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For any finite-dimensional vector space $\mathcal{F}$ of continuous functions $f : \mathbb{R}^1 \to \mathbb{R}^1$ we consider subspaces in $\mathcal{F}$ defined by systems of *equality conditions* $f(a_i) = f(b_i)$, where $\{a_i, b_i\},\ i = 1, \dots, n$, are some pairs of points in $\mathbb{R}^1$. It is proven that if $\dim \mathcal{F} < 2n - I(n)$, where $I(n)$ is the number of ones in the binary notation of n, then there necessarily exist independent systems of n equality conditions defining the subspaces of codimension greater than n in $\mathcal{F}$. We also prove lower estimates of the sizes of the inevitable drops of the codimensions of some of these subspaces.

Next, we apply these estimates to knot theory (in which systems of equality conditions are known as *chord diagrams*) and prove the inevitable presence of complicated nonstable terms in sequences of spectral sequences computing cohomology groups of spaces of knots.

1. Main results

Let $\mathcal{F}^N$ be an N-dimensional vector subspace of the space $C^0(\mathbb{R}^1, \mathbb{R}^1)$ of continuous functions $\mathbb{R}^1 \to \mathbb{R}^1$. Typically, a collection of n independent conditions of the form

$$f(a_i) = f(b_i), \tag{1}$$

where $a_i \neq b_i,\ i = 1, \dots, n$, defines a subspace of codimension n in $\mathcal{F}^N$ if $n \leq N$ and only the trivial subspace if $n \geq N$. However, for exceptional sets of such conditions, the codimensions of these subspaces can drop.

For example, if $\mathcal{F}^N$ is the space $\mathcal{P}^N$ of all polynomials of the form

$$\alpha_1 x^N + \alpha_2 x^{N-1} + \cdots + \alpha_N x \tag{2}$$

in the variable x, then all subspaces defined by arbitrarily many conditions

$$f(a_i) = f(-a_i)$$

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contain the $\left[\frac{N}{2}\right]$-dimensional subspace of even polynomials. Of course, the case of polynomials is very specific, but the situation when the dimensions of subspaces in $\mathcal{F}^N$ defined by some n independent conditions (1) are greater than $\max(N-n, 0)$ can be unavoidable by a choice of space $\mathcal{F}^N$.

Definition 1. An unordered pair $\{a, b\}$ of distinct points in $\mathbb{R}^1$ is called a *chord*. An unordered collection of n pairwise distinct chords is called an *n-chord diagram*.

The *subalgebra* of $C^0(\mathbb{R}^1, \mathbb{R}^1)$ corresponding to the n-chord diagram

$$\{\{a_1, b_1\}, \dots, \{a_n, b_n\}\} \tag{3}$$

consists of all functions satisfying n conditions (1) with these a_i, b_i. These n conditions (and the corresponding n chords) are *independent* if the codimension of this subalgebra in $C^0(\mathbb{R}^1, \mathbb{R}^1)$ is equal to n. (We say that an affine or vector subspace $\mathcal{T}$ of a function space $\mathcal{K}$ has codimension n if for any point $\varphi \in \mathcal{T}$ there exist n-dimensional affine subspaces in $\mathcal{K}$ intersecting $\mathcal{T}$ at this point only, and all affine subspaces of higher dimensions passing through φ intersect $\mathcal{T}$ along subspaces of positive dimensions.) Two independent n-chord diagrams are *equivalent* if the corresponding subalgebras in $C^0(\mathbb{R}^1, \mathbb{R}^1)$ coincide. A *resonance* of a chord diagram is a cyclic sequence of $k \geq 3$ its pairwise different chords such that one point of each chord also belongs to the preceding chord in this sequence, and the other its point also belongs to the next chord.

For example, two chord diagrams are equivalent if one of them contains the chords $\{a, b\}$ and $\{b, c\}$, the other contains the chords $\{a, b\}$ and $\{a, c\}$, and all other chords in them are common.

Proposition 2. *An n-chord diagram is independent if and only if it does not contain resonances. Two independent n-chord diagrams are equivalent if and only if they can be connected by a chain of elementary flips described in the previous paragraph. The space of independent n-chord diagrams is a smooth connected $2n$-dimensional manifold.*

Proof. The proof is elementary. □

1.1. *Results for the case of $N \geq n$.*

Proposition 3. *If $N \geq 2n-1$, then the codimension of the subspace in the space $\mathcal{P}^N$ of polynomials* (2), *defined by n conditions* (1) *of an arbitrary independent n-chord diagram* (3), *is equal to n.*

Proof. First, the assertion of our proposition will be true if we replace in it the space $\mathcal{P}^N$ by the $(N+1)$-dimensional space $\hat{\mathcal{P}}^N$ of all polynomials of degree N. Indeed, any n-chord diagram has at most $2n$ distinct endpoints a_i, b_i, therefore by interpolation theorem the evaluation morphism from the space of such polynomials to the space of real-valued functions on the set of these endpoints is epimorphic, and hence the preimage of any subspace of codimension n of the latter space also has codimension n in $\hat{\mathcal{P}}^N$. However, adding the constant functions preserves the subspace of $\hat{\mathcal{P}}^N$ defined by any chord diagram, therefore the codimension of the considered subspace in $\mathcal{P}^N$ is also equal to n. $\square$

Denote by $I(n)$ the number of ones in the binary notation of n.

Theorem 4. *If $n \le N < 2n - I(n)$, then for any N-dimensional vector subspace $\mathcal{F}^N \subset C^0(\mathbb{R}^1, \mathbb{R}^1)$ there exist independent n-chord diagrams* (3) *such that the codimension of the subspace in $\mathcal{F}^N$ consisting of functions satisfying all the corresponding conditions* (1) *is less than n. The dimension of the set of such exceptional n-chord diagrams is at least $3n - N - 1$ in the following exact sense*: *there exists a nontrivial element of the $(N - n + 1)$-dimensional homology group of the $2n$-dimensional manifold of all independent n-chord diagrams*, *such that each cycle representing this element necessarily intersects our set.*

In particular, if n is a power of 2 then the minimal dimension of the function spaces $\mathcal{F}^N$ in which any independent n-chord diagram defines a subspace of codimension exactly n is equal to $2n - 1$.

A more general result can be formulated in terms of configuration spaces; see, e.g., [1] for the current state of the theory of these spaces.

Definition 5. The *n-th configuration space $B(X, n)$* of a topological space X is the (naturally topologized) space of unordered subsets of cardinality n in X. The *regular bundle ξ_n* with base $B(X, n)$ is the vector bundle, whose fiber over an n-point configuration is the space of real-valued functions on the corresponding set of points.

Theorem 6. *Suppose that $N \ge n$ and for some natural r the cohomological product*

$$\prod_{i=1}^{r} w_{N-n+2i-1}(\xi_n) \tag{4}$$

of Stiefel–Whitney classes of the regular bundle ξ_n is not equal to 0 *in the ring $H^*(B(\mathbb{R}^2, n), \mathbb{Z}_2)$. Then for any N-dimensional vector subspace $\mathcal{F}^N$ of the space $C^0(\mathbb{R}^1, \mathbb{R}^1)$ there exists an independent system of n conditions* (1) *such that the subspace of $\mathcal{F}^N$ defined by this system has codimension $\le n - r$ in $\mathcal{F}^N$.*

The first statement of Theorem 4 follows immediately from this theorem (the case $r = 1$) and statement 5.3 of [3] asserting that the classes

$$w_k \in H^k(B(\mathbb{R}^2, n), \mathbb{Z}_2)$$

are nontrivial for all $k \le n - I(n)$; see also Proposition 30 in Section 6 below. The second statement of Theorem 4 will be proven at the end of Section 3.

Corollary 7. A. *If two natural numbers n and N satisfy one of the following pairs of conditions*:

(1) $n \geq 6,\ N = n$,

(2) $n \geq 10,\ N = n+1$,

(3) $n \geq 14,\ N = n+2$ *or* $n+3$,

(4) $n \geq 16,\ N = n+4$,

(5) $n \geq 18,\ N = n+5$,

(6) $n \geq 20,\ N = n+6$,

(7) $n \geq 24,\ N = n+7$,

(8) $n \geq 28,\ N = n+8$ *or* $n+9$,

(9) $n \geq 32,\ N = n+10$ *or* $n+11$,

then for any N-dimensional vector subspace $\mathcal{F}^N \subset C^0(\mathbb{R}^1, \mathbb{R}^1)$ there exists a system of n independent conditions (1) *defining a subspace of codimension $\leq n-2$ in $\mathcal{F}^N$*.

B. *If n and N satisfy one of the following pairs of conditions*:

(1) $n \geq 18,\ N = n$ *or* $n+1$,

(2) $n \geq 22,\ N = n+2$,

(3) $n \geq 26,\ N = n+3$,

(4) $n \geq 30,\ N = n+4$,

(5) $n \geq 36,\ N = n+5$,

(6) $n \geq 40,\ N = n+6$ *or* $n+7$,

(7) $n \geq 44,\ N = n+8$ *or* $n+9$,

then for any N-dimensional subspace $\mathcal{F}^N \subset C^0(\mathbb{R}^1, \mathbb{R}^1)$ there exists a system of n independent conditions (1) *defining a subspace of codimension $\leq n-3$ in $\mathcal{F}^N$*.

C. *If n and N satisfy one of the following conditions*:

(1) $n \geq 30$ *and* $N = n$ *or* $n+1$,

(2) $n \geq 44$ *and* $N = n+2$ *or* $n+3$,

(3) $n \geq 52$ *and* $N = n+4$ *or* $n+5$,

(4) $n \geq 56$ *and* $N = n+6$ *or* $n+7$,

then for any N-dimensional subspace $\mathcal{F}^N \subset C^0(\mathbb{R}^1, \mathbb{R}^1)$ there exists a system of n independent conditions (1) *defining a subspace of codimension $\leq n-4$ in $\mathcal{F}^N$*.

D. *If n and N satisfy one of the following conditions*:

(1) $n \geq 48$ *and* $N = n$ *or* $n+1$,

(2) $n \geq 60$ *and* $N = n+2$ *or* $n+3$,

(3) $n \geq 68$ *and* $N = n+4$ *or* $n+5$,

then for any N*-dimensional subspace* $\mathcal{F}^N \subset C^0(\mathbb{R}^1, \mathbb{R}^1)$ *there exists a system of* n *independent conditions* (1) *defining a subspace of codimension* $\leq n-5$ *in* $\mathcal{F}^N$.

E. *If* n *and* N *satisfy one of the following conditions*:

(1) $n \geq 64$ *and* $N = n$ *or* $n+1$,

(2) $n \geq 76$ *and* $N = n+2$ *or* $n+3$,

then for any N*-dimensional subspace* $\mathcal{F}^N \subset C^0(\mathbb{R}^1, \mathbb{R}^1)$ *there exists a system of* n *independent conditions* (1) *defining a subspace of codimension* $\leq n-6$ *in* $\mathcal{F}^N$.

F. *If we have* $n \geq 80$ *and* $N = n$ *or* $n+1$, *then for any* N*-dimensional subspace* $\mathcal{F}^N \subset C^0(\mathbb{R}^1, \mathbb{R}^1)$ *there exists a system of* n *independent conditions* (1) *defining a subspace of codimension* $\leq n-7$ *in* $\mathcal{F}^N$.

See Section 6 for the proof of this corollary. Its lists can easily be continued and the corresponding calculations can be programmed.

Remark. The first statement of Theorem 4 looks very similar (and is closely related) to the result of [2] estimating the dimensions of spaces of functions $\mathbb{R}^2 \to \mathbb{R}^1$ realizing n-regular embeddings of the plane. The main effort of our proof of Theorem 6 is a comparison of the configuration spaces used in these two problems, see Lemma 16 below.

1.2. *Results for the case of* $N \leq n$.

Theorem 8. *If* $N \leq n$ *and for some natural* r *the product*

$$\prod_{i=1}^{r} w_{n-N+2i-1}(\xi_n) \tag{5}$$

of Stiefel–Whitney classes of the bundle ξ_n *is not equal to* 0 *in the ring*

$$H^*(B(\mathbb{R}^2, n), \mathbb{Z}_2),$$

then for any N*-dimensional vector subspace* $\mathcal{F}^N \subset C^0(\mathbb{R}^1, \mathbb{R}^1)$ *there exists an independent* n*-chord diagram, such that the subspace of* $\mathcal{F}^N$ *consisting of functions satisfying the corresponding system of equality conditions is at least* r*-dimensional.*

If $N = n$, then Theorems 6 and 8 coincide tautologically.

Corollary 9. *If* $N \geq 2$, *then for any* N*-dimensional vector subspace* $\mathcal{F}^N \subset C^0(\mathbb{R}^1, \mathbb{R}^1)$ *there exist independent* n*-chord diagrams with arbitrarily large* n *such that the corresponding systems of equality conditions have nontrivial solutions in* $\mathcal{F}^N$.

Indeed, it is enough to prove this for $N = 2$ and numbers n equal to powers of 2. In this case $w_{n-N+1}(\xi_n) \neq 0$ by the previously mentioned result in [3]. $\square$

Remark. This corollary has also an elementary proof. Indeed, any 2-dimensional subspace of $C^0(\mathbb{R}^1, \mathbb{R}^1)$ contains a nonzero function taking equal values at some two different points $a, b \in \mathbb{R}^1$. Then this function necessarily satisfies the equality conditions $f(\tilde{a}) = f(\tilde{b})$ for a continuum of different pairs $\{\tilde{a}, \tilde{b}\} \subset [a, b]$.

Corollary 10. *All statements of Corollary 7 will remain valid if in each of its conditions we replace the value of N by $2n - N$ (e.g., $N = n + 4$ by $N = n - 4$) and simultaneously the corresponding conclusion "there exists a system of n independent conditions* (1) *defining a subspace of codimension $\leq n - r$ in $\mathcal{F}^N$" by "there exists a system of n independent conditions* (1) *defining a subspace of dimension $\geq r$ in $\mathcal{F}^N$".*

Remark. In terms of [6], the subspaces of anomalous codimensions defined by chord diagrams in finite-dimensional function spaces are responsible for the nonstable regions of the (p, q)-planes of the spectral sequences converging to cohomology groups of spaces of long knots $\mathbb{R}^1 \to \mathbb{R}^3$ defined by functions from these function spaces. These domains are the only possible sources of cohomology classes of the knot space (including 0-dimensional classes, i.e., knot invariants) not of finite-type. In Section 7 below, we prove some facts about filtrations of simplicial resolutions of discriminant spaces in finite-dimensional knot spaces, estimating the deviation of the corresponding spectral sequences from stable ones.

2. Scheme of proof of Theorem 6

Denote by CD_n the set of equivalence classes of independent n-chord diagrams. It has a natural topology induced by the topology of the variety of subalgebras of codimension n in $C^0(\mathbb{R}^1, \mathbb{R}^1)$. To describe this topology without infinite-dimensional considerations, let $\mathcal{A}$ be a sufficiently large finite-dimensional vector subspace of $C^0(\mathbb{R}, \mathbb{R})$, such that all subspaces of $\mathcal{A}$ defined by independent n-chord diagrams (that is, the intersections of $\mathcal{A}$ with subalgebras of $C^0(\mathbb{R}^1, \mathbb{R}^1)$ corresponding to these chord diagrams) have codimension exactly n in $\mathcal{A}$, and the nonequivalent n-chord diagrams define different subspaces. (For reasons similar to the proof of Proposition 3 we can take for such a space $\mathcal{A}$ the space $\mathcal{P}^M$, $M \geq 2n + 1$, or any space containing it; taking $\mathcal{P}^{2n-1}$ is not enough because nonequivalent n-chord diagrams can define equal subspaces in it). We can and will assume that $\mathcal{A}$ contains $\mathcal{F}^N$, because otherwise we can replace $\mathcal{A}$ by its sum with $\mathcal{F}^N$.

The set CD_n is embedded into the Grassmann manifold $G(\mathcal{A}, -n)$ of subspaces of codimension n in $\mathcal{A}$, and inherits a topology from this manifold. It is easy to see that this definition of a topology on CD_n does not depend on the choice of $\mathcal{A}$.

Suppose that $N \geq n$. Let $\Delta_r(\mathcal{F}^N) \subset G(\mathcal{A}, -n)$ be the set of all subspaces of codimension n in $\mathcal{A}$ whose sums with $\mathcal{F}^N$ have codimension at least r in $\mathcal{A}$.

Proposition 11. *The class in $H^*(G(\mathcal{A}, -n), \mathbb{Z}_2)$ Poincaré dual to the homology class of the algebraic variety $\Delta_r(\mathcal{F}^N)$ is equal to $r \times r$ determinant*:

$$
(6) \qquad \begin{vmatrix}
w_{N-n+r} & w_{N-n+r+1} & w_{N-n+r+2} & \dots & w_{N-n+2r-2} & w_{N-n+2r-1} \\
w_{N-n+r-1} & w_{N-n+r} & w_{N-n+r+1} & \dots & w_{N-n+2r-3} & w_{N-n+2r-2} \\
w_{N-n+r-2} & w_{N-n+r-1} & w_{N-n+r} & \dots & w_{N-n+2r-4} & w_{N-n+2r-3} \\
\dots & \dots & \dots & \dots & \dots & \dots \\
w_{N-n+2} & w_{N-n+3} & w_{N-n+4} & \dots & w_{N-n+r} & w_{N-n+r+1} \\
w_{N-n+1} & w_{N-n+2} & w_{N-n+3} & \dots & w_{N-n+r-1} & w_{N-n+r}
\end{vmatrix},
$$

where w_i are the Stiefel–Whitney classes of the tautological bundle on $G(\mathcal{A}, -n)$.

Proof. Let τ_n be the vector bundle on $G(\mathcal{A}, -n)$ whose fiber over the point $\{L\}$ corresponding to the subspace $L \subset \mathcal{A}$ is the space of linear functions on $\mathcal{A}$ vanishing on L. Consider the morphism of this bundle to the constant bundle with fiber $(\mathcal{F}^N)^*$, sending any such linear form to its restriction to $\mathcal{F}^N$. The variety $\Delta_r(\mathcal{F}^N)$ can be redefined as the set of points $\{L\}$ such that the rank of this morphism does not exceed $n - r$. By the real version of the Thom–Porteous formula (the proof of which literally repeats its complex analog given in [4, Section 14.4], after standard replacements of Chern classes by Stiefel–Whitney classes, $\mathbb{Z}$ by $\mathbb{Z}_2$, etc.) the class in $H^*(G(\mathcal{A}, -n), \mathbb{Z}_2)$ Poincaré dual to this variety is equal to the determinant of the form (6) in which all the symbols w_i are the Stiefel–Whitney classes of the virtual bundle $-\tau_n$.

The constant bundle on $G(\mathcal{A}, -n)$ with the fiber $\mathcal{A}^*$ is obviously isomorphic to the direct sum of τ_n and the bundle dual (and hence isomorphic) to the tautological bundle. Therefore $-\tau_n$ and this tautological bundle belong to the same class of the group $\tilde{K}(G(\mathcal{A}, -n))$, in particular have the same Stiefel–Whitney classes. □

These Stiefel–Whitney classes $w_i(-\tau_n)$ are equal to the i-dimensional components $\bar{w}_i(\tau_n) \in H^i(G(\mathcal{A}, -n), \mathbb{Z}_2)$ of the class $w^{-1}(\tau_n)$, where

$$w(\tau_n) = 1 + w_1(\tau_n) + \dots$$

is the total Stiefel–Whitney class of the bundle τ_n, see Section 4 in [5]. If the intersection of the subset $\mathrm{CD}_n \subset G(\mathcal{A}, -n)$ with $\Delta_r(\mathcal{F}^N)$ is empty, then the restriction homomorphism $H^*(G(\mathcal{A}, -n), \mathbb{Z}_2) \to H^*(\mathrm{CD}_n, \mathbb{Z}_2)$ maps the class (6) to zero. Theorem 6 therefore reduces to the following lemma.

Lemma 12. *If the class* (4) *is not equal to* 0, *then the restriction of the class* (6) *to the subvariety* $\mathrm{CD}_n \subset G(\mathcal{A}, -n)$ *is a nontrivial element of the group*

$$H^{r(N-n+r)}(\mathrm{CD}_n, \mathbb{Z}_2).$$

3. Proof of Lemma 12

Let $\mathbb{R}^2_+ \subset \mathbb{R}^2$ be the half-plane $\{(a, b) \mid a < b\} \subset \mathbb{R}^2$. Any point (a, b) of $\mathbb{R}^2_+$ can be identified with the chord $\{a, b\}$, and any element of the configuration space $B(\mathbb{R}^2_+, n)$ with an n-chord diagram. Let us denote by $\Xi \subset B(\mathbb{R}^2_+, n)$ the set of dependent (that is, containing resonances) n-chord diagrams. Consider the diagram

$$\begin{tikzcd} B(\mathbb{R}^2, n) & B(\mathbb{R}^2_+, n) \setminus \Xi \arrow[l, "\supset"'] \arrow[d, "\pi"] & \\ & \mathrm{CD}_n \arrow[r, "\subset"] & G(\mathcal{A}, -n) \end{tikzcd} \tag{7}$$

where π is the map sending any chord diagram to its equivalence class.

Lemma 13. *The restriction of the regular vector bundle ξ_n (see Definition 5) to the subset $B(\mathbb{R}^2_+, n) \setminus \Xi \subset B(\mathbb{R}^2, n)$ is isomorphic to the bundle pulled back by the map π from the bundle τ_n over CD_n.*

Proof. The bundle τ_n is isomorphic to its dual bundle τ_n^*, i.e., to the quotient of the trivial bundle with fiber $\mathcal{A}$ by the tautological bundle over $G(\mathcal{A}, -n)$.

Consider the following homomorphism from the trivial bundle with the fiber $\mathcal{A}$ over $B(\mathbb{R}^2_+, n) \setminus \Xi$ to ξ_n: over any n-chord diagram Γ it sends any function $f \in \mathcal{A} \subset C^0(\mathbb{R}^1, \mathbb{R}^1)$ to the function on the set of chords of this chord diagram, whose value on any chord $\{a_i, b_i\}$ is equal to the difference $f(b_i) - f(a_i)$. By the first characteristic property of the space $\mathcal{A}$ this morphism is surjective; by definition of inclusion $\mathrm{CD}_n \subset G(\mathcal{A}, -n)$ its kernel is equal to the fiber of the tautological bundle over the point $\pi(\Gamma) \in \mathrm{CD}_n$. Therefore our homomorphism induces an isomorphism between the bundles $\pi^*(\tau_n^*) \sim \pi^*(\tau_n)$ and ξ_n. □

Lemma 14 (see [3] or Proposition 31). *The square of any positive-dimensional element of the ring $H^*(B(\mathbb{R}^2, n), \mathbb{Z}_2)$ is equal to zero, in particular $w^{-1}(\xi_n) = w(\xi_n)$ and $w_i(\xi_n) = \bar{w}_i(\xi_n)$ for any i.* □

Lemma 15. *The determinant of the form* (6) *in which all classes w_i are replaced by $w_i(\xi_n)$ is equal to the product* (4) *in $H^*(B(\mathbb{R}^2, n), \mathbb{Z}_2)$.*

Proof. The matrix (6) is symmetrical with respect to the southwest/northeast diagonal, hence calculating its determinant mod 2 it suffices to count only those products of r matrix elements which are self-symmetric with respect to this diagonal. By Lemma 14 such products, not all factors of which lie in this diagonal, are also trivial. □

Lemma 16. *The inclusion $B(\mathbb{R}^2_+, n) \setminus \Xi \to B(\mathbb{R}^2, n)$ induces a monomorphism of cohomology groups $H^*(B(\mathbb{R}^2, n), \mathbb{Z}_2) \to H^*(B(\mathbb{R}^2_+, n) \setminus \Xi, \mathbb{Z}_2)$.*

Lemma 16 will be proved in Section 5. Lemma 12 follows from Lemmas 13–16 and the functoriality of Stiefel–Whitney classes. Namely, by Lemma 16 if the product (4) is nontrivial in $H^*(B(\mathbb{R}^2, n), \mathbb{Z}_2)$, then it is nontrivial also in $H^*(B(\mathbb{R}^2_+, n)\backslash\Xi, \mathbb{Z}_2)$. By the Lemmas 13–15 this element of $H^*(B(\mathbb{R}^2_+, n)\backslash\Xi, \mathbb{Z}_2)$ is equal to the class induced by the map π from the determinant (6), so this determinant is also nontrivial.

Proof of the last statement of Theorem 4. By Lemma 16 and statement 5.3 of [3], under the conditions of this theorem the class $w_{N-n+1}(\xi_n)$ is not trivial. We can then take an arbitrary element of the group $H_{N-n+1}(B(\mathbb{R}^2_+, n) \setminus \Xi, \mathbb{Z}_2)$ on which this class takes nonzero value: any cycle realizing such an element intersects the set $\pi^{-1}(\Delta_1(\mathcal{F}^N))$. □

4. Proof of Theorem 8

Now suppose that $N \leq n$. Let $\Lambda_r(\mathcal{F}^N)$ be the subset of $G(\mathcal{A}, -n)$ consisting of planes whose intersection with $\mathcal{F}^N$ is at least r-dimensional.

Proposition 17. *The class in $H^*(G(\mathcal{A}, -n), \mathbb{Z}_2)$ Poincaré dual to the variety $\Lambda_r(\mathcal{F}^N)$ is equal to $r \times r$ determinant similar to* (6), *in which $N - n$ in all lower indices is replaced by $n - N$, and w_i are Stiefel–Whitney classes of the bundle τ_n.*

Proof. The projection along the fibers of the tautological bundle over $G(\mathcal{A}, -n)$ defines a morphism from the constant bundle with fiber $\mathcal{F}^N$ and base $G(\mathcal{A}, -n)$ to the bundle dual (and hence isomorphic) to τ_n, i.e., to the quotient of the constant bundle with fiber $\mathcal{A}$ by the tautological bundle. The set $\Lambda_r(\mathcal{F}^N)$ can be defined as the set of points at which the rank of this morphism does not exceed $N - r$. Our proposition follows from the real version of Thom–Porteous formula applied to this morphism. □

The rest of the reduction of Theorem 8 to Lemma 16 repeats that of Theorem 6; the Stiefel–Whitney classes of the bundles $-\tau_n$ and τ_n participating in the corresponding Thom–Porteous formulas are the same by Lemma 14.

5. Proof of Lemma 16

5.1. ***Generators of Hopf algebra.*** We will prove the dual statement: the map $H_*(B(\mathbb{R}^2_+, n) \setminus \Xi, \mathbb{Z}_2) \to H_*(B(\mathbb{R}^2, n), \mathbb{Z}_2)$ induced by the identical embedding is epimorphic.

According to [3], all stabilization maps

$$H_*(B(\mathbb{R}^2, n), \mathbb{Z}_2) \to H_*(B(\mathbb{R}^2, n+m), \mathbb{Z}_2)$$

induced by the standard inclusions $B(\mathbb{R}^2, n) \hookrightarrow B(\mathbb{R}^2, n+m)$ are injective. Therefore, all elements of the group $H_*(B(\mathbb{R}^2, n), \mathbb{Z}_2)$ are given by polynomials in the multiplicative generators of the Hopf algebra $H_*(B(\mathbb{R}^2,\infty),\mathbb{Z}_2)$, and it suffices to prove that all these generators and their products participating in the construction of these elements can be realized by cycles lying in $B(\mathbb{R}^2_+,n)\setminus\Xi$.

These generators $[M_j]\in H_{2^j-1}(B(\mathbb{R}^2,2^j),\mathbb{Z}_2)$ were defined in Section 8 of [3] by the following cycles $M_j\subset B(\mathbb{R}^2,2^j)$. Arbitrarily choose two opposite points of the circle of radius 1 centered at the origin in $\mathbb{R}^2$. Take two circles of small radius ε with centers at these points and arbitrarily choose a pair of opposite points in each of them. Take circles of radius ε^2 centered at all obtained four points and choose a pair of opposite points in all of them. Continuing, after the j-th step we obtain a 2^j-configuration in $\mathbb{R}^2$. This construction involves $1+2+4+\cdots+2^{j-1}$ choices of opposite points in some circles, hence the set M_j of all possible 2^j-configurations that can be obtained in this way is (2^j-1)-dimensional. It is easy to see that this set is a closed submanifold in $B(\mathbb{R}^2,2^j)$, and therefore it defines an element $[M_j]$ of the group $H_{2^j-1}(B(\mathbb{R}^2,2^j),\mathbb{Z}_2)$, $j\geq 1$. Finally, define the element $[M_0]\in H_0(B(\mathbb{R}^2,1),\mathbb{Z}_2)$ as the class of a single point.

Unfortunately, these cycles with $j>1$ contain configurations with resonances, and, moreover, all configurations of class M_j do not lie in $\mathbb{R}^2_+$. To avoid these problems, we modify the previous construction by (1) replacing the circles with squares, (2) taking these squares of the same level (i.e., arising on the same stage of the construction) of varying sizes depending on their centers, and (3) shifting the resulting configurations into the half-plane $\mathbb{R}^2_+\subset\mathbb{R}^2$; see Section 5.4 below.

5.2. *Preparation for the construction.*

Definition 18. A segment in the plane $\mathbb{R}^2$ with coordinates a and b is called *vertical* (respectively, *horizontal*) if the coordinate a (respectively, b) is constant along it. A *straight resonance* of an n-point configuration in $\mathbb{R}^2$ is a closed chain of strictly alternating vertical and horizontal segments, all whose endpoints belong to our configuration.

Proposition 19. *Let $\bar a$ and $\bar b$ be two real numbers such that $\bar b-\bar a>8$. If all points $\{a_i,b_i\}$, $i=1,\dots,n$, of an n-chord diagram* (3) *satisfy the conditions*

$$|a_i-\bar a|<2,\quad |b_i-\bar b|<2, \tag{8}$$

and the corresponding n-configuration $\{(a_i,b_i)\}\subset B(\mathbb{R}^2_+,n)$ has a resonance, then it has a straight resonance.

Indeed, if two chords $\{a_i,b_i\}$ and $\{a_k,b_k\}$ satisfying the conditions (8) have common points, then either $a_i=a_k$ or $b_i=b_k$, but not $a_i=b_k$. □

We will construct our basic cycles in the set of configurations satisfying the condition of Proposition 19 for some $\bar a$ and $\bar b$, and prove that they do not have configurations with straight resonances.

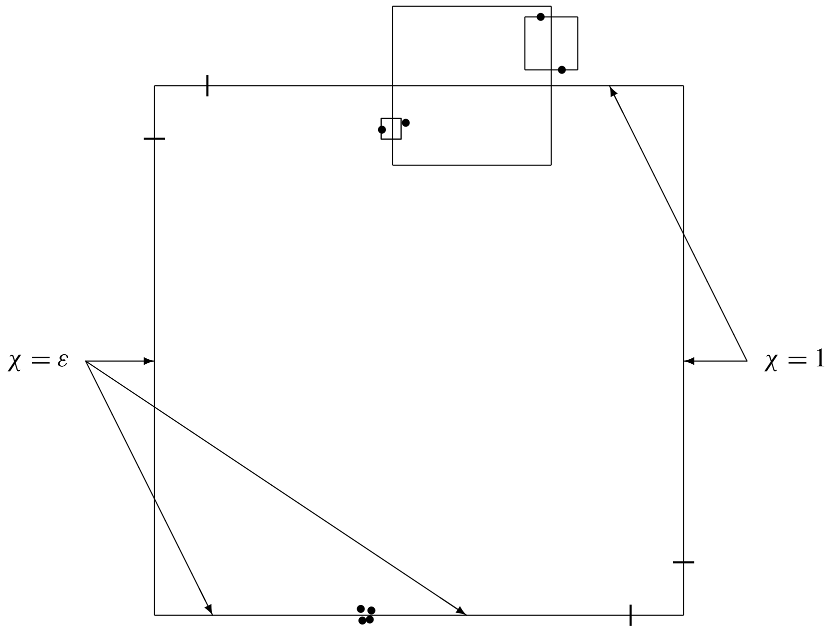

Figure 1. 8-configuration of the class $\tilde{M}_3$.

Let us fix a very small number $\varepsilon > 0$. Define the *basic square* $\square \subset \mathbb{R}^2$ as the union of four segments consecutively connecting the points

$$(-1, -1), (-1, 1), (1, 1), (1, -1)$$

and again $(-1, -1)$. Fix an arbitrary continuous function $\chi : \square \to [\varepsilon, 1]$ equal identically to 1 on segments

$$[(-1+\varepsilon, 1), (1, 1)] \quad \text{and} \quad [(1, -1+\varepsilon), (1, 1)], \tag{9}$$

equal to ε on segments

$$[(-1, -1), (-1, 1-\varepsilon)] \quad \text{and} \quad [(-1, -1), (1-\varepsilon, -1)], \tag{10}$$

and taking some intermediate values in the remaining ε-neighborhoods of corners $(-1, 1)$ and $(1, -1)$, see Figure 1.

5.3. ***First example.*** Let $n = 4$. Arbitrarily choose two opposite points A and $-A$ of the basic square. Consider two *squares of the second level* obtained from the basic square by the affine maps

$$X \mapsto A + \varepsilon\chi(A)X \quad \text{and} \quad X \mapsto -A + \varepsilon\chi(-A)X$$

(in particular, centered at the points A and $-A$), and arbitrarily choose two opposite points in each of these squares.

None of the 4-configurations thus obtained can have a straight resonance (i.e., to be the set of corners of a rectangle with vertical and horizontal sides). Indeed, two points of different squares of second level can be joined by a vertical (respectively, horizontal) segment only if the centers A and $-A$ of these squares are very close to the centers of the opposite horizontal (respectively, vertical) sides of the basic square. But in this case the values of the function χ at the points A and $-A$ are different, our two squares of second level have different sizes, and the segments connecting two opposite points in one of them and two opposite points in the other cannot be opposite sides of the same rectangle.

Therefore shifting all obtained 4-configurations by a fixed vector $(\bar{a}, \bar{b}) \in \mathbb{R}^2$ with $\bar{b} - \bar{a} > 8$, we get a 3-dimensional cycle in space $B(\mathbb{R}^2_+, 4) \setminus \Xi$.

5.4. *Construction of cycles $\tilde{M}_j$ (see Figure 1).* The general construction is an iteration of the previous one.

Namely, define two sequences of natural numbers u_j and T_j, $j \geq 2$, by recursion

$$(11)\quad u_2 = 1,\ T_2 = 2, \qquad u_j = u_{j-1} + T_{j-1} + 2, \quad T_j = u_j + T_{j-1} + 1 \qquad \text{for } j > 2.$$

Define the subset $\tilde{M}_1 \subset B(\mathbb{R}^2, 2)$ as the space of all choices of two opposite points in the basic square $\square$. Suppose we have defined the $(2^{j-1} - 1)$-dimensional subvariety $\tilde{M}_{j-1} \subset B(\mathbb{R}^2, 2^{j-1})$, $j \geq 2$. Then the $(2^j - 1)$-dimensional subvariety $\tilde{M}_j \subset B(\mathbb{R}^2, 2^j)$ is defined as the space of all 2^j-configurations consisting of

(i) a 2^{j-1}-configuration obtained from some configuration of the class $\tilde{M}_{j-1}$ by the affine map $\mathbb{R}^2 \to \mathbb{R}^2$ given by

$$\{X \mapsto A + \varepsilon \chi^{u_j}(A) X\}, \tag{12}$$

where A is some point of the basic square $\square$;

(ii) a 2^{j-1}-configuration obtained from some configuration of the class $\tilde{M}_{j-1}$ by the map

$$\{X \mapsto -A + \varepsilon \chi^{u_j}(-A) X\}, \tag{13}$$

with the same A, see Figure 1.

Any 2^j-configuration of the class $\tilde{M}_j$ uniquely determines the set of $2^j - 1$ *squares participating in its construction*: it consists of the basic square $\square$ and the images under maps (12), (13) of two collections of $2^{j-1} - 1$ squares participating in the construction of two 2^{j-1}-configurations of class $\tilde{M}_{j-1}$. This set is obviously organized into (the set of vertices of) an oriented binary tree. Namely, it has two squares of the second level, four squares of the third level, etc. Every square of the l-th level is centered at a point of some square of $(l-1)$-st level; we connect the two corresponding vertices by an edge of the tree oriented towards the vertex of level l. Any point of our configuration belongs to a square of the j-th level, i.e., to a leaf of the tree. We say that a point of the configuration is *subordinate* to this square of level j containing it, and also to all squares of lower levels connected with it by an oriented path in our binary tree. Conversely, we say that all these squares *dominate* such a point of the configuration; in particular, any square of level l dominates exactly 2^{j-l+1} points.

Lemma 20. *The length of the sides of the smallest square participating in the construction of a 2^j-configuration of class $\tilde{M}_j$ is not less than $2\varepsilon^{T_j}$.*

Proof. This follows by induction from the last condition (11) because both subsets constituting our 2^j-configuration are obtained from certain 2^{j-1}-configurations of the class $\tilde{M}_{j-1}$ by homotheties (12), (13) with coefficients $\geq \varepsilon^{u_j+1}$. $\square$

Lemma 21. *The absolute values of the coordinates a, b of all points of configurations of class $\tilde{M}_j$ do not exceed $1+\varepsilon+\varepsilon^2+\cdots+\varepsilon^{j-1} < \frac{1}{1-\varepsilon}$.*

Lemma 22. *If a square of level l participating in the construction of a configuration of the class $\tilde{M}_j$ has sides of length d, then all 2^{j-l+1} points of this configuration subordinate to this square lie in the $\frac{\varepsilon d}{\sqrt{2}(1-\varepsilon)}$-neighborhood of this square.*

Proof. Proofs of these two lemmas follow directly from the construction. $\square$

Finally, we move the obtained subvariety $\tilde{M}_j \subset B(\mathbb{R}^2, 2^j)$ into $B(\mathbb{R}^2_+, 2^j)$ shifting all its 2^j-configurations to $\mathbb{R}^2_+$ by some translation $\{X \mapsto X+(\bar{a}, \bar{b})\}$, where $\bar{b}-\bar{a} \geq 8$. Denote by ∇_j the resulting cycle in $B(\mathbb{R}^2_+, 2^j)$.

It is easy to see that ∇_j is the image of an embedding $M_j \to B(\mathbb{R}^2, 2^j)$ (where the manifold M_j was defined in Section 5.1), which is homotopic to the identical embedding; in particular, it defines the same homology class in $H_*(B(\mathbb{R}^2, 2^j), \mathbb{Z}_2)$. Indeed, such a homotopy is defined by (1) a family of functions connecting the function χ with the function equal identically to 1 in the space of positive functions $\square \to \mathbb{R}^1$, (2) a deformation of all circles to squares, and (3) the continuous shift of the plane by the vector $(\bar{a}, \bar{b})$. Therefore, to realize the homology class $[M_j]$ by a cycle from $B(\mathbb{R}^2_+) \setminus \Xi$ it remains to prove the following statement.

Theorem 23. *The variety ∇_j does not contain 2^j-chord diagrams with straight resonances.*

The proof of this theorem for arbitrary j is not as simple as for $j=2$, it uses the following generalization of straight resonances.

Definition 24. For any positive number δ, a segment in $\mathbb{R}^2$ is called δ-*horizontal* (respectively, δ-*vertical*) if the tangent of the angle between this segment and a horizontal (respectively, vertical) line belongs to the interval $[0, \delta)$. An n-configuration in $\mathbb{R}^2$ is δ-*resonant* if there exists a closed chain of strictly alternating δ-vertical and δ-horizontal segments in $\mathbb{R}^2$ such that the endpoints of any of its segments belong to our configuration.

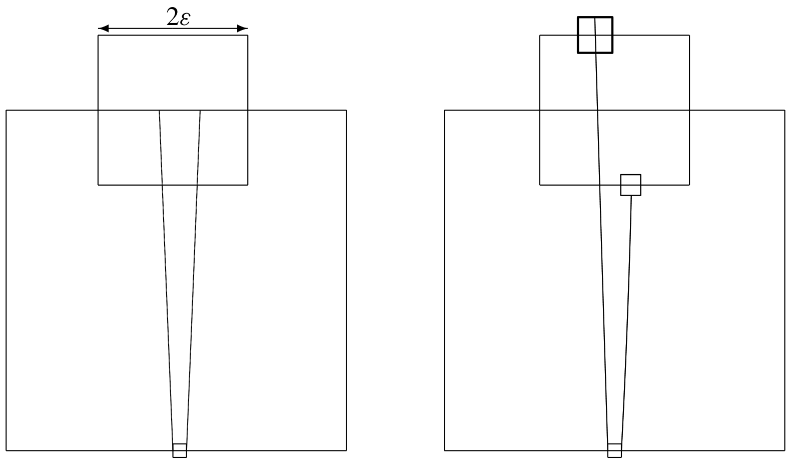

Figure 2. Discrepancy preventing resonances.

Theorem 23 now follows from the following statement.

Theorem 25. *If the number ε participating in the construction of the cycle $\tilde{M}_j$ is sufficiently small, then the set $\tilde{M}_j$ does not contain ε^{u_j+1}-resonant 2^j-configurations.*

5.5. ***Example and idea of the proof of Theorem 25.*** Let us again consider the case $j=2$. If a 4-configuration of the class $\tilde{M}_2$ is ε^2-resonant, then some points of opposite squares of second level participating in its construction are connected by segments almost parallel (up to angles with tangent $\le \varepsilon^2$) to a vertical or horizontal segment. Let us assume for certainty that these are almost vertical segments. Then the centers A and $-A$ of these squares are very close to the centers of the opposite horizontal sides of the basic square, in particular the function χ takes the value 1 at one of them and the value ε at the other. The a-coordinates of two points of our ε^2-resonant 4-configuration, placed in the bigger square of second level, differ by 2ε, see Figure 2 (left). On the other hand, these two points are connected by a chain of three segments of our ε^2-resonance passing through the smaller square, therefore this difference is estimated from above by the sum of (a) the length of the sides of the small square and (b) twice the maximal possible difference of a-coordinates of endpoints of the ε^2-vertical segments of our ε^2-resonance. The last difference is estimated from above by the maximal difference of b-coordinates of points of our configuration (which is at most $2+\varepsilon+\varepsilon^2$) multiplied by the allowed bending ε^2 of the segments of our ε^2-resonance. This sum is of order ε^2, a contradiction.

Further, let j be arbitrary; suppose that our configuration of class $\tilde{M}_j$ is ε^{u_j+1}-resonant, and two squares of second level participating in its construction are located near the centers of horizontal sides of the basic square. Our exponents (11) are chosen in such a way that the upper endpoints of any two ε^{u_j+1}-vertical segments connecting the points subordinate to different squares of the second level cannot lie too close to the opposite vertical sides of an arbitrary square subordinate to the bigger (upper) square of second level: indeed, by an estimate similar to that from the previous paragraph the difference of the a-coordinates of these endpoints is much smaller than the minimal distance between these vertical sides.

If $j > 2$, then there remains a possibility that these two endpoints lie in neighborhoods of opposite horizontal sides of such a square (see Figure 2, right) and are connected by some ε^{u_j+1}-resonant chain inside the upper square of second level. In this case, let us connect directly these two endpoints by a segment and forget about the part of our ε^{u_j+1}-resonance involving the points from the lower square. The exponents (11) are chosen so that the tangent of this segment with a vertical line is estimated from above by $\varepsilon^{u_{j-1}+1}$, and we obtain a $\varepsilon^{u_{j-1}+1}$-resonance inside the upper square only, which is prohibited by the induction hypothesis.

5.6. *Proof of Theorem 25.* Let us support this reasoning with strict estimates.

Suppose that Theorem 25 is proved for all cycles $\tilde{M}_i$, $i < j$. By the construction, any 2^j-configuration $\Gamma \in \tilde{M}_j$ splits into two subsets of cardinality 2^{j-1} with mass centers at some opposite points A and $-A$ of the basic square $\square$, any of these subsets lying in the $\sqrt{2}\varepsilon/(1-\varepsilon)$-neighborhood of the corresponding point A or $-A$.

Suppose that our configuration $\Gamma \in \tilde{M}_j$ is ε^{u_j+1}-resonant. If the entire chain of its points participating in this resonance is located in only one of these two subsets of Γ, then we get a contradiction with the induction hypothesis for $i = j-1$, because this subset is homothetic to a configuration of the class $\tilde{M}_{j-1}$, and $\varepsilon^{u_j+1} < \varepsilon^{u_{j-1}+1}$.

So, our chain should contain ε^{u_j+1}-vertical or ε^{u_j+1}-horizontal segments, which connect some points from these two subsets. Therefore, the corresponding points A and $-A$ are very close to either the center points of opposite horizontal sides of the basic square $\square$, or to the center points of its vertical sides. These two situations can be reduced to each other by the reflection in the diagonal $\{a = b\}$ of $\mathbb{R}^2$, it is therefore sufficient to consider only the first of them.

Consider a ε^{u_j+1}-vertical segment of our resonance chain which has endpoints in both these subsets; let A_0 be its endpoint in the upper subset. Starting from A_0, our chain somehow travels inside this upper subset and finally leaves it along some other ε^{u_j+1}-vertical segment; let B_0 be the upper point of the latter segment.

Lemma 26. (1) *The difference between the a-coordinates of points A_0 and B_0 is estimated from above by $7\varepsilon^{u_j+1}$.*

(2) *The difference between the b-coordinates of A_0 and B_0 is estimated from below by $\varepsilon^{T_{j-1}+1}$.*

Proof. (1) This difference is estimated from above by the sum of (a) the maximal difference of the a-coordinates of the points of the lower 2^{j-1}-subconfiguration of our 2^j-configuration Γ, (b) the difference of the a-coordinates of the point A_0 and the other endpoint of the segment of our chain connecting A_0 with this lower subconfiguration, and (c) the similar difference for the point B_0. By Lemma 21, formulas (12)–(13) and the definition of the function χ, the first of these differences is estimated from above by $2\varepsilon^{u_j+1}(1+O(\varepsilon))$; the other two are estimated by the heights of these two segments (which by Lemma 21 are smaller than $\frac{2}{1-\varepsilon}$) multiplied by their tangents with the vertical direction (which are estimated by ε^{u_j+1} since these segments are ε^{u_j+1}-vertical). Thus, the entire sum is estimated from above by $\varepsilon^{u_j+1}(6+O(\varepsilon)) < 7\varepsilon^{u_j+1}$.

(2) Let us consider two paths in the binary tree of squares participating in the construction of our configuration $\Gamma \in \tilde{M}_j$, starting from the basic square and consisting of all squares dominating the point A_0 (respectively, B_0). Let $\Box_k$ be the last (of highest level) common square of these two sequences. By Lemma 20, the length of its sides is at least $2\varepsilon^{T_{j-1}+1}$: indeed, this square is obtained from a square participating in the construction of a 2^{j-1}-configuration of class $\tilde{M}_{j-1}$ by a homothety with coefficient $\varepsilon\chi(A)$ for some point A from the central part of the upper side of the basic square (where $\chi \equiv 1$). The next two squares in these sequences (or their final points A_0 and B_0 if $\Box_k$ is a square of the last j-th level) are different, therefore these next squares (or points) are centered at (or coincide with) some points of opposite sides of $\Box_k$. These cannot be vertical sides: indeed, in this case by Lemma 22 the a-coordinates of our points A_0 and B_0 would differ by $2\varepsilon^{T_{j-1}+1}(1+O(\varepsilon))$, which contradicts statement (1) of our lemma, because by (11) $\varepsilon^{T_{j-1}+1} \gg \varepsilon^{u_j+1}$. Therefore, these are horizontal opposite sides, and hence the difference of their b-coordinates is estimated from below by the number $2\varepsilon^{T_{j-1}+1}(1+O(\varepsilon)) > \varepsilon^{T_{j-1}+1}$. Moreover, by Lemma 22 the last estimate is also valid for the difference of the b-coordinates of the points A_0 and B_0 subordinate to some squares centered at points of these sides. □

Corollary 27. *The segment* $[A_0, B_0]$ *is* $\varepsilon^{u_{j-1}+1}$*-vertical.*

Proof. By the previous lemma, the absolute value of the tangent of the angle between this segment and the vertical direction is estimated from above by $7\varepsilon^{u_j - T_{j-1}}$, which by (11) is less than $\varepsilon^{u_{j-1}+1}$ (since we can assume that $\varepsilon < \frac{1}{7}$). □

In particular, this segment $[A_0, B_0]$ is not ε^{u_j+1}-horizontal, so A_0 and B_0 cannot be neighboring points in our ε^{u_j+1}-resonance chain. Now consider the closed chain of segments in the upper subset of our $\tilde{M}_j$-configuration Γ, which consists of the segment $[A_0, B_0]$ and the part of our initial ε^{u_j+1}-resonance chain connecting these two points inside this upper subset of Γ. This closed chain is a $\varepsilon^{u_{j-1}+1}$-resonance, which contradicts the induction hypothesis over j. This contradiction finishes the proof of Theorem 25, and hence also of Theorem 23. □

Finally, every product $[M_{j_1}]\cdot[M_{j_2}]\cdots[M_{j_q}]$ of multiplicative generators of the Hopf algebra $H_*(B(\mathbb{R}^2,\infty),\mathbb{Z}_2)$ such that $2^{j_1}+2^{j_2}+\cdots+2^{j_q}=n$ can be realized by the set of all n-configurations, some 2^{j_1} points of which form a configuration of type $\tilde{M}_{j_1}$ shifted to $\mathbb{R}^2_+$ along the vector $(0,8)$, some other 2^{j_2} points form a configuration of type $\tilde{M}_{j_2}$ shifted along the vector $(56,64)\ldots$ and the last 2^{j_q} points form a configuration of type $\tilde{M}_{j_q}$ shifted along the vector $(8^q-8,8^q)$. The values of both coordinates of the points of any of these groups are very far from the coordinates of the points from any other group, thus all obtained n-configurations do not contain resonances. This finishes the proof of Lemma 16 and hence also of Theorems 6 and 8. □

6. Proof of Corollaries 7 and 10

Proposition 28. *All the statements of Corollary 7 (respectively, Corollary 10) are monotonic on N: if for a triplet of numbers (n,N,r), $n<N$ (respectively, $n>N$), it is true that for any N-dimensional subspace $\mathcal{F}^N\subset C^0(\mathbb{R}^1,\mathbb{R}^1)$ there exist systems of n independent equality conditions defining subspaces of codimension $\le n-r$ (respectively, of dimension r) in $\mathcal{F}^N$, then the same is true for the triplet $(n,N-1,r)$ (respectively, $(n,N+1,r)$).*

Proof. Apply the hypothesis of this proposition to an arbitrary N-dimensional space containing $\mathcal{F}^{N-1}$ (respectively, contained in $\mathcal{F}^{N+1}$). □

Let us recall several results of [3] on mod 2 cohomology of spaces $B(\mathbb{R}^2,n)$.

Proposition 29 (see [3, Section 4.8]). *For any k, the group $H^k(B(\mathbb{R}^2,n),\mathbb{Z}_2)$ has a canonical basis whose elements are in a one-to-one correspondence with unordered decompositions of the number n into $n-k$ powers of 2. In particular, this group is nontrivial if and only if $k\le n-I(n)$.*

The standard notation for such a basis element is $\langle 2^{l_1},2^{l_2},\ldots,2^{l_t}\rangle$, where $l_1\ge l_2\ge\cdots\ge l_t\ge 1$, $t\le n-k$: this is the list (in nonincreasing order) of all summands of such a decomposition which are *strictly greater* than 1.

Namely, such a basis element of $H^*(B(\mathbb{R}^2,n),\mathbb{Z}_2)$ is defined by the intersection index with the closure of the subvariety in $B(\mathbb{R}^2,n)$ consisting of all n-configurations such that there exist t distinct vertical lines in $\mathbb{R}^2$, one of which contains 2^{l_1} points of our configuration, some other one contains 2^{l_2} of them, etc.

We will also use the abbreviated notation $\langle 2^{s_1}_{v_1},2^{s_2}_{v_2},\ldots,2^{s_q}_{v_q}\rangle$ for these basis elements, where $s_1>s_2>\cdots>s_q\ge 1$ and $2^{s_i}_{v_i}$ means 2^{s_i} repeated v_i times; if some v_i is here equal to 1 then we write simply 2^{s_i} instead of $2^{s_i}_1$.

Proposition 30 (see [3, Section 5.2]). *The class $w_k(\xi_n)\in H^k(B(\mathbb{R}^2,n),\mathbb{Z}_2)$ for any $k<n$ is equal to the sum of all basic elements of this group described in the*

previous proposition. In particular, all classes $w_k(\xi_n)$ with $k \leq n - I(n)$ are not equal to 0.

So we have

$$w_1 = \langle 2 \rangle, \tag{14}$$

$$w_2 = \langle 2_2 \rangle, \tag{15}$$

$$w_3 = \langle 2_3 \rangle + \langle 4 \rangle, \tag{16}$$

$$w_4 = \langle 2_4 \rangle + \langle 4, 2 \rangle, \tag{17}$$

$$w_5 = \langle 2_5 \rangle + \langle 4, 2_2 \rangle, \tag{18}$$

$$w_6 = \langle 2_6 \rangle + \langle 4, 2_3 \rangle + \langle 4_2 \rangle, \tag{19}$$

$$w_7 = \langle 2_7 \rangle + \langle 4, 2_4 \rangle + \langle 4_2, 2 \rangle + \langle 8 \rangle, \tag{20}$$

$$w_8 = \langle 2_8 \rangle + \langle 4, 2_5 \rangle + \langle 4_2, 2_2 \rangle + \langle 8, 2 \rangle, \tag{21}$$

$$w_9 = \langle 2_9 \rangle + \langle 4, 2_6 \rangle + \langle 4_2, 2_3 \rangle + \langle 4_3 \rangle + \langle 8, 2_2 \rangle, \tag{22}$$

$$w_{10} = \langle 2_{10} \rangle + \langle 4, 2_7 \rangle + \langle 4_2, 2_4 \rangle + \langle 4_3, 2 \rangle + \langle 8, 2_3 \rangle + \langle 8, 4 \rangle, \tag{23}$$

$$w_{11} = \langle 2_{11} \rangle + \langle 4, 2_8 \rangle + \langle 4_2, 2_5 \rangle + \langle 4_3, 2_2 \rangle + \langle 8, 2_4 \rangle + \langle 8, 4, 2 \rangle, \tag{24}$$

$$w_{12} = \langle 2_{12} \rangle + \langle 4, 2_9 \rangle + \langle 4_2, 2_6 \rangle + \langle 4_3, 2_3 \rangle + \langle 4_4 \rangle + \langle 8, 2_5 \rangle + \langle 8, 4, 2_2 \rangle. \tag{25}$$

Proposition 31 (see [3, Sections 9 and 6]). *The cohomological product of two basis elements of the group $H^*(B(\mathbb{R}^2, n), \mathbb{Z}_2)$ having the form*

$$\langle 2^m, \dots, 2^m, 2^{m-1}, \dots, 2^{m-1}, \dots, 2, \dots, 2 \rangle,$$

where any number 2^i, $i \in \{1, 2, \dots, m\}$, occurs p_i times in the first factor and q_i times in the second and some of numbers p_i, q_i can be equal to 0, *is equal to*

$$\prod_{i=1}^{m} \binom{p_i + q_i}{p_i} \langle 2^m, \dots, 2^m, 2^{m-1}, \dots, 2^{m-1}, \dots, 2, \dots, 2 \rangle, \tag{26}$$

where any symbol 2^i in the angle brackets occurs $p_i + q_i$ times, all binomial coefficients are counted modulo 2, *and the entire expression* (26) *is assumed to be zero if $(p_m + q_m)2^m + (p_{m-1} + q_{m-1})2^{m-1} + \cdots + (p_1 + q_1)2 > n$.*

Now, all statements of Corollaries 7 and 10 follow immediately from Theorems 6, 8 and the following calculations.

A(1) By (14), (16) and (26), $w_1 w_3 = \langle 4, 2 \rangle$, which is nontrivial for $n \geq 6$.

A(2) By (15), (17) and (26),

$$w_2 w_4 = \langle 4, 2_3 \rangle + \langle 2_6 \rangle, \tag{27}$$

which is nontrivial if $n \geq 10$.

A(3) By (16), (18) and (26), $w_3 w_5 = \langle 4, 2_5 \rangle$, which is nontrivial if $n \geq 14$. By (17), (19) and (26),

(28) $$w_4 w_6 = \langle 4_2, 2_4 \rangle + \langle 4_3, 2 \rangle,$$

which is also nontrivial if $n \geq 14$.

A(4) By (18), (20) and (26),

(29) $$w_5 w_7 = \langle 4_3, 2_3 \rangle + \langle 8, 2_5 \rangle + \langle 8, 4, 2_2 \rangle,$$

which is nontrivial for $n \geq 16$.

A(5) By (19), (21) and (26),

(30) $$w_6 w_8 = \langle 2_{14} \rangle + \langle 4, 2_{11} \rangle + \langle 4_2, 2_8 \rangle + \langle 4_3, 2_5 \rangle + \langle 8, 2_7 \rangle + \langle 8, 4_2, 2 \rangle,$$

which is nontrivial for $n \geq 18$.

A(6) By (20), (22) and (26),

(31) $$w_7 w_9 = \langle 4, 2_{13} \rangle + \langle 4_3, 2_7 \rangle + \langle 8, 2_9 \rangle + \langle 8, 4_3 \rangle,$$

which is nontrivial for $n \geq 20$.

A(7) By (21), (23) and (26),

(32) $$w_8 w_{10} = \langle 4_2, 2_{12} \rangle + \langle 4_3, 2_9 \rangle + \langle 8, 4, 2_8 \rangle + \langle 8, 4_2, 2_5 \rangle + \langle 8, 4_3, 2_2 \rangle,$$

which is nontrivial if $n \geq 24$.

A(8) By (22), (24) and (26),

(33) $$w_9 w_{11} = \langle 4_3, 2_{11} \rangle + \langle 8, 4, 2_{10} \rangle + \langle 8, 4_3, 2_4 \rangle,$$

which is nontrivial if $n \geq 28$.

By (23), (25) and (26),

(34) $$w_{10} w_{12} \\ = \langle 4_4, 2_{10} \rangle + \langle 4_5, 2_7 \rangle + \langle 4_6, 2_4 \rangle + \langle 4_7, 2 \rangle + \langle 8, 4, 2_{12} \rangle + \langle 8, 4_4, 2_3 \rangle + \langle 8, 4_5 \rangle,$$

which also is nontrivial if $n \geq 28$.

A(9) The class w_{14} contains summand $\langle 8_2 \rangle$, therefore by (25) and (26) the product $w_{12} w_{14}$ contains summands $\langle 8_2, 4_4 \rangle$ and $\langle 8_3, 4, 2_2 \rangle$, each of which is nontrivial if $n \geq 32$.

B(1) By (15), (28) and (26),

(35) $$w_2 w_4 w_6 = \langle 4_2, 2_6 \rangle + \langle 4_3, 2_3 \rangle,$$

which is nontrivial for $n \geq 18$. By Theorem 6 this calculation proves the statement B(1) of Corollary 7 (respectively, Corollary 10) for $N = n+1$ (respectively, $N = n-1$), and the case $N = n$ follows by monotonicity, see Proposition 28.

B(2) By (16), (29) and (26), $w_3 w_5 w_7 = \langle 8, 4, 2_5 \rangle$, which is nontrivial for $n \geq 22$.

B(3) By (28), (21) and (26),

$$w_4 w_6 w_8 = \langle 4_2, 2_{12} \rangle + \langle 4_3, 2_9 \rangle + \langle 8, 4_2, 2_5 \rangle, \tag{36}$$

which is nontrivial for $n \geq 26$.

B(4) By (18), (31) and (26),

$$w_5 w_7 w_9 = \langle 8, 4, 2_{11} \rangle + \langle 8, 4_3, 2_5 \rangle, \tag{37}$$

which is nontrivial if $n \geq 30$.

B(5) By (19), (32) and (26),

$$w_6 w_8 w_{10} = \langle 8, 4, 2_{14} \rangle + \langle 8, 4_3, 2_8 \rangle, \tag{38}$$

which is nontrivial for $n \geq 36$.

B(6) By (32), (25) and (26),

$$\begin{aligned} &w_8 w_{10} w_{12} \\ &\quad = \langle 4_6, 2_{12} \rangle + \langle 4_7, 2_9 \rangle + \langle 8, 4_3, 2_{14} \rangle + \langle 8, 4_5, 2_8 \rangle + \langle 8, 4_6, 2_5 \rangle + \langle 8, 4_7, 2_2 \rangle, \end{aligned} \tag{39}$$

which is nontrivial for $n \geq 40$. By Theorems 6 and 8, this implies statements B(6) of Corollaries 7 and 10 for $N = n+7$ (respectively, $N = n-7$), and the cases $N = n+6$ (respectively, $N = n-6$) follow by monotonicity. Notice that the routine consideration for $N = n+6$ based on

$$w_7 w_9 w_{11} = \langle 4_3, 2_{11} \rangle + \langle 8, 4_3, 2_{11} \rangle, \tag{40}$$

gives the same result in more restrictive conditions, $n \geq 42$ only.

B(7) The class w_{14} contains the summand $\langle 8_2 \rangle$. Therefore by (34) and (26), the product $w_{10} w_{12} w_{14}$ contains the summand $\langle 8_3, 4_5 \rangle$, which is nontrivial if $n \geq 44$.

C(1) By (15), (36) and (26),

$$w_2 w_4 w_6 w_8 = \langle 4_2, 2_{14} \rangle + \langle 4_3, 2_{11} \rangle + \langle 8, 4_2, 2_7 \rangle, \tag{41}$$

which is nontrivial if $n \geq 30$. By Theorem 6, this proves statements C(1) of Corollaries 7 and 10 in the case $N = n+1$ (respectively, $N = n-1$), and the case $N = n$ follows by monotonicity.

C(2) By (17) and (38),

$$w_4 w_6 w_8 w_{10} = \langle 8, 4_3, 2_{12} \rangle, \tag{42}$$

which is nontrivial if $n \geq 44$. This proves statement C(2) of Corollary 7 (respectively, Corollary 10) for $N = n+3$ (respectively, $N = n-3$), which implies it also for $N = n+2$ (respectively, $N = n-2$).

C(3) By (19) and (39),

$$w_6 w_8 w_{10} w_{12} = \langle 8, 4_5, 2_{14} \rangle + \langle 8, 4_7, 2_8 \rangle, \tag{43}$$

which is nontrivial if $n \geq 52$. This proves statements C(3) for $N = n+5$ (respectively, $N = n-5$) and hence also for $N = n+4$ (respectively, $N = n-4$).

C(4) The class w_{14} contains the summand $\langle 8_2 \rangle$. Therefore by (39) and (26) the class $w_8 w_{10} w_{12} w_{14}$ contains the summand $\langle 8_3, 4_7, 2_2 \rangle$, which is nontrivial if $n \geq 56$. This proves statements C(4) for $N = n+7$ ($N = n-7$) and hence also for $N = n+6$ ($N = n-6$).

D(1) By (41), (23) and (26), $w_2 w_4 w_6 w_8 w_{10} = \langle 8, 4_3, 2_{14} \rangle$, which is nontrivial if $n \geq 48$.

D(2) By (17), (43) and (26), $w_4 w_6 w_8 w_{10} w_{12} = \langle 8, 4_7, 2_{12} \rangle$, which is nontrivial if $n \geq 60$.

D(3) Since w_{14} contains $\langle 8_2 \rangle$, by (43) and (26) the product $w_6 w_8 w_{10} w_{12} w_{14}$ contains the summand $\langle 8_3, 4_7, 2_8 \rangle$, which is nontrivial if $n \geq 68$.

E(1) By D(1) and formulas (25) and (26), $w_2 w_4 w_6 w_8 w_{10} w_{12} = \langle 8, 4_7, 2_{14} \rangle$, which is nontrivial if $n \geq 64$.

E(2) Since w_{14} contains $\langle 8_2 \rangle$, by D(2) and (26) the product $w_4 w_6 w_8 w_{10} w_{12} w_{14}$ contains the summand $\langle 8_3, 4_7, 2_{12} \rangle$, which is nontrivial if $n \geq 76$.

F. Since w_{14} contains the summand $\langle 8_2 \rangle$, by E(1) and formula (26) the class $w_2 w_4 w_6 w_8 w_{10} w_{12} w_{14}$ contains the summand $\langle 8_3, 4_7, 2_{14} \rangle$, nontrivial if $n \geq 80$. □

7. Equality conditions and homology of knot spaces

Let us denote by $\mathcal{K}$ the affine space of all C^∞-smooth maps $\mathbb{R}^1 \to \mathbb{R}^3$ coinciding with a fixed linear embedding outside a compact set in $\mathbb{R}^1$. Let Σ be the *discriminant subvariety* of $\mathcal{K}$ consisting of all maps which are not smooth embeddings, i.e., have either self-intersections or points of vanishing derivative. The elements of the set $\mathcal{K} \setminus \Sigma$ are called *long knots*. There is a natural one-to-one correspondence between the connected components of this set and the isotopy classes of the usual knots, i.e., of smooth embeddings $S^1 \to \mathbb{R}^3$ or $S^1 \to S^3$.

The variety Σ is swept out by affine subspaces $L(a, b)$ of codimension 3 in $\mathcal{K}$ corresponding to all chords $\{a, b\}$ in $\mathbb{R}^1$ (including degenerate chords with $a = b$) and consisting of maps $\varphi : \mathbb{R}^1 \to \mathbb{R}^3$ such that $\varphi(a) = \varphi(b)$ (or $\varphi'(a) = 0$ if $a = b$). Much of the topological structure of Σ can be described in terms of the order complex of the (naturally topologized) partially ordered set, whose elements correspond to these subspaces $L(a, b)$ and their finite intersections (defined by chord diagrams), and the order relation is the incidence of corresponding subspaces. For any n, the subspaces in $\mathcal{K}$ defined in this way by independent n-chord diagrams form an affine bundle over the space $\overline{\mathrm{CD}}_n$ of equivalence classes of such diagrams (including degenerate ones, containing chords of type $\{a, a\}$). The fibers of this bundle have codimension $3n$ in $\mathcal{K}$, and its normal bundle is isomorphic to the sum of three copies of the bundle τ_n^* considered in Section 2 (and continued to degenerate chord diagrams).

The topology of the space $\mathcal{K} \setminus \Sigma$ is related by a kind of Alexander duality to the topology of the complementary space Σ, in particular, the numerical knot invariants can be realized as linking numbers with infinite-dimensional cycles of codimension 1 in $\mathcal{K}$ contained in Σ. However, Alexander duality deals with finite-dimensional spaces only, therefore to apply it properly we use finite-dimensional approximations of the space $\mathcal{K}$. Namely, we consider infinite sequences $\mathcal{K}_1 \subset \mathcal{K}_2 \subset \dots$ of finite-dimensional affine subspaces of $\mathcal{K}$, such that any connected component of $\mathcal{K} \setminus \Sigma$ is represented by elements of subspaces $\mathcal{K}_j \setminus \Sigma$ with sufficiently large j, and moreover any homology class of $\mathcal{K} \setminus \Sigma$ is represented by cycles contained in such subspaces. (The existence of such sequences of subspaces $\mathcal{K}_j$ follows easily from Weierstrass approximation theorem). Then for any such subspace $\mathcal{K}_j$ of dimension d_j we have the Alexander isomorphisms

$$\tilde{H}^k(\mathcal{K}_j \setminus \Sigma) \simeq \overline{H}_{d_j-k-1}(\mathcal{K}_j \cap \Sigma), \tag{44}$$

where $\overline{H}_*$ denotes the Borel–Moore homology groups.

To study the left-hand groups in (44) (in particular, such a group with $k = 0$, i.e., the group of $\mathbb{Z}$-valued invariants of knots realizable in $\mathcal{K}_j$) a *simplicial resolution* of the space $\mathcal{K}_j \cap \Sigma$ is used in [6]. It is a certain topological space $\sigma(j)$ and a surjective map $\sigma(j) \to \mathcal{K}_j \cap \Sigma$ inducing an isomorphism of Borel–Moore homology groups. These groups $\overline{H}_*(\sigma(j)) \simeq \overline{H}_*(\mathcal{K}_j \cap \Sigma)$ can be calculated by a spectral sequence $\{E^r_{n,\beta}\}$ defined by a natural increasing filtration

$$\sigma_1(j) \subset \sigma_2(j) \subset \cdots \subset \sigma(j), \tag{45}$$

in particular, $E^1_{n,\beta} \simeq \overline{H}_{n+\beta}(\sigma_n(j) \setminus \sigma_{n-1}(j))$.

This filtration is finite if the subspace $\mathcal{K}_j$ is not very degenerate. Namely, any space $\sigma_n(j) \setminus \sigma_{n-1}(j)$ is constructed starting from the intersection sets of $\mathcal{K}_j$ with subspaces of codimension $3n$ in $\mathcal{K}$ defined by independent n-chord diagrams. Since the family of all such planes is $2n$-parametric, a *generic* d_j-dimensional affine subspace $\mathcal{K}_j$ meets only subspaces of this kind with $n \le d_j$, so $\sigma_{d_j}(j) = \sigma(j)$.

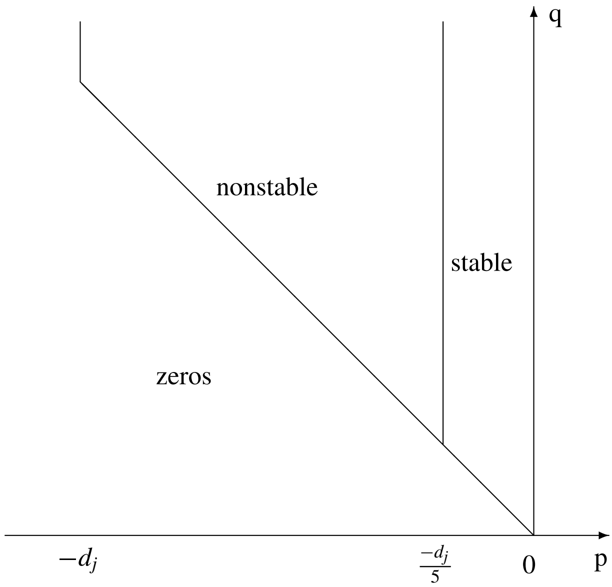

Figure 3. Spectral sequence for $H^*(\mathcal{K}_j \setminus \Sigma)$.

The formal change $E_r^{p,q} \equiv E^r_{-p,d_j-1-q}$ turns the homological spectral sequence defined by this filtration into a cohomological one, which by Alexander duality converges to the left-hand groups of (44). All nontrivial groups $E_r^{p,q}$, $r \geq 1$, of the last spectral sequence for a generic subspace $\mathcal{K}_j$ lie in the domain (see Figure 3)

$$\{(p,q) : p \in [-d_j, -1],\ p+q \geq 0\}.$$

If the approximating subspace $\mathcal{K}_j$ is generic and n is sufficiently small with respect to d_j (namely, $n \leq \frac{d_j}{5}$), then all subspaces of $\mathcal{K}$ defined by independent n-chord diagrams intersect $\mathcal{K}_j$ transversally along nonempty planes. Indeed, if $d_j > 3n$, then the codimension of the set of d_j-dimensional affine subspaces in $\mathcal{K}$, which are not generic with respect to a plane of codimension $3n$, is equal to $d_j - 3n + 1$; therefore the $2n$-parametric family of such sets corresponding to all subspaces defined by n-chord diagrams sweeps out a subset of codimension at least $d_j - 5n + 1$ (if this number is positive), and for $d_j \geq 5n$ we can choose $\mathcal{K}_j$ not from this subset.

If $\mathcal{K}_j$ is generic in this sense, then these intersection sets in $\mathcal{K}_j$ form an affine bundle of dimension $d_j - 3n$ with base $\overline{\mathrm{CD}}_n$. By the construction of the simplicial resolution, this implies that the topology of the sets $\sigma_n(j) \setminus \sigma_{n-1}(j)$ essentially stabilizes at this value of j: for all $j' > j$ the space $\sigma_n(j') \setminus \sigma_{n-1}(j')$ is homeomorphic to the direct product of spaces $\sigma_n(j) \setminus \sigma_{n-1}(j)$ and $\mathbb{R}^{d_{j'}-d_j}$. In particular, we have natural isomorphisms

$$E^1_{n,\beta}(j) \simeq E^1_{n,\beta+(d'_j-d_j)}(j') \quad \text{for all } j' > j,\ n \leq \tfrac{d_j}{5} \text{ and arbitrary } \beta.$$

Substitutions (44) turn them into natural isomorphisms $E_1^{p,q}(j') \simeq E_1^{p,q}(j)$ for all $p \geq -\frac{d_j}{5}$. Moreover, these isomorphisms commute with all the further differentials of our spectral sequence; the Borel–Moore homology groups of the spaces $\sigma_n(j)$ and $\sigma_n(j')$ for all $n \leq \frac{d_j}{5}$ and $j' \geq j$ are naturally isomorphic to each other up to the shift of dimensions by $d_{j'} - d_j$. The cohomology classes of $\mathcal{K} \setminus \Sigma$ arising from this area of the spectral sequence (i.e., the sequences of nontrivial cohomology classes of the spaces $\mathcal{K}_{j'} \setminus \Sigma$, $j' \geq j$, realizable by linking numbers with cycles located in $\sigma_n(j')$ for $n \leq \frac{d_j}{5}$ and corresponding to one another by these isomorphisms) are known as *finite-type* cohomology classes of the space of long knots. Therefore, the intriguing question about the completeness of the system of these classes in entire cohomology groups of $\mathcal{K} \setminus \Sigma$ (in particular, about the existence of nonequivalent knots not separated by finite-type invariants) depends on the groups $E_r^{p,q}(j)$ in the nonstable domains, on the deviation of these groups from stable ones, and on the way in which the nonstable groups $E_\infty^{p,q}(j)$ for different j correspond to the same cohomology classes of spaces $\mathcal{K}_j \setminus \Sigma$ with different j.

The arguments of the previous sections of this article allow us to say something about the nontriviality of this problem.

Proposition 32. *If* $4n - I(n) > d_j \geq 3n$, *then for any* d_j*-dimensional affine subspace* $\mathcal{K}_j \subset \mathcal{K}$ *there exist independent* n*-chord diagrams such that the corresponding affine subspaces of* $\mathcal{K}$ *have nongeneric* (*i.e.*, *either nontransversal or empty*) *intersections with the space* $\mathcal{K}_j$.

Proposition 33. *If* $2n + I(n) \leq d_j \leq 3n$, *then for almost any* d_j*-dimensional affine subspace* $\mathcal{K}_j \subset \mathcal{K}$ (*that is*, *for any subspace from a residual subset in the space of all such subspaces*) *there exist independent* n*-chord diagrams such that the corresponding affine subspaces of* $\mathcal{K}$ *have nonempty intersection with* $\mathcal{K}_j$.

Definition 34. Let $\mathcal{L}$ denote the affine bundle over the space $B(\mathbb{R}^2_+, n) \setminus \Xi$ of independent n-chord diagrams, whose fiber over any such diagram is the subspace of codimension $3n$ in $\mathcal{K}$ consisting of maps $\varphi : \mathbb{R}^1 \to \mathbb{R}^3$ taking the same values at endpoints of each chord of this diagram. For an affine subspace $\mathcal{K}_j \subset \mathcal{K}$ denote by $\|(\mathcal{K}_j)$ the subset in $B(\mathbb{R}^2_+, n) \setminus \Xi$ consisting of n-chord diagrams such that the corresponding fiber of bundle $\mathcal{L}$ contains lines parallel to some lines contained in the space $\mathcal{K}_j$.

Proof of Proposition 32. The normal bundle $\mathcal{L}^\perp$ of $\mathcal{L}$ in $\mathcal{K}$ is isomorphic to the direct sum of three copies of the regular bundle ξ_n. By Lemmas 13 and 14, its total Stiefel–Whitney class is then equal to $(w(\xi_n))^3 \equiv w(\xi_n)$, in particular, its i-dimensional component w_i is not trivial if $i \leq n - I(n)$.

Make $\mathcal{K}_j$ a vector space by arbitrarily choosing the "origin" point in it. If all fibers of the bundle $\mathcal{L}$ are in general position with respect to $\mathcal{K}_j$, then a $(d_j - 3n)$-dimensional vector bundle with the same base is defined, the fiber of which over a chord diagram is obtained from the intersection set of $\mathcal{K}_j$ and the corresponding fiber of the bundle $\mathcal{L}$ by a parallel translation, after which it passes through the origin point of $\mathcal{K}_j$. The total Stiefel–Whitney class of this bundle is equal to $w(\mathcal{L}^\perp)^{-1} \equiv w(\xi_n)^{-1}$, which by Lemma 14 is equal to $w(\xi_n)$. If $d_j - 3n < n - I(n)$, then this implies that $w_{n-I(n)}(\xi_n) = 0$, a contradiction. □

Lemma 35. *If $d_j \leq 3n$, then for almost any d_j-dimensional affine subspace $\mathcal{K}_j \subset \mathcal{K}$ the codimension of the set $\|(\mathcal{K}_j)$ in $B(\mathbb{R}^2_+, n) \setminus \Xi$ is at least $3n - d_j + 1$.*

Proof. Consider the space

$$\tilde{G}(\mathcal{K}, d_j) \times (B(\mathbb{R}^2_+, 2) \setminus \Xi) \tag{46}$$

of all pairs $\{\mathcal{K}_j, \Gamma\}$ where K_j is a d_j-dimensional affine subspace of $\mathcal{K}$ and Γ is an independent n-chord diagram. Denote by Λ the subset of this space consisting of pairs $\{\mathcal{K}_j, \Gamma\}$ such that $\Gamma \in \|(K_j)$. The space (46) and its subset Λ are both fibered over the space $B(\mathbb{R}^2_+, 2) \setminus \Xi$ of independent n-chord diagrams, and for any such diagram Γ the corresponding fiber of the latter fiber bundle has codimension $3n - d_j + 1$ in the fiber of the former. Therefore, the codimension of Λ in the space (46) is equal to $3n - d_j + 1$, and the typical fiber of the projection of Λ to the first factor of (46) has codimension at least $3n - d_j + 1$ in the corresponding fiber of the projection of entire space (46). □

Proof of Proposition 33. Let us fix a subspace $\mathcal{K}_j$ for which the condition of the previous lemma is satisfied. The complement of the set $\|(\mathcal{K}_j)$ in the manifold $B(\mathbb{R}^2_+, n) \setminus \Xi$ has then the same homology groups up to dimension $3n - d_j$ as entire $B(\mathbb{R}^2_+, n) \setminus \Xi$.

Consider the affine bundle $(\mathcal{L}^\perp)^*$ over the manifold $B(\mathbb{R}^2_+, n) \setminus \Xi$: its fibers consist of linear functions on $\mathcal{K}$ vanishing on the corresponding fibers of the bundle $\mathcal{L}$. Over the set $(B(\mathbb{R}^2_+, n) \setminus \Xi) \setminus \|(K_j)$ a $(3n - d_j)$-dimensional subbundle of $(\mathcal{L}^\perp)^*$ is defined, consisting of functions constant on $\mathcal{K}_j$. This subbundle has the same Stiefel–Whitney class (equal to $w(\xi_n)$) as the whole $(\mathcal{L}^\perp)^*$, since its normal bundle is isomorphic to the trivial bundle with fiber $(\mathcal{K}_j)^*$. If no fibers of the bundle $\mathcal{L}$ intersect the space $\mathcal{K}_j$, then this subbundle has a nowhere vanishing cross-section: indeed, we can define an arbitrary Euclidean structure on this subbundle, and choose in each fiber the linear function of unit norm taking the maximal value on $\mathcal{K}_j$. If $3n - d_j \leq n - I(n)$, then this contradicts the nontriviality of the class $w_{n-I(n)}(\xi_n)$. □

Remark. I hope that the further study of the characteristic classes of the bundle $\mathcal{L}$ (and of its analog defined on the space $\overline{\mathrm{CD}}_n$ of equivalence classes of chord diagrams, rather than on the resolution $B(\mathbb{R}^2_+, n) \setminus \Xi$ of this space) will provide not only the proofs of the inevitable troubles in the calculation of the cohomology classes of knot spaces, but also the construction of some such classes not reducible to classes of finite-type.

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