad \text{on } \mathbb{A}^1_{k^{\mathrm{ac}}}.$$

Further, we have

$$\pi_* \overline{\mathbb{Q}}_\ell \simeq \bigoplus_{\theta \in \mu_{p^e+1}(k^{\mathrm{ac}})^\vee} \mathcal{K}_\theta \quad \text{on } \mathbb{G}_{\mathrm{m},k^{\mathrm{ac}}},$$

since π is a finite étale $\mu_{p^e+1}(k^{\mathrm{ac}})$-covering over $\mathbb{G}_{\mathrm{m},k^{\mathrm{ac}}}$. Therefore, we have

$$\pi_* \pi^* \mathcal{L}_\psi \simeq \mathcal{L}_\psi \otimes \pi_* \overline{\mathbb{Q}}_\ell \simeq \bigoplus_{\theta \in \mu_{p^e+1}(k^{\mathrm{ac}})^\vee} \mathcal{L}_\psi \otimes \mathcal{K}_\theta \tag{2-4}$$

on $\mathbb{G}_{\mathrm{m},k^{\mathrm{ac}}}$. Let $\{0\}$ denote the origin of $\mathbb{A}^1_{k^{\mathrm{ac}}}$. Let $i : \{0\} \to \mathbb{A}^1_{k^{\mathrm{ac}}}$ and $j : \mathbb{G}_{\mathrm{m},k^{\mathrm{ac}}} \to \mathbb{A}^1_{k^{\mathrm{ac}}}$ be the natural immersions. From the exact sequence

$$0 \to j_! j^* \pi^* \mathcal{L}_\psi \to \pi^* \mathcal{L}_\psi \to i_* i^* \pi^* \mathcal{L}_\psi \to 0,$$

we have the exact sequence

$$0 \to H^0(\{0\}, i^* \pi^* \mathcal{L}_\psi) \to H_c^1(\mathbb{G}_{\mathrm{m},k^{\mathrm{ac}}}, \pi^* \mathcal{L}_\psi) \to H_c^1(\mathbb{A}^1_{k^{\mathrm{ac}}}, \pi^* \mathcal{L}_\psi) \to 0, \tag{2-5}$$

since

$$H_c^0(\mathbb{A}^1_{k^{\mathrm{ac}}}, \pi^* \mathcal{L}_\psi) = 0 \quad \text{and} \quad H^1(\{0\}, i^* \pi^* \mathcal{L}_\psi) = 0.$$

Note that $H^0(\{0\}, i^*\pi^*\mathcal{L}_\psi) \simeq \psi$. By (2-4), we have isomorphisms

$$H^1_{\mathrm{c}}(\mathbb{G}_{\mathrm{m},k^{\mathrm{ac}}}, \pi^*\mathcal{L}_\psi) \simeq H^1_{\mathrm{c}}(\mathbb{G}_{\mathrm{m},k^{\mathrm{ac}}}, \pi_*\pi^*\mathcal{L}_\psi) \simeq \bigoplus_{\theta \in \mu_{p^e+1}(k^{\mathrm{ac}})^\vee} H^1_{\mathrm{c}}(\mathbb{G}_{\mathrm{m},k^{\mathrm{ac}}}, \mathcal{L}_\psi \otimes \mathcal{K}_\theta). \tag{2-6}$$

We know that

$$\dim H^1_{\mathrm{c}}(\mathbb{G}_{\mathrm{m},k^{\mathrm{ac}}}, \mathcal{L}_\psi \otimes \mathcal{K}_\theta) = 1 \tag{2-7}$$

for any $\theta \in \mu_{p^e+1}(k^{\mathrm{ac}})^\vee$ by the proof of [Imai and Tsushima 2017, Lemma 7.1] (see [Imai and Tsushima 2023, (2.3)]). Since the composition of

$$H^0(\{0\}, i^*\pi^*\mathcal{L}_\psi) \to H^1_{\mathrm{c}}(\mathbb{G}_{\mathrm{m},k^{\mathrm{ac}}}, \pi^*\mathcal{L}_\psi)$$

and (2-6) is compatible with the actions of $\mu_{p^e+1}(k^{\mathrm{ac}}) \times \mathbb{F}_p$, it factors through an isomorphism $H^0(\{0\}, i^*\pi^*\mathcal{L}_\psi) \simeq H^1_{\mathrm{c}}(\mathbb{G}_{\mathrm{m},k^{\mathrm{ac}}}, \mathcal{L}_\psi)$ by (2-7). Then the claim follows from (2-5), (2-6) and (2-7). □

Let $\varrho : \mu_2(k) \hookrightarrow \mathbb{C}^\times$ be the nontrivial group homomorphism if $p \neq 2$. We define a character $\theta_0 \in \mu_{p^e+1}(k^{\mathrm{ac}})^\vee$ by

$$\theta_0(a) = \begin{cases} \varrho(a^{(p^e+1)/2}) & \text{if } p \neq 2, \\ 1 & \text{if } p = 2 \end{cases} \tag{2-8}$$

for $a \in \mu_{p^e+1}(k^{\mathrm{ac}})$. For an integer m and a positive odd integer m', let $\left(\frac{m}{m'}\right)$ denote the Jacobi symbol. For an odd prime p, we set

$$\epsilon(p) = \begin{cases} 1 & \text{if } p \equiv 1 \bmod 4, \\ \sqrt{-1} & \text{if } p \equiv 3 \bmod 4. \end{cases}$$

We have $\epsilon(p)^2 = \left(\frac{-1}{p}\right)$. We define a representation τ_n of $Q \rtimes \mathbb{Z}$ as the twist of τ_{ψ_0} by the character

$$\begin{aligned} &Q \rtimes \mathbb{Z} \to \mathbb{C}^\times, \\ &((a,b,c),m) \mapsto \begin{cases} \theta_0(a)^n \left(\left(-\epsilon(p)\left(\frac{-2n'}{p}\right)\right)^n p^{-\frac{1}{2}}\right)^m & \text{if } p \neq 2, \\ ((-1)^{\frac{n(n-2)}{8}} p^{-\frac{1}{2}})^m & \text{if } p = 2. \end{cases} \end{aligned} \tag{2-9}$$

The value of this character is related to a quadratic Gauss sum. A geometric origin of this character is given in (A-3). Let $(\zeta, \chi, c) \in \mu_{q-1}(K) \times (k^\times)^\vee \times \mathbb{C}^\times$. We take a uniformizer ϖ of K. We choose an element $\varphi'_\zeta \in K^{\mathrm{ac}}$ such that $\varphi'^{n'}_\zeta = \zeta\varpi$ and set $E_\zeta = K(\varphi'_\zeta)$. We choose elements $\alpha_\zeta, \beta_\zeta, \gamma_\zeta \in K^{\mathrm{ac}}$ such that

$$\alpha_\zeta^{p^e+1} = -\varphi'_\zeta, \quad \beta_\zeta^{p^{2e}} + \beta_\zeta = -\alpha_\zeta^{-1}, \quad \gamma_\zeta^p - \gamma_\zeta = \beta_\zeta^{p^e+1}. \tag{2-10}$$

For $\sigma \in W_{E_\zeta}$, we set

$$
\begin{aligned}
&a_\sigma = \sigma(\alpha_\zeta)/(\alpha_\zeta), \quad b_\sigma = a_\sigma \sigma(\beta_\zeta) - \beta_\zeta, \\
&c_\sigma = \sigma(\gamma_\zeta) - \gamma_\zeta + \sum_{i=0}^{e-1} (b_\sigma^{p^e}(\beta_\zeta + b_\sigma))^{p^i}.
\end{aligned} \tag{2-11}
$$

Then we have $a_\sigma, b_\sigma, c_\sigma \in \mathcal{O}_{K^{\mathrm{ac}}}$. For $\sigma \in W_{E_\zeta}$, we put $n_\sigma = v_{E_\zeta}(\mathrm{Art}_{E_\zeta}^{-1}(\sigma))$. We have the homomorphism

$$
\Theta_\zeta : W_{E_\zeta} \to Q \rtimes \mathbb{Z}, \quad \sigma \mapsto ((\bar{a}_\sigma, \bar{b}_\sigma, \bar{c}_\sigma), f n_\sigma). \tag{2-12}
$$

Lemma 2.3. *The image of the homomorphism Θ_ζ is $Q \rtimes (f\mathbb{Z})$.*

Proof. It suffices to show that the image of $I_{E_\zeta} \subset W_{E_\zeta}$ under Θ_ζ is equal to $Q \subset Q \rtimes \mathbb{Z}$, since the homomorphism $W_{E_\zeta} \to f\mathbb{Z}$, $\sigma \mapsto f n_\sigma$ is surjective. We put $N_\zeta = E_\zeta(\alpha_\zeta, \beta_\zeta, \gamma_\zeta)$. Then the kernel of Θ_ζ is equal to I_{N_ζ} by the definition. Hence we have an injection $I_{E_\zeta}/I_{N_\zeta} \hookrightarrow Q$. This injection is actually a bijection, since N_ζ is a totally ramified extension over E_ζ of degree $p^{2e+1}(p^e+1)$, which equals to $|Q|$. Therefore, we obtain the claim. $\square$

We write $\tau_{n,\zeta}$ for the representation of W_{E_ζ} given by Θ_ζ and τ_n. Recall that c is an element of $\mathbb{C}^\times$. Let $\phi_c : W_{E_\zeta} \to \mathbb{C}^\times$ be the character defined by $\phi_c(\sigma) = c^{n_\sigma}$. We have the isomorphism $\varphi_\zeta'^{\mathbb{Z}} \times \mathcal{O}_{E_\zeta}^\times \simeq E_\zeta{}^\times$ given by the multiplication. Let $\mathrm{Frob}_p : k^\times \to k^\times$ be the inverse of the p-th power map. We consider the following composition:

$$
\lambda_\zeta : W_{E_\zeta}^{\mathrm{ab}} \simeq E_\zeta{}^\times \simeq \varphi_\zeta'^{\mathbb{Z}} \times \mathcal{O}_{E_\zeta}^\times \xrightarrow{\mathrm{pr}_2} \mathcal{O}_{E_\zeta}^\times \xrightarrow{\text{can.}} k^\times \xrightarrow{\mathrm{Frob}_p^e} k^\times.
$$

We put

$$
\tau_{n,\zeta,\chi,c} = \tau_{n,\zeta} \otimes (\chi \circ \lambda_\zeta) \otimes \phi_c \quad \text{and} \quad \tau_{\zeta,\chi,c} = \mathrm{Ind}_{E_\zeta/K}\, \tau_{n,\zeta,\chi,c}. \tag{2-13}
$$

We will see that $\tau_{\zeta,\chi,c}$ is an irreducible representation of Swan conductor 1 in Proposition 10.8. This Galois representation $\tau_{\zeta,\chi,c}$ is our main object in this paper. We will study several invariants associated to this, for example, its determinant and epsilon factor.

2C. *Characterization.* We put

$$
Q_0 = \{(1, b, c) \in Q\}, \quad F = \{(1, 0, c) \in Q \mid c \in \mathbb{F}_p\}.
$$

We identify $\mathbb{F}_p$ with F by $c \mapsto (1, 0, c)$.

Lemma 2.4. *For any $g = (1, b, c) \in Q_0$ with $b \neq 0$, the map $Q_0 \to F$, $g' \mapsto [g, g']$ is surjective.*

Proof. For $(1, b_1, c_1), (1, b_2, c_2) \in Q_0$, we have

$$[(1, b_1, c_1), (1, b_2, c_2)] = \left(1, 0, \sum_{i=0}^{e-1} (b_1^{p^e} b_2 - b_1 b_2^{p^e})^{p^i}\right).$$

If $b_1 \neq 0$, then

$$\{b \in k^{\mathrm{ac}} \mid b^{p^{2e}} + b = 0\} \to \mathbb{F}_{p^e}, \quad b_2 \to b_1^{p^e} b_2 - b_1 b_2^{p^e}$$

is surjective. The claim follows from the surjectivity of $\mathrm{Tr}_{\mathbb{F}_{p^e}/\mathbb{F}_p}$. $\square$

By this lemma, we can apply the results from Section 1 to our situation with $G = Q_0$, $Z = F$ and $A = \mu_{p^e+1}(k^{\mathrm{ac}})$, where the action of $\mu_{p^e+1}(k^{\mathrm{ac}})$ on Q_0 is given by the embedding

$$\mu_{p^e+1}(k^{\mathrm{ac}}) \to Q, \quad a \mapsto (a, 0, 0)$$

and the conjugation. Let τ^0 denote the unique representation of Q characterized by

$$\tau^0|_F \simeq \psi_0^{\oplus p^e}, \quad \mathrm{Tr}\, \tau^0((a, 0, 0)) = -1 \tag{2-14}$$

for $a \in \mu_{p^e+1}(k^{\mathrm{ac}}) \setminus \{1\}$ (see Corollary 1.4).

We have a decomposition

$$\tau^0 = \bigoplus_{\theta \in \mu_{p^e+1}(k^{\mathrm{ac}})^\vee \setminus \{1\}} L_\theta \tag{2-15}$$

such that $a \in \mu_{p^e+1}(k^{\mathrm{ac}})$ acts on L_θ by $\theta(a)$, since the both sides of (2-15) have the same character as representations of $\mu_{p^e+1}(k^{\mathrm{ac}})$. For a positive integer m dividing p^e+1, we consider $\mu_m(k^{\mathrm{ac}})^\vee$ as a subset of $\mu_{p^e+1}(k^{\mathrm{ac}})^\vee$ by the dual of the surjection

$$\mu_{p^e+1}(k^{\mathrm{ac}}) \to \mu_m(k^{\mathrm{ac}}), \quad x \to x^{(p^e+1)/m}.$$

We simply write Q for the subgroup $Q \times \{0\} \subset Q \rtimes \mathbb{Z}$.

Lemma 2.5. *We have $\tau_{\psi_0}|_Q \simeq \tau^0$.*

Proof. The representation $\tau_{\psi_0}|_Q$ satisfies the characterization (2-14) by Lemma 2.2. Hence $\tau_{\psi_0}|_Q$ is isomorphic to τ^0. $\square$

Corollary 2.6. *The representation $\tau_{\psi_0}|_{Q_0}$ is irreducible.*

Proof. This follows from Corollary 1.4, equation (2-14) and Lemma 2.5. $\square$

For any odd prime p, we have

$$\sum_{x \in \mathbb{F}_p^\times} \psi_0(x^2) = \sum_{x \in \mathbb{F}_p^\times} \left(\frac{x}{p}\right) \psi_0(x) = \epsilon(p)\sqrt{p} \tag{2-16}$$

by Gauss.

Lemma 2.7. *We have*

$$\operatorname{Tr}\tau_{\psi_0}(\mathrm{Fr}(1)) = \begin{cases} -\epsilon(p)\sqrt{p} & \text{if } p \neq 2, \\ 0 & \text{if } p = 2. \end{cases}$$

Proof. By the Lefschetz trace formula, we have

$$\sum_{x \in \mathbb{A}^1(\mathbb{F}_p)} \operatorname{Tr}(\mathrm{Fr}_p, (\pi^*\mathcal{L}_{\psi_0})_x) = \sum_{i=0}^{2} (-1)^i \operatorname{Tr}(\mathrm{Fr}_p, H^i_{\mathrm{c}}(\mathbb{A}^1_{k^{\mathrm{ac}}}, \pi^*\mathcal{L}_{\psi})),$$

where Fr_p is the geometric p-th power Frobenius morphism. Since $H^i_{\mathrm{c}}(\mathbb{A}^1_{k^{\mathrm{ac}}}, \pi^*\mathcal{L}_{\psi})$ vanishes for $i = 0, 2$, we have

$$\begin{aligned} \operatorname{Tr}\tau_{\psi_0}(\mathrm{Fr}(1)) &= -\sum_{x \in \mathbb{A}^1(\mathbb{F}_p)} \operatorname{Tr}(\mathrm{Fr}_p, (\pi^*\mathcal{L}_{\psi_0})_x) \\ &= -\sum_{x \in \mathbb{F}_p} \psi_0(x^{p^e+1}) = -\sum_{x \in \mathbb{F}_p} \psi_0(x^2) = \begin{cases} -\epsilon(p)\sqrt{p} & \text{if } p \neq 2, \\ 0 & \text{if } p = 2, \end{cases} \end{aligned}$$

where we use (2-16) in the last equality. □

We assume $p = 2$ in this paragraph. We take $b_0 \in \mathbb{F}_{2^{2e}}$ such that $\operatorname{Tr}_{\mathbb{F}_{2^{2e}}/\mathbb{F}_2}(b_0) = 1$. Further, we put

$$c_0 = b_0^{2^e} + \sum_{0 \le i < j \le e-1} b_0^{2^{e+i}+2^j}. \tag{2-17}$$

Then we have

$$\begin{aligned} c_0^2 - c_0 &= b_0^{2^{e+1}} + b_0^{2^e} + \sum_{0 \le i < j \le e-1} b_0^{2^{e+i+1}+2^{j+1}} + \sum_{0 \le i < j \le e-1} b_0^{2^{e+i}+2^j} \\ &= b_0^{2^{e+1}} + b_0^{2^e} + \sum_{i=0}^{e-2} b_0^{2^{e+i+1}+2^e} + \sum_{j=1}^{e-1} b_0^{2^e+2^j} \\ &= b_0^{2^{e+1}} + b_0^{2^e} + b_0^{2^e}(1 + b_0 + b_0^{2^e}) = b_0^{2^e+1}, \end{aligned} \tag{2-18}$$

where we use $\operatorname{Tr}_{\mathbb{F}_{2^{2e}}/\mathbb{F}_2}(b_0) = 1$ at the third equality. We put

$$\boldsymbol{g} = ((1, b_0, c_0), -1) \in Q \rtimes \mathbb{Z}.$$

Lemma 2.8. *We assume that $p = 2$. Then we have $\operatorname{Tr}\tau_{\psi_0}(\boldsymbol{g}^{-1}) = -2$.*

Proof. We note that

$$\boldsymbol{g}^{-1} = \mathrm{Fr}(1)\left(\left(1, b_0, c_0 + \sum_{i=0}^{e-1} (b_0^{2^e+1})^{2^i}\right), 0\right). \tag{2-19}$$

For $y \in k^{\mathrm{ac}}$ satisfying $y^2 + b_0^{2^e} = y$, we take $x_y \in k^{\mathrm{ac}}$ such that $x_y^2 - x_y = y^{2^e+1}$. We take $y_0 \in k^{\mathrm{ac}}$ such that $y_0^2 + b_0^{2^e} = y_0$. Then, by the Lefschetz trace formula and (2-19), we have

$$\begin{aligned}
\operatorname{Tr} \tau_{\psi_0}(\boldsymbol{g}^{-1}) &= -\sum_{y^2+b_0^{2^e}=y} \operatorname{Tr}(\boldsymbol{g}^{-1}, (\pi^*\mathcal{L}_{\psi_0})_y) \\
&= -\sum_{y^2+b_0^{2^e}=y} \psi_0\biggl(x_y^2 - x_y + \sum_{i=0}^{e-1}(b_0 y^2)^{2^i} + c_0 + \sum_{i=0}^{e-1}(b_0^{2^e+1})^{2^i}\biggr) \\
&= -\sum_{z\in\mathbb{F}_2} \psi_0\biggl((y_0+z)^{2^e+1} + \sum_{i=0}^{e-1}(b_0(y_0+z))^{2^i} + c_0\biggr) = -2,
\end{aligned}$$

where we change a variable by $y = y_0 + z$ at the second equality, and use

$$y_0^{2^e+1} + \sum_{i=0}^{e-1}(b_0 y_0)^{2^i} = y_0\biggl(y_0 + \sum_{i=0}^{e-1} b_0^{2^{e+i}}\biggr) + \sum_{i=0}^{e-1} b_0^{2^i}\biggl(y_0 + \sum_{j=0}^{i-1} b_0^{2^{e+j}}\biggr) = c_0,$$

$$y_0^{2^e} + y_0 + \sum_{i=0}^{e-1} b_0^{2^i} = \sum_{i=0}^{e-1}(y_0^2 + y_0)^{2^i} + \sum_{i=0}^{e-1} b_0^{2^i} = \operatorname{Tr}_{\mathbb{F}_{2^{2e}}/\mathbb{F}_2}(b_0) = 1$$

at the last equality. □

Proposition 2.9. *The representation τ_{ψ_0} is characterized by $\tau_{\psi_0}|_Q \simeq \tau^0$ and*

$$\begin{cases} \operatorname{Tr} \tau_{\psi_0}(\mathrm{Fr}(1)) = -\epsilon(p)\sqrt{p} & \text{if } p \neq 2, \\ \operatorname{Tr} \tau_{\psi_0}(\boldsymbol{g}^{-1}) = -2 & \text{if } p = 2. \end{cases}$$

In particular, τ_{ψ_0} does not depend on the choice of ℓ and ι.

Proof. This follows from Lemmas 2.5, 2.7 and 2.8. □

3. Representations of general linear algebraic groups

3A. ***Simple supercuspidal representation.*** Let π be an irreducible supercuspidal representation of $\mathrm{GL}_n(K)$ over $\mathbb{C}$. Let $\varepsilon(\pi, s, \Psi)$ denote the Godement–Jacquet local constant of π with respect to the nontrivial character $\Psi : K \to \mathbb{C}^\times$. We simply write $\varepsilon(\pi, \Psi)$ for $\varepsilon\bigl(\pi, \frac{1}{2}, \Psi\bigr)$. By [Godement and Jacquet 1972, Theorem 3.3(4)], there exists an integer $\mathrm{sw}(\pi)$ such that

$$\varepsilon(\pi, s, \psi_K) = q^{-\mathrm{sw}(\pi)s}\varepsilon(\pi, 0, \psi_K).$$

We put $\mathrm{Sw}(\pi) = \max\{\mathrm{sw}(\pi), 0\}$, which we call the Swan conductor of π.

Definition 3.1. An irreducible supercuspidal representation π of $\mathrm{GL}_n(K)$ over $\mathbb{C}$ is called simple supercuspidal if $\mathrm{Sw}(\pi) = 1$.

This definition is equivalent to [Imai and Tsushima 2018, Definition 1.1] by [Imai and Tsushima 2018, Proposition 1.3].

3B. ***Construction.*** In the following, we construct a smooth representation $\pi_{\zeta,\chi,c}$ of $\mathrm{GL}_n(K)$ for each triple $(\zeta, \chi, c) \in \mu_{q-1}(K) \times (k^\times)^\vee \times \mathbb{C}^\times$.

Let $B \subset M_n(k)$ be the subring consisting of upper triangular matrices. Let $\mathfrak{I} \subset M_n(\mathcal{O}_K)$ be the inverse image of B under the reduction map $M_n(\mathcal{O}_K) \to M_n(k)$. Then $\mathfrak{I}$ is a hereditary $\mathcal{O}_K$-order (see [Bushnell and Kutzko 1993, (1.1)]). Let $\mathfrak{P}$ denote the Jacobson radical of the order $\mathfrak{I}$. We put $U_{\mathfrak{I}}^1 = 1 + \mathfrak{P} \subset \mathrm{GL}_n(\mathcal{O}_K)$. We set

$$\varphi_\zeta = \begin{pmatrix} \mathbf{0} & I_{n-1} \\ \zeta\varpi & \mathbf{0} \end{pmatrix} \in M_n(K) \quad \text{and} \quad L_\zeta = K(\varphi_\zeta).$$

Then, L_ζ is a totally ramified extension of K of degree n.

We put $\varphi_{\zeta,n} = n'\varphi_\zeta$ and

$$\epsilon_0 = \begin{cases} \frac{1}{2}(n'+1) & \text{if } p^e = 2, \\ 0 & \text{if } p^e \neq 2. \end{cases}$$

We define a character $\Lambda_{\zeta,\chi,c} : L_\zeta^\times U_{\mathfrak{I}}^1 \to \mathbb{C}^\times$ by

$$\Lambda_{\zeta,\chi,c}(\varphi_\zeta) = (-1)^{n-1+\epsilon_0 f} c, \qquad \Lambda_{\zeta,\chi,c}(x) = \chi(\bar{x}) \quad \text{for } x \in \mathcal{O}_K^\times,$$
$$\Lambda_{\zeta,\chi,c}(x) = (\psi_K \circ \mathrm{tr})(\varphi_{\zeta,n}^{-1}(x-1)) \quad \text{for } x \in U_{\mathfrak{I}}^1,$$

where tr means the trace as an element of $M_n(K)$. We put

$$\pi_{\zeta,\chi,c} = \text{c-Ind}_{L_\zeta^\times U_{\mathfrak{I}}^1}^{\mathrm{GL}_n(K)} \Lambda_{\zeta,\chi,c}.$$

Then, $\pi_{\zeta,\chi,c}$ is a simple supercuspidal representation of $\mathrm{GL}_n(K)$, and every simple supercuspidal representation is isomorphic to $\pi_{\zeta,\chi,c}$ for a uniquely determined $(\zeta, \chi, c) \in \mu_{q-1}(K) \times (k^\times)^\vee \times \mathbb{C}^\times$ by [Imai and Tsushima 2018, Proposition 1.3]. The representation $\pi_{\zeta,\chi,c}$ contains the m-simple stratum $[\mathfrak{I}, 1, 0, \varphi_{\zeta,n}^{-1}]$ in the sense of [Bushnell and Henniart 2014, Section 2.1].

Proposition 3.2. $\varepsilon(\pi_{\zeta,\chi,c}, \psi_K) = (-1)^{n-1+\epsilon_0 f} \chi(n')\, c.$

Proof. This follows from [Bushnell and Henniart 1999, Section 6.1, Lemma 2 and Section 6.3, Proposition 1]. □

4. Local Langlands correspondence

Our main theorem is the following.

Theorem 4.1. *The representations $\pi_{\zeta,\chi,c}$ and $\tau_{\zeta,\chi,c}$ correspond via the local Langlands correspondence.*

To prove this theorem, we recall a characterization of the local Langlands correspondence for epipelagic representations due to Bushnell–Henniart. Recall that $\Psi : K \to \mathbb{C}^\times$ is a nontrivial character. The following lemma is a special case of [Deligne and Henniart 1981, Proposition 4.13].

Lemma 4.2 [Bushnell and Henniart 2014, Lemma 2.3]. *Let τ be an irreducible smooth representation of W_K such that* $\mathrm{sw}(\tau) \geq 1$. *Then, there exists $\gamma_{\tau,\Psi} \in K^\times$ such that*

$$\varepsilon(\chi \otimes \tau, s, \Psi) = \chi(\gamma_{\tau,\Psi})^{-1} \varepsilon(\tau, s, \Psi)$$

for any tamely ramified character χ of W_K. This property determines the coset $\gamma_{\tau,\Psi} U_K^1$ uniquely.

Definition 4.3. Let τ be an irreducible smooth representation of W_K such that $\mathrm{sw}(\tau) \geq 1$. We take $\gamma_{\tau,\Psi}$ as in Lemma 4.2. We put

$$\mathrm{rsw}(\tau, \Psi) = \gamma_{\tau,\Psi}^{-1} \in K^\times / U_K^1,$$

which we call the refined Swan conductor of τ with respect to Ψ.

Remark 4.4. By (2-1), we have $v_K(\mathrm{rsw}(\tau, \psi_K)) = \mathrm{Sw}(\tau)$ in Definition 4.3.

Lemma 4.5. *Let π be an irreducible supercuspidal representation of* $\mathrm{GL}_n(K)$ *such that* $\mathrm{sw}(\pi) \geq 1$.

(1) *There exists $\gamma_{\pi,\Psi} \in K^\times$ such that*

$$\varepsilon(\chi \otimes \pi, s, \Psi) = \chi(\gamma_{\pi,\Psi})^{-1} \varepsilon(\pi, s, \Psi)$$

for any tamely ramified character χ of $K^\times$. This property determines the coset $\gamma_{\pi,\Psi} U_K^1$ uniquely.

(2) *Let* $[\mathfrak{A}, m, 0, \alpha]$ *be a simple stratum contained in π. Then we have* $\gamma_{\pi,\Psi} \equiv \det \alpha \mod U_K^1$.

Proof. The first statement is [Bushnell and Henniart 1999, Theorem 1.4(i)]. The second statement follows from [Bushnell and Henniart 1999, Remark 1.4]. □

Definition 4.6. Let π be an irreducible supercuspidal representation of $\mathrm{GL}_n(K)$ such that $\mathrm{sw}(\pi) \geq 1$. We take $\gamma_{\pi,\Psi}$ as in Lemma 4.5. Then we put

$$\mathrm{rsw}(\pi, \Psi) = \gamma_{\pi,\Psi}^{-1} \in K^\times / U_K^1,$$

which we call the refined Swan conductor of π with respect to Ψ.

Remark 4.7. We have $v_K(\mathrm{rsw}(\pi, \psi_K)) = \mathrm{Sw}(\pi)$ in Definition 4.6.

For an irreducible supercuspidal representation π of $\mathrm{GL}_n(K)$, let ω_π denote the central character of π.

Proposition 4.8 [Bushnell and Henniart 2014, Proposition 2.3]. *Let π be a simple supercuspidal representation of* $\mathrm{GL}_n(K)$. *The Langlands parameter for π is characterized as the n-dimensional irreducible smooth representation τ of W_K satisfying*

$$\det \tau = \omega_\pi, \quad \mathrm{rsw}(\tau, \psi_K) = \mathrm{rsw}(\pi, \psi_K), \quad \varepsilon(\tau, \psi_K) = \varepsilon(\pi, \psi_K).$$

We will show that $\tau_{\zeta,\chi,c}$ and $\pi_{\zeta,\chi,c}$ satisfy the conditions of Proposition 4.8 in Propositions 8.6, 10.5, Lemma 10.7 and Proposition 11.6.

5. General facts on epsilon factors

In this section, we recall some general facts on epsilon factors.

For a finite separable extension L over K, we put $\Psi_L = \Psi \circ \mathrm{Tr}_{L/K}$ and let

$$\lambda(L/K, \Psi) = \frac{\varepsilon(\mathrm{Ind}_{L/K} 1, s, \Psi)}{\varepsilon(1, s, \Psi_L)}$$

denote the Langlands constant which is independent of s, where 1 is the trivial representation of W_L (see [Bushnell and Henniart 2006, Section 30.4]).

Proposition 5.1. *Let τ be a finite dimensional smooth representation of W_K such that $\tau|_{P_K}$ is irreducible and nontrivial. Let L be a tamely ramified finite extension of K. Then we have*

$$\varepsilon(\tau|_{W_L}, \Psi_L) = \lambda(L/K, \Psi)^{-\dim\tau}\, \delta_{L/K}(\mathrm{rsw}(\tau, \Psi))\, \varepsilon(\tau, \Psi)^{[L:K]}.$$

Proof. This is proved by the same arguments as in [Bushnell and Henniart 2006, Proposition 48.3]. □

Proposition 5.2. *Let τ be a finite dimensional smooth representation of W_K such that $\tau|_{P_K}$ does not contain the trivial character.*

(1) *If ϕ is a tamely ramified character of W_K, then* $\mathrm{rsw}(\tau \otimes \phi, \Psi) = \mathrm{rsw}(\tau, \Psi)$.

(2) *Let L be a tamely ramified finite extension of K. Then we have*

$$\mathrm{rsw}(\tau|_{W_L}, \Psi_L) = \mathrm{rsw}(\tau, \Psi) \mod U_L^1.$$

Proof. This is [Bushnell and Henniart 2006, Theorem 48.1(2), (3)]. □

For a nontrivial character ξ of $K^\times$, the level of ξ means the least integer $m \geq 0$ such that ξ is trivial on U_K^{m+1}.

Proposition 5.3. *Let ξ be a character of $K^\times$ of level $m \geq 1$. Assume that $\gamma \in K^\times$ satisfies*

$$\xi(1+x) = \Psi(\gamma x) \quad \text{for } x \in \mathfrak{p}_K^{[m/2]+1}.$$

(1) *We have* $\mathrm{rsw}(\xi, \Psi) = \gamma^{-1}$.

(2) *We have*

$$\varepsilon(\xi, \Psi) = q^{[(m+1)/2]-(m+1)/2} \sum_{y \in U_K^{[(m+1)/2]}/U_K^{[m/2]+1}} \xi(\gamma y)^{-1} \Psi(\gamma y).$$

Proof. Claim (1) follows from [Bushnell and Henniart 2006, Stability theorem 23.8]. Claim (2) follows from [Bushnell and Henniart 2006, Section 23.5, Lemma 1, (23.6.2) and Proposition 23.6]. □

For a finite Galois extension L of K, let $\psi_{L/K}$ denote the Herbrand function of L/K and $\mathrm{Gal}(L/K)_i$ denote the lower numbering i-th ramification subgroup of $\mathrm{Gal}(L/K)$ for $i \geq 0$ (see [Serre 1968, Chapter IV]). We use the following lemmas to calculate the refined Swan conductor of a character of a Weil group.

Lemma 5.4. *Let m be a positive integer dividing f. Let h be a positive integer that is prime to p and less than $p^m v_K(p)/(p^m-1)$. Let L be a Galois extension of K defined by $x^{p^m} - x = 1/\varpi^h$. Then we have*

$$\mathrm{Gal}(L/K)_i = \begin{cases} \mathrm{Gal}(L/K) & \text{if } i \leq h, \\ \{1\} & \text{if } i > h \end{cases}$$

and

$$\psi_{L/K}(v) = \begin{cases} v & \text{if } v \leq h, \\ p^m(v-h)+h & \text{if } v > h. \end{cases}$$

Proof. Take an integer l such that $lh \equiv 1 \mod p^m$. Then we have

$$v_L\left(\frac{1}{x^l \varpi^{(lh-1)/p^m}}\right) = 1.$$

Hence, for $\sigma \in \mathrm{Gal}(L/K)$ and $i \geq 0$, we have $\sigma \in \mathrm{Gal}(L/K)_i$ if and only if

$$\text{(5-1)} \quad i+1 \leq v_L\left(\sigma\left(\frac{1}{x^l \varpi^{(lh-1)/p^m}}\right) - \frac{1}{x^l \varpi^{(lh-1)/p^m}}\right) = v_L(\sigma(x)^l - x^l) + hl + 1.$$

The right-hand side of (5-1) is $h+1$ if $\sigma \neq 1$. Hence the first claim follows. The second claim follows from the first claim. □

Lemma 5.5. *Let L be a totally ramified finite abelian extension of K. Let $m \geq 1$.*

(1) *We have*

$$\mathrm{Nr}_{L/K}(U_L^{\psi_{L/K}(m)}) \subset U_K^m, \quad \mathrm{Nr}_{L/K}(U_L^{\psi_{L/K}(m)+1}) \subset U_K^{m+1},$$
$$\mathrm{Art}_K(U_K^m) \subset \mathrm{Gal}(L/K)_{\psi_{L/K}(m)}.$$

(2) *We take $\alpha \in K$ and $\beta \in L$ such that $v_K(\alpha) = m$ and $v_L(\beta) = \psi_{L/K}(m)$. We put $P(z) = z^p - z$ for $z \in k$. Assume that*

$$\begin{tikzcd} U_L^{\psi_{L/K}(m)} \arrow[r, "\mathrm{Nr}_{L/K}"] \arrow[d, "p_{L,\beta}"] & U_K^m \arrow[d, "p_{K,\alpha}"] \\ k \arrow[r, "P"] & k \end{tikzcd}$$

is commutative, where

$$p_{K,\alpha} : U_K^m \to k, \qquad 1 + \alpha x \mapsto \bar{x},$$
$$p_{L,\beta} : U_L^{\psi_{L/K}(m)} \to k, \quad 1 + \beta x \mapsto \bar{x}.$$

Let ϖ_L be a uniformizer of L. Then we have

$$p_{L,\beta}\left(\frac{\mathrm{Art}_K(1+\alpha x)(\varpi_L)}{\varpi_L}\right)=\mathrm{Tr}_{k/\mathbb{F}_p}(\bar{x})$$

for $x \in \mathcal{O}_K$.

Proof. The first claim follows from [Serre 1968, Chapter V, Section 3, Proposition 4 and Chapter XV, Section 2, Corollaire 3 of Théorème 1]. We note that our normalization of the Artin reciprocity map is inverse to that in [Serre 1968, Chapter XIII, Section 4]. Let $x \in \mathcal{O}_K$. By [Serre 1968, Chapter XV, Section 3, Proposition 4] and the construction of the isomorphism of [Serre 1968, Chapter XV, Section 2, Proposition 3], we have

$$p_{L,\beta}\left(\frac{\mathrm{Art}_K(1+\alpha x)(\varpi_L)}{\varpi_L}\right)=z_x^q-z_x,$$

where we take $z_x \in k^{\mathrm{ac}}$ such that $z_x^p - z_x = \bar{x}$. Then we have the second claim, since

$$z_x^q - z_x = \mathrm{Tr}_{k/\mathbb{F}_p}(z_x^p - z_x) = \mathrm{Tr}_{k/\mathbb{F}_p}(\bar{x})$$

for such z_x. □

6. Stiefel–Whitney class and discriminant

6A. ***Stiefel–Whitney class.*** Let $R(W_K,\mathbb{R})$ be the Grothendieck group of finite-dimensional representations of W_K over $\mathbb{R}$ with finite images. For $V \in R(W_K,\mathbb{R})$, we put $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$ and define $\varepsilon(V_{\mathbb{C}},\Psi)$ by the additivity using the epsilon factors in Section 2A. For $V \in R(W_K,\mathbb{R})$, we define the i-th Stiefel–Whitney class $w_i(V) \in H^i(G_K,\mathbb{Z}/2\mathbb{Z})$ for $i \geq 0$ as in [Deligne 1976, (1.3)]. Let

$$\mathrm{cl}: H^2(G_K,\mathbb{Z}/2\mathbb{Z}) \to H^2(G_K, K^{\mathrm{ac},\times}) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z},$$

where the first map is induced by $\mathbb{Z}/2\mathbb{Z} \to K^{\mathrm{ac},\times}$, $m \mapsto (-1)^m$ and the second isomorphism is the invariant map.

Theorem 6.1 [Deligne 1976, Théorème 1.5]. *Assume that $V \in R(W_K,\mathbb{R})$ has dimension* 0 *and determinant* 1. *Then we have*

$$\varepsilon(V_{\mathbb{C}},\Psi)=\exp\bigl(2\pi\sqrt{-1}\,\mathrm{cl}(w_2(V))\bigr).$$

In particular, we have $\varepsilon(V_{\mathbb{C}},\Psi)=1$ if $\mathrm{ch}\,K=2$.

6B. ***Discriminant.*** Let L be a finite separable extension of K. We put

$$\delta_{L/K}=\det(\mathrm{Ind}_{L/K} 1).$$

6B1. *Multiplicative discriminant.* Assume $\operatorname{ch} K \neq 2$. We define $d_{L/K} \in K^\times/(K^\times)^2$ as the discriminant of the quadratic form $\operatorname{Tr}_{L/K}(x^2)$ on L. For $a \in K^\times/(K^\times)^2$, let $\{a\} \in H^1(G_K, \mathbb{Z}/2\mathbb{Z})$ and $\kappa_a \in \operatorname{Hom}(W_K, \{\pm 1\})$ be the elements corresponding to a under the natural isomorphisms

$$K^\times/(K^\times)^2 \simeq H^1(G_K, \mathbb{Z}/2\mathbb{Z}) \simeq \operatorname{Hom}(W_K, \{\pm 1\}).$$

We have

$$\delta_{L/K} = \kappa_{d_{L/K}} \tag{6-1}$$

by [Bourbaki 1981, Chapter V, Section 10, Example 2(6)] (see [Serre 1984, Section 1.4]). For $a, b \in K^\times/(K^\times)^2$, we put

$$\{a, b\} = \{a\} \cup \{b\} \in H^2(G_K, \mathbb{Z}/2\mathbb{Z}).$$

Proposition 6.2 [Abbes and Saito 2010, Proposition 6.5]. *Let m be the extension degree of L over K. We take a generator a of L over K. Let $f(x) \in K[x]$ be the minimal polynomial of a. We put $D = f'(a) \in L$. Then we have*

$$d_{L/K} = (-1)^{\binom{m}{2}} \operatorname{Nr}_{L/K}(D) \in K^\times/(K^\times)^2,$$
$$w_2(\operatorname{Ind}_{L/K} \kappa_D) = \binom{m}{4}\{-1, -1\} + \{d_{L/K}, 2\} \in H^2(G_K, \mathbb{Z}/2\mathbb{Z}).$$

6B2. *Additive discriminant.* We put $P_m(x) = x^m - x$ for any positive integer m. We assume that $\operatorname{ch} K = 2$.

Definition 6.3 [Bergé and Martinet 1985, Définition 2.7]. Let m be the extension degree of L over K. Let $f(x) \in K[x]$ be the minimal polynomial of a generator of L over K. We have a decomposition $f(x) = \prod_{1 \leq i \leq m}(x - a_i)$ over the Galois closure of L over K. We put

$$d^+_{L/K} = \sum_{1 \leq i < j \leq m} \frac{a_i a_j}{(a_i + a_j)^2} \in K/P_2(K),$$

which we call the additive discriminant of L over K.

Theorem 6.4 [Bergé and Martinet 1985, Théorème 2.7]. *Let L' be the subextension of K^{ac} over K corresponding to* $\operatorname{Ker} \delta_{L/K}$. *Then the extension L' over K corresponds to $d^+_{L/K} \in K/P_2(K)$ by the Artin–Schreier theory.*

7. Product formula of Deligne–Laumon

We recall a statement of the product formula of Deligne–Laumon. In this paper, we need only the rank one case, which is proved in [Deligne 1973, Proposition 10.12.1], but we follow the notation from [Laumon 1987].

7A. ***Local factor.*** We consider a triple $(T, \mathcal{F}, \omega)$ which consists of the following.

- The affine scheme $T = \operatorname{Spec} \mathcal{O}_{K_T}$ where $\mathcal{O}_{K_T}$ is the ring of integers in a local field K_T of characteristic p whose residue field contains k.
- A constructible $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{F}$ on T.
- A nonzero meromorphic 1-form ω on T.

Then we can associate $\varepsilon_{\psi_0}(T, \mathcal{F}, \omega) \in \mathbb{C}^\times$ to the triple $(T, \mathcal{F}, \omega)$ as in [Laumon 1987, Théorème 3.1.5.4] using ι.

Assume that $K_T = k((t))$. Let $\eta = \operatorname{Spec} k((t))$ be the generic point of T with the natural inclusion $j : \eta \to T$. We define a character $\Psi_\omega : k((t)) \to \mathbb{C}^\times$ by

$$\Psi_\omega(a) = (\psi_0 \circ \operatorname{Tr}_{k/\mathbb{F}_p})(\operatorname{Res}(a\omega)) \quad \text{for } a \in k((t)).$$

Let $l(\Psi_\omega)$ be the level of Ψ_ω in the sense of [Bushnell and Henniart 2006, Definition 1.7]. We fix an algebraic closure $k((t))^{\mathrm{ac}}$ of $k((t))$. For a rank 1 smooth $\overline{\mathbb{Q}}_\ell$-sheaf V on η corresponding to a character $\chi : G_{k((t))} \to \mathbb{C}^\times$ via ι, we have

$$\varepsilon_{\psi_0}(T, j_* V, \omega) = q^{-l(\Psi_\omega)/2} \varepsilon(\chi\omega_{-1/2}, \Psi_\omega) \tag{7-1}$$

by [Laumon 1987, Théorème 3.1.5.4(v); Tate 1979, (3.6.2)] and [Bushnell and Henniart 2006, Proposition 23.1(3)].

7B. ***Product formula.*** Let X be a geometrically connected proper smooth curve over k of genus g. Let $\mathcal{F}$ be a constructible $\overline{\mathbb{Q}}_\ell$-sheaf on X. Let $\operatorname{Frob}_q \in G_k$ be the geometric Frobenius element. We put

$$\varepsilon(X, \mathcal{F}) = \iota\left(\prod_{i=0}^{2} \det(-\operatorname{Frob}_q; H^i(X \otimes_k k^{\mathrm{ac}}, \mathcal{F}))^{(-1)^{i-1}} \right).$$

Let $\operatorname{rk}(\mathcal{F})$ be the generic rank of $\mathcal{F}$.

Theorem 7.1 [Laumon 1987, Théorème 3.2.1.1]. *Let ω be a nonzero meromorphic 1-form on X. Then we have*

$$\varepsilon(X, \mathcal{F}) = q^{\operatorname{rk}(\mathcal{F})(1-g)} \prod_{x \in |X|} \varepsilon_{\psi_0}(X_{(x)}, \mathcal{F}|_{X_{(x)}}, \omega|_{X_{(x)}}),$$

where $|X|$ is the set of closed points of X, and $X_{(x)}$ is the completion of X at x.

8. Determinant

In this section, we study $\det \tau_{\psi_0}$ to show the equality $\omega_{\pi_{\zeta,\chi,c}} = \det \tau_{\zeta,\chi,c}$ of the central character and the determinant. We use the product formula of Deligne–Laumon to study $\det \tau_{\psi_0}(\operatorname{Fr}(1))$, where $\operatorname{Fr}(1)$ is defined in (2-2).

Lemma 8.1. *We have $Q^{\mathrm{ab}} = Q/Q_0$.*

Proof. By Lemma 2.4, we have $Q^{\mathrm{ab}} = (Q/F)^{\mathrm{ab}}$. For $(a,b,c) \in Q$, let (a,b) be the image of (a,b,c) in Q/F. Then we have

$$(a,0)(a,b)(a,0)^{-1}(a,b)^{-1} = (1,(a-1)b).$$

Hence, we obtain the claim. □

We view θ_0 defined in (2-8) as a character of Q by $(a,b,c) \mapsto \theta_0(a)$. Recall that τ^0 is the representation of Q defined in (2-14).

Lemma 8.2. *We have* $\det \tau^0 = \theta_0$.

Proof. By Lemma 8.1, it suffices to show $\det \tau^0 = \theta_0$ on $\mu_{p^e+1}(k^{\mathrm{ac}})$. By Lemma 2.2 and Lemma 2.5, we have

$$\det \tau^0(a) = \prod_{\chi \in \mu_{p^e+1}(k^{\mathrm{ac}})^\vee \setminus \{1\}} \chi(a)$$

for $a \in \mu_{p^e+1}(k^{\mathrm{ac}})$. Hence, the claim follows. □

For $a \in k^\times$, let $\left(\frac{a}{k}\right)$ denote the quadratic residue symbol of k defined by

$$\left(\frac{a}{k}\right) = \begin{cases} 1 & \text{if } a \text{ is square in } k, \\ -1 & \text{if } a \text{ is not square in } k. \end{cases}$$

Lemma 8.3. *Let m be a positive integer that is prime to p. We take an m-th root $\varpi^{1/m}$ of ϖ, and put $L = K(\varpi^{1/m})$.*

(1) *If m is odd, then $\delta_{L/K}$ is the unramified character satisfying $\delta_{L/K}(\varpi) = \left(\frac{q}{m}\right)$.*

(2) *If m is even, we have $\delta_{L/K}(\varpi) = \left(\frac{-1}{q}\right)^{m/2}$ and $\delta_{L/K}(x) = \left(\frac{\bar{x}}{k}\right)$ for $x \in \mathcal{O}_K^\times$.*

Proof. These are proved in [Bushnell and Fröhlich 1983, (10.1.6)] if $\mathrm{ch}\, K = 0$. Actually, the same proof works also in the positive characteristic case. □

Lemma 8.4. *Let m, m' be positive integers that are prime to p. We take an m-th root $\varpi^{1/m}$ of ϖ, and put $L = K(\varpi^{1/m})$. Let $\psi'_K \colon K \to \mathbb{C}^\times$ be a character such that $\psi'_K(x) = \psi_0(\mathrm{Tr}_{k/\mathbb{F}_p}(m'\bar{x}))$ for $x \in \mathcal{O}_K$. Then we have*

$$\lambda(L/K, \psi'_K) = \begin{cases} \left(\frac{q}{m}\right) & \text{if } m \text{ is odd}, \\ -\left(-\epsilon(p)\left(\frac{2mm'}{p}\right)\left(\frac{-1}{p}\right)^{(m/2)-1}\right)^f & \text{if } m \text{ is even}. \end{cases}$$

Proof. If m is odd, we have

$$\lambda(L/K, \psi'_K) = \varepsilon(\delta_{L/K}, \psi'_K) = \left(\frac{q}{m}\right)$$

by [Henniart 1984, Proposition 2] and Lemma 8.3(1).

Assume that m is even. Note that $p \neq 2$ in this case. Then we have

$$d_{L/K} = (-1)^{m/2} \mathrm{Nr}_{L/K}(m(\varpi^{1/m})^{m-1}) = -(-1)^{m/2}\varpi \in K^\times/(K^\times)^2 \tag{8-1}$$

by Proposition 6.2. For $\chi \in (\mathbb{F}_q^\times)^\vee$ and $\psi \in \mathbb{F}_q^\vee \setminus \{1\}$, we set

$$\tau(\chi, \psi) = -\sum_{x \in \mathbb{F}_q^\times} \chi^{-1}(x)\psi(x)$$

and have the Hasse–Davenport formula

$$\tau(\chi \circ \mathrm{Nr}_{\mathbb{F}_{q^n}/\mathbb{F}_q}, \psi \circ \mathrm{Tr}_{\mathbb{F}_{q^n}/\mathbb{F}_q}) = \tau(\chi, \psi)^n. \tag{8-2}$$

Let

$$(\ ,\ )_K : K^\times/(K^\times)^2 \times K^\times/(K^\times)^2 \to \{\pm 1\}$$

denote the Hilbert symbol. By (6-1) and (8-1), we have

$$\delta_{L/K}(x) = \kappa_{d_{L/K}}(x) = (x, d_{L/K})_K = (x, \varpi)_K = \left(\frac{\bar{x}}{k}\right)$$

for $x \in \mathcal{O}_K^\times$. By [Bushnell and Henniart 2006, Theorem 23.5], we have

$$\varepsilon(\delta_{L/K}, \psi'_K) = q^{-1/2} \sum_{x \in \mathcal{O}_K^\times/U_K^1} \delta_{L/K}(x)\psi'_K(x) = q^{-1/2} \sum_{x \in k^\times} \left(\frac{x}{k}\right) \psi_0(\mathrm{Tr}_{k/\mathbb{F}_p}(m'x)).$$

By applying (8-2) to the extension k over $\mathbb{F}_p$ and using (2-16), we have

$$q^{-1/2} \sum_{x \in k^\times} \left(\frac{x}{k}\right) \psi_0(\mathrm{Tr}_{k/\mathbb{F}_p}(m'x)) = -\left(-\epsilon(p)\left(\frac{m'}{p}\right)\right)^f.$$

Hence, we have

$$\begin{aligned}\lambda(L/K, \psi'_K) &= \varepsilon(\delta_{L/K}, \psi'_K)\left(\frac{m}{q}\right)\left(\frac{-1}{q}\right)^{(m/2)-1}(d_{L/K}, 2)_K \\ &= -\left(-\epsilon(p)\left(\frac{2mm'}{p}\right)\left(\frac{-1}{p}\right)^{(m/2)-1}\right)^f\end{aligned}$$

by [Saito 1995, Theorem II.2B] and [Tate 1979, (3.6.1)]. □

Lemma 8.5. *We have*

$$\det \tau_{\psi_0}(\mathrm{Fr}(1)) = \begin{cases} \left(-\epsilon(p)\left(\frac{2}{p}\right)\right)^f q^{p^e/2} & \text{if } p \neq 2, \\ q^{2^{e-1}} & \text{if } p = 2. \end{cases}$$

Proof. Let x be the standard coordinate of $\mathbb{A}_k^1$. Let j be the open immersion $\mathbb{A}_k^1 \hookrightarrow \mathbb{P}_k^1$. We put $t = 1/x$. As in Section 7A, we put $T = \mathrm{Spec}\, k[[t]]$ and $\eta = \mathrm{Spec}\, k((t))$ with the open immersion $j : \eta \to T$.

We consider $k((s))$ as a subfield of $k((t))$ by $s = t^{p^e+1}$. Let $\tilde{\xi} : G_{k((s))} \to \mathbb{C}^\times$ be the Artin–Schreier character associated to $y^p - y = 1/s$ and ψ_0, which means the composite of

$$G_{k((s))} \to \mathbb{F}_p, \quad \sigma \mapsto \sigma(y) - y$$

and ψ_0^{-1} where y is an element of $k((t))^{\mathrm{ac}}$ such that $y^p - y = 1/s$.

We use the notation in Lemma 5.5. Note that $\psi_{k((s))(y)/k((s))}(1)=1$ by Lemma 5.4. We can check that

$$\mathrm{Nr}_{k((s))(y)/k((s))}(1+y^{-1}x)=1+s(x^p-x)$$

for $x \in k$. For $x \in \mathcal{O}_{k((s))}$, we have

$$\begin{aligned}\tilde{\xi}(1+sx)&=\psi_0^{-1}\bigl(\mathrm{Art}_{k((s))}(1+sx)(y)-y\bigr)\\&=\psi_0^{-1}\left(-p_{k((s))(y),y^{-1}}\left(\frac{\mathrm{Art}_{k((s))}(1+sx)(y^{-1})}{y^{-1}}\right)\right)=\psi_0(\mathrm{Tr}_{k/\mathbb{F}_p}(\bar{x})),\end{aligned}$$

where we use Lemma 5.5 with $\alpha=s$, $\beta=\varpi_{k((s))}=y^{-1}$. Hence, we have $\mathrm{rsw}(\tilde{\xi},\Psi_{s^{-1}ds})=s$ by Proposition 5.3(1).

Let $\xi : G_{k((t))} \to \mathbb{C}^\times$ be the restriction of $\tilde{\xi}$ to $G_{k((t))}$. Then ξ is the Artin–Schreier character associated to $y^p-y=1/t^{p^e+1}$ and ψ_0.

Let V_ξ be the smooth $\overline{\mathbb{Q}}_\ell$-sheaf on η corresponding to ξ via ι. Then we have $V_\xi \simeq \mathcal{L}_{\psi_0}|_\eta$ by [Deligne 1977, Définition 1.7 in Sommes trig.]. Let the notation be as in Lemma 2.2. We write ω for the meromorphic 1-form dx on $\mathbb{P}^1_k$. By [Laumon 1987, Théorème 3.1.5.4(v)], we have

$$\varepsilon_{\psi_0}\bigl(X_{(x)},(j_!\pi^*\mathcal{L}_{\psi_0})|_{X_{(x)}},\omega|_{X_{(x)}}\bigr)=1$$

for any $x \in |\mathbb{A}^1_k|$ with $X=\mathbb{P}^1_k$ in the notation of Theorem 7.1. We simply write ω for $\omega|_T$. Then we have

$$\det\tau_{\psi_0}(\mathrm{Fr}(1))=(-1)^{p^e}\varepsilon(\mathbb{P}^1_k,j_!\pi^*\mathcal{L}_{\psi_0})=(-1)^p q\,\varepsilon_{\psi_0}(T,j_!V_\xi,\omega)$$

by Theorem 7.1. Since ξ is a ramified character, we have $j_!V_\xi \simeq j_*V_\xi$. Hence,

$$\varepsilon_{\psi_0}(T,j_!V_\xi,\omega)=\varepsilon_{\psi_0}(T,j_*V_\xi,\omega)=q^{-1}\varepsilon(\xi\omega_{-1/2},\Psi_\omega)$$

by (7-1). Since $\omega=-t^{-2}dt$ on T, we have

$$\varepsilon(\xi\omega_{-1/2},\Psi_\omega)=(\xi\omega_{-1/2})(-t^{-1})\,\varepsilon(\xi\omega_{-1/2},\Psi_{t^{-1}dt})$$

by [Bushnell and Henniart 2006, 23.5 Lemma 1]. We have

$$\xi(-t^{-1})=\xi(-t^{p^e})=\xi(-t)^{p^e}=1,$$

since $\mathrm{Nr}_{k((t))(y)/k((t))}(y)=1/t^{p^e+1}$. Hence we obtain

$$(\xi\omega_{-1/2})(-t^{-1})\,\varepsilon(\xi\omega_{-1/2},\Psi_{t^{-1}dt})=q^{p^e/2}\varepsilon(\xi,\Psi_{t^{-1}dt})$$

by Lemma 4.2, since $\mathrm{rsw}(\xi,\Psi_{t^{-1}dt})=s$ by $\mathrm{rsw}(\tilde{\xi},\Psi_{s^{-1}ds})=s$ and Proposition 5.2(2). By Proposition 5.3(2), we have $\varepsilon(\tilde{\xi},\Psi_{s^{-1}ds})=\tilde{\xi}(s)=1$, since the level of $\tilde{\xi}$ is 1 and $\mathrm{Nr}_{k((s))(y)/k((s))}(y^{-1})=s$. Hence, we obtain

$$\varepsilon(\xi,\Psi_{t^{-1}dt})=\lambda\bigl(k((t))/k((s)),\Psi_{s^{-1}ds}\bigr)^{-1}\delta_{k((t))/k((s))}(\mathrm{rsw}(\tilde{\xi},\Psi_{s^{-1}ds}))$$

by Proposition 5.1. By Lemmas 8.4 and 8.3, we respectively have

$$\lambda\big(k((t))/k((s)),\,\Psi_{s^{-1}ds}\big)=\begin{cases}-\big(-\epsilon(p)\big(\frac{2}{p}\big)\big(\frac{-1}{p}\big)^{(p^e-1)/2}\big)^f & \text{if } p\neq 2,\\ \big(\frac{q}{p^e+1}\big) & \text{if } p=2\end{cases}$$

and

$$\delta_{k((t))/k((s))}(\mathrm{rsw}(\tilde{\xi},\,\Psi_{s^{-1}ds}))=\begin{cases}\big(\frac{-1}{q}\big)^{(p^e+1)/2} & \text{if } p\neq 2,\\ \big(\frac{q}{p^e+1}\big) & \text{if } p=2.\end{cases}$$

The claim follows from the above equalities. ☐

We simply write τ_ζ for $\tau_{\zeta,1,1}$.

Proposition 8.6. *We have* $\omega_{\pi_{\zeta,\chi,c}}=\det\tau_{\zeta,\chi,c}$.

Proof. By (2-13) and [Gallagher 1965, (1)], we have

$$\det\tau_{\zeta,\chi,c}=\delta_{E_\zeta/K}^{p^e}(\det\tau_{n,\zeta,\chi,c})|_{K^\times},\tag{8-3}$$

since $\delta_{E_\zeta/K}=\det(\mathrm{Ind}_{E_\zeta/K}1)$ and the transfer homomorphism $W_K^{\mathrm{ab}}\to W_{E_\zeta}^{\mathrm{ab}}$ is compatible with the natural inclusion $K^\times\to E_\zeta^\times$ under the Artin reciprocity maps. Hence, we may assume $\chi=1$ and $c=1$ by twist (see (2-13)). Then it suffices to show $\det\tau_\zeta=1$. We see that $\det\tau_\zeta$ is unramified by (2-9), Lemmas 2.5, 8.2, 8.3 and equation (8-3).

If p and n' are odd, then we have

$$\begin{aligned}\det\tau_\zeta(\varpi)&=\Big(\frac{q}{n'}\Big)^{p^e}\Big(-\epsilon(p)\Big(\frac{2}{p}\Big)p^{p^e/2}\Big)^{fn'}\Big(\Big(-\epsilon(p)\Big(\frac{-2n'}{p}\Big)\Big)^n p^{-\frac{1}{2}}\Big)^{fn'p^e}\\&=\Big(\Big(\frac{p}{n'}\Big)\Big(\frac{n'}{p}\Big)\epsilon(p)^{p^en-1}\Big)^{fn'}=\Big(\Big(\frac{p}{n'}\Big)\Big(\frac{n'}{p}\Big)(-1)^{\frac{1}{2}(p-1)\frac{1}{2}(n'-1)}\Big)^{fn'}=1\end{aligned}$$

by (8-3), Lemmas 8.3(1) and 8.5. We see that $\det\tau_\zeta(\varpi)=1$ similarly also in the other case using (8-3), Lemmas 8.3 and 8.5. ☐

9. Imprimitive field

In this section, we construct a field extension T_ζ^{u} of E_ζ such that $\tau_{n,\zeta}|_{W_{T_\zeta^{\mathrm{u}}}}$ is an induction of a character. We call T_ζ^{u} an imprimitive field of $\tau_{n,\zeta}$, since $\tau_{n,\zeta}|_{W_{T_\zeta^{\mathrm{u}}}}$ is not primitive.

9A. ***Construction of character.*** Here we construct subgroups $R\subset Q'\subset Q\rtimes\mathbb{Z}$ and a character ϕ_n of R. Later (see Section 9B) we will see that $\tau_n|_{Q'}\simeq\mathrm{Ind}_R^{Q'}\phi_n$. Our imprimitive field T_ζ^{u} will correspond to the subgroup $Q'\subset Q\rtimes\mathbb{Z}$.

Let e_0 be the positive integer such that $e_0\in 2^{\mathbb{N}}$ and e/e_0 is odd.

Lemma 9.1. *Assume $p\neq 2$. Then we have* $\mathrm{Tr}\,\tau_{\psi_0}(\mathrm{Fr}(2e_0))=p^{e_0}$.

Proof. For $a \in k^{\mathrm{ac}}$ and $b \in \mathbb{F}_{p^{2e_0}}$ such that $a^p - a = b^{p^e+1}$, we have that

$$a^{p^{2e_0}} - a = \mathrm{Tr}_{\mathbb{F}_{p^{2e_0}}/\mathbb{F}_p}(b^{p^e+1}). \tag{9-1}$$

By (9-1) and the Lefschetz trace formula, we see that

$$\begin{aligned}\mathrm{Tr}\,\tau_{\psi_0}(\mathrm{Fr}(2e_0)) &= -\sum_{b\in\mathbb{F}_{p^{2e_0}}}(\psi_0\circ\mathrm{Tr}_{\mathbb{F}_{p^{e_0}}/\mathbb{F}_p})(\mathrm{Tr}_{\mathbb{F}_{p^{2e_0}}/\mathbb{F}_{p^{e_0}}}(b^{p^{e_0}+1}))\\ &= -1-(p^{e_0}+1)\sum_{x\in\mathbb{F}_{p^{e_0}}^\times}(\psi_0\circ\mathrm{Tr}_{\mathbb{F}_{p^{e_0}}/\mathbb{F}_p})(x) = p^{e_0}\end{aligned}$$

using $(p^e+1,\, p^{2e_0}-1) = p^{e_0}+1$. □

Corollary 9.2. *Assume $p \neq 2$. Then we have* $\mathrm{Tr}\,\tau_n\big(\mathrm{Fr}(2e_0)\big) = (-1)^{ne_0(p-1)/2}$.

Proof. This follows from (2-9) and Lemma 9.1. □

Let n_0 be the biggest integer such that 2^{n_0} divides $p^{e_0}+1$. We take $r \in k^{\mathrm{ac}}$ such that $r^{2^{n_0}} = -1$. We define a subgroup R_0 of Q_0 by

$$R_0 = \{(1, b, c) \in Q_0 \mid b^{p^e} - rb = 0\}.$$

Lemma 9.3. (1) *If $p \neq 2$, then the action of $2e_0\mathbb{Z} \subset \mathbb{Z}$ on Q stabilizes R_0.*

(2) *If $p = 2$, then the action of $\boldsymbol{g}$ on $Q \rtimes \mathbb{Z}$ by conjugation stabilizes R_0.*

Proof. The first claim follows from $r^{p^{2e_0}-1} = 1$. We can see the second claim easily using (2-19). □

We put

$$Q' = \begin{cases} Q_0 \rtimes (2e_0\mathbb{Z}) & \text{if } p \neq 2,\\ Q_0 \rtimes \mathbb{Z} & \text{if } p = 2,\end{cases} \qquad R = \begin{cases} R_0 \rtimes (2e_0\mathbb{Z}) & \text{if } p \neq 2,\\ R_0 \cdot \langle \boldsymbol{g} \rangle & \text{if } p = 2\end{cases}$$

as subgroups of $Q \rtimes \mathbb{Z}$, which are well-defined by Lemma 9.3. We are going to construct a character ϕ_n of R in this subsection. Then, we will show that $\tau_n|_{Q'} \simeq \mathrm{Ind}_R^{Q'} \phi_n$ in the next subsection.

First, we consider the case where p is odd. We define a homomorphism $\phi_n : R \to \mathbb{C}^\times$ by

$$\begin{aligned}\phi_n\big(((1,b,c),0)\big) &= \psi_0\Big(c - \tfrac{1}{2}\sum_{i=0}^{e-1}(rb^2)^{p^i}\Big) \quad \text{for } (1,b,c) \in R_0,\\ \phi_n(\mathrm{Fr}(2e_0)) &= (-1)^{ne_0((p-1)/2)}.\end{aligned} \tag{9-2}$$

Then ϕ_n extends the character ψ_0 of F.

Next, we consider the case where $p = 2$. We define an abelian group R'_0 as

$$R'_0 = \{(b, c) \mid b \in \mathbb{F}_2,\ c \in \mathbb{F}_{2^{2e}},\ c^{2^e} - c = b\},$$

with the multiplication given by

$$(b_1, c_1)\cdot(b_2, c_2) = (b_1+b_2,\ c_1+c_2+b_1b_2).$$

We define $\phi : R_0 \to R_0'$ by

$$\phi((1, b, c)) = \Bigl(\mathrm{Tr}_{\mathbb{F}_{2^e}/\mathbb{F}_2}(b),\ c + \sum_{0\le i<j\le e-1} b^{2^i+2^j}\Bigr) \quad \text{for } (1, b, c)\in R_0,$$

which is a homomorphism by

$$\mathrm{Tr}_{\mathbb{F}_{2^e}/\mathbb{F}_2}(b)\,\mathrm{Tr}_{\mathbb{F}_{2^e}/\mathbb{F}_2}(b') = \mathrm{Tr}_{\mathbb{F}_{2^e}/\mathbb{F}_2}(bb') + \sum_{0\le i<j\le e-1}(b^{2^i}b'^{2^j} + b'^{2^i}b^{2^j}) \tag{9-3}$$

for $b, b'\in\mathbb{F}_{2^e}$. Let $b_0\in\mathbb{F}_{2^{2e}}$ be as before Lemma 2.8. Let F' be the kernel of the homomorphism

$$\mathbb{F}_{2^e}\to\mathbb{F}_2,\quad c\mapsto \mathrm{Tr}_{\mathbb{F}_{2^e}/\mathbb{F}_2}((b_0+b_0^{2^e})\,c).$$

We put $R_0'' = R_0'/F'$, where we consider F' as a subgroup of R_0' by $c\mapsto(0, c)$. Then R_0'' is a cyclic group of order 4. We write $\bar{g}(b, c)$ for the image of $(b, c)\in R_0'$ under the projection $R_0'\to R_0''$. Let $\phi' : R_0\to R_0''$ be the composite of ϕ and the projection $R_0'\to R_0''$. We put

$$s = \sum_{i=0}^{e-1} b_0^{2^i},\quad t = \mathrm{Tr}_{\mathbb{F}_{2^{2e}}/\mathbb{F}_{2^e}}(b_0). \tag{9-4}$$

We have $s^2+s=t$ and $\mathrm{Tr}_{\mathbb{F}_{2^e}/\mathbb{F}_2}(t) = \mathrm{Tr}_{\mathbb{F}_{2^{2e}}/\mathbb{F}_2}(b_0)=1$. We have

$$\Bigl(1, s^2 + \sum_{0\le i<j\le e-1} t^{2^i+2^j}\Bigr)\in R_0',$$

which is of order 4. The element $\bar{g}\bigl(1, s^2+\sum_{0\le i<j\le e-1}t^{2^i+2^j}\bigr)$ is a generator of R_0'', because

$$2\bar{g}\Bigl(1, s^2 + \sum_{0\le i<j\le e-1} t^{2^i+2^j}\Bigr) = \bar{g}(0, 1)\ne 0.$$

Let $\tilde{\psi}_0 : R_0''\to\mathbb{C}^\times$ be the faithful character satisfying

$$\tilde{\psi}_0\Bigl(\bar{g}\Bigl(1, s^2 + \sum_{0\le i<j\le e-1} t^{2^i+2^j}\Bigr)\Bigr) = -\sqrt{-1}.$$

We define a homomorphism $\phi_n : R\to\mathbb{C}^\times$ by

$$\begin{aligned} \phi_n\bigl(((1, b, c), 0)\bigr) &= (\tilde{\psi}_0\circ\phi')((1, b, c)) \quad \text{for } (1, b, c)\in R_0,\\ \phi_n(\boldsymbol{g}) &= (-1)^{\frac{1}{8}n(n-2)}\frac{-1+\sqrt{-1}}{\sqrt{2}}, \end{aligned} \tag{9-5}$$

which is a character of order 8. Then ϕ_n extends the character ψ_0 of F.

9B. *Induction of character.*

Lemma 9.4. *We have $\tau_n|_{Q'} \simeq \mathrm{Ind}_R^{Q'} \phi_n$.*

Proof. We write $\tilde{\psi}_n$ for $\phi_n|_{R_0}$. We know that $\tau_n|_{Q_0} \cong \mathrm{Ind}_{R_0}^{Q_0} \tilde{\psi}_n$ by Proposition 1.2, since R_0 is an abelian group such that $2\dim_{\mathbb{F}_p}(R_0/F) = \dim_{\mathbb{F}_p}(Q_0/F)$.

First, we consider the case where p is odd. The claim for general f follows from the claim for $f = 1$ by the restriction. Hence, we may assume that $f = 1$.

If $\tilde{\psi} \in R_0^\vee$ satisfies $\tilde{\psi}|_F = \psi_0$, then we have $\tau_n|_{Q_0} \cong \mathrm{Ind}_{R_0}^{Q_0} \tilde{\psi}$ by Proposition 1.2, and obtain an injective homomorphism $\tilde{\psi} \hookrightarrow \tau_n|_{R_0}$ as representations of R_0 by Frobenius reciprocity. Hence we have a decomposition

$$\tau_n|_{R_0} = \bigoplus_{\tilde{\psi} \in R_0^\vee,\, \tilde{\psi}|_F = \psi_0} \tilde{\psi}, \tag{9-6}$$

since the number of $\tilde{\psi} \in R_0^\vee$ such that $\tilde{\psi}|_F = \psi_0$ is p^e.

We put

$$\bar{R}_0 = \{b \in k^{\mathrm{ac}} \mid b^{p^e} - rb = 0\}.$$

The $\tilde{\psi}_n$-component in (9-6) is the unique component that is stable by the action of $((1, 0, 0), 2e_0)$, since the homomorphism

$$\bar{R}_0 \to \bar{R}_0, \quad b \mapsto b^{p^{2e_0}} - b$$

is an isomorphism. Hence, we have a nontrivial homomorphism $\phi_n \to \tau_n|_R$ by Corollary 9.2. Then we have a nontrivial homomorphism $\mathrm{Ind}_R^{Q'} \phi_n \to \tau_n|_{Q'}$ by Frobenius reciprocity. The representation $\tau_n|_{Q'}$ is irreducible by Corollary 2.6. Then we obtain the claim, since $[Q' : R] = p^e$.

Next we consider the case where $p = 2$. Then it suffices to show that

$$\mathrm{Tr}(\mathrm{Ind}_R^{Q'} \phi_n)(\boldsymbol{g}^{-1}) = -(-1)^{\frac{1}{8}n(n-2)}\sqrt{2}$$

by (2-9) and Proposition 2.9. We have a decomposition

$$(\mathrm{Ind}_R^{Q'} \phi_n)|_{R_0} = \bigoplus_{\phi \in R_0^\vee,\, \phi|_F = \psi_0} \phi. \tag{9-7}$$

Let $\tilde{\psi}_n'$ be the twist of $\tilde{\psi}_n$ by the character

$$R_0 \to \overline{\mathbb{Q}}_\ell^\times, \quad (1, b, c) \mapsto \psi_0(\mathrm{Tr}_{\mathbb{F}_{2^e}/\mathbb{F}_2}(b)).$$

Then only the $\tilde{\psi}_n$-component and the $\tilde{\psi}_n'$-component in (9-6) are stable by the action of $((1, b_0, c_0), 1)$, since the image of the homomorphism

$$\mathbb{F}_{2^e} \to \mathbb{F}_{2^e}, \quad b \mapsto b^2 - b$$

is equal to $\operatorname{Ker}\operatorname{Tr}_{\mathbb{F}_{2^e}/\mathbb{F}_2}$. The action of $\operatorname{Fr}(e)$ permutes the $\tilde{\psi}_n$-component and the $\tilde{\psi}'_n$-component. Hence, $\boldsymbol{g}$ acts on the $\tilde{\psi}'_n$-component by $\phi_n(\boldsymbol{g})$ times

$$\phi_n(\operatorname{Fr}(e)^{-1}\boldsymbol{g}\operatorname{Fr}(e)\boldsymbol{g}^{-1}) = \phi_n\left(\left(\left(1, t, c_0+c_0^{2^e}+\sum_{i=0}^{e-1}(b_0^{2^e+1}+b_0^{2^{e+1}})^{2^i}\right), 0\right)\right) = \sqrt{-1}.$$

Hence we have

$$\operatorname{Tr}(\operatorname{Ind}_R^{Q'}\phi_n)(\boldsymbol{g}^{-1}) = (1-\sqrt{-1})\,\phi_n(\boldsymbol{g}^{-1}) = -(-1)^{\frac{1}{8}n(n-2)}\sqrt{2}. \qquad \square$$

We use the notations from equation (2-10). We set $T_\zeta = E_\zeta(\alpha_\zeta)$, $M_\zeta = T_\zeta(\beta_\zeta)$ and $N_\zeta = M_\zeta(\gamma_\zeta)$. Let f_0 be the positive integer such that $f_0 \in 2^{\mathbb{N}}$ and f/f_0 is odd. We put

$$N = \begin{cases} 2e_0/f_0 & \text{if } p \neq 2 \text{ and } f_0 | 2e_0, \\ 1, & \text{otherwise.} \end{cases}$$

Let K^{ur} be the maximal unramified extension of K in K^{ac}. Let $K^{\mathrm{u}} \subset K^{\mathrm{ur}}$ be the unramified extension of degree N over K. Let k_N be the residue field of K^{u}. For a finite field extension L of K in K^{ac}, we write L^{u} for the composite field of L and K^{u} in K^{ac}. For $a \in k^{\mathrm{ac}}$, we write $\hat{a} \in \mathcal{O}_{K^{\mathrm{ur}}}$ for the Teichmüller lift of a. We put

$$\text{(9-8)} \qquad \delta'_\zeta = \begin{cases} \beta_\zeta^{p^e} - \hat{r}\beta_\zeta & \text{if } p \neq 2, \\ \beta_\zeta^{2^e} - \beta_\zeta + \sum_{i=0}^{e-1}\hat{b}_0^{2^i} & \text{if } p = 2, \end{cases} \qquad \epsilon_1 = \begin{cases} 0 & \text{if } p \neq 2, \\ 1 & \text{if } p = 2. \end{cases}$$

Then we have

$$\delta_\zeta'^{p^e} - \hat{r}^{-1}\delta'_\zeta \equiv -\alpha_\zeta^{-1} + \epsilon_1 \mod \mathfrak{p}_{T_\zeta^{\mathrm{u}}(\delta'_\zeta)}.$$

We take $\delta_\zeta \in T_\zeta^{\mathrm{u}}(\delta'_\zeta)$ such that

$$\text{(9-9)} \qquad \delta_\zeta^{p^e} - \hat{r}^{-1}\delta_\zeta = -\alpha_\zeta^{-1} + \epsilon_1, \qquad \delta_\zeta \equiv \delta'_\zeta \mod \mathfrak{p}_{T_\zeta^{\mathrm{u}}(\delta'_\zeta)}.$$

We put $M_\zeta'^{\mathrm{u}} = T_\zeta^{\mathrm{u}}(\delta_\zeta)$. The image of $\Theta_\zeta|_{W_{M_\zeta'^{\mathrm{u}}}}$ is contained in R. Let $\xi_{n,\zeta} : W_{M_\zeta'^{\mathrm{u}}} \to \mathbb{C}^\times$ be the composite of the restrictions $\Theta_\zeta|_{W_{M_\zeta'^{\mathrm{u}}}}$ and $\phi_n|_R$. By the local class field theory, we regard $\xi_{n,\zeta}$ as a character of $M_\zeta'^{\mathrm{u}\times}$.

Proposition 9.5. *We have* $\tau_{n,\zeta}|_{W_{T_\zeta^{\mathrm{u}}}} \simeq \operatorname{Ind}_{M_\zeta'^{\mathrm{u}}/T_\zeta^{\mathrm{u}}}\xi_{n,\zeta}$.

Proof. This follows from Lemma 9.4. $\square$

Remark 9.6. Our imprimitive field is different from that in [Bushnell and Henniart 2014, Section 5.1]. In our case, T_ζ^{u} need not be normal over K. This choice is technically important in our proof of the main result.

9C. ***Study of character.*** Here we study the character $\xi_{n,\zeta}$ in detail.

Assume that $\operatorname{ch} K = p$ and $f = 1$ in this subsection. We will use results in this subsection to compute the epsilon factor of $\xi_{n,\zeta}$ later after a reduction to the case where $\operatorname{ch} K = p$ and $f = 1$. By (2-10), (9-8), (9-9) and $\operatorname{ch} K = p$, we have that $\delta_\zeta = \delta'_\zeta$.

9C1. *Odd case.* Assume $p \neq 2$. We put

$$\theta_\zeta = \gamma_\zeta + \tfrac{1}{2} \sum_{i=0}^{e-1} (r\beta_\zeta^2)^{p^i}. \tag{9-10}$$

Since $r^{p^{e_0}+1} = -1$ and $(p^e+1)/(p^{e_0}+1)$ is an odd integer, we have $r^{p^e+1} = -1$. Then we have

$$\begin{aligned} \theta_\zeta^p - \theta_\zeta &= \beta_\zeta^{p^e+1} - \frac{1}{2r}(\beta_\zeta^{2p^e} + r^2\beta_\zeta^2) \\ &= -\frac{1}{2r}(\beta_\zeta^{2p^e} - 2r\beta_\zeta^{p^e+1} + r^2\beta_\zeta^2) = -\frac{1}{2r}\delta_\zeta^2. \end{aligned} \tag{9-11}$$

We put $N'^{\mathrm{u}}_\zeta = M'^{\mathrm{u}}_\zeta(\theta_\zeta)$. Let $\xi'_{n,\zeta}$ be the twist of $\xi_{n,\zeta}$ by the unramified character

$$W_{M'^{\mathrm{u}}_\zeta} \to \mathbb{C}^\times, \quad \sigma \mapsto \sqrt{-1}^{\,n n_\sigma (p-1)/2},$$

where n_σ is as before (2-12).

Lemma 9.7. *If $p \neq 2$, then $\xi'_{n,\zeta}$ factors through* $\mathrm{Gal}(N'^{\mathrm{u}}_\zeta / M'^{\mathrm{u}}_\zeta)$.

Proof. Let $\sigma \in \operatorname{Ker} \xi'_{n,\zeta}$. Recall that $a_\sigma, b_\sigma, c_\sigma$ are defined in (2-11). Then we have $(\bar{a}_\sigma, \bar{b}_\sigma, \bar{c}_\sigma) \in R_0$ and

$$\bar{c}_\sigma - \tfrac{1}{2} \sum_{i=0}^{e-1} (r\bar{b}_\sigma^2)^{p^i} = 0$$

by (9-2). Hence, we see that

$$\begin{aligned} \sigma(\theta_\zeta) - \theta_\zeta &= c_\sigma - \sum_{i=0}^{e-1} (rb_\sigma(\beta_\zeta + b_\sigma))^{p^i} + \tfrac{1}{2} \sum_{i=0}^{e-1} \bigl(r((\beta_\zeta + b_\sigma)^2 - \beta_\zeta^2)\bigr)^{p^i} \\ &= c_\sigma - \tfrac{1}{2} \sum_{i=0}^{e-1} (rb_\sigma^2)^{p^i} \equiv 0 \mod \mathfrak{p}_{N'^{\mathrm{u}}_\zeta} \end{aligned}$$

by (2-11). Therefore, we obtain the claim by $\sigma(\delta_\zeta) = \delta_\zeta$ and (9-11). □

9C2. *Even case.* Assume $p = 2$. Let $\xi'_{n,\zeta}$ be the twist of $\xi_{n,\zeta}$ by the character

$$W_{M'^{\mathrm{u}}_\zeta} \to \mathbb{C}^\times, \quad \sigma \mapsto \left((-1)^{\frac{1}{8}n(n-2)} \frac{-1+\sqrt{-1}}{\sqrt{2}}\right)^{n_\sigma}. \tag{9-12}$$

We take $b_1, b_2 \in k^{\mathrm{ac}}$ such that

$$b_1^2 - b_1 = s, \quad b_2^2 - b_2 = t\left(b_1^2 + \sum_{i=0}^{e-1} (b_1 s)^{2^i}\right). \tag{9-13}$$

We put

$$\eta_\zeta = \sum_{i=0}^{e-1} \beta_\zeta^{2^i} + b_1, \quad \gamma'_\zeta = \gamma_\zeta + \sum_{0 \le i < j \le e-1} \beta_\zeta^{2^i + 2^j}, \tag{9-14}$$

and

$$\theta'_\zeta=\sum_{i=0}^{e-1}(t\gamma'_\zeta)^{2^i}+\sum_{0\le i\le j\le e-2}t^{2^i}(\delta_\zeta\eta_\zeta)^{2^j}\\ +\sum_{0\le j<i\le e-1}t^{2^i}(b_1\delta_\zeta+s\eta_\zeta)^{2^j}+b_1^2\eta_\zeta+b_2. \tag{9-15}$$

Lemma 9.8. *We have* $\eta_\zeta^2-\eta_\zeta=\delta_\zeta$ *and* $\theta_\zeta'^2-\theta'_\zeta=(\delta_\zeta\eta_\zeta)^{2^{e-1}}$.

Proof. We can check the first claim easily. We show the second claim. We use P_m in Section 6B2. We have

$$P_2(\gamma'_\zeta)=(\beta_\zeta^{2^e}-\beta_\zeta)\sum_{i=0}^{e-1}\beta_\zeta^{2^i}+\beta_\zeta^2=(\delta_\zeta-s)(\eta_\zeta-b_1)+\beta_\zeta^2. \tag{9-16}$$

Hence, we have

$$P_{2^e}(\gamma'_\zeta)=\sum_{i=0}^{e-1}((\delta_\zeta-s)(\eta_\zeta-b_1))^{2^i}+(\eta_\zeta-b_1)^2. \tag{9-17}$$

By $b_1^4+b_1=s^2+s=t$ and $\eta_\zeta^2-\eta_\zeta=\delta_\zeta$, we have

$$(b_1^2\eta_\zeta)^2+b_1^2\eta_\zeta=t\eta_\zeta^2+b_1\eta_\zeta^2+b_1^2\eta_\zeta=t\eta_\zeta^2+b_1(\eta_\zeta^2+\eta_\zeta)+s\eta_\zeta=t\eta_\zeta^2+b_1\delta_\zeta+s\eta_\zeta.$$

Hence, by using $\sum_{i=1}^{e-1}t^{2^i}=1-t$ and $t\in\mathbb{F}_{2^e}$, we have

$$\begin{aligned}\theta_\zeta'^2-\theta'_\zeta&=tP_{2^e}(\gamma'_\zeta)+t\sum_{i=0}^{e-1}(\delta_\zeta\eta_\zeta+b_1\delta_\zeta+s\eta_\zeta)^{2^i}+(\delta_\zeta\eta_\zeta)^{2^{e-1}}+t\eta_\zeta^2+b_2^2-b_2\\ &=t\left(\sum_{i=0}^{e-1}(b_1s)^{2^i}+\eta_\zeta^2+b_1^2\right)+(\delta_\zeta\eta_\zeta)^{2^{e-1}}+t\eta_\zeta^2+b_2^2-b_2=(\delta_\zeta\eta_\zeta)^{2^{e-1}},\end{aligned}$$

where we use (9-17) at the second equality and (9-13) at the third one. □

We take $\theta_\zeta\in K^{\mathrm{ac}}$ such that $\theta'_\zeta=\theta_\zeta^{2^{e-1}}$. Then we have $\theta_\zeta^2-\theta_\zeta=\delta_\zeta\eta_\zeta$. We put $N_\zeta'^{\mathrm{u}}=M_\zeta'^{\mathrm{u}}(\eta_\zeta,\theta_\zeta)$, which is a cyclic extension of $M_\zeta'^{\mathrm{u}}$ of order 4 by Lemma 9.8.

Lemma 9.9. *The character* $\xi'_{n,\zeta}$ *factors through* $\mathrm{Gal}(N_\zeta'^{\mathrm{u}}/M_\zeta'^{\mathrm{u}})$.

Proof. Let $\sigma\in\operatorname{Ker}\xi'_{n,\zeta}$. We take $\sigma_1,\sigma_2\in\operatorname{Ker}\xi'_{n,\zeta}$ such that $\sigma=\sigma_1\sigma_2^{-n_\sigma}$, $\sigma_1\in I_{M_\zeta'^{\mathrm{u}}}$ and $\Theta_\zeta(\sigma_2)=((1,b_0,c_0),-1)$. Then we have $(\bar{a}_{\sigma_1},\bar{b}_{\sigma_1},\bar{c}_{\sigma_1})\in R_0$, $\mathrm{Tr}_{\mathbb{F}_{2^e}/\mathbb{F}_2}(\bar{b}_{\sigma_1})=0$ and

$$\mathrm{Tr}_{\mathbb{F}_{2^e}/\mathbb{F}_2}\left(t\left(\bar{c}_{\sigma_1}+\sum_{0\le i<j\le e-1}\bar{b}_{\sigma_1}^{2^i+2^j}\right)\right)=0 \tag{9-18}$$

by (9-5). It suffices to show that $\sigma_i(\eta_\zeta)=\eta_\zeta$ and $\sigma_i(\theta'_\zeta)=\theta'_\zeta$ for $i=1,2$.

We have

$$\sigma_1(\eta_\zeta)-\eta_\zeta \equiv \sum_{i=0}^{e-1} b_{\sigma_1}^{2^i} \equiv 0 \mod \mathfrak{p}_{N'^{\mathrm{u}}_\zeta},$$

$$\sigma_2(\eta_\zeta)-\eta_\zeta \equiv \sum_{i=0}^{e-1} b_0^{2^i}+b_1^2-b_1 \equiv 0 \mod \mathfrak{p}_{N'^{\mathrm{u}}_\zeta}$$

by $\mathrm{Tr}_{\mathbb{F}_{2^e}/\mathbb{F}_2}(\bar{b}_{\sigma_1})=0$ and $b_1^2-b_1=s$. By Lemma 9.8, we have

$$\sigma_i(\eta_\zeta)-\eta_\zeta \in \mathbb{F}_2 \quad \text{for } i=1,2.$$

Hence, we have $\sigma_i(\eta_\zeta)=\eta_\zeta$ for $i=1,2$. We have

$$\sigma_1(\theta'_\zeta)-\theta'_\zeta=\sum_{i=0}^{e-1}\Bigl(t(\sigma_1(\gamma'_\zeta)-\gamma'_\zeta)\Bigr)^{2^i}.$$

Further, we have

$$\begin{aligned}\sigma_1(\gamma'_\zeta)-\gamma'_\zeta &\equiv c_{\sigma_1}+\sum_{i=0}^{e-1}(b_{\sigma_1})^{2^{i+1}}+\sum_{i=0}^{e-1} b_{\sigma_1}^{2^i}\sum_{i=0}^{e-1}\beta_\zeta^{2^i}+\sum_{0\le i<j\le e-1} b_{\sigma_1}^{2^i+2^j}\\ &\equiv c_{\sigma_1}+\sum_{0\le i<j\le e-1} b_{\sigma_1}^{2^i+2^j} \mod \mathfrak{p}_{N'^{\mathrm{u}}_\zeta},\end{aligned}$$

where we use (2-11) and $\bar{b}_{\sigma_1}\in\mathbb{F}_{2^e}$ at the first equality, and use $\mathrm{Tr}_{\mathbb{F}_{2^e}/\mathbb{F}_2}(\bar{b}_{\sigma_1})=0$ at the second one. This implies $\sigma_1(\theta'_\zeta)\equiv\theta'_\zeta \mod \mathfrak{p}_{N'^{\mathrm{u}}_\zeta}$ by (9-18). By a similar argument as above using Lemma 9.8, we obtain $\sigma_1(\theta'_\zeta)=\theta'_\zeta$.

It remains to show $\sigma_2(\theta'_\zeta)=\theta'_\zeta$. Using (9-16) and $\mathrm{Tr}_{\mathbb{F}_{2^e}/\mathbb{F}_2}(t)=1$, we see that

$$\sum_{i=0}^{e-1}(t\gamma'_\zeta)^{2^i}=\gamma'_\zeta+\sum_{1\le i\le j\le e-1} t^{2^j}\beta_\zeta^{2^i}+\sum_{0\le i<j\le e-1} t^{2^j}((\delta_\zeta-s)(\eta_\zeta-b_1))^{2^i}. \tag{9-19}$$

We put

$$\gamma''_\zeta=\gamma'_\zeta+\sum_{1\le i\le j\le e-1} t^{2^j}\beta_\zeta^{2^i}.$$

By $c_0^2+c_0=b_0^{2^e+1}$ and $t=b_0+b_0^{2^e}$ (see (2-18), (9-4)), we have

$$\sigma_2(\gamma_\zeta)-\gamma_\zeta \equiv c_0+\sum_{i=0}^{e-1}(b_0^{2^e}(\beta_\zeta+b_0))^{2^i} \equiv c_0^{2^e}+\sum_{i=0}^{e-1}((b_0+t)\,\beta_\zeta)^{2^i} \mod \mathfrak{p}_{N'^{\mathrm{u}}_\zeta}.$$

Then we have

$$\sigma_2(\gamma'_\zeta)-\gamma'_\zeta \equiv c_0^{2^e}+s(\eta_\zeta-b_1)+\sum_{i=0}^{e-1}(t\beta_\zeta)^{2^i}+\sum_{0\le i<j\le e-1} b_0^{2^i+2^j} \mod \mathfrak{p}_{N'^{\mathrm{u}}_\zeta}$$

by (9-4) and (9-14). Hence, we have

$$\begin{aligned}\sigma_2(\gamma''_\zeta)-\gamma''_\zeta &\equiv \sigma_2(\gamma'_\zeta)-\gamma'_\zeta+\sum_{1\le i\le j\le e-1} t^{2^{j+1}}(\beta_\zeta+b_0)^{2^i}-\sum_{1\le i\le j\le e-1} t^{2^j}\beta_\zeta^{2^i}\\ &\equiv \sigma_2(\gamma'_\zeta)-\gamma'_\zeta+t(\eta_\zeta-b_1)+\sum_{i=0}^{e-1}(t\beta_\zeta)^{2^i}+\sum_{1\le i<j\le e} b_0^{2^i}t^{2^j}\\ &\equiv c_0^{2^e}+s^2(\eta_\zeta-b_1)+\sum_{0\le i<j\le e-1} b_0^{2^i+2^j}+\sum_{1\le i<j\le e} b_0^{2^i}t^{2^j} \mod \mathfrak{p}_{N'^{\mathrm{u}}_\zeta},\end{aligned}$$

where we use (9-14) and $t\in\mathbb{F}_{2^e}$ at the second equality and $s^2+s=t$ at the last equality. We can check that

$$c_0^{2^e}+\sum_{0\le i<j\le e-1} b_0^{2^i+2^j}+\sum_{1\le i<j\le e} b_0^{2^i}t^{2^j}=st$$

by (2-17), (9-4) and $\mathrm{Tr}_{\mathbb{F}_{2^{2e}}/\mathbb{F}_2}(b_0)=1$. As a result, we obtain

$$\sigma_2(\gamma''_\zeta)-\gamma''_\zeta\equiv s^2\eta_\zeta+b_1s^2+st \mod \mathfrak{p}_{N'^{\mathrm{u}}_\zeta}.$$

Hence, by (9-15) and (9-19), we have

$$\sigma_2(\theta'_\zeta)-\theta'_\zeta\equiv\sum_{i=0}^{2^{e-1}+2^{e-2}} d_i\eta_\zeta^i \mod \mathfrak{p}_{N'^{\mathrm{u}}_\zeta}$$

for some $d_i\in k^{\mathrm{ac}}$. We have

$$d_0=b_1s^2+st+t\sum_{j=1}^{e-1}(b_1s)^{2^j}+b_1s\sum_{l=1}^{e-1}t^{2^l}+b_2^2-b_2=0.$$

This implies $\sigma_2(\theta'_\zeta)=\theta'_\zeta$, since we know that $\sigma_2(\theta'_\zeta)-\theta'_\zeta\in\mathbb{F}_2$ by Lemma 9.8. □

10. Refined Swan conductor

Let $\widetilde{K}\subset K^{\mathrm{ur}}$ be the unramified extension of K^{u} generated by $\mu_{p^{4pe}-1}(K^{\mathrm{ur}})$. For a finite field extension L of K in K^{ac}, we write $\tilde{L}$ for the composite field of L and $\widetilde{K}$ in K^{ac}. We write $\widetilde{M}'_\zeta$ for $\widetilde{M}'^{\mathrm{u}}_\zeta$. Then $\widetilde{N}_\zeta$ is a Galois extension of $\widetilde{M}'_\zeta$. By equations (9-8) and (9-9), we can take $\beta'_\zeta\in\widetilde{M}_\zeta$ such that

$$\beta'^{p^e}_\zeta-\hat{r}\beta'_\zeta=\delta_\zeta,\qquad \beta'_\zeta\equiv\beta_\zeta \mod \mathfrak{p}_{\widetilde{M}_\zeta}, \tag{10-1}$$

since there is $x\in\mathbb{F}_{2^{4e}}$ such that $x^{2^e}-x=\sum_{i=0}^{e-1}b_0^{2^i}$ if $p=2$. Then we have $\widetilde{M}_\zeta=\widetilde{M}'_\zeta(\beta'_\zeta)$ by Krasner's lemma.

Lemma 10.1. (1) *We have*

$$\psi_{\widetilde{N}_\zeta/\widetilde{M}'_\zeta}(v)=\begin{cases} v & \text{if } v\le 1,\\ p^e(v-1)+1 & \text{if } 1<v\le 2,\\ p^{e+1}(v-2)+p^e+1 & \text{if } 2<v.\end{cases} \tag{10-2}$$

(2) *We have*

$$\mathrm{Gal}(\widetilde{N}_\zeta/\widetilde{M}'_\zeta)_i = \begin{cases} \mathrm{Gal}(\widetilde{N}_\zeta/\widetilde{M}'_\zeta) & \text{if } i \le 1, \\ \mathrm{Gal}(\widetilde{N}_\zeta/\widetilde{M}_\zeta) & \text{if } 2 \le i \le p^e+1, \\ \{1\} & \text{if } p^e+2 \le i. \end{cases}$$

Proof. We have

$$\psi_{\widetilde{M}_\zeta/\widetilde{M}'_\zeta}(v) = \begin{cases} v & \text{if } v \le 1, \\ p^e(v-1)+1 & \text{if } v > 1, \end{cases}$$

$$\psi_{\widetilde{N}_\zeta/\widetilde{M}_\zeta}(v) = \begin{cases} v & \text{if } v \le p^e+1, \\ p(v-p^e-1)+p^e+1 & \text{if } v > p^e+1 \end{cases}$$

by (2-10), (10-1) and Lemma 5.4 noting that $\hat{r}$ has a (p^e-1)-st root in $\widetilde{M}'_\zeta$. Hence, claim (1) follows from $\psi_{\widetilde{N}_\zeta/\widetilde{M}'_\zeta} = \psi_{\widetilde{N}_\zeta/\widetilde{M}_\zeta} \circ \psi_{\widetilde{M}_\zeta/\widetilde{M}'_\zeta}$. Claim (2) follows from claim (1) and

$$\mathrm{Gal}(\widetilde{N}_\zeta/\widetilde{M}'_\zeta)_{p^e+1} \supset \mathrm{Gal}(\widetilde{N}_\zeta/\widetilde{M}_\zeta)_{p^e+1} = \mathrm{Gal}(\widetilde{N}_\zeta/\widetilde{M}_\zeta). \qquad \square$$

We set

$$\varpi_{\widetilde{M}'_\zeta} = \delta_\zeta^{-1}, \quad \varpi_{\widetilde{M}_\zeta} = \beta_\zeta^{-1} \quad \text{and} \quad \varpi_{\widetilde{N}_\zeta} = (\gamma_\zeta \varpi_{\widetilde{M}_\zeta}^{p^{e-1}})^{-1}.$$

Then the elements $\varpi_{\widetilde{M}'_\zeta}$, $\varpi_{\widetilde{M}_\zeta}$ and $\varpi_{\widetilde{N}_\zeta}$ are uniformizers of $\widetilde{M}'_\zeta$, $\widetilde{M}_\zeta$ and $\widetilde{N}_\zeta$ respectively. Let $\tilde{k}$ be the residue field of $\widetilde{K}$.

Lemma 10.2. *We have a commutative diagram*

$$\begin{array}{ccc} U_{\widetilde{N}_\zeta}^{p^e+1} & \xrightarrow{\mathrm{Nr}_{\widetilde{N}_\zeta/\widetilde{M}'_\zeta}} & U_{\widetilde{M}'_\zeta}^2 \\ \downarrow & & \downarrow \\ \tilde{k} & \xrightarrow{P} & \tilde{k} \end{array}$$

where the map P is given by $x \mapsto x^p - x$ and the vertical maps are given by

$$p_{\widetilde{N}_\zeta, -\gamma_\zeta^{-1}} \colon U_{\widetilde{N}_\zeta}^{p^e+1} \to \tilde{k}, \qquad 1 - x\gamma_\zeta^{-1} \mapsto \bar{x},$$

$$p_{\widetilde{M}'_\zeta, \hat{r}\varpi_{\widetilde{M}'_\zeta}^2} \colon U_{\widetilde{M}'_\zeta}^2 \to \tilde{k}, \qquad 1 + x\hat{r}\varpi_{\widetilde{M}'_\zeta}^2 \mapsto \bar{x}.$$

Proof. The norm maps $\mathrm{Nr}_{\widetilde{N}_\zeta/\widetilde{M}_\zeta}$ and $\mathrm{Nr}_{\widetilde{M}_\zeta/\widetilde{M}'_\zeta}$ induce

$$U_{\widetilde{N}_\zeta}^{p^e+1}/U_{\widetilde{N}_\zeta}^{p^e+2} \to U_{\widetilde{M}_\zeta}^{p^e+1}/U_{\widetilde{M}_\zeta}^{p^e+2}, \quad 1 - u\gamma_\zeta^{-1} \mapsto 1 - (u^p - u)\varpi_{\widetilde{M}_\zeta}^{p^e+1},$$

$$U_{\widetilde{M}_\zeta}^{p^e+1}/U_{\widetilde{M}_\zeta}^{p^e+2} \to U_{\widetilde{M}'_\zeta}^2/U_{\widetilde{M}'_\zeta}^3, \quad 1 - u\varpi_{\widetilde{M}_\zeta}^{p^e+1} = 1 - u\beta_\zeta'^{-1}\varpi_{\widetilde{M}'_\zeta} \mapsto 1 + u\hat{r}\varpi_{\widetilde{M}'_\zeta}^2$$

respectively by Lemma 5.5(1) and calculations of the norms. Hence, the claim follows. $\square$

For any finite extension M of K, we write ψ_M for the composite $\psi_K \circ \mathrm{Tr}_{M/K}$.

Lemma 10.3. *We have* $\mathrm{rsw}(\xi_{n,\zeta}|_{W_{M'^{\mathrm{u}}_\zeta}}, \psi_{M'^{\mathrm{u}}_\zeta}) = -n'\delta_\zeta^{-(p^e+1)} \mod U^1_{M'^{\mathrm{u}}_\zeta}$.

Proof. We put $\tilde{\xi}_{n,\zeta} = \xi_{n,\zeta}|_{W_{\widetilde{M}'_\zeta}}$, and regard it as a character of $\widetilde{M}'^{\times}_\zeta$. By (2-12), Lemmas 5.5(1) and Lemma 10.1, the restriction of $\tilde{\xi}_{n,\zeta}$ to $U^2_{\widetilde{M}'_\zeta}$ is given by the composition

$$U^2_{\widetilde{M}'_\zeta} \xrightarrow{\mathrm{Art}_{\widetilde{M}'_\zeta}} \mathrm{Gal}(\widetilde{N}_\zeta/\widetilde{M}_\zeta) \simeq \mathbb{F}_p \xrightarrow{\psi_0} \overline{\mathbb{Q}}_\ell^{\times},$$

where the isomorphism $\mathrm{Gal}(\widetilde{N}_\zeta/\widetilde{M}_\zeta) \simeq \mathbb{F}_p$ is given by $\sigma \mapsto \overline{\sigma(\gamma_\zeta) - \gamma_\zeta}$. We define $p_{\widetilde{N}_\zeta, -\gamma_\zeta^{-1}}$ as in Lemma 10.2. For $u \in \mathcal{O}_{\widetilde{M}'_\zeta}$, we put $\sigma_u = \mathrm{Art}_{\widetilde{M}'_\zeta}(1 + u\hat{r}\varpi^2_{\widetilde{M}'_\zeta})$ and then have

$$\begin{aligned} \tilde{\xi}_{n,\zeta}(1 + u\hat{r}\varpi^2_{\widetilde{M}'_\zeta}) &= \psi_0(\overline{\sigma_u(\gamma_\zeta) - \gamma_\zeta}) \\ &= \psi_0\left(p_{\widetilde{N}_\zeta, -\gamma_\zeta^{-1}}\left(\frac{\gamma_\zeta}{\sigma_u(\gamma_\zeta)}\right)\right) \\ &= \psi_0\left(p_{\widetilde{N}_\zeta, -\gamma_\zeta^{-1}}\left(\frac{\sigma_u(\varpi_{\widetilde{N}_\zeta})}{\varpi_{\widetilde{N}_\zeta}}\right)\right) = \psi_0 \circ \mathrm{Tr}_{\tilde{k}/\mathbb{F}_p}(\bar{u}), \end{aligned} \tag{10-3}$$

where we use Lemmas 5.5(2) and 10.2 at the last equality. Since we have

$$\overline{\mathrm{Tr}_{\widetilde{M}'_\zeta/\widetilde{T}_\zeta}(\delta_\zeta^{p^e-1}u)} = -r^{-1}\bar{u}$$

for $u \in \mathcal{O}_{\widetilde{M}'_\zeta}$, we obtain

$$\tilde{\xi}_{n,\zeta}(1+x) = \psi_{\widetilde{M}'_\zeta}(-n'^{-1}\delta_\zeta^{p^e+1}x)$$

for $x \in \mathfrak{p}^2_{\widetilde{M}'_\zeta}$ by (10-3). This implies

$$\xi_{n,\zeta}(1+x) = \psi_{M'^{\mathrm{u}}_\zeta}(-n'^{-1}\delta_\zeta^{p^e+1}x) \tag{10-4}$$

for $x \in \mathfrak{p}^2_{M'^{\mathrm{u}}_\zeta}$, because $\mathrm{Tr}_{\tilde{k}/k_N} : \tilde{k} \to k_N$ is surjective. The claim follows from (10-4) and Proposition 5.3(1). □

Lemma 10.4. *We have* $\mathrm{rsw}(\tau_{n,\zeta,\chi,c}, \psi_{E_\zeta}) = n'\varphi'_\zeta \mod U^1_{E_\zeta}$.

Proof. By Proposition 5.2(1), we may assume that $\chi = 1$, $c = 1$. By Proposition 9.5 and Lemma 10.3, we have

$$\mathrm{rsw}(\tau_{n,\zeta}|_{W_{T^{\mathrm{u}}_\zeta}}, \psi_{T^{\mathrm{u}}_\zeta}) = \mathrm{Nr}_{M'^{\mathrm{u}}_\zeta/T^{\mathrm{u}}_\zeta}(\mathrm{rsw}(\xi_{n,\zeta}, \psi_{M'^{\mathrm{u}}_\zeta})) = n'\varphi'_\zeta \mod U^1_{T^{\mathrm{u}}_\zeta}. \tag{10-5}$$

Since T^{u}_ζ is a tamely ramified extension of E_ζ, we have

$$\mathrm{rsw}(\tau_{n,\zeta}, \psi_{E_\zeta}) = \mathrm{rsw}(\tau_{n,\zeta}|_{W_{T^{\mathrm{u}}_\zeta}}, \psi_{T^{\mathrm{u}}_\zeta}) \mod U^1_{T^{\mathrm{u}}_\zeta} \tag{10-6}$$

by Proposition 5.2(2). The claim follows from (10-5) and (10-6). □

Proposition 10.5. *We have* $\mathrm{rsw}(\tau_{\zeta,\chi,c}, \psi_K) = \mathrm{rsw}(\pi_{\zeta,\chi,c}, \psi_K)$.

Proof. By $\tau_{\zeta,\chi,c} = \mathrm{Ind}_{E_\zeta/K}\, \tau_{n,\zeta,\chi,c}$, we have

$$\mathrm{rsw}(\tau_{\zeta,\chi,c}, \psi_K) = \mathrm{Nr}_{E_\zeta/K}(\mathrm{rsw}(\tau_{n,\zeta,\chi,c}, \psi_{E_\zeta})). \tag{10-7}$$

Hence, the claim follows from Lemmas 4.5 and 10.4. ☐

Lemma 10.6. *We have* $\mathrm{Sw}(\tau_{\zeta,\chi,c}) = 1$.

Proof. This follows from Lemma 10.4 and (10-7). ☐

Lemma 10.7. *The representation* $\tau_{\zeta,\chi,c}$ *is irreducible.*

Proof. We know that the restriction of $\tau_{n,\zeta,\chi,c}$ to the wild inertia subgroup of W_{E_ζ} is irreducible by Corollary 2.6. Assume that $\tau_{\zeta,\chi,c}$ is not irreducible. Then we have an irreducible factor τ' of $\tau_{\zeta,\chi,c}$ such that $\mathrm{Sw}(\tau') = 0$, by Lemma 10.6 and the additivity of Sw. Then, the restriction of τ' to the wild inertia subgroup of W_K is trivial by $\mathrm{Sw}(\tau') = 0$. On the other hand, we have an injective homomorphism $\tau_{n,\zeta,\chi,c} \to \tau'|_{W_{E_\zeta}}$ by Frobenius reciprocity. This is a contradiction. ☐

Proposition 10.8. *The representation* $\tau_{\zeta,\chi,c}$ *is irreducible of Swan conductor* 1.

Proof. This follows from Lemmas 10.6 and 10.7. ☐

11. Epsilon factor

11A. ***Reduction to special cases.*** In this subsection, we show the equality

$$\varepsilon(\tau_{\zeta,\chi,c}, \psi_K) = \varepsilon(\pi_{\zeta,\chi,c}, \psi_K)$$

of epsilon factors assuming some results in the special case where $n = p^e$, $\mathrm{ch}\, K = p$ and $f = 1$. The results in the special case will be proved in the next subsection.

Lemma 11.1. *We have*

$$\lambda(E_\zeta/K, \psi_K) = \begin{cases} \left(\frac{q}{n'}\right) & \text{if } n' \text{ is odd}, \\ -\left(-\epsilon(p)\left(\frac{2n'}{p}\right)\left(\frac{-1}{p}\right)^{(n'/2)-1}\right)^f & \text{if } n' \text{ is even}, \end{cases}$$

$$\lambda(T_\zeta^{\mathrm{u}}/E_\zeta, \psi_{E_\zeta}) = \begin{cases} -(-1)^{\frac{1}{4}(p-1)fN} & \text{if } p \neq 2, \\ \left(\frac{q}{p^e+1}\right) & \text{if } p = 2. \end{cases}$$

Proof. We have

$$\lambda(T_\zeta^{\mathrm{u}}/E_\zeta, \psi_{E_\zeta}) = \lambda(T_\zeta^{\mathrm{u}}/E_\zeta^{\mathrm{u}}, \psi_{E_\zeta^{\mathrm{u}}})\, \lambda(E_\zeta^{\mathrm{u}}/E_\zeta, \psi_{E_\zeta})^{p^e+1} = \lambda(T_\zeta^{\mathrm{u}}/E_\zeta^{\mathrm{u}}, \psi_{E_\zeta^{\mathrm{u}}}).$$

If $p \neq 2$, then we have

$$\lambda(T_\zeta^{\mathrm{u}}/E_\zeta^{\mathrm{u}}, \psi_{E_\zeta^{\mathrm{u}}}) = -\left(-\epsilon(p)\left(\frac{2n'}{p}\right)\left(\frac{-1}{p}\right)^{(p^e-1)/2}\right)^{fN} = -(-1)^{\frac{1}{4}(p-1)fN}$$

by Lemma 8.4, since fN is even. The other assertions immediately follow from Lemma 8.4. ☐

Lemma 11.2. *We have*

$$\lambda(M'^{\mathrm{u}}_{\zeta}/T^{\mathrm{u}}_{\zeta}, \psi_{T^{\mathrm{u}}_{\zeta}}) = \begin{cases} (-1)^f & \text{if } p=2 \text{ and } e \leq 2, \\ \left(\frac{r}{k_N}\right), & \text{otherwise.} \end{cases}$$

Proof. Let $K_{(0)}$ and $K_{(p)}$ be nonarchimedean local fields of characteristic 0 and p respectively. Assume that the residue fields of $K_{(0)}$ and $K_{(p)}$ are isomorphic to k. We take uniformizers $\varpi_{(0)}$ and $\varpi_{(p)}$ of $K_{(0)}$ and $K_{(p)}$ respectively. We define $T^{\mathrm{u}}_{\zeta,(0)}$ similarly as T^{u}_{ζ} starting from $K_{(0)}$. We use similar notations also for other objects in the characteristic zero side and the positive characteristic side. We have the isomorphism

$$\mathcal{O}_{T^{\mathrm{u}}_{\zeta,(p)}}/\mathfrak{p}^2_{T^{\mathrm{u}}_{\zeta,(p)}} \xrightarrow{\sim} \mathcal{O}_{T^{\mathrm{u}}_{\zeta,(0)}}/\mathfrak{p}^2_{T^{\mathrm{u}}_{\zeta,(0)}}, \quad \xi_0 + \xi_1 \varpi_{T^{\mathrm{u}}_{\zeta,(p)}} \mapsto \hat{\xi}_0 + \hat{\xi}_1 \varpi_{T^{\mathrm{u}}_{\zeta,(0)}}$$

of algebras, where $\xi_0, \xi_1 \in k$. Hence, it suffices to show the claim in one of the characteristic zero side and the positive characteristic side by [Deligne 1984, Proposition 3.7.1], since $\mathrm{Gal}(M'^{\mathrm{u}}_{\zeta,(p)}/T^{\mathrm{u}}_{\zeta,(p)})^2 = 1$ and $\mathrm{Gal}(M'^{\mathrm{u}}_{\zeta,(0)}/T^{\mathrm{u}}_{\zeta,(0)})^2 = 1$, where we use upper numbering filtration of Galois groups.

First, we consider the case where $p \neq 2$ and $\mathrm{ch}\, K = p$. Then, we have $d_{M'^{\mathrm{u}}_{\zeta}/T^{\mathrm{u}}_{\zeta}} = \hat{r}$ by Proposition 6.2 and the fact that fN is even. Hence, $\delta_{M'^{\mathrm{u}}_{\zeta}/T^{\mathrm{u}}_{\zeta}}$ is unramified by (6-1). Hence, we have

$$\lambda(M'^{\mathrm{u}}_{\zeta}/T^{\mathrm{u}}_{\zeta}, \psi_{T^{\mathrm{u}}_{\zeta}}) = \varepsilon(\delta_{M'^{\mathrm{u}}_{\zeta}/T^{\mathrm{u}}_{\zeta}}, \psi_{T^{\mathrm{u}}_{\zeta}})^{p^e} = \left(\frac{r}{k_N}\right)$$

by [Henniart 1984, Proposition 2; Bushnell and Henniart 2006, Proposition 23.5] and (6-1).

We consider the case where $p=2$. Assume that $e \geq 3$ and $\mathrm{ch}\, K = 0$. We have $D = 2^e \delta_{\zeta}^{2^e-1} + 1$ in the notation of Proposition 6.2 with $(L, K, a) = (M'^{\mathrm{u}}_{\zeta}, T^{\mathrm{u}}_{\zeta}, \delta_{\zeta})$. Then, we have $D \in (M'^{\mathrm{u}\times}_{\zeta})^2$. Hence, we have $\kappa_D = 1$, $d_{M'^{\mathrm{u}}_{\zeta}/T^{\mathrm{u}}_{\zeta}} = 1$ and

$$w_2(\mathrm{Ind}_{M'^{\mathrm{u}}_{\zeta}/T^{\mathrm{u}}_{\zeta}} 1) = 1$$

by Proposition 6.2 and $\binom{p^e}{4} \equiv 0 \bmod 2$. Therefore we have

$$\lambda(M'^{\mathrm{u}}_{\zeta}/T^{\mathrm{u}}_{\zeta}, \psi_{T^{\mathrm{u}}_{\zeta}}) = \varepsilon(\mathrm{Ind}_{M'^{\mathrm{u}}_{\zeta}/T^{\mathrm{u}}_{\zeta}} 1, \psi_{T^{\mathrm{u}}_{\zeta}}) = \varepsilon(1^{\oplus p^e}, \psi_{T^{\mathrm{u}}_{\zeta}}) = 1$$

by Theorem 6.1.

Assume that $e=2$ and $\mathrm{ch}\, K = 2$. Then we see that $d^+_{M'^{\mathrm{u}}_{\zeta}/T^{\mathrm{u}}_{\zeta}} = 1$ by Definition 6.3. Hence, $\delta_{M'^{\mathrm{u}}_{\zeta}/T^{\mathrm{u}}_{\zeta}}$ is the unramified character satisfying

$$\delta_{M'^{\mathrm{u}}_{\zeta}/T^{\mathrm{u}}_{\zeta}}(\varpi_{T^{\mathrm{u}}_{\zeta}}) = (-1)^f$$

by Theorem 6.4. Then we see that

$$\lambda(M'^{\mathrm{u}}_{\zeta}/T^{\mathrm{u}}_{\zeta}, \psi_{T^{\mathrm{u}}_{\zeta}}) = \varepsilon(\mathrm{Ind}_{M'^{\mathrm{u}}_{\zeta}/T^{\mathrm{u}}_{\zeta}} 1, \psi_{T^{\mathrm{u}}_{\zeta}}) = \varepsilon(\delta_{M'^{\mathrm{u}}_{\zeta}/T^{\mathrm{u}}_{\zeta}} \oplus 1^{\oplus 3}, \psi_{T^{\mathrm{u}}_{\zeta}}) = (-1)^f,$$

where we use Theorem 6.1 at the second equality.

Assume that $e=1$ and $\mathrm{ch}\, K=2$. Let $\kappa_{M'^{\mathrm{u}}_\zeta/T^{\mathrm{u}}_\zeta}$ be the quadratic character associated to the extension M'^{u}_ζ over T^{u}_ζ. Then we have

$$\lambda(M'^{\mathrm{u}}_\zeta/T^{\mathrm{u}}_\zeta, \psi_{T^{\mathrm{u}}_\zeta}) = \varepsilon(\kappa_{M'^{\mathrm{u}}_\zeta/T^{\mathrm{u}}_\zeta}, \psi_{T^{\mathrm{u}}_\zeta})$$

by Theorem 6.1 similarly as above. We can check that the norm map $\mathrm{Nr}_{M'^{\mathrm{u}}_\zeta/T^{\mathrm{u}}_\zeta}$ induces

$$U^1_{M'^{\mathrm{u}}_\zeta}/U^2_{M'^{\mathrm{u}}_\zeta} \to U^1_{T^{\mathrm{u}}_\zeta}/U^2_{T^{\mathrm{u}}_\zeta}, \quad 1+u\delta_\zeta^{-1} \mapsto 1+(u^2-u)\,\alpha_\zeta.$$

Then, by Lemma 5.5, we have

$$\begin{aligned}\kappa_{M'^{\mathrm{u}}_\zeta/T^{\mathrm{u}}_\zeta}(1+\alpha_\zeta x) &= \psi_0\big(\mathrm{Art}_{T^{\mathrm{u}}_\zeta}(1+\alpha_\zeta x)(\delta_\zeta)-\delta_\zeta\big)\\ &= \psi_0\left(p_{M'^{\mathrm{u}}_\zeta,\delta_\zeta^{-1}}\left(\frac{\mathrm{Art}_{T^{\mathrm{u}}_\zeta}(1+\alpha_\zeta x)(\delta_\zeta^{-1})}{\delta_\zeta^{-1}}\right)\right)\\ &= \psi_0(\mathrm{Tr}_{k/\mathbb{F}_p}(\bar{x}))\end{aligned} \tag{11-1}$$

for $x \in \mathcal{O}_{T^{\mathrm{u}}_\zeta}$ noting that $k_N = k$. Hence, we have $\mathrm{rsw}(\kappa_{M'^{\mathrm{u}}_\zeta/T^{\mathrm{u}}_\zeta}, \psi_{T^{\mathrm{u}}_\zeta}) = \alpha_\zeta$ by Proposition 5.3(1). By Proposition 5.3(2), we have

$$\varepsilon(\kappa_{M'^{\mathrm{u}}_\zeta/T^{\mathrm{u}}_\zeta}, \psi_{T^{\mathrm{u}}_\zeta}) = \kappa_{M'^{\mathrm{u}}_\zeta/T^{\mathrm{u}}_\zeta}(\alpha_\zeta) = \kappa_{M'^{\mathrm{u}}_\zeta/T^{\mathrm{u}}_\zeta}(1+\alpha_\zeta) = (-1)^f,$$

where we use $\mathrm{Nr}_{M'^{\mathrm{u}}_\zeta/T^{\mathrm{u}}_\zeta}(\delta_\zeta) = \alpha_\zeta^{-1}+1$ and (11-1) at the last equality. □

Lemma 11.3. *We have*

$$\mathrm{Tr}_{M'^{\mathrm{u}}_\zeta/T^{\mathrm{u}}_\zeta}(\delta_\zeta^i) = \begin{cases} 0 & \text{if } 1 \le i \le p^e-2,\\ \hat{r}^{-1}(p^e-1) & \text{if } i = p^e-1.\end{cases}$$

Proof. Vanishing for $1 \le i \le p^e-2$ follows from (9-9). We have also

$$\mathrm{Tr}_{M'^{\mathrm{u}}_\zeta/T^{\mathrm{u}}_\zeta}(\delta_\zeta^{p^e-1}) = \mathrm{Tr}_{M'^{\mathrm{u}}_\zeta/T^{\mathrm{u}}_\zeta}(\hat{r}^{-1}+\delta_\zeta^{-1}(-\alpha_\zeta^{-1}+\epsilon_1)) = \hat{r}^{-1}(p^e-1)$$

by (9-9). □

Lemma 11.4. *We have*

$$\delta_{T^{\mathrm{u}}_\zeta/E_\zeta}(\mathrm{rsw}(\tau_{n,\zeta}, \psi_{E_\zeta})) = \begin{cases} 1 & \text{if } p \neq 2,\\ \left(\frac{q}{p^e+1}\right) & \text{if } p=2.\end{cases}$$

Proof. If $p=2$, the claim follows from Lemmas 8.3(1) and 10.4, since T^{u}_ζ is totally ramified over E_ζ.

Assume that $p \neq 2$. Then we have $d_{T^{\mathrm{u}}_\zeta/E^{\mathrm{u}}_\zeta} = (-1)^{(p^e+1)/2}\varphi'_\zeta$ by Proposition 6.2. Hence, we have $\delta_{T^{\mathrm{u}}_\zeta/E^{\mathrm{u}}_\zeta}((-1)^{(p^e-1)/2}\varphi'_\zeta) = 1$ by Lemma 8.3(2). Therefore, we have

$$\begin{aligned}\delta_{T^{\mathrm{u}}_\zeta/E_\zeta}(\mathrm{rsw}(\tau_{n,\zeta}, \psi_{E_\zeta})) &= \delta_{T^{\mathrm{u}}_\zeta/E^{\mathrm{u}}_\zeta}(n'\varphi'_\zeta)\\ &= \delta_{T^{\mathrm{u}}_\zeta/E^{\mathrm{u}}_\zeta}(n'(-1)^{(p^e-1)/2}) = \left(\frac{n'(-1)^{(p^e-1)/2}}{q^N}\right) = 1\end{aligned}$$

by [Gallagher 1965, (1)], Lemmas 8.3(2), 10.4 and the fact that fN is even. □

Lemma 11.5. *Assume that $n = p^e$. Then we have $\varepsilon(\tau_{\zeta,\chi,c}, \psi_K) \equiv \varepsilon(\pi_{\zeta,\chi,c}, \psi_K)$ mod $\mu_{p^e}(\mathbb{C})$.*

Proof. Let π be the representation of $\mathrm{GL}_n(K)$ corresponding to $\tau_{\zeta,\chi,c}$ by the local Langlands correspondence. By the proof of [Bushnell and Henniart 2014, Proposition 2.2], Propositions 8.6 and 10.5, we have

$$\pi \simeq c\text{-}\mathrm{Ind}_{L_\zeta^\times U_{\mathfrak{J}}^1}^{\mathrm{GL}_n(K)} \Lambda$$

for a character $\Lambda : L_\zeta^\times U_{\mathfrak{J}}^1 \to \mathbb{C}^\times$ which coincides with $\Lambda_{\zeta,\chi,c}$ on $K^\times U_{\mathfrak{J}}^1$. Then, the claim follows from [Bushnell and Henniart 2014, Lemma 2.2(1)], because $L_\zeta^\times U_{\mathfrak{J}}^1/(K^\times U_{\mathfrak{J}}^1)$ is the cyclic group of order p^e. □

Proposition 11.6. *We have $\varepsilon(\tau_{\zeta,\chi,c}, \psi_K) = \varepsilon(\pi_{\zeta,\chi,c}, \psi_K)$.*

Proof. By Proposition 3.2 and $\tau_{\zeta,\chi,c} \simeq \mathrm{Ind}_{E_\zeta/K}\, \tau_{n,\zeta,\chi,c}$, it suffices to show that

$$\lambda(E_\zeta/K, \psi_K)^{p^e} \varepsilon(\tau_{n,\zeta,\chi,c}, \psi_{E_\zeta}) = (-1)^{n-1+\epsilon_0 f} \chi(n')\, c.$$

By Lemma 10.4, we may assume $\chi = 1$ and $c = 1$. Hence, it suffices to show

$$\lambda(E_\zeta/K, \psi_K)^{p^e} \varepsilon(\tau_{n,\zeta}, \psi_{E_\zeta}) = (-1)^{n-1+\epsilon_0 f}. \tag{11-2}$$

Assuming that (11-2) is proved for $n = p^e$, we show (11-2) for general n. Let $\tau'_{n,\zeta}$ denote the representation of W_{E_ζ} given by Θ_ζ in (2-12) and τ_{p^e}. We put $\psi'_{E_\zeta} = n'^{-1}\psi_{E_\zeta}$. Applying the result for $n = p^e$ to E_ζ, φ'_ζ in place of K, ϖ, we have

$$\varepsilon(\tau'_{n,\zeta}, \psi'_{E_\zeta}) = (-1)^{p^e-1+\epsilon'_0 f},$$

where ϵ'_0 denotes ϵ_0 for $n = p^e$. Since $\det \tau'_{n,\zeta}$ is unramified as in the proof of Proposition 8.6, we have

$$\varepsilon(\tau'_{n,\zeta}, \psi_{E_\zeta}) = \det \tau'_{n,\zeta}(n')\, \varepsilon(\tau'_{n,\zeta}, \psi'_{E_\zeta}) = (-1)^{p^e-1+\epsilon'_0 f}. \tag{11-3}$$

We note that the inflation of the character in (2-9) by Θ_ζ factors through

$$W_{E_\zeta} \to \{\pm 1\} \times \mathbb{Z}, \quad \sigma \mapsto (a_\sigma^{(p^e+1)/2}, f n_\sigma).$$

If $p \neq 2$, then we have $(n'\varphi'_\zeta, -\varphi'_\zeta)_{E_\zeta} = \left(\frac{n'}{q}\right)$, where

$$(\ ,\ )_{E_\zeta} : E_\zeta^\times/(E_\zeta^\times)^2 \times E_\zeta^\times/(E_\zeta^\times)^2 \to \{\pm 1\}$$

denotes the Hilbert symbol. Hence, we have

$$\frac{\varepsilon(\tau_{n,\zeta}, \psi_{E_\zeta})}{\varepsilon(\tau'_{n,\zeta}, \psi_{E_\zeta})} = \begin{cases} \left(\frac{n'}{q}\right)^{n-p^e}\left(\left(\frac{n'}{p}\right)^n\left(-\epsilon(p)\left(\frac{-2}{p}\right)\right)^{n-p^e}\right)^f & \text{if } p \neq 2, \\ (-1)^{\left(\frac{1}{8}n(n-2)-\frac{1}{8}2^e(2^e-2)\right)f} & \text{if } p = 2 \end{cases} \tag{11-4}$$

by (2-9), Lemmas 4.2 and 10.4. Then we have (11-2) by Lemma 11.1, equations (11-3) and (11-4).

Therefore, we may assume that $n = p^e$. By Lemmas 11.1 and 11.5, it suffices to show that

$$\varepsilon(\tau_{n,\zeta}, \psi_{E_\zeta})^{N(p^e+1)} = \begin{cases} 1 & \text{if } p \neq 2, \\ (-1)^{1+\epsilon_0 f} & \text{if } p = 2. \end{cases}$$

By Proposition 5.1, we have

$$\varepsilon(\tau_{n,\zeta}, \psi_{E_\zeta})^{N(p^e+1)} = \delta_{T_\zeta^{\mathrm{u}}/E_\zeta}(\mathrm{rsw}(\tau_{n,\zeta}, \psi_{E_\zeta}))^{-1}\lambda(T_\zeta^{\mathrm{u}}/E_\zeta, \psi_{E_\zeta})^{p^e}\varepsilon(\tau_{n,\zeta}|_{W_{T_\zeta^{\mathrm{u}}}}, \psi_{T_\zeta^{\mathrm{u}}}).$$

By this, Lemmas 11.1 and 11.4, it suffices to show that

$$\varepsilon(\tau_{n,\zeta}|_{W_{T_\zeta^{\mathrm{u}}}}, \psi_{T_\zeta^{\mathrm{u}}}) = \begin{cases} -(-1)^{\frac{1}{4}(p-1)fN} & \text{if } p \neq 2, \\ (-1)^{1+\epsilon_0 f}\left(\frac{q}{p^e+1}\right) & \text{if } p = 2. \end{cases}$$

This follows from Lemma 11.2 and Proposition 11.7. □

We set $\varpi_{M_\zeta'^{\mathrm{u}}} = \delta_\zeta^{-1}$.

Proposition 11.7. *Assume that $n = p^e$. Then we have*

$$\varepsilon(\xi_{n,\zeta}, \psi_{M_\zeta'^{\mathrm{u}}}) = \begin{cases} -(-1)^{\frac{1}{4}(p-1)fN}\left(\frac{r}{k_N}\right) & \text{if } p \neq 2, \\ (-1)^{1+\epsilon_0 f} & \text{if } p = 2. \end{cases}$$

Proof. First, we reduce the problem to the positive characteristic case. Assume that $\mathrm{ch}\, K = 0$. Take a positive characteristic local field $K_{(p)}$ whose residue field is isomorphic to k. We define $M_{\zeta,(p)}'^{\mathrm{u}}$ similarly as $M_\zeta'^{\mathrm{u}}$ starting from $K_{(p)}$. We use similar notations also for other objects in the positive characteristic side. Then we have the isomorphism

$$\mathcal{O}_{M_{\zeta,(p)}'^{\mathrm{u}}}/\mathfrak{p}^3_{M_{\zeta,(p)}'^{\mathrm{u}}} \xrightarrow{\sim} \mathcal{O}_{M_\zeta'^{\mathrm{u}}}/\mathfrak{p}^3_{M_\zeta'^{\mathrm{u}}}, \quad \xi_0+\xi_1\varpi_{M_{\zeta,(p)}'^{\mathrm{u}}}+\xi_2\varpi^2_{M_{\zeta,(p)}'^{\mathrm{u}}} \mapsto \hat{\xi}_0+\hat{\xi}_1\varpi_{M_\zeta'^{\mathrm{u}}}+\hat{\xi}_2\varpi^2_{M_\zeta'^{\mathrm{u}}}$$

of algebras, where $\xi_1, \xi_2, \xi_3 \in k$. Hence, the problem is reduced to the positive characteristic case by [Deligne 1984, Proposition 3.7.1].

We may assume $K = \mathbb{F}_q((t))$. We put $K_{\langle 1\rangle} = \mathbb{F}_p((t))$. We define $M_{\zeta,\langle 1\rangle}'^{\mathrm{u}}$ similarly as $M_\zeta'^{\mathrm{u}}$ starting from $K_{\langle 1\rangle}$. We use similar notations also for other objects in the $K_{\langle 1\rangle}$-case. We put $f' = [M_\zeta'^{\mathrm{u}} : M_{\zeta,\langle 1\rangle}'^{\mathrm{u}}]$. We have

$$\delta_{M_\zeta'^{\mathrm{u}}/M_{\zeta,\langle 1\rangle}'^{\mathrm{u}}}(\mathrm{rsw}(\xi_{n,\zeta,\langle 1\rangle}, \psi_{M_{\zeta,\langle 1\rangle}'^{\mathrm{u}}})) = (-1)^{f'-1}$$

by Lemma 10.3. We have $\lambda(M_\zeta'^{\mathrm{u}}/M_{\zeta,\langle 1\rangle}'^{\mathrm{u}}, \psi_{M_{\zeta,\langle 1\rangle}'^{\mathrm{u}}}) = 1$, since the level of $\psi_{M_{\zeta,\langle 1\rangle}'^{\mathrm{u}}}$ is $2-p^e$ by Lemma 11.3. Then, we obtain

$$\varepsilon(\xi_{n,\zeta}, \psi_{M_\zeta'^{\mathrm{u}}}) = (-1)^{f'-1}\varepsilon(\xi_{n,\zeta,\langle 1\rangle}, \psi_{M_{\zeta,\langle 1\rangle}'^{\mathrm{u}}})^{f'} \tag{11-5}$$

by Proposition 5.1. By (11-5), the problem is reduced to the case where $f = 1$. In this case, the claim follows from Lemmas 11.11 and 11.16. □

11B. ***Special cases.*** We assume that $n = p^e$, $\mathrm{ch}\, K = p$ and $f = 1$ in this subsection.

11B1. *Odd case.* Assume that $p \neq 2$.

Lemma 11.8. *We have* $\psi_{M'^{\mathrm{u}}_\zeta}(-\delta_\zeta^{p^e+1}(1+x\varpi_{M'^{\mathrm{u}}_\zeta}))=1$ *for* $x \in k_N$.

Proof. For $x \in k_N$, we have

$$\psi_{M'^{\mathrm{u}}_\zeta}(-\delta_\zeta^{p^e+1}(1+x\varpi_{M'^{\mathrm{u}}_\zeta}))=\psi_{M'^{\mathrm{u}}_\zeta}(-(r^{-1}\delta_\zeta-\alpha_\zeta^{-1})(\delta_\zeta+x))=\psi_{M'^{\mathrm{u}}_\zeta}(-r^{-1}\delta_\zeta^2),$$

because $\mathrm{Tr}_{M'^{\mathrm{u}}_\zeta/T^{\mathrm{u}}_\zeta}(\delta_\zeta)=0$ and $[M'^{\mathrm{u}}_\zeta : T^{\mathrm{u}}_\zeta]=p^e$. If $p^e \neq 3$, then we have the claim, because $\mathrm{Tr}_{M'^{\mathrm{u}}_\zeta/T^{\mathrm{u}}_\zeta}(\delta_\zeta^2)=0$.

We assume that $p^e=3$. Then we have

$$\psi_{M'^{\mathrm{u}}_\zeta}(-r^{-1}\delta_\zeta^2)=\psi_{T^{\mathrm{u}}_\zeta}(-2r^{-2})=\psi_0(\mathrm{Tr}_{k_N/\mathbb{F}_p}(-2r^{-2}))=\psi_0(-N\,\mathrm{Tr}_{\mathbb{F}_{p^2}/\mathbb{F}_p}(r^{-2}))=1$$

by $\mathrm{Tr}_{M'^{\mathrm{u}}_\zeta/T^{\mathrm{u}}_\zeta}(\delta_\zeta^2)=2r^{-1}$ and $r^4=-1$. □

Let θ_ζ be as in (9-10).

Lemma 11.9. *We have*

$$\mathrm{Nr}_{N'^{\mathrm{u}}_\zeta/M'^{\mathrm{u}}_\zeta}(1+x\theta_\zeta^{(p-1)/2}\varpi_{M'^{\mathrm{u}}_\zeta})\equiv 1+(-2r)^{(1-p)/2}x^p\varpi_{M'^{\mathrm{u}}_\zeta}+\frac{x^2}{2}\varpi^2_{M'^{\mathrm{u}}_\zeta} \mod \mathfrak{p}^3_{M'^{\mathrm{u}}_\zeta}$$

for $x \in k_N$.

Proof. We put $T=1+x\theta_\zeta^{(p-1)/2}\varpi_{M'^{\mathrm{u}}_\zeta}$. By $\theta_\zeta^p-\theta_\zeta=(-2r)^{-1}\delta_\zeta^2$ in (9-11), we have

$$\theta_\zeta=-\frac{1}{2r}\delta_\zeta^2\big((x^{-1}(T-1)\,\delta_\zeta)^2-1\big)^{-1}.$$

Substituting this to $x^{-1}(T-1)\,\delta_\zeta=\theta_\zeta^{(p-1)/2}$, we have

$$(T^2-2T+1-x^2\varpi^2_{M'^{\mathrm{u}}_\zeta})^{(p-1)/2}(T-1)-(-2r)^{(1-p)/2}x^p\varpi_{M'^{\mathrm{u}}_\zeta}=0.$$

The claim follows from this. □

Lemma 11.10. *We have*

$$\sum_{x\in k_N}\xi_{n,\zeta}(1+x\varpi_{M'^{\mathrm{u}}_\zeta})^{-1}=-((-1)^{(p-1)/2}p)^{e_0}\Big(\frac{r}{k_N}\Big).$$

Proof. Let $\xi'_{n,\zeta}$ be as in Section 9C. We note that the left-hand side of the claim does not change even if we replace $\xi_{n,\zeta}$ by $\xi'_{n,\zeta}$. We have

$$\begin{aligned}\text{(11-6)}\qquad \sum_{x\in k_N}\xi'_{n,\zeta}(1+x\varpi_{M'^{\mathrm{u}}_\zeta})^{-1}&=\sum_{x\in k_N}\xi'_{n,\zeta}\big(1+(-2r)^{(1-p)/2}x^p\varpi_{M'^{\mathrm{u}}_\zeta}\big)^{-1}\\&=\sum_{x\in k_N}\xi'_{n,\zeta}\Big(1-\frac{x^2}{2}\varpi^2_{M'^{\mathrm{u}}_\zeta}\Big)^{-1}\\&=\sum_{x\in k_N}\psi_{M'^{\mathrm{u}}_\zeta}\Big(-\frac{x^2}{2}\delta_\zeta^{p^e-1}\Big),\end{aligned}$$

where we use Lemmas 9.7 and 11.9 at the second equality and (10-4) at the last equality. The last expression in (11-6) is equal to

$$\sum_{x\in k_N}\psi_{T_\zeta^{\mathrm{u}}}(-(2r)^{-1}(p^e-1)\,x^2)=\sum_{x\in k_N}\psi_0(\mathrm{Tr}_{k_N/\mathbb{F}_p}(rx^2))=-((-1)^{(p-1)/2}p)^{e_0}\left(\frac{r}{k_N}\right)$$

by (2-16), (8-2), Lemma 11.3 and $N=2e_0$. □

Lemma 11.11. *We have* $\varepsilon(\xi_{n,\zeta},\psi_{M_\zeta^{\prime\mathrm{u}}})=-(-1)^{\frac{1}{2}((p-1)\,e_0)}\left(\frac{r}{k_N}\right)$.

Proof. We have

$$\begin{aligned}\varepsilon(\xi_{n,\zeta},\psi_{M_\zeta^{\prime\mathrm{u}}})&=p^{-e_0}\sum_{x\in k_N}\xi_{n,\zeta}(-\delta_\zeta^{p^e+1}(1+x\varpi_{M_\zeta^{\prime\mathrm{u}}}))^{-1}\psi_{M_\zeta^{\prime\mathrm{u}}}(-\delta_\zeta^{p^e+1}(1+x\varpi_{M_\zeta^{\prime\mathrm{u}}}))\\&=-(-1)^{\frac{1}{2}((p-1)\,e_0)}\left(\frac{r}{k_N}\right)\xi_{n,\zeta}(-\delta_\zeta^{p^e+1})^{-1}\end{aligned}$$

by Proposition 5.3(2), Lemmas 11.8 and 11.10. We have

$$\begin{aligned}\xi_{n,\zeta}(-\delta_\zeta^{p^e+1})&=\xi'_{n,\zeta}(-\delta_\zeta^{p^e+1})(-1)^{\frac{1}{2}(p-1)\frac{1}{2}(p^e+1)N}\\&=\xi'_{n,\zeta}(-\delta_\zeta^{p^e+1})=\xi'_{n,\zeta}(-(-2r)^{(p^e+1)\frac{1}{2}(1-p)})=1,\end{aligned}$$

where we use

$$\mathrm{Nr}_{N_\zeta^{\prime\mathrm{u}}/M_\zeta^{\prime\mathrm{u}}}(\theta_\zeta^{(p-1)/2}\varpi_{M_\zeta^{\prime\mathrm{u}}})=(-2r)^{(1-p)/2}\varpi_{M_\zeta^{\prime\mathrm{u}}}$$

at the third equality and $k_N^\times\subset\mathrm{Nr}_{N_\zeta^{\prime\mathrm{u}}/M_\zeta^{\prime\mathrm{u}}}((N_\zeta^{\prime\mathrm{u}})^\times)$ at the last equality. Thus, we have the claim. □

11B2. *Even case.* Assume that $p=2$.

Lemma 11.12. *We have* $\mathrm{Tr}_{M_\zeta^{\prime\mathrm{u}}/K}(\delta_\zeta^{2^e+1})=0$ *and*

$$\mathrm{Tr}_{M_\zeta^{\prime\mathrm{u}}/K}(\delta_\zeta^{2^e})=\begin{cases}1 & \text{if } e=1,\\ 0 & \text{if } e\geq 2.\end{cases}$$

Proof. These follow from $\delta_\zeta^{2^e}-\delta_\zeta=\alpha_\zeta^{-1}+1$. □

Lemma 11.13. *We have* $\mathrm{Nr}_{N_\zeta^{\prime\mathrm{u}}/M_\zeta^{\prime\mathrm{u}}}(\theta_\zeta\delta_\zeta^{-1})=\delta_\zeta^{-1}$.

Proof. We have $\mathrm{Nr}_{N_\zeta^{\prime\mathrm{u}}/M_\zeta^{\prime\mathrm{u}}}(\theta_\zeta)=\delta_\zeta^3$ by $\theta_\zeta^2-\theta_\zeta=\delta_\zeta\eta_\zeta$ and $\eta_\zeta^2-\eta_\zeta=\delta_\zeta$. The claim follows from this. □

Let $\sigma_0\in\mathrm{Gal}(N_\zeta^{\prime\mathrm{u}}/M_\zeta^{\prime\mathrm{u}})$ be a generator of $\mathrm{Gal}(N_\zeta^{\prime\mathrm{u}}/M_\zeta^{\prime\mathrm{u}})$ determined by

$$\sigma_0(\eta_\zeta)-\eta_\zeta=1\quad\text{and}\quad\sigma_0(\theta_\zeta)-\theta_\zeta=\eta_\zeta.$$

Lemma 11.14. *Let* $\iota_{n,\zeta}:\mathrm{Gal}(N_\zeta^{\prime\mathrm{u}}/M_\zeta^{\prime\mathrm{u}})\to\mathbb{C}^\times$ *be the homomorphism induced by* $\xi'_{n,\zeta}$ *(see* Lemma 9.9*). Then we have* $\iota_{n,\zeta}(\sigma_0)=-\sqrt{-1}$.

Proof. Let s, t be as in (9-4). We take $\sigma \in I_{M'^{\mathrm{u}}_\zeta}$ such that $\Theta_\zeta(\sigma) = ((1, t, s^2), 0)$. Recall that

$$\phi'((1, t, s^2)) = \bar{g}\left(1, s^2 + \sum_{0 \le i < j \le e-1} t^{2^i + 2^j}\right) \in R''_0$$

is a generator. Then it suffices to show that $\sigma(\eta_\zeta) - \eta_\zeta = 1$ and $\sigma(\theta_\zeta) - \theta_\zeta = \eta_\zeta$. We can check the first equality easily. To show the second equality, it suffices to show that $\sigma(\theta'_\zeta) - \theta'_\zeta = \eta_\zeta^{2^{e-1}}$. By (2-11), we have

$$\sigma(\gamma_\zeta) - \gamma_\zeta \equiv s^2 + \sum_{i=0}^{e-1} (t\beta_\zeta + t^2)^{2^i} \mod \mathfrak{p}_{N'^{\mathrm{u}}_\zeta}.$$

By $t = \sigma(\beta_\zeta) - \beta_\zeta$, $\mathrm{Tr}_{\mathbb{F}_{2^e}/\mathbb{F}_2}(t) = 1$ and (9-14), we have

$$\sigma(\gamma'_\zeta) - \gamma'_\zeta = \eta_\zeta - b_1 + s^2 + \sum_{0 \le i \le j \le e-1} t^{2^i + 2^j} \mod \mathfrak{p}_{N'^{\mathrm{u}}_\zeta}.$$

Hence, by (9-15) and (9-19), we have

$$\sigma(\theta'_\zeta) - \theta'_\zeta \equiv \sum_{i=0}^{2^{e-1}} d_i \, \eta_\zeta^i \mod \mathfrak{p}_{N'^{\mathrm{u}}_\zeta}$$

with some $d_i \in k^{\mathrm{ac}}$. By (9-19), we have

$$\sum_{i=0}^{e-1} (t(\sigma(\gamma'_\zeta) - \gamma'_\zeta))^{2^i} = \sigma(\gamma'_\zeta) - \gamma'_\zeta + \sum_{1 \le i \le j \le e-1} t^{2^i + 2^j} + \sum_{0 \le i < j \le e-1} t^{2^j} (\delta_\zeta - s)^{2^i}.$$

Therefore, again by (9-15) and (9-19), we have

$$d_0 = b_1 + s^2 + \sum_{0 \le i \le j \le e-1} t^{2^i + 2^j} + \sum_{1 \le i \le j \le e-1} t^{2^i + 2^j} + b_1^2 = s + s^2 + t = 0.$$

This implies $\sigma(\theta'_\zeta) - \theta'_\zeta = \eta_\zeta^{2^{e-1}}$, since we know that $\sigma(\theta'_\zeta) - \theta'_\zeta - \eta_\zeta^{2^{e-1}} \in \mathbb{F}_2$ by Lemma 9.8 and $\sigma(\eta_\zeta) - \eta_\zeta = 1$. ☐

Lemma 11.15. *We have*

$$\varepsilon(\xi'_{n,\zeta}, \psi_{M'^{\mathrm{u}}_\zeta}) = \begin{cases} \frac{1+\sqrt{-1}}{\sqrt{2}} & \text{if } e = 1, \\ \frac{1-\sqrt{-1}}{\sqrt{2}} & \text{if } e \ge 2. \end{cases}$$

Proof. By Proposition 5.3, equation (10-4), Lemmas 11.3, 11.12 and 11.13, we have

$$(11\text{-}7) \quad \varepsilon(\xi'_{n,\zeta}, \psi_{M'^{\mathrm{u}}_\zeta}) = 2^{-1/2} \sum_{x \in \mathbb{F}_2} \xi'_{n,\zeta}(\delta_\zeta^{2^e+1}(1 + x\delta_\zeta^{-1}))^{-1} \psi_{M'^{\mathrm{u}}_\zeta}(\delta_\zeta^{2^e+1}(1 + x\delta_\zeta^{-1}))$$
$$= \begin{cases} 2^{-1/2}(1 - \xi'_{n,\zeta}(1 + \delta_\zeta^{-1})^{-1}) & \text{if } e = 1, \\ 2^{-1/2}(1 + \xi'_{n,\zeta}(1 + \delta_\zeta^{-1})^{-1}) & \text{if } e \ge 2. \end{cases}$$

First assume that $e=1$. Then we know the equality in the claim modulo $\mu_2(\mathbb{C})$ by Lemma 11.5. Hence it suffices to show the equality of the real parts. This follows from (11-7). In particular, we have $\xi'_{2,\zeta}(1+\delta_\zeta^{-1})=\sqrt{-1}$.

Next, we consider the general case. We put $\alpha'_1=1/(\delta_\zeta^2-\delta_\zeta+1)$ and $\varpi'=\alpha_1'^3$. Let $\xi'_{2,1,\zeta}$ denote $\xi'_{2,1}$ in the case where K and ϖ are replaced by $\mathbb{F}_2((\varpi'))$ and ϖ'. By applying Lemma 11.14 to $\xi'_{n,\zeta}$ and $\xi'_{2,1,\zeta}$, we have $\xi'_{n,\zeta}=\xi'_{2,1,\zeta}$. We know that $\xi'_{2,1,\zeta}(1+\delta_\zeta^{-1})=\sqrt{-1}$ by the result in the case $e=1$. Hence, we have $\xi'_{n,\zeta}(1+\delta_\zeta^{-1})=\sqrt{-1}$, which shows the claim. □

Lemma 11.16. *We have*

$$\varepsilon(\xi_{n,\zeta},\psi_{M_\zeta'^{\mathrm{u}}})=(-1)^{1+\epsilon_0}.$$

Proof. The epsilon factor $\varepsilon(\xi_{n,\zeta},\psi_{M_\zeta'^{\mathrm{u}}})$ equals $\varepsilon(\xi'_{n,\zeta},\psi_{M_\zeta'^{\mathrm{u}}})$ times

$$\begin{cases}\left(\frac{1+\sqrt{-1}}{\sqrt{2}}\right)^{-3(2^e+1)} & \text{if } e\neq 2,\\ -\left(\frac{1+\sqrt{-1}}{\sqrt{2}}\right)^{-3(2^e+1)} & \text{if } e=2\end{cases}$$

by Lemma 4.2, equation (9-12) and Lemma 10.3. Hence, the claim follows from Lemma 11.15. □

Appendix: Realization in cohomology of Artin–Schreier variety

We realize τ_n in the cohomology of an Artin–Schreier variety. Let ν_{n-2} be the quadratic form on $\mathbb{A}^{n-2}_{k^{\mathrm{ac}}}$ defined by

$$\nu_{n-2}((y_i)_{1\le i\le n-2})=-\frac{1}{n'}\sum_{1\le i\le j\le n-2}y_iy_j.$$

Let X be the smooth affine variety over k^{ac} defined by

$$x^p-x=y^{p^e+1}+\nu_{n-2}((y_i)_{1\le i\le n-2})\quad\text{in }\mathbb{A}^n_{k^{\mathrm{ac}}}.$$

We define a right action of $Q\rtimes\mathbb{Z}$ on X by

$$\begin{aligned}&(x,y,(y_i)_{1\le i\le n-2})((a,b,c),0)\\&\qquad=\left(x+\sum_{i=0}^{e-1}(by)^{p^i}+c,\,a(y+b^{p^e}),\,(a^{(p^e+1)/2}y_i)_{1\le i\le n-2}\right),\\&(x,y,(y_i)_{1\le i\le n-2})\,\mathrm{Fr}(1)=(x^p,y^p,(y_i^p)_{1\le i\le n-2}).\end{aligned}$$

We consider the morphism

$$\pi_{n-2}:\mathbb{A}^{n-1}_{k^{\mathrm{ac}}}\to\mathbb{A}^1_{k^{\mathrm{ac}}},\quad (y,(y_i)_{1\le i\le n-2})\mapsto y^{p^e+1}+\nu_{n-2}((y_i)_{1\le i\le n-2}).$$

Then we have a decomposition

(A-1)
$$H^{n-1}_{\mathrm{c}}(X, \overline{\mathbb{Q}}_\ell) \cong \bigoplus_{\psi \in \mathbb{F}_p^\vee \setminus \{1\}} H^{n-1}_{\mathrm{c}}(\mathbb{A}^{n-1}_{k^{\mathrm{ac}}}, \pi^*_{n-2}\mathcal{L}_\psi)$$

as $Q \rtimes \mathbb{Z}$ representations. Let ρ_n be the representation over $\mathbb{C}$ of $Q \rtimes \mathbb{Z}$ defined by

$$H^{n-1}_{\mathrm{c}}(\mathbb{A}^{n-1}_{k^{\mathrm{ac}}}, \pi^*_{n-2}\mathcal{L}_{\psi_0})\Big(\frac{n-1}{2}\Big)$$

and ι, where $\big(\frac{n-1}{2}\big)$ means the twist by the character $((a, b, c), m) \mapsto p^{m(n-1)/2}$.

Lemma A.1. *If $p \neq 2$, then we have* $\det \nu_{n-2} = -(-2n')^n \in \mathbb{F}_p^\times/(\mathbb{F}_p^\times)^2$.

Proof. This is an easy calculation. □

Proposition A.2. *We have $\tau_n \simeq \rho_n$.*

Proof. Let Y be the smooth affine variety over k^{ac} defined by

$$x^p - x = \nu_{n-2}((y_i)_{1 \le i \le n-2}) \quad \text{in } \mathbb{A}^{n-1}_{k^{\mathrm{ac}}}.$$

We define a right action of $Q \rtimes \mathbb{Z}$ on Y by

$$(x, (y_i)_{1 \le i \le n-2})((a, b, c), 0) = (x, (a^{(p^e+1)/2} y_i)_{1 \le i \le n-2}),$$
$$(x, (y_i)_{1 \le i \le n-2})\,\mathrm{Fr}(1) = (x^p, (y_i^p)_{1 \le i \le n-2}).$$

Using the action of $Q \rtimes \mathbb{Z}$ on Y, we can define an action of $Q \rtimes \mathbb{Z}$ on

$$H^{n-2}_{\mathrm{c}}(\mathbb{A}^{n-2}_{k^{\mathrm{ac}}}, \nu^*_{n-2}\mathcal{L}_{\psi_0}).$$

Then we have

(A-2)
$$H^{n-1}_{\mathrm{c}}(\mathbb{A}^{n-1}_{k^{\mathrm{ac}}}, \pi^*_{n-2}\mathcal{L}_{\psi_0}) \cong H^1_{\mathrm{c}}(\mathbb{A}^1_{k^{\mathrm{ac}}}, \pi^*\mathcal{L}_{\psi_0}) \otimes H^{n-2}_{\mathrm{c}}(\mathbb{A}^{n-2}_{k^{\mathrm{ac}}}, \nu^*_{n-2}\mathcal{L}_{\psi_0})$$

by the Künneth formula, where the isomorphism is compatible with the actions of $Q \rtimes \mathbb{Z}$. By (A-2), it suffices to show the action of $Q \rtimes \mathbb{Z}$ on

(A-3)
$$H^{n-2}_{\mathrm{c}}(\mathbb{A}^{n-2}_{k^{\mathrm{ac}}}, \nu^*_{n-2}\mathcal{L}_{\psi_0})\Big(\frac{n-1}{2}\Big)$$

is equal to the character (2-9) via ι.

First, consider the case where $p \neq 2$. The equality of the actions of Q follows from [Denef and Loeser 1998, Lemma 2.2.3]. We have

(A-4)
$$\begin{aligned}(-1)^{n-2} \sum_{\boldsymbol{y} \in \mathbb{F}_p^{n-2}} \psi_0(\nu_{n-2}(\boldsymbol{y})) &= \Big(\frac{-1}{p}\Big)\Big(-\Big(\frac{-2n'}{p}\Big)\Big)^n (\epsilon(p)\sqrt{p})^{n-2} \\ &= \Big(-\epsilon(p)\Big(\frac{-2n'}{p}\Big)\Big)^n \sqrt{p}^{\,n-2}\end{aligned}$$

by Lemma A.1. The equality of the actions of $\mathrm{Fr}(1) \in Q \rtimes \mathbb{Z}$ follows from [Deligne 1977, Sommes trig. Scholie 1.9] and (A-4).

If $p = 2$, the equality follows from [Imai and Tsushima 2020, Proposition 4.5] and $\big(\frac{2}{n-1}\big) = (-1)^{\frac{1}{8}n(n-2)}$. □

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DIVISORS OF FOURIER COEFFICIENTS OF TWO NEWFORMS

ARVIND KUMAR AND MONI KUMARI

For a pair of distinct non-CM newforms of weights at least 2 and having rational integral Fourier coefficients $a_1(n)$ and $a_2(n)$, under GRH, we obtain an estimate for the set of primes p such that

$$\omega(a_1(p) - a_2(p)) \leq \left[7k + \tfrac{1}{2} + k^{1/5}\right],$$

where $\omega(n)$ denotes the number of distinct prime divisors of an integer n and k is the maximum of their weights. As an application, under GRH, we show that the number of primes giving congruences between two such newforms is bounded by $\left[7k + \frac{1}{2} + k^{1/5}\right]$. We also obtain a multiplicity-one result for newforms via congruences.

1. Introduction and statement of the results

For an elliptic curve $E/\mathbb{Q}$ and a prime p of good reduction, let $N_p(E) := p+1-a(p)$ be the number of points of the reduction of E modulo p. Assume that E is not $\mathbb{Q}$-isogenous to an elliptic curve with torsion. Then Koblitz's conjecture [7] says that the number of primes $p \leq X$ for which $N_p(E)$ is prime is asymptotically equal to $C_E(X/(\log X)^2)$, where C_E is a positive constant depending on E. In particular, $N_p(E)$ is prime infinitely often when p runs over the set of primes. This conjecture is still open but there are many results towards this in the literature (see [17]). Indeed, Koblitz's conjecture can be seen as a variant of the twin prime conjecture (for more details, see [7]).

Inspired by Koblitz's conjecture, Kirti Joshi [6] studied the prime divisors of $N_p(f) := p^{k-1} + 1 - a(p)$, where $a(p)$ is the (integer) p-th Fourier coefficient of a newform $f \in S_k(N)$, the space of cusp forms of weight k and level N. Note that for the Ramanujan delta function $\Delta \in S_{12}(1)$, $\omega(N_p(\Delta)) \geq 3$ for any $p \geq 5$, where $\omega(n)$ is the number of distinct prime divisors of an integer n. This shows that, in general, the obvious variant of Koblitz's conjecture is not true for modular forms of higher weights. In fact, Joshi shows that there exist infinitely many cusp forms f_{k_i}

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(which need not be eigenforms) of increasing weight k_i and of level 1 such that $\omega(N_p(f_{k_i})) \geq 2$ for all primes p.

If f is a non-CM newform of weight $k \geq 4$ then in the same paper Joshi gives an estimate for the primes p for which $N_p(f)$ is an almost prime, i.e., has few prime divisors. More precisely, under GRH and Artin's holomorphy conjecture, he uses a suitably weighted sieve due to Richert to prove that

$$\left|\left\{p \leq X : \omega(N_p(f)) \leq \left[5k + 1 + \sqrt{\log k}\,\right]\right\}\right| \gg \frac{X}{(\log X)^2}, \tag{1-1}$$

where $[\,\cdot\,]$ is the greatest integer function. He also proves a similar result for the function $\Omega(N_p(f))$, where $\Omega(n)$ counts the number of prime divisors of n with multiplicity.

One can interpret $N_p(f)$ as the difference of p-th Fourier coefficients of the normalized Eisenstein series E_k and the newform f. This leads us to study the number of prime divisors of the difference between the p-th Fourier coefficients of any two distinct cuspidal newforms which allows us to deduce many interesting consequences about congruences between newforms, multiplicity-one results, etc. More precisely, we prove the following.

Theorem 1.1. *Let $f_1 \in S_{k_1}(N_1)$ and $f_2 \in S_{k_2}(N_2)$ be non-CM newforms with integer Fourier coefficients $a_1(n)$ and $a_2(n)$, respectively, of weights at least* 2. *We also assume that f_1 and f_2 are not character twists of each other if $k_1 = k_2$. Put*

$$k = \max\{k_1, k_2\}.$$

Then under GRH, we have

$$\left|\left\{p \leq X : a_1(p) \neq a_2(p),\ \omega(a_1(p) - a_2(p)) \leq \left[7k + \tfrac{1}{2} + k^{1/5}\right]\right\}\right| \gg \frac{X}{(\log X)^2}. \tag{1-2}$$

If $k \geq 6$ then the term $k^{1/5}$ appearing in (1-2) *can be replaced with the smaller term $\sqrt{\log k}$.*

We remark that because of Deligne's estimate of Fourier coefficients, for any p, $N_p(f)$ in (1-1) never vanishes, whereas $a_1(p) - a_2(p)$ may be zero. Therefore we remove such primes from (1-2).

Remark 1.2. In Theorem 1.1 and all the subsequent results in this section, by GRH, we mean the generalized Riemann hypothesis holds for all the number fields L_h, $h \geq 1$ (see Section 2B for the definition of L_h), i.e., the Dedekind zeta functions associated with L_h have no zeros in the complex region $\mathrm{Re}(s) > \frac{1}{2}$ for all h.

We now state a few applications of our main result. Unless stated otherwise, throughout the paper we shall work with forms f_1 and f_2 as in Theorem 1.1. We also assume that a newform is always normalized so that its first Fourier coefficient is 1. An immediate consequence of Theorem 1.1 is the following.

Corollary 1.3. *Let f_1 and f_2 be newforms as in Theorem 1.1. Then under GRH there exist infinitely many primes p such that $a_1(p) \neq a_2(p)$ and*

$$\omega(a_1(p) - a_2(p)) \leq \left[7k + \tfrac{1}{2} + k^{1/5}\right].$$

We now recall a multiplicity-one result which says that if $a_1(p) = a_2(p)$ for all but finitely many primes p, then $f_1 = f_2$. Rajan [14] has extensively generalized this result by proving that if $a_1(p) = a_2(p)$ for a set of primes p of positive upper density, then f_1 is a character twist of f_2. This is known as a strong multiplicity-one result. Recently, in [12], a variant of this result for normalized Fourier coefficients has been obtained. In this direction, we prove in Proposition 4.2 that, under GRH, if

$$\left|\{p \leq X : a_1(p) = a_2(p)\}\right| \gg X^{13/14+\epsilon} \tag{1-3}$$

for any $\epsilon > 0$, then f_1 is a character twist of f_2. As a consequence of Theorem 1.1, we obtain the following interesting result that can be seen as a variant of a multiplicity-one result in terms of congruences.

Corollary 1.4. *Let f_1 and f_2 be non-CM normalized newforms of weight k_1 and k_2 with integer Fourier coefficients $a_1(n)$ and $a_2(n)$, respectively. Put $k = \max\{k_1, k_2\}$ and assume GRH. If there exist primes $\ell_1, \ell_2, \ldots, \ell_n$ such that $n > \left[7k + \frac{1}{2} + k^{1/5}\right]$ and, for each $1 \leq i \leq n$,*

$$a_1(p) \equiv a_2(p) \pmod{\ell_i}, \tag{1-4}$$

for all p except for a set of primes of order $o(X/(\log X)^2)$, then $k_1 = k_2$ and f_1 is a character twist of f_2.

Proof. On the contrary, assume that f_1 is not a character twist of f_2. For $1 \leq i \leq n$, let $B_i(X) = \{p \leq X : a_1(p) \not\equiv a_2(p) \pmod{\ell_i}\}$. Put $B(X) = \bigcup_{i=0}^{n} B_i(X)$. Then, for $p \notin B(X)$,

$$\ell_1 \ell_2 \cdots \ell_n \mid (a_1(p) - a_2(p)) \implies \omega(a_1(p) - a_2(p)) \geq n.$$

In particular,

$$\left\{p \leq X : a_1(p) \neq a_2(p) \text{ and } \omega(a_1(p) - a_2(p)) \leq \left[7k + \tfrac{1}{2} + k^{1/5}\right]\right\} \subset B(X).$$

But from our assumptions in (1-4) we have $|B(X)| = o(X/(\log X)^2)$ and this contradicts Theorem 1.1. □

We now mention the last application of Theorem 1.1 which is related to the number of congruence primes of a newform. Recall that for a newform $f_1 \in S_k(N_1)$ with integer Fourier coefficients $a_1(n)$, a positive integer D is called a congruence divisor if there exists another newform $f_2 \in S_k(N_2)$ with integer Fourier coefficients $a_2(n)$ which is not a character twist of f_1 such that f_1 and f_2 are congruent modulo D, that is, $a_1(n) \equiv a_2(n) \pmod{D}$ for all $(n, N_1 N_2) = 1$. Indeed, this is equivalent to the condition that $a_1(p) \equiv a_2(p) \pmod{D}$ for all $(p, N_1N_2) = 1$. If D is a prime, then D is called a congruence prime and we refer to [3] for a nice overview of the subject. A congruence divisor of a newform is an important object to study as it is connected to many well-known problems. To name a few, a bound of the largest congruence divisor is related to the ABC conjecture, and if $k = 2$, then the congruence primes for f_1 are related to the prime divisors of the minimal degree of the modular parametrization to the elliptic curve attached to f_1 via the Eichler–Shimura mapping (see [11, p. 179–180]). It would be also of great interest to bound the number of congruence primes of a newform (see remark on page 180 of [11]). However, if we fix two newforms, then the following result gives a bound on the number of congruence primes which is immediate by Corollary 1.3.

Corollary 1.5. *Let f_1 and f_2 be newforms as in Theorem 1.1. Suppose there exists a positive integer D such that f_1 and f_2 are congruent modulo D. Then under GRH*

$$\omega(D) \le \left[7k + \tfrac{1}{2} + k^{1/5}\right].$$

Each prime divisor of D gives a congruence between f_1 and f_2; therefore, Corollary 1.5 ensures that the number of primes giving congruences between two newforms is bounded uniformly in terms of their weights and not on the levels. This is the novelty of this result.

We now discuss some results about the function $\Omega(a_1(p) - a_2(p))$, where p varies over the set of primes. Using a similar idea as the proof of Theorem 1.1, we obtain the following.

Theorem 1.6. *Let f_1 and f_2 be as in Theorem 1.1. Then under GRH, we have*

$$\left|\left\{p \le X : a_1(p) \ne a_2(p) \text{ and } \Omega(a_1(p) - a_2(p)) \le \left[13k + \tfrac{1}{2} + \sqrt{\log k}\,\right]\right\}\right| \gg \frac{X}{(\log X)^2}.$$

It is clear that Theorem 1.6 also has applications of similar nature to that of Theorem 1.1 mentioned above and we would not repeat it here.

Remark 1.7. It is possible to obtain an upper bound of the right order of magnitude for the estimate in Theorem 1.6. In fact, we can do so by using Selberg's sieve and the ideas used in the proof of [6, Theorem 2.3.1]. More precisely, under GRH, one can obtain that if f_1 and f_2 are as in Theorem 1.1, then

$$\left|\left\{p \le X : a_1(p) \ne a_2(p) \text{ and } \Omega(a_1(p) - a_2(p)) \le \left[\tfrac{1}{2}(29k - 13)\right]\right\}\right| \ll \frac{X}{(\log X)^2}.$$

From the above estimate, it follows that under GRH

$$\left|\{p \le X : a_1(p) - a_2(p) \text{ is prime}\}\right| \ll \frac{X}{(\log X)^2}.$$

In particular, the natural density of the set $\{p : a_1(p) - a_2(p) \text{ is prime}\}$ is zero. It would be interesting to obtain a suitable lower bound of this set or at least to know whether there are infinitely many primes p for which $a_1(p) - a_2(p)$ is a prime.

In fact, all the above results are valid even if we replace $a_1(p) - a_2(p)$ with $a_1(p) + a_2(p)$. Also, similar results but with better bounds hold in Theorems 1.1 and 1.6 if we assume Artin's holomorphicity conjecture in addition to GRH. It is also worth mentioning that the full strength of GRH is not essential to prove our theorems. Rather, a quasi-GRH, which assumes a zero-free region for the associated Dedekind zeta functions in the region $\mathrm{Re}(s) = 1-\epsilon$ for some $\epsilon \in \left(0, \frac{1}{2}\right)$, is sufficient for our purpose (see the discussion and results proved in [13]). In this case, we obtain similar results to Propositions 2.1 and 4.1, with the only difference being that the exponent of x becomes $1-\epsilon$ instead of $\frac{1}{2}$. However, it then requires a more careful analysis of handling the error terms in the subsequent part of the proof of our results, which will not be carried out here.

Contents and structure of the paper. The theorem of Deligne connecting the theory of ℓ-adic Galois representations to Fourier coefficients of newforms opens the door for obtaining many new results regarding the arithmetical nature of these coefficients. This connection and the Chebotarev density theorem play a prominent role in this paper. These are recalled in Section 2. To prove our results, we first establish Proposition 4.3 which gives an asymptotic formula for the number of primes p up to X for which $a_1(p) \neq a_2(p)$ and $a_1(p) - a_2(p)$ is divisible by a fixed positive integer. Proof of Proposition 4.3 requires computations of the image of the product Galois representations attached to f_1 and f_2 and this is obtained in Section 3. Finally, we apply a suitably weighted sieve due to Richert, recalled in Section 5, to prove our results. To establish the sieve conditions with the required uniformity of parameters, Proposition 4.3 plays a crucial role. We use the ideas employed in [6; 17] to prove our main results in Sections 6 and 7.

Notation. For any real number $X \geq 2$, $\pi(X)$ denotes the number of primes less than or equal to X. Along with the standard analytic notation $\ll$, $\gg$, O, o, $\sim$ (the implied constants will often depend on the pair of forms under consideration), we use the letters p, ℓ, q, ℓ_1, ℓ_2, etc. to denote prime numbers throughout the paper.

2. Preliminaries

We summarize some standard results without proofs which will be used throughout the paper. We closely follow [2] for our exposition.

2A. ***Chebotarev density theorem.*** We recall the Chebotarev density theorem which is one of the principal tools needed for proving the main theorems of this paper.

Let K be a finite Galois extension of $\mathbb{Q}$ with the Galois group G and degree n_K. For an unramified prime p, we denote by Frob_p, a Frobenius element of K at p in G. For a subset C of G, stable under conjugation, we define

$$\pi_C(X) := \{p \le X : p \text{ unramified in } K \text{ and } \mathrm{Frob}_p \in C\}.$$

The Chebotarev density theorem states that

$$\pi_C(X) \sim \frac{|C|}{|G|}\pi(X).$$

We will use the following conditional effective version of this theorem which was first obtained by Lagarias and Odlyzko [8] and was subsequently refined by Serre [16]. To state this, let d_K be the absolute value of the discriminant of $K/\mathbb{Q}$ and $\zeta_K(s)$ be the Dedekind zeta function associated with K.

Proposition 2.1. *Suppose $\zeta_K(s)$ satisfies GRH. Then*

$$\pi_C(X) = \frac{|C|}{|G|}\pi(X) + O\left(\frac{|C|}{|G|}X^{1/2}(\log d_K + n_K \log X)\right).$$

In addition to GRH for $\zeta_K(s)$, by assuming Artin's holomorphy conjecture (which states that the Artin L-function associated to any nontrivial representation of the Galois group $\mathrm{Gal}(K/\mathbb{Q})$ has an analytic continuation on the whole complex plane) one can improve the error term in the above asymptotic formula for $\pi_C(X)$.

2B. ***mod-h Galois representations.*** Let $G_{\mathbb{Q}} = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ be the absolute Galois group of an algebraic closure $\bar{\mathbb{Q}}$ of $\mathbb{Q}$. Let $k \ge 2$, $N \ge 1$ and ℓ be a prime. Suppose $f \in S_k(N)$ is a newform with integer Fourier coefficients $a(n)$. The work of Eichler, Shimura and Deligne (see [1]) give the existence of a two-dimensional continuous, odd and irreducible Galois representation

$$\rho_{f,\ell} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{Z}_\ell),$$

which is unramified at $p \nmid N\ell$. If Frob_p denotes a Frobenius element corresponding to such a prime, then the representation $\rho_{f,\ell}$ has the property that

$$\mathrm{tr}(\rho_{f,\ell}(\mathrm{Frob}_p)) = a(p), \quad \det(\rho_{f,\ell}(\mathrm{Frob}_p)) = p^{k-1}.$$

By reduction and semisimplification, we obtain a mod-ℓ Galois representation,

$$\bar{\rho}_{f,\ell} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{F}_\ell),$$

where $\mathbb{F}_\ell := \mathbb{Z}/\ell\mathbb{Z}$.

Let $h = \prod_{j=1}^{t} \ell_j^{n_j}$ be a positive integer. Using the ℓ_j-adic representations attached to f, we consider an h-adic representation given by products of mod-ℓ_j's representations

$$\rho_{f,h} : G_{\mathbb{Q}} \to \mathrm{GL}_2\Big(\prod_{1\le j\le t} \mathbb{Z}_{\ell_j}\Big).$$

For each $1 \le j \le t$, we have the natural projection $\mathbb{Z}_{\ell_j} \twoheadrightarrow \mathbb{Z}/\ell_j^{n_j}\mathbb{Z}$, and hence we obtain a mod-h Galois representation given by

$$\bar{\rho}_{f,h} : G_{\mathbb{Q}} \to \mathrm{GL}_2\Big(\prod_{1\le j\le t} \mathbb{Z}/\ell_j^{n_j}\mathbb{Z}\Big) \xrightarrow{\cong} \mathrm{GL}_2(\mathbb{Z}/h\mathbb{Z}).$$

If $p \nmid Nh$ is a prime, then $\bar{\rho}_{f,h}$ is unramified at p and

$$\mathrm{tr}(\bar{\rho}_{f,h}(\mathrm{Frob}_p)) \equiv a(p) \pmod{h}, \qquad \det(\bar{\rho}_{f,h}(\mathrm{Frob}_p)) \equiv p^{k-1} \pmod{h}.$$

Let $f_1 \in S_{k_1}(N_1)$ and $f_2 \in S_{k_2}(N_2)$ be newforms having integer Fourier coefficients $a_1(n)$ and $a_2(n)$, respectively. Then one can consider the product representation $\bar{\rho}_h$ of $\bar{\rho}_{f_1,h}$ and $\bar{\rho}_{f_2,h}$, defined by

$$\begin{aligned} \bar{\rho}_h : G_{\mathbb{Q}} &\to \mathrm{GL}_2(\mathbb{Z}/h\mathbb{Z}) \times \mathrm{GL}_2(\mathbb{Z}/h\mathbb{Z}),\\ \sigma &\mapsto (\bar{\rho}_{f_1,h}(\sigma), \bar{\rho}_{f_2,h}(\sigma)). \end{aligned}$$

Let $\mathscr{A}_h$ denote the image of $G_{\mathbb{Q}}$ under $\bar{\rho}_h$. By the fundamental theorem of Galois theory, the fixed field of $\ker(\bar{\rho}_h)$, say L_h, is a finite Galois extension of $\mathbb{Q}$ and

$$\mathrm{Gal}(L_h/\mathbb{Q}) \cong \mathscr{A}_h. \tag{2-1}$$

Let $\mathscr{C}_h$ be the subset of $\mathscr{A}_h$ defined by

$$\mathscr{C}_h = \{(A, B) \in \mathscr{A}_h : \mathrm{tr}(A) = \mathrm{tr}(B)\}.$$

We now define the following function on the set of positive integers which will play an important role throughout the paper. For an integer $h > 1$, define

$$\delta(h) := \frac{|\mathscr{C}_h|}{|\mathscr{A}_h|} \tag{2-2}$$

and $\delta(1) := 1$. Since the trace of the image of complex conjugation is always zero, $\mathscr{C}_h \neq \phi$, and hence $\delta(h) > 0$ for every integer h.

3. Technical results

Let f_1 and f_2 be newforms as before. The main aim of this section is to obtain an asymptotic size of $\delta(\ell^n)$ for $n = 1, 2$ and this requires the computation of the cardinalities of $\mathscr{A}_{\ell^n}$ and $\mathscr{C}_{\ell^n}$. Building on the work of Ribet [15] and Momose [10], Loeffler [9] has determined the image $\mathscr{A}_{\ell^n}$ of the product Galois representations.

More precisely, Loeffler [9, Theorem 3.2.2] has proved that that there exists a positive constant $M(f_1, f_2)$ such that, for all primes $\ell \geq M(f_1, f_2)$ and $n \geq 1$,

$$\mathscr{A}_{\ell^n} = \big\{(A, B) \in \mathrm{GL}_2(\mathbb{Z}/\ell^n\mathbb{Z}) \times \mathrm{GL}_2(\mathbb{Z}/\ell^n\mathbb{Z}) : \det(A) = v^{k_1-1}, \det(B) = v^{k_2-1}, v \in (\mathbb{Z}/\ell^n\mathbb{Z})^\times\big\}. \tag{3-1}$$

In other words, the mod-ℓ^n representations of two newforms (that are not character twists of each other) are as independent as possible. In the rest of the paper, we denote the constant $M(f_1, f_2)$ by M and without loss of generality, we assume that $M \geq 3$. Clearly, for $\ell \geq M$,

$$\mathscr{C}_{\ell^n} = \{(A, B) \in \mathscr{A}_{\ell^n} : \mathrm{tr}(A) = \mathrm{tr}(B)\}. \tag{3-2}$$

3A. ***Combinatorial lemmas.*** Here we obtain results about cardinalities of $\mathscr{A}_{\ell^n}$ and $\mathscr{C}_{\ell^n}$ for any $\ell \geq M$. We first assume that

$$\lambda_n = \gcd(\ell^n - \ell^{n-1}, k_1 - 1, k_2 - 1)$$

and

$$\Lambda_n = \{(v^{k_1-1}, v^{k_2-1}) : v \in (\mathbb{Z}/\ell^n\mathbb{Z})^\times\}. \tag{3-3}$$

Recall that $\ell \geq 3$. We now consider the group homomorphism

$$\phi : (\mathbb{Z}/\ell^n\mathbb{Z})^\times \to \Lambda_n \quad \text{defined by } \phi(v) = (v^{k_1-1}, v^{k_2-1}).$$

Since ϕ is surjective and its kernel $\{v \in (\mathbb{Z}/\ell^n\mathbb{Z})^\times : v^{\lambda_n} = 1\}$ is a cyclic subgroup of $(\mathbb{Z}/\ell^n\mathbb{Z})^\times$ of order λ_n, we obtain

$$|\Lambda_n| = \frac{|(\mathbb{Z}/\ell^n\mathbb{Z})^\times|}{\lambda_n} = \frac{\ell^n - \ell^{n-1}}{\lambda_n}. \tag{3-4}$$

We first recall the following result proved in [2, Lemma 3.3].

Lemma 3.1. *For any prime $\ell \geq M$,*

$$|\mathscr{A}_\ell| = \frac{1}{\lambda_1}(\ell - 1)^3(\ell^2 + \ell)^2.$$

Using Lemma 3.1, we now compute $|\mathscr{A}_{\ell^n}|$ for any $n \geq 1$.

Lemma 3.2. *For any prime $\ell \geq M$ and integer $n \geq 1$,*

$$|\mathscr{A}_{\ell^n}| = \frac{1}{\lambda_n}\ell^{7(n-1)}(\ell - 1)^3(\ell^2 + \ell)^2.$$

Proof. Let $\psi : \mathscr{A}_{\ell^n} \to \mathscr{A}_\ell$ be the natural map defined by

$$(A, B) \mapsto (A \pmod{\ell}, B \pmod{\ell}). \tag{3-5}$$

Since it is a surjective group homomorphism, we have

$$|\mathscr{A}_{\ell^n}| = |\ker(\psi)|\,|\mathscr{A}_\ell|.$$

Therefore, in view of Lemma 3.1, to evaluate $|\mathscr{A}_{\ell^n}|$ it is sufficient to compute $|\ker(\psi)|$. For that we first compute the cardinality of the set

$$\{\gamma \in \mathrm{GL}_2(\mathbb{Z}/\ell^n\mathbb{Z}) : \det(\gamma) = d,\ \gamma \equiv \mathrm{Id} \pmod{\ell}\},$$

where $d \in (\mathbb{Z}/\ell^n\mathbb{Z})^\times$ such that $d \equiv 1 \pmod{\ell}$ is fixed and Id is the identity element in $\mathrm{GL}_2(\mathbb{F}_\ell)$. Any general element of the above set will be of the form

$$\begin{pmatrix} 1+x\ell & y\ell \\ z\ell & 1+w\ell \end{pmatrix},$$

where $0 \le x, y, z, w < \ell^{n-1}$ with the condition that

$$(1+x\ell)(1+w\ell) - yz\ell^2 = d.$$

As $d \equiv 1 \pmod{\ell}$, the above equation reduces to

$$x(1+w\ell) = \frac{d-1}{\ell} + yz\ell - w.$$

Since $1 + w\ell \in (Z/\ell^n\mathbb{Z})^\times$ for such w, for any choices of $0 \le y, z, w < \ell^{n-1}$ the above equation gives a unique x. Therefore

$$\bigl|\{\gamma \in \mathrm{GL}_2(\mathbb{Z}/\ell^n\mathbb{Z}) : \det(\gamma) = d,\ \gamma \equiv \mathrm{Id} \pmod{\ell}\}\bigr| = \ell^{3(n-1)}. \tag{3-6}$$

Now, we note that

$$\ker(\psi) = \bigl|\{(A, B) \in \mathscr{A}_{\ell^n} : (A, B) \equiv (\mathrm{Id}, \mathrm{Id}) \pmod{\ell}\}\bigr|;$$

therefore from (3-1)

$$|\ker(\psi)| = \sum_{(d_1,d_2)\in\Lambda_n} \sum_{\substack{A\in\mathrm{GL}_2(\mathbb{Z}/\ell^n\mathbb{Z}) \\ \det(A)=d_1 \\ A\equiv\mathrm{Id} \pmod{\ell}}} 1 \sum_{\substack{B\in\mathrm{GL}_2(\mathbb{Z}/\ell^n\mathbb{Z}) \\ \det(b)=d_2 \\ b\equiv\mathrm{Id} \pmod{\ell}}} 1.$$

In the above, congruence conditions on A and B compel that $d_1 \equiv d_2 \equiv 1 \pmod{\ell}$, and hence using (3-6) gives

$$|\ker(\psi)| = \ell^{6(n-1)} \sum_{\substack{(d_1,d_2)\in\Lambda_n \\ d_1\equiv d_2\equiv 1 \pmod{\ell}}} 1.$$

Since the sum appearing on the right side of the above equation is the cardinality of the kernel of the natural (surjective) reduction map $\Lambda_n \to \Lambda_1$ given in (3-5), we have

$$|\ker(\psi)| = \frac{|\Lambda_n|}{|\Lambda_1|}\ell^{6(n-1)}.$$

Now using (3-4) in the above yields the desired result. □

Our next aim is to compute the cardinalities of $\mathscr{C}_\ell$ and $\mathscr{C}_{\ell^2}$. Though an explicit computation is possible, we only obtain asymptotic formulas here and that is enough for our purpose. To simplify our notation, we denote the set of quadratic and nonquadratic residue elements in $(\mathbb{Z}/\ell^n\mathbb{Z})^\times$ by Q_n and Q_n^c, respectively.

Lemma 3.3. *For any prime $\ell \geq M$,*

$$|\mathscr{C}_\ell| = \frac{\ell^6}{\lambda_1} + O(\ell^5).$$

Proof. From the definition of $\mathscr{C}_\ell$

$$\begin{aligned}|\mathscr{C}_\ell| &= \sum_{(d_1,d_2)\in\Lambda_1} \Big|\{(A,B)\in \mathrm{GL}_2(\mathbb{F}_\ell)\times \mathrm{GL}_2(\mathbb{F}_\ell):\\ &\qquad\qquad \det(A)=d_1,\ \det(B)=d_2,\ \mathrm{tr}(A)=\mathrm{tr}(B)\}\Big|\\ &= \sum_{t\in\mathbb{F}_\ell}\sum_{(d_1,d_2)\in\Lambda_1}\sum_{\substack{A\in \mathrm{GL}_2(\mathbb{F}_\ell)\\ \det(A)=d_1\\ \mathrm{tr}(A)=t}} 1 \sum_{\substack{B\in \mathrm{GL}_2(\mathbb{F}_\ell)\\ \det(B)=d_2\\ \mathrm{tr}(B)=t}} 1.\end{aligned}$$

Split the sum over Λ_1 into three parts, namely

$$|\mathscr{C}_\ell| = \sum_{t\in\mathbb{F}_\ell}\Bigg[\sum_{\substack{(d_1,d_2)\in\Lambda_1\\ t^2-4d_1\in Q_1}} + \sum_{\substack{(d_1,d_2)\in\Lambda_1\\ t^2=4d_1}} + \sum_{\substack{(d_1,d_2)\in\Lambda_1\\ t^2-4d_1\in Q_1^c}}\Bigg] \sum_{\substack{A\in \mathrm{GL}_2(\mathbb{F}_\ell)\\ \det(A)=d_1\\ \mathrm{tr}(A)=t}} 1 \sum_{\substack{B\in \mathrm{GL}_2(\mathbb{F}_\ell)\\ \det(B)=d_2\\ \mathrm{tr}(B)=t}} 1 \tag{3-7}$$

and we denote the corresponding sums by S_1, S_2 and S_3, respectively. Thus

$$S_1 = \sum_{t\in\mathbb{F}_\ell}\sum_{\substack{(d_1,d_2)\in\Lambda_1\\ t^2-4d_1\in Q_1}}\sum_{\substack{A\in \mathrm{GL}_2(\mathbb{F}_\ell)\\ \det(A)=d_1\\ \mathrm{tr}(A)=t}} 1 \sum_{\substack{B\in \mathrm{GL}_2(\mathbb{F}_\ell)\\ \det(B)=d_2\\ \mathrm{tr}(B)=t}} 1.$$

To proceed further, note that for given $d \in \mathbb{F}_\ell^\times$ and $t \in \mathbb{F}_\ell$ one can obtain the following result by employing an elementary counting argument:

$$\big|\{\gamma\in \mathrm{GL}_2(\mathbb{F}_\ell):\det(\gamma)=d,\ \mathrm{tr}(\gamma)=t\}\big| = \begin{cases}\ell^2+\ell & \text{if } t^2-4d\in Q_1,\\ \ell^2 & \text{if } t^2=4d,\\ \ell^2-\ell & \text{if } t^2-4d\in Q_1^c.\end{cases} \tag{3-8}$$

Using (3-8) gives

$$\begin{aligned} S_1 &= (\ell^2+\ell)\sum_{t\in\mathbb{F}_\ell}\sum_{\substack{(d_1,d_2)\in\Lambda_1\\ t^2-4d_1\in Q_1}}\sum_{\substack{B\in \mathrm{GL}_2(\mathbb{F}_\ell)\\ \det(B)=d_2\\ \mathrm{tr}(B)=t}} 1\\ &= (\ell^2+\ell)\sum_{t\in\mathbb{F}_\ell}\Bigg[\sum_{\substack{(d_1,d_2)\in\Lambda_1\\ t^2-4d_1\in Q_1\\ t^2-4d_2\in Q_1}} + \sum_{\substack{(d_1,d_2)\in\Lambda_1\\ t^2-4d_1\in Q_1\\ t^2=4d_2}} + \sum_{\substack{(d_1,d_2)\in\Lambda_1\\ t^2-4d_1\in Q_1\\ t^2-4d_2\in Q_1^c}}\Bigg]\sum_{\substack{B\in \mathrm{GL}_2(\mathbb{F}_\ell)\\ \det(B)=t_2\\ \mathrm{tr}(B)=t}} 1.\end{aligned}$$

Again using (3-8)

$$S_1 = (\ell^2+\ell)\sum_{t\in\mathbb{F}_\ell}\Bigg[(\ell^2+\ell)\sum_{\substack{(d_1,d_2)\in\Lambda_1\\ t^2-4d_1\in Q_1\\ t^2-4d_2\in Q_1}} 1+\ell^2\sum_{\substack{(d_1,d_2)\in\Lambda_1\\ t^2-4d_1\in Q_1\\ t^2=4d_2}} 1+(\ell^2-\ell)\sum_{\substack{(d_1,d_2)\in\Lambda_1\\ t^2-4d_1\in Q_1\\ t^2-4d_2\in Q_1^c}} 1\Bigg].$$

Collecting the terms containing ℓ^4 gives

$$S_1 = \ell^4\sum_{t\in\mathbb{F}_\ell}\sum_{\substack{(d_1,d_2)\in\Lambda_1\\ t^2-4d_1\in Q_1}} 1+O(\ell^5).$$

Similarly, we have

$$S_2 = \ell^4\sum_{t\in\mathbb{F}_\ell}\sum_{\substack{(d_1,d_2)\in\Lambda_1\\ t^2=4d_1}} 1+O(\ell^5),\qquad S_3 = \ell^4\sum_{t\in\mathbb{F}_\ell}\sum_{\substack{(d_1,d_2)\in\Lambda_1\\ t^2-4d_1\in Q_1^c}} 1+O(\ell^5).$$

Combining all together, we have, from (3-7),

$$|\mathscr{C}_\ell| = \ell^4\sum_{t\in\mathbb{F}_\ell}\sum_{(d_1,d_2)\in\Lambda_1} 1+O(\ell^5)$$

and now using (3-4) completes the proof. □

To compute $|\mathscr{C}_{\ell^2}|$, we first prove the following result which is a generalization of (3-8) for the ring $\mathbb{Z}/\ell^2\mathbb{Z}$.

Lemma 3.4. *For any $d\in(\mathbb{Z}/\ell^2\mathbb{Z})^\times$ and $t\in\mathbb{Z}/\ell^2\mathbb{Z}$, we have*

$$\begin{aligned}(3\text{-}9)\quad &\big|\{\gamma\in \mathrm{GL}_2(\mathbb{Z}/\ell^2\mathbb{Z}) : \det(\gamma)=d,\ \mathrm{tr}(\gamma)=t\}\big|\\ &= \begin{cases} \ell^4+\ell^3-\ell^2 & \textit{if } t^2-4d=0,\\ \ell^4-\ell^2 & \textit{if } 0\neq t^2-4d\equiv 0 \pmod{\ell},\\ \ell^4+\ell^3 & \textit{if } t^2-4d\in Q_2,\\ \ell^4-\ell^3 & \textit{if } t^2-4d\in Q_2^c.\end{cases}\end{aligned}$$

Proof. It is clear that

$$\big|\{\gamma\in \mathrm{GL}_2(\mathbb{Z}/\ell^2\mathbb{Z}) : \det(\gamma)=d,\ \mathrm{tr}(\gamma)=t\}\big| = |\mathscr{N}|,$$

where $\mathscr{N} := \{(a,b,c)\in(\mathbb{Z}/\ell^2\mathbb{Z})^3 : a^2-at+bc=-d\}$. To compute $|\mathscr{N}|$ we divide the set $\mathscr{N}$ into three disjoint subsets $\mathscr{N}_1$, $\mathscr{N}_2$ and $\mathscr{N}_3$ based on the following three cases, respectively. Hence

$$|\mathscr{N}| = |\mathscr{N}_1|+|\mathscr{N}_2|+|\mathscr{N}_3|. \tag{3-10}$$

Case (i): $a = 0$. Then the condition $bc = -d$ forces that b and c both have to be units and for any b there exists a unique c. Hence

$$|\mathscr{N}_1| = \ell^2 - \ell.$$

Case (ii): $a \neq 0$ and $bc = 0$. The latter condition implies that either b or c is 0, or both are (nonzero) zero-divisors of $\mathbb{Z}/\ell^2\mathbb{Z}$. The total number of such pairs is $2\ell^2 - 1 + (\ell - 1)^2 = 3\ell^2 - 2\ell$. Therefore,

$$|\mathscr{N}_2| = |\{a \in \mathbb{Z}/\ell^2\mathbb{Z} : a^2 - at + d = 0\}| \times (3\ell^2 - 2\ell). \tag{3-11}$$

We now claim that

$$|\{a \in \mathbb{Z}/\ell^2\mathbb{Z} : a^2 - at + d = 0\}| = \begin{cases} \ell & \text{if } t^2 - 4d = 0, \\ 0 & \text{if } 0 \neq t^2 - 4d \equiv 0 \pmod{\ell}, \\ 2 & \text{if } t^2 - 4d \in Q_2, \\ 0 & \text{if } t^2 - 4d \in Q_2^c. \end{cases} \tag{3-12}$$

To prove this, we see that if $t^2 - 4d = 0$, then any $a \equiv \frac{t}{2} \pmod{\ell}$ is a solution of $a^2 - at + d = 0$ and there are ℓ such choices for a. Next, assume that $0 \neq t^2 - 4d \equiv 0 \pmod{\ell}$. If $a^2 - at + d = 0$ has solutions, say x and y, then

$$(x - y)^2 = (x + y)^2 - 4xy = t^2 - 4d \equiv 0 \pmod{\ell}.$$

Therefore, $x - y \equiv 0 \pmod{\ell} \implies t^2 - 4d = (x - y)^2 = 0$, which is a contradiction. The last two cases are clear.

Thus using (3-12) in (3-11) gives the cardinality of $\mathscr{N}_2$.

Case (iii): $a \neq 0$ and $bc \neq 0$. In this case, bc can be either a (nonzero) zero-divisor or a unit. Clearly, the number of choices for b and c such that bc is a given nonzero zero-divisor is $2\ell(\ell - 1)$ and for a given unit the number of such choices is $\ell^2 - \ell$. Therefore, we have

$$\begin{aligned} |\mathscr{N}_3| = \big|\{a \in \mathbb{Z}/\ell^2\mathbb{Z} : 0 \neq a^2 - at + d \equiv 0 \pmod{\ell})\}\big| \times 2\ell(\ell - 1) \\ + \big|\{a \in \mathbb{Z}/\ell^2\mathbb{Z} : a^2 - at + d \in (\mathbb{Z}/\ell^2\mathbb{Z})^\times\}\big| \times (\ell^2 - \ell). \end{aligned} \tag{3-13}$$

If $a^2 - at + d = m\ell$ for some $m \in \mathbb{F}_\ell^\times$, then from (3-12)

$$|\{a \in \mathbb{Z}/\ell^2\mathbb{Z} : a^2 - at + d = m\ell\}| = \begin{cases} \ell & \text{if } t^2 - 4(d - m\ell) = 0, \\ 0 & \text{if } 0 \neq t^2 - 4(d - m\ell) \equiv 0 \pmod{\ell}, \\ 2 & \text{if } t^2 - 4(d - m\ell) \in Q_2 \iff t^2 - 4d \in Q_2, \\ 0 & \text{if } t^2 - 4(d - m\ell) \in Q_2^c \iff t^2 - 4d \in Q_2^c. \end{cases}$$

Note that there exists a unique $m \in \mathbb{F}_\ell^\times$ such that $t^2 - 4(d - m\ell) = 0$, and in that case $0 \neq t^2 - 4d \equiv 0 \pmod{\ell}$. Therefore

$$\text{(3-14)} \quad \left|\{a \in \mathbb{Z}/\ell^2\mathbb{Z} : 0 \neq a^2 - at + d \equiv 0 \pmod{\ell}\}\right| = \begin{cases} 0 & \text{if } t^2 - 4d = 0, \\ \ell & \text{if } 0 \neq t^2 - 4d \equiv 0 \pmod{\ell}, \\ 2(\ell - 1) & \text{if } t^2 - 4d \in Q_2, \\ 0 & \text{if } t^2 - 4d \in Q_2^c. \end{cases}$$

As we have $\ell^2 - 1$ choices of a in this case, (3-12) and (3-14) immediately gives

$$\text{(3-15)} \quad \left|\{a \in \mathbb{Z}/\ell^2\mathbb{Z} : a^2 - at + d \in (\mathbb{Z}/\ell^2\mathbb{Z})^\times\}\right| = \begin{cases} \ell^2 - \ell - 1 & \text{if } t^2 - 4d = 0, \\ \ell^2 - \ell - 1 & \text{if } 0 \neq t^2 - 4d \equiv 0 \pmod{\ell}, \\ \ell^2 - 2\ell - 1 & \text{if } t^2 - 4d \in Q_2, \\ \ell^2 - 1 & \text{if } t^2 - 4d \in Q_2^c. \end{cases}$$

Substituting (3-14) and (3-15) in (3-13) and then combining all the above three cases in (3-10) gives the desired result. □

We are now ready to give a desirable estimate for $|\mathscr{C}_{\ell^2}|$.

Lemma 3.5. *For any prime $\ell \geq M$,*

$$|\mathscr{C}_{\ell^2}| = \frac{\ell^{12}}{\lambda_2} + O(\ell^{11}).$$

Proof. We use similar arguments as in the proof of Lemma 3.3, and hence we will only give an outline of the proof here. We write

$$|\mathscr{C}_{\ell^2}| = \sum_{t \in \mathbb{Z}/\ell^2\mathbb{Z}} \sum_{(d_1, d_2) \in \Lambda_2} \sum_{\substack{A \in \mathrm{GL}_2(\mathbb{Z}/\ell^2\mathbb{Z}) \\ \det(A) = d_1 \\ \mathrm{tr}(A) = t}} 1 \sum_{\substack{B \in \mathrm{GL}_2(\mathbb{Z}/\ell^2\mathbb{Z}) \\ \det(B) = d_2 \\ \mathrm{tr}(B) = t}} 1.$$

We split the sum over Λ_2 into four parts, namely

$$|\mathscr{C}_{\ell^2}| = \sum_{t \in \mathbb{Z}/\ell^2\mathbb{Z}} \Bigg[\sum_{\substack{(d_1, d_2) \in \Lambda_2 \\ t^2 - 4d_1 \in Q_2}} + \sum_{\substack{(d_1, d_2) \in \Lambda_1 \\ t^2 = 4d_1}} + \sum_{\substack{(d_1, d_2) \in \Lambda_1 \\ 0 \neq t^2 - 4d_1 \equiv 0 \pmod{\ell}}} + \sum_{\substack{(d_1, d_2) \in \Lambda_1 \\ t^2 - 4d_1 \in Q_2^c}} \Bigg] \sum_{\substack{A \in \mathrm{GL}_2(\mathbb{Z}/\ell^2\mathbb{Z}) \\ \det(A) = d_1 \\ \mathrm{tr}(A) = t}} 1 \sum_{\substack{B \in \mathrm{GL}_2(\mathbb{Z}/\ell^2\mathbb{Z}) \\ \det(B) = d_2 \\ \mathrm{tr}(B) = t}} 1$$

and denote the corresponding sums by S_1', S_2', S_3' and S_4' so that

$$\text{(3-16)} \qquad |\mathscr{C}_{\ell^2}| = S_1' + S_2' + S_3' + S_4'.$$

Now applying Lemma 3.4, we obtain

$$S_1' = (\ell^4 + \ell^3) \sum_{t \in \mathbb{Z}/\ell^2\mathbb{Z}} \sum_{\substack{(d_1,d_2)\in\Lambda_2 \\ t^2-4d_1\in Q_2}} \sum_{\substack{B\in \mathrm{GL}_2(\mathbb{Z}/\ell^2\mathbb{Z}) \\ \det(B)=d_2 \\ \mathrm{tr}(B)=t}} 1.$$

As before, splitting the middle sum into four parts and applying Lemma 3.4 yields

$$\begin{aligned} S_1' = (\ell^4 + \ell^3) \sum_{t \in \mathbb{Z}/\ell^2\mathbb{Z}} \Bigg[&(\ell^4+\ell^3) \sum_{\substack{(d_1,d_2)\in\Lambda_2 \\ t^2-4d_1\in Q_2 \\ t^2-4d_2\in Q_2}} + (\ell^4+\ell^3-\ell^2) \sum_{\substack{(d_1,d_2)\in\Lambda_2 \\ t^2-4d_1\in Q_2 \\ t^2=4d_2}} \\ &+ (\ell^4-\ell^2) \sum_{\substack{(d_1,d_2)\in\Lambda_2 \\ t^2-4d_1\in Q_2 \\ 0\equiv t^2-4d_2\equiv 0 \pmod{\ell}}} + (\ell^4-\ell^3) \sum_{\substack{(d_1,d_2)\in\Lambda_2 \\ t^2-4d_1\in Q_2 \\ t^2-4d_2\in Q_2^c}} \Bigg] 1 \end{aligned}$$

and then collecting the terms containing ℓ^8 gives

$$S_1' = \ell^8 \sum_{t \in \mathbb{Z}/\ell^2\mathbb{Z}} \sum_{\substack{(d_1,d_2)\in\Lambda_2 \\ t^2-4d_1\in Q_2}} 1 + O(\ell^{11}).$$

Computing S_2', S_3' and S_4' in a similar manner and substituting in (3-16), we have

$$|\mathscr{C}_{\ell^2}| = \ell^8 \sum_{t \in \mathbb{Z}/\ell^2\mathbb{Z}} \sum_{(d_1,d_2)\in\Lambda_2} 1 + O(\ell^{11})$$

and finally using (3-4) completes the proof. □

Let $h = \ell_1^{n_1}\ell_2^{n_2}\cdots\ell_t^{n_t}$. Since the fixed field of $\ker(\bar{\rho}_h)$ is contained in the compositum of fixed fields of $\ker(\bar{\rho}_{\ell_i^{n_i}})$, from (2-1)

$$|\mathscr{A}_h| \le |\mathscr{A}_{\ell_1^{n_1}}||\mathscr{A}_{\ell_2^{n_2}}|\cdots|\mathscr{A}_{\ell_t^{n_t}}| \quad \text{and} \quad |\mathscr{C}_h| \le |\mathscr{C}_{\ell_1^{n_1}}||\mathscr{C}_{\ell_2^{n_2}}|\cdots|\mathscr{C}_{\ell_t^{n_t}}|.$$

For any prime ℓ and integer $n \ge 1$, $\mathscr{A}_{\ell^n}$ is contained in the set

$$\big\{(A, B) \in \mathrm{GL}_2(\mathbb{Z}/\ell^n\mathbb{Z}) \times \mathrm{GL}_2(\mathbb{Z}/\ell^n\mathbb{Z}) : \\ \det(A) = v^{k_1-1}, \det(B) = v^{k_2-1}, v \in (\mathbb{Z}/\ell^n\mathbb{Z})^\times\big\},$$

and hence a simple counting argument gives

$$|\mathscr{A}_{\ell^n}| \ll \ell^{7n} \quad \text{and} \quad |\mathscr{C}_{\ell^n}| \ll \ell^{6n}.$$

Therefore now it is clear that, for any integer $h \ge 1$,

$$|\mathscr{A}_h| \ll h^7 \quad \text{and} \quad |\mathscr{C}_h| \ll h^6. \tag{3-17}$$

3B. ***Asymptotic size of $\delta(\ell)$.*** Recall that, for any positive integer $h > 1$,

$$\delta(h) = \frac{|\mathscr{C}_h|}{|\mathscr{A}_h|}.$$

An immediate consequence of the results in the previous section is the following.

Proposition 3.6. *If ℓ varies over primes then, for $n = 1, 2$,*

$$\delta(\ell^n) \sim \frac{1}{\ell^n} \quad \textit{as } \ell \to \infty.$$

Finally, we state the following multiplicative property which plays an important role to prove our main result.

Proposition 3.7 [2, Proposition 3.6]. *For primes $\ell_1, \ell_2 > M$ with $\ell_1 \neq \ell_2$, we have*

$$\delta(\ell_1 \ell_2) = \delta(\ell_1)\delta(\ell_2).$$

4. Analytic results on primes

Recall that f_1 and f_2 are non-CM newforms with integer Fourier coefficients which are not character twists of each other. For a positive integer $h \geq 1$ and a real number $X \geq 2$, consider the function

$$\pi_{f_1, f_2}(X, h) := \sum_{\substack{p \leq X, (p, hN)=1 \\ h | (a_1(p) - a_2(p))}} 1. \tag{4-1}$$

The representation $\bar{\rho}_h$, defined in Section 2, is unramified outside hN. Also, it is ramified at all the primes $\ell \mid h$ because its determinant contains a nontrivial power of the mod-ℓ cyclotomic character which is ramified at ℓ. However, there may exist some primes dividing N at which $\bar{\rho}_h$ is unramified. It follows that a prime p is unramified in L_h only if either $(p, hN) = 1$ or $p \mid N$. Since the image of Frobenius elements under $\bar{\rho}_h$ generate $\mathscr{A}_h$, we can write

$$\pi_{f_1, f_2}(X, h) = \big|\{p \leq X : p \text{ unramified in } L_h,\ \bar{\rho}_h(\mathrm{Frob}_p) \in \mathscr{C}_h\}\big| + O(1),$$

where the error term is due to the possible primes divisors of N which are unramified in L_h. Since the trace and the determinant maps are stable under conjugation, the group $\mathscr{A}_h$ and the set $\mathscr{C}_h$ are also stable under conjugation. Now applying the Chebotarev density theorem (see Proposition 2.1) for the field L_h, we obtain the following.

Proposition 4.1. *Let $f_1 \in S_{k_1}(N_1)$ and $f_2 \in S_{k_2}(N_2)$ be non-CM newforms with rational integral coefficients $a_1(n)$ and $a_2(n)$, respectively. Assume that f_1 and f_2*

are not character twists of each other. Let $N = \mathrm{lcm}(N_1, N_2)$ *and* $h \geq 1$ *be an integer. If GRH holds for the field* L_h, *then*

$$\pi_{f_1,f_2}(X, h) = \delta(h)\pi(X) + O(h^6 X^{1/2} \log(hNX)). \tag{4-2}$$

We remark that to establish Proposition 4.1, we need to use (3-17) and the following variation of a result of Hensel (see [16, Proposition 5, p. 129]):

$$\log d_{L_h} \leq \mathscr{A}_h \log(hN\mathscr{A}_h). \tag{4-3}$$

For our purpose, we now use Proposition 4.1 to obtain the following result giving an upper bound for the set of primes p with $a_1(p) = a_2(p)$. This may be also of independent interest.

Proposition 4.2. *Let* f_1 *and* f_2 *be newforms as before. Then under GRH*

$$\left|\{p \leq X : a_1(p) = a_2(p)\}\right| = O(X^{13/14}).$$

Proof. Clearly, for any prime ℓ,

$$\left|\{p \leq X : a_1(p) = a_2(p)\}\right| \leq \pi_{f_1,f_2}(X, \ell) + O(1).$$

Hence using Proposition 4.1, for a large prime ℓ,

$$\left|\{p \leq X : a_1(p) = a_2(p)\}\right| = O\left(\frac{\pi(X)}{\ell}\right) + O(\ell^6 X^{1/2} \log(\ell N X)).$$

Now by Bertrand's postulate, we choose a prime ℓ between $X^{1/14}/\log X$ and $2(X^{1/14}/\log X)$ and this proves the result. □

Note that in Proposition 4.2, GRH is used for the field L_ℓ, for all but finitely many primes ℓ.

We remark that for newforms of weight 2 and by making use of various abelian extensions, in [13, Theorem 10], a better estimate in Proposition 4.2 is obtained.

We now define

$$\pi^*_{f_1,f_2}(X, h) = \sum_{\substack{p \leq X \\ h \mid (a_1(p) - a_2(p)) \\ a_1(p) \neq a_2(p)}} 1. \tag{4-4}$$

Using Propositions 4.1 and 4.2 we deduce the following.

Proposition 4.3. *Let* f_1 *and* f_2 *be newforms as in Proposition 4.1 and* $h \geq 1$ *be an integer. If GRH holds for the field* L_h, *then*

$$\pi^*_{f_1,f_2}(X, h) = \delta(h)\pi(X) + O(h^6 X^{1/2} \log(hNX)) + O(X^{13/14}). \tag{4-5}$$

Remark 4.4. Indeed, the estimates given in Propositions 4.1 and 4.3 are also valid for the set of primes $p \le X$ with $h \mid (a_1(p) + a_2(p))$. This can be achieved by considering the set $\mathscr{C}_h' = \{(A, B) \in \mathscr{A}_h : \mathrm{tr}(A) = -\mathrm{tr}(B)\}$ instead of $\mathscr{C}_h$ in Section 3 and following the same arguments.

Remark 4.5. In the above propositions, if one assumes Artin's holomorphy conjecture in addition to GRH, then an improved error term can be obtained. More precisely, in Propositions 4.1 and 4.3, we have $O(h^3 X^{1/2} \log(hNX))$ instead of $O(h^6 X^{1/2} \log(hNX))$ which gives the estimate for Proposition 4.2

$$\bigl|\{p \le X : a_1(p) = a_2(p)\}\bigr| = O(X^{7/8}). \tag{4-6}$$

5. Sieving tool: Richert's weighted one-dimensional sieve form

We will prove Theorem 1.1 by using a suitably weighted sieve due to Richert [4]. The sieve problem we encounter here is a one-dimensional sieve problem in the parlance of "sieve methods". We will use notation and conventions from [4].

Let $\mathcal{A}$ be a finite set of integers not necessarily positive or distinct. Let $\mathcal{P}$ be an infinite set of prime numbers. For each prime $\ell \in \mathcal{P}$, let $\mathcal{A}_\ell := \{a \in \mathcal{A} : a \equiv 0 \pmod{\ell}\}$. We write

$$|\mathcal{A}| = X + r_1 \quad \text{and} \quad |\mathcal{A}_\ell| = \delta(\ell)\, X + r_\ell, \tag{5-1}$$

where X (resp. $\delta(\ell)\, X$) and r_1 (resp. r_ℓ) are a close approximation and remainder to $\mathcal{A}$ (resp. $\mathcal{A}_\ell$), respectively. For a square free integer d composed of primes of $\mathcal{P}$, let

$$\mathcal{A}_d = \{a \in \mathcal{A} : a \equiv 0 \pmod d\}, \quad \delta(d) = \prod_{\ell \mid d} \delta(\ell) \quad \text{and} \quad r_d = |\mathcal{A}_d| - \delta(d) X.$$

Notice that the function δ depends on both $\mathcal{A}$ and $\mathcal{P}$. For a real number $z > 0$, let

$$P(z) = \prod_{\ell \in \mathcal{P}, \ell < z} \ell \quad \text{and} \quad W(z) = \prod_{\ell \in \mathcal{P}, \ell < z} (1 - \delta(\ell)).$$

Hypothesis 5.1 [4, p. 29, 142, 219]. *For the above setup, we now state a series of hypotheses.*

Ω_1: *There exists a constant $A_1 > 0$ such that*

$$0 \le \delta(\ell) \le 1 - \frac{1}{A_1} \quad \textit{for all } \ell \in \mathcal{P}.$$

$\Omega_2(1, L)$: *If $2 \le w \le z$, then*

$$-L \le \sum_{w \le \ell \le z} \delta(\ell) \log \ell - \log \frac{z}{w} \le A_2,$$

where $A_2 \ge 1$ and $L \ge 1$ are some constants independent of z and w.

$R(1,\alpha)$: *There exist* $0 < \alpha < 1$ *and* $A_3, A_4 \geq 1$ *such that, for* $X \geq 2$,

$$\sum_{d \leq \frac{X^\alpha}{(\log X)^{A_3}}} \mu(d)^2\, 3^{\omega(d)}\, |r_d| \leq A_4 \frac{X}{(\log X)^2}.$$

For $\mathcal{A}$ and $\mathcal{P}$ as above and for real numbers u, v and λ with $u \leq v$, define the weighted sum

$$\mathcal{W}(\mathcal{A}, \mathcal{P}, v, u, \lambda) = \sum_{\substack{a \in \mathcal{A} \\ (a, P(X^{1/v}))=1}} \Biggl(1 - \sum_{\substack{X^{1/v} \leq q < X^{1/u} \\ q \mid a, q \in \mathcal{P}}} \lambda \Bigl(1 - u \frac{\log q}{\log X}\Bigr)\Biggr). \tag{5-2}$$

We now state the following form of Richert's weighted one-dimensional sieve.

Theorem 5.2 [4, Theorem 9.1, Lemma 9.1]. *With notation as above, assume that Hypothesis 5.1 for* Ω_1, $\Omega_2(1, L)$ *and* $R(1,\alpha)$ *hold for suitable constants* L *and* α. *Suppose further that there exists* $u, v, \lambda \in \mathbb{R}$ *and* $A_5 \geq 1$ *such that*

$$\frac{1}{\alpha} < u < v, \quad \frac{2}{\alpha} \leq v \leq \frac{4}{\alpha}, \quad 0 < \lambda < A_5.$$

Then

$$\mathcal{W}(\mathcal{A}, \mathcal{P}, v, u, \lambda) \geq X W(X^{1/v}) \biggl(F(\alpha, v, u, \lambda) - \frac{cL}{(\log X)^{1/14}}\biggr),$$

where c *is a constant depends at most on* u *and* v (*as well as on the* A_i*'s and* α) *and*

$$F(\alpha, v, u, \lambda) = \frac{2e^\gamma}{\alpha v}\biggl(\log(\alpha v - 1) - \lambda \alpha u \log \frac{v}{u} + \lambda(\alpha u - 1) \log \frac{\alpha v - 1}{\alpha u - 1}\biggr). \tag{5-3}$$

Here γ *is Euler's constant and* X *is the approximation of* $\mathscr{A}$ *given in* (5-1).

6. Proof of Theorem 1.1

We shall closely follow the arguments of [6]. The idea is to apply Theorem 5.2 to the situation

$$\mathcal{A} := \bigl\{|a_1(p) - a_2(p)| : p \leq X,\, a_1(p) \neq a_2(p)\bigr\} \quad \text{and} \quad \mathcal{P} := \{\ell : \ell \geq M\},$$

where $M = M(f_1, f_2)$ is the constant in Section 3. It is clear that, for any $\ell \in \mathcal{P}$,

$$|\mathcal{A}_\ell| = \bigl|\bigl\{p \leq X : a_1(p) \neq a_2(p),\, \ell \mid (a_1(p) - a_2(p))\bigr\}\bigr| = \pi^*_{f_1, f_2}(X, \ell).$$

Applying Proposition 4.3, under GRH, we obtain

$$|\mathcal{A}_\ell| = \delta(\ell) \frac{X}{\log X} + r_\ell,$$

where $r_\ell = O(\ell^6 X^{1/2} \log(\ell N X)) + O(X^{13/14})$. If d is a square free integer composed of primes from $\mathcal{P}$, then from Propositions 3.7 and 4.3 we have

$$\delta(d) = \prod_{\ell \mid d, \ell \in \mathcal{P}} \delta(\ell) \quad \text{and} \quad r_d = O(d^6 X^{1/2} \log(dNX)) + O(X^{13/14}). \tag{6-1}$$

To apply Theorem 5.2, we now verify that hypotheses Ω_1, $\Omega_2(L, 1)$ and $R(1, \alpha)$, given in Hypothesis 5.1, hold for our choice of $\mathcal{A}$ and $\mathcal{P}$.

Lemma 6.1. *Let f_1 and f_2 be newforms as before. Then we have the following*:

(1) *Hypothesis Ω_1 holds with a suitable A_1.*

(2) *Hypothesis $\Omega_2(1, L)$ holds with a suitable L.*

(3) *Under GRH, the hypothesis $R(1, \alpha)$ holds with any $\alpha < \frac{1}{14}$.*

Proof. By Proposition 3.6 the validity of hypotheses Ω_1 and $\Omega_2(1, L)$ are immediate because if $\ell \in P$ then $\delta(\ell) \sim \frac{1}{\ell}$ and this proves hypothesis Ω_1 while the latter one can be achieved by using Mertens's theorem (see [6, Lemmas 4.6.1, 4.6.2, 4.6.3]). So we only give a proof of part (3). From [5, p. 260], we know that $3^{\omega(n)} \leq d(n)^{3\log 3/\log 2} \ll n^\epsilon$. Therefore, for any positive constant A_3, from (6-1), we have

$$\sum_{d \leq \frac{X^\alpha}{(\log X)^{A_3}}} \mu(d)^2\, 3^{\omega(d)}\, |r_d| \ll \sum_{d \leq \frac{X^\alpha}{(\log X)^{A_3}}} (d^{6+\epsilon} X^{1/2} \log(dNX) + X^{13/14}).$$

We now see that, for any $\alpha < \frac{1}{14}$,

$$\sum_{d \leq \frac{X^\alpha}{(\log X)^{A_3}}} \mu(d)^2\, 3^{\omega(d)}\, |r_d| \ll \frac{X}{(\log X)^2}$$

and this completes the proof. $\square$

Next we need to choose sieve parameters α, u, v, λ satisfying conditions in Theorem 5.2. For $k \geq 2$ we take

$$\alpha = \frac{k-1}{14k}, \quad u = \frac{14k+1}{k-1}, \quad v = \frac{56k}{k-1}, \quad \lambda = \frac{1}{k^{1/5}}. \tag{6-2}$$

Clearly, $\frac{1}{\alpha} < u < v$, $\frac{2}{\alpha} \leq v \leq \frac{4}{\alpha}$ and $0 < \lambda < 1$. This shows that these parameters satisfy the conditions required for applying Theorem 5.2, and hence for our choices of $\mathcal{A}$ and $\mathcal{P}$, we obtain

$$\mathcal{W}(\mathcal{A}, \mathcal{P}, v, u, \lambda) \gg \frac{X}{(\log X)^2}\left(F(\alpha, v, u, \lambda) - \frac{cL}{(\log X)^{1/14}}\right).$$

Note that here we have used the fact that $|\mathcal{A}| \gg X/\log X$ and $W(X) \gg 1/\log X$ for $X \gg 0$ which follows immediately by using Proposition 3.6. Also for the choices of sieve parameters α, u, v, λ given in (6-2), the function $F(\alpha, v, u, \lambda)$, defined by (5-3), can be computed explicitly and is given by

$$F\Big(\frac{k-1}{14k}, \frac{56k}{k-1}, \frac{14k+1}{k-1}, \frac{1}{k^{1/5}}\Big) = \frac{e^{\gamma}\big(14k^{6/5}\log 3+\log 42k-(1+14k)\log\big(\frac{56k}{14k+1}\big)\big)}{28k^{6/5}}.$$

Also, $F(\alpha, v, u, \lambda) > 0$ for $k > 1.71\ldots$. Therefore for a fixed weight $k \geq 2$ one can choose X, sufficiently large, such that $F(\alpha, v, u, \lambda) - (cL)/(\log X)^{1/14} > 0$. In other words, we have

$$\mathcal{W}(\mathcal{A}, \mathcal{P}, v, u, \lambda) \gg \frac{X}{(\log X)^2}. \tag{6-3}$$

There are at least $X/(\log X)^2$ many primes $p \leq X$ which make a positive contribution to the left-hand side of (6-3). Therefore to complete the proof of the first part of Theorem 1.1 it is sufficient to show that, for any such prime p,

$$\omega(a_1(p) - a_2(p)) \leq \big[7k + \tfrac{1}{2} + k^{1/5}\big].$$

Let p be such a prime. Then $(a_1(p) - a_2(p), X^{1/v}) = 1$ and

$$1 - \sum_{\substack{X^{1/v} \leq q < X^{1/u} \\ q \mid (a_1(p)-a_2(p))}} \lambda\left(1 - u\frac{\log q}{\log X}\right) > 0. \tag{6-4}$$

Therefore, we write

$$\omega(a_1(p) - a_2(p)) = \sum_{q \mid (a_1(p)-a_2(p))} 1 = \sum_{\substack{X^{1/v} < q < X^{1/u} \\ q \mid (a_1(p)-a_2(p))}} 1 + \sum_{\substack{q \geq X^{1/u} \\ q \mid (a_1(p)-a_2(p))}} 1. \tag{6-5}$$

Now to estimate the first sum on the right of (6-5) we use (6-4) and obtain

$$\sum_{\substack{X^{1/v} < q < X^{1/u} \\ q \mid (a_1(p)-a_2(p))}} 1 < \frac{1}{\lambda} + u \sum_{\substack{X^{1/v} < q < X^{1/u} \\ q \mid (a_1(p)-a_2(p))}} \frac{\log q}{\log X}.$$

For the second sum we observe that if $q \geq X^{1/u}$ then $\log q/\log X \geq \frac{1}{u}$ that gives

$$\sum_{\substack{q \geq X^{1/u} \\ q \mid (a_1(p)-a_2(p))}} 1 \leq u \sum_{\substack{q \geq X^{1/u} \\ q \mid (a_1(p)-a_2(p))}} \frac{\log q}{\log X}.$$

Substituting the last two inequalities in (6-5) yields

$$\omega(a_1(p) - a_2(p)) \leq \frac{1}{\lambda} + u \sum_{q \mid (a_1(p)-a_2(p))} \frac{\log q}{\log X} \leq \frac{1}{\lambda} + u\frac{\log |a_1(p) - a_2(p)|}{\log X}.$$

Using Deligne's estimate we know $|a_1(p)-a_2(p)| \le 4p^{(k-1)/2}$. Therefore for any $p \le X$ as above, we have

$$\omega(a_1(p)-a_2(p)) \le \frac{1}{\lambda} + u\frac{k-1}{2} + u\frac{\log 4}{\log X}.$$

Substituting the values of u and λ from (6-2) and choosing X large enough completes the proof of the first part of the theorem.

Finally, the last assertion of the theorem when $k \ge 6$ can be achieved by taking $\lambda = 1/\sqrt{\log k}$ instead of $\lambda = 1/k^{1/5}$ in the above proof and then following the same arguments.

7. Proof of Theorem 1.6

The idea of the proof is similar to the proof of Theorem 1.1 with minor modifications. We shall apply Theorem 5.2 with the same setting as in Section 6. For $k \ge 2$, we choose the sieve parameters as

$$\alpha = \frac{k-1}{14k}, \quad u = \frac{26k+1}{k-1}, \quad v = \frac{30k}{k-1}, \quad \lambda = \frac{1}{\sqrt{\log k}}.$$

Again, these parameters satisfy the conditions required for Theorem 5.2 and the corresponding function $F(\alpha, v, u, \lambda) > 0$ for $k > 1.006$. Hence as in the proof of Theorem 1.1, the corresponding weighted sum satisfies

$$\mathcal{W}(\mathcal{A}, \mathcal{P}, v, u, \lambda) \gg \frac{X}{(\log X)^2}. \tag{7-1}$$

Next we observe that

$$\left|\{p \le X : \ell^2 \mid (a_1(p)-a_2(p)),\ X^{1/v} \le \ell \le X^{1/u}\}\right| = \sum_{X^{1/v} \le \ell \le X^{1/u}} (\pi_{f_1,f_2}(X,\ell^2) + O(1)),$$

where the error term is due to the presence of those primes p such that $p \mid \ell N$ and $\ell^2 \mid (a_1(p)-a_2(p))$. Applying Proposition 4.1 gives that the left side of the above equality is equal to

$$\pi(X) \sum_{X^{1/v} \le \ell \le X^{1/u}} \frac{1}{\ell^2} + O\Bigg(X^{1/2+\epsilon} \sum_{X^{1/v} \le \ell \le X^{1/u}} \ell^{12}\Bigg).$$

Since $u > 26$, we have

$$\left|\{p \le X : \ell^2 \mid (a_1(p)-a_2(p)),\ X^{1/v} \le \ell \le X^{1/u}\}\right| = o\left(\frac{X}{(\log X)^2}\right). \tag{7-2}$$

We conclude, by combining (7-1) and (7-2), that there are at least $X/(\log X)^2$ many primes $p \le X$ such that

(a) $a_1(p) - a_2(p)$ does not have any prime divisors less than $X^{1/v}$,

(b) for primes $\ell \mid (a_1(p) - a_2(p))$ with $X^{1/v} < \ell < X^{1/u}$, $\ell^2 \nmid (a_1(p) - a_2(p))$,

(c) the contribution of p to the sifting function $\mathcal{W}(\mathcal{A}, \mathcal{P}, v, u, \lambda)$ is positive, i.e.,

$$1 - \sum_{\substack{X^{1/v} \le q < X^{1/u} \\ q \| (a_1(p) - a_2(p))}} \lambda \left(1 - u \frac{\log q}{\log X}\right) > 0.$$

In order to complete the proof, we will show that if $p \le X$ is a prime satisfying the three conditions above then

$$\Omega(a_1(p) - a_2(p)) \le \left[13k + \tfrac{1}{2} + \sqrt{\log k}\,\right].$$

Let $p \le X$ be a prime satisfying (a), (b) and (c). Then, as in the proof of Theorem 1.1,

$$\Omega(a_1(p) - a_2(p)) = \sum_{\substack{X^{1/v} < q < X^{1/u} \\ q \| (a_1(p) - a_2(p))}} 1 + \sum_{\substack{q \ge X^{1/u} \\ q^m | (a_1(p) - a_2(p))}} 1 < \frac{1}{\lambda} + u \sum_{q^m | (a_1(p) - a_2(p))} \frac{\log q}{\log X}$$

which gives

$$\Omega(a_1(p) - a_2(p)) \le \frac{1}{\lambda} + u \frac{\log |a_1(p) - a_2(p)|}{\log X}.$$

Now applying Deligne's estimate and arguing as in the proof of Theorem 1.1, we get the desired result.

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DESINGULARIZATIONS OF QUIVER GRASSMANNIANS FOR THE EQUIORIENTED CYCLE QUIVER

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We construct torus equivariant desingularizations of quiver Grassmannians for arbitrary nilpotent representations of an equioriented cycle quiver. We apply this to the computation of their torus equivariant cohomology.

1. Introduction

Quiver Grassmannians are projective varieties parametrizing subrepresentations of quiver representations. Originating in the geometric study of quiver representations [Schofield 1992] and in cluster algebra theory [Caldero and Chapoton 2006], they have been applied extensively in recent years in a Lie-theoretic context, namely as a fruitful source for degenerations of (affine) flag varieties [Cerulli Irelli et al. 2013; 2017; Feigin et al. 2017; Pütz 2022]. This approach allows for an application of homological methods from the representation theory of quivers to the study of such degenerate structures.

The resulting varieties being typically singular, a construction of natural desingularizations is very desirable. For quiver Grassmannians of representations of Dynkin quivers this was accomplished in [Cerulli Irelli et al. 2013] building on [Feigin and Finkelberg 2013], and for Grassmannians of subrepresentations of loop quivers in [Feigin et al. 2017] (in other directions, this construction was generalized to representations of large classes of finite dimensional algebras in [Crawley-Boevey and Sauter 2017; Keller and Scherotzke 2014; Leclerc and Plamondon 2013]).

In the present paper, we synthesize the approaches of [Cerulli Irelli et al. 2013; Feigin et al. 2017] and construct desingularizations of quiver Grassmannians for nilpotent representations of equioriented cycle quivers, thereby, in particular, desingularizing degenerate affine flag varieties [Pütz 2022].

As an important application, this allows us to describe the equivariant cohomology of degenerate affine flag varieties and more general quiver Grassmannians, in continuation of the program started in [Lanini and Pütz 2023a; 2023b].

In the first section, we recall some background material on quiver Grassmannians for nilpotent representations of the equioriented cycle quiver. In Section 3 we give

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an explicit construction for desingularizations of quiver Grassmannians for nilpotent representations of the equioriented cycle, along the lines of [Cerulli Irelli et al. 2013]. We prove that the desingularization has a particularly favorable geometric structure, namely it is isomorphic to a tower of Grassmann bundle. Consequently, it, admits a cellular decomposition which is compatible with the cellular decomposition of the singular quiver Grassmannian. In Section 4, we recall the definition of certain torus actions on cyclic quiver Grassmannians, together with the necessary framework to compute torus equivariant cohomology. Finally, in Section 5 we prove that the desingularization is equivariant with respect to the torus action as introduced in [Lanini and Pütz 2023a]. This allows to use the construction from that paper for the computation of torus equivariant cohomology to all quiver Grassmannians for nilpotent representations of the equioriented cycle.

2. Quiver Grassmannians for the equioriented cycle

In this section we recall some definitions concerning quiver Grassmannians and representations of the equioriented cycle. We refer to [Kirillov Jr. 2016; Schiffler 2014] for general representation theoretic properties, and to [Cerulli Irelli et al. 2012] for basic properties of quiver Grassmannians.

Generalities on quiver representations. Let Q be a quiver, consisting of a set of vertices Q_0 and a set of arrows Q_1 between the vertices. A Q-representation M consists of a tuple of $\mathbb{C}$-vector spaces $M^{(i)}$ for $i \in Q_0$ and tuple of linear maps $M_\alpha : M^{(i)} \to M^{(j)}$ for $(\alpha : i \to j) \in Q_1$. We denote the category of finite dimensional Q-representations by $\mathrm{rep}_{\mathbb{C}}(Q)$. The morphisms between two objects M and N are tuples of linear maps $\varphi_i : M^{(i)} \to N^{(i)}$ for $i \in Q_0$ such that $\varphi_j \circ M_\alpha = N_\alpha \circ \varphi_i$ holds for all $(\alpha : i \to j) \in Q_1$.

Definition 2.1. For $M \in \mathrm{rep}_{\mathbb{C}}(Q)$ and $\boldsymbol{e} \in \mathbb{N}^{Q_0}$, the **quiver Grassmannian** $\mathrm{Gr}_{\boldsymbol{e}}(M)$ is the closed subvariety of $\prod_{i \in Q_0} \mathrm{Gr}_{e_i}(M^{(i)})$ of all subrepresentations U of M such that $\dim_{\mathbb{C}} U^{(i)} = e_i$ for $i \in Q_0$.

For an isomorphism class $[N]$ of Q-representations, the stratum $\mathcal{S}_{[N]}$ is defined as the set of all points (that is, subrepresentations) $U \in \mathrm{Gr}_{\boldsymbol{e}}(M)$ such that U is isomorphic to N. By [Cerulli Irelli et al. 2012, Lemma 2.4], $\mathcal{S}_{[N]}$ is locally closed and irreducible. If there are only finitely many isomorphism classes of subrepresentations of M, as will be the case in the following, the $\mathcal{S}_{[N]}$ thus define a finite stratification of the quiver Grassmannians.

Every basis B of $M \in \mathrm{rep}_{\mathbb{C}}(Q)$ consists of bases

$$B^{(i)} = \{v_k^{(i)} : k \in [m_i]\}$$

for each vector space $M^{(i)}$ of the Q-representation M, where $m_i := \dim_{\mathbb{C}} M^{(i)}$ for all $i \in Q_0$, and $[m] := \{1, \ldots, m\}$.

Definition 2.2. Let $M \in \mathrm{rep}_{\mathbb{C}}(Q)$ and B a basis of M. The **coefficient quiver** $Q(M, B)$ consists of:

(QM0) The vertex set $Q(M, B)_0 = B$.

(QM1) The set of arrows $Q(M, B)_1$, containing $(\tilde{\alpha} : v_k^{(i)} \to v_\ell^{(j)})$ if and only if $(\alpha : i \to j) \in Q_1$ and the coefficient of $v_\ell^{(j)}$ in $M_\alpha v_k^{(i)}$ is nonzero.

Representations of the equioriented cycle. For $n \in \mathbb{N}$, by Δ_n we denote the equioriented cycle quiver on n vertices. Hence the set of arrows and the sets of vertices are in bijection with $\mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z}$; more precisely, we have $(\Delta_n)_0 = \mathbb{Z}/n\mathbb{Z}$ and arrows $\alpha_i : i \to i+1$ for all $i \in \mathbb{Z}/n\mathbb{Z}$. Here and in the following, we consider all indices modulo n unless specified differently.

A Δ_n-representation M is called N**-nilpotent** for $N \in \mathbb{N}$ if

$$M_{\alpha_{i+N-1}} \circ M_{\alpha_{i+N-2}} \circ \cdots \circ M_{\alpha_{i+1}} \circ M_{\alpha_i} = 0$$

for all $i \in \mathbb{Z}_n$, i.e., all concatenations of the maps of M along the arrows of Δ_n vanish after at most N steps. M is called **nilpotent** if it is N-nilpotent for some N. From now on we use the short hand notation M_i for the map along the arrow α_i.

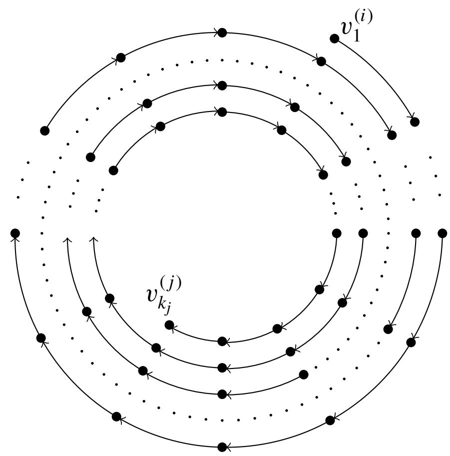
Example 2.3. Let $i \in \mathbb{Z}_n$ and let $\ell \in \mathbb{Z}_{\geq 1}$. Consider the $\mathbb{C}$-vector space V with basis $B = \{b_1, \ldots, b_\ell\}$ equipped with the $\mathbb{Z}_n$-grading given by $\deg(b_k) = i + k - 1 \in \mathbb{Z}_n$. Take the operator $A \in \mathrm{End}(V)$ uniquely determined by setting $Ab_k = b_{k+1}$ for any $k < \ell$ and $Aw_\ell = 0$. The vector space of the corresponding Δ_n-representation over the j-th vertex is spanned by the elements of B of degree j. Let m_j be the number of these basis elements. In this basis, the map from vertex j to $j+1$ is given by a $m_j \times m_{j+1}$ matrix with ones on the diagonal below the main diagonal and all other entries equal to zero. It is immediate to check that this Δ_n-representation is nilpotent. We denote this representation by $U_i(\ell)$.

Proposition 2.4 [Kirillov Jr. 2016, Theorem 7.6.(1)]. *Every indecomposable nilpotent representation of Δ_n is isomorphic to some $U_i(\ell)$.*

Example 2.5. Observe that the basis B from Example 2.3 can be obviously rearranged into the union of ordered bases

$$B^{(i)} = \{v_r^{(i)} : r \in [k_i]\} \quad \text{for } i \in \mathbb{Z}_n,$$

where k_i is the number of elements $b \in B$ with $\deg(b) = i$. With respect to B, the coefficient quiver of $U_i(\ell)$ has the form:



By [Kirillov Jr. 2016, Theorem 1.11], every nilpotent Δ_n-representation is isomorphic to a Δ_n-representation of the form

$$M := \bigoplus_{i \in \mathbb{Z}_n} \bigoplus_{\ell \in [N]} U_i(\ell) \otimes \mathbb{C}^{d_{i,\ell}},$$

with $d_{i,\ell} \in \mathbb{Z}_{\geq 0}$ for all $i \in \mathbb{Z}_n$ and $\ell \in [N]$. Here N is the nilpotence parameter of the representation and the tensor product with the $\mathbb{C}$-vector spaces counts the multiplicities of the indecomposable summands.

Let $\mathbb{C}\Delta_n$ be the path algebra of Δ_n and define the path

$$p_i(N) := (i + N \,|\alpha_{i+N-1} \circ \alpha_{i+N-2} \circ \cdots \circ \alpha_{i+1} \circ \alpha_i \,|\, i)$$

for all $i \in \mathbb{Z}_n$ and some fixed $N \in \mathbb{N}$. We define the path algebra ideal

$$\mathrm{I}_N := \langle p_i(N) : i \in \mathbb{Z}_n \rangle \subset \mathbb{C}\Delta_n,$$

generated by all paths of length N, and we denote the truncated path algebra $\mathbb{C}\Delta_n/\mathrm{I}_N$ by $A_n^{(N)}$. The following is a special case of [Schiffler 2014, Theorem 5.4].

Proposition 2.6. *The category* $\mathrm{rep}_{\mathbb{C}}(\Delta_n, \mathrm{I}_N)$ *of bounded quiver representations is equivalent to the category* $\mathrm{mod}_{\mathbb{C}}(A_n^{(N)})$ *of modules over the truncated path algebra.*

Remark 2.7. The $U_i(N)$ for $i \in \mathbb{Z}_n$ are the longest indecomposable nilpotent representations in $\mathrm{rep}_{\mathbb{C}}(\Delta_n, \mathrm{I}_N)$.

Let $P_i \in \mathrm{rep}_{\mathbb{C}}(\Delta_n, \mathrm{I}_N)$ be the Δ_n-representation starting at vertex $i \in \mathbb{Z}_n$ corresponding to the projective indecomposable $A_n^{(N)}$-module, and let $I_j \in \mathrm{rep}_{\mathbb{C}}(\Delta_n, \mathrm{I}_N)$ be the Δ_n-representation ending at vertex $j \in \mathbb{Z}_n$ corresponding to the injective indecomposable $A_n^{(N)}$-representation. We can identify bounded projective and bounded injective representations of the cycle, via indecomposable nilpotent representations (see [Pütz 2022, Proposition 4.2]):

Proposition 2.8. *For $n, N \in \mathbb{N}$ and all $i, j \in \mathbb{Z}_n$ the projective and injective representations P_i and I_j of the bound quiver (Δ_n, I_N) satisfy*

$$P_i \cong U_i(N) \cong I_{i+N-1} \quad \text{and} \quad I_j \cong U_{j-N+1}(N) \cong P_{j-N+1}.$$

Remark 2.9. In particular, $U_i(N)$ is projective and injective in $\mathrm{rep}_{\mathbb{C}}(\Delta_n, \mathrm{I}_N)$. If we want to emphasize the injective nature of an indecomposable Δ_n-representation we sometimes use the notation $U(j; \ell) := U_{j-\ell+1}(\ell)$.

Parametrization of irreducible components. In Section 3 we will construct desingularizations of all quiver Grassmannians associated to nilpotent representations of the equioriented cycle, which requires knowledge of their irreducible components. Let us first recall the approach: since there are only finitely many isomorphism classes of nilpotent Δ_n-representations in any fixed dimension, the stratification of every quiver Grassmannian into strata $\mathcal{S}_{[N]}$ is finite. Since the strata are irreducible, the irreducible components of quiver Grassmannians are therefore of the form $\overline{\mathcal{S}_{[N]}}$ for certain isomorphism classes $[N]$, which provide a natural labeling (and a canonical representative) of the components.

For arbitrary nilpotent representations of the equioriented cycle the structure of the irreducible components of the associated quiver Grassmannians is not known. In the special case that all indecomposable direct summands of the Δ_n-representation M have length $N = \omega n$ (for $\omega \in \mathbb{N}$) and $\boldsymbol{e} = (\omega k, \dots, \omega k) \in \mathbb{Z}^n$, we have an explicit description of the irreducible components of the quiver Grassmannian $\mathrm{Gr}_{\boldsymbol{e}}(M)$ [Pütz 2022, Lemma 4.10]:

Lemma 2.10. *Let M denote the Δ_n-representation $\oplus_{i \in \mathbb{Z}_n} U(i; \omega n) \otimes \mathbb{C}^{d_i}$ with $\omega \in \mathbb{N}$ and $d_i \in \mathbb{Z}_{\geq 0}$ for all $i \in \mathbb{Z}_n$, define $m := \sum_{i \in \mathbb{Z}_n} d_i$ and $\boldsymbol{e} := (\omega k, \dots, \omega k) \in \mathbb{Z}^n$. The irreducible components of $\mathrm{Gr}_{\boldsymbol{e}}(M)$ are in bijection with the set*

$$C_k(\boldsymbol{d}) := \left\{ \boldsymbol{p} \in \mathbb{Z}^n_{\geq 0} : p_i \leq d_i \ \text{ for all } \ i \in \mathbb{Z}_n, \sum_{i \in \mathbb{Z}_n} p_i = k \right\}$$

and they all have dimension $\omega k(m-k)$.

Remark 2.11. A representative of the open dense stratum in the irreducible component corresponding to $\boldsymbol{p} \in C_k(\boldsymbol{d})$ is

$$U_{\boldsymbol{p}} := \bigoplus_{i \in \mathbb{Z}_n} U(i; \omega n) \otimes \mathbb{C}^{p_i}.$$

Example 2.12. Let $d_i = 1$ for all $i \in \mathbb{Z}_n$. Then by Lemma 2.10 the irreducible components are parametrized by the k-element subsets of $[n]$ and the representatives are

$$\bigoplus_{j \in I} U(j; \omega n)$$

for $I \in \binom{[n]}{k}$. The dimension of the irreducible components is $\omega k(n-k)$.

3. Construction of the desingularization

The approach to the construction of desingularizations of quiver Grassmannians for the equioriented cycle quiver carried out in this section is a synthesis of the approach of Cerulli Irelli et al. [2013] for Dynkin quivers and the approach of Feigin et al. [2017] for the loop quiver. We will construct another quiver for which certain quiver Grassmannians yield desingularizations, which relies on certain favorable homological properties similar to those in [Cerulli Irelli et al. 2013, Section 4]. Note that the present case is not immediately covered by the generalizations [Crawley-Boevey and Sauter 2017; Keller and Scherotzke 2014; Leclerc and Plamondon 2013], and our construction has the advantage of being of a very explicit linear algebra nature.

Bounded representations of the equioriented cylinder. In this subsection we introduce a map $\Lambda : \mathrm{rep}_{\mathbb{C}}(\Delta_n, \mathrm{I}_N) \to \mathrm{rep}_{\mathbb{C}}(Q, \mathrm{I})$ for some bound quiver (Q, I) such that each quiver Grassmannian associated to $\Lambda(M)$ is smooth for all $M \in \mathrm{rep}_{\mathbb{C}}(\Delta_n, \mathrm{I}_N)$. We start with the definition of Q and the ideal I. Let $\hat{\Delta}_{n,N}$ be the quiver with vertices $(\hat{\Delta}_{n,N})_0 = \{(i,k) : i \in \mathbb{Z}_n,\ k \in [N]\}$ and arrows

$$\begin{aligned}\left(\hat{\Delta}_{n,N}\right)_1 = &\left\{\alpha_{i,k} : (i,k) \to (i,k+1) \text{ such that } i \in \mathbb{Z}_n,\ k \in [N-1]\right\}\\ &\cup \left\{\beta_{i,k} : (i,k) \to (i+1,k-1) \text{ such that } i \in \mathbb{Z}_n,\ k \in [N] \setminus \{1\}\right\},\end{aligned}$$

which we call an equioriented cylinder quiver. We define $\hat{\mathrm{I}}_{n,N}$ as the ideal in the path algebra $\mathbb{C}\hat{\Delta}_{n,N}$ generated by the relations

$$\beta_{i,k+1} \circ \alpha_{i,k} \equiv \alpha_{i+1,k-1} \circ \beta_{i,k} \quad \text{and} \quad \alpha_{i+1,N-1} \circ \beta_{i,N} \equiv 0$$

for all $i \in \mathbb{Z}_n$ and all $k \in [N-1] \setminus \{1\}$.

Example 3.1. $\hat{\Delta}_{4,4}$ is the following quiver:

$$\begin{tikzcd}
(3,4) \arrow[rr, "\beta_{3,4}"] & & (4,3) \arrow[rr, "\alpha_{4,3}"] \arrow[dr, "\beta_{4,3}"] & & (4,4) \arrow[dd, "\beta_{4,4}"] \\
& (4,2) \arrow[r, "\beta_{4,2}"'] \arrow[ur, "\alpha_{4,2}"] & (1,1) \arrow[r, "\alpha_{1,1}"'] & (1,2) \arrow[d, "\beta_{1,2}"'] \arrow[dr, "\alpha_{1,2}"] & \\
(3,3) \arrow[uu, "\alpha_{3,3}"] \arrow[ur, "\beta_{3,3}"] & (4,1) \arrow[u, "\alpha_{4,1}"'] & & (2,1) \arrow[d, "\alpha_{2,1}"'] & (1,3) \arrow[dd, "\alpha_{1,3}"] \arrow[dl, "\beta_{1,3}"] \\
& (3,2) \arrow[ul, "\alpha_{3,2}"] \arrow[u, "\beta_{3,2}"'] & (3,1) \arrow[l, "\alpha_{3,1}"'] & (2,2) \arrow[l, "\beta_{2,2}"'] \arrow[dl, "\alpha_{2,2}"] & \\
(2,4) \arrow[uu, "\beta_{2,4}"] & & (2,3) \arrow[ll, "\alpha_{2,3}"] \arrow[ul, "\beta_{2,3}"] & & (1,4) \arrow[ll, "\beta_{1,4}"]
\end{tikzcd}$$

We define a functor $\Lambda : \mathrm{rep}_{\mathbb{C}}(\Delta_n, \mathrm{I}_N) \to \mathrm{rep}_{\mathbb{C}}(\hat{\Delta}_{n,N}, \hat{\mathrm{I}}_{n,N})$ on objects by

$$\Lambda(M) := \hat{M} = \big((\hat{M}^{(i,k)})_{i\in\mathbb{Z}_n, k\in[N]}, (\hat{M}_{\alpha_{i,k}}, \hat{M}_{\beta_{i,k+1}})_{i\in\mathbb{Z}_n, k\in[N-1]}\big),$$

with

$$\begin{aligned}
\hat{M}^{(i,1)} &:= M^{(i)} && \text{for } k = 1,\\
\hat{M}^{(i,k)} &:= M_{i+k-2} \circ M_{i+k-3} \circ \cdots \circ M_{i+1} \circ M_i(M^{(i)}) && \text{for } k \geq 2,\\
\hat{M}_{\alpha_{i,k}} &:= M_{i+k-1}|_{\hat{M}^{(i,k)}} && \text{for } k \geq 1,\\
\hat{M}_{\beta_{i,k}} &:= \iota : \hat{M}^{(i,k)} \hookrightarrow \hat{M}^{(i+1,k-1)} && \text{for } k \geq 2,
\end{aligned}$$

where the inclusions in the last row arise naturally from the definition of the vector spaces of the representation $\hat{M}$. Here M_i denotes the map along the arrow α_i.

Example 3.2. Let $n = N = 2$ and $M = U(1; 2) \oplus U(2; 2)$. The $\hat{\Delta}_{2,2}$-representation $\Lambda(M)$ is

$$\begin{tikzcd}
& \mathbb{C} \arrow[dr, "\binom{0}{1}"] & \\
\mathbb{C}^2 \arrow[ur, "(10)"] & & \mathbb{C}^2 \arrow[dl, "(10)"] \\
& \mathbb{C} \arrow[ul, "\binom{0}{1}"] &
\end{tikzcd}$$

Proposition 3.3. *$\Lambda : \mathrm{rep}_{\mathbb{C}}(\Delta_n, \mathrm{I}_N) \to \mathrm{rep}_{\mathbb{C}}(\hat{\Delta}_{n,N}, \hat{\mathrm{I}}_{n,N})$ as defined above induces a bijection $\Lambda_{N,M} : \mathrm{Hom}_{\Delta_n}(N, M) \to \mathrm{Hom}_{\hat{\Delta}_{n,N}}(\hat{N}, \hat{M})$ for all $N, M \in \mathrm{rep}_{\mathbb{C}}(\Delta_n, \mathrm{I}_N)$ and hence is a fully faithful functor.*

Proof. By construction of Λ, the vector spaces constituting $\hat{M} \in \mathrm{rep}_{\mathbb{C}}(\hat{\Delta}_{n,N}, \hat{\mathrm{I}}_{n,N})$ are subspaces of the corresponding vector spaces constituting M. Hence each morphism in $\mathrm{Hom}_{\Delta_n}(N, M)$ induces a morphism in $\mathrm{Hom}_{\hat{\Delta}_{n,N}}(\hat{N}, \hat{M})$ whose components at the additional vertices are obtained by restriction. It is immediate to check that this induces the desired bijection $\Lambda_{N,M}$, $\Lambda_{N,M}(\mathrm{id}_M) = \mathrm{id}_{\hat{M}}$ and that $\Lambda_{N,M}(\phi) \circ \Lambda_{N,M}(\psi) = \Lambda_{N,M}(\phi \circ \psi)$ holds for all $\phi, \psi \in \mathrm{Hom}_{\Delta_n}(N, M)$ and all $N, M \in \mathrm{rep}_{\mathbb{C}}(\Delta_n, \mathrm{I}_N)$. □

Now we want to describe the image of the indecomposable $U_i(\ell)$ under Λ. Let $A_{\infty\times N}$ be the infinite band quiver of height N, that is, the quiver with vertices (i, k) for $i \in \mathbb{Z}$ and $k \in [N]$ and arrows $\alpha_{i,k} : (i, k) \to (i, k+1)$ and $\beta_{i,k} : (i, k) \to (i+1, k-1)$ whenever both vertices exist. Define a map of quivers $\phi : A_{\infty\times N} \to \hat{\Delta}_{n,N}$, induced by sending each index $i \in \mathbb{Z}$ to its equivalence class $i \in \mathbb{Z}_n$. This extends to a push-down functor $\Phi : \mathrm{rep}_{\mathbb{C}}(A_{\infty\times N}) \to \mathrm{rep}_{\mathbb{C}}(\hat{\Delta}_{n,N})$ with

$$(\Phi(V))^{(i,k)} = \bigoplus_{r\in\mathbb{Z}} V^{(i+rn,k)}, \quad (\Phi(V))_{\alpha_{i,k}} = \bigoplus_{r\in\mathbb{Z}} V_{\alpha_{i+rn,k}}, \quad (\Phi(V))_{\beta_{i,k}} = \bigoplus_{r\in\mathbb{Z}} V_{\beta_{i+rn,k}}$$

for all $V \in \mathrm{rep}_{\mathbb{C}}(A_{\infty\times N})$. Consider the $A_{\infty\times N}$-representation $V(i; \ell)$ with vector spaces $V(i; \ell)^{(j,k)} = \mathbb{C}$ for $(j, k) \in (A_{\infty\times N})_0$ with $i \leq j \leq i+\ell-1$ and $1 \leq k \leq i+\ell-j$

and zero otherwise. The maps along the arrows of $A_{\infty\times N}$ are identities if both the source and target space are one-dimensional and zero otherwise. Using the explicit definitions of the functors Λ and Φ, and the explicit descriptions of the representations $U_i(\ell)$ and $V(i;\ell)$, we can now directly verify that

$$\Lambda(U_i(\ell)) = \Phi(V(i;\ell)).$$

Analogously, we define $A_{\infty\times N}$-representations $V(i,k;\ell)$ consisting of vector spaces $V(i,k;\ell)^{(j,r)} = \mathbb{C}$ for $(j,r)\in(A_{\infty\times N})_0$ with $j\geq i$ and $i+k\leq j+r\leq i+k+\ell$ and $W(i,k;\ell)$ with vector spaces $W(i,k;\ell)^{(j,r)} = \mathbb{C}$ for $(j,r)\in(A_{\infty\times N})_0$ with $j\leq i$ and $i-\ell\leq j+r\leq i+k$.

Example 3.4. For $N=4$ the quiver $A_{\infty\times N}$ is

$$\begin{array}{c}
\cdots\quad (-5,4)\ (-4,4)\ (-3,4)\ (-2,4)\ (-1,4)\ \ (0,4)\ \ (1,4)\ \ (2,4)\ \ (3,4)\quad\cdots\\
\nearrow\ \searrow\ \nearrow\ \searrow\ \nearrow\ \searrow\ \nearrow\ \searrow\ \nearrow\ \searrow\ \nearrow\ \searrow\ \nearrow\ \searrow\ \nearrow\ \searrow\ \nearrow\\
\cdots\quad (-5,3)\ (-4,3)\ (-3,3)\ (-2,3)\ (-1,3)\ \ (0,3)\ \ (1,3)\ \ (2,3)\ \ (3,3)\quad\cdots\\
\searrow\ \nearrow\ \searrow\ \nearrow\ \searrow\ \nearrow\ \searrow\ \nearrow\ \searrow\ \nearrow\ \searrow\ \nearrow\ \searrow\ \nearrow\ \searrow\ \nearrow\ \searrow\\
\cdots\quad (-4,2)\ (-3,2)\ (-2,2)\ (-1,2)\ \ (0,2)\ \ (1,2)\ \ (2,2)\ \ (3,2)\ \ (4,2)\quad\cdots\\
\nearrow\ \searrow\ \nearrow\ \searrow\ \nearrow\ \searrow\ \nearrow\ \searrow\ \nearrow\ \searrow\ \nearrow\ \searrow\ \nearrow\ \searrow\ \nearrow\ \searrow\ \nearrow\\
\cdots\quad (-4,1)\ (-3,1)\ (-2,1)\ (-1,1)\ \ (0,1)\ \ (1,1)\ \ (2,1)\ \ (3,1)\ \ (4,1)\quad\cdots
\end{array}$$

Its representation $V(1,3)$ is of the form

$$\begin{array}{c}
\cdots\quad 0\quad 0\quad 0\quad 0\quad 0\quad 0\quad 0\quad 0\quad 0\quad\cdots\\
\nearrow\ \searrow\ \nearrow\ \searrow\ \nearrow\ \searrow\ \nearrow\ \searrow\ \nearrow\ \searrow\ \nearrow\ \searrow\ \nearrow\ \searrow\ \nearrow\ \searrow\ \nearrow\\
\cdots\quad 0\quad 0\quad 0\quad 0\quad 0\quad \mathbb{C}\quad 0\quad 0\quad 0\quad\cdots\\
\searrow\ \nearrow\ \searrow\ \nearrow\ \searrow\ \nearrow\ \searrow\ \nearrow\ \searrow\ \nearrow\ \searrow\ \nearrow\ \searrow\ \nearrow\ \searrow\ \nearrow\ \searrow\\
\cdots\quad 0\quad 0\quad 0\quad 0\quad \mathbb{C}\quad \mathbb{C}\quad 0\quad 0\quad 0\quad\cdots\\
\nearrow\ \searrow\ \nearrow\ \searrow\ \nearrow\ \searrow\ \nearrow\ \searrow\ \nearrow\ \searrow\ \nearrow\ \searrow\ \nearrow\ \searrow\ \nearrow\ \searrow\ \nearrow\\
\cdots\quad 0\quad 0\quad 0\quad 0\quad \mathbb{C}\quad \mathbb{C}\quad \mathbb{C}\quad 0\quad 0\quad\cdots
\end{array}$$

From now on we erase all zeros and arrows connected to zeros from the picture. Hence we obtain

$$V(i,2,2) = \begin{array}{ccccccc}
 & & \mathbb{C} & & & & \\
 & \nearrow & & \searrow & & & \\
\mathbb{C} & & & & \mathbb{C} & & \\
 & \searrow & & \nearrow & & \searrow & \\
 & & \mathbb{C} & & & & \mathbb{C}
\end{array}
\qquad\text{and}\qquad
W(i,3,2) = \begin{array}{ccccccccc}
 & & & & & & \mathbb{C} & & \\
 & & & & & \nearrow & & \searrow & \\
 & & & & \mathbb{C} & & & & \mathbb{C} \\
 & & & \nearrow & & \searrow & & \nearrow & \\
 & & \mathbb{C} & & & & \mathbb{C} & & \\
 & \nearrow & & \searrow & & \nearrow & & & \\
\mathbb{C} & & & & \mathbb{C} & & & &
\end{array}$$

Homological properties of the category of cylinder representations. In this section, we follow closely the approach of Cerulli Irelli et al. [2013] to establish certain favorable homological properties of the image of the functor Λ.

Proposition 3.5. *The simple, projective and injective objects in* $\operatorname{rep}_{\mathbb{C}}(\hat{\Delta}_{n,N}, \hat{\mathrm{I}}_{n,N})$ *are given by*

$$S_{i,k} := \Phi(V(i,k;0)), \quad P_{i,k} := \Phi(V(i,k;N-k)), \quad I_{i,k} := \Phi(W(i,k;N-k)),$$

respectively, for all $(i,k) \in (\hat{\Delta}_{n,N})_0$.

Proof. For the simple objects this is immediate. The parametrization of the projective and injective representations is a direct computation using the formula based on paths in the quiver $\hat{\Delta}_{n,N}$ (see [Schiffler 2014, Definition 5.3]) and their relations from $\hat{\mathrm{I}}_{n,N}$ as described in the beginning of this section. $\square$

Theorem 3.6. *The category* $\operatorname{rep}_{\mathbb{C}}(\hat{\Delta}_{n,N}, \hat{\mathrm{I}}_{n,N})$ *has global dimension at most two.*

Proof. It suffices to construct projective resolutions of length at most two for all simple representations in $\operatorname{rep}_{\mathbb{C}}(\hat{\Delta}_{n,N}, \hat{\mathrm{I}}_{n,N})$. These representations are denoted by $S_{i,k}$ and consist of a single copy of $\mathbb{C}$ at vertex (i,k) and all other vector spaces and the maps are zero. The projective resolutions of $S_{i,1}$ are of the form

$$0 \to P_{i,2} \to P_{i,1} \to S_{i,1} \to 0$$

and for $S_{i,k}$ with $k \geq 2$ this generalizes to

$$0 \to P_{i+1,k} \to P_{i+1,k-1} \oplus P_{i,k+1} \to P_{i,k} \to S_{i,k} \to 0. \qquad \square$$

Example 3.7. For $N = 4$ and $S_{i,3}$ we obtain the following projective resolution:

$$\begin{array}{ccccccc}
P_{i+1,3} & & P_{1+1,2} \oplus P_{i,4} & & P_{i,3} & & S_{i,3} \\
\begin{array}{ccccccccc}
 & & \mathbb{C} & & & & & & \\
 & \nearrow & & \searrow & & & & & \\
\mathbb{C} & & & & \mathbb{C} & & & & \\
 & \searrow & & \nearrow & & \searrow & & & \\
 & & \mathbb{C} & & & & \mathbb{C} & & \\
 & & & \searrow & & \nearrow & & \searrow & \\
 & & & & \mathbb{C} & & & & \mathbb{C}
\end{array}
& \hookrightarrow &
\begin{array}{ccccccccccc}
\mathbb{C} & & & & \mathbb{C} & & & & & & \\
 & \searrow & & \nearrow & & \searrow & & & & & \\
 & & \mathbb{C}^2 & & & & \mathbb{C} & & & & \\
 & \nearrow & & \searrow & & \nearrow & & \searrow & & & \\
\mathbb{C} & & & & \mathbb{C}^2 & & & & \mathbb{C} & & \\
 & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & \\
 & & \mathbb{C} & & & & \mathbb{C}^2 & & & & \mathbb{C}
\end{array}
& \longrightarrow &
\begin{array}{ccccccccc}
 & & \mathbb{C} & & & & & & \\
 & \nearrow & & \searrow & & & & & \\
\mathbb{C} & & & & \mathbb{C} & & & & \\
 & \searrow & & \nearrow & & \searrow & & & \\
 & & \mathbb{C} & & & & \mathbb{C} & & \\
 & & & \searrow & & \nearrow & & \searrow & \\
 & & & & \mathbb{C} & & & & \mathbb{C}
\end{array}
& \twoheadrightarrow & \mathbb{C}
\end{array}$$

Lemma 3.8. *For* $M \in \operatorname{rep}_{\mathbb{C}}(\Delta_n, \mathrm{I}_N)$ *the injective and projective dimension of* $\hat{M}$ *is at most one and* $\operatorname{Ext}^1_{\hat{\Delta}_{n,N}, \hat{\mathrm{I}}_{n,N}}(\hat{M}, \hat{M}) = 0$.

Proof. It suffices to compute the projective and injective dimension of the image of all indecomposable representations $U_i(\ell) \in \operatorname{rep}_{\mathbb{C}}(\Delta_n, \mathrm{I}_N)$, by exhibiting projective resp. injective resolutions, namely

$$0 \to P_{i,\ell+1} \to P_{i,1} \to \hat{U}_i(\ell) \to 0, \quad 0 \to \hat{U}(j;\ell) \to I_{j,1} \to I_{j-\ell,\ell+1} \to 0,$$

where $j := i + \ell - 1$ and hence $U(j;\ell) = U_i(\ell)$.

It remains to prove vanishing of all

$$\mathrm{Ext}^1_{\hat{\Delta}_{n,N},\hat{\mathrm{I}}_{n,N}}(\hat{U}_i(\ell), \hat{U}_j(\ell')).$$

We apply the functor

$$\mathrm{Hom}_{\hat{\Delta}_{n,N},\hat{\mathrm{I}}_{n,N}}(\_, \hat{U}_j(\ell'))$$

to the above projective resolution of $\hat{U}_i(\ell)$, simplify the terms involving projectives, and obtain the exact sequence

$$0 \to \mathrm{Hom}(\hat{U}_i(\ell), \hat{U}_j(\ell')) \to \hat{U}_j(\ell')^{(i,1)} \xrightarrow{\alpha} \hat{U}_j(\ell')^{(i,\ell+1)} \to \mathrm{Ext}^1(\hat{U}_i(\ell), \hat{U}_j(\ell')) \to 0.$$

By definition of Λ, the map α is the canonical surjection

$$U_j(\ell')^{(i)} \to \big(U_j(\ell')_{i+\ell-1} \circ \cdots \circ U_j(\ell')_i\big)(U_j(\ell')^{(i)}),$$

proving the desired Ext^1-vanishing. □

Example 3.9. For $N = 4$ we obtain the following projective and injective resolutions of $U_i(3) = U(i-2;3)$:

$$\begin{tikzcd}[column sep=tiny]
P_{i,4} &&&&&&&& P_{i,1} &&&&&&& \hat{U}_i(3) \\
\mathbb{C}\arrow[dr] &&&&&&&& \mathbb{C}\arrow[dr] \\
& \mathbb{C}\arrow[dr] &&& \hookrightarrow &&& \mathbb{C}\arrow[ur]\arrow[dr] && \mathbb{C}\arrow[dr] &&& \twoheadrightarrow &&& \mathbb{C}\arrow[dr] && \\
&& \mathbb{C}\arrow[dr] &&&& \mathbb{C}\arrow[ur]\arrow[dr] && \mathbb{C}\arrow[ur]\arrow[dr] && \mathbb{C}\arrow[dr] &&&& \mathbb{C}\arrow[ur]\arrow[dr] && \mathbb{C}\arrow[dr] & \\
&&& \mathbb{C} && \mathbb{C}\arrow[ur] && \mathbb{C}\arrow[ur] && \mathbb{C}\arrow[ur] && \mathbb{C} && \mathbb{C}\arrow[ur] && \mathbb{C}\arrow[ur] && \mathbb{C}
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=tiny]
\hat{U}(i-2;3) &&&&&&&&& I_{i-2,1} &&&&&&&& I_{i-5,4} \\
&&&&&&&&& \mathbb{C}\arrow[dr] &&&&&&&& \mathbb{C} \\
&& \mathbb{C}\arrow[dr] &&& \hookrightarrow &&& \mathbb{C}\arrow[ur]\arrow[dr] && \mathbb{C}\arrow[dr] &&& \twoheadrightarrow &&& \mathbb{C}\arrow[ur] & \\
& \mathbb{C}\arrow[ur]\arrow[dr] && \mathbb{C}\arrow[dr] &&&& \mathbb{C}\arrow[ur]\arrow[dr] && \mathbb{C}\arrow[ur]\arrow[dr] && \mathbb{C}\arrow[dr] &&&& \mathbb{C}\arrow[ur] && \\
\mathbb{C}\arrow[ur] && \mathbb{C}\arrow[ur] && \mathbb{C} && \mathbb{C}\arrow[ur] && \mathbb{C}\arrow[ur] && \mathbb{C}\arrow[ur] && \mathbb{C} && \mathbb{C}\arrow[ur] &&&
\end{tikzcd}$$

The restriction functor. For each $W \in \mathrm{rep}_{\mathbb{C}}(\hat{\Delta}_{n,N}, \hat{\mathrm{I}}_{n,N})$ we define the representation $\mathrm{res}\, W \in \mathrm{rep}_{\mathbb{C}}(\Delta_n, \mathrm{I}_N)$ by

$$\mathrm{res}\, W := \big((W^{(i,1)})_{i\in\mathbb{Z}_n}, (W_{\beta_{i,2}} \circ W_{\alpha_{i,1}})_{i\in\mathbb{Z}_n}\big).$$

This induces maps $\mathrm{res}_{V,W} : \mathrm{Hom}_{\hat{\Delta}_{n,N}}(V, W) \to \mathrm{Hom}_{\Delta_n}(\mathrm{res}\, V, \mathrm{res}\, W)$, by forgetting the components of the morphisms at the vertices (i, k) with $k \geq 2$. Hence we obtain a functor $\mathrm{res} : \mathrm{rep}_{\mathbb{C}}(\hat{\Delta}_{n,N}, \hat{\mathrm{I}}_{n,N}) \to \mathrm{rep}_{\mathbb{C}}(\Delta_n, \mathrm{I}_N)$. The proof of the following proposition is immediate by the construction of Λ and res.

Proposition 3.10. $\mathrm{res} \circ \Lambda(M) = M$ *holds for all* $M \in \mathrm{rep}_{\mathbb{C}}(\Delta_n, \mathrm{I}_N)$.

The desingularization map. In this subsection we provide the construction of the desingularization map, again closely following [Cerulli Irelli et al. 2013]. An example is given below.

Definition 3.11. An isomorphism class $[N]$ of Δ_n-representations is called a **generic subrepresentation type** of $M \in \mathrm{rep}_{\mathbb{C}}(\Delta_n, \mathrm{I}_N)$ to dimension vector $\boldsymbol{e}$, if the stratum $\mathcal{S}_{[N]}$ is open in $\mathrm{Gr}_{\boldsymbol{e}}(M)$. The set of generic subrepresentation types is denoted by $\mathrm{gsub}_{\boldsymbol{e}}(M)$.

Remark 3.12. By construction, for some $[N] \in \mathrm{gsub}_{\boldsymbol{e}}(M)$ the closure of the stratum $\mathcal{S}_{[N]}$ is an irreducible component of $\mathrm{Gr}_{\boldsymbol{e}}(M)$, and all irreducible components are obtained in this way.

Remark 3.13. In general, there is no explicit description of the $\mathrm{gsub}_{\boldsymbol{e}}(M)$. But if the indecomposable summands of M are all of length ωn for $n, \omega \in \mathbb{N}$, we can apply Lemma 2.10.

Example 3.14. Let $n = 3$, $N = 2$ and consider the quiver Grassmannian for $M = U_1(2)^2 \oplus U_2(2)^3 \oplus U_3(2)$ and $\boldsymbol{e} = (1, 2, 3)$. It has eight isomorphism classes of subrepresentations but only two irreducible components. Namely $\overline{\mathcal{S}_{[N_{1,2}]}}$ for $N_1 = U_2(2)^2 \oplus U_3(2)$ and $N_2 = U_1(2) \oplus U_2(2) \oplus U_3(1)^2$. The stratum of N_1 is seven-dimensional whereas the stratum of N_2 is only five-dimensional.

For $[N] \in \mathrm{gsub}_{\boldsymbol{e}}(M)$ we define a map from a quiver Grassmannian of the cylinder quiver to a quiver Grassmannian of the cycle quiver

$$\pi_N : \mathrm{Gr}_{\dim \hat{N}}(\hat{M}) \to \mathrm{Gr}_{\boldsymbol{e}}(M)$$

by $\pi_N(U) := \mathrm{res}\, U$ for all $U \in \mathrm{Gr}_{\dim \hat{N}}(\hat{M})$.

Proposition 3.15. *For each* $[N] \in \mathrm{gsub}_{\boldsymbol{e}}(M)$ *the map*

$$\pi_N : \mathrm{Gr}_{\dim \hat{N}}(\hat{M}) \to \mathrm{Gr}_{\boldsymbol{e}}(M)$$

is injective over $\mathcal{S}_{[N]}$.

Proof. Let $U \in \mathcal{S}_{[N]} \subseteq \mathrm{Gr}_{\boldsymbol{e}}(M)$, then $\dim \hat{U} = \dim \hat{N}$ and

$$\pi_N^{-1}(U) = \left\{ V \in \mathrm{Gr}_{\dim \hat{N}}(\hat{M}) : V^{(i,1)} = U^{(i)} \text{ for all } i \in \mathbb{Z}_n \right\}.$$

In particular $\hat{U}$ is contained in $\pi_N^{-1}(U) \subset \mathrm{Gr}_{\dim \hat{N}}(\hat{M})$. It remains to show that $\pi_N^{-1}(U) = \{\hat{U}\}$: By construction of res and Λ it follows that $\hat{U}^{(i,2)} \subset V^{(i,2)}$ and $\dim_{\mathbb{C}} \hat{U}^{(i,2)} = \dim_{\mathbb{C}} V^{(i,2)}$ holds for all $V \in \pi_N^{-1}(U)$ since U and N are isomorphic. This implies that $\hat{U}^{(i,2)} = V^{(i,2)}$ holds for all $i \in \mathbb{Z}_n$. Inductively, it follows that $V = \hat{U}$. □

Proposition 3.16. *For each* $[N] \in \mathrm{gsub}_{\boldsymbol{e}}(M)$ *the fiber of* π_N *over* $U \in \mathrm{Gr}_{\boldsymbol{e}}(M)$ *is*

$$\pi_N^{-1}(U) = \mathcal{F}_U := \left\{ F \in \mathrm{Gr}_{\dim \hat{N}}(\hat{M}) : \hat{U} \subseteq F \right\} \cong \mathrm{Gr}_{\dim \hat{N} - \dim \hat{U}}(\hat{M}/\hat{U}).$$

Proof. Observe that $\dim U = \dim N$, so that $\dim_{\mathbb{C}} \hat{U}^{(i,1)} = \dim_{\mathbb{C}} \hat{N}^{(i,1)}$ for all $i \in \mathbb{Z}_n$ and the first nontrivial choice of a subspace $F^{(i,k)}$ is over vertices (i, k) with $k \geq 2$. The inclusion $\mathcal{F}_U \subseteq \pi_N^{-1}(U)$ holds since $\pi_N(F) = U$ is clear by definition of $\mathcal{F}_U$ and the construction of the restriction functor. The other inclusion follows since every point V of the fiber $\pi_N^{-1}(U)$ has to contain the vector spaces of $\hat{U}$ in its vector spaces $V^{(i,k)}$ over each vertex (i, k) of $\hat{\Delta}_{n,N}$, in order to map to U. The isomorphism between $\mathcal{F}_U$ and the quiver Grassmannian is a direct consequence of the explicit description of the fiber. □

We are now ready to state the main result of the paper, which is proved after the next proposition.

Theorem 3.17. *Let* $M \in \mathrm{rep}_{\mathbb{C}}(\Delta_n, \mathrm{I}_N)$. *The map*

$$\pi := \bigsqcup_{[N] \in \mathrm{gsub}_{\boldsymbol{e}}(M)} \pi_N : \bigsqcup_{[N] \in \mathrm{gsub}_{\boldsymbol{e}}(M)} \mathrm{Gr}_{\dim \hat{N}}(\hat{M}) \to \mathrm{Gr}_{\boldsymbol{e}}(M)$$

is a desingularization of $\mathrm{Gr}_{\boldsymbol{e}}(M)$.

Remark 3.18. Using Proposition 3.16, we can compute the fiber dimensions for the desingularization to examine whether it is small, in the spirit of [Feigin and Finkelberg 2013, Section 2]. This is the case for the quiver Grassmannian $\mathrm{Gr}_2(M)$ from [Lanini and Pütz 2023b, Example 3.13] where $Q = \Delta_1$ and $M = U_1(2) \oplus S_1^2$. In general, desingularizations of quiver Grassmannians for the cycle are not small. It already fails for the loop quiver (i.e., Δ_1) and the quiver Grassmannian $\mathrm{Gr}_2(N)$ where $N = U_1(2)^2$.

For the proof of Theorem 3.17 we recollect the main properties of the maps π_N:

Proposition 3.19. *Let* $M \in \mathrm{rep}_{\mathbb{C}}(\Delta_n, \mathrm{I}_N)$ *and* $[N] \in \mathrm{gsub}_{\boldsymbol{e}}(M)$. *Then*:

(i) *The variety* $\mathrm{Gr}_{\dim \hat{N}}(\hat{M})$ *is smooth with irreducible equidimensional connected components.*

(ii) *The map* π_N *is one-to-one over* $\mathcal{S}_{[N]}$.

(iii) *The image of* π_N *is closed in* $\mathrm{Gr}_{\boldsymbol{e}}(M)$ *and contains* $\overline{\mathcal{S}_{[N]}}$.

(iv) *The map* π_N *is projective.*

Proof. By Theorem 3.6 and Lemma 3.8 we can apply [Cerulli Irelli et al. 2013, Proposition 7.1] to each quiver Grassmannian $\mathrm{Gr}_{\dim \hat{N}}(\hat{M})$ and obtain the properties stated in (i). Proposition 3.15 is exactly part (ii). The remaining parts are proven analogous to [Cerulli Irelli et al. 2013, Theorem 7.5] since the functor Λ is fully faithful by Proposition 3.3. $\square$

Proof of Theorem 3.17. By [Cerulli Irelli et al. 2021, Proposition 37], we obtain that

$$\mathrm{Gr}_{\dim \hat{N}}(\hat{M}) = \overline{\mathcal{S}_{[\hat{N}]}}$$

since $\hat{M}$ is rigid by Lemma 3.8. With the properties of $\hat{\Delta}_{n,N}$-representations from Theorem 3.6 and Lemma 3.8, the maps π_N as in Proposition 3.19 and Λ as in Proposition 3.3, the rest of the proof is the same as for [Cerulli Irelli et al. 2013, Corollary 7.7]. $\square$

Remark 3.20. In particular, Cerulli Irelli et al. [2021, Proposition 37] proves the conjecture from [Cerulli Irelli et al. 2013, Remark 7.8], about the irreducibility of $\mathrm{Gr}_{\boldsymbol{e}}(\hat{M})$ in [Cerulli Irelli et al. 2013, Corollary 7.7] for arbitrary representations M of a Dynkin quiver.

The following result generalizes [Feigin et al. 2023b, Theorem 7.10].

Theorem 3.21. *For each* $[N] \in \mathrm{gsub}_{\boldsymbol{e}}(M)$ *the quiver Grassmannian* $\mathrm{Gr}_{\dim \hat{N}}(\hat{M})$ *is isomorphic to a tower of fibrations*

$$\mathrm{Gr}_{\dim \hat{N}}(\hat{M}) = X_1 \to X_2 \to \cdots \to X_N = \prod_{i \in \mathbb{Z}_n} \mathrm{Gr}_{\hat{n}^{(i,N)}}(\mathbb{C}^{\hat{m}^{(i,N)}}),$$

where $\hat{\boldsymbol{n}} := \dim \hat{N}$ *and* $\hat{\boldsymbol{m}} := \dim \hat{M}$ *and each map* $X_k \to X_k + 1$ *for* $k \in [N-1]$ *is a fibration with fiber isomorphic to a product of ordinary Grassmannians of subspaces.*

Proof. Every point U of the quiver Grassmannian $\mathrm{Gr}_{\dim \hat{N}}(\hat{M})$ is parametrized by a collection of subspaces $U^{(i,k)} \subseteq M^{(i,k)}$ for $i \in \mathbb{Z}_n$ and $k \in [N]$. In particular it is a point in

$$\mathrm{Gr}_{\hat{\boldsymbol{n}}}(\mathbb{C}^{\hat{\boldsymbol{m}}}) := \prod_{i \in \mathbb{Z}_n} \prod_{k \in [N]} \mathrm{Gr}_{\hat{n}^{(i,k)}}(\mathbb{C}^{\hat{m}^{(i,k)}}).$$

Define X_k as the image of $\mathrm{Gr}_{\dim \hat{N}}(\hat{M})$ in the variety $\mathrm{Gr}_{\hat{\boldsymbol{n}}}(\hat{\boldsymbol{m}})^{(k)}$ which is defined analogous to $\mathrm{Gr}_{\hat{\boldsymbol{n}}}(\mathbb{C}^{\hat{\boldsymbol{m}}})$, with the only difference that the second product runs over $\{k, k+1, \ldots, N\}$ instead of $[N]$. Hence $\mathrm{Gr}_{\dim \hat{N}}(\hat{M}) = X_1$ follows by construction.

We proceed by decreasing induction starting from $k = N$. Every point in the product of Grassmannians of subspaces $\mathrm{Gr}_{\hat{\boldsymbol{n}}}(\hat{\boldsymbol{m}})^{(N)}$ can be extended to an element of $\mathrm{Gr}_{\dim \hat{N}}(\hat{M})$ since the upper vector spaces of an element in the quiver Grassmannian are not related. This implies $X_N = \mathrm{Gr}_{\hat{\boldsymbol{n}}}(\hat{\boldsymbol{m}})^{(N)}$ as desired.

Now assume that the vector spaces $U^{(i,k')}$ are fixed for all $i \in \mathbb{Z}_n$ and $k' > k$. Since U has to be contained in the quiver Grassmannian it has to satisfy the relations

$$\hat{M}_{\alpha_{i+1,k}} \circ \hat{M}_{\beta_{i,k+1}} U^{(i,k+1)} \subseteq U^{(i+1,k+1)} \quad \text{for all } i \in \mathbb{Z}_n.$$

Hence the next layer of vector spaces $U^{(i,k)}$ requires

$$\hat{M}_{\beta_{i-1,k+1}} U^{(i-1,k+1)} \subseteq U^{(i,k)} \quad \text{and} \quad \hat{M}_{\alpha_{i,k}} U^{(i,k)} \subseteq U^{(i,k+1)} \qquad \text{for all } i \in \mathbb{Z}_n.$$

This is equivalent to the choice of a point in the Grassmannian

$$\mathrm{Gr}_{\hat{n}^{(i,k)} - \hat{n}^{(i-1,k+1)}}(U^{(i,k)} / \hat{M}_{\beta_{i-1,k+1}} U^{(i-1,k+1)})$$

because every map $\hat{M}_{\alpha_{i,k}}$ is a projection where the last $\hat{m}^{(i,k)} - \hat{m}^{(i,k+1)}$ coordinates are sent to zero and each $\hat{M}_{\beta_{i,k}}$ is an inclusion. □

Remark 3.22. The explicit description of the desingularization in Theorem 3.21 allows to construct a cellular decomposition of $\mathrm{Gr}_{\dim \hat{N}}(\hat{M})$ (see Theorem 5.5). In particular, it implies that $\mathrm{Gr}_{\dim \hat{N}}(\hat{M})$ is smooth.

4. Torus equivariant cohomology and equivariant Euler classes

In this section we briefly recall definitions and constructions concerning torus actions on quiver Grassmannians, torus equivariant cohomology and torus equivariant Euler classes. More details on the general theory is found in [Arabia 1998; Brion 1998; Goresky et al. 1998; Gonzales 2014]. The application to quiver Grassmannians is introduced in [Lanini and Pütz 2023a; 2023b]. In Section 5 we provide examples and apply our desingularizations to the computation of equivariant cohomology of quiver Grassmannians for the equioriented cycle.

Moment graph and torus equivariant cohomology. Let X be a projective algebraic variety over $\mathbb{C}$. The action of an algebraic torus $T \cong (\mathbb{C}^*)^r$ on X is **skeletal** if the number of T-fixed points and the number of one-dimensional T-orbits in X is finite. We call a cocharacter $\chi \in \mathfrak{X}_*(T)$ **generic** for the T-action on X if $X^T = X^{\chi(\mathbb{C}^*)}$. By $\mathfrak{X}^*(T)$ we denote the character lattice of T. The T-equivariant cohomology of X with rational coefficients is denoted by $H_T^\bullet(X)$.

Definition 4.1. The pair (X, T) is a **GKM-variety** if the T-action on X is skeletal and the rational cohomology of X vanishes in odd degrees.

Remark 4.2. By [Brion 2000, Lemma 2] this is equivalent to [Lanini and Pütz 2023a, Definition 1.4].

The closure $\bar{E}$ of every one-dimensional T-orbit E in a projective GKM-variety admits an T-equivariant isomorphism to $\mathbb{CP}^1$. Thus each one-dimensional T-orbit connects two distinct T-fixed points of X.

Definition 4.3. Let (X, T) be a GKM-variety, and let $\chi \in \mathfrak{X}_*(T)$ be a generic cocharacter. The corresponding **moment graph** $\mathcal{G} = \mathcal{G}(X, T, \chi)$ of a GKM-variety is given by the following data:

(MG0) The T-fixed points as vertices, i.e., $\mathcal{G}_0 = X^T$.

(MG1) The closures of one-dimensional T-orbits $\bar{E} = E \cup \{x, y\}$ as edges in $\mathcal{G}_1$, oriented from x to y if $\lim_{\lambda\to 0} \chi(\lambda).p = x$ for $p \in E$.

(MG2) Every $\bar{E}$ is labeled by $\alpha_E \in \mathfrak{X}^*(T)$ describing the T-action on E.

Theorem 4.4 [Goresky et al. 1998, Theorem 1.2.2]. *Let (X, T) be a GKM-variety with moment graph $\mathcal{G} = \mathcal{G}(X, T, \chi)$ and set $R := H_T^\bullet(pt)$. Then*

$$H_T^\bullet(X) \cong \left\{ (f_x) \in \bigoplus_{x\in\mathcal{G}_0} R \;\middle|\; f_{x_E} - f_{y_E} \in \alpha_E R \text{ for any } \bar{E} = E \cup \{x_E, y_E\} \in \mathcal{G}_1 \right\}.$$

Remark 4.5. The characters from (MG2) are only unique up to a sign. This sign does not play a role in Theorem 4.4. Hence we can fix our favorite convention.

BB-filterable varieties. In this subsection we describe a class of varieties which admit an explicit formula for the computation of their equivariant cohomology. Let X be a $\mathbb{C}^*$-variety. By $X^{\mathbb{C}^*}$ we denote its fixed point set and $X_1, \dots, X_m$ denote the connected components of $X^{\mathbb{C}^*}$. This induces a decomposition

$$X = \bigcup_{i\in[m]} W_i, \quad \text{with } W_i := \left\{ x \in X : \lim_{z\to 0} z.x \in X_i \right\}, \tag{4.6}$$

where W_i is called attracting set of X_i. Since decompositions of this type were first studied by Bialynicki-Birula [1973], we call it a **BB-decomposition**.

Definition 4.7. We say that W_i from (4.6) is a **rational cell** if it is rationally smooth at all $w \in W_i$. This in turn holds if

$$H^{2\dim_{\mathbb{C}}(W_i)}(W_i, W_i \setminus \{w\}) \simeq \mathbb{Q} \quad \text{and} \quad H^m(W_i, W_i \setminus \{w\}) = 0$$

for any $m \neq 2\dim_{\mathbb{C}}(W_i)$ (see [Gonzales 2014, p. 292, Definition 3.4]).

Definition 4.8. A projective T-variety X is **BB-filterable** if:

(BB1) The fixed point set X^T is finite.

(BB2) There exists a generic cocharacter $\chi : \mathbb{C}^* \to T$, i.e., $X^{\chi(\mathbb{C}^*)} = X^T$, such that the associated BB-decomposition consists of rational cells.

Theorem 4.9 (see [Lanini and Pütz 2023a, Theorem 1.15]). *Let X be a BB-filterable projective T-variety. Then*:

(1) *X admits a filtration into T-stable closed subvarieties Z_i such that*

$$\varnothing = Z_0 \subset Z_1 \subset \cdots \subset Z_{m-1} \subset Z_m = X.$$

(2) *Each $W_i = Z_i \setminus Z_{i-1}$ is a rational cell, for all $i \in [m]$.*

(3) *The singular rational cohomology of Z_i vanishes in odd degrees, for $i \in [m]$.*

(4) *If, additionally, the T-action on X is skeletal, each Z_i is a GKM-variety.*

Euler classes and cohomology module bases. For the precise definition of Euler classes we refer the reader to [Arabia 1998, Section 2.2.1]. Instead we give three properties which are enough to determine the equivariant Euler classes in our setting.

Lemma 4.10 (see [Brion 1998, Corollary 15, Lemma 16, Theorem 18]). *Let Y be a T-variety and $y \in Y^T$.*

(1) *If Y is smooth at y then* $\mathrm{Eu}_T(y, Y) = (-1)^{\dim(Y)} \det T_y Y$, *where* $\det T_y Y$ *is the product of the characters by which T acts on the tangent space $T_y Y$.*

(2) *If Y is rationally smooth at y then* $\mathrm{Eu}_T(y, Y) = z \cdot \det T_y Y$, *for some $z \in \mathbb{Q} \setminus \{0\}$.*

(3) *If $\pi : Y \to X$ is a T-equivariant resolution of singularities and $|Y^T| < \infty$, then*
$$\mathrm{Eu}_T(x, X)^{-1} = \sum_{y \in Y^T, \pi(y) = x} \mathrm{Eu}_T(y, Y)^{-1}.$$

Remark 4.11. Lemma 4.10 differs from [Brion 1998] by using Euler classes instead of equivariant multiplicities which are inverse to each other up to a sign.

Definition 4.12 (see [Gonzales 2014, Lemma 6.7]). Let $X^T = \{x_1, \dots, x_m\}$. For $i \in [m]$, the local index of $f \in H_T^\bullet(X)$ at $x_i \in X^T$ is

$$I_i(f) = \sum_{j \in [m] : x_j \in Z_i} \frac{f_{x_j}}{\mathrm{Eu}_T(x_j, Z_i)}.$$

The next theorem gives an explicit formula to compute a basis for $H_T^\bullet(X)$ as free module over $H_T^\bullet(pt)$. Observe that everything depends on the order of the fixed points which is in general not unique.

Theorem 4.13 (see [Lanini and Pütz 2023a, Theorem 2.12]). *Let (X, T) be a BB-filterable GKM-variety with filtration*

$$\varnothing = Z_0 \subset Z_1 \subset \cdots \subset Z_m = X$$

as in Theorem 4.9. Let $X^T = \{x_1, \dots, x_m\}$ with $x_i \in W_i = Z_i \setminus Z_{i-1}$. There exists a unique basis $\{\theta^{(i)}\}_{i \in [m]}$ of $H_T^\bullet(X)$ as free module over $H_T^\bullet(pt)$, such that for any $i \in [m]$ the following properties hold:

(1) $\theta^{(i)}_{x_j} = 0$ *for all* $j < i$.

(2) $\theta^{(i)}_{x_i} = \mathrm{Eu}_T(x_i, Z_i)$.

(3) $I_j(\theta^{(i)}) = 0$ *for all* $j \neq i$.

Remark 4.14. Observe that (1) and (2) imply $I_i(\theta^{(i)}) = 1$ by Definition 4.12.

Torus action on cyclic quiver Grassmannians. We briefly recall torus actions on quiver Grassmannians for the equioriented cycle (see [Lanini and Pütz 2023a, Section 5]).

Remark 4.15. From now on we assume the choice of a basis B of M such that the connected components of $Q(M, B)$ are in bijection with the indecomposable direct summands of M. Such a choice is always possible by [Kirillov Jr. 2016, Theorem 1.11].

A **grading** of $M \in \mathrm{rep}_{\mathbb{C}}(\Delta_n)$ with respect to a fixed basis is a map $\mathrm{wt} : B \to \mathbb{Z}^B$. This induces an action of $\lambda \in \mathbb{C}^*$ by

$$\lambda.b := \lambda^{\mathrm{wt}(b)} \cdot b.$$

Remark 4.16. Combining several weight functions $\mathrm{wt}_1, \dots, \mathrm{wt}_D : B \to \mathbb{Z}^B$, we can define the action of $\lambda = (\lambda_j)_{j\in[D]} \in (\mathbb{C}^*)^D$ by

$$\lambda.b := \prod_{j\in[D]} \lambda_j^{\mathrm{wt}_j(b)} \cdot b = \lambda_1^{\mathrm{wt}_1(b)} \cdot \dots \cdot \lambda_D^{\mathrm{wt}_D(b)} \cdot b.$$

Observe that this action extends to the quiver Grassmannian $\mathrm{Gr}_{\boldsymbol{e}}(M)$ only under some additional assumptions about the grading (see [Lanini and Pütz 2023a, Lemma 5.12]).

Theorem 4.17 (see [Lanini and Pütz 2023a, Theorem 6.6]). *Let M be a nilpotent representation of Δ_n with d-many indecomposable direct summands, and let $\boldsymbol{e} \leq \dim M$ be such that* $\mathrm{Gr}_{\boldsymbol{e}}(M)$ *is nonempty. Let* $T := (\mathbb{C}^*)^{d+1}$ *act on* $\mathrm{Gr}_{\boldsymbol{e}}(M)$ *as in* [Lanini and Pütz 2023a, Lemma 5.12]. *Then* $(\mathrm{Gr}_{\boldsymbol{e}}(M), T)$ *is a projective BB-filterable GKM-variety.*

Remark 4.18. If the desingularizations constructed in Theorem 3.17 are T-equivariant, this theorem implies that we can compute the T-equivariant cohomology of all quiver Grassmannians for nilpotent representations of Δ_n, using Theorem 4.13.

From now on we assume that $T := (\mathbb{C}^*)^{d+1}$ acts on $\mathrm{Gr}_{\boldsymbol{e}}(M)$ as in [Lanini and Pütz 2023a, Lemma 5.12]. Here d is the number of connected components in $Q(M, B)$ and the additional parameter comes from cyclic symmetry. The weight functions of the action are defined implicitly by the formula used in [Lanini and Pütz 2023a, Section 5.2].

5. Torus equivariant desingularization and application

In this section we apply the methods from the previous section to compute Euler classes at singular points and torus equivariant cohomology of quiver Grassmannians for the equioriented cycle using their desingularizations as constructed in Theorem 3.17. It remains to show that these desingularizations are torus equivariant.

Torus action on the desingularization. Let $M \in \mathrm{rep}_{\mathbb{C}}(\Delta_n, \mathrm{I}_N)$ be nilpotent with d-many indecomposable direct summands, and let $T := (\mathbb{C}^*)^{d+1}$ act on $\mathrm{Gr}_{\boldsymbol{e}}(M)$ as in [Lanini and Pütz 2023a, Lemma 5.12].

Remark 5.1. A choice of basis B of $M \in \mathrm{rep}_{\mathbb{C}}(\Delta_n, \mathrm{I}_N)$ induces a basis $\hat{B}$ of $\hat{M} \in \mathrm{rep}_{\mathbb{C}}(\hat{\Delta}_{n,N}, \hat{\mathrm{I}}_{n,N})$ such that the connected components of $Q(\hat{M}, \hat{B})$ are in bijection with the images of the indecomposable summands of M. In particular the basis $\hat{B}^{(i,k)}$ over the vertex (i,k) of $\hat{\Delta}_{n,N}$ is a subset in the basis $B^{(i+k-1)}$ of cardinality $m_{i+k-1} - c$ where c is the corank of the map $M_{\alpha_{i+k-2}} \circ \cdots \circ M_{\alpha_i}$ if $k \geq 2$ and $\hat{B}^{(i,k)} = B^{(i)}$ for $k = 1$. This allows us to extend the T-action to the vector spaces of $\hat{M}$ by extending the weight functions according to the inclusions of the basis described above. In other words, all basis vectors of $\hat{B}$ which have the same image in B get the same weight.

Proposition 5.2. *The T-action on the vector spaces of $\hat{M}$ as defined in Remark 5.1 extends to every quiver Grassmannian* $\mathrm{Gr}_{\boldsymbol{k}}(\hat{M})$.

Proof. We have to show that the T-action is compatible with the maps of the quiver representation $\hat{M}$. By construction of the action and the representation $\hat{M}$, this follows immediately from the compatibility of the T-action (on the vector spaces of M) with the maps of M as shown in [Lanini and Pütz 2023a, Lemma 5.12]. □

Lemma 5.3. *The desingularization of Theorem 3.17 is T-equivariant.*

Proof. With Proposition 5.2, the statement follows immediately from the construction of the grading as in Remark 5.1 together with the description of the desingularization in Theorem 3.17. □

Remark 5.4. The T-equivariance of the desingularization allows us to use [Lanini and Pütz 2023a, Lemma 2.1(3)] for the computation of equivariant Euler classes at the singular points of $\mathrm{Gr}_{\boldsymbol{e}}(M)$. This allows us to apply [Lanini and Pütz 2023a, Theorem 2.12] about the construction of a basis for the T-equivariant cohomology to all quiver Grassmannians for nilpotent representations of the cycle.

Cellular decomposition of the desingularization.

Theorem 5.5. *For $[N] \in \mathrm{gsub}_{\boldsymbol{e}}(M)$ the T-fixed points of $\mathrm{Gr}_{\dim \hat{N}}(\hat{M})$ are exactly the preimages of the T-fixed points of $\overline{\mathcal{S}_{[N]}} \subset \mathrm{Gr}_{\boldsymbol{e}}(M)$ under π_N. The $\mathbb{C}^*$-attracting sets of these points provide a cellular decomposition of* $\mathrm{Gr}_{\dim \hat{N}}(\hat{M})$.

Proof. The T-equivariance of π_N from Lemma 5.3 gives the desired description of the fixed points. Now we prove that the $\mathbb{C}^*$-attracting sets of these fixed points from the BB-decomposition are cells. By [Carrell 2002, Lemma 4.12], they provide an α-partition, i.e., there exists a total order of the fixed points

$$\mathrm{Gr}_{\dim \hat{N}}(\hat{M})^{\mathbb{C}^*} = \{p_1, \dots, p_r\}$$

such that $\bigsqcup_{j=1}^{s} W_i$ is closed in $\mathrm{Gr}_{\dim \hat{N}}(\hat{M})$ for all $s \in [r]$. It remains to show that they are isomorphic to affine spaces. This is induced by the cellular decomposition of $\mathrm{Gr}_{\boldsymbol{e}}(M)$ and the T-equivariance of the desingularization:

Assume $p \in \mathrm{Gr}_{\dim \hat{N}}(\hat{M})$ is a T-fixed point. The vector space $p^{(i,k)}$ over the vertex (i,k) of $\hat{\Delta}_{n,N}$ is a point in the Grassmannian of subspaces $\mathrm{Gr}_{\hat{n}^{(i,k)}}(\mathbb{C}^{\hat{m}^{(i,k)}})$. By construction of the $\mathbb{C}^*$-action (as in Remark 5.1), the attracting set of $p^{(i,k)}$ in $\mathrm{Gr}_{\hat{n}^{(i,k)}}(\mathbb{C}^{\hat{m}^{(i,k)}})$ is a cell. The attracting set of p in the whole quiver Grassmannian is the intersection of these cells along the maps of $\hat{M}$. We proceed by induction on k. For $k = 1$ there is nothing to show because there are no maps between the vector spaces. If $k = 2$, we have the original vector spaces of the representation M and one additional layer of subspaces therein. The relations between the coordinates in the attracting sets are the same as for $\mathrm{Gr}_{\boldsymbol{e}}(M)$. Hence they are cells by [Lanini and Pütz 2023a, Theorem 5.7]. The maps of $\hat{M}$ along the arrows $\beta_{i,k}$ of $\hat{\Delta}_{n,N}$ are inclusions and the maps along $\alpha_{i,k}$ are projections where the last $m^{(i,k)} - m^{(i,k+1)}$ coordinates are sent to zero (see [Lanini and Pütz 2023a, Proposition 4.8]). Thus we obtain that the intersecting relations for each $k \in [N]$ are of the form as described in [Lanini and Pütz 2023a, Theorem 5.7]. This implies the desired isomorphisms to affine spaces. □

Remark 5.6. In the setting that

$$M = \bigoplus_{i \in \mathbb{Z}_n} U_i(\omega n) \quad \text{and} \quad \boldsymbol{e} = (\omega k, \dots, \omega k) \in \mathbb{Z}^n$$

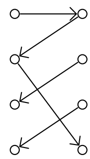
it is possible to strengthen the results concerning the desingularization (see [Feigin et al. 2023a, Sections 2.5 and 2.6]). Namely, $\mathrm{Gr}_{\boldsymbol{e}}(M)$ has $\binom{n}{k}$ explicitly described irreducible components (see Example 2.12) and the cells of $\mathrm{Gr}_{\dim \hat{N}}(\hat{M})$ are the strata of the corresponding T-fixed points.

Example. Now, we provide an explicit example for the constructions from the previous sections. Let $M := U_1(4) \oplus U_2(2) \oplus U_2(2)$ be a Δ_2-representation and fix the dimension vector $\boldsymbol{e} = (2, 2)$. The quiver Grassmannian $\mathrm{Gr}_{\boldsymbol{e}}(M)$ has five strata (i.e., isomorphism classes of subrepresentations) with the representatives:

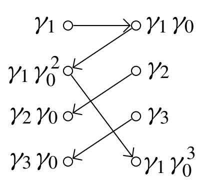
$$V_1 := U_1(4), \quad V_2 := S_2 \oplus U_1(3), \quad V_3 := U_2(2) \oplus U_2(2),$$
$$V_4 := U_1(2) \oplus U_2(2), \quad V_5 := S_1 \oplus S_2 \oplus U_2(2).$$

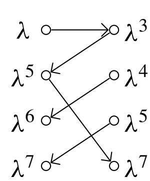
The stratum of V_2 is three-dimensional, the strata of V_1, V_3 and V_4 are two-dimensional and the stratum of V_5 is one-dimensional. This is computed using [Pütz 2022, Proposition 4.4] and [Cerulli Irelli et al. 2012, Lemma 2.4].

Let the basis B of M be the union of the standard basis for each indecomposable summand of M. Then its coefficient quiver is

$Q(M, B) =$ 

where the arrows from left to right have $\alpha : 1 \to 2$ as underlying arrow in Δ_2. The arrows from right to left have $\beta : 2 \to 1$ as underlying arrow. We define the action of $\gamma := (\gamma_0, \gamma_1, \gamma_2, \gamma_3) \in T := (\mathbb{C}^*)^{3+1}$ and $\lambda \in \mathbb{C}^*$ on B as

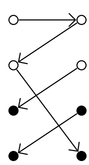
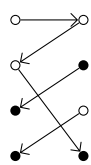
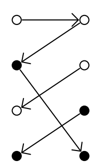
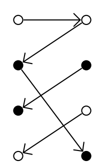


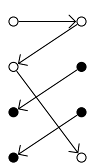
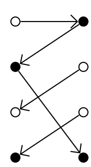
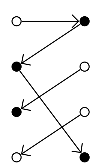
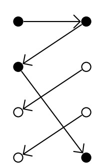


These actions extend linearly to the vector spaces of M and to the whole quiver Grassmannian by [Lanini and Pütz 2023a, Lemma 5.12]. Moreover,

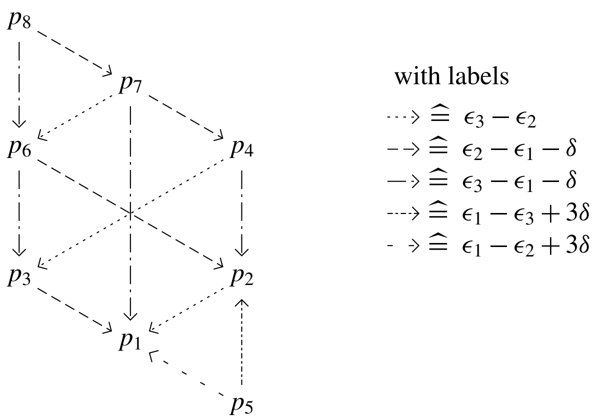
$$\chi : \mathbb{C}^* \to T, \quad \lambda \mapsto (\lambda^2, \lambda, \lambda, \lambda)$$

is a generic cocharacter by [Lanini and Pütz 2023a, Theorem 5.14]. We apply [Cerulli Irelli 2011, Theorem 1] to compute the fixed points of both actions:

$p_1 =$  $p_2 =$  $p_3 =$  $p_4 =$ 

$p_5 =$  $p_6 =$  $p_7 =$  $p_8 =$ 

Here the black vertices indicate the corresponding subrepresentation of M. The pairs p_1 and p_2, p_3 and p_4, and p_6, p_7 are each isomorphic as subrepresentations of M. The attracting sets of the fixed points are cells by [Pütz 2022, Theorem 4.13]. Their dimension equals the number of out going arrows in the following moment graph which is computed using [Lanini and Pütz 2023a, Theorem 6.15].



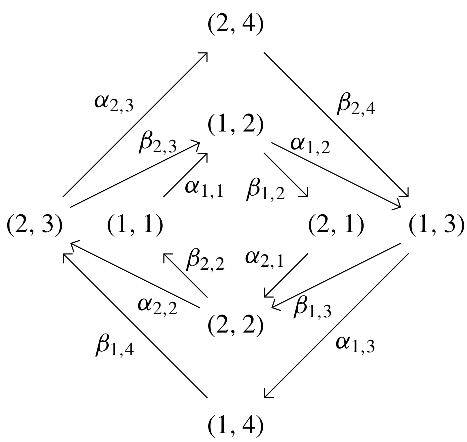
The labels are expressed as linear combination of the characters

$$\begin{aligned} \epsilon_i : T \to \mathbb{C}^*, \qquad & (\gamma_0, \gamma_1, \dots, \gamma_d) \mapsto \gamma_i \quad \text{for } i \in [d], \\ \delta : T \to \mathbb{C}^*, \qquad & (\gamma_0, \gamma_1, \dots, \gamma_d) \mapsto \gamma_0. \end{aligned}$$

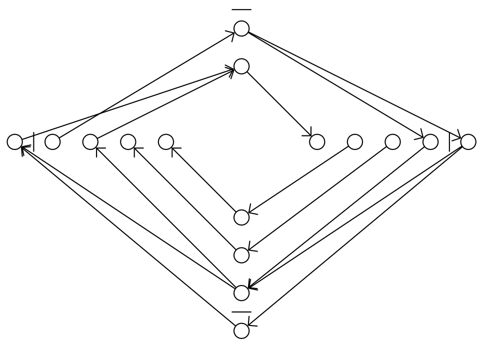
Here the dashed lines were used to highlight the symmetries of the labeling and avoid to write the labels in the picture.

There are four points which are not rationally smooth. Namely the tangent spaces at p_1, p_2, p_6 and p_7 are four-dimensional, whereas $\mathrm{Gr}_{\boldsymbol{e}}(M)$ itself is three-dimensional. We can read this from the picture as follows: the number of edges adjacent to a point is the dimension of its tangent space and the number of outgoing edges is the cell dimension. The irreducible components are obtained as closure of the strata of the points p_8, p_7 and p_5, because their strata are not contained in the closure of any other stratum. These are generic subrepresentation types of M for dimension vector $\boldsymbol{e} = (2, 2)$. Hence the desingularization of $\mathrm{Gr}_{\boldsymbol{e}}(M)$ consists of three components.

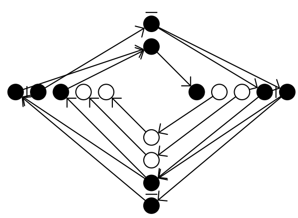
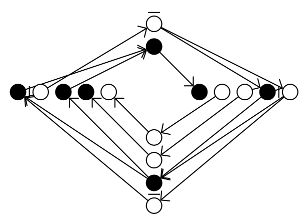
The extended quiver $\hat{\Delta}_{2,4}$ is

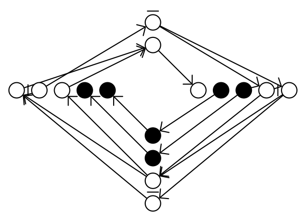


For the basis induced by the basis B of M, the coefficient quiver of $\hat{M}$ is

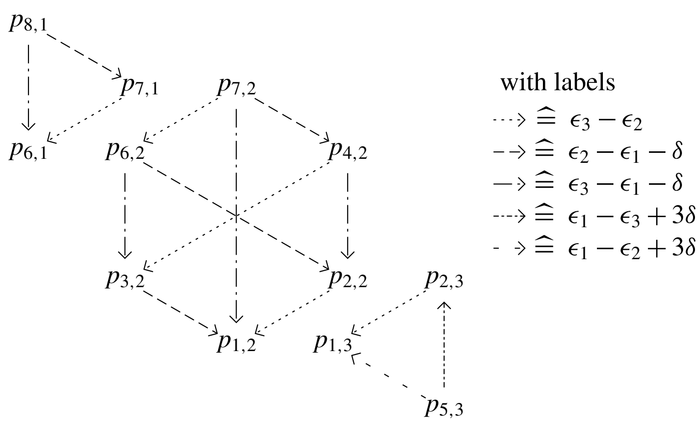


Here the separating lines between the vertices indicate if they live over the inner or outer vertex of $\hat{\Delta}_{2,4}$ in that position. Representatives for the extended representations of the generic subrepresentation types are

$\hat{V}_1 =$  $\hat{V}_2 =$ 

$\hat{V}_3 =$ 

With the explicit description of the cellular decomposition of the quiver Grassmannians $\mathrm{Gr}_{\dim \hat{V}_1}(\hat{M})$, $\mathrm{Gr}_{\dim \hat{V}_2}(\hat{M})$ and $\mathrm{Gr}_{\dim \hat{V}_3}(\hat{M})$ from Theorem 5.5, it is a straightforward computation that their moment graphs are



Here $\hat{p}_{i,j}$ is the preimage of p_i in $\mathrm{Gr}_{\dim \hat{V}_j}(\hat{M})$. Moreover, from the cellular decompositions we obtain the isomorphisms

$$\mathrm{Gr}_{\dim \hat{V}_1}(\hat{M}) \cong \mathrm{Gr}_1(\mathbb{C}^3), \quad \mathrm{Gr}_{\dim \hat{V}_2}(\hat{M}) \cong \mathcal{F}l(\mathrm{SL}_3), \quad \mathrm{Gr}_{\dim \hat{V}_3}(\hat{M}) \cong \mathrm{Gr}_2(\mathbb{C}^3).$$

With the moment graph of the desingularization as described above, it is possible to compute the Euler classes at the singular points of $\mathrm{Gr}_{\boldsymbol{e}}(M)$ using Lemma 4.10. For example we obtain

$$\begin{aligned}\mathrm{Eu}_T(p_1, Z_5) &= \frac{1}{(\epsilon_3-\epsilon_2)(\epsilon_2-\epsilon_1-\delta)} + \frac{1}{(\epsilon_3-\epsilon_2)(\epsilon_1-\epsilon_2+3\delta)} \\ &= \frac{2\delta}{(\epsilon_3-\epsilon_2)(\epsilon_2-\epsilon_1-\delta)(\epsilon_1-\epsilon_2+3\delta)},\end{aligned}$$

where $Z_5 = \bigcup_{i=1}^{5} W_i$.

We compute the following basis of $H_T^\bullet(\mathrm{Gr}_{\boldsymbol{e}}(M))$ as free module over $H_T^\bullet(pt)$:

$$\begin{aligned}
\varphi^{(1)} &= (1, 1, 1, 1, 1, 1, 1, 1),\\
\varphi^{(2)} &= (0, \epsilon_3-\epsilon_2, 0, \epsilon_3-\epsilon_2, \epsilon_1-\epsilon_2+3\delta, \epsilon_3-\epsilon_1-\delta, \epsilon_3-\epsilon_1-\delta, \epsilon_3-\epsilon_1-\delta),\\
\varphi^{(3)} &= (0, 0, \epsilon_2-\epsilon_1-\delta, \epsilon_3-\epsilon_1-\delta, 0, \epsilon_2-\epsilon_1-\delta, \epsilon_3-\epsilon_2, \epsilon_3-\epsilon_1-\delta),\\
\varphi^{(4)} &= (\epsilon_3-\epsilon_2)(\epsilon_3-\epsilon_1-\delta)\cdot(0, 0, 0, 1, 0, 0, 1, 0),\\
\varphi^{(5)} &= (\epsilon_1-\epsilon_2+3\delta)(\epsilon_1-\epsilon_3+3\delta)\cdot(0, 0, 0, 0, 1, 0, 0, 0),\\
\varphi^{(6)} &= (\epsilon_2-\epsilon_1-\delta)(\epsilon_3-\epsilon_1-\delta)\cdot(0, 0, 0, 0, 0, 1, 1, 0),\\
\varphi^{(7)} &= (\epsilon_3-\epsilon_2)(\epsilon_2-\epsilon_1-\delta)(\epsilon_3-\epsilon_1-\delta)\cdot(0, 0, 0, 0, 0, 0, 1, 0),\\
\varphi^{(8)} &= (\epsilon_2-\epsilon_1-\delta)(\epsilon_3-\epsilon_1-\delta)\cdot(0, 0, 0, 0, 0, 0, 0, 1).
\end{aligned}$$

Observe that the special role of p_8 in this example allows to generate more zero-entries as in the general setting of Theorem 4.13.

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VARIETIES OF CHORD DIAGRAMS, BRAID GROUP COHOMOLOGY AND DEGENERATION OF EQUALITY CONDITIONS

VICTOR A. VASSILIEV

For any finite-dimensional vector space $\mathcal{F}$ of continuous functions $f : \mathbb{R}^1 \to \mathbb{R}^1$ we consider subspaces in $\mathcal{F}$ defined by systems of *equality conditions* $f(a_i) = f(b_i)$, where $\{a_i, b_i\}$, $i = 1, \dots, n$, are some pairs of points in $\mathbb{R}^1$. It is proven that if $\dim \mathcal{F} < 2n - I(n)$, where $I(n)$ is the number of ones in the binary notation of n, then there necessarily exist independent systems of n equality conditions defining the subspaces of codimension greater than n in $\mathcal{F}$. We also prove lower estimates of the sizes of the inevitable drops of the codimensions of some of these subspaces.

Next, we apply these estimates to knot theory (in which systems of equality conditions are known as *chord diagrams*) and prove the inevitable presence of complicated nonstable terms in sequences of spectral sequences computing cohomology groups of spaces of knots.

1. Main results

Let $\mathcal{F}^N$ be an N-dimensional vector subspace of the space $C^0(\mathbb{R}^1, \mathbb{R}^1)$ of continuous functions $\mathbb{R}^1 \to \mathbb{R}^1$. Typically, a collection of n independent conditions of the form

$$f(a_i) = f(b_i), \tag{1}$$

where $a_i \neq b_i,\ i = 1, \dots, n$, defines a subspace of codimension n in $\mathcal{F}^N$ if $n \leq N$ and only the trivial subspace if $n \geq N$. However, for exceptional sets of such conditions, the codimensions of these subspaces can drop.

For example, if $\mathcal{F}^N$ is the space $\mathcal{P}^N$ of all polynomials of the form

$$\alpha_1 x^N + \alpha_2 x^{N-1} + \cdots + \alpha_N x \tag{2}$$

in the variable x, then all subspaces defined by arbitrarily many conditions

$$f(a_i) = f(-a_i)$$

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Keywords: chord diagram, configuration space, characteristic class.

contain the $\left[\frac{N}{2}\right]$-dimensional subspace of even polynomials. Of course, the case of polynomials is very specific, but the situation when the dimensions of subspaces in $\mathcal{F}^N$ defined by some n independent conditions (1) are greater than $\max(N-n, 0)$ can be unavoidable by a choice of space $\mathcal{F}^N$.

Definition 1. An unordered pair $\{a, b\}$ of distinct points in $\mathbb{R}^1$ is called a *chord*. An unordered collection of n pairwise distinct chords is called an *n-chord diagram*.

The *subalgebra* of $C^0(\mathbb{R}^1, \mathbb{R}^1)$ corresponding to the n-chord diagram

$$\{\{a_1, b_1\}, \dots, \{a_n, b_n\}\} \tag{3}$$

consists of all functions satisfying n conditions (1) with these a_i, b_i. These n conditions (and the corresponding n chords) are *independent* if the codimension of this subalgebra in $C^0(\mathbb{R}^1, \mathbb{R}^1)$ is equal to n. (We say that an affine or vector subspace $\mathcal{T}$ of a function space $\mathcal{K}$ has codimension n if for any point $\varphi \in \mathcal{T}$ there exist n-dimensional affine subspaces in $\mathcal{K}$ intersecting $\mathcal{T}$ at this point only, and all affine subspaces of higher dimensions passing through φ intersect $\mathcal{T}$ along subspaces of positive dimensions.) Two independent n-chord diagrams are *equivalent* if the corresponding subalgebras in $C^0(\mathbb{R}^1, \mathbb{R}^1)$ coincide. A *resonance* of a chord diagram is a cyclic sequence of $k \geq 3$ its pairwise different chords such that one point of each chord also belongs to the preceding chord in this sequence, and the other its point also belongs to the next chord.

For example, two chord diagrams are equivalent if one of them contains the chords $\{a, b\}$ and $\{b, c\}$, the other contains the chords $\{a, b\}$ and $\{a, c\}$, and all other chords in them are common.

Proposition 2. *An n-chord diagram is independent if and only if it does not contain resonances. Two independent n-chord diagrams are equivalent if and only if they can be connected by a chain of elementary flips described in the previous paragraph. The space of independent n-chord diagrams is a smooth connected $2n$-dimensional manifold.*

Proof. The proof is elementary. □

1.1. *Results for the case of $N \geq n$.*

Proposition 3. *If $N \geq 2n-1$, then the codimension of the subspace in the space $\mathcal{P}^N$ of polynomials* (2), *defined by n conditions* (1) *of an arbitrary independent n-chord diagram* (3), *is equal to n.*

Proof. First, the assertion of our proposition will be true if we replace in it the space $\mathcal{P}^N$ by the $(N+1)$-dimensional space $\hat{\mathcal{P}}^N$ of all polynomials of degree N. Indeed, any n-chord diagram has at most $2n$ distinct endpoints a_i, b_i, therefore by interpolation theorem the evaluation morphism from the space of such polynomials to the space of real-valued functions on the set of these endpoints is epimorphic, and hence the preimage of any subspace of codimension n of the latter space also has codimension n in $\hat{\mathcal{P}}^N$. However, adding the constant functions preserves the subspace of $\hat{\mathcal{P}}^N$ defined by any chord diagram, therefore the codimension of the considered subspace in $\mathcal{P}^N$ is also equal to n. □

Denote by $I(n)$ the number of ones in the binary notation of n.

Theorem 4. *If $n \leq N < 2n - I(n)$, then for any N-dimensional vector subspace $\mathcal{F}^N \subset C^0(\mathbb{R}^1, \mathbb{R}^1)$ there exist independent n-chord diagrams* (3) *such that the codimension of the subspace in $\mathcal{F}^N$ consisting of functions satisfying all the corresponding conditions* (1) *is less than n. The dimension of the set of such exceptional n-chord diagrams is at least $3n - N - 1$ in the following exact sense*: *there exists a nontrivial element of the $(N - n + 1)$-dimensional homology group of the $2n$-dimensional manifold of all independent n-chord diagrams*, *such that each cycle representing this element necessarily intersects our set.*

In particular, if n is a power of 2 then the minimal dimension of the function spaces $\mathcal{F}^N$ in which any independent n-chord diagram defines a subspace of codimension exactly n is equal to $2n - 1$.

A more general result can be formulated in terms of configuration spaces; see, e.g., [1] for the current state of the theory of these spaces.

Definition 5. The *n-th configuration space $B(X, n)$* of a topological space X is the (naturally topologized) space of unordered subsets of cardinality n in X. The *regular bundle ξ_n* with base $B(X, n)$ is the vector bundle, whose fiber over an n-point configuration is the space of real-valued functions on the corresponding set of points.

Theorem 6. *Suppose that $N \geq n$ and for some natural r the cohomological product*

$$\prod_{i=1}^{r} w_{N-n+2i-1}(\xi_n) \tag{4}$$

of Stiefel–Whitney classes of the regular bundle ξ_n is not equal to 0 *in the ring $H^*(B(\mathbb{R}^2, n), \mathbb{Z}_2)$. Then for any N-dimensional vector subspace $\mathcal{F}^N$ of the space $C^0(\mathbb{R}^1, \mathbb{R}^1)$ there exists an independent system of n conditions* (1) *such that the subspace of $\mathcal{F}^N$ defined by this system has codimension $\leq n - r$ in $\mathcal{F}^N$.*

The first statement of Theorem 4 follows immediately from this theorem (the case $r = 1$) and statement 5.3 of [3] asserting that the classes

$$w_k \in H^k(B(\mathbb{R}^2, n), \mathbb{Z}_2)$$

are nontrivial for all $k \leq n - I(n)$; see also Proposition 30 in Section 6 below. The second statement of Theorem 4 will be proven at the end of Section 3.

Corollary 7. A. *If two natural numbers n and N satisfy one of the following pairs of conditions*:

(1) $n \geq 6,\ N = n$,

(2) $n \geq 10,\ N = n+1$,

(3) $n \geq 14,\ N = n+2$ *or* $n+3$,

(4) $n \geq 16,\ N = n+4$,

(5) $n \geq 18,\ N = n+5$,

(6) $n \geq 20,\ N = n+6$,

(7) $n \geq 24,\ N = n+7$,

(8) $n \geq 28,\ N = n+8$ *or* $n+9$,

(9) $n \geq 32,\ N = n+10$ *or* $n+11$,

then for any N-dimensional vector subspace $\mathcal{F}^N \subset C^0(\mathbb{R}^1, \mathbb{R}^1)$ there exists a system of n independent conditions (1) *defining a subspace of codimension $\leq n-2$ in $\mathcal{F}^N$.*

B. *If n and N satisfy one of the following pairs of conditions*:

(1) $n \geq 18,\ N = n$ *or* $n+1$,

(2) $n \geq 22,\ N = n+2$,

(3) $n \geq 26,\ N = n+3$,

(4) $n \geq 30,\ N = n+4$,

(5) $n \geq 36,\ N = n+5$,

(6) $n \geq 40,\ N = n+6$ *or* $n+7$,

(7) $n \geq 44,\ N = n+8$ *or* $n+9$,

then for any N-dimensional subspace $\mathcal{F}^N \subset C^0(\mathbb{R}^1, \mathbb{R}^1)$ there exists a system of n independent conditions (1) *defining a subspace of codimension $\leq n-3$ in $\mathcal{F}^N$.*

C. *If n and N satisfy one of the following conditions*:

(1) $n \geq 30$ *and* $N = n$ *or* $n+1$,

(2) $n \geq 44$ *and* $N = n+2$ *or* $n+3$,

(3) $n \geq 52$ *and* $N = n+4$ *or* $n+5$,

(4) $n \geq 56$ *and* $N = n+6$ *or* $n+7$,

then for any N-dimensional subspace $\mathcal{F}^N \subset C^0(\mathbb{R}^1, \mathbb{R}^1)$ there exists a system of n independent conditions (1) *defining a subspace of codimension $\leq n-4$ in $\mathcal{F}^N$.*

D. *If n and N satisfy one of the following conditions*:

(1) $n \geq 48$ *and* $N = n$ *or* $n+1$,

(2) $n \geq 60$ *and* $N = n+2$ *or* $n+3$,

(3) $n \geq 68$ *and* $N = n+4$ *or* $n+5$,

then for any N*-dimensional subspace* $\mathcal{F}^N \subset C^0(\mathbb{R}^1, \mathbb{R}^1)$ *there exists a system of* n *independent conditions* (1) *defining a subspace of codimension* $\leq n-5$ *in* $\mathcal{F}^N$.

E. *If* n *and* N *satisfy one of the following conditions*:

(1) $n \geq 64$ *and* $N = n$ *or* $n+1$,

(2) $n \geq 76$ *and* $N = n+2$ *or* $n+3$,

then for any N*-dimensional subspace* $\mathcal{F}^N \subset C^0(\mathbb{R}^1, \mathbb{R}^1)$ *there exists a system of* n *independent conditions* (1) *defining a subspace of codimension* $\leq n-6$ *in* $\mathcal{F}^N$.

F. *If we have* $n \geq 80$ *and* $N = n$ *or* $n+1$, *then for any* N*-dimensional subspace* $\mathcal{F}^N \subset C^0(\mathbb{R}^1, \mathbb{R}^1)$ *there exists a system of* n *independent conditions* (1) *defining a subspace of codimension* $\leq n-7$ *in* $\mathcal{F}^N$.

See Section 6 for the proof of this corollary. Its lists can easily be continued and the corresponding calculations can be programmed.

Remark. The first statement of Theorem 4 looks very similar (and is closely related) to the result of [2] estimating the dimensions of spaces of functions $\mathbb{R}^2 \to \mathbb{R}^1$ realizing n-regular embeddings of the plane. The main effort of our proof of Theorem 6 is a comparison of the configuration spaces used in these two problems, see Lemma 16 below.

1.2. *Results for the case of* $N \leq n$.

Theorem 8. *If* $N \leq n$ *and for some natural* r *the product*

$$\prod_{i=1}^{r} w_{n-N+2i-1}(\xi_n) \tag{5}$$

of Stiefel–Whitney classes of the bundle ξ_n *is not equal to* 0 *in the ring*

$$H^*(B(\mathbb{R}^2, n), \mathbb{Z}_2),$$

then for any N*-dimensional vector subspace* $\mathcal{F}^N \subset C^0(\mathbb{R}^1, \mathbb{R}^1)$ *there exists an independent* n*-chord diagram, such that the subspace of* $\mathcal{F}^N$ *consisting of functions satisfying the corresponding system of equality conditions is at least* r*-dimensional.*

If $N = n$, then Theorems 6 and 8 coincide tautologically.

Corollary 9. *If* $N \geq 2$, *then for any* N*-dimensional vector subspace* $\mathcal{F}^N \subset C^0(\mathbb{R}^1, \mathbb{R}^1)$ *there exist independent* n*-chord diagrams with arbitrarily large* n *such that the corresponding systems of equality conditions have nontrivial solutions in* $\mathcal{F}^N$.

Indeed, it is enough to prove this for $N = 2$ and numbers n equal to powers of 2. In this case $w_{n-N+1}(\xi_n) \neq 0$ by the previously mentioned result in [3]. □

Remark. This corollary has also an elementary proof. Indeed, any 2-dimensional subspace of $C^0(\mathbb{R}^1, \mathbb{R}^1)$ contains a nonzero function taking equal values at some two different points $a, b \in \mathbb{R}^1$. Then this function necessarily satisfies the equality conditions $f(\tilde{a}) = f(\tilde{b})$ for a continuum of different pairs $\{\tilde{a}, \tilde{b}\} \subset [a, b]$.

Corollary 10. *All statements of Corollary 7 will remain valid if in each of its conditions we replace the value of N by* $2n - N$ (*e.g.*, $N = n+4$ *by* $N = n-4$) *and simultaneously the corresponding conclusion "there exists a system of n independent conditions* (1) *defining a subspace of codimension* $\leq n - r$ *in* $\mathcal{F}^N$*" by "there exists a system of n independent conditions* (1) *defining a subspace of dimension* $\geq r$ *in* $\mathcal{F}^N$*".*

Remark. In terms of [6], the subspaces of anomalous codimensions defined by chord diagrams in finite-dimensional function spaces are responsible for the nonstable regions of the (p, q)-planes of the spectral sequences converging to cohomology groups of spaces of long knots $\mathbb{R}^1 \to \mathbb{R}^3$ defined by functions from these function spaces. These domains are the only possible sources of cohomology classes of the knot space (including 0-dimensional classes, i.e., knot invariants) not of finite-type. In Section 7 below, we prove some facts about filtrations of simplicial resolutions of discriminant spaces in finite-dimensional knot spaces, estimating the deviation of the corresponding spectral sequences from stable ones.

2. Scheme of proof of Theorem 6

Denote by CD_n the set of equivalence classes of independent n-chord diagrams. It has a natural topology induced by the topology of the variety of subalgebras of codimension n in $C^0(\mathbb{R}^1, \mathbb{R}^1)$. To describe this topology without infinite-dimensional considerations, let $\mathcal{A}$ be a sufficiently large finite-dimensional vector subspace of $C^0(\mathbb{R}, \mathbb{R})$, such that all subspaces of $\mathcal{A}$ defined by independent n-chord diagrams (that is, the intersections of $\mathcal{A}$ with subalgebras of $C^0(\mathbb{R}^1, \mathbb{R}^1)$ corresponding to these chord diagrams) have codimension exactly n in $\mathcal{A}$, and the nonequivalent n-chord diagrams define different subspaces. (For reasons similar to the proof of Proposition 3 we can take for such a space $\mathcal{A}$ the space $\mathcal{P}^M$, $M \geq 2n + 1$, or any space containing it; taking $\mathcal{P}^{2n-1}$ is not enough because nonequivalent n-chord diagrams can define equal subspaces in it). We can and will assume that $\mathcal{A}$ contains $\mathcal{F}^N$, because otherwise we can replace $\mathcal{A}$ by its sum with $\mathcal{F}^N$.

The set CD_n is embedded into the Grassmann manifold $G(\mathcal{A}, -n)$ of subspaces of codimension n in $\mathcal{A}$, and inherits a topology from this manifold. It is easy to see that this definition of a topology on CD_n does not depend on the choice of $\mathcal{A}$.

Suppose that $N \geq n$. Let $\Delta_r(\mathcal{F}^N) \subset G(\mathcal{A}, -n)$ be the set of all subspaces of codimension n in $\mathcal{A}$ whose sums with $\mathcal{F}^N$ have codimension at least r in $\mathcal{A}$.

Proposition 11. *The class in $H^*(G(\mathcal{A}, -n), \mathbb{Z}_2)$ Poincaré dual to the homology class of the algebraic variety $\Delta_r(\mathcal{F}^N)$ is equal to $r \times r$ determinant*:

$$(6) \qquad \begin{vmatrix} w_{N-n+r} & w_{N-n+r+1} & w_{N-n+r+2} & \cdots & w_{N-n+2r-2} & w_{N-n+2r-1} \\ w_{N-n+r-1} & w_{N-n+r} & w_{N-n+r+1} & \cdots & w_{N-n+2r-3} & w_{N-n+2r-2} \\ w_{N-n+r-2} & w_{N-n+r-1} & w_{N-n+r} & \cdots & w_{N-n+2r-4} & w_{N-n+2r-3} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ w_{N-n+2} & w_{N-n+3} & w_{N-n+4} & \cdots & w_{N-n+r} & w_{N-n+r+1} \\ w_{N-n+1} & w_{N-n+2} & w_{N-n+3} & \cdots & w_{N-n+r-1} & w_{N-n+r} \end{vmatrix},$$

where w_i are the Stiefel–Whitney classes of the tautological bundle on $G(\mathcal{A}, -n)$.

Proof. Let τ_n be the vector bundle on $G(\mathcal{A}, -n)$ whose fiber over the point $\{L\}$ corresponding to the subspace $L \subset \mathcal{A}$ is the space of linear functions on $\mathcal{A}$ vanishing on L. Consider the morphism of this bundle to the constant bundle with fiber $(\mathcal{F}^N)^*$, sending any such linear form to its restriction to $\mathcal{F}^N$. The variety $\Delta_r(\mathcal{F}^N)$ can be redefined as the set of points $\{L\}$ such that the rank of this morphism does not exceed $n - r$. By the real version of the Thom–Porteous formula (the proof of which literally repeats its complex analog given in [4, Section 14.4], after standard replacements of Chern classes by Stiefel–Whitney classes, $\mathbb{Z}$ by $\mathbb{Z}_2$, etc.) the class in $H^*(G(\mathcal{A}, -n), \mathbb{Z}_2)$ Poincaré dual to this variety is equal to the determinant of the form (6) in which all the symbols w_i are the Stiefel–Whitney classes of the virtual bundle $-\tau_n$.

The constant bundle on $G(\mathcal{A}, -n)$ with the fiber $\mathcal{A}^*$ is obviously isomorphic to the direct sum of τ_n and the bundle dual (and hence isomorphic) to the tautological bundle. Therefore $-\tau_n$ and this tautological bundle belong to the same class of the group $\tilde{K}(G(\mathcal{A}, -n))$, in particular have the same Stiefel–Whitney classes. $\square$

These Stiefel–Whitney classes $w_i(-\tau_n)$ are equal to the i-dimensional components $\bar{w}_i(\tau_n) \in H^i(G(\mathcal{A}, -n), \mathbb{Z}_2)$ of the class $w^{-1}(\tau_n)$, where

$$w(\tau_n) = 1 + w_1(\tau_n) + \ldots$$

is the total Stiefel–Whitney class of the bundle τ_n, see Section 4 in [5]. If the intersection of the subset $\mathrm{CD}_n \subset G(\mathcal{A}, -n)$ with $\Delta_r(\mathcal{F}^N)$ is empty, then the restriction homomorphism $H^*(G(\mathcal{A}, -n), \mathbb{Z}_2) \to H^*(\mathrm{CD}_n, \mathbb{Z}_2)$ maps the class (6) to zero. Theorem 6 therefore reduces to the following lemma.

Lemma 12. *If the class* (4) *is not equal to* 0, *then the restriction of the class* (6) *to the subvariety* $\mathrm{CD}_n \subset G(\mathcal{A}, -n)$ *is a nontrivial element of the group*

$$H^{r(N-n+r)}(\mathrm{CD}_n, \mathbb{Z}_2).$$

3. Proof of Lemma 12

Let $\mathbb{R}^2_+ \subset \mathbb{R}^2$ be the half-plane $\{(a, b) \mid a < b\} \subset \mathbb{R}^2$. Any point (a, b) of $\mathbb{R}^2_+$ can be identified with the chord $\{a, b\}$, and any element of the configuration space $B(\mathbb{R}^2_+, n)$ with an n-chord diagram. Let us denote by $\Xi \subset B(\mathbb{R}^2_+, n)$ the set of dependent (that is, containing resonances) n-chord diagrams. Consider the diagram

$$\begin{tikzcd} B(\mathbb{R}^2, n) & B(\mathbb{R}^2_+, n) \setminus \Xi \arrow[l, hook', "\supset"'] \arrow[d, "\pi"'] & \\ & \mathrm{CD}_n \arrow[r, hook, "\subset"] & G(\mathcal{A}, -n) \end{tikzcd} \tag{7}$$

where π is the map sending any chord diagram to its equivalence class.

Lemma 13. *The restriction of the regular vector bundle* ξ_n (*see Definition* 5) *to the subset* $B(\mathbb{R}^2_+, n) \setminus \Xi \subset B(\mathbb{R}^2, n)$ *is isomorphic to the bundle pulled back by the map* π *from the bundle* τ_n *over* CD_n.

Proof. The bundle τ_n is isomorphic to its dual bundle τ_n^*, i.e., to the quotient of the trivial bundle with fiber $\mathcal{A}$ by the tautological bundle over $G(\mathcal{A}, -n)$.

Consider the following homomorphism from the trivial bundle with the fiber $\mathcal{A}$ over $B(\mathbb{R}^2_+, n) \setminus \Xi$ to ξ_n: over any n-chord diagram Γ it sends any function $f \in \mathcal{A} \subset C^0(\mathbb{R}^1, \mathbb{R}^1)$ to the function on the set of chords of this chord diagram, whose value on any chord $\{a_i, b_i\}$ is equal to the difference $f(b_i) - f(a_i)$. By the first characteristic property of the space $\mathcal{A}$ this morphism is surjective; by definition of inclusion $\mathrm{CD}_n \subset G(\mathcal{A}, -n)$ its kernel is equal to the fiber of the tautological bundle over the point $\pi(\Gamma) \in \mathrm{CD}_n$. Therefore our homomorphism induces an isomorphism between the bundles $\pi^*(\tau_n^*) \sim \pi^*(\tau_n)$ and ξ_n. □

Lemma 14 (see [3] or Proposition 31). *The square of any positive-dimensional element of the ring* $H^*(B(\mathbb{R}^2, n), \mathbb{Z}_2)$ *is equal to zero, in particular* $w^{-1}(\xi_n) = w(\xi_n)$ *and* $w_i(\xi_n) = \bar{w}_i(\xi_n)$ *for any* i. □

Lemma 15. *The determinant of the form* (6) *in which all classes* w_i *are replaced by* $w_i(\xi_n)$ *is equal to the product* (4) *in* $H^*(B(\mathbb{R}^2, n), \mathbb{Z}_2)$.

Proof. The matrix (6) is symmetrical with respect to the southwest/northeast diagonal, hence calculating its determinant mod 2 it suffices to count only those products of r matrix elements which are self-symmetric with respect to this diagonal. By Lemma 14 such products, not all factors of which lie in this diagonal, are also trivial. □

Lemma 16. *The inclusion* $B(\mathbb{R}^2_+, n) \setminus \Xi \to B(\mathbb{R}^2, n)$ *induces a monomorphism of cohomology groups* $H^*(B(\mathbb{R}^2, n), \mathbb{Z}_2) \to H^*(B(\mathbb{R}^2_+, n) \setminus \Xi, \mathbb{Z}_2)$.

Lemma 16 will be proved in Section 5. Lemma 12 follows from Lemmas 13–16 and the functoriality of Stiefel–Whitney classes. Namely, by Lemma 16 if the product (4) is nontrivial in $H^*(B(\mathbb{R}^2, n), \mathbb{Z}_2)$, then it is nontrivial also in $H^*(B(\mathbb{R}^2_+, n)\setminus \Xi, \mathbb{Z}_2)$. By the Lemmas 13–15 this element of $H^*(B(\mathbb{R}^2_+, n)\setminus \Xi, \mathbb{Z}_2)$ is equal to the class induced by the map π from the determinant (6), so this determinant is also nontrivial.

Proof of the last statement of Theorem 4. By Lemma 16 and statement 5.3 of [3], under the conditions of this theorem the class $w_{N-n+1}(\xi_n)$ is not trivial. We can then take an arbitrary element of the group $H_{N-n+1}(B(\mathbb{R}^2_+, n) \setminus \Xi, \mathbb{Z}_2)$ on which this class takes nonzero value: any cycle realizing such an element intersects the set $\pi^{-1}(\Delta_1(\mathcal{F}^N))$. □

4. Proof of Theorem 8

Now suppose that $N \leq n$. Let $\Lambda_r(\mathcal{F}^N)$ be the subset of $G(\mathcal{A}, -n)$ consisting of planes whose intersection with $\mathcal{F}^N$ is at least r-dimensional.

Proposition 17. *The class in $H^*(G(\mathcal{A}, -n), \mathbb{Z}_2)$ Poincaré dual to the variety $\Lambda_r(\mathcal{F}^N)$ is equal to $r \times r$ determinant similar to* (6), *in which $N - n$ in all lower indices is replaced by $n - N$, and w_i are Stiefel–Whitney classes of the bundle τ_n.*

Proof. The projection along the fibers of the tautological bundle over $G(\mathcal{A}, -n)$ defines a morphism from the constant bundle with fiber $\mathcal{F}^N$ and base $G(\mathcal{A}, -n)$ to the bundle dual (and hence isomorphic) to τ_n, i.e., to the quotient of the constant bundle with fiber $\mathcal{A}$ by the tautological bundle. The set $\Lambda_r(\mathcal{F}^N)$ can be defined as the set of points at which the rank of this morphism does not exceed $N - r$. Our proposition follows from the real version of Thom–Porteous formula applied to this morphism. □

The rest of the reduction of Theorem 8 to Lemma 16 repeats that of Theorem 6; the Stiefel–Whitney classes of the bundles $-\tau_n$ and τ_n participating in the corresponding Thom–Porteous formulas are the same by Lemma 14.

5. Proof of Lemma 16

5.1. ***Generators of Hopf algebra.*** We will prove the dual statement: the map $H_*(B(\mathbb{R}^2_+, n) \setminus \Xi, \mathbb{Z}_2) \to H_*(B(\mathbb{R}^2, n), \mathbb{Z}_2)$ induced by the identical embedding is epimorphic.

According to [3], all stabilization maps

$$H_*(B(\mathbb{R}^2, n), \mathbb{Z}_2) \to H_*(B(\mathbb{R}^2, n+m), \mathbb{Z}_2)$$

induced by the standard inclusions $B(\mathbb{R}^2, n) \hookrightarrow B(\mathbb{R}^2, n+m)$ are injective. Therefore, all elements of the group $H_*(B(\mathbb{R}^2, n), \mathbb{Z}_2)$ are given by polynomials in the multiplicative generators of the Hopf algebra $H_*(B(\mathbb{R}^2, \infty), \mathbb{Z}_2)$, and it suffices to prove that all these generators and their products participating in the construction of these elements can be realized by cycles lying in $B(\mathbb{R}^2_+, n) \setminus \Xi$.

These generators $[M_j] \in H_{2^j-1}(B(\mathbb{R}^2, 2^j), \mathbb{Z}_2)$ were defined in Section 8 of [3] by the following cycles $M_j \subset B(\mathbb{R}^2, 2^j)$. Arbitrarily choose two opposite points of the circle of radius 1 centered at the origin in $\mathbb{R}^2$. Take two circles of small radius ε with centers at these points and arbitrarily choose a pair of opposite points in each of them. Take circles of radius ε^2 centered at all obtained four points and choose a pair of opposite points in all of them. Continuing, after the j-th step we obtain a 2^j-configuration in $\mathbb{R}^2$. This construction involves $1+2+4+\cdots+2^{j-1}$ choices of opposite points in some circles, hence the set M_j of all possible 2^j-configurations that can be obtained in this way is (2^j-1)-dimensional. It is easy to see that this set is a closed submanifold in $B(\mathbb{R}^2, 2^j)$, and therefore it defines an element $[M_j]$ of the group $H_{2^j-1}(B(\mathbb{R}^2, 2^j), \mathbb{Z}_2)$, $j \geq 1$. Finally, define the element $[M_0] \in H_0(B(\mathbb{R}^2, 1), \mathbb{Z}_2)$ as the class of a single point.

Unfortunately, these cycles with $j > 1$ contain configurations with resonances, and, moreover, all configurations of class M_j do not lie in $\mathbb{R}^2_+$. To avoid these problems, we modify the previous construction by (1) replacing the circles with squares, (2) taking these squares of the same level (i.e., arising on the same stage of the construction) of varying sizes depending on their centers, and (3) shifting the resulting configurations into the half-plane $\mathbb{R}^2_+ \subset \mathbb{R}^2$; see Section 5.4 below.

5.2. *Preparation for the construction.*

Definition 18. A segment in the plane $\mathbb{R}^2$ with coordinates a and b is called *vertical* (respectively, *horizontal*) if the coordinate a (respectively, b) is constant along it. A *straight resonance* of an n-point configuration in $\mathbb{R}^2$ is a closed chain of strictly alternating vertical and horizontal segments, all whose endpoints belong to our configuration.

Proposition 19. *Let $\bar{a}$ and $\bar{b}$ be two real numbers such that $\bar{b} - \bar{a} > 8$. If all points $\{a_i, b_i\}$, $i = 1, \ldots, n$, of an n-chord diagram* (3) *satisfy the conditions*

$$|a_i - \bar{a}| < 2, \quad |b_i - \bar{b}| < 2, \tag{8}$$

and the corresponding n-configuration $\{(a_i, b_i)\} \subset B(\mathbb{R}^2_+, n)$ has a resonance, then it has a straight resonance.

Indeed, if two chords $\{a_i, b_i\}$ and $\{a_k, b_k\}$ satisfying the conditions (8) have common points, then either $a_i = a_k$ or $b_i = b_k$, but not $a_i = b_k$. □

We will construct our basic cycles in the set of configurations satisfying the condition of Proposition 19 for some $\bar{a}$ and $\bar{b}$, and prove that they do not have configurations with straight resonances.

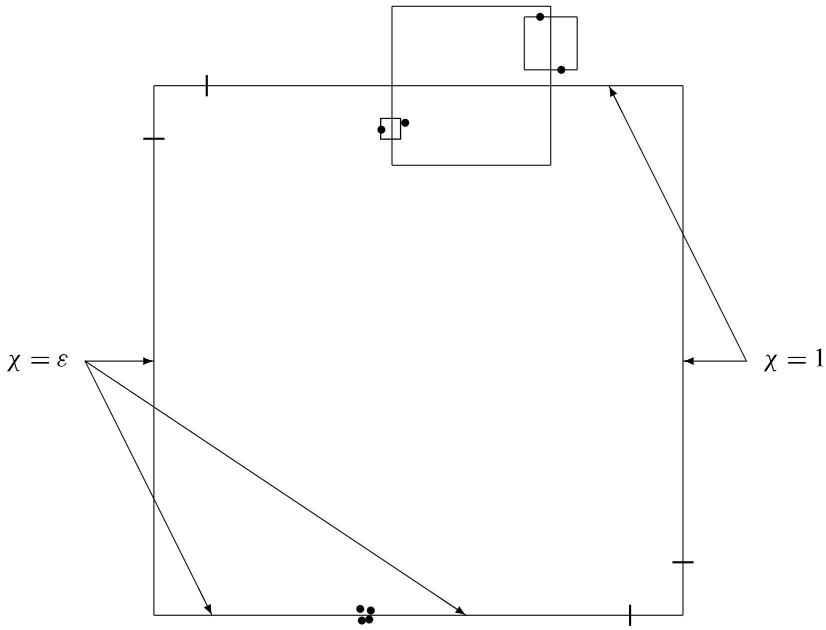

Figure 1. 8-configuration of the class $\tilde{M}_3$.

Let us fix a very small number $\varepsilon > 0$. Define the *basic square* $\Box \subset \mathbb{R}^2$ as the union of four segments consecutively connecting the points

$$(-1, -1), (-1, 1), (1, 1), (1, -1)$$

and again $(-1, -1)$. Fix an arbitrary continuous function $\chi : \Box \to [\varepsilon, 1]$ equal identically to 1 on segments

$$[(-1+\varepsilon, 1), (1, 1)] \quad \text{and} \quad [(1, -1+\varepsilon), (1, 1)], \tag{9}$$

equal to ε on segments

$$[(-1, -1), (-1, 1-\varepsilon)] \quad \text{and} \quad [(-1, -1), (1-\varepsilon, -1)], \tag{10}$$

and taking some intermediate values in the remaining ε-neighborhoods of corners $(-1, 1)$ and $(1, -1)$, see Figure 1.

5.3. ***First example.*** Let $n = 4$. Arbitrarily choose two opposite points A and $-A$ of the basic square. Consider two *squares of the second level* obtained from the basic square by the affine maps

$$X \mapsto A + \varepsilon\chi(A)X \quad \text{and} \quad X \mapsto -A + \varepsilon\chi(-A)X$$

(in particular, centered at the points A and $-A$), and arbitrarily choose two opposite points in each of these squares.

None of the 4-configurations thus obtained can have a straight resonance (i.e., to be the set of corners of a rectangle with vertical and horizontal sides). Indeed, two points of different squares of second level can be joined by a vertical (respectively, horizontal) segment only if the centers A and $-A$ of these squares are very close to the centers of the opposite horizontal (respectively, vertical) sides of the basic square. But in this case the values of the function χ at the points A and $-A$ are different, our two squares of second level have different sizes, and the segments connecting two opposite points in one of them and two opposite points in the other cannot be opposite sides of the same rectangle.

Therefore shifting all obtained 4-configurations by a fixed vector $(\bar{a}, \bar{b}) \in \mathbb{R}^2$ with $\bar{b} - \bar{a} > 8$, we get a 3-dimensional cycle in space $B(\mathbb{R}^2_+, 4) \setminus \Xi$.

5.4. ***Construction of cycles $\tilde{M}_j$ (see Figure 1).*** The general construction is an iteration of the previous one.

Namely, define two sequences of natural numbers u_j and T_j, $j \geq 2$, by recursion

$$u_2 = 1,\ T_2 = 2, \qquad u_j = u_{j-1} + T_{j-1} + 2, \quad T_j = u_j + T_{j-1} + 1 \qquad \text{for } j > 2. \tag{11}$$

Define the subset $\tilde{M}_1 \subset B(\mathbb{R}^2, 2)$ as the space of all choices of two opposite points in the basic square $\Box$. Suppose we have defined the $(2^{j-1} - 1)$-dimensional subvariety $\tilde{M}_{j-1} \subset B(\mathbb{R}^2, 2^{j-1})$, $j \geq 2$. Then the $(2^j - 1)$-dimensional subvariety $\tilde{M}_j \subset B(\mathbb{R}^2, 2^j)$ is defined as the space of all 2^j-configurations consisting of

(i) a 2^{j-1}-configuration obtained from some configuration of the class $\tilde{M}_{j-1}$ by the affine map $\mathbb{R}^2 \to \mathbb{R}^2$ given by

$$\{X \mapsto A + \varepsilon \chi^{u_j}(A) X\}, \tag{12}$$

where A is some point of the basic square $\Box$;

(ii) a 2^{j-1}-configuration obtained from some configuration of the class $\tilde{M}_{j-1}$ by the map

$$\{X \mapsto -A + \varepsilon \chi^{u_j}(-A) X\}, \tag{13}$$

with the same A, see Figure 1.

Any 2^j-configuration of the class $\tilde{M}_j$ uniquely determines the set of $2^j - 1$ *squares participating in its construction*: it consists of the basic square $\Box$ and the images under maps (12), (13) of two collections of $2^{j-1} - 1$ squares participating in the construction of two 2^{j-1}-configurations of class $\tilde{M}_{j-1}$. This set is obviously organized into (the set of vertices of) an oriented binary tree. Namely, it has two squares of the second level, four squares of the third level, etc. Every square of the l-th level is centered at a point of some square of $(l-1)$-st level; we connect the two corresponding vertices by an edge of the tree oriented towards the vertex of level l. Any point of our configuration belongs to a square of the j-th level, i.e., to a leaf of the tree. We say that a point of the configuration is *subordinate* to this square of level j containing it, and also to all squares of lower levels connected with it by an oriented path in our binary tree. Conversely, we say that all these squares *dominate* such a point of the configuration; in particular, any square of level l dominates exactly 2^{j-l+1} points.

Lemma 20. *The length of the sides of the smallest square participating in the construction of a 2^j-configuration of class $\tilde{M}_j$ is not less than $2\varepsilon^{T_j}$.*

Proof. This follows by induction from the last condition (11) because both subsets constituting our 2^j-configuration are obtained from certain 2^{j-1}-configurations of the class $\tilde{M}_{j-1}$ by homotheties (12), (13) with coefficients $\geq \varepsilon^{u_j+1}$. □

Lemma 21. *The absolute values of the coordinates a, b of all points of configurations of class $\tilde{M}_j$ do not exceed $1+\varepsilon+\varepsilon^2+\cdots+\varepsilon^{j-1} < \frac{1}{1-\varepsilon}$.*

Lemma 22. *If a square of level l participating in the construction of a configuration of the class $\tilde{M}_j$ has sides of length d, then all 2^{j-l+1} points of this configuration subordinate to this square lie in the $\frac{\varepsilon d}{\sqrt{2}(1-\varepsilon)}$-neighborhood of this square.*

Proof. Proofs of these two lemmas follow directly from the construction. □

Finally, we move the obtained subvariety $\tilde{M}_j \subset B(\mathbb{R}^2, 2^j)$ into $B(\mathbb{R}^2_+, 2^j)$ shifting all its 2^j-configurations to $\mathbb{R}^2_+$ by some translation $\{X \mapsto X+(\bar{a}, \bar{b})\}$, where $\bar{b}-\bar{a} \geq 8$. Denote by ∇_j the resulting cycle in $B(\mathbb{R}^2_+, 2^j)$.

It is easy to see that ∇_j is the image of an embedding $M_j \to B(\mathbb{R}^2, 2^j)$ (where the manifold M_j was defined in Section 5.1), which is homotopic to the identical embedding; in particular, it defines the same homology class in $H_*(B(\mathbb{R}^2, 2^j), \mathbb{Z}_2)$. Indeed, such a homotopy is defined by (1) a family of functions connecting the function χ with the function equal identically to 1 in the space of positive functions $\square \to \mathbb{R}^1$, (2) a deformation of all circles to squares, and (3) the continuous shift of the plane by the vector $(\bar{a}, \bar{b})$. Therefore, to realize the homology class $[M_j]$ by a cycle from $B(\mathbb{R}^2_+) \setminus \Xi$ it remains to prove the following statement.

Theorem 23. *The variety ∇_j does not contain 2^j-chord diagrams with straight resonances.*

The proof of this theorem for arbitrary j is not as simple as for $j=2$, it uses the following generalization of straight resonances.

Definition 24. For any positive number δ, a segment in $\mathbb{R}^2$ is called *δ-horizontal* (respectively, *δ-vertical*) if the tangent of the angle between this segment and a horizontal (respectively, vertical) line belongs to the interval $[0, \delta)$. An n-configuration in $\mathbb{R}^2$ is *δ-resonant* if there exists a closed chain of strictly alternating δ-vertical and δ-horizontal segments in $\mathbb{R}^2$ such that the endpoints of any of its segments belong to our configuration.

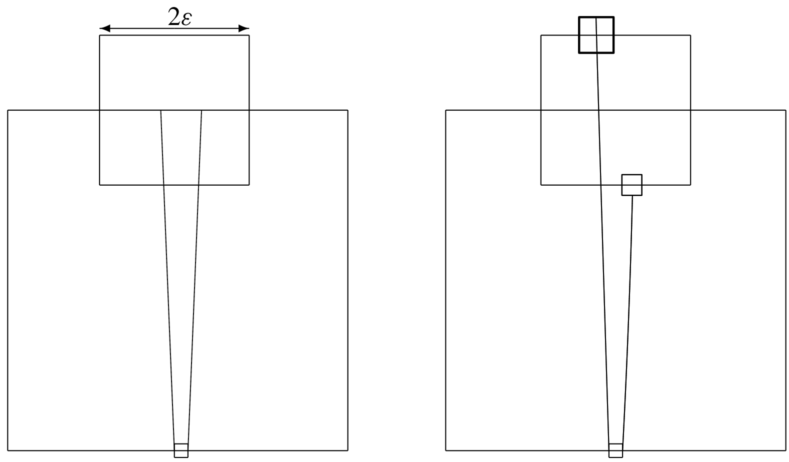

Figure 2. Discrepancy preventing resonances.

Theorem 23 now follows from the following statement.

Theorem 25. *If the number ε participating in the construction of the cycle $\tilde{M}_j$ is sufficiently small, then the set $\tilde{M}_j$ does not contain ε^{u_j+1}-resonant 2^j-configurations.*

5.5. ***Example and idea of the proof of Theorem 25.*** Let us again consider the case $j = 2$. If a 4-configuration of the class $\tilde{M}_2$ is ε^2-resonant, then some points of opposite squares of second level participating in its construction are connected by segments almost parallel (up to angles with tangent $\leq \varepsilon^2$) to a vertical or horizontal segment. Let us assume for certainty that these are almost vertical segments. Then the centers A and $-A$ of these squares are very close to the centers of the opposite horizontal sides of the basic square, in particular the function χ takes the value 1 at one of them and the value ε at the other. The a-coordinates of two points of our ε^2-resonant 4-configuration, placed in the bigger square of second level, differ by 2ε, see Figure 2 (left). On the other hand, these two points are connected by a chain of three segments of our ε^2-resonance passing through the smaller square, therefore this difference is estimated from above by the sum of (a) the length of the sides of the small square and (b) twice the maximal possible difference of a-coordinates of endpoints of the ε^2-vertical segments of our ε^2-resonance. The last difference is estimated from above by the maximal difference of b-coordinates of points of our configuration (which is at most $2+\varepsilon+\varepsilon^2$) multiplied by the allowed bending ε^2 of the segments of our ε^2-resonance. This sum is of order ε^2, a contradiction.

Further, let j be arbitrary; suppose that our configuration of class $\tilde{M}_j$ is ε^{u_j+1}-resonant, and two squares of second level participating in its construction are located near the centers of horizontal sides of the basic square. Our exponents (11) are chosen in such a way that the upper endpoints of any two ε^{u_j+1}-vertical segments connecting the points subordinate to different squares of the second level cannot lie too close to the opposite vertical sides of an arbitrary square subordinate to the bigger (upper) square of second level: indeed, by an estimate similar to that from the previous paragraph the difference of the a-coordinates of these endpoints is much smaller than the minimal distance between these vertical sides.

If $j > 2$, then there remains a possibility that these two endpoints lie in neighborhoods of opposite horizontal sides of such a square (see Figure 2, right) and are connected by some ε^{u_j+1}-resonant chain inside the upper square of second level. In this case, let us connect directly these two endpoints by a segment and forget about the part of our ε^{u_j+1}-resonance involving the points from the lower square. The exponents (11) are chosen so that the tangent of this segment with a vertical line is estimated from above by $\varepsilon^{u_{j-1}+1}$, and we obtain a $\varepsilon^{u_{j-1}+1}$-resonance inside the upper square only, which is prohibited by the induction hypothesis.

5.6. ***Proof of Theorem 25.*** Let us support this reasoning with strict estimates.

Suppose that Theorem 25 is proved for all cycles $\tilde{M}_i$, $i < j$. By the construction, any 2^j-configuration $\Gamma \in \tilde{M}_j$ splits into two subsets of cardinality 2^{j-1} with mass centers at some opposite points A and $-A$ of the basic square $\square$, any of these subsets lying in the $\sqrt{2}\varepsilon/(1-\varepsilon)$-neighborhood of the corresponding point A or $-A$.

Suppose that our configuration $\Gamma \in \tilde{M}_j$ is ε^{u_j+1}-resonant. If the entire chain of its points participating in this resonance is located in only one of these two subsets of Γ, then we get a contradiction with the induction hypothesis for $i = j-1$, because this subset is homothetic to a configuration of the class $\tilde{M}_{j-1}$, and $\varepsilon^{u_j+1} < \varepsilon^{u_{j-1}+1}$.

So, our chain should contain ε^{u_j+1}-vertical or ε^{u_j+1}-horizontal segments, which connect some points from these two subsets. Therefore, the corresponding points A and $-A$ are very close to either the center points of opposite horizontal sides of the basic square $\square$, or to the center points of its vertical sides. These two situations can be reduced to each other by the reflection in the diagonal $\{a = b\}$ of $\mathbb{R}^2$, it is therefore sufficient to consider only the first of them.

Consider a ε^{u_j+1}-vertical segment of our resonance chain which has endpoints in both these subsets; let A_0 be its endpoint in the upper subset. Starting from A_0, our chain somehow travels inside this upper subset and finally leaves it along some other ε^{u_j+1}-vertical segment; let B_0 be the upper point of the latter segment.

Lemma 26. (1) *The difference between the a-coordinates of points A_0 and B_0 is estimated from above by $7\varepsilon^{u_j+1}$.*

(2) *The difference between the b-coordinates of A_0 and B_0 is estimated from below by $\varepsilon^{T_{j-1}+1}$.*

Proof. (1) This difference is estimated from above by the sum of (a) the maximal difference of the a-coordinates of the points of the lower 2^{j-1}-subconfiguration of our 2^j-configuration Γ, (b) the difference of the a-coordinates of the point A_0 and the other endpoint of the segment of our chain connecting A_0 with this lower subconfiguration, and (c) the similar difference for the point B_0. By Lemma 21, formulas (12)–(13) and the definition of the function χ, the first of these differences is estimated from above by $2\varepsilon^{u_j+1}(1+O(\varepsilon))$; the other two are estimated by the heights of these two segments (which by Lemma 21 are smaller than $\frac{2}{1-\varepsilon}$) multiplied by their tangents with the vertical direction (which are estimated by ε^{u_j+1} since these segments are ε^{u_j+1}-vertical). Thus, the entire sum is estimated from above by $\varepsilon^{u_j+1}(6+O(\varepsilon)) < 7\varepsilon^{u_j+1}$.

(2) Let us consider two paths in the binary tree of squares participating in the construction of our configuration $\Gamma \in \tilde{M}_j$, starting from the basic square and consisting of all squares dominating the point A_0 (respectively, B_0). Let $\Box_k$ be the last (of highest level) common square of these two sequences. By Lemma 20, the length of its sides is at least $2\varepsilon^{T_{j-1}+1}$: indeed, this square is obtained from a square participating in the construction of a 2^{j-1}-configuration of class $\tilde{M}_{j-1}$ by a homothety with coefficient $\varepsilon\chi(A)$ for some point A from the central part of the upper side of the basic square (where $\chi \equiv 1$). The next two squares in these sequences (or their final points A_0 and B_0 if $\Box_k$ is a square of the last j-th level) are different, therefore these next squares (or points) are centered at (or coincide with) some points of opposite sides of $\Box_k$. These cannot be vertical sides: indeed, in this case by Lemma 22 the a-coordinates of our points A_0 and B_0 would differ by $2\varepsilon^{T_{j-1}+1}(1+O(\varepsilon))$, which contradicts statement (1) of our lemma, because by (11) $\varepsilon^{T_{j-1}+1} \gg \varepsilon^{u_j+1}$. Therefore, these are horizontal opposite sides, and hence the difference of their b-coordinates is estimated from below by the number $2\varepsilon^{T_{j-1}+1}(1+O(\varepsilon)) > \varepsilon^{T_{j-1}+1}$. Moreover, by Lemma 22 the last estimate is also valid for the difference of the b-coordinates of the points A_0 and B_0 subordinate to some squares centered at points of these sides. $\Box$

Corollary 27. *The segment* $[A_0, B_0]$ *is* $\varepsilon^{u_{j-1}+1}$*-vertical.*

Proof. By the previous lemma, the absolute value of the tangent of the angle between this segment and the vertical direction is estimated from above by $7\varepsilon^{u_j-T_{j-1}}$, which by (11) is less than $\varepsilon^{u_{j-1}+1}$ (since we can assume that $\varepsilon < \frac{1}{7}$). $\Box$

In particular, this segment $[A_0, B_0]$ is not ε^{u_j+1}-horizontal, so A_0 and B_0 cannot be neighboring points in our ε^{u_j+1}-resonance chain. Now consider the closed chain of segments in the upper subset of our $\tilde{M}_j$-configuration Γ, which consists of the segment $[A_0, B_0]$ and the part of our initial ε^{u_j+1}-resonance chain connecting these two points inside this upper subset of Γ. This closed chain is a $\varepsilon^{u_{j-1}+1}$-resonance, which contradicts the induction hypothesis over j. This contradiction finishes the proof of Theorem 25, and hence also of Theorem 23. $\Box$

Finally, every product $[M_{j_1}] \cdot [M_{j_2}] \cdots [M_{j_q}]$ of multiplicative generators of the Hopf algebra $H_*(B(\mathbb{R}^2, \infty), \mathbb{Z}_2)$ such that $2^{j_1} + 2^{j_2} + \cdots + 2^{j_q} = n$ can be realized by the set of all n-configurations, some 2^{j_1} points of which form a configuration of type $\tilde{M}_{j_1}$ shifted to $\mathbb{R}^2_+$ along the vector $(0, 8)$, some other 2^{j_2} points form a configuration of type $\tilde{M}_{j_2}$ shifted along the vector $(56, 64)\ldots$ and the last 2^{j_q} points form a configuration of type $\tilde{M}_{j_q}$ shifted along the vector $(8^q - 8, 8^q)$. The values of both coordinates of the points of any of these groups are very far from the coordinates of the points from any other group, thus all obtained n-configurations do not contain resonances. This finishes the proof of Lemma 16 and hence also of Theorems 6 and 8. □

6. Proof of Corollaries 7 and 10

Proposition 28. *All the statements of Corollary 7 (respectively, Corollary 10) are monotonic on N: if for a triplet of numbers (n, N, r), $n < N$ (respectively, $n > N$), it is true that for any N-dimensional subspace $\mathcal{F}^N \subset C^0(\mathbb{R}^1, \mathbb{R}^1)$ there exist systems of n independent equality conditions defining subspaces of codimension $\leq n - r$ (respectively, of dimension r) in $\mathcal{F}^N$, then the same is true for the triplet $(n, N-1, r)$ (respectively, $(n, N+1, r)$).*

Proof. Apply the hypothesis of this proposition to an arbitrary N-dimensional space containing $\mathcal{F}^{N-1}$ (respectively, contained in $\mathcal{F}^{N+1}$). □

Let us recall several results of [3] on mod 2 cohomology of spaces $B(\mathbb{R}^2, n)$.

Proposition 29 (see [3, Section 4.8]). *For any k, the group $H^k(B(\mathbb{R}^2, n), \mathbb{Z}_2)$ has a canonical basis whose elements are in a one-to-one correspondence with unordered decompositions of the number n into $n - k$ powers of 2. In particular, this group is nontrivial if and only if $k \leq n - I(n)$.*

The standard notation for such a basis element is $\langle 2^{l_1}, 2^{l_2}, \ldots, 2^{l_t} \rangle$, where $l_1 \geq l_2 \geq \cdots \geq l_t \geq 1$, $t \leq n - k$: this is the list (in nonincreasing order) of all summands of such a decomposition which are *strictly greater* than 1.

Namely, such a basis element of $H^*(B(\mathbb{R}^2, n), \mathbb{Z}_2)$ is defined by the intersection index with the closure of the subvariety in $B(\mathbb{R}^2, n)$ consisting of all n-configurations such that there exist t distinct vertical lines in $\mathbb{R}^2$, one of which contains 2^{l_1} points of our configuration, some other one contains 2^{l_2} of them, etc.

We will also use the abbreviated notation $\langle 2^{s_1}_{v_1}, 2^{s_2}_{v_2}, \ldots, 2^{s_q}_{v_q} \rangle$ for these basis elements, where $s_1 > s_2 > \cdots > s_q \geq 1$ and $2^{s_i}_{v_i}$ means 2^{s_i} repeated v_i times; if some v_i is here equal to 1 then we write simply 2^{s_i} instead of $2^{s_i}_1$.

Proposition 30 (see [3, Section 5.2]). *The class $w_k(\xi_n) \in H^k(B(\mathbb{R}^2, n), \mathbb{Z}_2)$ for any $k < n$ is equal to the sum of all basic elements of this group described in the* *previous proposition. In particular, all classes $w_k(\xi_n)$ with $k \le n - I(n)$ are not equal to* 0.

So we have

$$w_1 = \langle 2 \rangle, \tag{14}$$

$$w_2 = \langle 2_2 \rangle, \tag{15}$$

$$w_3 = \langle 2_3 \rangle + \langle 4 \rangle, \tag{16}$$

$$w_4 = \langle 2_4 \rangle + \langle 4, 2 \rangle, \tag{17}$$

$$w_5 = \langle 2_5 \rangle + \langle 4, 2_2 \rangle, \tag{18}$$

$$w_6 = \langle 2_6 \rangle + \langle 4, 2_3 \rangle + \langle 4_2 \rangle, \tag{19}$$

$$w_7 = \langle 2_7 \rangle + \langle 4, 2_4 \rangle + \langle 4_2, 2 \rangle + \langle 8 \rangle, \tag{20}$$

$$w_8 = \langle 2_8 \rangle + \langle 4, 2_5 \rangle + \langle 4_2, 2_2 \rangle + \langle 8, 2 \rangle, \tag{21}$$

$$w_9 = \langle 2_9 \rangle + \langle 4, 2_6 \rangle + \langle 4_2, 2_3 \rangle + \langle 4_3 \rangle + \langle 8, 2_2 \rangle, \tag{22}$$

$$w_{10} = \langle 2_{10} \rangle + \langle 4, 2_7 \rangle + \langle 4_2, 2_4 \rangle + \langle 4_3, 2 \rangle + \langle 8, 2_3 \rangle + \langle 8, 4 \rangle, \tag{23}$$

$$w_{11} = \langle 2_{11} \rangle + \langle 4, 2_8 \rangle + \langle 4_2, 2_5 \rangle + \langle 4_3, 2_2 \rangle + \langle 8, 2_4 \rangle + \langle 8, 4, 2 \rangle, \tag{24}$$

$$w_{12} = \langle 2_{12} \rangle + \langle 4, 2_9 \rangle + \langle 4_2, 2_6 \rangle + \langle 4_3, 2_3 \rangle + \langle 4_4 \rangle + \langle 8, 2_5 \rangle + \langle 8, 4, 2_2 \rangle. \tag{25}$$

Proposition 31 (see [3, Sections 9 and 6]). *The cohomological product of two basis elements of the group $H^*(B(\mathbb{R}^2, n), \mathbb{Z}_2)$ having the form*

$$\langle 2^m, \dots, 2^m, 2^{m-1}, \dots, 2^{m-1}, \dots, 2, \dots, 2 \rangle,$$

where any number 2^i, $i \in \{1, 2, \dots, m\}$, occurs p_i times in the first factor and q_i times in the second and some of numbers p_i, q_i can be equal to 0, *is equal to*

$$\prod_{i=1}^{m} \binom{p_i + q_i}{p_i} \langle 2^m, \dots, 2^m, 2^{m-1}, \dots, 2^{m-1}, \dots, 2, \dots, 2 \rangle, \tag{26}$$

where any symbol 2^i in the angle brackets occurs $p_i + q_i$ times, all binomial coefficients are counted modulo 2, *and the entire expression* (26) *is assumed to be zero if $(p_m + q_m)2^m + (p_{m-1} + q_{m-1})2^{m-1} + \cdots + (p_1 + q_1)2 > n$.*

Now, all statements of Corollaries 7 and 10 follow immediately from Theorems 6, 8 and the following calculations.

A(1) By (14), (16) and (26), $w_1 w_3 = \langle 4, 2 \rangle$, which is nontrivial for $n \ge 6$.

A(2) By (15), (17) and (26),

$$w_2 w_4 = \langle 4, 2_3 \rangle + \langle 2_6 \rangle, \tag{27}$$

which is nontrivial if $n \ge 10$.

A(3) By (16), (18) and (26), $w_3 w_5 = \langle 4, 2_5\rangle$, which is nontrivial if $n \geq 14$. By (17), (19) and (26),

$$w_4 w_6 = \langle 4_2, 2_4\rangle + \langle 4_3, 2\rangle, \tag{28}$$

which is also nontrivial if $n \geq 14$.

A(4) By (18), (20) and (26),

$$w_5 w_7 = \langle 4_3, 2_3\rangle + \langle 8, 2_5\rangle + \langle 8, 4, 2_2\rangle, \tag{29}$$

which is nontrivial for $n \geq 16$.

A(5) By (19), (21) and (26),

$$w_6 w_8 = \langle 2_{14}\rangle + \langle 4, 2_{11}\rangle + \langle 4_2, 2_8\rangle + \langle 4_3, 2_5\rangle + \langle 8, 2_7\rangle + \langle 8, 4_2, 2\rangle, \tag{30}$$

which is nontrivial for $n \geq 18$.

A(6) By (20), (22) and (26),

$$w_7 w_9 = \langle 4, 2_{13}\rangle + \langle 4_3, 2_7\rangle + \langle 8, 2_9\rangle + \langle 8, 4_3\rangle, \tag{31}$$

which is nontrivial for $n \geq 20$.

A(7) By (21), (23) and (26),

$$w_8 w_{10} = \langle 4_2, 2_{12}\rangle + \langle 4_3, 2_9\rangle + \langle 8, 4, 2_8\rangle + \langle 8, 4_2, 2_5\rangle + \langle 8, 4_3, 2_2\rangle, \tag{32}$$

which is nontrivial if $n \geq 24$.

A(8) By (22), (24) and (26),

$$w_9 w_{11} = \langle 4_3, 2_{11}\rangle + \langle 8, 4, 2_{10}\rangle + \langle 8, 4_3, 2_4\rangle, \tag{33}$$

which is nontrivial if $n \geq 28$.

By (23), (25) and (26),

$$\begin{aligned} &w_{10} w_{12} \\ &= \langle 4_4, 2_{10}\rangle + \langle 4_5, 2_7\rangle + \langle 4_6, 2_4\rangle + \langle 4_7, 2\rangle + \langle 8, 4, 2_{12}\rangle + \langle 8, 4_4, 2_3\rangle + \langle 8, 4_5\rangle, \end{aligned} \tag{34}$$

which also is nontrivial if $n \geq 28$.

A(9) The class w_{14} contains summand $\langle 8_2\rangle$, therefore by (25) and (26) the product $w_{12} w_{14}$ contains summands $\langle 8_2, 4_4\rangle$ and $\langle 8_3, 4, 2_2\rangle$, each of which is nontrivial if $n \geq 32$.

B(1) By (15), (28) and (26),

$$w_2 w_4 w_6 = \langle 4_2, 2_6\rangle + \langle 4_3, 2_3\rangle, \tag{35}$$

which is nontrivial for $n \geq 18$. By Theorem 6 this calculation proves the statement B(1) of Corollary 7 (respectively, Corollary 10) for $N = n+1$ (respectively, $N = n-1$), and the case $N = n$ follows by monotonicity, see Proposition 28.

B(2) By (16), (29) and (26), $w_3 w_5 w_7 = \langle 8, 4, 2_5 \rangle$, which is nontrivial for $n \geq 22$.

B(3) By (28), (21) and (26),

$$w_4 w_6 w_8 = \langle 4_2, 2_{12} \rangle + \langle 4_3, 2_9 \rangle + \langle 8, 4_2, 2_5 \rangle, \tag{36}$$

which is nontrivial for $n \geq 26$.

B(4) By (18), (31) and (26),

$$w_5 w_7 w_9 = \langle 8, 4, 2_{11} \rangle + \langle 8, 4_3, 2_5 \rangle, \tag{37}$$

which is nontrivial if $n \geq 30$.

B(5) By (19), (32) and (26),

$$w_6 w_8 w_{10} = \langle 8, 4, 2_{14} \rangle + \langle 8, 4_3, 2_8 \rangle, \tag{38}$$

which is nontrivial for $n \geq 36$.

B(6) By (32), (25) and (26),

$$\begin{aligned} &w_8 w_{10} w_{12} \\ &= \langle 4_6, 2_{12} \rangle + \langle 4_7, 2_9 \rangle + \langle 8, 4_3, 2_{14} \rangle + \langle 8, 4_5, 2_8 \rangle + \langle 8, 4_6, 2_5 \rangle + \langle 8, 4_7, 2_2 \rangle, \end{aligned} \tag{39}$$

which is nontrivial for $n \geq 40$. By Theorems 6 and 8, this implies statements B(6) of Corollaries 7 and 10 for $N = n+7$ (respectively, $N = n-7$), and the cases $N = n+6$ (respectively, $N = n-6$) follow by monotonicity. Notice that the routine consideration for $N = n+6$ based on

$$w_7 w_9 w_{11} = \langle 4_3, 2_{11} \rangle + \langle 8, 4_3, 2_{11} \rangle, \tag{40}$$

gives the same result in more restrictive conditions, $n \geq 42$ only.

B(7) The class w_{14} contains the summand $\langle 8_2 \rangle$. Therefore by (34) and (26), the product $w_{10} w_{12} w_{14}$ contains the summand $\langle 8_3, 4_5 \rangle$, which is nontrivial if $n \geq 44$.

C(1) By (15), (36) and (26),

$$w_2 w_4 w_6 w_8 = \langle 4_2, 2_{14} \rangle + \langle 4_3, 2_{11} \rangle + \langle 8, 4_2, 2_7 \rangle, \tag{41}$$

which is nontrivial if $n \geq 30$. By Theorem 6, this proves statements C(1) of Corollaries 7 and 10 in the case $N = n+1$ (respectively, $N = n-1$), and the case $N = n$ follows by monotonicity.

C(2) By (17) and (38),

$$w_4 w_6 w_8 w_{10} = \langle 8, 4_3, 2_{12} \rangle, \tag{42}$$

which is nontrivial if $n \geq 44$. This proves statement C(2) of Corollary 7 (respectively, Corollary 10) for $N = n+3$ (respectively, $N = n-3$), which implies it also for $N = n+2$ (respectively, $N = n-2$).

C(3) By (19) and (39),

$$w_6 w_8 w_{10} w_{12} = \langle 8, 4_5, 2_{14}\rangle + \langle 8, 4_7, 2_8\rangle, \tag{43}$$

which is nontrivial if $n \geq 52$. This proves statements C(3) for $N = n+5$ (respectively, $N = n-5$) and hence also for $N = n+4$ (respectively, $N = n-4$).

C(4) The class w_{14} contains the summand $\langle 8_2\rangle$. Therefore by (39) and (26) the class $w_8 w_{10} w_{12} w_{14}$ contains the summand $\langle 8_3, 4_7, 2_2\rangle$, which is nontrivial if $n \geq 56$. This proves statements C(4) for $N = n+7$ ($N = n-7$) and hence also for $N = n+6$ ($N = n-6$).

D(1) By (41), (23) and (26), $w_2 w_4 w_6 w_8 w_{10} = \langle 8, 4_3, 2_{14}\rangle$, which is nontrivial if $n \geq 48$.

D(2) By (17), (43) and (26), $w_4 w_6 w_8 w_{10} w_{12} = \langle 8, 4_7, 2_{12}\rangle$, which is nontrivial if $n \geq 60$.

D(3) Since w_{14} contains $\langle 8_2\rangle$, by (43) and (26) the product $w_6 w_8 w_{10} w_{12} w_{14}$ contains the summand $\langle 8_3, 4_7, 2_8\rangle$, which is nontrivial if $n \geq 68$.

E(1) By D(1) and formulas (25) and (26), $w_2 w_4 w_6 w_8 w_{10} w_{12} = \langle 8, 4_7, 2_{14}\rangle$, which is nontrivial if $n \geq 64$.

E(2) Since w_{14} contains $\langle 8_2\rangle$, by D(2) and (26) the product $w_4 w_6 w_8 w_{10} w_{12} w_{14}$ contains the summand $\langle 8_3, 4_7, 2_{12}\rangle$, which is nontrivial if $n \geq 76$.

F. Since w_{14} contains the summand $\langle 8_2\rangle$, by E(1) and formula (26) the class $w_2 w_4 w_6 w_8 w_{10} w_{12} w_{14}$ contains the summand $\langle 8_3, 4_7, 2_{14}\rangle$, nontrivial if $n \geq 80$. □

7. Equality conditions and homology of knot spaces

Let us denote by $\mathcal{K}$ the affine space of all C^∞-smooth maps $\mathbb{R}^1 \to \mathbb{R}^3$ coinciding with a fixed linear embedding outside a compact set in $\mathbb{R}^1$. Let Σ be the *discriminant subvariety* of $\mathcal{K}$ consisting of all maps which are not smooth embeddings, i.e., have either self-intersections or points of vanishing derivative. The elements of the set $\mathcal{K} \setminus \Sigma$ are called *long knots*. There is a natural one-to-one correspondence between the connected components of this set and the isotopy classes of the usual knots, i.e., of smooth embeddings $S^1 \to \mathbb{R}^3$ or $S^1 \to S^3$.

The variety Σ is swept out by affine subspaces $L(a,b)$ of codimension 3 in $\mathcal{K}$ corresponding to all chords $\{a,b\}$ in $\mathbb{R}^1$ (including degenerate chords with $a=b$) and consisting of maps $\varphi : \mathbb{R}^1 \to \mathbb{R}^3$ such that $\varphi(a) = \varphi(b)$ (or $\varphi'(a) = 0$ if $a = b$). Much of the topological structure of Σ can be described in terms of the order complex of the (naturally topologized) partially ordered set, whose elements correspond to these subspaces $L(a,b)$ and their finite intersections (defined by chord diagrams), and the order relation is the incidence of corresponding subspaces. For any n, the subspaces in $\mathcal{K}$ defined in this way by independent n-chord diagrams form an affine bundle over the space $\overline{\mathrm{CD}}_n$ of equivalence classes of such diagrams (including degenerate ones, containing chords of type $\{a,a\}$). The fibers of this bundle have codimension $3n$ in $\mathcal{K}$, and its normal bundle is isomorphic to the sum of three copies of the bundle τ_n^* considered in Section 2 (and continued to degenerate chord diagrams).

The topology of the space $\mathcal{K}\setminus\Sigma$ is related by a kind of Alexander duality to the topology of the complementary space Σ, in particular, the numerical knot invariants can be realized as linking numbers with infinite-dimensional cycles of codimension 1 in $\mathcal{K}$ contained in Σ. However, Alexander duality deals with finite-dimensional spaces only, therefore to apply it properly we use finite-dimensional approximations of the space $\mathcal{K}$. Namely, we consider infinite sequences $\mathcal{K}_1\subset\mathcal{K}_2\subset\dots$ of finite-dimensional affine subspaces of $\mathcal{K}$, such that any connected component of $\mathcal{K}\setminus\Sigma$ is represented by elements of subspaces $\mathcal{K}_j\setminus\Sigma$ with sufficiently large j, and moreover any homology class of $\mathcal{K}\setminus\Sigma$ is represented by cycles contained in such subspaces. (The existence of such sequences of subspaces $\mathcal{K}_j$ follows easily from Weierstrass approximation theorem). Then for any such subspace $\mathcal{K}_j$ of dimension d_j we have the Alexander isomorphisms

$$\tilde{H}^k(\mathcal{K}_j\setminus\Sigma)\simeq\overline{H}_{d_j-k-1}(\mathcal{K}_j\cap\Sigma), \tag{44}$$

where $\overline{H}_*$ denotes the Borel–Moore homology groups.

To study the left-hand groups in (44) (in particular, such a group with $k=0$, i.e., the group of $\mathbb{Z}$-valued invariants of knots realizable in $\mathcal{K}_j$) a *simplicial resolution* of the space $\mathcal{K}_j\cap\Sigma$ is used in [6]. It is a certain topological space $\sigma(j)$ and a surjective map $\sigma(j)\to\mathcal{K}_j\cap\Sigma$ inducing an isomorphism of Borel–Moore homology groups. These groups $\overline{H}_*(\sigma(j))\simeq\overline{H}_*(\mathcal{K}_j\cap\Sigma)$ can be calculated by a spectral sequence $\{E^r_{n,\beta}\}$ defined by a natural increasing filtration

$$\sigma_1(j)\subset\sigma_2(j)\subset\cdots\subset\sigma(j), \tag{45}$$

in particular, $E^1_{n,\beta}\simeq\overline{H}_{n+\beta}(\sigma_n(j)\setminus\sigma_{n-1}(j))$.

This filtration is finite if the subspace $\mathcal{K}_j$ is not very degenerate. Namely, any space $\sigma_n(j)\setminus\sigma_{n-1}(j)$ is constructed starting from the intersection sets of $\mathcal{K}_j$ with subspaces of codimension $3n$ in $\mathcal{K}$ defined by independent n-chord diagrams. Since the family of all such planes is $2n$-parametric, a *generic* d_j-dimensional affine subspace $\mathcal{K}_j$ meets only subspaces of this kind with $n\le d_j$, so $\sigma_{d_j}(j)=\sigma(j)$.

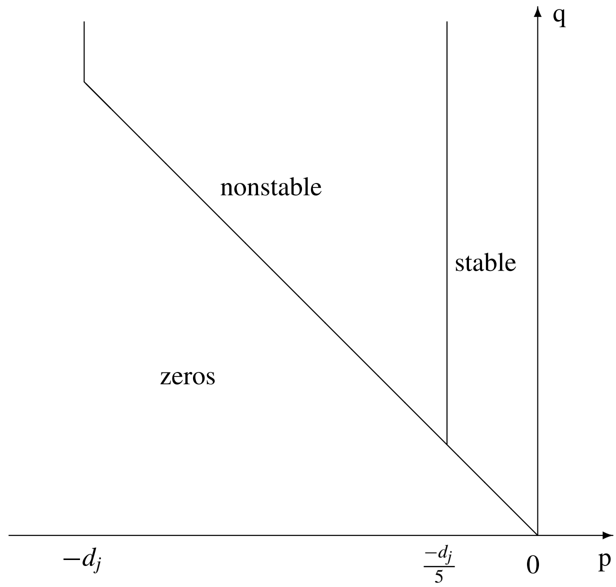

Figure 3. Spectral sequence for $H^*(\mathcal{K}_j \setminus \Sigma)$.

The formal change $E_r^{p,q} \equiv E^r_{-p,d_j-1-q}$ turns the homological spectral sequence defined by this filtration into a cohomological one, which by Alexander duality converges to the left-hand groups of (44). All nontrivial groups $E_r^{p,q}$, $r \geq 1$, of the last spectral sequence for a generic subspace $\mathcal{K}_j$ lie in the domain (see Figure 3)

$$\{(p,q) : p \in [-d_j, -1],\, p+q \geq 0\}.$$

If the approximating subspace $\mathcal{K}_j$ is generic and n is sufficiently small with respect to d_j (namely, $n \leq \frac{d_j}{5}$), then all subspaces of $\mathcal{K}$ defined by independent n-chord diagrams intersect $\mathcal{K}_j$ transversally along nonempty planes. Indeed, if $d_j > 3n$, then the codimension of the set of d_j-dimensional affine subspaces in $\mathcal{K}$, which are not generic with respect to a plane of codimension $3n$, is equal to $d_j - 3n + 1$; therefore the $2n$-parametric family of such sets corresponding to all subspaces defined by n-chord diagrams sweeps out a subset of codimension at least $d_j - 5n + 1$ (if this number is positive), and for $d_j \geq 5n$ we can choose $\mathcal{K}_j$ not from this subset.

If $\mathcal{K}_j$ is generic in this sense, then these intersection sets in $\mathcal{K}_j$ form an affine bundle of dimension $d_j - 3n$ with base $\overline{\mathrm{CD}}_n$. By the construction of the simplicial resolution, this implies that the topology of the sets $\sigma_n(j) \setminus \sigma_{n-1}(j)$ essentially stabilizes at this value of j: for all $j' > j$ the space $\sigma_n(j') \setminus \sigma_{n-1}(j')$ is homeomorphic to the direct product of spaces $\sigma_n(j) \setminus \sigma_{n-1}(j)$ and $\mathbb{R}^{d_{j'}-d_j}$. In particular, we have natural isomorphisms

$$E^1_{n,\beta}(j) \simeq E^1_{n,\beta+(d_j'-d_j)}(j') \quad \text{for all } j' > j,\; n \leq \tfrac{d_j}{5} \text{ and arbitrary } \beta.$$

Substitutions (44) turn them into natural isomorphisms $E_1^{p,q}(j') \simeq E_1^{p,q}(j)$ for all $p \geq -\frac{d_j}{5}$. Moreover, these isomorphisms commute with all the further differentials of our spectral sequence; the Borel–Moore homology groups of the spaces $\sigma_n(j)$ and $\sigma_n(j')$ for all $n \leq \frac{d_j}{5}$ and $j' \geq j$ are naturally isomorphic to each other up to the shift of dimensions by $d_{j'} - d_j$. The cohomology classes of $\mathcal{K} \setminus \Sigma$ arising from this area of the spectral sequence (i.e., the sequences of nontrivial cohomology classes of the spaces $\mathcal{K}_{j'} \setminus \Sigma,\ j' \geq j$, realizable by linking numbers with cycles located in $\sigma_n(j')$ for $n \leq \frac{d_j}{5}$ and corresponding to one another by these isomorphisms) are known as *finite-type* cohomology classes of the space of long knots. Therefore, the intriguing question about the completeness of the system of these classes in entire cohomology groups of $\mathcal{K} \setminus \Sigma$ (in particular, about the existence of nonequivalent knots not separated by finite-type invariants) depends on the groups $E_r^{p,q}(j)$ in the nonstable domains, on the deviation of these groups from stable ones, and on the way in which the nonstable groups $E_\infty^{p,q}(j)$ for different j correspond to the same cohomology classes of spaces $\mathcal{K}_j \setminus \Sigma$ with different j.

The arguments of the previous sections of this article allow us to say something about the nontriviality of this problem.

Proposition 32. *If* $4n - I(n) > d_j \geq 3n$, *then for any* d_j*-dimensional affine subspace* $\mathcal{K}_j \subset \mathcal{K}$ *there exist independent* n*-chord diagrams such that the corresponding affine subspaces of* $\mathcal{K}$ *have nongeneric* (*i.e.*, *either nontransversal or empty*) *intersections with the space* $\mathcal{K}_j$.

Proposition 33. *If* $2n + I(n) \leq d_j \leq 3n$, *then for almost any* d_j*-dimensional affine subspace* $\mathcal{K}_j \subset \mathcal{K}$ (*that is*, *for any subspace from a residual subset in the space of all such subspaces*) *there exist independent* n*-chord diagrams such that the corresponding affine subspaces of* $\mathcal{K}$ *have nonempty intersection with* $\mathcal{K}_j$.

Definition 34. Let $\mathcal{L}$ denote the affine bundle over the space $B(\mathbb{R}^2_+, n) \setminus \Xi$ of independent n-chord diagrams, whose fiber over any such diagram is the subspace of codimension $3n$ in $\mathcal{K}$ consisting of maps $\varphi : \mathbb{R}^1 \to \mathbb{R}^3$ taking the same values at endpoints of each chord of this diagram. For an affine subspace $\mathcal{K}_j \subset \mathcal{K}$ denote by $\|(\mathcal{K}_j)$ the subset in $B(\mathbb{R}^2_+, n) \setminus \Xi$ consisting of n-chord diagrams such that the corresponding fiber of bundle $\mathcal{L}$ contains lines parallel to some lines contained in the space $\mathcal{K}_j$.

Proof of Proposition 32. The normal bundle $\mathcal{L}^\perp$ of $\mathcal{L}$ in $\mathcal{K}$ is isomorphic to the direct sum of three copies of the regular bundle ξ_n. By Lemmas 13 and 14, its total Stiefel–Whitney class is then equal to $(w(\xi_n))^3 \equiv w(\xi_n)$, in particular, its i-dimensional component w_i is not trivial if $i \leq n - I(n)$.

Make $\mathcal{K}_j$ a vector space by arbitrarily choosing the "origin" point in it. If all fibers of the bundle $\mathcal{L}$ are in general position with respect to $\mathcal{K}_j$, then a $(d_j - 3n)$-dimensional vector bundle with the same base is defined, the fiber of which over a chord diagram is obtained from the intersection set of $\mathcal{K}_j$ and the corresponding fiber of the bundle $\mathcal{L}$ by a parallel translation, after which it passes through the origin point of $\mathcal{K}_j$. The total Stiefel–Whitney class of this bundle is equal to $w(\mathcal{L}^\perp)^{-1} \equiv w(\xi_n)^{-1}$, which by Lemma 14 is equal to $w(\xi_n)$. If $d_j - 3n < n - I(n)$, then this implies that $w_{n-I(n)}(\xi_n) = 0$, a contradiction. $\square$

Lemma 35. *If $d_j \leq 3n$, then for almost any d_j-dimensional affine subspace $\mathcal{K}_j \subset \mathcal{K}$ the codimension of the set $\|(\mathcal{K}_j)$ in $B(\mathbb{R}^2_+, n) \setminus \Xi$ is at least $3n - d_j + 1$.*

Proof. Consider the space

$$\tilde{G}(\mathcal{K}, d_j) \times (B(\mathbb{R}^2_+, 2) \setminus \Xi) \tag{46}$$

of all pairs $\{\mathcal{K}_j, \Gamma\}$ where K_j is a d_j-dimensional affine subspace of $\mathcal{K}$ and Γ is an independent n-chord diagram. Denote by Λ the subset of this space consisting of pairs $\{\mathcal{K}_j, \Gamma\}$ such that $\Gamma \in \|(K_j)$. The space (46) and its subset Λ are both fibered over the space $B(\mathbb{R}^2_+, 2) \setminus \Xi$ of independent n-chord diagrams, and for any such diagram Γ the corresponding fiber of the latter fiber bundle has codimension $3n - d_j + 1$ in the fiber of the former. Therefore, the codimension of Λ in the space (46) is equal to $3n - d_j + 1$, and the typical fiber of the projection of Λ to the first factor of (46) has codimension at least $3n - d_j + 1$ in the corresponding fiber of the projection of entire space (46). $\square$

Proof of Proposition 33. Let us fix a subspace $\mathcal{K}_j$ for which the condition of the previous lemma is satisfied. The complement of the set $\|(\mathcal{K}_j)$ in the manifold $B(\mathbb{R}^2_+, n) \setminus \Xi$ has then the same homology groups up to dimension $3n - d_j$ as entire $B(\mathbb{R}^2_+, n) \setminus \Xi$.

Consider the affine bundle $(\mathcal{L}^\perp)^*$ over the manifold $B(\mathbb{R}^2_+, n) \setminus \Xi$: its fibers consist of linear functions on $\mathcal{K}$ vanishing on the corresponding fibers of the bundle $\mathcal{L}$. Over the set $(B(\mathbb{R}^2_+, n) \setminus \Xi) \setminus \|(K_j)$ a $(3n - d_j)$-dimensional subbundle of $(\mathcal{L}^\perp)^*$ is defined, consisting of functions constant on $\mathcal{K}_j$. This subbundle has the same Stiefel–Whitney class (equal to $w(\xi_n)$) as the whole $(\mathcal{L}^\perp)^*$, since its normal bundle is isomorphic to the trivial bundle with fiber $(\mathcal{K}_j)^*$. If no fibers of the bundle $\mathcal{L}$ intersect the space $\mathcal{K}_j$, then this subbundle has a nowhere vanishing cross-section: indeed, we can define an arbitrary Euclidean structure on this subbundle, and choose in each fiber the linear function of unit norm taking the maximal value on $\mathcal{K}_j$. If $3n - d_j \leq n - I(n)$, then this contradicts the nontriviality of the class $w_{n-I(n)}(\xi_n)$. $\square$

Remark. I hope that the further study of the characteristic classes of the bundle $\mathcal{L}$ (and of its analog defined on the space $\overline{\mathrm{CD}}_n$ of equivalence classes of chord diagrams, rather than on the resolution $B(\mathbb{R}^2_+, n) \setminus \Xi$ of this space) will provide not only the proofs of the inevitable troubles in the calculation of the cohomology classes of knot spaces, but also the construction of some such classes not reducible to classes of finite-type.

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POSITIVELY CURVED FINSLER METRICS ON VECTOR BUNDLES, II

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We show that if E is an ample vector bundle of rank at least two with some curvature bound on $O_{P(E^*)}(1)$, then $E^* \otimes \det E$ is Kobayashi positive. The proof relies on comparing the curvature of $(\det E^*)^k$ and $S^k E$ for large k and using duality of convex Finsler metrics. Following the same thread of thought, we show if E is ample with similar curvature bounds on $O_{P(E^*)}(1)$ and $O_{P(E\otimes \det E^*)}(1)$, then E is Kobayashi positive. With additional assumptions, we can furthermore show that $E^* \otimes \det E$ and E are Griffiths positive.

1. Introduction

Let E be a holomorphic vector bundle of rank r over a compact complex manifold X of dimension n. We denote the dual bundle by E^* and its projectivized bundle by $P(E^*)$. The vector bundle E is said to be ample if the line bundle $O_{P(E^*)}(1)$ over $P(E^*)$ is ample. On the other hand, E is called Griffiths positive if E carries a Griffiths positive Hermitian metric. Moreover, E is called Kobayashi positive if E carries a strongly pseudoconvex Finsler metric whose Kobayashi curvature is positive (we will give a quick review on Finsler metrics and Kobayashi curvature in Section 2A; or see [Wu 2022, Section 2]).

There are two conjectures made by Griffiths [1969] and Kobayashi [1975] regarding the equivalence of ampleness and positivity:

(1) If E is ample, then E is Griffiths positive.

(2) If E is ample, then E is Kobayashi positive.

These two conjectures are still open, save for $n = 1$, in [Umemura 1973; Campana and Flenner 1990] (for recent progress, see [Berndtsson 2009a; Mourougane and Takayama 2007; Hering et al. 2010; Liu et al. 2013; Liu and Yang 2015; Naumann 2021; Feng et al. 2020; Demailly 2021; Finski 2022; Pingali 2021; Ma and Zhang 2023]). Note that the converse of each conjecture is true [Feng et al. 2020; Wu 2022].

By Kodaira's embedding theorem, ampleness of a line bundle is equivalent to the existence of a positively curved metric on the line bundle. So, the conjectures

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of Griffiths and Kobayashi can be rephrased: *Given a positively curved metric on* $O_{P(E^*)}(1)$, *can we construct a positively curved Hermitian/Finsler metric on* E? In this paper, we show that it is so, by imposing curvature bounds on tautological line bundles of $P(E^*)$ and $P(E)$. Since Hermitian metrics on $O_{P(E^*)}(1)$ are in one-to-one correspondence with Finsler metrics on E^*, these curvature bounds can also be written in terms of Kobayashi curvature.

We first consider a relevant case where the picture is clearer. It is known that, for rank of E at least 2:

(1) If E is Griffiths positive, then $E^* \otimes \det E$ with the induced metric is Griffiths positive.

(2) If E is ample, then $E^* \otimes \det E$ is ample.

The first fact can be found in [Demailly 2012, p. 346, Theorem 9.2] and the second in [Hartshorne 1966, Corollary 5.3] together with the isomorphism (see Appendix)

$$\bigwedge^{r-1} E \simeq E^* \otimes \det E.$$

If we follow the guidance of Griffiths and Kobayashi, we would ask whether or not the ampleness of E implies Griffiths/Kobayashi positivity of $E^* \otimes \det E$ for $r \geq 2$. Our first result is that this can be achieved by imposing curvature bounds on $O_{P(E^*)}(1)$.

Let $q : P(E^*) \to X$ be the projection. Let g be a metric on $O_{P(E^*)}(1)$ whose curvature restricted to a fiber $\Theta(g)|_{P(E_z^*)}$ is positive for all $z \in X$. For a tangent vector $\eta \in T_z^{1,0}X$ and a point $[\zeta] \in P(E_z^*)$, we consider tangent vectors $\tilde{\eta}$ to $P(E^*)$ at $(z, [\zeta])$ such that $q_*(\tilde{\eta}) = \eta$, namely the lifts of η to $T^{1,0}_{(z,[\zeta])}P(E^*)$. Then we define the function

$$(\eta, [\zeta]) \mapsto \inf_{q_*(\tilde{\eta})=\eta} \Theta(g)(\tilde{\eta}, \bar{\tilde{\eta}}) := m(\eta, [\zeta]), \tag{1-1}$$

where the infimum taken over all the lifts of η to $T^{1,0}_{(z,[\zeta])}P(E^*)$. This infimum is actually a minimum, see (2-3). On the other hand, since such a metric g corresponds to a strongly pseudoconvex Finsler metric on E^*, and if we denote its Kobayashi curvature by $\theta(g)$ a (1, 1)-form on $P(E^*)$, then

$$m(\eta, [\zeta]) = -\theta(g)(\tilde{\eta}, \bar{\tilde{\eta}}). \tag{1-2}$$

The term on the right is independent of the choice of lifts $\tilde{\eta}$ (we will prove (1-2) in Section 2A).

Theorem 1. *Assume $r \geq 2$ and the line bundle $O_{P(E^*)}(1)$ has a positively curved metric h and a metric g with $\Theta(g)|_{P(E_z^*)} > 0$ for all $z \in X$. If there exist a Hermitian* *metric Ω on X and a constant $M \in [1, r)$ such that the following inequalities of $(1, 1)$-forms hold*:

$$Mq^*\Omega \geq -\theta(g), \tag{1-3}$$

$$q^*\Omega \leq -\theta(h), \tag{1-4}$$

then $E^* \otimes \det E$ is Kobayashi positive.

We can of course choose g to be h in Theorem 1, but having two different metrics seems more flexible. The proof of Theorem 1 relies on two observations. First, starting with g and h on $O_{P(E^*)}(1)$, we construct two Hermitian metrics on $S^k E$ and $\det E$ respectively. The curvature of the induced metric on $S^k E \otimes (\det E^*)^k$ can be shown to be Griffiths negative for k large (see Section 3 for details). The second observation (see [Wu 2022]) is that since the induced metric on $S^k E \otimes (\det E^*)^k$ is basically an L^2-metric, its k-th root is a convex Finsler metric on $E \otimes \det E^*$ which is also strongly plurisubharmonic on the total space minus the zero section. After perturbing this Finsler metric and taking duality, we get a convex and strongly pseudoconvex Finsler metric on $E^* \otimes \det E$ whose Kobayashi curvature is positive. So the bundle $E^* \otimes \det E$ is Kobayashi positive. Notice that the Finsler metric we find is actually convex.

The reason why we impose Ω, M and inequalities (1-3) and (1-4) in Theorem 1 is the following. On the bundle $S^k E \otimes (\det E^*)^k$, the curvature of the induced metric is roughly bounded above by $k \sum_m a_m c_m - rk \sum_m b_m c_m$ where a_m and b_m are some positive integrals with $\sum_m a_m = \sum_m b_m = 1$, and c_m are positive numbers related to the curvature of h. It does not seem possible to us that the upper bound $k \sum_m a_m c_m - rk \sum_m b_m c_m$ can be made negative without any assumption. So we introduce Ω and M to control the upper bound.

With small changes on the proof, one can write down a variant of Theorem 1 where the conclusion is about the Kobayashi positivity of $E^* \otimes (\det E)^l$ (see the end of Section 3).

Now let us go back to the original conjecture of Kobayashi and adapt the proof of Theorem 1 to this case. Let $p : P(E) \to X$ be the projection. We recall under the canonical isomorphism $P(E \otimes \det E^*) \simeq P(E)$, the line bundle $O_{P(E \otimes \det E^*)}(1)$ corresponds to the line bundle $O_{P(E)}(1) \otimes p^* \det E$ (see [Kobayashi 1987, p. 86, Proposition 3.6.21]). Let g be a metric on $O_{P(E)}(1) \otimes p^* \det E$ with $\Theta(g)|_{P(E_z)} > 0$ for all $z \in X$. For a tangent vector $\eta \in T^{1,0}_z X$ and a point $[\xi] \in P(E_z)$, we similarly have

$$(\eta, [\xi]) \mapsto \inf_{p_*(\eta')=\eta} \Theta(g)(\eta', \bar{\eta}'),$$

where η' are the lifts of η to $T^{1,0}_{(z,[\xi])} P(E)$. Meanwhile, such a metric g corresponds to a strongly pseudoconvex Finsler metric on $E \otimes \det E^*$ and we denote its Kobayashi curvature by $\theta(g)$ a (1, 1)-form on $P(E)$. As before,

$$\inf_{p_*(\eta')=\eta} \Theta(g)(\eta', \bar{\eta}') = -\theta(g)(\eta', \bar{\eta}'). \tag{1-5}$$

Theorem 2. *Assume $r \geq 2$ and $O_{P(E^*)}(1)$ has a positively curved metric h and $O_{P(E)}(1) \otimes p^* \det E$ has a metric g with $\Theta(g)|_{P(E_z)} > 0$ for all $z \in X$. If there exist a Hermitian metric Ω on X and a constant $M \in [1, r)$ such that*

$$Mp^*\Omega \geq -\theta(g), \tag{1-6}$$

$$q^*\Omega \leq -\theta(h), \tag{1-7}$$

then E is Kobayashi positive.

Since the ampleness of E implies ampleness of $E^* \otimes \det E$, one choice for g in Theorem 2 is a positively curved metric on $O_{P(E)}(1) \otimes p^* \det E$, but how much this choice helps is unknown to us. The proof of Theorem 2 follows the same scheme as in Theorem 1. We first use h and g to construct Hermitian metrics on $\det E$ and $S^k E^* \otimes (\det E)^k$ respectively. The induced metric on $[S^k E^* \otimes (\det E)^k] \otimes (\det E^*)^k$ is Griffiths negative for k large (see Section 4). Then by taking k-th root, perturbing, and taking duality, we obtain a convex, strongly pseudoconvex, and Kobayashi positive Finsler metric on E.

1A. *Griffiths positivity.* The conclusions in Theorems 1 and 2 are about Finsler metrics. For their Hermitian counterpart, we need additional assumptions. The reason is that in Theorems 1 and 2, taking large tensor power of various bundles helps us eliminate the curvature of the relative canonical bundles $K_{P(E^*)/X}$ and $K_{P(E)/X}$, and after getting the desired estimates we take k-th root to produce Finsler metrics. However, the step of taking k-th root produces only Finsler, not Hermitian metrics. So the first step of taking large tensor power is not allowed if one wants Hermitian metrics.

Let us be more precise. For a metric g on $O_{P(E^*)}(1)$ with $\Theta(g)|_{P(E_z^*)} > 0$ for all $z \in X$, we denote $\Theta(g)|_{P(E_z^*)}$ by ω_z for the moment. The relative canonical bundle $K_{P(E^*)/X}$ has a metric induced from $\{\omega_z^{r-1}\}_{z \in X}$ and we denote the corresponding curvature by γ_g, a (1, 1)-form on $P(E^*)$. For $\eta \in T_z^{1,0}X$ and $[\zeta] \in P(E_z^*)$, we consider

$$(\eta, [\zeta]) \mapsto \sup_{q_*(\tilde{\eta})=\eta} \gamma_g(\tilde{\eta}, \bar{\tilde{\eta}}),$$

where the supremum taken over all the lifts of η to $T^{1,0}_{(z,[\zeta])} P(E^*)$. The supremum is a maximum under a suitable assumption, see (2-9). Moreover, for $z \in X$, the restriction $\gamma_g|_{P(E_z^*)}$ is actually the negative of Ricci curvature $-\mathrm{Ric}_{\omega_z}$ of the metric ω_z on $P(E_z^*)$.

Any Hermitian metric G on E^* will induce a metric g on $O_{P(E^*)}(1)$ with $\Theta(g)|_{P(E_z^*)} > 0$ and $\gamma_g|_{P(E_z^*)} < 0$ for all $z \in X$. Indeed, in this case, $\Theta(g)|_{P(E_z^*)}$

is the Fubini–Study metric and its Ricci curvature is positive, so $\gamma_g|_{P(E_z^*)} < 0$. Furthermore, for any $\eta \in T_z^{1,0}X$ and any $[\zeta] \in P(E_z^*)$,

$$\sup_{q_*(\tilde{\eta})=\eta} \gamma_g(\tilde{\eta}, \bar{\tilde{\eta}}) = r\theta(g)(\tilde{\eta}, \bar{\tilde{\eta}}) - q^*\Theta(\det G)(\tilde{\eta}, \bar{\tilde{\eta}}) \tag{1-8}$$

(we will prove (1-8) in Section 2B).

Theorem 3. *Assume $r \geq 2$ and the line bundle $O_{P(E^*)}(1)$ has a positively curved metric h and a metric g induced from a Hermitian metric G on E^*. If there exist a Hermitian metric Ω on X and a constant $M \in [1, r)$ such that*

$$Mq^*\Omega \geq -(r+1)\,\theta(g) + q^*\Theta(\det G), \tag{1-9}$$

$$q^*\Omega \leq -\theta(h), \tag{1-10}$$

then $E^* \otimes \det E$ is Griffiths positive.

Theorem 3 could be seen as a Hermitian analogue of Theorem 1. To state a Hermitian analogue of Theorem 2, we use again the isomorphism between $O_{P(E\otimes\det E^*)}(1) \to P(E \otimes \det E^*)$ and $O_{P(E)}(1) \otimes p^* \det E \to P(E)$.

Theorem 4. *Suppose that $r \geq 2$ and $O_{P(E^*)}(1)$ has a positively curved metric h and $O_{P(E)}(1) \otimes p^* \det E$ has a metric g induced from a Hermitian metric G on $E \otimes \det E^*$. If there exist a Hermitian metric Ω on X and a constant $M \in [1, r)$ such that*

$$Mp^*\Omega \geq -(r+1)\,\theta(g) + p^*\Theta(\det G), \tag{1-11}$$

$$q^*\Omega \leq -\theta(h), \tag{1-12}$$

then E is Griffiths positive.

In all the theorems above, the existence of the metric h comes from ampleness of E. So the real assumptions lie in (g, Ω, M) and the inequalities they have to satisfy. To weaken or remove these inequalities, one possible direction is to use geometric flows as in [Naumann 2021; Wan 2022; Ustinovskiy 2019; Li et al. 2021]. Another possible direction is to use the interplay between the optimal L^2-estimates and the positivity of curvature (see [Guan and Zhou 2015; Berndtsson and Lempert 2016; Lempert 2017; Hacon et al. 2018; Zhou and Zhu 2018]).

One example where the assumptions of all the theorems above are satisfied is given by $E = L^9 \oplus L^8 \oplus L^7$ with L a positive line bundle. The triple $(9, 8, 7)$ or the rank $r = 3$ is not that important; the point is to make sure the eigenvalues of the curvature with respect to some positive $(1, 1)$-form do not spread out too far. This example also indicates that a reasonable choice for Ω is probably related to $c_1(\det E)$.

A more sophisticated example, related to approximate Hermitian–Yang–Mills metrics [Jacob 2014; Misra and Ray 2021; Li et al. 2021], is semistable ample vector bundles over Riemann surfaces (see Section 7 for details of the examples).

The proof of Theorem 1 is given in Section 3 and almost as a corollary we prove Theorem 2 in Section 4. The proof of Theorem 3 in Section 5 is a modification of Theorem 1 but we still write out the details. In Section 6, we prove Theorem 4 based on Section 5.

2. Preliminaries

2A. *Finsler metrics.* We will use some facts about Finsler metrics on vector bundles which can be found in [Kobayashi 1975; 1996; Cao and Wong 2003; Aikou 2004; Wu 2022]. First, we recall the definition of Finsler metrics. Let E^* be a holomorphic vector bundle of rank r over a compact complex manifold X. For a vector $\zeta \in E^*_z$, we symbolically write $(z, \zeta) \in E^*$. A Finsler metric G on the vector bundle $E^* \to X$ is a real-valued function on E^* such that:

(1) G is smooth away from the zero section of E^*.

(2) For $(z, \zeta) \in E^*$, $G(z, \zeta) \geq 0$, and equality holds if and only if $\zeta = 0$.

(3) $G(z, \lambda\zeta) = |\lambda|^2 G(z, \zeta)$ for $\lambda \in \mathbb{C}$.

A Finsler metric G on E^* is said to be:

(1) Strongly pseudoconvex if the fiberwise complex Hessian of G is positive definite on $E^* \setminus \{\text{zero section}\}$, namely $(\sqrt{-1}\partial\bar{\partial}G)|_{E^*_z} > 0$ for all $z \in X$.

(2) Convex if $G^{1/2}$ restricted to each fiber E^*_z is convex.

Let g be a Hermitian metric on $\mathcal{O}_{P(E^*)}(1)$ with $\Theta(g)|_{P(E^*_z)} > 0$ for all $z \in X$. Such a g corresponds to a strongly pseudoconvex Finsler metric G on E^*. Since $(\sqrt{-1}\partial\bar{\partial}G)|_{E^*_z} > 0$, we can define a Hermitian metric $\tilde{G}$ on the pull-back bundle q^*E^*, where $q : P(E^*) \to X$ is the projection, as follows. For a vector Z in the fiber $q^*E^*_{(z,[\zeta])}$, we define

$$\tilde{G}_{(z,[\zeta])}(Z, Z) = (\sqrt{-1}\partial\bar{\partial}G)|_{E^*_z}(Z, \bar{Z}),$$

where the Z on the right-hand side is viewed as a tangent vector to E^*_z at ζ by the identification of vector spaces $q^*E^*_{(z,[\zeta])} = E^*_z$ and $E^*_z = T_\zeta E^*_z$ (see [Wu 2022, Section 2.2] for a local coordinate description).

Now $(q^*E^*, \tilde{G})$ is a Hermitian holomorphic vector bundle, so we can talk about its Chern curvature Θ, an $\operatorname{End} q^*E^*$-valued $(1, 1)$-form on $P(E^*)$. With respect to the metric $\tilde{G}$, the bundle q^*E^* has a fiberwise orthogonal decomposition

$$\mathcal{O}_{P(E^*)}(-1) \oplus \mathcal{O}_{P(E^*)}(-1)^{\perp},$$

and so Θ can be written as a block matrix. Let $\Theta|_{O_{P(E^*)}(-1)}$ denote the block in the matrix Θ corresponding to $\mathrm{End}(O_{P(E^*)}(-1))$. Since $O_{P(E^*)}(-1)$ is a line bundle, $\Theta|_{O_{P(E^*)}(-1)}$ is a $(1,1)$-form on $P(E^*)$, and it is called the Kobayashi curvature of the Finsler metric G. We will use $\theta(g)$ to denote the Kobayashi curvature

$$\theta(g) := \Theta|_{O_{P(E^*)}(-1)}. \tag{2-1}$$

In order to relate the Kobayashi curvature $\theta(g)$ to the curvature $\Theta(g)$ of g, we consider coordinates normal at one point. Given a point $(z_0, [\zeta_0]) \in P(E^*)$, there exists a holomorphic frame $\{s_i\}$ for E^* around $z_0 \in X$ such that

$$\begin{gathered} G_{\zeta_i\bar{\zeta}_j}(z_0,\zeta_0) = \delta_{ij}, \\ G_{\zeta_i\bar{\zeta}_j z_\alpha}(z_0,\zeta_0) = G_{\zeta_i\bar{\zeta}_j \bar{z}_\beta}(z_0,\zeta_0) = G_{\bar{\zeta}_j z_\alpha}(z_0,\zeta_0) = G_{z_\alpha}(z_0,\zeta_0) = 0, \end{gathered} \tag{2-2}$$

where we use $\{\zeta_i\}$ for the fiber coordinates on E^* with respect to the frame $\{s_i\}$ and $\{z_\alpha\}$ for the local coordinates on X (such a frame can be obtained by (5.11) in [Kobayashi 1996]). Moreover if Ω is a Hermitian metric on X, then by a linear transformation in the z-coordinates, we can make

$$\Omega\left(\frac{\partial}{\partial z_\alpha}, \frac{\partial}{\partial \bar{z}_\beta}\right)(z_0) = \delta_{\alpha\beta}$$

without affecting (2-2). We will call this coordinate system normal at the point $(z_0, [\zeta_0]) \in P(E^*)$.

Around the point $(z_0, [\zeta_0]) \in P(E^*)$, we assume the local coordinates

$$(z_1, \dots, z_n, w_1, \dots, w_{r-1})$$

are given by $w_i = \zeta_i/\zeta_r$ for $i = 1 \sim r-1$. So

$$e := \frac{\zeta_1 s_1 + \cdots + \zeta_r s_r}{\zeta_r} = w_1 s_1 + \cdots + w_{r-1} s_{r-1} + s_r$$

is a holomorphic frame for $O_{P(E^*)}(-1)$. Let e^* be the dual frame of $O_{P(E^*)}(1)$ around $(z_0, [\zeta_0]) \in P(E^*)$ and $g(e^*, e^*) = e^{-\phi}$. Then, the curvature $\Theta(g)$ can be written locally as

$$\begin{aligned} \sum_{\alpha,\beta} \frac{\partial^2\phi}{\partial z_\alpha \partial \bar{z}_\beta} dz_\alpha \wedge d\bar{z}_\beta + \sum_{\alpha,j} \frac{\partial^2\phi}{\partial z_\alpha \partial \bar{w}_j} dz_\alpha \wedge d\bar{w}_j \\ + \sum_{i,\beta} \frac{\partial^2\phi}{\partial w_i \partial \bar{z}_\beta} dw_i \wedge d\bar{z}_\beta + \sum_{i,j} \frac{\partial^2\phi}{\partial w_i \partial \bar{w}_j} dw_i \wedge d\bar{w}_j. \end{aligned}$$

Note that the terms $\frac{\partial^2\phi}{\partial z_\alpha \partial \bar{w}_j} := \phi_{\alpha\bar{j}}$ vanish at $(z_0, [\zeta_0])$ by (2-2) and the fact

$$e^\phi = \frac{1}{g(e^*, e^*)} = G(w_1 s_1 + \cdots + w_{r-1} s_{r-1} + s_r).$$

For a tangent vector $\eta \in T^{1,0}_{z_0}X$, we can write $\eta = \sum_\alpha \eta_\alpha \frac{\partial}{\partial z_\alpha}$. For the lifts $\tilde{\eta}$ of η to $T^{1,0}_{(z_0,[\zeta_0])}P(E^*)$, we have

$$\inf_{q_*(\tilde{\eta})=\eta} \Theta(g)(\tilde{\eta}, \bar{\tilde{\eta}}) = \sum_{\alpha,\beta} \phi_{\alpha\bar{\beta}}|_{(z_0,[\zeta_0])}\, \eta_\alpha \bar{\eta}_\beta \tag{2-3}$$

because $\phi_{\alpha\bar{j}} = 0$ at $(z_0, [\zeta_0])$ and the matrix $(\phi_{i\bar{j}})$ is positive. On the other hand, using the same coordinate system, the curvature Θ of $\tilde{G}$ can be written as

$$\Theta = \sum_{\alpha,\beta} R_{\alpha\bar{\beta}}\, dz_\alpha \wedge d\bar{z}_\beta + \sum_{\alpha,l} P_{\alpha\bar{l}}\, dz_\alpha \wedge d\bar{w}_l + \sum_{k,\beta} \mathcal{P}_{k\bar{\beta}}\, dw_k \wedge d\bar{z}_\beta + \sum_{k,l} Q_{k\bar{l}}\, dw_k \wedge d\bar{w}_l,$$

where $R_{\alpha\bar{\beta}}$, $P_{\alpha\bar{l}}$, $\mathcal{P}_{k\bar{\beta}}$, and $Q_{k\bar{l}}$ are endomorphisms of q^*E^*. By [Wu 2022, (2.4)], for any lift $\tilde{\eta}$ of η to $T^{1,0}_{(z_0,[\zeta_0])}P(E^*)$, we have

$$\begin{aligned} &\theta(g)(\tilde{\eta}, \bar{\tilde{\eta}}) \\ &\quad = \Theta|_{O_{P(E^*)}(-1)}(\tilde{\eta}, \bar{\tilde{\eta}}) = \sum_{\alpha,\beta} \frac{\tilde{G}(R_{\alpha\bar{\beta}}\, \zeta_0, \zeta_0)}{\tilde{G}(\zeta_0, \zeta_0)} \eta_\alpha \bar{\eta}_\beta = -\sum_{\alpha,\beta} \phi_{\alpha\bar{\beta}}|_{(z_0,[\zeta_0])}\, \eta_\alpha \bar{\eta}_\beta, \end{aligned} \tag{2-4}$$

where the last equality is by [Kobayashi 1996, (5.16)].

From (2-3) and (2-4), we see

$$\inf_{q_*(\tilde{\eta})=\eta} \Theta(g)(\tilde{\eta}, \bar{\tilde{\eta}}) = -\theta(g)(\tilde{\eta}, \bar{\tilde{\eta}}),$$

which is formula (1-2) we claim in the introduction, and when evaluated using normal coordinates they are $\sum_{\alpha,\beta} \phi_{\alpha\bar{\beta}}|_{(z_0,[\zeta_0])}\, \eta_\alpha \bar{\eta}_\beta$.

2B. *Hermitian metrics.* This subsection is a special case of Section 2A and it will be used in the proofs of Theorems 3 and 4. Let G be a Hermitian metric on the bundle E^*. The pull-back bundle $q^*E^* \to P(E^*)$ with the pull-back metric q^*G induces a metric g^* on the subbundle $O_{P(E^*)}(-1)$. We denote the dual metric on $O_{P(E^*)}(1)$ by g.

Let Ω be a Hermitian metric on X and z_0 a point in X with local coordinates $\{z_\alpha\}$ such that

$$\Omega\left(\frac{\partial}{\partial z_\alpha}, \frac{\partial}{\partial \bar{z}_\beta}\right)(z_0) = \delta_{\alpha\beta}.$$

There exists a holomorphic frame $\{s_i\}$ for E^* around z_0 such that

$$G(s_i, s_j) = \delta_{ij} + O(|z|^2),$$

where z_0 corresponds to the origin in the local coordinates. We use $\{\zeta_i\}$ for the fiber coordinates with respect to the frame $\{s_i\}$. For a point $(z_0, [\zeta_0]) \in P(E^*)$, we assume the local coordinates $(z_1, \ldots, z_n, w_1, \ldots, w_{r-1})$ around $(z_0, [\zeta_0])$ are given by $w_i = \zeta_i/\zeta_r$ for $i = 1 \sim r-1$. So

$$e := \frac{\zeta_1 s_1 + \cdots + \zeta_r s_r}{\zeta_r} = w_1 s_1 + \cdots + w_{r-1} s_{r-1} + s_r$$

is a holomorphic frame for $O_{P(E^*)}(-1)$ and

$$g^*(e,e) = q^*G(w_1 s_1 + \cdots + w_{r-1}s_{r-1} + s_r,\, w_1 s_1 + \cdots + w_{r-1}s_{r-1} + s_r)$$
$$= 1 + O(|z|^2) + O(|w|^2) + O(|w|\,|z|^2) + O(|w|^2|z|^2).$$

The z_α-derivative of $g^*(e,e)$ is $g^*(e,e)_{z_\alpha} = O((1+|w|+|w|^2)|z|)$, and hence the w_i-derivatives of $g^*(e,e)_{z_\alpha}$ of any order are zero when evaluated at z_0. Therefore, if we denote $g^*(e,e)$ by e^ϕ, then at z_0

$$\phi_{\alpha\bar{j}} = \phi_{\alpha i\bar{j}} = \phi_{\alpha i\bar{j}\bar{k}} = 0 \quad \text{and} \quad (\log\det(\phi_{i\bar{j}}))_{\alpha\bar{k}} = 0. \tag{2-5}$$

In this coordinate system, the curvature $\Theta(g)$ is

$$\sum_{\alpha,\beta} \frac{\partial^2\phi}{\partial z_\alpha\,\partial\bar{z}_\beta} dz_\alpha \wedge d\bar{z}_\beta + \sum_{\alpha,j} \frac{\partial^2\phi}{\partial z_\alpha\,\partial\bar{w}_j} dz_\alpha \wedge d\bar{w}_j + \sum_{i,\beta} \frac{\partial^2\phi}{\partial w_i\,\partial\bar{z}_\beta} dw_i \wedge d\bar{z}_\beta + \sum_{i,j} \frac{\partial^2\phi}{\partial w_i\,\partial\bar{w}_j} dw_i \wedge d\bar{w}_j.$$

For a tangent vector $\eta \in T^{1,0}_{z_0}X$, we can write $\eta = \sum_\alpha \eta_\alpha \frac{\partial}{\partial z_\alpha}$. For the lifts $\tilde{\eta}$ of η to $T^{1,0}_{(z_0,[\zeta_0])}P(E^*)$, we have

$$\inf_{q_*(\tilde{\eta})=\eta} \Theta(g)(\tilde{\eta},\bar{\tilde{\eta}}) = \sum_{\alpha,\beta} \phi_{\alpha\bar{\beta}}|_{(z_0,[\zeta_0])}\,\eta_\alpha\,\bar{\eta}_\beta \tag{2-6}$$

because $\phi_{\alpha\bar{j}} = 0$ at z_0 and the matrix $(\phi_{i\bar{j}})$ is positive. Since G is a Hermitian metric, the corresponding Kobayashi curvature is

$$\theta(g) = q^*\Theta(G)|_{O_{P(E^*)}(-1)}, \tag{2-7}$$

which is equal to the negative of (2-6) by Section 2A.

Using the same coordinate system, the restriction $\Theta(g)|_{P(E^*_z)}$ is $\sum \phi_{i\bar{j}}\,dw_i \wedge d\bar{w}_j$, so the metric on $K_{P(E^*)/X}$ induced from $\{(\Theta(g)|_{P(E^*_z)})^{r-1}\}_{z\in X}$ has its curvature γ_g equal to

$$\sum_{\alpha,\beta} (\log\det(\phi_{i\bar{j}}))_{\alpha\bar{\beta}}\,dz_\alpha \wedge d\bar{z}_\beta + \sum_{\alpha,j} (\log\det(\phi_{i\bar{j}}))_{\alpha\bar{j}}\,dz_\alpha \wedge d\bar{w}_j + \sum_{i,\beta} (\log\det(\phi_{i\bar{j}}))_{i\bar{\beta}}\,dw_i \wedge d\bar{z}_\beta + \sum_{i,j} (\log\det(\phi_{i\bar{j}}))_{i\bar{j}}\,dw_i \wedge d\bar{w}_j. \tag{2-8}$$

The matrix $\big((\log\det(\phi_{i\bar{j}}))_{i\bar{j}}\big)$ is negative because it represents the negative of the Ricci curvature of the Fubini–Study metric on $P(E^*_z)$. Moreover, the terms $(\log\det(\phi_{i\bar{j}}))_{\alpha\bar{j}} = 0$ at z_0 by (2-5). As a result, for a tangent vector $\eta \in T^{1,0}_{z_0}X$ with $\eta = \sum \eta_\alpha \frac{\partial}{\partial z_\alpha}$ in this coordinate system, we have

$$\sup_{q_*(\tilde{\eta})=\eta} \gamma_g(\tilde{\eta},\bar{\tilde{\eta}}) = \sum_{\alpha,\beta} (\log\det(\phi_{i\bar{j}}))_{\alpha\bar{\beta}}|_{(z_0,[\zeta_0])}\,\eta_\alpha\,\bar{\eta}_\beta, \tag{2-9}$$

where $\tilde{\eta}$ are the lifts of η to $T^{1,0}_{(z_0,[\zeta_0])}P(E^*)$.

Finally, the metric on $K_{P(E^*)/X}$ induced from $\{(\Theta(g)|_{P(E_z^*)})^{r-1}\}_{z\in X}$ can be identified with the metric $(g^*)^r \otimes q^*(\det G^*)$ under the isomorphism

$$K_{P(E^*)/X} \simeq O_{P(E^*)}(-r) \otimes q^* \det E$$

(see [Kobayashi 1987, p. 85, Proposition 3.6.20]). This fact can be verified at one point using the normal coordinates above. Therefore,

$$\gamma_g = -r\Theta(g) - q^*\Theta(\det G). \tag{2-10}$$

So, for any $\eta \in T_z^{1,0}X$ and any $[\zeta] \in P(E_z^*)$,

$$\begin{aligned}\sup_{q_*(\tilde{\eta})=\eta} \gamma_g(\tilde{\eta}, \bar{\tilde{\eta}}) &= -r \inf_{q_*(\tilde{\eta})=\eta} \Theta(g)(\tilde{\eta}, \bar{\tilde{\eta}}) - \Theta(\det G)(\eta, \bar{\eta}) \\ &= r\theta(g)(\tilde{\eta}, \bar{\tilde{\eta}}) - \Theta(\det G)(\eta, \bar{\eta}).\end{aligned}$$

This is formula (1-8) that we promise to prove in the introduction.

2C. ***Convexity.*** Let E be a holomorphic vector bundle of rank r over a compact complex manifold X. Given a Hermitian metric H_k on the symmetric power S^kE, we can define a Finsler metric on E by assigning to $u \in E$ length $H_k(u^k, u^k)^{1/2k}$. We will denote this Finsler metric by $H_k^{1/2k}$, namely $H_k^{1/2k}(u) = H_k(u^k, u^k)^{1/2k}$.

Lemma 5. *Let F_1 be a vector bundle and F_2 a line bundle over X. Assume F_2 carries a Hermitian metric H. We also assume, for some k, S^kF_1 carries a Hermitian metric H_k such that the induced Finsler metric $H_k^{1/2k}$ on F_1 is convex*:

$$H_k^{1/2k}(u+v) \le H_k^{1/2k}(u) + H_k^{1/2k}(v) \quad \textit{for } u, v \in F_1.$$

Then the Finsler metric $(H_k \otimes H^k)^{1/2k}$ on $F_1 \otimes F_2$ is convex.

Since F_2 is a line bundle, there is a canonical isomorphism between the bundles $S^k(F_1 \otimes F_2)$ and $S^kF_1 \otimes F_2^k$ which we use implicitly in the statement of Lemma 5. Roughly speaking, Lemma 5 indicates that convexity is not affected by tensoring with a line bundle.

Proof. Fix $p \in X$. The fiber $F_2|_p$ is a one dimensional vector space and we let e be a basis. For x and $y \in F_1 \otimes F_2|_p$, we can write $x = \tilde{x} \otimes e$ and $y = \tilde{y} \otimes e$ where $\tilde{x}, \tilde{y} \in F_1|_p$. By definition,

$$\begin{aligned}(H_k \otimes H^k)^{1/2k}(x+y) &= H_k \otimes H^k((x+y)^k, (x+y)^k)^{1/2k} \\ &= H_k \otimes H^k((\tilde{x}+\tilde{y})^k \otimes e^k, (\tilde{x}+\tilde{y})^k \otimes e^k)^{1/2k} \\ &= H_k((\tilde{x}+\tilde{y})^k, (\tilde{x}+\tilde{y})^k)^{1/2k} H^k(e^k, e^k)^{1/2k} \\ &\le [H_k(\tilde{x}^k, \tilde{x}^k)^{1/2k} + H_k(\tilde{y}^k, \tilde{y}^k)^{1/2k}]\, H^k(e^k, e^k)^{1/2k} \\ &= (H_k \otimes H^k)^{1/2k}(x) + (H_k \otimes H^k)^{1/2k}(y).\end{aligned}$$

Therefore the Finsler metric $(H_k \otimes H^k)^{1/2k}$ is convex. □

2D. ***Direct image bundles.*** We recall how to construct Hermitian metrics on direct image bundles and compute their curvature. Let g be a Hermitian metric on $O_{P(E^*)}(1)$ with curvature $\Theta(g)$. Denote the restriction of the curvature to a fiber, $\Theta(g)|_{P(E_z^*)}$ by ω_z for $z \in X$, and assume $\omega_z > 0$ for all $z \in X$. With the canonical isomorphism

$$\Phi_{k,z} : S^k E_z \to H^0(P(E_z^*), O_{P(E_z^*)}(k)) \quad \text{for } k \geq 0$$

(see [Demailly 2012, p. 278, Theorem 15.5]), we define a Hermitian metric H_k on $S^k E$ by

$$H_k(u, v) := \int_{P(E_z^*)} g^k(\Phi_{k,z}(u), \Phi_{k,z}(v))\, \omega_z^{r-1} \quad \text{for } u \text{ and } v \in S^k E_z. \tag{2-11}$$

Let us denote by Θ_k the curvature of H_k. Fixing $z \in X$ and $u \in S^k E_z$, in order to estimate the $(1,1)$-form $H_k(\Theta_k u, u)$, we first extend the vector u to a local holomorphic section $\tilde{u}$ whose covariant derivative at z with respect to H_k equals zero. A straightforward computation shows

$$\partial\bar{\partial} H_k(\tilde{u}, \tilde{u})|_z = -H_k(\Theta_k u, u).$$

But $H_k(\tilde{u}, \tilde{u})(z)$ for z near z can also be written as the push-forward

$$q_*\big(g^k(\Phi_{k,z}(\tilde{u}), \Phi_{k,z}(\tilde{u}))\, \Theta(g)^{r-1}\big),$$

where $q : P(E^*) \to X$ is the projection, so

$$-H_k(\Theta_k u, u) = \partial\bar{\partial} H_k(\tilde{u}, \tilde{u})|_z = q_*\, \partial\bar{\partial}\big(g^k(\Phi_{k,z}(\tilde{u}), \Phi_{k,z}(\tilde{u}))\, \Theta(g)^{r-1}\big)|_z. \tag{2-12}$$

Similarly, we can use a metric on $O_{P(E)}(1) \otimes p^* \det E$ to construct Hermitian metrics on $S^k E^* \otimes (\det E)^k$. The formula is similar to (2-11), and we use bold symbols to highlight the change. Let $\boldsymbol{g}$ be a metric on $O_{P(E)}(1) \otimes p^* \det E$ with curvature $\Theta(\boldsymbol{g})$. Denote the restriction of the curvature to a fiber $\Theta(\boldsymbol{g})|_{P(E_z)}$ by $\boldsymbol{\omega}_z$ for $z \in X$. Assume $\boldsymbol{\omega}_z > 0$ for all $z \in X$. With the canonical isomorphism

$$\boldsymbol{\Phi}_{k,z} : S^k E_z^* \otimes (\det E_z)^k \to H^0\big(P(E_z), O_{P(E_z)}(k) \otimes (p^* \det E_z)^k\big) \quad \text{for } k \geq 0,$$

we define a Hermitian metric $\boldsymbol{H}_k$ on $S^k E^* \otimes (\det E)^k$ by

$$\boldsymbol{H}_k(u, v) := \int_{P(E_z)} \boldsymbol{g}^k(\boldsymbol{\Phi}_{k,z}(u), \boldsymbol{\Phi}_{k,z}(v))\, \boldsymbol{\omega}_z^{r-1} \tag{2-13}$$

for u and $v \in S^k E_z^* \otimes (\det E_z)^k$. We also have a curvature formula similar to (2-12).

2E. *Berndtsson's positivity theorem.* Let h be a metric on $O_{P(E^*)}(1)$ with curvature $\Theta(h) > 0$. Denote $\Theta(h)|_{P(E_z^*)}$ by ω_z for $z \in X$. We are going to define a Hermitian metric on $\det E$ using the metric h. The relative canonical bundle $K_{P(E^*)/X}$ has a metric induced from $\{\omega_z^{r-1}\}_{z\in X}$. With h^r on $O_{P(E^*)}(r)$ and the isomorphism $K_{P(E^*)/X} \otimes O_{P(E^*)}(r) \simeq q^* \det E$, there is an induced metric ρ on $q^* \det E$. Using the canonical isomorphism

$$\Psi_z : \det E_z \to H^0(P(E_z^*), q^* \det E_z),$$

we define a Hermitian metric H on $\det E$ by

$$H(u, v) := \int_{P(E_z^*)} \rho(\Psi_z(u), \Psi_z(v))\, \omega_z^{r-1} \quad \text{for } u \text{ and } v \in \det E_z. \tag{2-14}$$

By Berndtsson's theorem [Berndtsson 2009a], this metric H is Griffiths positive, but it is the inequality that leads to this fact we will use. We follow the presentation in [Liu et al. 2013, Section 4.1] (see also [Berndtsson 2009b, Section 2]). Denote the curvature of H by Θ. Fix $z \in X$, $v \in \det E_z$ and $\eta \in T_z^{1,0} X$. For a local holomorphic frame of E^* around z, we denote by $\{\zeta_i\}$ the fiber coordinates with respect to this frame, and by $\{z_\alpha\}$ the local coordinates on X. Around $P(E_z^*)$ in $P(E^*)$, we have homogeneous coordinates $[\zeta_1, \dots, \zeta_r]$ which induce local coordinates $(w_1, \dots, w_{r-1})$. For a local frame e^* of $O_{P(E^*)}(1)$, we denote $h(e^*, e^*)$ by $e^{-\phi}$ and write the tangent vector $\eta = \sum \eta_\alpha \frac{\partial}{\partial z_\alpha}$. The inequality that leads to Berndtsson's theorem is

$$\begin{aligned} &-H(\Theta v, v)(\eta, \bar\eta) \\ &\quad \le \int_{P(E_z^*)} \rho(\Psi_z(v), \Psi_z(v))\, r \sum_{\alpha,\beta} \Bigl(\sum_{i,j} \phi_{\alpha\bar j}\, \phi^{i\bar j}\, \phi_{i\bar\beta} - \phi_{\alpha\bar\beta} \Bigr) \eta_\alpha\, \bar\eta_\beta\, \omega_z^{r-1}, \end{aligned} \tag{2-15}$$

where

$$\phi_{i\bar j} := \frac{\partial^2 \phi}{\partial w_i\, \partial \overline{w}_j}, \quad \phi_{\alpha\bar j} := \frac{\partial^2 \phi}{\partial z_\alpha\, \partial \overline{w}_j}, \quad \phi_{\alpha\bar\beta} := \frac{\partial^2 \phi}{\partial z_\alpha\, \partial \bar z_\beta},$$

and $(\phi^{i\bar j})$ is the inverse matrix of $(\phi_{i\bar j})$. Since $\det E$ is a line bundle, the curvature Θ is a $(1, 1)$-form, and so $H(\Theta v, v)(\eta, \bar\eta) = H(v, v)\, \Theta(\eta, \bar\eta)$. If we further assume $H(v, v) = 1$, then the left-hand side of (2-15) becomes $-\Theta(\eta, \bar\eta)$.

3. Proof of Theorem 1

Recall that h and g are metrics on $O_{P(E^*)}(1)$ that satisfy the assumptions in Theorem 1 and the inequalities (1-3) and (1-4). We use the metric h to construct a Hermitian metric H on $\det E$ as in (2-14), and the metric g to construct Hermitian metrics H_k on $S^k E$ as in (2-11). The number k is yet to be determined.

We start with the metric g. Given a point $(z_0, [\zeta_0]) \in P(E^*)$, we have the normal coordinate system from Section 2A. In this coordinate system, let us introduce the following n-by-n matrix-valued function:

$$B_k = ((B_k)_{\alpha\beta}) := \big(k\phi_{\alpha\bar\beta} - (\log\det(\phi_{i\bar j}))_{\alpha\bar\beta}\big),$$

where $g(e^*, e^*) = e^{-\phi}$. By continuity, there is a neighborhood U of $(z_0, [\zeta_0])$ in $P(E^*)$ such that in U

$$(\phi_{\alpha\bar\beta})|_{(z_0,[\zeta_0])} + \frac{r-M}{4}\,\mathrm{Id}_{n\times n} \geq (\phi_{\alpha\bar\beta}). \tag{3-1}$$

For this U, there is a positive integer k_0 such that for $k \geq k_0$ and in U

$$(\phi_{\alpha\bar\beta}) + \frac{r-M}{4}\,\mathrm{Id}_{n\times n} \geq \frac{B_k}{k}. \tag{3-2}$$

Let us summarize what we have done so far:

Lemma 6. *Given a point $(z_0, [\zeta_0]) \in P(E^*)$, there exist a coordinate neighborhood U of $(z_0, [\zeta_0])$ in $P(E^*)$ and a positive integer k_0 such that in U and for $k \geq k_0$*

$$(\phi_{\alpha\bar\beta})|_{(z_0,[\zeta_0])} + \frac{r-M}{2}\,\mathrm{Id}_{n\times n} \geq \frac{B_k}{k}. \tag{3-3}$$

By Lemma 6, since $P(E^*_{z_0})$ is compact, we can find on $P(E^*_{z_0})$ finitely many points $\{(z_0, [\zeta_l])\}_l$ each of which corresponds to a coordinate neighborhood U_l in $P(E^*)$ and a positive integer k_l such that the corresponding (3-3) holds, and $P(E^*_{z_0}) \subset \bigcup_l U_l$. Denote $\max_l k_l$ by $k_{\max}$. The point z_0 has a neighborhood W in X such that for $z \in W$, the fiber $P(E^*_z)$ can be partitioned as $\bigcup_m V_m$ with each V_m in U_l for some l. By shrinking W, we can assume that for each U_l the corresponding $\Omega\big(\frac{\partial}{\partial z_\alpha}, \frac{\partial}{\partial \bar z_\beta}\big) := \Omega_{\alpha\bar\beta}$ satisfies

$$-\varepsilon\delta_{\alpha\beta} < \Omega_{\alpha\bar\beta}(z) - \delta_{\alpha\beta} < \varepsilon\delta_{\alpha\beta} \quad \text{for } z \in W, \tag{3-4}$$

where $\varepsilon := \frac{r-M}{5(r+M)}$.

Recall the Hermitian metrics H_k on $S^k E$ in (2-11) constructed using the metric g. Denote by Θ_k the curvature of H_k. We claim the following lemma (one can also use the asymptotic expansion in [Ma and Zhang 2023] to deduce the lemma).

Lemma 7. *For $k \geq k_{\max}$, $z \in W$, $0 \neq \eta \in T^{1,0}_z X$, and $u \in S^k E_z$ with $H_k(u, u) = 1$, we have*

$$H_k(\Theta_k u, u)(\eta, \bar\eta) \leq \Big(M + \frac{r-M}{2}\Big)\, k\,\frac{\Omega(\eta, \bar\eta)}{(1-\varepsilon)}. \tag{3-5}$$

Proof. As in Section 2D, we extend the vector $u \in S^k E_z$ to a local holomorphic section $\tilde{u}$ whose covariant derivative at z equals zero, and we have

$$-H_k(\Theta_k u, u) = \partial\bar\partial H_k(\tilde u, \tilde u)|_z = \int_{P(E_z^*)} \partial\bar\partial\big(g^k(\Phi_{k,z}(\tilde u), \Phi_{k,z}(\tilde u))\,\Theta(g)^{r-1}\big)$$
$$= \sum_m \int_{V_m} \partial\bar\partial\big(g^k(\Phi_{k,z}(\tilde u), \Phi_{k,z}(\tilde u))\,\Theta(g)^{r-1}\big).$$

In the last equality, we partition the fiber $P(E_z^*)$ as $\bigcup_m V_m$ with each V_m in U_l for some l. In a fixed $V_m \subset U_l$, using the coordinate system of U_l, we can write $\Phi_{k,z}(\tilde u)$ as $f(e^*)^k$ with f a scalar-valued holomorphic function and e^* a local frame for $O_{P(E^*)}(1)$. So, $g^k(\Phi_{k,z}(\tilde u), \Phi_{k,z}(\tilde u)) = |f|^2 e^{-k\phi}$. Meanwhile, recall the curvature $\Theta(g) = \partial\bar\partial\phi$. By Stokes' theorem and a count on degrees, we have

$$\sum_m \int_{V_m} \partial\bar\partial\big(g^k(\Phi_{k,z}(\tilde u), \Phi_{k,z}(\tilde u))\,\Theta(g)^{r-1}\big)$$
$$= \sum_m \int_{V_m} \sum_{\alpha,\beta} \frac{\partial^2 |f|^2 e^{-k\phi} \det(\phi_{i\bar j})}{\partial z_\alpha \partial \bar z_\beta} dz_\alpha \wedge d\bar z_\beta \bigwedge_j dw_j \wedge d\bar w_j.$$

So, if the tangent vector $\eta = \sum_\alpha \eta_\alpha \frac{\partial}{\partial z_\alpha}$ in the coordinate neighborhood U_l, then

$$\text{(3-6)}\quad -H_k(\Theta_k u, u)(\eta, \bar\eta) = \sum_m \int_{V_m} \sum_{\alpha,\beta} \frac{\partial^2 |f|^2 e^{-k\phi} \det(\phi_{i\bar j})}{\partial z_\alpha \partial \bar z_\beta} \eta_\alpha \bar\eta_\beta \bigwedge_j dw_j \wedge d\bar w_j.$$

Note that the integrands in (3-6) are written in the local coordinates of corresponding U_l. A direct computation shows

$$\sum_{\alpha,\beta} \frac{\partial^2 |f|^2 e^{-k\phi} \det(\phi_{i\bar j})}{\partial z_\alpha \partial \bar z_\beta} \eta_\alpha \bar\eta_\beta$$
$$= e^{-k\phi}\det(\phi_{i\bar j}) \Big| \sum_\alpha \frac{\partial f}{\partial z_\alpha}\eta_\alpha - f \sum_\alpha \big(k\phi_\alpha - (\log\det\phi_{i\bar j})_\alpha\big)\eta_\alpha \Big|^2$$
$$- |f|^2 e^{-k\phi}\det(\phi_{i\bar j}) \sum_{\alpha,\beta} \big(k\phi_{\alpha\bar\beta} - (\log\det\phi_{i\bar j})_{\alpha\bar\beta}\big)\eta_\alpha\bar\eta_\beta$$
$$\geq -|f|^2 e^{-k\phi}\det(\phi_{i\bar j}) \sum_{\alpha,\beta} (B_k)_{\alpha\beta}\eta_\alpha\bar\eta_\beta.$$

By (3-3),

$$\text{(3-7)}\qquad \frac{1}{k}\sum_{\alpha,\beta}(B_k)_{\alpha\beta}\eta_\alpha\bar\eta_\beta \leq \sum_{\alpha,\beta}\phi_{\alpha\bar\beta}|_{(z_0,[\zeta_l])}\eta_\alpha\bar\eta_\beta + \frac{r-M}{2}\sum_\alpha |\eta_\alpha|^2.$$

Using the coordinate system of U_l, the tangent vector $\eta = \sum_\alpha \eta_\alpha \frac{\partial}{\partial z_\alpha}$ at z induces a tangent vector $\eta_l = \sum_\alpha \eta_\alpha \frac{\partial}{\partial z_\alpha}|_{z_0}$ at z_0. Denote the lifts of η_l to $T^{1,0}_{(z_0,[\zeta_l])}P(E^*)$

by $\tilde{\eta}_l$. According to (1-2), (1-3), and (2-3), we see

$$M\sum_{\alpha}|\eta_\alpha|^2 \ge -\theta(g)(\tilde{\eta}_l,\bar{\tilde{\eta}}_l) = \inf_{q_*(\tilde{\eta}_l)=\eta_l}\Theta(g)(\tilde{\eta}_l,\bar{\tilde{\eta}}_l) = \sum_{\alpha,\beta}\phi_{\alpha\bar{\beta}}|_{(z_0,[\zeta_l])}\,\eta_\alpha\,\bar{\eta}_\beta. \tag{3-8}$$

Therefore, (3-7) becomes

$$\frac{1}{k}\sum_{\alpha,\beta}(B_k)_{\alpha\beta}\,\eta_\alpha\,\bar{\eta}_\beta \le \left(M+\frac{r-M}{2}\right)\sum_{\alpha}|\eta_\alpha|^2 \le \left(M+\frac{r-M}{2}\right)\frac{\Omega(\eta,\bar{\eta})}{(1-\varepsilon)}, \tag{3-9}$$

where we use (3-4) in the second inequality. So, (3-6) becomes

$$\begin{aligned} &-H_k(\Theta_k u,u)(\eta,\bar{\eta}) \\ &\quad\ge \sum_m\int_{V_m} -|f|^2 e^{-k\phi}\det(\phi_{i\bar{j}})\bigwedge_j dw_j\wedge d\bar{w}_j\left(M+\frac{r-M}{2}\right)k\frac{\Omega(\eta,\bar{\eta})}{(1-\varepsilon)} \\ &\quad= -\left(M+\frac{r-M}{2}\right)k\frac{\Omega(\eta,\bar{\eta})}{(1-\varepsilon)} \end{aligned} \tag{3-10}$$

since $H_k(u,u)=1$. □

We turn now to the metric h. The argument about h is similar to that about g, and it will be used in Theorems 2, 3, and 4. Given a point $(z_0,[\zeta_0])\in P(E^*)$, we have the normal coordinate system from Section 2A with respect to the metric h. In this coordinate system, let us introduce the n-by-n matrix-valued function

$$A=(A_{\alpha\beta}):=\left(\phi_{\alpha\bar{\beta}}-\sum_{i,j}\phi_{\alpha\bar{j}}\,\phi^{i\bar{j}}\,\phi_{i\bar{\beta}}\right),$$

where $h(e^*,e^*)=e^{-\phi}$ and $(\phi^{i\bar{j}})$ is the inverse matrix of $(\phi_{i\bar{j}})$. By continuity, there is a neighborhood U of $(z_0,[\zeta_0])$ in $P(E^*)$ such that in U

$$rA+\frac{r-M}{4}\,\mathrm{Id}_{n\times n}\ge rA|_{(z_0,[\zeta_0])}. \tag{3-11}$$

In summary:

Lemma 8. *Given a point* $(z_0,[\zeta_0])\in P(E^*)$, *there exists a coordinate neighborhood* U *of* $(z_0,[\zeta_0])$ *in* $P(E^*)$ *such that in* U

$$rA+\frac{r-M}{4}\,\mathrm{Id}_{n\times n}\ge r(\phi_{\alpha\bar{\beta}})|_{(z_0,[\zeta_0])}. \tag{3-12}$$

By Lemma 8, since $P(E^*_{z_0})$ is compact, we can find on $P(E^*_{z_0})$ finitely many points $\{(z_0,[\zeta_l])\}_l$ each of which corresponds to a coordinate neighborhood U_l in $P(E^*)$ such that the corresponding (3-12) holds, and $P(E^*_{z_0})\subset\bigcup_l U_l$. The point z_0 has a neighborhood W' in X such that for $z\in W'$, the fiber $P(E^*_z)$ can be partitioned as $\bigcup_m V_m$ with each V_m in U_l for some l. By shrinking W', we can assume that for each U_l the corresponding $\Omega\big(\frac{\partial}{\partial z_\alpha}, \frac{\partial}{\partial \bar{z}_\beta}\big) := \Omega_{\alpha\bar{\beta}}$ satisfies

$$-\varepsilon\delta_{\alpha\beta} < \Omega_{\alpha\bar{\beta}}(z) - \delta_{\alpha\beta} < \varepsilon\delta_{\alpha\beta} \quad \text{for } z \in W', \tag{3-13}$$

where $\varepsilon := \frac{r-M}{5(r+M)}$.

Recall the Hermitian metric H on $\det E$ in (2-14) constructed using the metric h. Denote by Θ the curvature of H. We claim:

Lemma 9. *For $z \in W'$ and $\eta \in T_z^{1,0}X$, we have*

$$-\Theta(\eta, \bar{\eta}) \le -\left(r - \frac{r-M}{4}\right)\frac{\Omega(\eta, \bar{\eta})}{(1+\varepsilon)}. \tag{3-14}$$

Proof. Using (2-15) and assuming $H(v, v) = 1$, we get

$$\begin{aligned} -\Theta(\eta, \bar{\eta}) & \\ \le \sum_m \int_{V_m} & \rho(\Psi_z(v), \Psi_z(v))\, r \sum_{\alpha,\beta}\Big(\sum_{i,j} \phi_{\alpha\bar{j}}\, \phi^{i\bar{j}}\, \phi_{i\bar{\beta}} - \phi_{\alpha\bar{\beta}}\Big)\eta_\alpha\, \bar{\eta}_\beta\, \omega_z^{r-1}, \end{aligned} \tag{3-15}$$

where we again partition $P(E_z^*)$ as $\bigcup_m V_m$ with each V_m in U_l for some l. Note that the integrands in (3-15) are written in the local coordinates of corresponding U_l. In a fixed $V_m \subset U_l$, we have $\eta = \sum_\alpha \eta_\alpha \frac{\partial}{\partial z_\alpha}$, and by (3-12) we see

$$r \sum_{\alpha,\beta} A_{\alpha\beta}\, \eta_\alpha\, \bar{\eta}_\beta + \frac{r-M}{4} \sum_\alpha |\eta_\alpha|^2 \ge r \sum_{\alpha,\beta} \phi_{\alpha\bar{\beta}}|_{(z_0,[\zeta_l])}\, \eta_\alpha\, \bar{\eta}_\beta. \tag{3-16}$$

In U_l, the tangent vector

$$\eta = \sum_\alpha \eta_\alpha \frac{\partial}{\partial z_\alpha} \quad \text{at } z$$

induces a tangent vector

$$\eta_l = \sum_\alpha \eta_\alpha \frac{\partial}{\partial z_\alpha}\Big|_{z_0} \quad \text{at } z_0.$$

Denote the lifts of η_l to $T^{1,0}_{(z_0,[\zeta_l])}P(E^*)$ by $\tilde{\eta}_l$. By (1-2), (1-4), and (2-3), we see

$$\sum_\alpha |\eta_\alpha|^2 \le -\theta(h)(\tilde{\eta}_l, \bar{\tilde{\eta}}_l) = \inf_{q_*(\tilde{\eta}_l)=\eta_l} \Theta(h)(\tilde{\eta}_l, \bar{\tilde{\eta}}_l) = \sum_{\alpha,\beta} \phi_{\alpha\bar{\beta}}|_{(z_0,[\zeta_l])}\, \eta_\alpha\, \bar{\eta}_\beta. \tag{3-17}$$

Therefore, (3-16) becomes

$$r \sum_{\alpha,\beta} A_{\alpha\beta}\, \eta_\alpha\, \bar{\eta}_\beta \ge \left(r - \frac{r-M}{4}\right) \sum_\alpha |\eta_\alpha|^2 \ge \left(r - \frac{r-M}{4}\right) \frac{\Omega(\eta, \bar{\eta})}{1+\varepsilon},$$

where we use (3-13) in the second inequality. So, (3-15) becomes

$$\begin{aligned}-\Theta(\eta,\bar\eta) &\le -\sum_m \int_{V_m} \rho(\Psi_z(v),\Psi_z(v))\,\omega_z^{r-1}\left(r-\frac{r-M}{4}\right)\frac{\Omega(\eta,\bar\eta)}{1+\varepsilon}\\ &= -\left(r-\frac{r-M}{4}\right)\frac{\Omega(\eta,\bar\eta)}{(1+\varepsilon)}\end{aligned}$$

because $H(v,v)=1$. □

Now we put together the L^2-metrics H_k on S^kE in (2-11), and H on $\det E$ in (2-14). Since $(\det E^*)^k$ is a line bundle, we can identify $\mathrm{End}(S^kE\otimes(\det E^*)^k)$ with $\mathrm{End}(S^kE)$, and the curvature of the metric $H_k\otimes(H^*)^k$ on $S^kE\otimes(\det E^*)^k$ can be written as

$$\Theta_k - k\Theta\otimes \mathrm{Id}_{S^kE},$$

where Θ_k and Θ are the curvature of H_k and H respectively. We claim that for $k\ge k_{\max}$ and in $W\cap W'$ a neighborhood of z_0, the metric $H_k\otimes(H^*)^k$ is Griffiths negative. Indeed, as a result of Lemmas 7 and 9, for $k\ge k_{\max}$, $z\in W\cap W'$, $0\ne\eta\in T_z^{1,0}X$, and $u\in S^kE_z$ with $H_k(u,u)=1$, we see

$$H_k(\Theta_k u,u)(\eta,\bar\eta)-k\Theta(\eta,\bar\eta)\le k\left(M+\frac{r-M}{2}\right)\frac{\Omega(\eta,\bar\eta)}{(1-\varepsilon)}-k\left(r-\frac{r-M}{4}\right)\frac{\Omega(\eta,\bar\eta)}{(1+\varepsilon)}.$$

The term on the right is negative after some computation using $\varepsilon=\frac{r-M}{5(r+M)}$. So, we have proved the claim that for $k\ge k_{\max}$ and in $W\cap W'\subset X$, the metric $H_k\otimes(H^*)^k$ is Griffiths negative. Since X is compact, $H_k\otimes(H^*)^k$ is Griffiths negative on the entire X for k large enough.

Now we fix k such that the Hermitian metric $H_k\otimes(H^*)^k$ on the bundle

$$S^kE\otimes(\det E^*)^k$$

is Griffiths negative on X. The Hermitian metric H_k by construction is an L^2-integral, so its k-th root is a convex Finsler metric on E (see [Wu 2022, proof of Theorem 1] for details). By Lemma 5, the k-th root of $H_k\otimes(H^*)^k$ is a convex Finsler metric on $E\otimes\det E^*$ which we denote by F. Moreover, this Finsler metric F is strongly plurisubharmonic on $E\otimes\det E^*\setminus\{\text{zero section}\}$ due to Griffiths negativity of $H_k\otimes(H^*)^k$. By adding a small Hermitian metric, we can assume F is strongly convex and strongly plurisubharmonic.

In general, the Kobayashi curvature of Finsler metrics do not behave well under duality [Demailly 1999, Remark 2.7]. But since our Finsler metric F is strongly convex, the dual Finsler metric of F is in fact strongly pseudoconvex and Kobayashi positive (this duality result is originally due to Sommese [1978] and Demailly [1999, Theorem 2.5]. See also [Wu 2022, proof of Theorem 1 and Lemma 6]). In summary, the dual Finsler metric of F is a convex, strongly pseudoconvex, and Kobayashi positive Finsler metric on $E^* \otimes \det E$. Hence the proof of Theorem 1 is complete.

With slight modification on the proof, one has the following variant of Theorem 1.

Theorem 10. *Assume $r \geq 2$ and the line bundle $O_{P(E^*)}(1)$ has a positively curved metric h and a metric g with $\Theta(g)|_{P(E_z^*)} > 0$ for all $z \in X$. If there exist a Hermitian metric Ω on X and a constant $M \geq 1$ such that the following inequalities of* $(1, 1)$*-forms hold*

$$Mq^*\Omega \geq -\theta(g), \tag{3-18}$$

$$q^*\Omega \leq -\theta(h), \tag{3-19}$$

then for any positive integer $l > M/r$, the bundle $E^* \otimes (\det E)^l$ is Kobayashi positive.

4. Proof of Theorem 2

The proof is similar to what we do in Section 3 except that we are dealing with not only $P(E^*)$ but $P(E)$ here. The metric h is used to define a Hermitian metric H on $\det E$ as in (2-14). The metric g is used to define Hermitian metrics $\boldsymbol{H}_k$ on $S^k E^* \otimes (\det E)^k$ as in (2-13).

Fix z_0 in X. For the metric h on $O_{P(E^*)}(1)$, we follow the path that leads to Lemma 9 in Section 3 to deduce a neighborhood W' of z_0 in X such that for $z \in W'$ and $\eta \in T_z^{1,0}X$, the curvature Θ of H satisfies

$$-\Theta(\eta, \bar{\eta}) \leq -\left(r - \frac{r-M}{4}\right)\frac{\Omega(\eta, \bar{\eta})}{(1+\varepsilon)}, \tag{4-1}$$

with $\varepsilon = \frac{r-M}{5(r+M)}$.

For the metric g on $O_{P(E)}(1) \otimes p^* \det E$, we replace $O_{P(E^*)}(1) \to P(E^*)$ in Section 3 with $O_{P(E \otimes \det E^*)}(1) \to P(E \otimes \det E^*)$ and use the canonical isomorphism between $O_{P(E \otimes \det E^*)}(1) \to P(E \otimes \det E^*)$ and $O_{P(E)}(1) \otimes p^* \det E \to P(E)$. Then following the argument leading to Lemma 7, we obtain a positive integer $k_{\max}$ and a neighborhood W of z_0 in X such that for $k \geq k_{\max}$, $z \in W$, $\eta \in T_z^{1,0}X$, and $u \in S^k E_z^* \otimes (\det E_z)^k$ with $\boldsymbol{H}_k(u, u) = 1$, the curvature $\boldsymbol{\Theta}_k$ of $\boldsymbol{H}_k$ satisfies

$$\boldsymbol{H}_k(\boldsymbol{\Theta}_k u, u)(\eta, \bar{\eta}) \leq \left(M + \frac{r-M}{2}\right) k \frac{\Omega(\eta, \bar{\eta})}{(1-\varepsilon)}. \tag{4-2}$$

On the bundle $[S^k E^* \otimes (\det E)^k] \otimes (\det E^*)^k$, there is a Hermitian metric $\boldsymbol{H}_k \otimes (H^*)^k$ with curvature $\boldsymbol{\Theta}_k - k\Theta \otimes \mathrm{Id}_{S^k E^* \otimes (\det E)^k}$. As a result of (4-1) and (4-2), we deduce that, for $k \geq k_{\max}$, $z \in W \cap W'$, $\eta \in T_z^{1,0}X$, and $u \in S^k E_z^* \otimes (\det E_z)^k$ with $\boldsymbol{H}_k(u, u) = 1$,

$$\boldsymbol{H}_k(\boldsymbol{\Theta}_k u, u)(\eta, \bar{\eta}) - k\Theta(\eta, \bar{\eta}) \leq k\left(M + \frac{r-M}{2}\right)\frac{\Omega(\eta, \bar{\eta})}{(1-\varepsilon)} - k\left(r - \frac{r-M}{4}\right)\frac{\Omega(\eta, \bar{\eta})}{(1+\varepsilon)}.$$

Again, the term on the right is negative using $\varepsilon = \frac{r-M}{5(r+M)}$. So we have proved that for $k \ge k_{\max}$ and in $W \cap W'$, the metric $\boldsymbol{H}_k \otimes (H^*)^k$ is Griffiths negative. Since X is compact, $\boldsymbol{H}_k \otimes (H^*)^k$ is Griffiths negative on X for k large.

Now we fix k such that $\boldsymbol{H}_k \otimes (H^*)^k$ on the bundle

$$[S^k E^* \otimes (\det E)^k] \otimes (\det E^*)^k \simeq S^k E^*$$

is Griffiths negative. Using the same argument as those at the end of Section 3, we obtain a convex, strongly pseudoconvex, Kobayashi positive Finsler metric on E.

5. Proof of Theorem 3

We use the metric h to construct a Hermitian metric H on $\det E$ as in (2-14), and the metric g to construct a Hermitian metric H_1 on $S^1 E = E$ as in (2-11).

We start with the metric g. For $(z_0, [\zeta_0])$ in $P(E^*)$, there is a special coordinate system given in Section 2B. In this coordinate system, we define the following n-by-n matrix-valued function:

$$B = (B_{\alpha\beta}) := \big(\phi_{\alpha\bar{\beta}} - (\log\det(\phi_{i\bar{j}}))_{\alpha\bar{\beta}}\big),$$

where $g(e^*, e^*) = e^{-\phi}$. By continuity, there is a neighborhood U of $(z_0, [\zeta_0])$ in $P(E^*)$ such that in U

$$B|_{(z_0,[\zeta_0])} + \frac{r-M}{4}\,\mathrm{Id}_{n\times n} \ge B.$$

In summary:

Lemma 11. *Given a point $(z_0, [\zeta_0]) \in P(E^*)$, there exists a coordinate neighborhood U of $(z_0, [\zeta_0])$ in $P(E^*)$ such that in U*

$$B|_{(z_0,[\zeta_0])} + \frac{r-M}{4}\,\mathrm{Id}_{n\times n} \ge B. \tag{5-1}$$

By Lemma 11, since $P(E^*_{z_0})$ is compact, we can find finitely many points $\{(z_0, [\zeta_l])\}_l$ on $P(E^*_{z_0})$ each of which corresponds to a coordinate neighborhood U_l in $P(E^*)$ such that the corresponding (5-1) holds, and $P(E^*_{z_0}) \subset \bigcup_l U_l$. The fiber $P(E^*_{z_0})$ can be partitioned as $\bigcup_m V_m$ with each V_m in U_l for some l.

Recall the Hermitian metric H_1 on E in (2-11) constructed using the metric g. Denote by Θ_1 the curvature of H_1. We claim:

Lemma 12. *For $0 \ne \eta \in T^{1,0}_{z_0}X$ and $u \in E_{z_0}$ with $H_1(u, u) = 1$, we have*

$$H_1(\Theta_1 u, u)(\eta, \bar{\eta}) \le \Big(M + \frac{r-M}{4}\Big)\,\Omega(\eta, \bar{\eta}). \tag{5-2}$$

Proof. As in Section 2D, we extend the vector $u \in E_{z_0}$ to a local holomorphic section $\tilde{u}$ whose covariant derivative at z_0 equals zero, and we have

$$
\begin{aligned}
-H_1(\Theta_1 u, u) = \partial\bar\partial H_1(\tilde u, \tilde u)|_{z_0} &= \int_{P(E^*_{z_0})} \partial\bar\partial\big(g(\Phi_{1,z}(\tilde u), \Phi_{1,z}(\tilde u))\,\Theta(g)^{r-1}\big)\\
&= \sum_m \int_{V_m} \partial\bar\partial\big(g(\Phi_{1,z}(\tilde u), \Phi_{1,z}(\tilde u))\,\Theta(g)^{r-1}\big).
\end{aligned}
$$

In a fixed $V_m \subset U_l$, we can write $\Phi_{1,z}(\tilde u)$ as fe^* with f a scalar-valued holomorphic function and e^* a local frame for $O_{P(E^*)}(1)$. So,

$$
g(\Phi_{1,z}(\tilde u), \Phi_{1,z}(\tilde u)) = |f|^2 e^{-\phi}.
$$

Meanwhile, recall the curvature $\Theta(g) = \partial\bar\partial\phi$. By Stokes' theorem and a count on degrees, we have

$$
\begin{aligned}
&\sum_m \int_{V_m} \partial\bar\partial\big(g(\Phi_{1,z}(\tilde u), \Phi_{1,z}(\tilde u))\,\Theta(g)^{r-1}\big)\\
&\qquad = \sum_m \int_{V_m} \sum_{\alpha,\beta} \frac{\partial^2 |f|^2 e^{-\phi} \det(\phi_{i\bar j})}{\partial z_\alpha \partial \bar z_\beta} dz_\alpha \wedge d\bar z_\beta \bigwedge_j dw_j \wedge d\bar w_j.
\end{aligned}
$$

So,

$$
-H_1(\Theta_1 u, u)(\eta, \bar\eta) = \sum_m \int_{V_m} \sum_{\alpha,\beta} \frac{\partial^2 |f|^2 e^{-\phi} \det(\phi_{i\bar j})}{\partial z_\alpha \partial \bar z_\beta} \eta_\alpha \bar\eta_\beta \bigwedge_j dw_j \wedge d\bar w_j \tag{5-3}
$$

for $T^{1,0}_{z_0} X \ni \eta = \sum_\alpha \eta_\alpha \frac{\partial}{\partial z_\alpha}$. A direct computation shows

$$
\begin{aligned}
&\sum_{\alpha,\beta} \frac{\partial^2 |f|^2 e^{-\phi} \det(\phi_{i\bar j})}{\partial z_\alpha \partial \bar z_\beta} \eta_\alpha \bar\eta_\beta\\
&\qquad = e^{-\phi} \det(\phi_{i\bar j}) \Big| \sum_\alpha \frac{\partial f}{\partial z_\alpha} \eta_\alpha - f \sum_\alpha \big(\phi_\alpha - (\log\det\phi_{i\bar j})_\alpha\big) \eta_\alpha \Big|^2\\
&\qquad\qquad - |f|^2 e^{-\phi} \det(\phi_{i\bar j}) \sum_{\alpha,\beta} \big(\phi_{\alpha\bar\beta} - (\log\det\phi_{i\bar j})_{\alpha\bar\beta}\big) \eta_\alpha \bar\eta_\beta\\
&\qquad \ge -|f|^2 e^{-\phi} \det(\phi_{i\bar j}) \sum_{\alpha,\beta} B_{\alpha\beta}\, \eta_\alpha \bar\eta_\beta.
\end{aligned}
$$

By (1-8), (1-9), (2-6), and (2-9), we see

$$
\begin{aligned}
M \sum_\alpha |\eta_\alpha|^2 &\ge -(r+1)\,\theta(g)(\tilde\eta, \bar{\tilde\eta}) + q^*\Theta(\det G)(\tilde\eta, \bar{\tilde\eta})\\
&= \inf_{q_*(\tilde\eta)=\eta} \Theta(g)(\tilde\eta, \bar{\tilde\eta}) - \sup_{q_*(\tilde\eta)=\eta} \gamma_g(\tilde\eta, \bar{\tilde\eta})\\
&= \sum_{\alpha,\beta} \phi_{\alpha\bar\beta}|_{(z_0,[\zeta_l])}\, \eta_\alpha \bar\eta_\beta - \sum_{\alpha,\beta} (\log\det(\phi_{i\bar j}))_{\alpha\bar\beta}|_{(z_0,[\zeta_l])}\, \eta_\alpha \bar\eta_\beta\\
&= \sum_{\alpha,\beta} B_{\alpha\beta}|_{(z_0,[\zeta_l])}\, \eta_\alpha \bar\eta_\beta.
\end{aligned}
$$

Therefore, (5-1) becomes

$$\sum_{\alpha,\beta} B_{\alpha\beta}\,\eta_\alpha\,\bar\eta_\beta \le \Big(M+\frac{r-M}{4}\Big)\sum_\alpha |\eta_\alpha|^2 = \Big(M+\frac{r-M}{4}\Big)\,\Omega(\eta,\bar\eta). \tag{5-4}$$

So, (5-3) becomes

$$\begin{aligned} &-H_1(\Theta_1 u,u)(\eta,\bar\eta) \\ &\quad\ge \sum_m \int_{V_m} -|f|^2 e^{-\phi}\det(\phi_{i\bar j}) \bigwedge_j dw_j\wedge d\bar w_j \Big(M+\frac{r-M}{4}\Big)\,\Omega(\eta,\bar\eta) \\ &\quad= -\Big(M+\frac{r-M}{4}\Big)\,\Omega(\eta,\bar\eta) \end{aligned} \tag{5-5}$$

since $H_1(u,u)=1$. □

For the metric h on $O_{P(E^*)}(1)$, as in Lemma 9 from Section 3 with slight modification, we deduce that for $\eta\in T^{1,0}_{z_0}X$, the curvature Θ of H satisfies

$$-\Theta(\eta,\bar\eta) \le -\Big(r-\frac{r-M}{4}\Big)\,\Omega(\eta,\bar\eta). \tag{5-6}$$

Finally, we consider the metric $H_1\otimes H^*$ on $E\otimes\det E^*$. Since $\det E^*$ is a line bundle, we can identify $\operatorname{End}(E\otimes\det E^*)$ with $\operatorname{End} E$, and the curvature of the metric $H_1\otimes H^*$ can be written as $\Theta_1-\Theta\otimes \mathrm{Id}_E$, where Θ_1 and Θ are the curvature of H_1 and H respectively. As a result of Lemma 12 and (5-6), we see for $0\ne\eta\in T^{1,0}_{z_0}X$ and $u\in E_{z_0}$ with $H_1(u,u)=1$,

$$H_1(\Theta_1 u,u)(\eta,\bar\eta)-\Theta(\eta,\bar\eta) \le \Big(M+\frac{r-M}{4}\Big)\,\Omega(\eta,\bar\eta)-\Big(r-\frac{r-M}{4}\Big)\,\Omega(\eta,\bar\eta),$$

the term on the right is negative. Hence we have proved that at z_0 the metric $H_1\otimes H^*$ is Griffiths negative. The point z_0 is arbitrary, so $H_1\otimes H^*$ is Griffiths negative on X. As a result, the dual bundle $E^*\otimes\det E$ is Griffiths positive.

6. Proof of Theorem 4

The metric h is used to define a Hermitian metric H on $\det E$ as in (2-14). The metric g is used to define Hermitian metric $\boldsymbol{H}_1$ on $E^*\otimes\det E$ as in (2-13).

Given z_0 in X. For the metric h on $O_{P(E^*)}(1)$, as in the formula (5-6) from Section 5, for $\eta\in T^{1,0}_{z_0}X$ we have

$$-\Theta(\eta,\bar\eta) \le -\Big(r-\frac{r-M}{4}\Big)\,\Omega(\eta,\bar\eta). \tag{6-1}$$

For the metric g on $O_{P(E)}(1)\otimes p^*\det E$, we replace $O_{P(E^*)}(1)\to P(E^*)$ in Section 5 with $O_{P(E\otimes\det E^*)}(1)\to P(E\otimes\det E^*)$ and use the canonical isomorphism between $O_{P(E\otimes\det E^*)}(1)\to P(E\otimes\det E^*)$ and $O_{P(E)}(1)\otimes p^*\det E\to P(E)$. Then as in Lemma 12, we get for $\eta \in T^{1,0}_{z_0}X$, and $u \in E^*_{z_0} \otimes (\det E_{z_0})$ with $\boldsymbol{H}_1(u, u) = 1$, the curvature $\boldsymbol{\Theta}_1$ of $\boldsymbol{H}_1$ satisfies

$$\boldsymbol{H}_1(\boldsymbol{\Theta}_1 u, u)(\eta, \bar{\eta}) \leq \left(M + \frac{r-M}{4}\right) \Omega(\eta, \bar{\eta}). \tag{6-2}$$

On the bundle $(E^* \otimes \det E) \otimes \det E^*$, there is a Hermitian metric $\boldsymbol{H}_1 \otimes H^*$ with curvature $\boldsymbol{\Theta}_1 - \Theta \otimes \mathrm{Id}_{E^* \otimes \det E}$. As a result of (6-1) and (6-2), we deduce that for $\eta \in T^{1,0}_{z_0}X$, and $u \in E^*_{z_0} \otimes (\det E_{z_0})$ with $\boldsymbol{H}_1(u, u) = 1$,

$$\begin{aligned} \boldsymbol{H}_1(\boldsymbol{\Theta}_1 u, u)(\eta, \bar{\eta}) - \Theta(\eta, \bar{\eta}) \\ \leq \left(M + \frac{r-M}{4}\right) \Omega(\eta, \bar{\eta}) - \left(r - \frac{r-M}{4}\right) \Omega(\eta, \bar{\eta}), \end{aligned} \tag{6-3}$$

the term on the right is negative. So the Hermitian metric $\boldsymbol{H}_1 \otimes H^*$ is Griffiths negative at z_0 an arbitrary point. Hence $\boldsymbol{H}_1 \otimes H^*$ is Griffiths negative on X, and the bundle E is Griffiths positive.

7. Examples

Example 13. We provide here an example where the assumptions in Theorems 1, 2, 3, and 4 are satisfied. Let L be a line bundle with a metric H whose curvature $\Theta > 0$. Let $E = L^9 \oplus L^8 \oplus L^7$ a vector bundle of rank $r = 3$. The induced metric $(H^*)^9 \oplus (H^*)^8 \oplus (H^*)^7$ on the dual bundle E^* has curvature

$$\Theta(E^*) = (-9\Theta) \oplus (-8\Theta) \oplus (-7\Theta),$$

which is Griffiths negative, so the corresponding metric h on $O_{P(E^*)}(1)$ is positively curved. According to (2-7), we see

$$-\theta(h) = -q^* \Theta(E^*)|_{O_{P(E^*)}(-1)}.$$

Hence we have

$$7q^*\Theta \leq -\theta(h) \leq 9q^*\Theta. \tag{7-1}$$

For all four theorems, we will use this metric h on $O_{P(E^*)}(1)$ and take Ω to be 7Θ. So $q^*\Omega \leq -\theta(h)$ always holds. The choice of g will be different from case to case.

For Theorem 1, we choose g to be h, and hence by (7-1) and $\Omega = 7\Theta$ we get

$$q^*\Omega \leq -\theta(h) = -\theta(g) \leq \tfrac{9}{7} q^*\Omega. \tag{7-2}$$

To fulfill the assumption of Theorem 1, we can choose $M = \frac{9}{7}$ which is in the interval $[1, 3)$.

For Theorem 2, since $E \otimes \det E^* = (L^*)^{15} \oplus (L^*)^{16} \oplus (L^*)^{17}$ has induced curvature $(-15\Theta) \oplus (-16\Theta) \oplus (-17\Theta)$ which is Griffiths negative, the corresponding metric g on $O_{P(E)}(1) \otimes p^* \det E$ is positively curved and satisfies

$$15p^*\Theta \leq -\theta(g) \leq 17p^*\Theta. \tag{7-3}$$

Together with (7-1) and $\Omega = 7\Theta$, we have

$$\tfrac{17}{7} p^* \Omega \geq -\theta(g) \quad \text{and} \quad q^* \Omega \leq -\theta(h). \tag{7-4}$$

We can choose $M = \frac{17}{7}$ which is in $[1, 3)$.

For Theorem 3, notice that h is induced from $(H^*)^9 \oplus (H^*)^8 \oplus (H^*)^7$ on E^*, so if we use $(H^*)^9 \oplus (H^*)^8 \oplus (H^*)^7$ for the Hermitian metric G, then the corresponding g is actually h. Since $\Theta(\det G) = -24\Theta$, by using (7-1) we have

$$-(r+1)\,\theta(g) + q^*\Theta(\det G) = -4\theta(h) - 24q^*\Theta \leq 12q^*\Theta = \tfrac{12}{7} q^*\Omega. \tag{7-5}$$

We choose $M = \frac{12}{7}$ which is in $[1, 3)$.

Finally for Theorem 4, on $E \otimes \det E^* = (L^*)^{15} \oplus (L^*)^{16} \oplus (L^*)^{17}$, we will use the metric $(H^*)^{15} \oplus (H^*)^{16} \oplus (H^*)^{17}$ for G, so $\Theta(\det G) = -48\Theta$. Moreover, the corresponding metric g on $O_{P(E)}(1) \otimes p^* \det E$ satisfies

$$15 p^*\Theta \leq -\theta(g) \leq 17 p^*\Theta, \tag{7-6}$$

so we get

$$-(r+1)\,\theta(g) + p^*\Theta(\det G) \leq 20 p^*\Theta = \tfrac{20}{7} p^*\Omega. \tag{7-7}$$

We choose $M = \frac{20}{7}$ which is in $[1, 3)$.

Example 14. Let X be a compact Riemann surface with a Hermitian metric ω. Let E be an ω-semistable ample vector bundle of rank r over X. The assumptions in Theorems 1, 2, 3, and 4 are all satisfied in this case. We will explain for only Theorems 2 and 4. Theorems 1 and 3 can be verified similarly. By [Li et al. 2021, Theorem 1.7, Remark 1.8, and Theorem 1.11], there exists a constant $c > 0$ such that for any $\delta > 0$, there exists a Hermitian metric H_δ on E satisfying

$$(c - \delta)\,\mathrm{Id}_E \leq \sqrt{-1}\Lambda_\omega\,\Theta(H_\delta) \leq (c + \delta)\,\mathrm{Id}_E, \tag{7-8}$$

where Λ_ω is the contraction with respect to ω. Since X is a Riemann surface, Λ_ω locally is multiplication by a positive function.

For Theorem 2, we choose $\delta = \frac{c}{5r}$. The Hermitian metric H_δ^* on E^* induces a metric h on $O_{P(E^*)}(1)$. Due to (2-7), we see

$$-\theta(h) = -q^*\Theta(H_\delta^*)|_{O_{P(E^*)}(-1)}; \tag{7-9}$$

combining with (7-8), we have

$$(c - \delta)\, q^*\omega \leq -\theta(h) \leq (c + \delta)\, q^*\omega. \tag{7-10}$$

The Hermitian metric $H_\delta \otimes \det H_\delta^*$ on $E \otimes \det E^*$ induces on $O_{P(E)}(1) \otimes p^* \det E$ a metric g. Similar to (7-10), we have

$$-\theta(g) \leq [-(c - \delta) + r(c + \delta)]\, p^*\omega. \tag{7-11}$$

If we choose $\Omega = (c-\delta)\,\omega$ and $M = r - \frac{1}{2}$, then

$$[-(c-\delta) + r(c+\delta)]\, p^*\omega \le M p^*\Omega.$$

As a result, we achieve the assumption in Theorem 2:

$$q^*\Omega \le -\theta(h) \quad \text{and} \quad -\theta(g) \le M p^*\Omega.$$

For Theorem 4, we choose $\delta = \frac{c}{9r}$. We still have (7-10). The Hermitian metric G on $E \otimes \det E^*$ is taken to be $H_\delta \otimes \det H_\delta^*$, so we get

$$\begin{aligned}
&-(r+1)\,\theta(g) + p^*\Theta(\det G)\\
&\qquad = -(r+1)[p^*\Theta(H_\delta)|_{O_{P(E)}(-1)} - p^*\Theta(\det H_\delta)] - (r-1)\,p^*\Theta(\det H_\delta)\\
&\qquad \le [-(r+1)(c-\delta) + 2r(c+\delta)]\, p^*\omega.
\end{aligned}$$

If we choose $\Omega = (c-\delta)\,\omega$ and $M = r - \frac{1}{2}$, then

$$[-(r+1)(c-\delta) + 2r(c+\delta)]\, p^*\omega \le M p^*\Omega.$$

So the assumption of Theorem 4 is satisfied.

In light of [Li et al. 2021, Theorem 1.7], it is possible to modify our theorems so that semistability is not needed in this example.

Appendix

Here we prove the isomorphism $\bigwedge^{r-1} E \simeq E^* \otimes \det E$ where r is the rank of E.

Proof. Let $\{e_1, \dots, e_r\}$ and $\{f_1, \dots, f_r\}$ be two sets of local frames for E with the transition matrix $g = (g_{ij})$; namely, on the intersection of the two frames, we have $f_i = \sum_j g_{ij}\, e_j$. On the bundle $\bigwedge^{r-1} E$, we have the induced local frame $\{\hat{e}_1, \dots, \hat{e}_r\}$ where $\hat{e}_k$ is $e_1 \wedge \cdots \wedge e_r$ with e_k removed. Similarly, we have another frame $\{\hat{f}_1, \dots, \hat{f}_r\}$. Let $\hat{g} = (\hat{g}_{ij})$ be the corresponding transition matrix for the bundle $\bigwedge^{r-1} E$, namely, $\hat{f}_i = \sum_j \hat{g}_{ij}\, \hat{e}_j$. It is not hard to verify that $\hat{g}_{ij}$ is the determinant of the matrix g with the i-th row and j-th column removed.

For the dual bundle E^*, the corresponding transition matrix for the dual frames $\{e_1^*, \dots, e_r^*\}$ and $\{f_1^*, \dots, f_r^*\}$ is the transpose of g^{-1}. Therefore, the transition matrix for the bundle $E^* \otimes \det E$ is $c = (c_{ij})$ where $c_{ij} = (-1)^{i+j}\,\hat{g}_{ij}$.

Now, let us denote by A the diagonal matrix whose i-th diagonal entry is $(-1)^i$. Notice that the inverse of A is still A. Also, after a straightforward computation, we have $AcA^{-1} = \hat{g}$. So, the two bundles $\bigwedge^{r-1} E$ and $E^* \otimes \det E$ are isomorphic. □

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