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## POSITIVELY CURVED FINSLER METRICS ON VECTOR BUNDLES, II

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# POSITIVELY CURVED FINSLER METRICS ON VECTOR BUNDLES, II

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**We show that if $E$ is an ample vector bundle of rank at least two with some curvature bound on $O_{P(E^*)}(1)$, then $E^* \otimes \det E$ is Kobayashi positive. The proof relies on comparing the curvature of $(\det E^*)^k$ and $S^k E$ for large $k$ and using duality of convex Finsler metrics. Following the same thread of thought, we show if $E$ is ample with similar curvature bounds on $O_{P(E^*)}(1)$ and $O_{P(E\otimes \det E^*)}(1)$, then $E$ is Kobayashi positive. With additional assumptions, we can furthermore show that $E^* \otimes \det E$ and $E$ are Griffiths positive.**

## 1. Introduction

Let $E$ be a holomorphic vector bundle of rank $r$ over a compact complex manifold $X$ of dimension $n$. We denote the dual bundle by $E^*$ and its projectivized bundle by $P(E^*)$. The vector bundle $E$ is said to be ample if the line bundle $O_{P(E^*)}(1)$ over $P(E^*)$ is ample. On the other hand, $E$ is called Griffiths positive if $E$ carries a Griffiths positive Hermitian metric. Moreover, $E$ is called Kobayashi positive if $E$ carries a strongly pseudoconvex Finsler metric whose Kobayashi curvature is positive (we will give a quick review on Finsler metrics and Kobayashi curvature in Section 2A; or see [Wu 2022, Section 2]).

There are two conjectures made by Griffiths [1969] and Kobayashi [1975] regarding the equivalence of ampleness and positivity:

(1) If $E$ is ample, then $E$ is Griffiths positive.

(2) If $E$ is ample, then $E$ is Kobayashi positive.

These two conjectures are still open, save for $n = 1$, in [Umemura 1973; Campana and Flenner 1990] (for recent progress, see [Berndtsson 2009a; Mourougane and Takayama 2007; Hering et al. 2010; Liu et al. 2013; Liu and Yang 2015; Naumann 2021; Feng et al. 2020; Demailly 2021; Finski 2022; Pingali 2021; Ma and Zhang 2023]). Note that the converse of each conjecture is true [Feng et al. 2020; Wu 2022].

By Kodaira's embedding theorem, ampleness of a line bundle is equivalent to the existence of a positively curved metric on the line bundle. So, the conjectures

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of Griffiths and Kobayashi can be rephrased: *Given a positively curved metric on $O_{P(E^*)}(1)$, can we construct a positively curved Hermitian/Finsler metric on $E$?* In this paper, we show that it is so, by imposing curvature bounds on tautological line bundles of $P(E^*)$ and $P(E)$. Since Hermitian metrics on $O_{P(E^*)}(1)$ are in one-to-one correspondence with Finsler metrics on $E^*$, these curvature bounds can also be written in terms of Kobayashi curvature.

We first consider a relevant case where the picture is clearer. It is known that, for rank of $E$ at least 2:

(1) If $E$ is Griffiths positive, then $E^* \otimes \det E$ with the induced metric is Griffiths positive.

(2) If $E$ is ample, then $E^* \otimes \det E$ is ample.

The first fact can be found in [Demailly 2012, p. 346, Theorem 9.2] and the second in [Hartshorne 1966, Corollary 5.3] together with the isomorphism (see Appendix)

$$\bigwedge^{r-1} E \simeq E^* \otimes \det E.$$

If we follow the guidance of Griffiths and Kobayashi, we would ask whether or not the ampleness of $E$ implies Griffiths/Kobayashi positivity of $E^* \otimes \det E$ for $r \geq 2$. Our first result is that this can be achieved by imposing curvature bounds on $O_{P(E^*)}(1)$.

Let $q : P(E^*) \to X$ be the projection. Let $g$ be a metric on $O_{P(E^*)}(1)$ whose curvature restricted to a fiber $\Theta(g)|_{P(E^*_z)}$ is positive for all $z \in X$. For a tangent vector $\eta \in T^{1,0}_z X$ and a point $[\zeta] \in P(E^*_z)$, we consider tangent vectors $\tilde{\eta}$ to $P(E^*)$ at $(z, [\zeta])$ such that $q_*(\tilde{\eta}) = \eta$, namely the lifts of $\eta$ to $T^{1,0}_{(z,[\zeta])} P(E^*)$. Then we define the function

$$(\eta, [\zeta]) \mapsto \inf_{q_*(\tilde{\eta})=\eta} \Theta(g)(\tilde{\eta}, \bar{\tilde{\eta}}) := m(\eta, [\zeta]), \tag{1-1}$$

where the infimum taken over all the lifts of $\eta$ to $T^{1,0}_{(z,[\zeta])} P(E^*)$. This infimum is actually a minimum, see (2-3). On the other hand, since such a metric $g$ corresponds to a strongly pseudoconvex Finsler metric on $E^*$, and if we denote its Kobayashi curvature by $\theta(g)$ a (1, 1)-form on $P(E^*)$, then

$$m(\eta, [\zeta]) = -\theta(g)(\tilde{\eta}, \bar{\tilde{\eta}}). \tag{1-2}$$

The term on the right is independent of the choice of lifts $\tilde{\eta}$ (we will prove (1-2) in Section 2A).

**Theorem 1.** *Assume $r \geq 2$ and the line bundle $O_{P(E^*)}(1)$ has a positively curved metric $h$ and a metric $g$ with $\Theta(g)|_{P(E^*_z)} > 0$ for all $z \in X$. If there exist a Hermitian* *metric $\Omega$ on $X$ and a constant $M \in [1, r)$ such that the following inequalities of $(1, 1)$-forms hold*:

$$Mq^*\Omega \geq -\theta(g), \tag{1-3}$$

$$q^*\Omega \leq -\theta(h), \tag{1-4}$$

*then $E^* \otimes \det E$ is Kobayashi positive.*

We can of course choose $g$ to be $h$ in Theorem 1, but having two different metrics seems more flexible. The proof of Theorem 1 relies on two observations. First, starting with $g$ and $h$ on $O_{P(E^*)}(1)$, we construct two Hermitian metrics on $S^k E$ and $\det E$ respectively. The curvature of the induced metric on $S^k E \otimes (\det E^*)^k$ can be shown to be Griffiths negative for $k$ large (see Section 3 for details). The second observation (see [Wu 2022]) is that since the induced metric on $S^k E \otimes (\det E^*)^k$ is basically an $L^2$-metric, its $k$-th root is a convex Finsler metric on $E \otimes \det E^*$ which is also strongly plurisubharmonic on the total space minus the zero section. After perturbing this Finsler metric and taking duality, we get a convex and strongly pseudoconvex Finsler metric on $E^* \otimes \det E$ whose Kobayashi curvature is positive. So the bundle $E^* \otimes \det E$ is Kobayashi positive. Notice that the Finsler metric we find is actually convex.

The reason why we impose $\Omega$, $M$ and inequalities (1-3) and (1-4) in Theorem 1 is the following. On the bundle $S^k E \otimes (\det E^*)^k$, the curvature of the induced metric is roughly bounded above by $k \sum_m a_m c_m - rk \sum_m b_m c_m$ where $a_m$ and $b_m$ are some positive integrals with $\sum_m a_m = \sum_m b_m = 1$, and $c_m$ are positive numbers related to the curvature of $h$. It does not seem possible to us that the upper bound $k \sum_m a_m c_m - rk \sum_m b_m c_m$ can be made negative without any assumption. So we introduce $\Omega$ and $M$ to control the upper bound.

With small changes on the proof, one can write down a variant of Theorem 1 where the conclusion is about the Kobayashi positivity of $E^* \otimes (\det E)^l$ (see the end of Section 3).

Now let us go back to the original conjecture of Kobayashi and adapt the proof of Theorem 1 to this case. Let $p : P(E) \to X$ be the projection. We recall under the canonical isomorphism $P(E \otimes \det E^*) \simeq P(E)$, the line bundle $O_{P(E\otimes\det E^*)}(1)$ corresponds to the line bundle $O_{P(E)}(1) \otimes p^* \det E$ (see [Kobayashi 1987, p. 86, Proposition 3.6.21]). Let $g$ be a metric on $O_{P(E)}(1) \otimes p^* \det E$ with $\Theta(g)|_{P(E_z)} > 0$ for all $z \in X$. For a tangent vector $\eta \in T^{1,0}_z X$ and a point $[\xi] \in P(E_z)$, we similarly have

$$(\eta, [\xi]) \mapsto \inf_{p_*(\eta')=\eta} \Theta(g)(\eta', \bar{\eta}'),$$

where $\eta'$ are the lifts of $\eta$ to $T^{1,0}_{(z,[\xi])} P(E)$. Meanwhile, such a metric $g$ corresponds to a strongly pseudoconvex Finsler metric on $E \otimes \det E^*$ and we denote its Kobayashi curvature by $\theta(g)$ a $(1,1)$-form on $P(E)$. As before,

$$\inf_{p_*(\eta')=\eta} \Theta(g)(\eta', \bar{\eta}') = -\theta(g)(\eta', \bar{\eta}'). \tag{1-5}$$

**Theorem 2.** *Assume $r \geq 2$ and $O_{P(E^*)}(1)$ has a positively curved metric $h$ and $O_{P(E)}(1) \otimes p^* \det E$ has a metric $g$ with $\Theta(g)|_{P(E_z)} > 0$ for all $z \in X$. If there exist a Hermitian metric $\Omega$ on $X$ and a constant $M \in [1, r)$ such that*

$$Mp^*\Omega \geq -\theta(g), \tag{1-6}$$

$$q^*\Omega \leq -\theta(h), \tag{1-7}$$

*then $E$ is Kobayashi positive.*

Since the ampleness of $E$ implies ampleness of $E^* \otimes \det E$, one choice for $g$ in Theorem 2 is a positively curved metric on $O_{P(E)}(1) \otimes p^* \det E$, but how much this choice helps is unknown to us. The proof of Theorem 2 follows the same scheme as in Theorem 1. We first use $h$ and $g$ to construct Hermitian metrics on $\det E$ and $S^k E^* \otimes (\det E)^k$ respectively. The induced metric on $[S^k E^* \otimes (\det E)^k] \otimes (\det E^*)^k$ is Griffiths negative for $k$ large (see Section 4). Then by taking $k$-th root, perturbing, and taking duality, we obtain a convex, strongly pseudoconvex, and Kobayashi positive Finsler metric on $E$.

**1A. *Griffiths positivity.*** The conclusions in Theorems 1 and 2 are about Finsler metrics. For their Hermitian counterpart, we need additional assumptions. The reason is that in Theorems 1 and 2, taking large tensor power of various bundles helps us eliminate the curvature of the relative canonical bundles $K_{P(E^*)/X}$ and $K_{P(E)/X}$, and after getting the desired estimates we take $k$-th root to produce Finsler metrics. However, the step of taking $k$-th root produces only Finsler, not Hermitian metrics. So the first step of taking large tensor power is not allowed if one wants Hermitian metrics.

Let us be more precise. For a metric $g$ on $O_{P(E^*)}(1)$ with $\Theta(g)|_{P(E_z^*)} > 0$ for all $z \in X$, we denote $\Theta(g)|_{P(E_z^*)}$ by $\omega_z$ for the moment. The relative canonical bundle $K_{P(E^*)/X}$ has a metric induced from $\{\omega_z^{r-1}\}_{z \in X}$ and we denote the corresponding curvature by $\gamma_g$, a $(1,1)$-form on $P(E^*)$. For $\eta \in T_z^{1,0}X$ and $[\zeta] \in P(E_z^*)$, we consider

$$(\eta, [\zeta]) \mapsto \sup_{q_*(\tilde{\eta})=\eta} \gamma_g(\tilde{\eta}, \bar{\tilde{\eta}}),$$

where the supremum taken over all the lifts of $\eta$ to $T^{1,0}_{(z,[\zeta])}P(E^*)$. The supremum is a maximum under a suitable assumption, see (2-9). Moreover, for $z \in X$, the restriction $\gamma_g|_{P(E_z^*)}$ is actually the negative of Ricci curvature $-\mathrm{Ric}_{\omega_z}$ of the metric $\omega_z$ on $P(E_z^*)$.

Any Hermitian metric $G$ on $E^*$ will induce a metric $g$ on $O_{P(E^*)}(1)$ with $\Theta(g)|_{P(E_z^*)} > 0$ and $\gamma_g|_{P(E_z^*)} < 0$ for all $z \in X$. Indeed, in this case, $\Theta(g)|_{P(E_z^*)}$

is the Fubini–Study metric and its Ricci curvature is positive, so $\gamma_g|_{P(E_z^*)} < 0$. Furthermore, for any $\eta \in T_z^{1,0}X$ and any $[\zeta] \in P(E_z^*)$,

$$\sup_{q_*(\tilde{\eta})=\eta} \gamma_g(\tilde{\eta}, \bar{\tilde{\eta}}) = r\theta(g)(\tilde{\eta}, \bar{\tilde{\eta}}) - q^*\Theta(\det G)(\tilde{\eta}, \bar{\tilde{\eta}}) \tag{1-8}$$

(we will prove (1-8) in Section 2B).

**Theorem 3.** *Assume $r \geq 2$ and the line bundle $O_{P(E^*)}(1)$ has a positively curved metric $h$ and a metric $g$ induced from a Hermitian metric $G$ on $E^*$. If there exist a Hermitian metric $\Omega$ on $X$ and a constant $M \in [1, r)$ such that*

$$Mq^*\Omega \geq -(r+1)\,\theta(g) + q^*\Theta(\det G), \tag{1-9}$$

$$q^*\Omega \leq -\theta(h), \tag{1-10}$$

*then $E^* \otimes \det E$ is Griffiths positive.*

Theorem 3 could be seen as a Hermitian analogue of Theorem 1. To state a Hermitian analogue of Theorem 2, we use again the isomorphism between $O_{P(E\otimes\det E^*)}(1) \to P(E \otimes \det E^*)$ and $O_{P(E)}(1) \otimes p^* \det E \to P(E)$.

**Theorem 4.** *Suppose that $r \geq 2$ and $O_{P(E^*)}(1)$ has a positively curved metric $h$ and $O_{P(E)}(1) \otimes p^* \det E$ has a metric $g$ induced from a Hermitian metric $G$ on $E \otimes \det E^*$. If there exist a Hermitian metric $\Omega$ on $X$ and a constant $M \in [1, r)$ such that*

$$Mp^*\Omega \geq -(r+1)\,\theta(g) + p^*\Theta(\det G), \tag{1-11}$$

$$q^*\Omega \leq -\theta(h), \tag{1-12}$$

*then $E$ is Griffiths positive.*

In all the theorems above, the existence of the metric $h$ comes from ampleness of $E$. So the real assumptions lie in $(g, \Omega, M)$ and the inequalities they have to satisfy. To weaken or remove these inequalities, one possible direction is to use geometric flows as in [Naumann 2021; Wan 2022; Ustinovskiy 2019; Li et al. 2021]. Another possible direction is to use the interplay between the optimal $L^2$-estimates and the positivity of curvature (see [Guan and Zhou 2015; Berndtsson and Lempert 2016; Lempert 2017; Hacon et al. 2018; Zhou and Zhu 2018]).

One example where the assumptions of all the theorems above are satisfied is given by $E = L^9 \oplus L^8 \oplus L^7$ with $L$ a positive line bundle. The triple $(9, 8, 7)$ or the rank $r = 3$ is not that important; the point is to make sure the eigenvalues of the curvature with respect to some positive $(1, 1)$-form do not spread out too far. This example also indicates that a reasonable choice for $\Omega$ is probably related to $c_1(\det E)$.

A more sophisticated example, related to approximate Hermitian–Yang–Mills metrics [Jacob 2014; Misra and Ray 2021; Li et al. 2021], is semistable ample vector bundles over Riemann surfaces (see Section 7 for details of the examples).

The proof of Theorem 1 is given in Section 3 and almost as a corollary we prove Theorem 2 in Section 4. The proof of Theorem 3 in Section 5 is a modification of Theorem 1 but we still write out the details. In Section 6, we prove Theorem 4 based on Section 5.

## 2. Preliminaries

**2A.** ***Finsler metrics.*** We will use some facts about Finsler metrics on vector bundles which can be found in [Kobayashi 1975; 1996; Cao and Wong 2003; Aikou 2004; Wu 2022]. First, we recall the definition of Finsler metrics. Let $E^*$ be a holomorphic vector bundle of rank $r$ over a compact complex manifold $X$. For a vector $\zeta \in E^*_z$, we symbolically write $(z, \zeta) \in E^*$. A Finsler metric $G$ on the vector bundle $E^* \to X$ is a real-valued function on $E^*$ such that:

(1) $G$ is smooth away from the zero section of $E^*$.

(2) For $(z, \zeta) \in E^*$, $G(z, \zeta) \geq 0$, and equality holds if and only if $\zeta = 0$.

(3) $G(z, \lambda\zeta) = |\lambda|^2 G(z, \zeta)$ for $\lambda \in \mathbb{C}$.

A Finsler metric $G$ on $E^*$ is said to be:

(1) Strongly pseudoconvex if the fiberwise complex Hessian of $G$ is positive definite on $E^* \setminus \{\text{zero section}\}$, namely $(\sqrt{-1}\partial\bar\partial G)|_{E^*_z} > 0$ for all $z \in X$.

(2) Convex if $G^{1/2}$ restricted to each fiber $E^*_z$ is convex.

Let $g$ be a Hermitian metric on $O_{P(E^*)}(1)$ with $\Theta(g)|_{P(E^*_z)} > 0$ for all $z \in X$. Such a $g$ corresponds to a strongly pseudoconvex Finsler metric $G$ on $E^*$. Since $(\sqrt{-1}\partial\bar\partial G)|_{E^*_z} > 0$, we can define a Hermitian metric $\tilde{G}$ on the pull-back bundle $q^*E^*$, where $q : P(E^*) \to X$ is the projection, as follows. For a vector $Z$ in the fiber $q^*E^*_{(z,[\zeta])}$, we define

$$\tilde{G}_{(z,[\zeta])}(Z, Z) = (\sqrt{-1}\partial\bar\partial G)|_{E^*_z}(Z, \bar{Z}),$$

where the $Z$ on the right-hand side is viewed as a tangent vector to $E^*_z$ at $\zeta$ by the identification of vector spaces $q^*E^*_{(z,[\zeta])} = E^*_z$ and $E^*_z = T_\zeta E^*_z$ (see [Wu 2022, Section 2.2] for a local coordinate description).

Now $(q^*E^*, \tilde{G})$ is a Hermitian holomorphic vector bundle, so we can talk about its Chern curvature $\Theta$, an $\operatorname{End} q^*E^*$-valued $(1, 1)$-form on $P(E^*)$. With respect to the metric $\tilde{G}$, the bundle $q^*E^*$ has a fiberwise orthogonal decomposition

$$O_{P(E^*)}(-1) \oplus O_{P(E^*)}(-1)^{\perp},$$

and so $\Theta$ can be written as a block matrix. Let $\Theta|_{O_{P(E^*)}(-1)}$ denote the block in the matrix $\Theta$ corresponding to $\operatorname{End}(O_{P(E^*)}(-1))$. Since $O_{P(E^*)}(-1)$ is a line bundle, $\Theta|_{O_{P(E^*)}(-1)}$ is a (1, 1)-form on $P(E^*)$, and it is called the Kobayashi curvature of the Finsler metric $G$. We will use $\theta(g)$ to denote the Kobayashi curvature

$$\theta(g) := \Theta|_{O_{P(E^*)}(-1)}. \tag{2-1}$$

In order to relate the Kobayashi curvature $\theta(g)$ to the curvature $\Theta(g)$ of $g$, we consider coordinates normal at one point. Given a point $(z_0, [\zeta_0]) \in P(E^*)$, there exists a holomorphic frame $\{s_i\}$ for $E^*$ around $z_0 \in X$ such that

$$\begin{gathered} G_{\zeta_i\bar{\zeta}_j}(z_0, \zeta_0) = \delta_{ij}, \\ G_{\zeta_i\bar{\zeta}_j z_\alpha}(z_0, \zeta_0) = G_{\zeta_i\bar{\zeta}_j\bar{z}_\beta}(z_0, \zeta_0) = G_{\bar{\zeta}_j z_\alpha}(z_0, \zeta_0) = G_{z_\alpha}(z_0, \zeta_0) = 0, \end{gathered} \tag{2-2}$$

where we use $\{\zeta_i\}$ for the fiber coordinates on $E^*$ with respect to the frame $\{s_i\}$ and $\{z_\alpha\}$ for the local coordinates on $X$ (such a frame can be obtained by (5.11) in [Kobayashi 1996]). Moreover if $\Omega$ is a Hermitian metric on $X$, then by a linear transformation in the $z$-coordinates, we can make

$$\Omega\left(\frac{\partial}{\partial z_\alpha}, \frac{\partial}{\partial \bar{z}_\beta}\right)(z_0) = \delta_{\alpha\beta}$$

without affecting (2-2). We will call this coordinate system normal at the point $(z_0, [\zeta_0]) \in P(E^*)$.

Around the point $(z_0, [\zeta_0]) \in P(E^*)$, we assume the local coordinates

$$(z_1, \dots, z_n, w_1, \dots, w_{r-1})$$

are given by $w_i = \zeta_i/\zeta_r$ for $i = 1 \sim r-1$. So

$$e := \frac{\zeta_1 s_1 + \cdots + \zeta_r s_r}{\zeta_r} = w_1 s_1 + \cdots + w_{r-1} s_{r-1} + s_r$$

is a holomorphic frame for $O_{P(E^*)}(-1)$. Let $e^*$ be the dual frame of $O_{P(E^*)}(1)$ around $(z_0, [\zeta_0]) \in P(E^*)$ and $g(e^*, e^*) = e^{-\phi}$. Then, the curvature $\Theta(g)$ can be written locally as

$$\begin{aligned} \sum_{\alpha,\beta} \frac{\partial^2\phi}{\partial z_\alpha \partial \bar{z}_\beta} dz_\alpha \wedge d\bar{z}_\beta + \sum_{\alpha,j} \frac{\partial^2\phi}{\partial z_\alpha \partial \overline{w}_j} dz_\alpha \wedge d\overline{w}_j \\ + \sum_{i,\beta} \frac{\partial^2\phi}{\partial w_i \partial \bar{z}_\beta} dw_i \wedge d\bar{z}_\beta + \sum_{i,j} \frac{\partial^2\phi}{\partial w_i \partial \overline{w}_j} dw_i \wedge d\overline{w}_j. \end{aligned}$$

Note that the terms $\frac{\partial^2\phi}{\partial z_\alpha \partial \overline{w}_j} := \phi_{\alpha\bar{j}}$ vanish at $(z_0, [\zeta_0])$ by (2-2) and the fact

$$e^\phi = \frac{1}{g(e^*, e^*)} = G(w_1 s_1 + \cdots + w_{r-1} s_{r-1} + s_r).$$

For a tangent vector $\eta \in T^{1,0}_{z_0}X$, we can write $\eta = \sum_\alpha \eta_\alpha \frac{\partial}{\partial z_\alpha}$. For the lifts $\tilde{\eta}$ of $\eta$ to $T^{1,0}_{(z_0,[\zeta_0])}P(E^*)$, we have

$$\inf_{q_*(\tilde{\eta})=\eta} \Theta(g)(\tilde{\eta}, \bar{\tilde{\eta}}) = \sum_{\alpha,\beta} \phi_{\alpha\bar{\beta}}|_{(z_0,[\zeta_0])}\, \eta_\alpha \bar{\eta}_\beta \tag{2-3}$$

because $\phi_{\alpha\bar{j}} = 0$ at $(z_0, [\zeta_0])$ and the matrix $(\phi_{i\bar{j}})$ is positive. On the other hand, using the same coordinate system, the curvature $\Theta$ of $\tilde{G}$ can be written as

$$\Theta = \sum_{\alpha,\beta} R_{\alpha\bar{\beta}}\, dz_\alpha \wedge d\bar{z}_\beta + \sum_{\alpha,l} P_{\alpha\bar{l}}\, dz_\alpha \wedge d\bar{w}_l + \sum_{k,\beta} \mathcal{P}_{k\bar{\beta}}\, dw_k \wedge d\bar{z}_\beta + \sum_{k,l} Q_{k\bar{l}}\, dw_k \wedge d\bar{w}_l,$$

where $R_{\alpha\bar{\beta}}$, $P_{\alpha\bar{l}}$, $\mathcal{P}_{k\bar{\beta}}$, and $Q_{k\bar{l}}$ are endomorphisms of $q^*E^*$. By [Wu 2022, (2.4)], for any lift $\tilde{\eta}$ of $\eta$ to $T^{1,0}_{(z_0,[\zeta_0])}P(E^*)$, we have

$$\begin{aligned} &\theta(g)(\tilde{\eta}, \bar{\tilde{\eta}}) \\ &\quad = \Theta|_{O_{P(E^*)}(-1)}(\tilde{\eta}, \bar{\tilde{\eta}}) = \sum_{\alpha,\beta} \frac{\tilde{G}(R_{\alpha\bar{\beta}}\,\zeta_0, \zeta_0)}{\tilde{G}(\zeta_0, \zeta_0)}\, \eta_\alpha \bar{\eta}_\beta = -\sum_{\alpha,\beta} \phi_{\alpha\bar{\beta}}|_{(z_0,[\zeta_0])}\, \eta_\alpha \bar{\eta}_\beta, \end{aligned} \tag{2-4}$$

where the last equality is by [Kobayashi 1996, (5.16)].

From (2-3) and (2-4), we see

$$\inf_{q_*(\tilde{\eta})=\eta} \Theta(g)(\tilde{\eta}, \bar{\tilde{\eta}}) = -\theta(g)(\tilde{\eta}, \bar{\tilde{\eta}}),$$

which is formula (1-2) we claim in the introduction, and when evaluated using normal coordinates they are $\sum_{\alpha,\beta} \phi_{\alpha\bar{\beta}}|_{(z_0,[\zeta_0])}\, \eta_\alpha \bar{\eta}_\beta$.

**2B.** ***Hermitian metrics.*** This subsection is a special case of Section 2A and it will be used in the proofs of Theorems 3 and 4. Let $G$ be a Hermitian metric on the bundle $E^*$. The pull-back bundle $q^*E^* \to P(E^*)$ with the pull-back metric $q^*G$ induces a metric $g^*$ on the subbundle $O_{P(E^*)}(-1)$. We denote the dual metric on $O_{P(E^*)}(1)$ by $g$.

Let $\Omega$ be a Hermitian metric on $X$ and $z_0$ a point in $X$ with local coordinates $\{z_\alpha\}$ such that

$$\Omega\left(\frac{\partial}{\partial z_\alpha}, \frac{\partial}{\partial \bar{z}_\beta}\right)(z_0) = \delta_{\alpha\beta}.$$

There exists a holomorphic frame $\{s_i\}$ for $E^*$ around $z_0$ such that

$$G(s_i, s_j) = \delta_{ij} + O(|z|^2),$$

where $z_0$ corresponds to the origin in the local coordinates. We use $\{\zeta_i\}$ for the fiber coordinates with respect to the frame $\{s_i\}$. For a point $(z_0, [\zeta_0]) \in P(E^*)$, we assume the local coordinates $(z_1, \ldots, z_n, w_1, \ldots, w_{r-1})$ around $(z_0, [\zeta_0])$ are given by $w_i = \zeta_i/\zeta_r$ for $i = 1 \sim r-1$. So

$$e := \frac{\zeta_1 s_1 + \cdots + \zeta_r s_r}{\zeta_r} = w_1 s_1 + \cdots + w_{r-1} s_{r-1} + s_r$$

is a holomorphic frame for $O_{P(E^*)}(-1)$ and

$$\begin{aligned} g^*(e,e) &= q^*G(w_1 s_1+\cdots+w_{r-1}s_{r-1}+s_r,\, w_1 s_1+\cdots+w_{r-1}s_{r-1}+s_r)\\ &= 1+O(|z|^2)+O(|w|^2)+O(|w|\,|z|^2)+O(|w|^2|z|^2). \end{aligned}$$

The $z_\alpha$-derivative of $g^*(e,e)$ is $g^*(e,e)_{z_\alpha}=O((1+|w|+|w|^2)|z|)$, and hence the $w_i$-derivatives of $g^*(e,e)_{z_\alpha}$ of any order are zero when evaluated at $z_0$. Therefore, if we denote $g^*(e,e)$ by $e^\phi$, then at $z_0$

$$\phi_{\alpha\bar{j}}=\phi_{\alpha i\bar{j}}=\phi_{\alpha i\bar{j}\bar{k}}=0 \quad\text{and}\quad (\log\det(\phi_{i\bar{j}}))_{\alpha\bar{k}}=0. \tag{2-5}$$

In this coordinate system, the curvature $\Theta(g)$ is

$$\sum_{\alpha,\beta}\frac{\partial^2\phi}{\partial z_\alpha\partial\bar{z}_\beta}dz_\alpha\wedge d\bar{z}_\beta+\sum_{\alpha,j}\frac{\partial^2\phi}{\partial z_\alpha\partial\bar{w}_j}dz_\alpha\wedge d\bar{w}_j +\sum_{i,\beta}\frac{\partial^2\phi}{\partial w_i\partial\bar{z}_\beta}dw_i\wedge d\bar{z}_\beta+\sum_{i,j}\frac{\partial^2\phi}{\partial w_i\partial\bar{w}_j}dw_i\wedge d\bar{w}_j.$$

For a tangent vector $\eta\in T^{1,0}_{z_0}X$, we can write $\eta=\sum_\alpha\eta_\alpha\frac{\partial}{\partial z_\alpha}$. For the lifts $\tilde{\eta}$ of $\eta$ to $T^{1,0}_{(z_0,[\zeta_0])}P(E^*)$, we have

$$\inf_{q_*(\tilde{\eta})=\eta}\Theta(g)(\tilde{\eta},\bar{\tilde{\eta}})=\sum_{\alpha,\beta}\phi_{\alpha\bar{\beta}}|_{(z_0,[\zeta_0])}\eta_\alpha\bar{\eta}_\beta \tag{2-6}$$

because $\phi_{\alpha\bar{j}}=0$ at $z_0$ and the matrix $(\phi_{i\bar{j}})$ is positive. Since $G$ is a Hermitian metric, the corresponding Kobayashi curvature is

$$\theta(g)=q^*\Theta(G)|_{O_{P(E^*)}(-1)}, \tag{2-7}$$

which is equal to the negative of (2-6) by Section 2A.

Using the same coordinate system, the restriction $\Theta(g)|_{P(E^*_z)}$ is $\sum\phi_{i\bar{j}}\,dw_i\wedge d\bar{w}_j$, so the metric on $K_{P(E^*)/X}$ induced from $\{(\Theta(g)|_{P(E^*_z)})^{r-1}\}_{z\in X}$ has its curvature $\gamma_g$ equal to

$$\sum_{\alpha,\beta}(\log\det(\phi_{i\bar{j}}))_{\alpha\bar{\beta}}\,dz_\alpha\wedge d\bar{z}_\beta+\sum_{\alpha,j}(\log\det(\phi_{i\bar{j}}))_{\alpha\bar{j}}\,dz_\alpha\wedge d\bar{w}_j +\sum_{i,\beta}(\log\det(\phi_{i\bar{j}}))_{i\bar{\beta}}\,dw_i\wedge d\bar{z}_\beta+\sum_{i,j}(\log\det(\phi_{i\bar{j}}))_{i\bar{j}}\,dw_i\wedge d\bar{w}_j. \tag{2-8}$$

The matrix $\big((\log\det(\phi_{i\bar{j}}))_{i\bar{j}}\big)$ is negative because it represents the negative of the Ricci curvature of the Fubini–Study metric on $P(E^*_z)$. Moreover, the terms $(\log\det(\phi_{i\bar{j}}))_{\alpha\bar{j}}=0$ at $z_0$ by (2-5). As a result, for a tangent vector $\eta\in T^{1,0}_{z_0}X$ with $\eta=\sum\eta_\alpha\frac{\partial}{\partial z_\alpha}$ in this coordinate system, we have

$$\sup_{q_*(\tilde{\eta})=\eta}\gamma_g(\tilde{\eta},\bar{\tilde{\eta}})=\sum_{\alpha,\beta}(\log\det(\phi_{i\bar{j}}))_{\alpha\bar{\beta}}|_{(z_0,[\zeta_0])}\eta_\alpha\bar{\eta}_\beta, \tag{2-9}$$

where $\tilde{\eta}$ are the lifts of $\eta$ to $T^{1,0}_{(z_0,[\zeta_0])}P(E^*)$.

Finally, the metric on $K_{P(E^*)/X}$ induced from $\{(\Theta(g)|_{P(E_z^*)})^{r-1}\}_{z\in X}$ can be identified with the metric $(g^*)^r \otimes q^*(\det G^*)$ under the isomorphism

$$K_{P(E^*)/X} \simeq O_{P(E^*)}(-r) \otimes q^* \det E$$

(see [Kobayashi 1987, p. 85, Proposition 3.6.20]). This fact can be verified at one point using the normal coordinates above. Therefore,

$$\gamma_g = -r\Theta(g) - q^*\Theta(\det G). \tag{2-10}$$

So, for any $\eta \in T_z^{1,0}X$ and any $[\zeta] \in P(E_z^*)$,

$$\begin{aligned}\sup_{q_*(\tilde\eta)=\eta} \gamma_g(\tilde\eta, \bar{\tilde\eta}) &= -r \inf_{q_*(\tilde\eta)=\eta} \Theta(g)(\tilde\eta, \bar{\tilde\eta}) - \Theta(\det G)(\eta, \bar\eta)\\ &= r\theta(g)(\tilde\eta, \bar{\tilde\eta}) - \Theta(\det G)(\eta, \bar\eta).\end{aligned}$$

This is formula (1-8) that we promise to prove in the introduction.

**2C. *Convexity.*** Let $E$ be a holomorphic vector bundle of rank $r$ over a compact complex manifold $X$. Given a Hermitian metric $H_k$ on the symmetric power $S^kE$, we can define a Finsler metric on $E$ by assigning to $u \in E$ length $H_k(u^k, u^k)^{1/2k}$. We will denote this Finsler metric by $H_k^{1/2k}$, namely $H_k^{1/2k}(u) = H_k(u^k, u^k)^{1/2k}$.

**Lemma 5.** *Let $F_1$ be a vector bundle and $F_2$ a line bundle over $X$. Assume $F_2$ carries a Hermitian metric $H$. We also assume, for some $k$, $S^kF_1$ carries a Hermitian metric $H_k$ such that the induced Finsler metric $H_k^{1/2k}$ on $F_1$ is convex*:

$$H_k^{1/2k}(u+v) \le H_k^{1/2k}(u) + H_k^{1/2k}(v) \quad \text{for } u, v \in F_1.$$

*Then the Finsler metric $(H_k \otimes H^k)^{1/2k}$ on $F_1 \otimes F_2$ is convex.*

Since $F_2$ is a line bundle, there is a canonical isomorphism between the bundles $S^k(F_1 \otimes F_2)$ and $S^kF_1 \otimes F_2^k$ which we use implicitly in the statement of Lemma 5. Roughly speaking, Lemma 5 indicates that convexity is not affected by tensoring with a line bundle.

*Proof.* Fix $p \in X$. The fiber $F_2|_p$ is a one dimensional vector space and we let $e$ be a basis. For $x$ and $y \in F_1 \otimes F_2|_p$, we can write $x = \tilde x \otimes e$ and $y = \tilde y \otimes e$ where $\tilde x, \tilde y \in F_1|_p$. By definition,

$$\begin{aligned}(H_k \otimes H^k)^{1/2k}(x+y) &= H_k \otimes H^k((x+y)^k, (x+y)^k)^{1/2k}\\ &= H_k \otimes H^k((\tilde x+\tilde y)^k \otimes e^k, (\tilde x+\tilde y)^k \otimes e^k)^{1/2k}\\ &= H_k((\tilde x+\tilde y)^k, (\tilde x+\tilde y)^k)^{1/2k} H^k(e^k, e^k)^{1/2k}\\ &\le [H_k(\tilde x^k, \tilde x^k)^{1/2k} + H_k(\tilde y^k, \tilde y^k)^{1/2k}]\, H^k(e^k, e^k)^{1/2k}\\ &= (H_k \otimes H^k)^{1/2k}(x) + (H_k \otimes H^k)^{1/2k}(y).\end{aligned}$$

Therefore the Finsler metric $(H_k \otimes H^k)^{1/2k}$ is convex. □

**2D.** ***Direct image bundles.*** We recall how to construct Hermitian metrics on direct image bundles and compute their curvature. Let $g$ be a Hermitian metric on $O_{P(E^*)}(1)$ with curvature $\Theta(g)$. Denote the restriction of the curvature to a fiber, $\Theta(g)|_{P(E^*_z)}$ by $\omega_z$ for $z \in X$, and assume $\omega_z > 0$ for all $z \in X$. With the canonical isomorphism

$$\Phi_{k,z} : S^k E_z \to H^0(P(E^*_z), O_{P(E^*_z)}(k)) \quad \text{for } k \geq 0$$

(see [Demailly 2012, p. 278, Theorem 15.5]), we define a Hermitian metric $H_k$ on $S^k E$ by

$$H_k(u, v) := \int_{P(E^*_z)} g^k(\Phi_{k,z}(u), \Phi_{k,z}(v))\, \omega_z^{r-1} \quad \text{for } u \text{ and } v \in S^k E_z. \tag{2-11}$$

Let us denote by $\Theta_k$ the curvature of $H_k$. Fixing $z \in X$ and $u \in S^k E_z$, in order to estimate the $(1, 1)$-form $H_k(\Theta_k u, u)$, we first extend the vector $u$ to a local holomorphic section $\tilde{u}$ whose covariant derivative at $z$ with respect to $H_k$ equals zero. A straightforward computation shows

$$\partial\bar{\partial} H_k(\tilde{u}, \tilde{u})|_z = -H_k(\Theta_k u, u).$$

But $H_k(\tilde{u}, \tilde{u})(z)$ for $z$ near $z$ can also be written as the push-forward

$$q_*\big(g^k(\Phi_{k,z}(\tilde{u}), \Phi_{k,z}(\tilde{u}))\, \Theta(g)^{r-1}\big),$$

where $q : P(E^*) \to X$ is the projection, so

$$-H_k(\Theta_k u, u) = \partial\bar{\partial} H_k(\tilde{u}, \tilde{u})|_z = q_*\, \partial\bar{\partial}\big(g^k(\Phi_{k,z}(\tilde{u}), \Phi_{k,z}(\tilde{u}))\, \Theta(g)^{r-1}\big)|_z. \tag{2-12}$$

Similarly, we can use a metric on $O_{P(E)}(1) \otimes p^* \det E$ to construct Hermitian metrics on $S^k E^* \otimes (\det E)^k$. The formula is similar to (2-11), and we use bold symbols to highlight the change. Let $\boldsymbol{g}$ be a metric on $O_{P(E)}(1) \otimes p^* \det E$ with curvature $\Theta(\boldsymbol{g})$. Denote the restriction of the curvature to a fiber $\Theta(\boldsymbol{g})|_{P(E_z)}$ by $\boldsymbol{\omega}_z$ for $z \in X$. Assume $\boldsymbol{\omega}_z > 0$ for all $z \in X$. With the canonical isomorphism

$$\boldsymbol{\Phi}_{k,z} : S^k E^*_z \otimes (\det E_z)^k \to H^0\big(P(E_z), O_{P(E_z)}(k) \otimes (p^* \det E_z)^k\big) \quad \text{for } k \geq 0,$$

we define a Hermitian metric $\boldsymbol{H}_k$ on $S^k E^* \otimes (\det E)^k$ by

$$\boldsymbol{H}_k(u, v) := \int_{P(E_z)} \boldsymbol{g}^k(\boldsymbol{\Phi}_{k,z}(u), \boldsymbol{\Phi}_{k,z}(v))\, \boldsymbol{\omega}_z^{r-1} \tag{2-13}$$

for $u$ and $v \in S^k E^*_z \otimes (\det E_z)^k$. We also have a curvature formula similar to (2-12).

**2E.** ***Berndtsson's positivity theorem.*** Let $h$ be a metric on $O_{P(E^*)}(1)$ with curvature $\Theta(h) > 0$. Denote $\Theta(h)|_{P(E_z^*)}$ by $\omega_z$ for $z \in X$. We are going to define a Hermitian metric on $\det E$ using the metric $h$. The relative canonical bundle $K_{P(E^*)/X}$ has a metric induced from $\{\omega_z^{r-1}\}_{z\in X}$. With $h^r$ on $O_{P(E^*)}(r)$ and the isomorphism $K_{P(E^*)/X} \otimes O_{P(E^*)}(r) \simeq q^* \det E$, there is an induced metric $\rho$ on $q^* \det E$. Using the canonical isomorphism

$$\Psi_z : \det E_z \to H^0(P(E_z^*), q^* \det E_z),$$

we define a Hermitian metric $H$ on $\det E$ by

$$H(u, v) := \int_{P(E_z^*)} \rho(\Psi_z(u), \Psi_z(v))\, \omega_z^{r-1} \quad \text{for } u \text{ and } v \in \det E_z. \tag{2-14}$$

By Berndtsson's theorem [Berndtsson 2009a], this metric $H$ is Griffiths positive, but it is the inequality that leads to this fact we will use. We follow the presentation in [Liu et al. 2013, Section 4.1] (see also [Berndtsson 2009b, Section 2]). Denote the curvature of $H$ by $\Theta$. Fix $z \in X$, $v \in \det E_z$ and $\eta \in T_z^{1,0} X$. For a local holomorphic frame of $E^*$ around $z$, we denote by $\{\zeta_i\}$ the fiber coordinates with respect to this frame, and by $\{z_\alpha\}$ the local coordinates on $X$. Around $P(E_z^*)$ in $P(E^*)$, we have homogeneous coordinates $[\zeta_1, \dots, \zeta_r]$ which induce local coordinates $(w_1, \dots, w_{r-1})$. For a local frame $e^*$ of $O_{P(E^*)}(1)$, we denote $h(e^*, e^*)$ by $e^{-\phi}$ and write the tangent vector $\eta = \sum \eta_\alpha \frac{\partial}{\partial z_\alpha}$. The inequality that leads to Berndtsson's theorem is

$$\begin{aligned} &-H(\Theta v, v)(\eta, \bar\eta) \\ &\qquad \le \int_{P(E_z^*)} \rho(\Psi_z(v), \Psi_z(v))\, r \sum_{\alpha,\beta} \Bigl( \sum_{i,j} \phi_{\alpha\bar j}\, \phi^{i\bar j}\, \phi_{i\bar\beta} - \phi_{\alpha\bar\beta} \Bigr) \eta_\alpha\, \bar\eta_\beta\, \omega_z^{r-1}, \end{aligned} \tag{2-15}$$

where

$$\phi_{i\bar j} := \frac{\partial^2 \phi}{\partial w_i\, \partial \bar w_j}, \quad \phi_{\alpha\bar j} := \frac{\partial^2 \phi}{\partial z_\alpha\, \partial \bar w_j}, \quad \phi_{\alpha\bar\beta} := \frac{\partial^2 \phi}{\partial z_\alpha\, \partial \bar z_\beta},$$

and $(\phi^{i\bar j})$ is the inverse matrix of $(\phi_{i\bar j})$. Since $\det E$ is a line bundle, the curvature $\Theta$ is a $(1,1)$-form, and so $H(\Theta v, v)(\eta, \bar\eta) = H(v, v)\, \Theta(\eta, \bar\eta)$. If we further assume $H(v, v) = 1$, then the left-hand side of (2-15) becomes $-\Theta(\eta, \bar\eta)$.

## 3. Proof of Theorem 1

Recall that $h$ and $g$ are metrics on $O_{P(E^*)}(1)$ that satisfy the assumptions in Theorem 1 and the inequalities (1-3) and (1-4). We use the metric $h$ to construct a Hermitian metric $H$ on $\det E$ as in (2-14), and the metric $g$ to construct Hermitian metrics $H_k$ on $S^k E$ as in (2-11). The number $k$ is yet to be determined.

We start with the metric $g$. Given a point $(z_0, [\zeta_0]) \in P(E^*)$, we have the normal coordinate system from Section 2A. In this coordinate system, let us introduce the following $n$-by-$n$ matrix-valued function:

$$B_k = ((B_k)_{\alpha\beta}) := \big(k\phi_{\alpha\bar{\beta}} - (\log\det(\phi_{i\bar{j}}))_{\alpha\bar{\beta}}\big),$$

where $g(e^*, e^*) = e^{-\phi}$. By continuity, there is a neighborhood $U$ of $(z_0, [\zeta_0])$ in $P(E^*)$ such that in $U$

$$(\phi_{\alpha\bar{\beta}})|_{(z_0,[\zeta_0])} + \frac{r-M}{4}\,\mathrm{Id}_{n\times n} \geq (\phi_{\alpha\bar{\beta}}). \tag{3-1}$$

For this $U$, there is a positive integer $k_0$ such that for $k \geq k_0$ and in $U$

$$(\phi_{\alpha\bar{\beta}}) + \frac{r-M}{4}\,\mathrm{Id}_{n\times n} \geq \frac{B_k}{k}. \tag{3-2}$$

Let us summarize what we have done so far:

**Lemma 6.** *Given a point $(z_0, [\zeta_0]) \in P(E^*)$, there exist a coordinate neighborhood $U$ of $(z_0, [\zeta_0])$ in $P(E^*)$ and a positive integer $k_0$ such that in $U$ and for $k \geq k_0$*

$$(\phi_{\alpha\bar{\beta}})|_{(z_0,[\zeta_0])} + \frac{r-M}{2}\,\mathrm{Id}_{n\times n} \geq \frac{B_k}{k}. \tag{3-3}$$

By Lemma 6, since $P(E^*_{z_0})$ is compact, we can find on $P(E^*_{z_0})$ finitely many points $\{(z_0, [\zeta_l])\}_l$ each of which corresponds to a coordinate neighborhood $U_l$ in $P(E^*)$ and a positive integer $k_l$ such that the corresponding (3-3) holds, and $P(E^*_{z_0}) \subset \bigcup_l U_l$. Denote $\max_l k_l$ by $k_{\max}$. The point $z_0$ has a neighborhood $W$ in $X$ such that for $z \in W$, the fiber $P(E^*_z)$ can be partitioned as $\bigcup_m V_m$ with each $V_m$ in $U_l$ for some $l$. By shrinking $W$, we can assume that for each $U_l$ the corresponding $\Omega\big(\frac{\partial}{\partial z_\alpha}, \frac{\partial}{\partial \bar{z}_\beta}\big) := \Omega_{\alpha\bar{\beta}}$ satisfies

$$-\varepsilon\delta_{\alpha\beta} < \Omega_{\alpha\bar{\beta}}(z) - \delta_{\alpha\beta} < \varepsilon\delta_{\alpha\beta} \quad \text{for } z \in W, \tag{3-4}$$

where $\varepsilon := \frac{r-M}{5(r+M)}$.

Recall the Hermitian metrics $H_k$ on $S^k E$ in (2-11) constructed using the metric $g$. Denote by $\Theta_k$ the curvature of $H_k$. We claim the following lemma (one can also use the asymptotic expansion in [Ma and Zhang 2023] to deduce the lemma).

**Lemma 7.** *For $k \geq k_{\max}$, $z \in W$, $0 \neq \eta \in T^{1,0}_z X$, and $u \in S^k E_z$ with $H_k(u, u) = 1$, we have*

$$H_k(\Theta_k u, u)(\eta, \bar{\eta}) \leq \Big(M + \frac{r-M}{2}\Big)\, k\, \frac{\Omega(\eta, \bar{\eta})}{(1-\varepsilon)}. \tag{3-5}$$

*Proof.* As in Section 2D, we extend the vector $u \in S^k E_z$ to a local holomorphic section $\tilde{u}$ whose covariant derivative at $z$ equals zero, and we have

$$-H_k(\Theta_k u, u) = \partial\bar{\partial} H_k(\tilde{u}, \tilde{u})|_z = \int_{P(E_z^*)} \partial\bar{\partial}\big(g^k(\Phi_{k,z}(\tilde{u}), \Phi_{k,z}(\tilde{u}))\,\Theta(g)^{r-1}\big)$$
$$= \sum_m \int_{V_m} \partial\bar{\partial}\big(g^k(\Phi_{k,z}(\tilde{u}), \Phi_{k,z}(\tilde{u}))\,\Theta(g)^{r-1}\big).$$

In the last equality, we partition the fiber $P(E_z^*)$ as $\bigcup_m V_m$ with each $V_m$ in $U_l$ for some $l$. In a fixed $V_m \subset U_l$, using the coordinate system of $U_l$, we can write $\Phi_{k,z}(\tilde{u})$ as $f(e^*)^k$ with $f$ a scalar-valued holomorphic function and $e^*$ a local frame for $O_{P(E^*)}(1)$. So, $g^k(\Phi_{k,z}(\tilde{u}), \Phi_{k,z}(\tilde{u})) = |f|^2 e^{-k\phi}$. Meanwhile, recall the curvature $\Theta(g) = \partial\bar{\partial}\phi$. By Stokes' theorem and a count on degrees, we have

$$\sum_m \int_{V_m} \partial\bar{\partial}\big(g^k(\Phi_{k,z}(\tilde{u}), \Phi_{k,z}(\tilde{u}))\,\Theta(g)^{r-1}\big)$$
$$= \sum_m \int_{V_m} \sum_{\alpha,\beta} \frac{\partial^2 |f|^2 e^{-k\phi} \det(\phi_{i\bar{j}})}{\partial z_\alpha \partial \bar{z}_\beta} dz_\alpha \wedge d\bar{z}_\beta \bigwedge_j dw_j \wedge d\bar{w}_j.$$

So, if the tangent vector $\eta = \sum_\alpha \eta_\alpha \frac{\partial}{\partial z_\alpha}$ in the coordinate neighborhood $U_l$, then

$$\text{(3-6)} \quad -H_k(\Theta_k u, u)(\eta, \bar{\eta}) = \sum_m \int_{V_m} \sum_{\alpha,\beta} \frac{\partial^2 |f|^2 e^{-k\phi} \det(\phi_{i\bar{j}})}{\partial z_\alpha \partial \bar{z}_\beta} \eta_\alpha \bar{\eta}_\beta \bigwedge_j dw_j \wedge d\bar{w}_j.$$

Note that the integrands in (3-6) are written in the local coordinates of corresponding $U_l$. A direct computation shows

$$\sum_{\alpha,\beta} \frac{\partial^2 |f|^2 e^{-k\phi} \det(\phi_{i\bar{j}})}{\partial z_\alpha \partial \bar{z}_\beta} \eta_\alpha \bar{\eta}_\beta$$
$$= e^{-k\phi} \det(\phi_{i\bar{j}}) \Big| \sum_\alpha \frac{\partial f}{\partial z_\alpha} \eta_\alpha - f \sum_\alpha \big(k\phi_\alpha - (\log\det\phi_{i\bar{j}})_\alpha\big)\eta_\alpha \Big|^2$$
$$- |f|^2 e^{-k\phi} \det(\phi_{i\bar{j}}) \sum_{\alpha,\beta} \big(k\phi_{\alpha\bar{\beta}} - (\log\det\phi_{i\bar{j}})_{\alpha\bar{\beta}}\big)\eta_\alpha \bar{\eta}_\beta$$
$$\geq -|f|^2 e^{-k\phi} \det(\phi_{i\bar{j}}) \sum_{\alpha,\beta} (B_k)_{\alpha\beta} \eta_\alpha \bar{\eta}_\beta.$$

By (3-3),

$$\text{(3-7)} \qquad \frac{1}{k} \sum_{\alpha,\beta} (B_k)_{\alpha\beta} \eta_\alpha \bar{\eta}_\beta \leq \sum_{\alpha,\beta} \phi_{\alpha\bar{\beta}}|_{(z_0,[\zeta_l])} \eta_\alpha \bar{\eta}_\beta + \frac{r-M}{2} \sum_\alpha |\eta_\alpha|^2.$$

Using the coordinate system of $U_l$, the tangent vector $\eta = \sum_\alpha \eta_\alpha \frac{\partial}{\partial z_\alpha}$ at $z$ induces a tangent vector $\eta_l = \sum_\alpha \eta_\alpha \frac{\partial}{\partial z_\alpha}|_{z_0}$ at $z_0$. Denote the lifts of $\eta_l$ to $T^{1,0}_{(z_0,[\zeta_l])} P(E^*)$

by $\tilde{\eta}_l$. According to (1-2), (1-3), and (2-3), we see

$$M\sum_{\alpha}|\eta_\alpha|^2 \geq -\theta(g)(\tilde{\eta}_l,\bar{\tilde{\eta}}_l)=\inf_{q_*(\tilde{\eta}_l)=\eta_l}\Theta(g)(\tilde{\eta}_l,\bar{\tilde{\eta}}_l)=\sum_{\alpha,\beta}\phi_{\alpha\bar{\beta}}|_{(z_0,[\zeta_l])}\,\eta_\alpha\,\bar{\eta}_\beta. \tag{3-8}$$

Therefore, (3-7) becomes

$$\frac{1}{k}\sum_{\alpha,\beta}(B_k)_{\alpha\beta}\,\eta_\alpha\,\bar{\eta}_\beta \leq \left(M+\frac{r-M}{2}\right)\sum_{\alpha}|\eta_\alpha|^2 \leq \left(M+\frac{r-M}{2}\right)\frac{\Omega(\eta,\bar{\eta})}{(1-\varepsilon)}, \tag{3-9}$$

where we use (3-4) in the second inequality. So, (3-6) becomes

$$\begin{aligned} &-H_k(\Theta_k u,u)(\eta,\bar{\eta}) \\ &\quad\geq \sum_m \int_{V_m} -|f|^2 e^{-k\phi}\det(\phi_{i\bar{j}})\bigwedge_j dw_j\wedge d\bar{w}_j\left(M+\frac{r-M}{2}\right)k\frac{\Omega(\eta,\bar{\eta})}{(1-\varepsilon)} \\ &\quad= -\left(M+\frac{r-M}{2}\right)k\frac{\Omega(\eta,\bar{\eta})}{(1-\varepsilon)} \end{aligned} \tag{3-10}$$

since $H_k(u,u)=1$. □

We turn now to the metric $h$. The argument about $h$ is similar to that about $g$, and it will be used in Theorems 2, 3, and 4. Given a point $(z_0,[\zeta_0])\in P(E^*)$, we have the normal coordinate system from Section 2A with respect to the metric $h$. In this coordinate system, let us introduce the $n$-by-$n$ matrix-valued function

$$A=(A_{\alpha\beta}):=\left(\phi_{\alpha\bar{\beta}}-\sum_{i,j}\phi_{\alpha\bar{j}}\,\phi^{i\bar{j}}\,\phi_{i\bar{\beta}}\right),$$

where $h(e^*,e^*)=e^{-\phi}$ and $(\phi^{i\bar{j}})$ is the inverse matrix of $(\phi_{i\bar{j}})$. By continuity, there is a neighborhood $U$ of $(z_0,[\zeta_0])$ in $P(E^*)$ such that in $U$

$$rA+\frac{r-M}{4}\,\mathrm{Id}_{n\times n}\geq rA|_{(z_0,[\zeta_0])}. \tag{3-11}$$

In summary:

**Lemma 8.** *Given a point $(z_0,[\zeta_0])\in P(E^*)$, there exists a coordinate neighborhood $U$ of $(z_0,[\zeta_0])$ in $P(E^*)$ such that in $U$*

$$rA+\frac{r-M}{4}\,\mathrm{Id}_{n\times n}\geq r(\phi_{\alpha\bar{\beta}})|_{(z_0,[\zeta_0])}. \tag{3-12}$$

By Lemma 8, since $P(E^*_{z_0})$ is compact, we can find on $P(E^*_{z_0})$ finitely many points $\{(z_0,[\zeta_l])\}_l$ each of which corresponds to a coordinate neighborhood $U_l$ in $P(E^*)$ such that the corresponding (3-12) holds, and $P(E^*_{z_0})\subset\bigcup_l U_l$. The point $z_0$ has a neighborhood $W'$ in $X$ such that for $z\in W'$, the fiber $P(E^*_z)$ can be partitioned as $\bigcup_m V_m$ with each $V_m$ in $U_l$ for some $l$. By shrinking $W'$, we can assume that for each $U_l$ the corresponding $\Omega\big(\frac{\partial}{\partial z_\alpha}, \frac{\partial}{\partial \bar{z}_\beta}\big) := \Omega_{\alpha\bar{\beta}}$ satisfies

$$-\varepsilon\delta_{\alpha\beta} < \Omega_{\alpha\bar{\beta}}(z) - \delta_{\alpha\beta} < \varepsilon\delta_{\alpha\beta} \quad \text{for } z \in W', \tag{3-13}$$

where $\varepsilon := \frac{r-M}{5(r+M)}$.

Recall the Hermitian metric $H$ on $\det E$ in (2-14) constructed using the metric $h$. Denote by $\Theta$ the curvature of $H$. We claim:

**Lemma 9.** *For $z \in W'$ and $\eta \in T^{1,0}_z X$, we have*

$$-\Theta(\eta, \bar{\eta}) \le -\left(r - \frac{r-M}{4}\right)\frac{\Omega(\eta, \bar{\eta})}{(1+\varepsilon)}. \tag{3-14}$$

*Proof.* Using (2-15) and assuming $H(v, v) = 1$, we get

$$\begin{aligned} &-\Theta(\eta, \bar{\eta}) \\ &\quad \le \sum_m \int_{V_m} \rho(\Psi_z(v), \Psi_z(v))\, r \sum_{\alpha,\beta}\Big(\sum_{i,j} \phi_{\alpha\bar{j}}\,\phi^{i\bar{j}}\,\phi_{i\bar{\beta}} - \phi_{\alpha\bar{\beta}}\Big)\eta_\alpha\,\bar{\eta}_\beta\,\omega_z^{r-1}, \end{aligned} \tag{3-15}$$

where we again partition $P(E^*_z)$ as $\bigcup_m V_m$ with each $V_m$ in $U_l$ for some $l$. Note that the integrands in (3-15) are written in the local coordinates of corresponding $U_l$. In a fixed $V_m \subset U_l$, we have $\eta = \sum_\alpha \eta_\alpha \frac{\partial}{\partial z_\alpha}$, and by (3-12) we see

$$r\sum_{\alpha,\beta} A_{\alpha\beta}\,\eta_\alpha\,\bar{\eta}_\beta + \frac{r-M}{4}\sum_\alpha |\eta_\alpha|^2 \ge r\sum_{\alpha,\beta} \phi_{\alpha\bar{\beta}}|_{(z_0,[\zeta_l])}\,\eta_\alpha\,\bar{\eta}_\beta. \tag{3-16}$$

In $U_l$, the tangent vector

$$\eta = \sum_\alpha \eta_\alpha \frac{\partial}{\partial z_\alpha} \quad \text{at } z$$

induces a tangent vector

$$\eta_l = \sum_\alpha \eta_\alpha \frac{\partial}{\partial z_\alpha}\Big|_{z_0} \quad \text{at } z_0.$$

Denote the lifts of $\eta_l$ to $T^{1,0}_{(z_0,[\zeta_l])}P(E^*)$ by $\tilde{\eta}_l$. By (1-2), (1-4), and (2-3), we see

$$\sum_\alpha |\eta_\alpha|^2 \le -\theta(h)(\tilde{\eta}_l, \bar{\tilde{\eta}}_l) = \inf_{q_*(\tilde{\eta}_l)=\eta_l} \Theta(h)(\tilde{\eta}_l, \bar{\tilde{\eta}}_l) = \sum_{\alpha,\beta} \phi_{\alpha\bar{\beta}}|_{(z_0,[\zeta_l])}\,\eta_\alpha\,\bar{\eta}_\beta. \tag{3-17}$$

Therefore, (3-16) becomes

$$r\sum_{\alpha,\beta} A_{\alpha\beta}\,\eta_\alpha\,\bar{\eta}_\beta \ge \left(r - \frac{r-M}{4}\right)\sum_\alpha |\eta_\alpha|^2 \ge \left(r - \frac{r-M}{4}\right)\frac{\Omega(\eta, \bar{\eta})}{1+\varepsilon},$$

where we use (3-13) in the second inequality. So, (3-15) becomes

$$\begin{aligned}-\Theta(\eta,\bar\eta) &\le -\sum_m \int_{V_m} \rho(\Psi_z(v),\Psi_z(v))\,\omega_z^{r-1}\left(r-\frac{r-M}{4}\right)\frac{\Omega(\eta,\bar\eta)}{1+\varepsilon}\\ &= -\left(r-\frac{r-M}{4}\right)\frac{\Omega(\eta,\bar\eta)}{(1+\varepsilon)}\end{aligned}$$

because $H(v,v)=1$. □

Now we put together the $L^2$-metrics $H_k$ on $S^kE$ in (2-11), and $H$ on $\det E$ in (2-14). Since $(\det E^*)^k$ is a line bundle, we can identify $\mathrm{End}(S^kE\otimes(\det E^*)^k)$ with $\mathrm{End}(S^kE)$, and the curvature of the metric $H_k\otimes(H^*)^k$ on $S^kE\otimes(\det E^*)^k$ can be written as

$$\Theta_k - k\Theta\otimes \mathrm{Id}_{S^kE},$$

where $\Theta_k$ and $\Theta$ are the curvature of $H_k$ and $H$ respectively. We claim that for $k\ge k_{\max}$ and in $W\cap W'$ a neighborhood of $z_0$, the metric $H_k\otimes(H^*)^k$ is Griffiths negative. Indeed, as a result of Lemmas 7 and 9, for $k\ge k_{\max}$, $z\in W\cap W'$, $0\ne\eta\in T_z^{1,0}X$, and $u\in S^kE_z$ with $H_k(u,u)=1$, we see

$$H_k(\Theta_k u,u)(\eta,\bar\eta)-k\Theta(\eta,\bar\eta)\le k\left(M+\frac{r-M}{2}\right)\frac{\Omega(\eta,\bar\eta)}{(1-\varepsilon)}-k\left(r-\frac{r-M}{4}\right)\frac{\Omega(\eta,\bar\eta)}{(1+\varepsilon)}.$$

The term on the right is negative after some computation using $\varepsilon=\frac{r-M}{5(r+M)}$. So, we have proved the claim that for $k\ge k_{\max}$ and in $W\cap W'\subset X$, the metric $H_k\otimes(H^*)^k$ is Griffiths negative. Since $X$ is compact, $H_k\otimes(H^*)^k$ is Griffiths negative on the entire $X$ for $k$ large enough.

Now we fix $k$ such that the Hermitian metric $H_k\otimes(H^*)^k$ on the bundle

$$S^kE\otimes(\det E^*)^k$$

is Griffiths negative on $X$. The Hermitian metric $H_k$ by construction is an $L^2$-integral, so its $k$-th root is a convex Finsler metric on $E$ (see [Wu 2022, proof of Theorem 1] for details). By Lemma 5, the $k$-th root of $H_k\otimes(H^*)^k$ is a convex Finsler metric on $E\otimes\det E^*$ which we denote by $F$. Moreover, this Finsler metric $F$ is strongly plurisubharmonic on $E\otimes\det E^*\setminus\{\text{zero section}\}$ due to Griffiths negativity of $H_k\otimes(H^*)^k$. By adding a small Hermitian metric, we can assume $F$ is strongly convex and strongly plurisubharmonic.

In general, the Kobayashi curvature of Finsler metrics do not behave well under duality [Demailly 1999, Remark 2.7]. But since our Finsler metric $F$ is strongly convex, the dual Finsler metric of $F$ is in fact strongly pseudoconvex and Kobayashi positive (this duality result is originally due to Sommese [1978] and Demailly [1999, Theorem 2.5]. See also [Wu 2022, proof of Theorem 1 and Lemma 6]). In summary, the dual Finsler metric of $F$ is a convex, strongly pseudoconvex, and Kobayashi positive Finsler metric on $E^* \otimes \det E$. Hence the proof of Theorem 1 is complete.

With slight modification on the proof, one has the following variant of Theorem 1.

**Theorem 10.** *Assume $r \geq 2$ and the line bundle $O_{P(E^*)}(1)$ has a positively curved metric $h$ and a metric $g$ with $\Theta(g)|_{P(E_z^*)} > 0$ for all $z \in X$. If there exist a Hermitian metric $\Omega$ on $X$ and a constant $M \geq 1$ such that the following inequalities of $(1, 1)$-forms hold*

$$Mq^*\Omega \geq -\theta(g), \tag{3-18}$$

$$q^*\Omega \leq -\theta(h), \tag{3-19}$$

*then for any positive integer $l > M/r$, the bundle $E^* \otimes (\det E)^l$ is Kobayashi positive.*

## 4. Proof of Theorem 2

The proof is similar to what we do in Section 3 except that we are dealing with not only $P(E^*)$ but $P(E)$ here. The metric $h$ is used to define a Hermitian metric $H$ on $\det E$ as in (2-14). The metric $g$ is used to define Hermitian metrics $\boldsymbol{H}_k$ on $S^k E^* \otimes (\det E)^k$ as in (2-13).

Fix $z_0$ in $X$. For the metric $h$ on $O_{P(E^*)}(1)$, we follow the path that leads to Lemma 9 in Section 3 to deduce a neighborhood $W'$ of $z_0$ in $X$ such that for $z \in W'$ and $\eta \in T_z^{1,0}X$, the curvature $\Theta$ of $H$ satisfies

$$-\Theta(\eta, \bar{\eta}) \leq -\left(r - \frac{r-M}{4}\right)\frac{\Omega(\eta, \bar{\eta})}{(1+\varepsilon)}, \tag{4-1}$$

with $\varepsilon = \frac{r-M}{5(r+M)}$.

For the metric $g$ on $O_{P(E)}(1) \otimes p^* \det E$, we replace $O_{P(E^*)}(1) \to P(E^*)$ in Section 3 with $O_{P(E \otimes \det E^*)}(1) \to P(E \otimes \det E^*)$ and use the canonical isomorphism between $O_{P(E \otimes \det E^*)}(1) \to P(E \otimes \det E^*)$ and $O_{P(E)}(1) \otimes p^* \det E \to P(E)$. Then following the argument leading to Lemma 7, we obtain a positive integer $k_{\max}$ and a neighborhood $W$ of $z_0$ in $X$ such that for $k \geq k_{\max}$, $z \in W$, $\eta \in T_z^{1,0}X$, and $u \in S^k E_z^* \otimes (\det E_z)^k$ with $\boldsymbol{H}_k(u, u) = 1$, the curvature $\boldsymbol{\Theta}_k$ of $\boldsymbol{H}_k$ satisfies

$$\boldsymbol{H}_k(\boldsymbol{\Theta}_k u, u)(\eta, \bar{\eta}) \leq \left(M + \frac{r-M}{2}\right) k \frac{\Omega(\eta, \bar{\eta})}{(1-\varepsilon)}. \tag{4-2}$$

On the bundle $[S^k E^* \otimes (\det E)^k] \otimes (\det E^*)^k$, there is a Hermitian metric $\boldsymbol{H}_k \otimes (H^*)^k$ with curvature $\boldsymbol{\Theta}_k - k\Theta \otimes \mathrm{Id}_{S^k E^* \otimes (\det E)^k}$. As a result of (4-1) and (4-2), we deduce that, for $k \geq k_{\max}$, $z \in W \cap W'$, $\eta \in T_z^{1,0}X$, and $u \in S^k E_z^* \otimes (\det E_z)^k$ with $\boldsymbol{H}_k(u, u) = 1$,

$$\boldsymbol{H}_k(\boldsymbol{\Theta}_k u, u)(\eta, \bar{\eta}) - k\Theta(\eta, \bar{\eta}) \leq k\left(M + \frac{r-M}{2}\right)\frac{\Omega(\eta, \bar{\eta})}{(1-\varepsilon)} - k\left(r - \frac{r-M}{4}\right)\frac{\Omega(\eta, \bar{\eta})}{(1+\varepsilon)}.$$

Again, the term on the right is negative using $\varepsilon = \frac{r-M}{5(r+M)}$. So we have proved that for $k \geq k_{\max}$ and in $W \cap W'$, the metric $\boldsymbol{H}_k \otimes (H^*)^k$ is Griffiths negative. Since $X$ is compact, $\boldsymbol{H}_k \otimes (H^*)^k$ is Griffiths negative on $X$ for $k$ large.

Now we fix $k$ such that $\boldsymbol{H}_k \otimes (H^*)^k$ on the bundle

$$[S^k E^* \otimes (\det E)^k] \otimes (\det E^*)^k \simeq S^k E^*$$

is Griffiths negative. Using the same argument as those at the end of Section 3, we obtain a convex, strongly pseudoconvex, Kobayashi positive Finsler metric on $E$.

## 5. Proof of Theorem 3

We use the metric $h$ to construct a Hermitian metric $H$ on $\det E$ as in (2-14), and the metric $g$ to construct a Hermitian metric $H_1$ on $S^1 E = E$ as in (2-11).

We start with the metric $g$. For $(z_0, [\zeta_0])$ in $P(E^*)$, there is a special coordinate system given in Section 2B. In this coordinate system, we define the following $n$-by-$n$ matrix-valued function:

$$B = (B_{\alpha\beta}) := \big(\phi_{\alpha\bar{\beta}} - (\log\det(\phi_{i\bar{j}}))_{\alpha\bar{\beta}}\big),$$

where $g(e^*, e^*) = e^{-\phi}$. By continuity, there is a neighborhood $U$ of $(z_0, [\zeta_0])$ in $P(E^*)$ such that in $U$

$$B|_{(z_0,[\zeta_0])} + \frac{r-M}{4}\,\mathrm{Id}_{n\times n} \geq B.$$

In summary:

**Lemma 11.** *Given a point* $(z_0, [\zeta_0]) \in P(E^*)$, *there exists a coordinate neighborhood $U$ of* $(z_0, [\zeta_0])$ *in* $P(E^*)$ *such that in $U$*

$$B|_{(z_0,[\zeta_0])} + \frac{r-M}{4}\,\mathrm{Id}_{n\times n} \geq B. \tag{5-1}$$

By Lemma 11, since $P(E^*_{z_0})$ is compact, we can find finitely many points $\{(z_0, [\zeta_l])\}_l$ on $P(E^*_{z_0})$ each of which corresponds to a coordinate neighborhood $U_l$ in $P(E^*)$ such that the corresponding (5-1) holds, and $P(E^*_{z_0}) \subset \bigcup_l U_l$. The fiber $P(E^*_{z_0})$ can be partitioned as $\bigcup_m V_m$ with each $V_m$ in $U_l$ for some $l$.

Recall the Hermitian metric $H_1$ on $E$ in (2-11) constructed using the metric $g$. Denote by $\Theta_1$ the curvature of $H_1$. We claim:

**Lemma 12.** *For* $0 \neq \eta \in T^{1,0}_{z_0}X$ *and* $u \in E_{z_0}$ *with* $H_1(u, u) = 1$, *we have*

$$H_1(\Theta_1 u, u)(\eta, \bar{\eta}) \leq \Big(M + \frac{r-M}{4}\Big)\,\Omega(\eta, \bar{\eta}). \tag{5-2}$$

*Proof.* As in Section 2D, we extend the vector $u \in E_{z_0}$ to a local holomorphic section $\tilde{u}$ whose covariant derivative at $z_0$ equals zero, and we have

$$
\begin{aligned}
-H_1(\Theta_1 u, u) = \partial\bar\partial H_1(\tilde u, \tilde u)|_{z_0} &= \int_{P(E^*_{z_0})} \partial\bar\partial\big(g(\Phi_{1,z}(\tilde u), \Phi_{1,z}(\tilde u))\,\Theta(g)^{r-1}\big)\\
&= \sum_m \int_{V_m} \partial\bar\partial\big(g(\Phi_{1,z}(\tilde u), \Phi_{1,z}(\tilde u))\,\Theta(g)^{r-1}\big).
\end{aligned}
$$

In a fixed $V_m \subset U_l$, we can write $\Phi_{1,z}(\tilde u)$ as $f e^*$ with $f$ a scalar-valued holomorphic function and $e^*$ a local frame for $O_{P(E^*)}(1)$. So,

$$
g(\Phi_{1,z}(\tilde u), \Phi_{1,z}(\tilde u)) = |f|^2 e^{-\phi}.
$$

Meanwhile, recall the curvature $\Theta(g) = \partial\bar\partial\phi$. By Stokes' theorem and a count on degrees, we have

$$
\begin{aligned}
&\sum_m \int_{V_m} \partial\bar\partial\big(g(\Phi_{1,z}(\tilde u), \Phi_{1,z}(\tilde u))\,\Theta(g)^{r-1}\big)\\
&\qquad = \sum_m \int_{V_m} \sum_{\alpha,\beta} \frac{\partial^2 |f|^2 e^{-\phi}\det(\phi_{i\bar j})}{\partial z_\alpha \partial \bar z_\beta} dz_\alpha \wedge d\bar z_\beta \bigwedge_j dw_j \wedge d\bar w_j.
\end{aligned}
$$

So,

$$
-H_1(\Theta_1 u, u)(\eta, \bar\eta) = \sum_m \int_{V_m} \sum_{\alpha,\beta} \frac{\partial^2 |f|^2 e^{-\phi}\det(\phi_{i\bar j})}{\partial z_\alpha\, \partial \bar z_\beta} \eta_\alpha\, \bar\eta_\beta \bigwedge_j dw_j \wedge d\bar w_j \tag{5-3}
$$

for $T^{1,0}_{z_0} X \ni \eta = \sum_\alpha \eta_\alpha \frac{\partial}{\partial z_\alpha}$. A direct computation shows

$$
\begin{aligned}
&\sum_{\alpha,\beta} \frac{\partial^2 |f|^2 e^{-\phi}\det(\phi_{i\bar j})}{\partial z_\alpha \partial \bar z_\beta} \eta_\alpha\, \bar\eta_\beta\\
&\qquad = e^{-\phi}\det(\phi_{i\bar j}) \Big| \sum_\alpha \frac{\partial f}{\partial z_\alpha}\eta_\alpha - f \sum_\alpha \big(\phi_\alpha - (\log\det\phi_{i\bar j})_\alpha\big)\,\eta_\alpha \Big|^2\\
&\qquad\qquad - |f|^2 e^{-\phi}\det(\phi_{i\bar j}) \sum_{\alpha,\beta} \big(\phi_{\alpha\bar\beta} - (\log\det\phi_{i\bar j})_{\alpha\bar\beta}\big)\,\eta_\alpha\,\bar\eta_\beta\\
&\qquad \ge -|f|^2 e^{-\phi}\det(\phi_{i\bar j}) \sum_{\alpha,\beta} B_{\alpha\beta}\,\eta_\alpha\,\bar\eta_\beta.
\end{aligned}
$$

By (1-8), (1-9), (2-6), and (2-9), we see

$$
\begin{aligned}
M \sum_\alpha |\eta_\alpha|^2 &\ge -(r+1)\,\theta(g)(\tilde\eta, \bar{\tilde\eta}) + q^*\Theta(\det G)(\tilde\eta, \bar{\tilde\eta})\\
&= \inf_{q_*(\tilde\eta)=\eta} \Theta(g)(\tilde\eta, \bar{\tilde\eta}) - \sup_{q_*(\tilde\eta)=\eta} \gamma_g(\tilde\eta, \bar{\tilde\eta})\\
&= \sum_{\alpha,\beta} \phi_{\alpha\bar\beta}|_{(z_0,[\zeta_l])}\,\eta_\alpha\,\bar\eta_\beta - \sum_{\alpha,\beta} (\log\det(\phi_{i\bar j}))_{\alpha\bar\beta}|_{(z_0,[\zeta_l])}\,\eta_\alpha\,\bar\eta_\beta\\
&= \sum_{\alpha,\beta} B_{\alpha\beta}|_{(z_0,[\zeta_l])}\,\eta_\alpha\,\bar\eta_\beta.
\end{aligned}
$$

Therefore, (5-1) becomes

$$\sum_{\alpha,\beta} B_{\alpha\beta}\,\eta_\alpha\,\bar\eta_\beta \le \Big(M+\frac{r-M}{4}\Big)\sum_\alpha |\eta_\alpha|^2 = \Big(M+\frac{r-M}{4}\Big)\,\Omega(\eta,\bar\eta). \tag{5-4}$$

So, (5-3) becomes

$$\begin{aligned} -H_1(\Theta_1 u,u)(\eta,\bar\eta) &\ge \sum_m \int_{V_m} -|f|^2 e^{-\phi}\det(\phi_{i\bar j}) \bigwedge_j dw_j\wedge d\bar w_j \Big(M+\frac{r-M}{4}\Big)\,\Omega(\eta,\bar\eta)\\ &= -\Big(M+\frac{r-M}{4}\Big)\,\Omega(\eta,\bar\eta) \end{aligned} \tag{5-5}$$

since $H_1(u,u)=1$. □

For the metric $h$ on $O_{P(E^*)}(1)$, as in Lemma 9 from Section 3 with slight modification, we deduce that for $\eta\in T^{1,0}_{z_0}X$, the curvature $\Theta$ of $H$ satisfies

$$-\Theta(\eta,\bar\eta) \le -\Big(r-\frac{r-M}{4}\Big)\,\Omega(\eta,\bar\eta). \tag{5-6}$$

Finally, we consider the metric $H_1\otimes H^*$ on $E\otimes\det E^*$. Since $\det E^*$ is a line bundle, we can identify $\mathrm{End}(E\otimes\det E^*)$ with $\mathrm{End}\,E$, and the curvature of the metric $H_1\otimes H^*$ can be written as $\Theta_1-\Theta\otimes\mathrm{Id}_E$, where $\Theta_1$ and $\Theta$ are the curvature of $H_1$ and $H$ respectively. As a result of Lemma 12 and (5-6), we see for $0\ne\eta\in T^{1,0}_{z_0}X$ and $u\in E_{z_0}$ with $H_1(u,u)=1$,

$$H_1(\Theta_1 u,u)(\eta,\bar\eta)-\Theta(\eta,\bar\eta) \le \Big(M+\frac{r-M}{4}\Big)\,\Omega(\eta,\bar\eta)-\Big(r-\frac{r-M}{4}\Big)\,\Omega(\eta,\bar\eta),$$

the term on the right is negative. Hence we have proved that at $z_0$ the metric $H_1\otimes H^*$ is Griffiths negative. The point $z_0$ is arbitrary, so $H_1\otimes H^*$ is Griffiths negative on $X$. As a result, the dual bundle $E^*\otimes\det E$ is Griffiths positive.

## 6. Proof of Theorem 4

The metric $h$ is used to define a Hermitian metric $H$ on $\det E$ as in (2-14). The metric $g$ is used to define Hermitian metric $\boldsymbol{H}_1$ on $E^*\otimes\det E$ as in (2-13).

Given $z_0$ in $X$. For the metric $h$ on $O_{P(E^*)}(1)$, as in the formula (5-6) from Section 5, for $\eta\in T^{1,0}_{z_0}X$ we have

$$-\Theta(\eta,\bar\eta) \le -\Big(r-\frac{r-M}{4}\Big)\,\Omega(\eta,\bar\eta). \tag{6-1}$$

For the metric $g$ on $O_{P(E)}(1)\otimes p^*\det E$, we replace $O_{P(E^*)}(1)\to P(E^*)$ in Section 5 with $O_{P(E\otimes\det E^*)}(1)\to P(E\otimes\det E^*)$ and use the canonical isomorphism between $O_{P(E\otimes\det E^*)}(1)\to P(E\otimes\det E^*)$ and $O_{P(E)}(1)\otimes p^*\det E\to P(E)$. Then as in Lemma 12, we get for $\eta \in T^{1,0}_{z_0}X$, and $u \in E^*_{z_0} \otimes (\det E_{z_0})$ with $\boldsymbol{H}_1(u, u) = 1$, the curvature $\boldsymbol{\Theta}_1$ of $\boldsymbol{H}_1$ satisfies

$$\boldsymbol{H}_1(\boldsymbol{\Theta}_1 u, u)(\eta, \bar{\eta}) \leq \Big(M + \frac{r-M}{4}\Big)\,\Omega(\eta, \bar{\eta}). \tag{6-2}$$

On the bundle $(E^* \otimes \det E) \otimes \det E^*$, there is a Hermitian metric $\boldsymbol{H}_1 \otimes H^*$ with curvature $\boldsymbol{\Theta}_1 - \Theta \otimes \mathrm{Id}_{E^* \otimes \det E}$. As a result of (6-1) and (6-2), we deduce that for $\eta \in T^{1,0}_{z_0}X$, and $u \in E^*_{z_0} \otimes (\det E_{z_0})$ with $\boldsymbol{H}_1(u, u) = 1$,

$$\begin{aligned} &\boldsymbol{H}_1(\boldsymbol{\Theta}_1 u, u)(\eta, \bar{\eta}) - \Theta(\eta, \bar{\eta}) \\ &\qquad \leq \Big(M + \frac{r-M}{4}\Big)\,\Omega(\eta, \bar{\eta}) - \Big(r - \frac{r-M}{4}\Big)\,\Omega(\eta, \bar{\eta}), \end{aligned} \tag{6-3}$$

the term on the right is negative. So the Hermitian metric $\boldsymbol{H}_1 \otimes H^*$ is Griffiths negative at $z_0$ an arbitrary point. Hence $\boldsymbol{H}_1 \otimes H^*$ is Griffiths negative on $X$, and the bundle $E$ is Griffiths positive.

## 7. Examples

**Example 13.** We provide here an example where the assumptions in Theorems 1, 2, 3, and 4 are satisfied. Let $L$ be a line bundle with a metric $H$ whose curvature $\Theta > 0$. Let $E = L^9 \oplus L^8 \oplus L^7$ a vector bundle of rank $r = 3$. The induced metric $(H^*)^9 \oplus (H^*)^8 \oplus (H^*)^7$ on the dual bundle $E^*$ has curvature

$$\Theta(E^*) = (-9\Theta) \oplus (-8\Theta) \oplus (-7\Theta),$$

which is Griffiths negative, so the corresponding metric $h$ on $O_{P(E^*)}(1)$ is positively curved. According to (2-7), we see

$$-\theta(h) = -q^*\Theta(E^*)|_{O_{P(E^*)}(-1)}.$$

Hence we have

$$7q^*\Theta \leq -\theta(h) \leq 9q^*\Theta. \tag{7-1}$$

For all four theorems, we will use this metric $h$ on $O_{P(E^*)}(1)$ and take $\Omega$ to be $7\Theta$. So $q^*\Omega \leq -\theta(h)$ always holds. The choice of $g$ will be different from case to case.

For Theorem 1, we choose $g$ to be $h$, and hence by (7-1) and $\Omega = 7\Theta$ we get

$$q^*\Omega \leq -\theta(h) = -\theta(g) \leq \tfrac{9}{7}q^*\Omega. \tag{7-2}$$

To fulfill the assumption of Theorem 1, we can choose $M = \frac{9}{7}$ which is in the interval $[1, 3)$.

For Theorem 2, since $E \otimes \det E^* = (L^*)^{15} \oplus (L^*)^{16} \oplus (L^*)^{17}$ has induced curvature $(-15\Theta) \oplus (-16\Theta) \oplus (-17\Theta)$ which is Griffiths negative, the corresponding metric $g$ on $O_{P(E)}(1) \otimes p^* \det E$ is positively curved and satisfies

$$15p^*\Theta \leq -\theta(g) \leq 17p^*\Theta. \tag{7-3}$$

Together with (7-1) and $\Omega = 7\Theta$, we have

$$\tfrac{17}{7} p^*\Omega \geq -\theta(g) \quad \text{and} \quad q^*\Omega \leq -\theta(h). \tag{7-4}$$

We can choose $M = \frac{17}{7}$ which is in $[1, 3)$.

For Theorem 3, notice that $h$ is induced from $(H^*)^9 \oplus (H^*)^8 \oplus (H^*)^7$ on $E^*$, so if we use $(H^*)^9 \oplus (H^*)^8 \oplus (H^*)^7$ for the Hermitian metric $G$, then the corresponding $g$ is actually $h$. Since $\Theta(\det G) = -24\Theta$, by using (7-1) we have

$$-(r+1)\,\theta(g) + q^*\Theta(\det G) = -4\theta(h) - 24q^*\Theta \leq 12q^*\Theta = \tfrac{12}{7} q^*\Omega. \tag{7-5}$$

We choose $M = \frac{12}{7}$ which is in $[1, 3)$.

Finally for Theorem 4, on $E \otimes \det E^* = (L^*)^{15} \oplus (L^*)^{16} \oplus (L^*)^{17}$, we will use the metric $(H^*)^{15} \oplus (H^*)^{16} \oplus (H^*)^{17}$ for $G$, so $\Theta(\det G) = -48\Theta$. Moreover, the corresponding metric $g$ on $O_{P(E)}(1) \otimes p^* \det E$ satisfies

$$15p^*\Theta \leq -\theta(g) \leq 17p^*\Theta, \tag{7-6}$$

so we get

$$-(r+1)\,\theta(g) + p^*\Theta(\det G) \leq 20p^*\Theta = \tfrac{20}{7} p^*\Omega. \tag{7-7}$$

We choose $M = \frac{20}{7}$ which is in $[1, 3)$.

**Example 14.** Let $X$ be a compact Riemann surface with a Hermitian metric $\omega$. Let $E$ be an $\omega$-semistable ample vector bundle of rank $r$ over $X$. The assumptions in Theorems 1, 2, 3, and 4 are all satisfied in this case. We will explain for only Theorems 2 and 4. Theorems 1 and 3 can be verified similarly. By [Li et al. 2021, Theorem 1.7, Remark 1.8, and Theorem 1.11], there exists a constant $c > 0$ such that for any $\delta > 0$, there exists a Hermitian metric $H_\delta$ on $E$ satisfying

$$(c-\delta)\,\mathrm{Id}_E \leq \sqrt{-1}\Lambda_\omega\,\Theta(H_\delta) \leq (c+\delta)\,\mathrm{Id}_E, \tag{7-8}$$

where $\Lambda_\omega$ is the contraction with respect to $\omega$. Since $X$ is a Riemann surface, $\Lambda_\omega$ locally is multiplication by a positive function.

For Theorem 2, we choose $\delta = \frac{c}{5r}$. The Hermitian metric $H_\delta^*$ on $E^*$ induces a metric $h$ on $O_{P(E^*)}(1)$. Due to (2-7), we see

$$-\theta(h) = -q^*\Theta(H_\delta^*)|_{O_{P(E^*)}(-1)}; \tag{7-9}$$

combining with (7-8), we have

$$(c-\delta)\,q^*\omega \leq -\theta(h) \leq (c+\delta)\,q^*\omega. \tag{7-10}$$

The Hermitian metric $H_\delta \otimes \det H_\delta^*$ on $E \otimes \det E^*$ induces on $O_{P(E)}(1) \otimes p^* \det E$ a metric $g$. Similar to (7-10), we have

$$-\theta(g) \leq [-(c-\delta) + r(c+\delta)]\,p^*\omega. \tag{7-11}$$

If we choose $\Omega = (c-\delta)\,\omega$ and $M = r - \frac{1}{2}$, then

$$[-(c-\delta) + r(c+\delta)]\, p^*\omega \le Mp^*\Omega.$$

As a result, we achieve the assumption in Theorem 2:

$$q^*\Omega \le -\theta(h) \quad \text{and} \quad -\theta(g) \le Mp^*\Omega.$$

For Theorem 4, we choose $\delta = \frac{c}{9r}$. We still have (7-10). The Hermitian metric $G$ on $E \otimes \det E^*$ is taken to be $H_\delta \otimes \det H_\delta^*$, so we get

$$\begin{aligned}
&-(r+1)\,\theta(g) + p^*\Theta(\det G)\\
&\qquad = -(r+1)[p^*\Theta(H_\delta)|_{O_{P(E)}(-1)} - p^*\Theta(\det H_\delta)] - (r-1)p^*\Theta(\det H_\delta)\\
&\qquad \le [-(r+1)(c-\delta) + 2r(c+\delta)]\, p^*\omega.
\end{aligned}$$

If we choose $\Omega = (c-\delta)\,\omega$ and $M = r - \frac{1}{2}$, then

$$[-(r+1)(c-\delta) + 2r(c+\delta)]\, p^*\omega \le Mp^*\Omega.$$

So the assumption of Theorem 4 is satisfied.

In light of [Li et al. 2021, Theorem 1.7], it is possible to modify our theorems so that semistability is not needed in this example.

## Appendix

Here we prove the isomorphism $\bigwedge^{r-1} E \simeq E^* \otimes \det E$ where $r$ is the rank of $E$.

*Proof.* Let $\{e_1, \dots, e_r\}$ and $\{f_1, \dots, f_r\}$ be two sets of local frames for $E$ with the transition matrix $g = (g_{ij})$; namely, on the intersection of the two frames, we have $f_i = \sum_j g_{ij}\, e_j$. On the bundle $\bigwedge^{r-1} E$, we have the induced local frame $\{\hat{e}_1, \dots, \hat{e}_r\}$ where $\hat{e}_k$ is $e_1 \wedge \cdots \wedge e_r$ with $e_k$ removed. Similarly, we have another frame $\{\hat{f}_1, \dots, \hat{f}_r\}$. Let $\hat{g} = (\hat{g}_{ij})$ be the corresponding transition matrix for the bundle $\bigwedge^{r-1} E$, namely, $\hat{f}_i = \sum_j \hat{g}_{ij}\, \hat{e}_j$. It is not hard to verify that $\hat{g}_{ij}$ is the determinant of the matrix $g$ with the $i$-th row and $j$-th column removed.

For the dual bundle $E^*$, the corresponding transition matrix for the dual frames $\{e_1^*, \dots, e_r^*\}$ and $\{f_1^*, \dots, f_r^*\}$ is the transpose of $g^{-1}$. Therefore, the transition matrix for the bundle $E^* \otimes \det E$ is $c = (c_{ij})$ where $c_{ij} = (-1)^{i+j}\, \hat{g}_{ij}$.

Now, let us denote by $A$ the diagonal matrix whose $i$-th diagonal entry is $(-1)^i$. Notice that the inverse of $A$ is still $A$. Also, after a straightforward computation, we have $AcA^{-1} = \hat{g}$. So, the two bundles $\bigwedge^{r-1} E$ and $E^* \otimes \det E$ are isomorphic. □

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