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Let ξ be an irrational algebraic real number and let $(p_k/q_k)_{k\geq 1}$ denote the sequence of its convergents. Let $(u_n)_{n\geq 1}$ be a nondegenerate linear recurrence sequence of integers, which is not a polynomial sequence. We show that if the intersection of the sequences $(q_k)_{k\geq 1}$ and $(u_n)_{n\geq 1}$ is infinite, then ξ is a quadratic number. This extends an earlier work of Lenstra and Shallit (1993). We also discuss several arithmetical properties of the base-b representation of the integers q_k, $k \geq 1$, where $b \geq 2$ is an integer. Finally, when ξ is a (possibly transcendental) non-Liouville number, we prove a result implying the existence of a large prime factor of $q_{k-1}\, q_k\, q_{k+1}$ for large k. This is related to earlier results of Erdős and Mahler (1939), Shorey and Stewart (1983), and Shparlinskii (1987).

1. introduction

Let θ be an arbitrary irrational real number and $(p_k(\theta)/q_k(\theta))_{k\geq 1}$ (we will use the shorter notation p_k/q_k when no confusion is possible and ξ instead of θ if the number is known to be algebraic) denote the sequence of its convergents.

Let $\mathcal{N}$ be an infinite set of positive integers. It follows from a result of Borosh and Fraenkel [6] that the set

$$\mathcal{K}(\mathcal{N}) = \{\theta \in \mathbb{R} : q_k(\theta) \text{ is in } \mathcal{N} \text{ for arbitrarily large } k\}$$

has always Hausdorff dimension at least $\frac{1}{2}$ and its Lebesgue measure is zero if there is some positive δ such that the series $\sum_{q\in\mathcal{N}} q^{-1+\delta}$ converges. Examples of sets $\mathcal{N}$ (or integer sequences $(u_n)_{n\geq 1}$) with the latter property include nondegenerate linear recurrence sequences, the set of integers having a bounded number of nonzero digits in their base-10 representation, sets of positive values taken at integer values by a given integer polynomial of degree at least 2, and sets of positive integers divisible only by prime numbers from a given, finite set.

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Our main purpose is to discuss whether $\mathcal{K}(\mathcal{N})$ contains algebraic numbers for some special sets $\mathcal{N}$ for which $\mathcal{K}(\mathcal{N})$ has zero Lebesgue measure. Said differently, for an arbitrary irrational real algebraic number ξ, we investigate various arithmetical properties of the sequence $(q_k(\xi))_{k\geq 1}$. We consider the following questions:

A. Does the greatest prime factor of $q_k(\xi)$ tends to infinity with k? If yes, how rapidly?

B. Does the number of nonzero digits in the base-10 representation of $q_k(\xi)$ tends to infinity with k? If yes, how rapidly?

C. Are there infinitely many squares (cubes, perfect powers) in $(q_k(\xi))_{k\geq 1}$?

D. Is the intersection of $(q_k(\xi))_{k\geq 1}$ with a given linear recurrence sequence of integers finite or infinite?

First, let us recall that very few is known on the continued fraction expansion of an algebraic number of degree at least 3, while the continued fraction expansion of a quadratic real number ξ is ultimately periodic and takes the form

$$\xi = [a_0; a_1, \ldots, a_r, \overline{a_{r+1}, \ldots, a_{r+s}}].$$

Consequently, we have $q_{k+2s} = tq_{k+s} - (-1)^s\, q_k$ for $k > r$, where t is the trace of

$$\begin{pmatrix} a_{r+1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{r+2} & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_{r+s} & 1 \\ 1 & 0 \end{pmatrix};$$

see [18; 19]. This shows that $(q_k(\xi))_{k\geq 1}$ is the union of s binary recurrences whose roots are the roots of the polynomial $X^2 - tX + (-1)^s$, that is, the real numbers $\frac{1}{2}(t \pm \sqrt{t^2 - 4(-1)^s})$. Thus, for a quadratic real number ξ, we immediately derive Diophantine results on $(q_k(\xi))_{k\geq 1}$ from results on binary recurrences of the above form.

Question A has already been discussed in [7] and earlier works. Let us mention that it easily follows from Ridout's theorem [23] that the greatest prime factor of $q_k(\xi)$ tends to infinity with n, but we have no estimate of the rate of growth, except when ξ is quadratic (by known effective results on binary recurrences, see [28]). Furthermore, the theory of linear forms in logarithms gives a lower bound for the greatest prime factor of the product $p_k(\xi)\, q_k(\xi)$, which tends to infinity at least as fast as some constant times $\log_2 q_k(\xi) \log_3 q_k(\xi)/\log_4 q_k(\xi)$, where $\log_j$ denotes the j-th iterated logarithm function. Although we have no new contribution to Question A as stated for algebraic numbers ξ, we obtain new results on prime factors of $q_k(\theta)$ for a transcendental number θ. In 1939, Erdős and Mahler [16] proved that the greatest prime factor of $q_{k-1}(\theta)\, q_k(\theta)\, q_{k+1}(\theta)$ tends to infinity as k tends to infinity. In this paper, we obtain a more explicit result involving the irrationality exponent of θ.

We give a partial answer to Question B, which has not been investigated up to now. Question C is solved when ξ is quadratic: there are only finitely many perfect powers in the sequence $(q_k(\xi))_{k\ge1}$ thanks to results of Pethő [22] and Shorey and Stewart [25] stating that there are only finitely many perfect powers in binary recurrence sequences of integers. This result is effective. When ξ has degree at least 3, Question C appears to be very difficult. Since $(n^d)_{n\ge1}$ is a linear recurrence sequence for any given positive integer d, a large part of Question C is contained in Question D.

Question D is interesting for several reasons. First, some assumption on the linear recurrence must be added, since the linear recurrence $(n)_{n\ge1}$ has infinite intersection with the sequence $(q_k(\xi))_{k\ge1}$. Second, as already mentioned, when ξ is quadratic, its continued fraction expansion is ultimately periodic and the sequence $(q_k(\xi))_{k\ge1}$ is the union of a finite set of binary recurrences. Among our results, we show that if a "nonsingular" linear recurrence has an infinite intersection with $(q_k(\xi))_{k\ge1}$, then ξ must be quadratic. Unfortunately we must exclude linear recurrences of the form $(n^d)_{n\ge1}$, and hence we do not have any contribution to Question C.

Recall that any nonzero linear recurrence sequence $(u_n)_{n\ge1}$ of complex numbers can be expressed as

$$u_n = P_1(n)\,\alpha_1^n + \cdots + P_r(n)\,\alpha_r^n \quad \text{for } n \ge 1,$$

where $r \ge 1$, $\alpha_1, \ldots, \alpha_r$ are distinct nonzero complex numbers (called the roots of the recurrence), and $P_1, \ldots, P_r$ are nonzero polynomials with complex coefficients. This expression is unique up to rearranging the terms. The sequence $(u_n)_{n\ge1}$ is called nondegenerate if α_i/α_j is not a root of unity for $1 \le i \ne j \le r$. For most problems about linear recurrence sequences, it is harmless to assume that $(u_n)_{n\ge1}$ is nondegenerate. Indeed, if $(u_n)_{n\ge1}$ is degenerate and L denotes the lcm of the orders of the roots of unity of the form α_i/α_j, then each of the subsequences $(u_{nL+m})_{n\ge1}$ with $m \in \{0, \ldots, L-1\}$ is either identically zero or nondegenerate.

The proofs of our results rest on the p-adic Schmidt subspace theorem. This powerful tool was first applied to the study of continued fraction expansions of algebraic numbers by Corvaja and Zannier in [13; 14]. They proved in [13] that, for any positive real quadratic irrational α which is neither the square root of a rational number, nor a unit in the ring of integers of $\mathbb{Q}(\alpha)$, the period length of the continued fraction for α^n tends to infinity with n. They established in [14] that if $\alpha(n)$ and $\beta(n)$ are power sums over the rationals satisfying suitable necessary assumptions, then the length of the continued fraction for $\alpha(n)/\beta(n)$ tends to infinity with n; see also [12; 17; 24] for related questions. The Schmidt subspace theorem has also been used by Adamczewski and Bugeaud in [1; 2; 3; 9] to prove that the continued fraction expansion of an algebraic number of degree at least 3 cannot have arbitrary long repetitions nor quasipalindromes close to its start.

Let $(u_n)_{n\geq 1}$ be a nonconstant linear recurrence sequence with integral roots greater than 1 and rational coefficients. It follows from [15, Theorem 4.16] that the intersection of the sequences $(u_n)_{n\geq 1}$ and $(q_k)_{k\geq 1}$ is finite. This gives a first partial result toward Question D. For a real number θ, we let $\|\theta\|$ denote the distance from θ to the nearest integer. Our first main result gives a full answer to Question D. Its proof uses results of Kulkarni, Mavraki, and Nguyen [20], which extend a seminal work of Corvaja and Zannier [13], who showed that, if a real algebraic number $\alpha > 1$ and ℓ in $(0, 1)$ are such that $\|\alpha^n\| < \ell^n$ for infinitely many positive integers n, then there is a positive integer d such that α^d is a Pisot number (observe that this conclusion is best possible).

Theorem 1.1. *Let $(p_k/q_k)_{k\geq 1}$ be the sequence of convergents to an irrational real algebraic number ξ of degree d. Let $\varepsilon > 0$. Let $(u_n)_{n\geq 1}$ be a nondegenerate linear recurrence sequence of integers, which is not a polynomial sequence. Then the set*

$$\left\{n \in \mathbb{N} : u_n \neq 0 \text{ and } \|u_n \xi\| < \frac{1}{|u_n|^{(1/(d-1))+\varepsilon}}\right\}$$

is finite. In particular, if $d \geq 3$, then there are only finitely many pairs (n, k) such that $u_n = q_k$.

The case $d = 2$ of Theorem 1.1 is immediate, since quadratic real numbers have bounded partial quotients in their continued fraction expansion. Consequently, we restrict our attention to the case $d \geq 3$. Theorem 1.1 is a special case of Theorem 3.6, which deals with a larger class of integer sequences than that of recurrence sequences.

When $d = 3$, the exponent $\frac{1}{d-1} = \frac{1}{2}$ is best possible, as can be seen with the following example. Let $K \subset \mathbb{R}$ be a cubic field with a pair of complex-conjugate embeddings. Let $\xi \in K$ with $|\xi| > 1$ be a unit of the ring of integers. Let α and $\bar{\alpha}$ denote the remaining Galois conjugates of ξ. We have $|\alpha| = |\xi|^{-1/2}$ and, setting $u_n = \xi^n + \alpha^n + \bar{\alpha}^n$ for $n \geq 1$, we check that

$$|u_n \xi - u_{n+1}| \ll_\xi |\alpha^n| \ll_\xi |u_n|^{-1/2} \quad \text{for } n \geq 1,$$

where $\ll_\xi$ means that the implicit constant is positive and depends only on ξ. When $d \geq 4$, we do not know if Theorem 1.1 remains valid with a smaller exponent than $\frac{1}{d-1}$.

Theorem 1.1 allows us to complement the result of Lenstra and Shallit [21]:

Theorem 1.2 (Lenstra and Shallit [21]). *Let θ be an irrational real number, whose continued fraction expansion is given by $\theta = [a_0; a_1, a_2, \dots]$, and let $(p_k)_{k\geq 1}$ and $(q_k)_{k\geq 1}$ be the sequence of numerators and denominators of the convergents to θ. Then the following four conditions are equivalent:*

(i) *The sequence* $(p_k)_{k\ge 1}$ *satisfies a linear recurrence with constant complex coefficients.*

(ii) *The sequence* $(q_k)_{k\ge 1}$ *satisfies a linear recurrence with constant complex coefficients.*

(iii) *The sequence* $(a_n)_{n\ge 0}$ *is ultimately periodic.*

(iv) θ *is a quadratic irrational.*

The proof of Theorem 1.2 rests on the Hadamard quotient theorem. A simpler proof of a more general statement has been given by Bézivin [4], who instead of (ii) only assumes that $(q_k)_{k\ge 1}$ satisfies a linear recurrence with coefficients being polynomials in k and that the series $\sum_{k\ge 1} q_k z^k$ has a nonzero convergence radius.

We strengthen Theorem 1.2 for convergents of algebraic numbers as follows.

Corollary 1.3. *Let* $\xi = [a_0; a_1, a_2, \dots]$ *be an irrational real algebraic number, and let* $(p_k)_{k\ge 1}$ *and* $(q_k)_{k\ge 1}$ *be the sequence of numerators and denominators of the convergents to* ξ. *Then the following four conditions are equivalent*:

(i) *The sequence* $(p_k)_{k\ge 1}$ *has an infinite intersection with some nondegenerate linear recurrence sequence that is not a polynomial sequence.*

(ii) *The sequence* $(q_k)_{k\ge 1}$ *has an infinite intersection with some nondegenerate linear recurrence sequence that is not a polynomial sequence.*

(iii) *The sequence* $(a_n)_{n\ge 0}$ *is ultimately periodic.*

(iv) ξ *is a quadratic irrational.*

Now we present our results concerning Question B. Let $b \ge 2$ be an integer. Every positive integer N can be written uniquely as

$$N = d_k\, b^k + \cdots + d_1 b + b_0,$$

where

$$d_0, d_1, \dots, d_k \in \{0, 1, \dots, b-1\}, \quad d_k \ne 0.$$

We define the length

$$\mathcal{L}(N, b) = \mathrm{Card}\{0 \le j \le k : d_j \ne 0\}$$

of the b-ary representation of N. We also define the number of digit changes by

$$\mathcal{DC}(N, b) = \mathrm{Card}\{2 \le j \le k : d_j \ne d_{j-1}\}.$$

Theorem 1.4. *Let* ξ *be an irrational real algebraic number and let* $b \ge 2$ *be an integer. Let* $(u_n)_{n\ge 1}$ *be a strictly increasing sequence of positive integers and* $\lambda \in (0, 1]$ *such that for every* $\varepsilon > 0$, *the inequality*

$$\|u_n\, \xi\| < u_n^{-\lambda+\varepsilon}$$

holds for all but finitely many n. *We have*:

(i) *Let k be a positive integer and let $\varepsilon > 0$. For all sufficiently large n, if δ is a divisor of u_n with $\mathcal{L}(\delta, b) \le k$ then $\delta < u_n^{(k-\lambda)/k+\varepsilon}$.*

(ii) *Let k be a nonnegative integer and let $\varepsilon > 0$. For all sufficiently large n, if δ is a divisor of u_n with $\mathcal{DC}(\delta, b) \le k$ then $\delta < u_n^{(k+2-\lambda)/(k+2)+\varepsilon}$.*

Consequently, let $(p_k/q_k)_{k\ge1}$ denote the sequence of convergents to ξ then each one of the limits $\lim_{k\to+\infty} \mathcal{L}(q_k, b)$, $\lim_{k\to+\infty} \mathcal{DC}(q_k, b)$, $\lim_{k\to+\infty} \mathcal{L}(p_k, b)$, *and* $\lim_{k\to+\infty} \mathcal{DC}(p_k, b)$ *is infinite.*

Except for certain quadratic numbers, it seems to be a very difficult problem to get an effective version of the last assertion of Theorem 1.4. Stewart [27, Theorem 2] established that if $(u_n)_{n\ge1}$ is a binary sequence of integers, whose roots ξ, ξ' are quadratic numbers with $|\xi| > \max\{1, |\xi'|\}$, then there exists a positive real number C such that

$$\mathcal{L}(u_n, b) > \frac{\log n}{\log\log n + C} - 1, \quad n \ge 5.$$

Consequently, if $(p_k/q_k)_{k\ge1}$ denote the sequence of convergents to a quadratic real algebraic number, then for $k \ge 4$ we have

$$\mathcal{L}(q_k, b) > \frac{\log k}{\log\log k + C} - 1 \quad \text{and} \quad \mathcal{DC}(q_k, b) > \frac{\log k}{\log\log k + C} - 1.$$

A similar question can be asked for the Zeckendorf representation [30] of q_k. Let $(F_n)_{n\ge0}$ denote the Fibonacci sequence defined by

$$F_0 = 0, \quad F_1 = 1, \quad \text{and} \quad F_{n+2} = F_{n+1} + F_n \qquad \text{for } n \ge 0.$$

Every positive integer N can be written uniquely as a sum:

$$N = \varepsilon_\ell F_\ell + \varepsilon_{\ell-1} F_{\ell-1} + \cdots + \varepsilon_2 F_2 + \varepsilon_1 F_1,$$

with $\varepsilon_\ell = 1$, ε_j in $\{0, 1\}$, and $\varepsilon_j \varepsilon_{j+1} = 0$ for $j = 1, \ldots, \ell - 1$. This representation of N is called its Zeckendorf representation. The number of digits of N in its Zeckendorf representation is the number of positive integers j for which ε_j is equal to 1. By using the Schmidt subspace theorem we can in a similar way prove that the number of digits of $q_k(\xi)$ in its Zeckendorf representation tends to infinity with k, we omit the details (but see [10]).

Our last result is motivated by a theorem of Erdős and Mahler [16] on convergents to real numbers. Let S be a set of prime numbers. For a nonzero integer N, let $[N]_S$ denote the largest divisor of N composed solely of primes from S. Set $[0]_S = 0$. Recall that the irrationality exponent $\mu(\theta)$ of an irrational real number θ is the supremum of the real numbers μ such that there exist infinitely many rational numbers r/s with $s \ge 1$ and $|\theta - r/s| < 1/s^\mu$. It is always at least equal to 2 and, by definition, θ is called a Liouville number when $\mu(\theta)$ is infinite. Erdős and Mahler [16] established that, when θ is irrational and not a Liouville number, then the greatest prime factor of $q_{k-1}\,q_k\,q_{k+1}$ tends to infinity with k. We obtain the following more precise version of their result.

Theorem 1.5. *Let θ be an irrational real number and μ its irrationality exponent. Let $(p_k/q_k)_{k\geq 1}$ denote the sequence of convergents to θ. Let S be a finite set of prime numbers. If μ is finite, then, for every $\varepsilon > 0$ and every k sufficiently large (depending on ε), we have*

$$[q_{k-1}\,q_k\,q_{k+1}]_S < (q_{k-1}\,q_k\,q_{k+1})^{\mu/(\mu+1)+\varepsilon}. \tag{1-1}$$

The same conclusion holds when the sequence $(q_k)_{k\geq 1}$ is replaced by $(|p_k|)_{k\geq 1}$.

When θ is algebraic irrational and $\varepsilon > 0$, we have $[q_k]_S < q_k^{\varepsilon}$ for all large k by Ridout's theorem. The interesting feature of Theorem 1.5 is that it holds for all transcendental non-Liouville numbers.

Theorem 1.5 is ineffective. Under its assumption, it is proved in [11] that there exists a (large) positive, effectively computable $c = c(S)$ such that

$$[q_{k-1}\,q_k\,q_{k+1}]_S < (q_{k-1}\,q_k\,q_{k+1})^{1-1/(c\mu\log\mu)}, \quad k \geq 2.$$

For $\mu = 2$ (that is, for almost all θ), the exponent in (1-1) becomes $\frac{2}{3} + \varepsilon$. It is an interesting question to determine whether it is best possible. It cannot be smaller than $\frac{1}{3}$. Indeed, the Folding lemma (see, e.g., [8, Section 7.6]) allows one, for any given integer $b \geq 2$, to construct explicitly real numbers θ with $\mu(\theta) = 2$ and having infinitely many convergents whose denominator is a power of b.

Furthermore, there exist irrational real numbers $\theta = [a_0; a_1, a_2, \ldots]$ with convergents p_k/q_k such that the q_k's are alternating among powers of 2 and 3. Indeed, let $k \geq 2$ and assume that $q_{k-1} = 2^c$ and $q_k = 3^d$ for positive integers c, d. Then, we have to find a positive integer a_{k+1} such that $2^c + a_{k+1}3^d$ is a power of 2. To do this, it is sufficient to take for m the smallest integer greater than c such that 2^{m-c} is congruent to 1 modulo 3^d and then define $a_{k+1} = (2^{m-c} - 1)/3^d$. The sequence $(a_k)_{k\geq 1}$ increases very fast and θ is a Liouville number.

We are grateful to Professor Igor Shparlinskii for bringing our attention to [26]. Suppose the irrational number θ has the property that $\log q_n \ll n$ for every n; the set of all such θ's is strictly smaller than the set of all non-Liouville numbers. Then [26, Theorem 5] implies $P[q_1 \cdots q_n] \gg n$ for all sufficiently large n where $P[\,\cdot\,]$ denotes the largest prime factor. It seems possible to relax the condition $\log q_n \ll n$ at the expense of a weaker lower bound for $P[q_1 \cdots q_n]$ in order to allow θ to be certain Liouville numbers. On the other hand, it seems possible to extend the proof of Theorem 1.5 to get a lower bound for $P[q_1 \cdots q_n]$ in terms of n and the irrationality exponent of θ. We leave this further discussion for future work.

The outline of this paper is as follows. The proof of Theorem 1.5 and additional remarks on [16] are given in Section 4. Theorem 1.4 is established in Section 2 and the other results are proved in Section 3.

2. Proof of Theorem 1.4

For a prime number ℓ, we let $v_\ell : \mathbb{Q} \to \mathbb{Z} \cup \{\infty\}$ be the additive ℓ-adic valuation and let $|\cdot|_\ell = \ell^{-v_\ell(\cdot)}$ be the ℓ-adic absolute value.

Proof of Theorem 1.4. First, we prove part (i). Let $\mathcal{N}_1$ be the set of tuples $(m, n_1, \dots, n_a)$ such that:

- $1 \le a \le k$ and $n_1 < n_2 < \cdots < n_a$ are nonnegative integers.
- There exist $d_1, \dots, d_a$ in $\{1, \dots, b-1\}$ such that $\delta := d_a\, b^{n_a} + \cdots + d_1 b^{n_1}$ is a divisor of u_m and $\delta \ge u_m^{(k-\lambda)/k+\varepsilon}$.

Assume that $\mathcal{N}_1$ is infinite. Then, there exist an integer h with $1 \le h \le k$, positive integers $D_1, \dots, D_h$, an infinite set $\mathcal{N}_2$ of $(h+1)$-tuples $(m_i, n_{1,i}, \dots, n_{h,i})$ for $i \ge 1$ such that:

- $n_{1,i} < \cdots < n_{h,i}$ are nonnegative integers.
- For $i \ge 1$, we have a divisor of u_{m_i}:

$$\delta_{m_i} := D_h\, b^{n_{h,i}} + \cdots + D_1\, b^{n_{1,i}},$$

with $\delta_{m_i} \ge u_{m_i}^{(k-\lambda)/k+\varepsilon}$.
- We have

$$\lim_{i \to +\infty} (n_{j,i} - n_{j-1,i}) = +\infty, \quad j = 2, \dots, h. \tag{2-1}$$

For $i \ge 1$, let w_{m_i} denote the nearest integer to $u_{m_i}\xi$ and let

$$v_{m_i} := u_{m_i}/\delta_{m_i} \le u_m^{\lambda/k-\varepsilon}. \tag{2-2}$$

When m_i is sufficiently large, we have

$$|\xi D_h\, v_{m_i} b^{n_{h,i}} + \cdots + \xi D_1\, v_{m_i} b^{n_{1,i}} - w_{m_i}| = \|\xi u_{m_i}\| < |u_{m_i}|^{-\lambda+\varepsilon/2}, \tag{2-3}$$

thanks to the given properties of $(u_m)_{m\ge 1}$ and λ. We are in position to apply the Schmidt subspace theorem.

Let S denote the set of prime divisors of b. Consider the linear forms in

$$\boldsymbol{X} = (X_0, X_1, \dots, X_h)$$

given by

$$L_{j,\infty}(\boldsymbol{X}) := X_j, \quad j = 1, \dots, h,$$
$$L_{0,\infty}(\boldsymbol{X}) := \xi D_h\, X_h + \cdots + \xi D_1 X_1 - X_0,$$

and, for every prime number ℓ in S,

$$L_{j,\ell}(\boldsymbol{X}) := X_j, \quad j = 0, \dots, h.$$

For the tuple

$$\boldsymbol{b}_i = (w_{m_i}, v_{m_i} b^{n_{h,i}}, \dots, v_{m_i} b^{n_{2,i}}, v_{m_i} b^{n_{1,i}}),$$

with a sufficiently large m_i, we use (2-2) and (2-3) to obtain

$$\begin{aligned}\prod_{j=0}^{h} |L_{j,\infty}(\boldsymbol{b}_i)| \times \prod_{\ell \in S} \prod_{j=0}^{h} |L_{j,\ell}(\boldsymbol{b}_i)|_\ell &\leq \|\xi u_{m_i}\| \cdot |v_{m_i}|^h \\ &< |u_{m_i}|^{-(h-1/2)\,\varepsilon} \ll H(\boldsymbol{b}_i)^{-(h-1/2)\,\varepsilon},\end{aligned}$$

where the implied constant is independent of i and $H(\boldsymbol{b}_i)$ is the Weil height of the projective point $[w_{m_i} : v_{m_i} b^{n_{h,i}} : \cdots : v_{m_i} b^{n_{1,i}}]$.

The subspace theorem [5, Corollary 7.2.5] implies that there exist integers $t_0, t_1, \dots, t_h$, not all zero, and an infinite subset $\mathcal{N}_3$ of $\mathcal{N}_2$ such that

$$\text{(2-4)}\quad v_{m_i}(t_h\, b^{n_{h,i}} + \cdots + t_1\, b^{n_{1,i}}) + t_0\, w_{m_i} = 0 \quad \text{for } (w_{m_i}, n_{h,i}, \dots, n_{1,i}) \in \mathcal{N}_3.$$

Dividing the above equation by u_{m_i} and letting i tend to infinity, we deduce that

$$\frac{t_h}{D_h} + t_0\, \xi = 0.$$

Since ξ is irrational, we must have $t_0 = t_h = 0$. Then, we use (2-1) and (2-4) to derive that $t_1 = \cdots = t_{h-1} = 0$, a contradiction. This finishes the proof of (i).

We now prove part (ii) using a similar method. Let $s \geq 0$ and let x be a positive integer such that $\mathcal{DC}(x, b) = s$. If $s = 0$, we can write

$$x = d + db + \cdots + db^n = \frac{db^{n+1} - d}{b-1},$$

with $n \geq 0$ and $d \in \{1, \dots, b-1\}$. If $s > 0$, let $0 < c_1 < c_2 < \cdots < c_s$ denote the exponents of b where digit changes take place:

$$\begin{aligned}x &= d_0(1 + \cdots + b^{c_1 - 1}) + d_1(b^{c_1} + \cdots + b^{c_2 - 1}) + \cdots + d_s(b^{c_s} + \cdots + b^n) \\ &= \frac{-d_0 + (d_0 - d_1)\, b^{c_1} + (d_1 - d_2)\, b^{c_2} + \cdots + (d_{s-1} - d_s)\, b^{c_s} + d_s\, b^{n+1}}{b-1},\end{aligned}$$

with $n \geq c_s$, $d_0, \dots, d_s \in \{0, \dots, b-1\}$, and $d_{i+1} \neq d_i$ for $0 \leq i \leq s-1$.

Let $\mathcal{N}_4$ be the set of tuples $(m, n_0, n_1, \dots, n_a)$ such that:

- $0 \leq a \leq k+1$ and $n_0 < \ldots < n_a$ are nonnegative integers.
- There exist integers $e_0, \dots, e_a$ in $[-(b-1), b-1]$ such that

  $$\delta := \frac{e_0\, b^{n_0} + \cdots + e_{k+1}\, b^{n_{k+1}}}{b-1}$$

  is a divisor of u_m and $\delta \geq u_m^{(k+2-\lambda)/(k+2)+\varepsilon}$.

Assume that $\mathcal{N}_4$ is infinite. Then, there exist an integer h with $0 \le h \le k+1$, nonzero integers $E_0, \ldots, E_h$, an infinite set $\mathcal{N}_5$ of $(h+2)$-tuples $(m_i, n_{h,i}, \ldots, n_{0,i})$ for $i \ge 1$ such that:

- $n_{0,i} < \ldots < n_{h,i}$ are nonnegative integers.
- For $i \ge 1$,
$$\delta_{m_i} := \frac{E_h\, b^{n_{h,i}} + \cdots + E_0\, b^{n_{0,i}}}{b-1}$$
is a divisor of u_{m_i} with $\delta_{m_i} \ge u_{m_i}^{(k+2-\lambda)/(k+2)+\varepsilon}$.
- We have
$$\lim_{i \to +\infty} (n_{j,i} - n_{j-1,i}) = +\infty, \qquad j = 1, \ldots, h.$$

We can now apply the subspace theorem in essentially the same way as before to finish the proof. □

3. Proof of Theorem 1.1 and Corollary 1.3

In Corollary 1.3, the equivalence (iii) ⇔ (iv) and the implications (iv) ⇒ (i) and (iv) ⇒ (ii) are well known and have already appeared in Theorem 1.2. The implication (ii) ⇒ (iv) is essentially the last assertion of Theorem 1.1 while the remaining implication (i) ⇒ (iv) follows from the inequality $\| p_k/\xi \| \ll_\xi |p_k|^{-1}$ and Theorem 1.1 again. We spend the rest of this section to discuss Theorem 1.1.

From now on $\mathbb{N}$ is the set of positive integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\boldsymbol{\mu}$ is the group of roots of unity, and $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Let h denote the absolute logarithmic Weil height on $\overline{\mathbb{Q}}$. Let $k \in \mathbb{N}$, a tuple $(\alpha_1, \ldots, \alpha_k)$ of nonzero complex numbers is called nondegenerate if $\alpha_i/\alpha_j \notin \boldsymbol{\mu}$ for $1 \le i \ne j \le k$. We consider the following more general family of sequences than (nondegenerate) linear recurrence sequences:

Definition 3.1. Let K be a number field. Let $\mathscr{S}(K)$ be the set of all sequences $(u_n)_{n\ge 1}$ of complex numbers with the following property. There exist $k \in \mathbb{N}_0$ together with a nondegenerate tuple $(\alpha_1, \ldots, \alpha_k) \in (K^*)^k$ such that, when n is sufficiently large, we can express

$$u_n = q_{n,1}\, \alpha_1^n + \cdots + q_{n,k}\, \alpha_k^n \tag{3-1}$$

for $q_{n,1}, \ldots, q_{n,k} \in K^*$ and $\max_{1\le i\le k} h(q_{n,i}) = o(n)$.

In Definition 3.1, we allow $k = 0$ for which the empty sum in the right-hand side of (3-1) means 0. Any sequence $(u_n)_{n\ge 1}$ that is eventually 0 is in $\mathscr{S}(K)$.

Example 3.2. Consider a linear recurrence sequence $(v_n)_{n\ge 1}$ of the form

$$v_n = P_1(n)\, r_1^n + \cdots + P_k(n)\, r_k^n,$$

with $k \in \mathbb{N}$, distinct $r_1, \ldots, r_k \in K^*$, and nonzero $P_1, \ldots, P_k \in K[X]$. Let L be the lcm of the order of the roots of unity that appear among the r_i/r_j for $1 \le i, j \le k$. Then each one of the L sequences $(v_{nL+r})_{n\ge 1}$ for $r = 0, \ldots, L-1$ is a member of $\mathscr{S}(K)$.

As an explicit example, consider $v_n = 2^n + (-2)^n + n$ for $n \in \mathbb{N}$. The sequence $(v_{2n} = 2\cdot 4^n + 2n)_{n\ge 1}$ is in $\mathscr{S}(\mathbb{Q})$ and a tuple $(\alpha_1, \ldots, \alpha_k)$ satisfying the requirement in Definition 3.1 is $(\alpha_1 = 4,\ \alpha_2 = 1)$. The sequence $(v_{2n+1} = 2n+1)_{n\ge 1}$ is in $\mathscr{S}(\mathbb{Q})$ and a tuple $(\alpha_1, \ldots, \alpha_k)$ satisfying the requirement in Definition 3.1 is $(\alpha_1 = 1)$.

Lemma 3.3. *Let K be a number field and let $(u_n)_{n\ge 1}$ be an element of $\mathscr{S}(K)$. Let $k, \ell \in \mathbb{N}_0$ and let $(\alpha_1, \ldots, \alpha_k)$ and $(\beta_1, \ldots, \beta_\ell)$ be nondegenerate tuples of nonzero elements of K. Suppose that when n is sufficiently large, we can express*

$$u_n = q_{n,1}\, \alpha_1^n + \cdots + q_{n,k}\, \alpha_k^n = r_{n,1}\, \beta_1^n + \cdots + r_{n,\ell}\, \beta_\ell^n$$

for $q_{n,1}, \ldots, r_{n,\ell} \in K^*$ such that

$$\max\{h(q_{n,i}), h(r_{n,j}) : 1 \le i \le k,\ 1 \le j \le \ell\} = o(n)$$

as n tends to infinity. Then $k = \ell$ and there exist a permutation σ of $\{1, \ldots, k\}$ together with roots of unity $\zeta_1, \ldots, \zeta_k$ in K such that $\alpha_i = \zeta_i\, \beta_{\sigma(i)}$ for $1 \le i \le k$ and $q_{n,i}\, \zeta_i^n = r_{n,\sigma(i)}$ for every sufficiently large n and for every $1 \le i \le k$.

Proof. This follows from [20, Proposition 2.2]. □

Definition 3.4. Let K be a number field and let $(u_n)_{n\ge 1}$ be in $\mathscr{S}(K)$. Let $(\alpha_1, \ldots, \alpha_k)$ satisfy the requirement in Definition 3.1. We call k the number of $\mathscr{S}(K)$-roots of $(u_n)_{n\ge 1}$; this is well defined, thanks to Lemma 3.3. We call $(\alpha_1, \ldots, \alpha_k)$ a tuple of $\mathscr{S}(K)$-roots of $(u_n)_{n\ge 1}$; this is well defined up to permuting the α_i's and multiplying each α_i by a root of unity in K.

Here is the reason why we use the strange terminology "$\mathscr{S}(K)$-roots" instead of the usual "characteristic roots". In the theory of linear recurrence sequences, we have the well defined notion of characteristic roots. For example, *the* characteristic roots of $(u_n = 2^n + 1)_{n\ge 1}$ are 2 and 1. When regarding $(u_n)_{n\ge 1}$ as an element of $\mathscr{S}(K)$, we may say that any tuple $(2\zeta, \zeta')$ where ζ and ζ' are roots of unity in K is a tuple of $\mathscr{S}(K)$-roots of $(u_n)_{n\ge 1}$.

Definition 3.5. Let K be a number field. Let $(u_n)_{n\ge 1}$ be an element of $\mathscr{S}(K)$ and let $k \in \mathbb{N}_0$ be its number of $\mathscr{S}(K)$-roots. We say that $(u_n)_{n\ge 1}$ is admissible if

- either $k = 0$, i.e., $(u_n)_{n\ge 1}$ is eventually 0,
- or $k > 0$ and at least one entry in a tuple of $\mathscr{S}(K)$-roots of $(u_n)_{n\ge 1}$ is not a root of unity.

Since every nondegenerate linear recurrence sequence of algebraic numbers that is not a polynomial sequence is an admissible element of $\mathscr{S}(K)$ for some number field K, Theorem 1.1 follows from the below theorem.

Theorem 3.6. *Let ξ be an algebraic number of degree $d \geq 3$. Let $\varepsilon > 0$ and let K be a number field. Let $(u_n)_{n\geq 1}$ be a sequence of integers that is also an admissible element of $\mathscr{S}(K)$. Then the set*

$$\left\{ n \in \mathbb{N} : u_n \neq 0 \ \text{and} \ \|u_n \xi\| < \frac{1}{|u_n|^{(1/(d-1))+\varepsilon}} \right\}$$

is finite.

The proof of Theorem 3.6 relies on a result of Kulkarni et al. [20], which extends a seminal work of Corvaja and Zannier [13]. By a sublinear function, we mean a function $f : \mathbb{N} \to (0, \infty)$ such that $\lim_{n\to\infty} f(n)/n = 0$, that is, $f(n) = o(n)$. We need the following slightly more flexible version of [20, Theorem 1.4]:

Theorem 3.7. *Let $C \in (0, 1]$. Let K be a number field, let $k \in \mathbb{N}$, let $(\alpha_1, \ldots, \alpha_k)$ be a nondegenerate tuple of algebraic numbers satisfying $|\alpha_i| \geq C$ for $1 \leq i \leq k$, and let f be a sublinear function. Assume that for some $\theta \in (0, C)$, the set $\mathcal{M}$ of $(n, q_1, \ldots, q_k) \in \mathbb{N} \times (K^*)^k$ satisfying*

$$\left\| \sum_{i=1}^{k} q_i \alpha_i^n \right\| < \theta^n \ \text{and} \ \max_{1\leq i\leq k} h(q_i) < f(n)$$

is infinite. Then:

(i) *α_i is an algebraic integer for $i = 1, \ldots, k$.*
(ii) *For each $\sigma \in G_{\mathbb{Q}}$ and $i = 1, \ldots, k$ such that $\frac{\sigma(\alpha_i)}{\alpha_j} \notin \boldsymbol{\mu}$ for $j = 1, \ldots, k$, we have $|\sigma(\alpha_i)| < C$.*

Moreover, for all but finitely many $(n, q_1, \ldots, q_k) \in \mathcal{M}$ we have

$$\text{for } (\sigma, i, j) \in G_{\mathbb{Q}} \times \{1, \ldots, k\}^2, \quad \sigma(q_i \alpha_i^n) = q_j \alpha_j^n \ \text{if and only if} \ \frac{\sigma(\alpha_i)}{\alpha_j} \in \boldsymbol{\mu}.$$

Remark 3.8. Theorem 3.7 in the case $C = 1$ is exactly [20, Theorem 1.4].

Proof of Theorem 3.7. When n is fixed, there are only finitely many tuples $(n, q_1, \ldots, q_k)$ in $\mathcal{M}$, thanks to the upper bound on $\max h(q_i)$ and Northcott's property. In the following, for $(n, q_1, \ldots, q_k)$ in $\mathcal{M}$, we tacitly assume that n is sufficiently large.

For N large enough, we have $1/\theta^N > 3/C^N$ and the interval $[1/C^N, 1/\theta^N)$ contains a prime number D which does not divide the denominator of α_i for $i = 1, \ldots, k$. We have

$$D\theta^N < 1 \leq DC^N.$$

Fix $\theta' \in (D\theta^N, 1)$. Let $\beta_i = D\alpha_i^N$ for $1 \le i \le k$. We now define the set $\mathcal{M}'$ as follows. Consider $(n, q_1, \dots, q_k) \in \mathcal{M}$ with $n \equiv r \bmod N$, write $n = mN + r$ with $r \in \{0, \dots, N-1\}$, then we have

$$\Big\| \sum_{i=1}^{k} q_i\, \alpha_i^r \beta_i^m \Big\| = \Big\| \sum_{i=1}^{k} q_i\, \alpha_i^r (D\alpha_i^N)^m \Big\| < D^m \theta^n = \theta^r (D\theta^N)^m < \theta'^m,$$

assuming n and hence m are sufficiently large so that the last inequality holds, thanks to the choice $\theta' \in (D\theta^n, 1)$. We include the tuple $(m, q_1\, \alpha_1^r, \dots, q_k\, \alpha_k^r)$ in $\mathcal{M}'$. Finally, consider the sublinear function

$$g(n) = f(n) + (N-1) \max_{1 \le i \le k} h(\alpha_i),$$

so that $\max_{1 \le i \le k} h(q_i\, \alpha_i^r) < g(n)$.

We apply [20, Theorem 1.4] for the tuple $(\beta_1, \dots, \beta_k)$, the function g, the number θ', and the set $\mathcal{M}'$ to conclude that:

- $D\alpha_i^N$ is an algebraic integer for $1 \le i \le k$. Our choice of D implies that α_i is an algebraic integer for $1 \le i \le k$.
- For each $\sigma \in G_{\mathbb{Q}}$ and $i \in \{1, \dots, k\}$ such that $\frac{\sigma(\alpha_i)}{\sigma(\alpha_j)} \notin \boldsymbol{\mu}$ for every $j \in \{1, \dots, k\}$, we have $\sigma(D\alpha_i^N) < 1$ consequently $\sigma(\alpha_i) < 1/D^{1/N} \le C$.
- The last assertion of Theorem 3.7 holds.

This finishes the proof. □

Proof of Theorem 3.6. Let k denote the number of $\mathscr{S}(K)$-roots of $(u_n)_{n\ge1}$. The case $k = 0$ (i.e., $(u_n)_{n\ge1}$ is eventually 0) is obvious. Assume $k > 0$ and let $(\alpha_1, \dots, \alpha_k)$ be a tuple of $\mathscr{S}(K)$-roots of $(u_n)_{n\ge1}$. For $L \in \mathbb{N}$ and $r \in \{0, \dots, L-1\}$, each sequence $(u_{nL+r})_{n\ge1}$ is an admissible element of $\mathscr{S}(K)$ and admits $(\alpha_1^L, \dots, \alpha_k^L)$ as a tuple of $\mathscr{S}(K)$-roots. Let L be the lcm of the order of roots of unity among the $\sigma(\alpha_i)/\tau(\alpha_j)$ for $\sigma, \tau \in G_{\mathbb{Q}}$ and $1 \le i, j \le k$ and replace $(u_n)_{n\ge1}$ by each $(u_{nL+r})_{n\ge1}$, we may assume

$$\text{(3-2)}\quad \text{for } \sigma, \tau \in G_{\mathbb{Q}} \text{ and } 1 \le i, j \le k, \quad \frac{\sigma(\alpha_i)}{\tau(\alpha_j)} \in \boldsymbol{\mu} \text{ if and only if } \sigma(\alpha_i) = \tau(\alpha_j).$$

We first prove that the set $\{\alpha_1, \dots, \alpha_k\}$ is Galois invariant.

For sufficiently large n, express

$$u_n = q_{n,1}\, \alpha_1^n + \cdots + q_{n,k}\, \alpha_k^n$$

as in Definition 3.1. Let $\sigma \in G_{\mathbb{Q}}$, since $u_n \in \mathbb{Z}$ we have

$$q_{n,1}\, \alpha_1^n + \cdots + q_{n,k}\, \alpha_k^n = \sigma(q_{n,1})\, \sigma(\alpha_1)^n + \cdots + \sigma(q_{n,k})\, \sigma(\alpha_k)^n \quad \text{for all large } n.$$

From [20, Proposition 2.2], we have that for every $i \in \{1, \dots, k\}$ there exists $j \in \{1, \dots, k\}$ such that $\sigma(\alpha_i)/\alpha_j \in \boldsymbol{\mu}$ and this gives $\sigma(\alpha_i) = \alpha_j$, thanks to (3-2). Theorem 3.7 implies that the α_i's are algebraic integers and for every sufficiently large n, for $(\sigma, i, j) \in G_{\mathbb{Q}} \times \{1, \dots, k\}^2$ we have

$$\sigma(q_{n,i}) = q_{n,j}, \quad \text{whenever } \sigma(\alpha_i) = \alpha_j. \tag{3-3}$$

Since $(u_n)_{n\geq 1}$ is admissible, at least one of the α_i's is not a root of unity and hence

$$M := \max_{1\leq i\leq k} |\alpha_i| > 1. \tag{3-4}$$

Suppose the set

$$T := \left\{ n \in \mathbb{N} : u_n \neq 0 \text{ and } \|u_n\,\xi\| < \frac{1}{|u_n|^{(1/(d-1))+\varepsilon}} \right\}$$

is infinite, then we will arrive at a contradiction. Let δ denote a sufficiently small positive real number that will be specified later. By [20, Section 2], we have

$$|u_n| > M^{(1-\delta)\,n} \tag{3-5}$$

for all large n. Therefore

$$\|\xi\, q_{n,1}\, \alpha_1^n + \cdots + \xi\, q_{n,k}\, \alpha_k^n\| < \frac{1}{M^{(1-\delta)(1/(d-1)+\varepsilon)\,n}} \tag{3-6}$$

for all large n in T.

We relabel the α_i's and let $m \leq \ell \leq k$ such that

(i) $|\alpha_1| = M$.

(ii) $|\alpha_i| \geq \frac{1}{M^{1/(d-1)+\delta}}$ for $1 \leq i \leq \ell$ while $|\alpha_i| < \frac{1}{M^{1/(d-1)+\delta}}$ for $\ell+1 \leq i \leq k$.

(iii) Among the $\alpha_1, \dots, \alpha_\ell$, we have that $\alpha_1, \dots, \alpha_m$ are exactly the Galois conjugates of α_1. When combining with (ii), this means that $\alpha_1, \dots, \alpha_m$ are precisely the Galois conjugates of α_1 with modulus at least $M^{-(1/(d-1)+\delta)}$.

We require δ small enough so that

$$\frac{1}{d-1} + \delta < (1-\delta)\Bigl(\frac{1}{d-1} + \varepsilon\Bigr). \tag{3-7}$$

Choose the real number c such that:

$$\frac{1}{d-1} + \delta < c < (1-\delta)\Bigl(\frac{1}{d-1} + \varepsilon\Bigr) \quad \text{and} \quad |\alpha_i| < \frac{1}{M^c} \text{ for } \ell+1 \leq i \leq k. \tag{3-8}$$

Thanks to this choice of c and the assumption that $h(q_{n,i}) = o(n)$ for $1 \leq i \leq k$, we have

$$|\xi\, q_{n,\ell+1}\, \alpha_{\ell+1}^n + \cdots + \xi\, q_{n,k}\, \alpha_k^n| < \frac{1}{2M^{cn}} \tag{3-9}$$

for all sufficiently large n. From (3-6) and (3-8), we have

$$\|\xi\, q_{n,1}\, \alpha_1^n + \cdots + \xi\, q_{n,k}\, \alpha_k^n\| < \frac{1}{2M^{cn}} \tag{3-10}$$

for all large n in T. Combining the above inequalities, we have

$$\|\xi\, q_{n,1}\, \alpha_1^n + \cdots + \xi\, q_{n,\ell}\, \alpha_\ell^n\| < \frac{1}{M^{cn}} \tag{3-11}$$

for all large n in T.

Let F be the Galois closure of $K(\xi)$. We apply Theorem 3.7 for the tuple $(\alpha_1, \dots, \alpha_\ell)$, $C = M^{-(1/(d-1)+\delta)}$, $\theta = M^{-c}$, and the inequality (3-11) then use (3-2) and (3-3) to have that for every large n in T, $\sigma \in \mathrm{Gal}(F/\mathbb{Q})$, and $1 \le i, j \le \ell$,

$$\text{if } \sigma(\alpha_i) = \alpha_j, \text{ then } \sigma(\xi\, q_{n,i}\, \alpha_i^n) = \xi\, q_{n,j}\, \alpha_j^n \text{ and hence } \sigma(\xi) = \xi. \tag{3-12}$$

Since $\alpha_1, \dots, \alpha_m$ are exactly the Galois conjugates of α_1 among the $\alpha_1, \dots, \alpha_\ell$, equation (3-12) implies that ξ is fixed by at least $m|\,\mathrm{Gal}(F/\mathbb{Q}(\alpha_1))| = m[F : \mathbb{Q}(\alpha_1)]$ many automorphisms in $\mathrm{Gal}(F/\mathbb{Q})$. Put $d' = [\mathbb{Q}(\alpha_1) : \mathbb{Q}]$, we have

$$[F : \mathbb{Q}(\xi)] = |\,\mathrm{Gal}(F/\mathbb{Q}(\xi))| \ge m[F : \mathbb{Q}(\alpha_1)] = \frac{m}{d'}[F : \mathbb{Q}]. \tag{3-13}$$

Since $[\mathbb{Q}(\xi) : \mathbb{Q}] = d$, equation (3-13) implies $m \le d'/d$. This means α_1 has at least $d'(d-1)/d$ many Galois conjugates with modulus less than $M^{-(1/(d-1)+\delta)}$. Combining with the fact that all Galois conjugates of α_1 have modulus at most M, we have

$$|\,\mathrm{N}_{\mathbb{Q}(\alpha_1)/\mathbb{Q}}(\alpha_1)| \le M^{d'/d} M^{-(1/(d-1)+\delta)\, d'(d-1)/d} < 1,$$

since $M > 1$ and $\delta > 0$. This contradicts the fact that α_1 is a nonzero algebraic integer and we finish the proof. □

4. Proof of Theorem 1.5 and further discussion on Erdős and Mahler [16]

Proof of Theorem 1.5. We assume that θ is not a Liouville number, that is, we assume that μ is finite. Define

$$Q_k = q_{k-1}\, q_k\, q_{k+1}, \quad k \ge 2.$$

Let S be a finite set of prime numbers. Write $\theta = [a_0; a_1, a_2, \dots]$ and recall that

$$q_{k+1} = a_{k+1}\, q_k + q_{k-1}, \quad k \ge 2.$$

Let $k \ge 2$ and set $d_k = \gcd(q_{k-1}, q_{k+1})$. Since q_{k-1} and q_k are coprime, we see that d_k divides a_{k+1}. Define

$$q_{k-1}^* = q_{k-1}/d_k, \quad q_{k+1}^* = q_{k+1}/d_k, \quad a_{k+1}^* = a_{k+1}/d_k.$$

Then, we have

$$q_{k+1}^* = a_{k+1}^* \, q_k + q_{k-1}^*, \quad k \geq 2.$$

Let $\varepsilon > 0$. By the Schmidt subspace theorem, the set of points (q_{k-1}^*, q_{k+1}^*) such that

$$q_{k-1}^* \, q_{k+1}^* \prod_{p \in S} |q_{k-1}^* \, q_{k+1}^* (q_{k+1}^* - q_{k-1}^*)|_p < (q_{k+1}^*)^{-\varepsilon}$$

is contained in a union of finitely many proper subspaces. Since q_{k-1}^* and q_{k+1}^* are coprime, this set is finite. We deduce that, for k large enough, we get

$$\prod_{p \in S} |q_{k-1}^* \, q_{k+1}^* (q_{k+1}^* - q_{k-1}^*)|_p > (q_{k-1}^* \, q_{k+1}^*)^{-1} (q_{k+1}^*)^{-\varepsilon},$$

thus

$$\prod_{p \in S} |q_{k-1} \, q_{k+1} (a_{k+1}^* \, q_k)|_p > (q_{k-1} \, q_{k+1})^{-1} (q_{k+1}^*)^{-\varepsilon},$$

and hence

$$\prod_{p \in S} |q_{k-1} \, q_{k+1} \, q_k|_p > (q_{k-1} \, q_{k+1})^{-1} (q_{k+1})^{-\varepsilon}.$$

Recalling that $q_{k-1} < q_k$ and $q_{k+1} < q_k^{\mu-1+\varepsilon}$ for k large enough, we get

$$[Q_k]_S < q_{k-1} \, q_{k+1}^{1+\varepsilon} < q_k^{\mu+\varepsilon} \, Q_k^{\varepsilon}.$$

Since

$$Q_k < q_k^2 \, q_{k+1} < q_k^{\mu+1+\varepsilon},$$

we get

$$[Q_k]_S < Q_k^{(\mu+\varepsilon)/(\mu+1+\varepsilon)} \, Q_k^{\varepsilon}.$$

This proves (1-1). The last assertion can be proved in the same manner, thanks to the identity $p_{k+1} = a_{k+1} \, p_k + p_{k-1}$ and the inequalities

$$|p_{k-1}| < |p_k| \quad \text{and} \quad |p_{k+1}| < |p_k|^{\mu-1+\varepsilon}$$

for large k. □

The following was suggested at the end of [16]:

Question 4.1 (Erdős and Mahler [16]). *Let θ be an irrational real number such that the largest prime factor of $p_n(\theta) \, q_n(\theta)$ is bounded for infinitely many n. Is it true that θ is a Liouville number*?

Without further details, Erdős and Mahler stated the existence of θ with the given properties in Question 4.1. We provide a construction here for the sake of completeness.

Let S and T be disjoint nonempty sets of prime numbers such that S has at least two elements. We construct uncountably many θ such that for infinitely many n the prime factors of $p_n(\theta)$ belong to S while the prime factors of $q_n(\theta)$ belong to T. To simplify the notation, we consider the case $S = \{2, 3\}$ and $T = \{5\}$. The construction for general S and T follows the same method. The constructed numbers θ have the form

$$\theta=\sum_{i=1}^{+\infty}\frac{a_i}{5^{3^i}}.$$

Let Γ be the set of positive integers with only prime factors in $\{2, 3\}$. For every positive integer m, let $\gamma(m)$ denote the smallest element of Γ that is greater than m. Let $f(m):=\frac{\gamma(m)-m}{m}$. By [29], we have

$$\lim_{m\to+\infty} f(m)=0. \tag{4-1}$$

First, we construct the sequence of positive integers $s(1)<s(2)<\dots$ recursively:

- $s(1)=1$.
- After having $s(1),\dots,s(k)$, let N_k be a positive integer depending on $s(k)$ such that

$$f(m)<\frac{1}{5^{3^{s(k)+1}}}\quad\text{for } m\geq N_k. \tag{4-2}$$

Then we choose $s(k+1)$ so that

$$5^{2\cdot 3^{s(k+1)-1}}\geq N_k\quad\text{and}\quad s(k+1)>s(k)+1. \tag{4-3}$$

Now we construct the a_i's:

- $a_1=1$.
- Choose arbitrary $a_i\in\{1,2\}$ for $i\notin\{s(1),s(2),\dots\}$. Since $s(k+1)>s(k)+1$ for every k, the set $\mathbb{N}\setminus\{s(1),s(2),\dots\}$ is infinite. Hence there are uncountably many choices here.
- Since $s(1)=1$, we already had $a_{s(1)}$. Suppose we have $a_{s(1)},\dots,a_{s(k)}$ positive integers with the following properties:

  (i) For $1\leq j\leq k$, we have $\sum_{i=1}^{s(j)}\frac{a_i}{5^{3^i}}=\frac{u_{s(j)}}{5^{3^{s(j)}}}$ with $u_{s(j)}\in\Gamma$.

  (ii) For $2\leq j\leq k$, we have $\frac{a_{s(j)}}{5^{3^{s(j)}}}<\frac{1}{5^{3^{s(j-1)+1}}}$.

We now define $a_{s(k+1)}$ so that the above two properties continue to hold with $j=k+1$ as well. Thanks to property (ii) and the fact that $a_i\leq 2$ for $i\notin\{s(1),s(2),\dots\}$, we have the rough estimate

$$\frac{u}{5^{3^{s(k+1)-1}}}:=\sum_{i=1}^{s(k+1)-1}\frac{a_i}{5^{3^i}}\leq\sum_{i=1}^{s(k+1)-1}\frac{2}{5^{3^i}}+\sum_{j=1}^{k-1}\frac{1}{5^{3^{s(j)+1}}}<1,$$

and hence, $u < 5^{3^{s(k+1)-1}}$. From

$$\sum_{i=1}^{s(k+1)} \frac{a_i}{5^{3^i}} = \frac{u}{5^{3^{s(k+1)-1}}} + \frac{a_{s(k+1)}}{5^{3^{s(k+1)}}} = \frac{u \cdot 5^{2 \cdot 3^{s(k+1)-1}} + a_{s(k+1)}}{5^{3^{s(k+1)}}},$$

we now define

$$a_{s(k+1)} = \gamma(u \cdot 5^{2 \cdot 3^{s(k+1)-1}}) - u \cdot 5^{2 \cdot 3^{s(k+1)-1}}.$$

Recall that $\gamma(u \cdot 5^{2 \cdot 3^{s(k+1)-1}})$ is the smallest element of Γ that is greater than $u \cdot 5^{2 \cdot 3^{s(k+1)-1}}$. This verifies property (i) for $j = k+1$. To verify (ii) for $j = k+1$, we have

$$\begin{aligned}
\frac{a_{s(k+1)}}{5^{3^{s(k+1)}}} &= \frac{\gamma(u \cdot 5^{2 \cdot 3^{s(k+1)-1}}) - u \cdot 5^{2 \cdot 3^{s(k+1)-1}}}{u \cdot 5^{2 \cdot 3^{s(k+1)-1}}} \cdot \frac{u \cdot 5^{2 \cdot 3^{s(k+1)-1}}}{5^{3^{s(k+1)}}} \\
&< \frac{\gamma(u \cdot 5^{2 \cdot 3^{s(k+1)-1}}) - u \cdot 5^{2 \cdot 3^{s(k+1)-1}}}{u \cdot 5^{2 \cdot 3^{s(k+1)-1}}} \qquad [\text{since } u < 5^{3^{s(k+1)-1}}] \\
&= f(u \cdot 5^{2 \cdot 3^{s(k+1)-1}}) \\
&< \frac{1}{5^{3^{s(k)+1}}} \qquad \text{by (4-2) and (4-3).}
\end{aligned}$$

By the principle of recursive definition, we have a_i for $i \in \{s(1), s(2), \dots\}$ such that property (i) holds for every $j \geq 1$ and property (ii) holds for every $j \geq 2$.

Write $u_n/v_n = \sum_{i \leq n} \frac{a_i}{5^{3^i}}$ with $v_n = 5^{3^n}$. We have

$$|\theta - u_{s(k)}/v_{s(k)}| = \sum_{i=s(k)+1}^{\infty} \frac{a_i}{5^{3^i}} < \sum_{i=s(k)+1}^{\infty} \frac{2}{5^{3^i}} + \sum_{j=k}^{\infty} \frac{1}{5^{3^{s(j)+1}}} < \frac{4}{5^{3^{s(k)+1}}} = \frac{4}{v_{s(k)}^3}.$$

Therefore the $u_{s(k)}/v_{s(k)}$ are among the convergents to θ.

It is not clear to us whether the above numbers θ are always Liouville numbers. However, we suspect that this is the case. In order to construct Liouville numbers, we can use a similar method for numbers of the form $\sum_{i \geq 1} \frac{b_i}{5^{i!}}$.

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