

*Pacific
Journal of
Mathematics*

**POSITIVELY CURVED FINSLER METRICS
ON VECTOR BUNDLES, II**

KUANG-RU WU

POSITIVELY CURVED FINSLER METRICS ON VECTOR BUNDLES, II

KUANG-RU WU

We show that if E is an ample vector bundle of rank at least two with some curvature bound on $O_{P(E^*)}(1)$, then $E^* \otimes \det E$ is Kobayashi positive. The proof relies on comparing the curvature of $(\det E^*)^k$ and $S^k E$ for large k and using duality of convex Finsler metrics. Following the same thread of thought, we show if E is ample with similar curvature bounds on $O_{P(E^*)}(1)$ and $O_{P(E \otimes \det E^*)}(1)$, then E is Kobayashi positive. With additional assumptions, we can furthermore show that $E^* \otimes \det E$ and E are Griffiths positive.

1. Introduction

Let E be a holomorphic vector bundle of rank r over a compact complex manifold X of dimension n . We denote the dual bundle by E^* and its projectivized bundle by $P(E^*)$. The vector bundle E is said to be ample if the line bundle $O_{P(E^*)}(1)$ over $P(E^*)$ is ample. On the other hand, E is called Griffiths positive if E carries a Griffiths positive Hermitian metric. Moreover, E is called Kobayashi positive if E carries a strongly pseudoconvex Finsler metric whose Kobayashi curvature is positive (we will give a quick review on Finsler metrics and Kobayashi curvature in [Section 2A](#); or see [\[Wu 2022, Section 2\]](#)).

There are two conjectures made by Griffiths [\[1969\]](#) and Kobayashi [\[1975\]](#) regarding the equivalence of ampleness and positivity:

- (1) If E is ample, then E is Griffiths positive.
- (2) If E is ample, then E is Kobayashi positive.

These two conjectures are still open, save for $n = 1$, in [\[Umemura 1973; Campana and Flenner 1990\]](#) (for recent progress, see [\[Berndtsson 2009a; Mourougane and Takayama 2007; Hering et al. 2010; Liu et al. 2013; Liu and Yang 2015; Naumann 2021; Feng et al. 2020; Demailly 2021; Finski 2022; Pingali 2021; Ma and Zhang 2023\]](#)). Note that the converse of each conjecture is true [\[Feng et al. 2020; Wu 2022\]](#).

By Kodaira's embedding theorem, ampleness of a line bundle is equivalent to the existence of a positively curved metric on the line bundle. So, the conjectures

MSC2020: 32F17, 32J25.

Keywords: Kobayashi positivity, Griffiths positivity, L^2 -metrics.

of Griffiths and Kobayashi can be rephrased: *Given a positively curved metric on $O_{P(E^*)}(1)$, can we construct a positively curved Hermitian/Finsler metric on E ?* In this paper, we show that it is so, by imposing curvature bounds on tautological line bundles of $P(E^*)$ and $P(E)$. Since Hermitian metrics on $O_{P(E^*)}(1)$ are in one-to-one correspondence with Finsler metrics on E^* , these curvature bounds can also be written in terms of Kobayashi curvature.

We first consider a relevant case where the picture is clearer. It is known that, for rank of E at least 2:

- (1) If E is Griffiths positive, then $E^* \otimes \det E$ with the induced metric is Griffiths positive.
- (2) If E is ample, then $E^* \otimes \det E$ is ample.

The first fact can be found in [Demailly 2012, p. 346, Theorem 9.2] and the second in [Hartshorne 1966, Corollary 5.3] together with the isomorphism (see Appendix)

$$\bigwedge^{r-1} E \simeq E^* \otimes \det E.$$

If we follow the guidance of Griffiths and Kobayashi, we would ask whether or not the ampleness of E implies Griffiths/Kobayashi positivity of $E^* \otimes \det E$ for $r \geq 2$. Our first result is that this can be achieved by imposing curvature bounds on $O_{P(E^*)}(1)$.

Let $q : P(E^*) \rightarrow X$ be the projection. Let g be a metric on $O_{P(E^*)}(1)$ whose curvature restricted to a fiber $\Theta(g)|_{P(E_z^*)}$ is positive for all $z \in X$. For a tangent vector $\eta \in T_z^{1,0}X$ and a point $[\zeta] \in P(E_z^*)$, we consider tangent vectors $\tilde{\eta}$ to $P(E^*)$ at $(z, [\zeta])$ such that $q_*(\tilde{\eta}) = \eta$, namely the lifts of η to $T_{(z, [\zeta])}^{1,0}P(E^*)$. Then we define the function

$$(1-1) \quad (\eta, [\zeta]) \mapsto \inf_{q_*(\tilde{\eta})=\eta} \Theta(g)(\tilde{\eta}, \bar{\tilde{\eta}}) := m(\eta, [\zeta]),$$

where the infimum taken over all the lifts of η to $T_{(z, [\zeta])}^{1,0}P(E^*)$. This infimum is actually a minimum, see (2-3). On the other hand, since such a metric g corresponds to a strongly pseudoconvex Finsler metric on E^* , and if we denote its Kobayashi curvature by $\theta(g)$ a $(1, 1)$ -form on $P(E^*)$, then

$$(1-2) \quad m(\eta, [\zeta]) = -\theta(g)(\tilde{\eta}, \bar{\tilde{\eta}}).$$

The term on the right is independent of the choice of lifts $\tilde{\eta}$ (we will prove (1-2) in Section 2A).

Theorem 1. *Assume $r \geq 2$ and the line bundle $O_{P(E^*)}(1)$ has a positively curved metric h and a metric g with $\Theta(g)|_{P(E_z^*)} > 0$ for all $z \in X$. If there exist a Hermitian*

metric Ω on X and a constant $M \in [1, r)$ such that the following inequalities of $(1, 1)$ -forms hold:

$$(1-3) \quad Mq^* \Omega \geq -\theta(g),$$

$$(1-4) \quad q^* \Omega \leq -\theta(h),$$

then $E^* \otimes \det E$ is Kobayashi positive.

We can of course choose g to be h in [Theorem 1](#), but having two different metrics seems more flexible. The proof of [Theorem 1](#) relies on two observations. First, starting with g and h on $O_{P(E^*)}(1)$, we construct two Hermitian metrics on $S^k E$ and $\det E$ respectively. The curvature of the induced metric on $S^k E \otimes (\det E^*)^k$ can be shown to be Griffiths negative for k large (see [Section 3](#) for details). The second observation (see [\[Wu 2022\]](#)) is that since the induced metric on $S^k E \otimes (\det E^*)^k$ is basically an L^2 -metric, its k -th root is a convex Finsler metric on $E \otimes \det E^*$ which is also strongly plurisubharmonic on the total space minus the zero section. After perturbing this Finsler metric and taking duality, we get a convex and strongly pseudoconvex Finsler metric on $E^* \otimes \det E$ whose Kobayashi curvature is positive. So the bundle $E^* \otimes \det E$ is Kobayashi positive. Notice that the Finsler metric we find is actually convex.

The reason why we impose Ω , M and inequalities (1-3) and (1-4) in [Theorem 1](#) is the following. On the bundle $S^k E \otimes (\det E^*)^k$, the curvature of the induced metric is roughly bounded above by $k \sum_m a_m c_m - rk \sum_m b_m c_m$ where a_m and b_m are some positive integrals with $\sum_m a_m = \sum_m b_m = 1$, and c_m are positive numbers related to the curvature of h . It does not seem possible to us that the upper bound $k \sum_m a_m c_m - rk \sum_m b_m c_m$ can be made negative without any assumption. So we introduce Ω and M to control the upper bound.

With small changes on the proof, one can write down a variant of [Theorem 1](#) where the conclusion is about the Kobayashi positivity of $E^* \otimes (\det E)^l$ (see the end of [Section 3](#)).

Now let us go back to the original conjecture of Kobayashi and adapt the proof of [Theorem 1](#) to this case. Let $p : P(E) \rightarrow X$ be the projection. We recall under the canonical isomorphism $P(E \otimes \det E^*) \simeq P(E)$, the line bundle $O_{P(E \otimes \det E^*)}(1)$ corresponds to the line bundle $O_{P(E)}(1) \otimes p^* \det E$ (see [\[Kobayashi 1987, p. 86, Proposition 3.6.21\]](#)). Let g be a metric on $O_{P(E)}(1) \otimes p^* \det E$ with $\Theta(g)|_{P(E_z)} > 0$ for all $z \in X$. For a tangent vector $\eta \in T_z^{1,0} X$ and a point $[\xi] \in P(E_z)$, we similarly have

$$(\eta, [\xi]) \mapsto \inf_{p_*(\eta')=\eta} \Theta(g)(\eta', \bar{\eta}'),$$

where η' are the lifts of η to $T_{(z, [\xi])}^{1,0} P(E)$. Meanwhile, such a metric g corresponds to a strongly pseudoconvex Finsler metric on $E \otimes \det E^*$ and we denote its Kobayashi

curvature by $\theta(g)$ a $(1, 1)$ -form on $P(E)$. As before,

$$(1-5) \quad \inf_{p_*(\eta')=\eta} \Theta(g)(\eta', \bar{\eta}') = -\theta(g)(\eta', \bar{\eta}').$$

Theorem 2. *Assume $r \geq 2$ and $O_{P(E^*)}(1)$ has a positively curved metric h and $O_{P(E)}(1) \otimes p^* \det E$ has a metric g with $\Theta(g)|_{P(E_z)} > 0$ for all $z \in X$. If there exist a Hermitian metric Ω on X and a constant $M \in [1, r)$ such that*

$$(1-6) \quad Mp^* \Omega \geq -\theta(g),$$

$$(1-7) \quad q^* \Omega \leq -\theta(h),$$

then E is Kobayashi positive.

Since the ampleness of E implies ampleness of $E^* \otimes \det E$, one choice for g in Theorem 2 is a positively curved metric on $O_{P(E)}(1) \otimes p^* \det E$, but how much this choice helps is unknown to us. The proof of Theorem 2 follows the same scheme as in Theorem 1. We first use h and g to construct Hermitian metrics on $\det E$ and $S^k E^* \otimes (\det E)^k$ respectively. The induced metric on $[S^k E^* \otimes (\det E)^k] \otimes (\det E^*)^k$ is Griffiths negative for k large (see Section 4). Then by taking k -th root, perturbing, and taking duality, we obtain a convex, strongly pseudoconvex, and Kobayashi positive Finsler metric on E .

1A. Griffiths positivity. The conclusions in Theorems 1 and 2 are about Finsler metrics. For their Hermitian counterpart, we need additional assumptions. The reason is that in Theorems 1 and 2, taking large tensor power of various bundles helps us eliminate the curvature of the relative canonical bundles $K_{P(E^*)/X}$ and $K_{P(E)/X}$, and after getting the desired estimates we take k -th root to produce Finsler metrics. However, the step of taking k -th root produces only Finsler, not Hermitian metrics. So the first step of taking large tensor power is not allowed if one wants Hermitian metrics.

Let us be more precise. For a metric g on $O_{P(E^*)}(1)$ with $\Theta(g)|_{P(E_z^*)} > 0$ for all $z \in X$, we denote $\Theta(g)|_{P(E_z^*)}$ by ω_z for the moment. The relative canonical bundle $K_{P(E^*)/X}$ has a metric induced from $\{\omega_z^{r-1}\}_{z \in X}$ and we denote the corresponding curvature by γ_g , a $(1, 1)$ -form on $P(E^*)$. For $\eta \in T_z^{1,0} X$ and $[\zeta] \in P(E_z^*)$, we consider

$$(\eta, [\zeta]) \mapsto \sup_{q_*(\tilde{\eta})=\eta} \gamma_g(\tilde{\eta}, \bar{\tilde{\eta}}),$$

where the supremum taken over all the lifts of η to $T_{(z, [\zeta])}^{1,0} P(E^*)$. The supremum is a maximum under a suitable assumption, see (2-9). Moreover, for $z \in X$, the restriction $\gamma_g|_{P(E_z^*)}$ is actually the negative of Ricci curvature $-\text{Ric}_{\omega_z}$ of the metric ω_z on $P(E_z^*)$.

Any Hermitian metric G on E^* will induce a metric g on $O_{P(E^*)}(1)$ with $\Theta(g)|_{P(E_z^*)} > 0$ and $\gamma_g|_{P(E_z^*)} < 0$ for all $z \in X$. Indeed, in this case, $\Theta(g)|_{P(E_z^*)}$

is the Fubini–Study metric and its Ricci curvature is positive, so $\gamma_g|_{P(E_z^*)} < 0$. Furthermore, for any $\eta \in T_z^{1,0}X$ and any $[\zeta] \in P(E_z^*)$,

$$(1-8) \quad \sup_{q_*(\tilde{\eta})=\eta} \gamma_g(\tilde{\eta}, \tilde{\eta}) = r\theta(g)(\tilde{\eta}, \tilde{\eta}) - q^*\Theta(\det G)(\tilde{\eta}, \tilde{\eta})$$

(we will prove (1-8) in Section 2B).

Theorem 3. *Assume $r \geq 2$ and the line bundle $O_{P(E^*)}(1)$ has a positively curved metric h and a metric g induced from a Hermitian metric G on E^* . If there exist a Hermitian metric Ω on X and a constant $M \in [1, r)$ such that*

$$(1-9) \quad Mq^*\Omega \geq -(r + 1)\theta(g) + q^*\Theta(\det G),$$

$$(1-10) \quad q^*\Omega \leq -\theta(h),$$

then $E^* \otimes \det E$ is Griffiths positive.

Theorem 3 could be seen as a Hermitian analogue of Theorem 1. To state a Hermitian analogue of Theorem 2, we use again the isomorphism between $O_{P(E \otimes \det E^*)}(1) \rightarrow P(E \otimes \det E^*)$ and $O_{P(E)}(1) \otimes p^*\det E \rightarrow P(E)$.

Theorem 4. *Suppose that $r \geq 2$ and $O_{P(E^*)}(1)$ has a positively curved metric h and $O_{P(E)}(1) \otimes p^*\det E$ has a metric g induced from a Hermitian metric G on $E \otimes \det E^*$. If there exist a Hermitian metric Ω on X and a constant $M \in [1, r)$ such that*

$$(1-11) \quad Mp^*\Omega \geq -(r + 1)\theta(g) + p^*\Theta(\det G),$$

$$(1-12) \quad q^*\Omega \leq -\theta(h),$$

then E is Griffiths positive.

In all the theorems above, the existence of the metric h comes from ampleness of E . So the real assumptions lie in (g, Ω, M) and the inequalities they have to satisfy. To weaken or remove these inequalities, one possible direction is to use geometric flows as in [Naumann 2021; Wan 2022; Ustinovskiy 2019; Li et al. 2021]. Another possible direction is to use the interplay between the optimal L^2 -estimates and the positivity of curvature (see [Guan and Zhou 2015; Berndtsson and Lempert 2016; Lempert 2017; Hacon et al. 2018; Zhou and Zhu 2018]).

One example where the assumptions of all the theorems above are satisfied is given by $E = L^9 \oplus L^8 \oplus L^7$ with L a positive line bundle. The triple $(9, 8, 7)$ or the rank $r = 3$ is not that important; the point is to make sure the eigenvalues of the curvature with respect to some positive $(1, 1)$ -form do not spread out too far. This example also indicates that a reasonable choice for Ω is probably related to $c_1(\det E)$.

A more sophisticated example, related to approximate Hermitian–Yang–Mills metrics [Jacob 2014; Misra and Ray 2021; Li et al. 2021], is semistable ample vector bundles over Riemann surfaces (see Section 7 for details of the examples).

The proof of Theorem 1 is given in Section 3 and almost as a corollary we prove Theorem 2 in Section 4. The proof of Theorem 3 in Section 5 is a modification of Theorem 1 but we still write out the details. In Section 6, we prove Theorem 4 based on Section 5.

2. Preliminaries

2A. Finsler metrics. We will use some facts about Finsler metrics on vector bundles which can be found in [Kobayashi 1975; 1996; Cao and Wong 2003; Aikou 2004; Wu 2022]. First, we recall the definition of Finsler metrics. Let E^* be a holomorphic vector bundle of rank r over a compact complex manifold X . For a vector $\zeta \in E_z^*$, we symbolically write $(z, \zeta) \in E^*$. A Finsler metric G on the vector bundle $E^* \rightarrow X$ is a real-valued function on E^* such that:

- (1) G is smooth away from the zero section of E^* .
- (2) For $(z, \zeta) \in E^*$, $G(z, \zeta) \geq 0$, and equality holds if and only if $\zeta = 0$.
- (3) $G(z, \lambda\zeta) = |\lambda|^2 G(z, \zeta)$ for $\lambda \in \mathbb{C}$.

A Finsler metric G on E^* is said to be:

- (1) Strongly pseudoconvex if the fiberwise complex Hessian of G is positive definite on $E^* \setminus \{\text{zero section}\}$, namely $(\sqrt{-1}\partial\bar{\partial}G)|_{E_z^*} > 0$ for all $z \in X$.
- (2) Convex if $G^{1/2}$ restricted to each fiber E_z^* is convex.

Let g be a Hermitian metric on $O_{P(E^*)}(1)$ with $\Theta(g)|_{P(E^*)} > 0$ for all $z \in X$. Such a g corresponds to a strongly pseudoconvex Finsler metric G on E^* . Since $(\sqrt{-1}\partial\bar{\partial}G)|_{E_z^*} > 0$, we can define a Hermitian metric \tilde{G} on the pull-back bundle q^*E^* , where $q : P(E^*) \rightarrow X$ is the projection, as follows. For a vector Z in the fiber $q^*E_{(z, [\zeta])}^*$, we define

$$\tilde{G}_{(z, [\zeta])}(Z, Z) = (\sqrt{-1}\partial\bar{\partial}G)|_{E_z^*}(Z, \bar{Z}),$$

where the Z on the right-hand side is viewed as a tangent vector to E_z^* at ζ by the identification of vector spaces $q^*E_{(z, [\zeta])}^* = E_z^*$ and $E_z^* = T_\zeta E_z^*$ (see [Wu 2022, Section 2.2] for a local coordinate description).

Now (q^*E^*, \tilde{G}) is a Hermitian holomorphic vector bundle, so we can talk about its Chern curvature Θ , an $\text{End } q^*E^*$ -valued $(1, 1)$ -form on $P(E^*)$. With respect to the metric \tilde{G} , the bundle q^*E^* has a fiberwise orthogonal decomposition

$$O_{P(E^*)}(-1) \oplus O_{P(E^*)}(-1)^\perp,$$

and so Θ can be written as a block matrix. Let $\Theta|_{O_{P(E^*)}(-1)}$ denote the block in the matrix Θ corresponding to $\text{End}(O_{P(E^*)}(-1))$. Since $O_{P(E^*)}(-1)$ is a line bundle, $\Theta|_{O_{P(E^*)}(-1)}$ is a $(1, 1)$ -form on $P(E^*)$, and it is called the Kobayashi curvature of the Finsler metric G . We will use $\theta(g)$ to denote the Kobayashi curvature

$$(2-1) \quad \theta(g) := \Theta|_{O_{P(E^*)}(-1)}.$$

In order to relate the Kobayashi curvature $\theta(g)$ to the curvature $\Theta(g)$ of g , we consider coordinates normal at one point. Given a point $(z_0, [\zeta_0]) \in P(E^*)$, there exists a holomorphic frame $\{s_i\}$ for E^* around $z_0 \in X$ such that

$$(2-2) \quad \begin{aligned} G_{\zeta_i \bar{\zeta}_j}(z_0, \zeta_0) &= \delta_{ij}, \\ G_{\zeta_i \bar{\zeta}_j z_\alpha}(z_0, \zeta_0) &= G_{\zeta_i \bar{\zeta}_j \bar{z}_\beta}(z_0, \zeta_0) = G_{\bar{\zeta}_j z_\alpha}(z_0, \zeta_0) = G_{z_\alpha}(z_0, \zeta_0) = 0, \end{aligned}$$

where we use $\{\zeta_i\}$ for the fiber coordinates on E^* with respect to the frame $\{s_i\}$ and $\{z_\alpha\}$ for the local coordinates on X (such a frame can be obtained by (5.11) in [Kobayashi 1996]). Moreover if Ω is a Hermitian metric on X , then by a linear transformation in the z -coordinates, we can make

$$\Omega\left(\frac{\partial}{\partial z_\alpha}, \frac{\partial}{\partial \bar{z}_\beta}\right)(z_0) = \delta_{\alpha\beta}$$

without affecting (2-2). We will call this coordinate system normal at the point $(z_0, [\zeta_0]) \in P(E^*)$.

Around the point $(z_0, [\zeta_0]) \in P(E^*)$, we assume the local coordinates

$$(z_1, \dots, z_n, w_1, \dots, w_{r-1})$$

are given by $w_i = \zeta_i/\zeta_r$ for $i = 1 \sim r-1$. So

$$e := \frac{\zeta_1 s_1 + \dots + \zeta_r s_r}{\zeta_r} = w_1 s_1 + \dots + w_{r-1} s_{r-1} + s_r$$

is a holomorphic frame for $O_{P(E^*)}(-1)$. Let e^* be the dual frame of $O_{P(E^*)}(1)$ around $(z_0, [\zeta_0]) \in P(E^*)$ and $g(e^*, e^*) = e^{-\phi}$. Then, the curvature $\Theta(g)$ can be written locally as

$$\begin{aligned} \sum_{\alpha, \beta} \frac{\partial^2 \phi}{\partial z_\alpha \partial \bar{z}_\beta} dz_\alpha \wedge d\bar{z}_\beta + \sum_{\alpha, j} \frac{\partial^2 \phi}{\partial z_\alpha \partial \bar{w}_j} dz_\alpha \wedge d\bar{w}_j \\ + \sum_{i, \beta} \frac{\partial^2 \phi}{\partial w_i \partial \bar{z}_\beta} dw_i \wedge d\bar{z}_\beta + \sum_{i, j} \frac{\partial^2 \phi}{\partial w_i \partial \bar{w}_j} dw_i \wedge d\bar{w}_j. \end{aligned}$$

Note that the terms $\frac{\partial^2 \phi}{\partial z_\alpha \partial \bar{w}_j} := \phi_{\alpha \bar{j}}$ vanish at $(z_0, [\zeta_0])$ by (2-2) and the fact

$$e^\phi = \frac{1}{g(e^*, e^*)} = G(w_1 s_1 + \dots + w_{r-1} s_{r-1} + s_r).$$

For a tangent vector $\eta \in T_{z_0}^{1,0} X$, we can write $\eta = \sum_{\alpha} \eta_{\alpha} \frac{\partial}{\partial z_{\alpha}}$. For the lifts $\tilde{\eta}$ of η to $T_{(z_0, [\zeta_0])}^{1,0} P(E^*)$, we have

$$(2-3) \quad \inf_{q^*(\tilde{\eta})=\eta} \Theta(g)(\tilde{\eta}, \bar{\tilde{\eta}}) = \sum_{\alpha, \beta} \phi_{\alpha\bar{\beta}}|_{(z_0, [\zeta_0])} \eta_{\alpha} \bar{\eta}_{\beta}$$

because $\phi_{\alpha\bar{j}} = 0$ at $(z_0, [\zeta_0])$ and the matrix $(\phi_{i\bar{j}})$ is positive. On the other hand, using the same coordinate system, the curvature Θ of \tilde{G} can be written as

$$\Theta = \sum_{\alpha, \beta} R_{\alpha\bar{\beta}} dz_{\alpha} \wedge d\bar{z}_{\beta} + \sum_{\alpha, l} P_{\alpha\bar{l}} dz_{\alpha} \wedge d\bar{w}_l + \sum_{k, \beta} \mathcal{P}_{k\bar{\beta}} dw_k \wedge d\bar{z}_{\beta} + \sum_{k, l} Q_{k\bar{l}} dw_k \wedge d\bar{w}_l,$$

where $R_{\alpha\bar{\beta}}$, $P_{\alpha\bar{l}}$, $\mathcal{P}_{k\bar{\beta}}$, and $Q_{k\bar{l}}$ are endomorphisms of q^*E^* . By [Wu 2022, (2.4)], for any lift $\tilde{\eta}$ of η to $T_{(z_0, [\zeta_0])}^{1,0} P(E^*)$, we have

$$(2-4) \quad \theta(g)(\tilde{\eta}, \bar{\tilde{\eta}}) = \Theta|_{O_{P(E^*)}(-1)}(\tilde{\eta}, \bar{\tilde{\eta}}) = \sum_{\alpha, \beta} \frac{\tilde{G}(R_{\alpha\bar{\beta}} \zeta_0, \zeta_0)}{\tilde{G}(\zeta_0, \zeta_0)} \eta_{\alpha} \bar{\eta}_{\beta} = - \sum_{\alpha, \beta} \phi_{\alpha\bar{\beta}}|_{(z_0, [\zeta_0])} \eta_{\alpha} \bar{\eta}_{\beta},$$

where the last equality is by [Kobayashi 1996, (5.16)].

From (2-3) and (2-4), we see

$$\inf_{q^*(\tilde{\eta})=\eta} \Theta(g)(\tilde{\eta}, \bar{\tilde{\eta}}) = -\theta(g)(\tilde{\eta}, \bar{\tilde{\eta}}),$$

which is formula (1-2) we claim in the introduction, and when evaluated using normal coordinates they are $\sum_{\alpha, \beta} \phi_{\alpha\bar{\beta}}|_{(z_0, [\zeta_0])} \eta_{\alpha} \bar{\eta}_{\beta}$.

2B. Hermitian metrics. This subsection is a special case of Section 2A and it will be used in the proofs of Theorems 3 and 4. Let G be a Hermitian metric on the bundle E^* . The pull-back bundle $q^*E^* \rightarrow P(E^*)$ with the pull-back metric q^*G induces a metric g^* on the subbundle $O_{P(E^*)}(-1)$. We denote the dual metric on $O_{P(E^*)}(1)$ by g .

Let Ω be a Hermitian metric on X and z_0 a point in X with local coordinates $\{z_{\alpha}\}$ such that

$$\Omega\left(\frac{\partial}{\partial z_{\alpha}}, \frac{\partial}{\partial \bar{z}_{\beta}}\right)(z_0) = \delta_{\alpha\beta}.$$

There exists a holomorphic frame $\{s_i\}$ for E^* around z_0 such that

$$G(s_i, s_j) = \delta_{ij} + O(|z|^2),$$

where z_0 corresponds to the origin in the local coordinates. We use $\{\zeta_i\}$ for the fiber coordinates with respect to the frame $\{s_i\}$. For a point $(z_0, [\zeta_0]) \in P(E^*)$, we assume the local coordinates $(z_1, \dots, z_n, w_1, \dots, w_{r-1})$ around $(z_0, [\zeta_0])$ are given by $w_i = \zeta_i/\zeta_r$ for $i = 1 \sim r-1$. So

$$e := \frac{\zeta_1 s_1 + \dots + \zeta_r s_r}{\zeta_r} = w_1 s_1 + \dots + w_{r-1} s_{r-1} + s_r$$

is a holomorphic frame for $O_{P(E^*)}(-1)$ and

$$\begin{aligned} g^*(e, e) &= q^*G(w_1s_1 + \cdots + w_{r-1}s_{r-1} + s_r, w_1s_1 + \cdots + w_{r-1}s_{r-1} + s_r) \\ &= 1 + O(|z|^2) + O(|w|^2) + O(|w||z|^2) + O(|w|^2|z|^2). \end{aligned}$$

The z_α -derivative of $g^*(e, e)$ is $g^*(e, e)_{z_\alpha} = O((1 + |w| + |w|^2)|z|)$, and hence the w_i -derivatives of $g^*(e, e)_{z_\alpha}$ of any order are zero when evaluated at z_0 . Therefore, if we denote $g^*(e, e)$ by e^ϕ , then at z_0

$$(2-5) \quad \phi_{\alpha\bar{j}} = \phi_{\alpha i\bar{j}} = \phi_{\alpha i\bar{j}\bar{k}} = 0 \quad \text{and} \quad (\log \det(\phi_{i\bar{j}}))_{\alpha\bar{k}} = 0.$$

In this coordinate system, the curvature $\Theta(g)$ is

$$\begin{aligned} \sum_{\alpha, \beta} \frac{\partial^2 \phi}{\partial z_\alpha \partial \bar{z}_\beta} dz_\alpha \wedge d\bar{z}_\beta + \sum_{\alpha, j} \frac{\partial^2 \phi}{\partial z_\alpha \partial \bar{w}_j} dz_\alpha \wedge d\bar{w}_j \\ + \sum_{i, \beta} \frac{\partial^2 \phi}{\partial w_i \partial \bar{z}_\beta} dw_i \wedge d\bar{z}_\beta + \sum_{i, j} \frac{\partial^2 \phi}{\partial w_i \partial \bar{w}_j} dw_i \wedge d\bar{w}_j. \end{aligned}$$

For a tangent vector $\eta \in T_{z_0}^{1,0}X$, we can write $\eta = \sum_\alpha \eta_\alpha \frac{\partial}{\partial z_\alpha}$. For the lifts $\tilde{\eta}$ of η to $T_{(z_0, [\zeta_0])}^{1,0}P(E^*)$, we have

$$(2-6) \quad \inf_{q^*(\tilde{\eta})=\eta} \Theta(g)(\tilde{\eta}, \tilde{\eta}) = \sum_{\alpha, \beta} \phi_{\alpha\bar{\beta}}|_{(z_0, [\zeta_0])} \eta_\alpha \bar{\eta}_\beta$$

because $\phi_{\alpha\bar{j}} = 0$ at z_0 and the matrix $(\phi_{i\bar{j}})$ is positive. Since G is a Hermitian metric, the corresponding Kobayashi curvature is

$$(2-7) \quad \theta(g) = q^*\Theta(G)|_{O_{P(E^*)}(-1)},$$

which is equal to the negative of (2-6) by Section 2A.

Using the same coordinate system, the restriction $\Theta(g)|_{P(E^*_z)}$ is $\sum \phi_{i\bar{j}} dw_i \wedge d\bar{w}_j$, so the metric on $K_{P(E^*)/X}$ induced from $\{(\Theta(g)|_{P(E^*_z)})^{r-1}\}_{z \in X}$ has its curvature γ_g equal to

$$(2-8) \quad \sum_{\alpha, \beta} (\log \det(\phi_{i\bar{j}}))_{\alpha\bar{\beta}} dz_\alpha \wedge d\bar{z}_\beta + \sum_{\alpha, j} (\log \det(\phi_{i\bar{j}}))_{\alpha\bar{j}} dz_\alpha \wedge d\bar{w}_j \\ + \sum_{i, \beta} (\log \det(\phi_{i\bar{j}}))_{i\bar{\beta}} dw_i \wedge d\bar{z}_\beta + \sum_{i, j} (\log \det(\phi_{i\bar{j}}))_{i\bar{j}} dw_i \wedge d\bar{w}_j.$$

The matrix $((\log \det(\phi_{i\bar{j}}))_{i\bar{j}})$ is negative because it represents the negative of the Ricci curvature of the Fubini–Study metric on $P(E^*_z)$. Moreover, the terms $(\log \det(\phi_{i\bar{j}}))_{\alpha\bar{j}} = 0$ at z_0 by (2-5). As a result, for a tangent vector $\eta \in T_{z_0}^{1,0}X$ with $\eta = \sum \eta_\alpha \frac{\partial}{\partial z_\alpha}$ in this coordinate system, we have

$$(2-9) \quad \sup_{q^*(\tilde{\eta})=\eta} \gamma_g(\tilde{\eta}, \tilde{\eta}) = \sum_{\alpha, \beta} (\log \det(\phi_{i\bar{j}}))_{\alpha\bar{\beta}}|_{(z_0, [\zeta_0])} \eta_\alpha \bar{\eta}_\beta,$$

where $\tilde{\eta}$ are the lifts of η to $T_{(z_0, [\zeta_0])}^{1,0}P(E^*)$.

Finally, the metric on $K_{P(E^*)/X}$ induced from $\{(\Theta(g)|_{P(E_z^*)})^{r-1}\}_{z \in X}$ can be identified with the metric $(g^*)^r \otimes q^*(\det G^*)$ under the isomorphism

$$K_{P(E^*)/X} \simeq O_{P(E^*)}(-r) \otimes q^* \det E$$

(see [Kobayashi 1987, p. 85, Proposition 3.6.20]). This fact can be verified at one point using the normal coordinates above. Therefore,

$$(2-10) \quad \gamma_g = -r\Theta(g) - q^*\Theta(\det G).$$

So, for any $\eta \in T_z^{1,0}X$ and any $[\zeta] \in P(E_z^*)$,

$$\begin{aligned} \sup_{q_*(\bar{\eta})=\eta} \gamma_g(\bar{\eta}, \bar{\eta}) &= -r \inf_{q_*(\bar{\eta})=\eta} \Theta(g)(\bar{\eta}, \bar{\eta}) - \Theta(\det G)(\eta, \bar{\eta}) \\ &= r\theta(g)(\bar{\eta}, \bar{\eta}) - \Theta(\det G)(\eta, \bar{\eta}). \end{aligned}$$

This is formula (1-8) that we promise to prove in the introduction.

2C. Convexity. Let E be a holomorphic vector bundle of rank r over a compact complex manifold X . Given a Hermitian metric H_k on the symmetric power $S^k E$, we can define a Finsler metric on E by assigning to $u \in E$ length $H_k(u^k, u^k)^{1/2k}$. We will denote this Finsler metric by $H_k^{1/2k}$, namely $H_k^{1/2k}(u) = H_k(u^k, u^k)^{1/2k}$.

Lemma 5. *Let F_1 be a vector bundle and F_2 a line bundle over X . Assume F_2 carries a Hermitian metric H . We also assume, for some k , $S^k F_1$ carries a Hermitian metric H_k such that the induced Finsler metric $H_k^{1/2k}$ on F_1 is convex:*

$$H_k^{1/2k}(u+v) \leq H_k^{1/2k}(u) + H_k^{1/2k}(v) \quad \text{for } u, v \in F_1.$$

Then the Finsler metric $(H_k \otimes H^k)^{1/2k}$ on $F_1 \otimes F_2$ is convex.

Since F_2 is a line bundle, there is a canonical isomorphism between the bundles $S^k(F_1 \otimes F_2)$ and $S^k F_1 \otimes F_2^k$ which we use implicitly in the statement of Lemma 5. Roughly speaking, Lemma 5 indicates that convexity is not affected by tensoring with a line bundle.

Proof. Fix $p \in X$. The fiber $F_2|_p$ is a one dimensional vector space and we let e be a basis. For x and $y \in F_1 \otimes F_2|_p$, we can write $x = \tilde{x} \otimes e$ and $y = \tilde{y} \otimes e$ where $\tilde{x}, \tilde{y} \in F_1|_p$. By definition,

$$\begin{aligned} (H_k \otimes H^k)^{1/2k}(x+y) &= H_k \otimes H^k((x+y)^k, (x+y)^k)^{1/2k} \\ &= H_k \otimes H^k((\tilde{x} + \tilde{y})^k \otimes e^k, (\tilde{x} + \tilde{y})^k \otimes e^k)^{1/2k} \\ &= H_k((\tilde{x} + \tilde{y})^k, (\tilde{x} + \tilde{y})^k)^{1/2k} H^k(e^k, e^k)^{1/2k} \\ &\leq [H_k(\tilde{x}^k, \tilde{x}^k)^{1/2k} + H_k(\tilde{y}^k, \tilde{y}^k)^{1/2k}] H^k(e^k, e^k)^{1/2k} \\ &= (H_k \otimes H^k)^{1/2k}(x) + (H_k \otimes H^k)^{1/2k}(y). \end{aligned}$$

Therefore the Finsler metric $(H_k \otimes H^k)^{1/2k}$ is convex. \square

2D. Direct image bundles. We recall how to construct Hermitian metrics on direct image bundles and compute their curvature. Let g be a Hermitian metric on $O_{P(E^*)}(1)$ with curvature $\Theta(g)$. Denote the restriction of the curvature to a fiber, $\Theta(g)|_{P(E_z^*)}$ by ω_z for $z \in X$, and assume $\omega_z > 0$ for all $z \in X$. With the canonical isomorphism

$$\Phi_{k,z} : S^k E_z \rightarrow H^0(P(E_z^*), O_{P(E_z^*)}(k)) \quad \text{for } k \geq 0$$

(see [Demailly 2012, p. 278, Theorem 15.5]), we define a Hermitian metric H_k on $S^k E$ by

$$(2-11) \quad H_k(u, v) := \int_{P(E_z^*)} g^k(\Phi_{k,z}(u), \Phi_{k,z}(v)) \omega_z^{r-1} \quad \text{for } u \text{ and } v \in S^k E_z.$$

Let us denote by Θ_k the curvature of H_k . Fixing $z \in X$ and $u \in S^k E_z$, in order to estimate the $(1, 1)$ -form $H_k(\Theta_k u, u)$, we first extend the vector u to a local holomorphic section \tilde{u} whose covariant derivative at z with respect to H_k equals zero. A straightforward computation shows

$$\partial \bar{\partial} H_k(\tilde{u}, \tilde{u})|_z = -H_k(\Theta_k u, u).$$

But $H_k(\tilde{u}, \tilde{u})(z)$ for z near z can also be written as the push-forward

$$q_* (g^k(\Phi_{k,z}(\tilde{u}), \Phi_{k,z}(\tilde{u})) \Theta(g)^{r-1}),$$

where $q : P(E^*) \rightarrow X$ is the projection, so

$$(2-12) \quad -H_k(\Theta_k u, u) = \partial \bar{\partial} H_k(\tilde{u}, \tilde{u})|_z = q_* \partial \bar{\partial} (g^k(\Phi_{k,z}(\tilde{u}), \Phi_{k,z}(\tilde{u})) \Theta(g)^{r-1})|_z.$$

Similarly, we can use a metric on $O_{P(E)}(1) \otimes p^* \det E$ to construct Hermitian metrics on $S^k E^* \otimes (\det E)^k$. The formula is similar to (2-11), and we use bold symbols to highlight the change. Let g be a metric on $O_{P(E)}(1) \otimes p^* \det E$ with curvature $\Theta(g)$. Denote the restriction of the curvature to a fiber $\Theta(g)|_{P(E_z)}$ by ω_z for $z \in X$. Assume $\omega_z > 0$ for all $z \in X$. With the canonical isomorphism

$$\Phi_{k,z} : S^k E_z^* \otimes (\det E_z)^k \rightarrow H^0(P(E_z), O_{P(E_z)}(k) \otimes (p^* \det E_z)^k) \quad \text{for } k \geq 0,$$

we define a Hermitian metric \mathbf{H}_k on $S^k E^* \otimes (\det E)^k$ by

$$(2-13) \quad \mathbf{H}_k(u, v) := \int_{P(E_z)} g^k(\Phi_{k,z}(u), \Phi_{k,z}(v)) \omega_z^{r-1}$$

for u and $v \in S^k E_z^* \otimes (\det E_z)^k$. We also have a curvature formula similar to (2-12).

2E. Berndtsson's positivity theorem. Let h be a metric on $O_{P(E^*)}(1)$ with curvature $\Theta(h) > 0$. Denote $\Theta(h)|_{P(E_z^*)}$ by ω_z for $z \in X$. We are going to define a Hermitian metric on $\det E$ using the metric h . The relative canonical bundle $K_{P(E^*)/X}$ has a metric induced from $\{\omega_z^{r-1}\}_{z \in X}$. With h^r on $O_{P(E^*)}(r)$ and the isomorphism $K_{P(E^*)/X} \otimes O_{P(E^*)}(r) \simeq q^* \det E$, there is an induced metric ρ on $q^* \det E$. Using the canonical isomorphism

$$\Psi_z : \det E_z \rightarrow H^0(P(E_z^*), q^* \det E_z),$$

we define a Hermitian metric H on $\det E$ by

$$(2-14) \quad H(u, v) := \int_{P(E_z^*)} \rho(\Psi_z(u), \Psi_z(v)) \omega_z^{r-1} \quad \text{for } u \text{ and } v \in \det E_z.$$

By Berndtsson's theorem [Berndtsson 2009a], this metric H is Griffiths positive, but it is the inequality that leads to this fact we will use. We follow the presentation in [Liu et al. 2013, Section 4.1] (see also [Berndtsson 2009b, Section 2]). Denote the curvature of H by Θ . Fix $z \in X$, $v \in \det E_z$ and $\eta \in T_z^{1,0} X$. For a local holomorphic frame of E^* around z , we denote by $\{\zeta_i\}$ the fiber coordinates with respect to this frame, and by $\{z_\alpha\}$ the local coordinates on X . Around $P(E_z^*)$ in $P(E^*)$, we have homogeneous coordinates $[\zeta_1, \dots, \zeta_r]$ which induce local coordinates (w_1, \dots, w_{r-1}) . For a local frame e^* of $O_{P(E^*)}(1)$, we denote $h(e^*, e^*)$ by $e^{-\phi}$ and write the tangent vector $\eta = \sum \eta_\alpha \frac{\partial}{\partial z_\alpha}$. The inequality that leads to Berndtsson's theorem is

$$(2-15) \quad -H(\Theta v, v)(\eta, \bar{\eta}) \leq \int_{P(E_z^*)} \rho(\Psi_z(v), \Psi_z(v)) r \sum_{\alpha, \beta} \left(\sum_{i, j} \phi_{\alpha \bar{j}} \phi^{i \bar{j}} \phi_{i \bar{\beta}} - \phi_{\alpha \bar{\beta}} \right) \eta_\alpha \bar{\eta}_\beta \omega_z^{r-1},$$

where

$$\phi_{i \bar{j}} := \frac{\partial^2 \phi}{\partial w_i \partial \bar{w}_j}, \quad \phi_{\alpha \bar{j}} := \frac{\partial^2 \phi}{\partial z_\alpha \partial \bar{w}_j}, \quad \phi_{\alpha \bar{\beta}} := \frac{\partial^2 \phi}{\partial z_\alpha \partial \bar{z}_\beta},$$

and $(\phi^{i \bar{j}})$ is the inverse matrix of $(\phi_{i \bar{j}})$. Since $\det E$ is a line bundle, the curvature Θ is a $(1, 1)$ -form, and so $H(\Theta v, v)(\eta, \bar{\eta}) = H(v, v) \Theta(\eta, \bar{\eta})$. If we further assume $H(v, v) = 1$, then the left-hand side of (2-15) becomes $-\Theta(\eta, \bar{\eta})$.

3. Proof of Theorem 1

Recall that h and g are metrics on $O_{P(E^*)}(1)$ that satisfy the assumptions in Theorem 1 and the inequalities (1-3) and (1-4). We use the metric h to construct a Hermitian metric H on $\det E$ as in (2-14), and the metric g to construct Hermitian metrics H_k on $S^k E$ as in (2-11). The number k is yet to be determined.

We start with the metric g . Given a point $(z_0, [\zeta_0]) \in P(E^*)$, we have the normal coordinate system from [Section 2A](#). In this coordinate system, let us introduce the following n -by- n matrix-valued function:

$$B_k = ((B_k)_{\alpha\beta}) := (k\phi_{\alpha\bar{\beta}} - (\log \det(\phi_{i\bar{j}}))_{\alpha\bar{\beta}}),$$

where $g(e^*, e^*) = e^{-\phi}$. By continuity, there is a neighborhood U of $(z_0, [\zeta_0])$ in $P(E^*)$ such that in U

$$(3-1) \quad (\phi_{\alpha\bar{\beta}})|_{(z_0, [\zeta_0])} + \frac{r-M}{4} \text{Id}_{n \times n} \geq (\phi_{\alpha\bar{\beta}}).$$

For this U , there is a positive integer k_0 such that for $k \geq k_0$ and in U

$$(3-2) \quad (\phi_{\alpha\bar{\beta}}) + \frac{r-M}{4} \text{Id}_{n \times n} \geq \frac{B_k}{k}.$$

Let us summarize what we have done so far:

Lemma 6. *Given a point $(z_0, [\zeta_0]) \in P(E^*)$, there exist a coordinate neighborhood U of $(z_0, [\zeta_0])$ in $P(E^*)$ and a positive integer k_0 such that in U and for $k \geq k_0$*

$$(3-3) \quad (\phi_{\alpha\bar{\beta}})|_{(z_0, [\zeta_0])} + \frac{r-M}{2} \text{Id}_{n \times n} \geq \frac{B_k}{k}.$$

By [Lemma 6](#), since $P(E_{z_0}^*)$ is compact, we can find on $P(E_{z_0}^*)$ finitely many points $\{(z_0, [\zeta_l])\}_l$ each of which corresponds to a coordinate neighborhood U_l in $P(E^*)$ and a positive integer k_l such that the corresponding (3-3) holds, and $P(E_{z_0}^*) \subset \bigcup_l U_l$. Denote $\max_l k_l$ by k_{\max} . The point z_0 has a neighborhood W in X such that for $z \in W$, the fiber $P(E_z^*)$ can be partitioned as $\bigcup_m V_m$ with each V_m in U_l for some l . By shrinking W , we can assume that for each U_l the corresponding $\Omega(\frac{\partial}{\partial z_\alpha}, \frac{\partial}{\partial \bar{z}_\beta}) := \Omega_{\alpha\bar{\beta}}$ satisfies

$$(3-4) \quad -\varepsilon \delta_{\alpha\beta} < \Omega_{\alpha\bar{\beta}}(z) - \delta_{\alpha\beta} < \varepsilon \delta_{\alpha\beta} \quad \text{for } z \in W,$$

where $\varepsilon := \frac{r-M}{5(r+M)}$.

Recall the Hermitian metrics H_k on $S^k E$ in (2-11) constructed using the metric g . Denote by Θ_k the curvature of H_k . We claim the following lemma (one can also use the asymptotic expansion in [\[Ma and Zhang 2023\]](#) to deduce the lemma).

Lemma 7. *For $k \geq k_{\max}$, $z \in W$, $0 \neq \eta \in T_z^{1,0} X$, and $u \in S^k E_z$ with $H_k(u, u) = 1$, we have*

$$(3-5) \quad H_k(\Theta_k u, u)(\eta, \bar{\eta}) \leq \left(M + \frac{r-M}{2}\right) k \frac{\Omega(\eta, \bar{\eta})}{(1-\varepsilon)}.$$

Proof. As in Section 2D, we extend the vector $u \in S^k E_z$ to a local holomorphic section \tilde{u} whose covariant derivative at z equals zero, and we have

$$\begin{aligned} -H_k(\Theta_k u, u) &= \partial \bar{\partial} H_k(\tilde{u}, \tilde{u})|_z = \int_{P(E_z^*)} \partial \bar{\partial} (g^k(\Phi_{k,z}(\tilde{u}), \Phi_{k,z}(\tilde{u})) \Theta(g)^{r-1}) \\ &= \sum_m \int_{V_m} \partial \bar{\partial} (g^k(\Phi_{k,z}(\tilde{u}), \Phi_{k,z}(\tilde{u})) \Theta(g)^{r-1}). \end{aligned}$$

In the last equality, we partition the fiber $P(E_z^*)$ as $\bigcup_m V_m$ with each V_m in U_l for some l . In a fixed $V_m \subset U_l$, using the coordinate system of U_l , we can write $\Phi_{k,z}(\tilde{u})$ as $f(e^*)^k$ with f a scalar-valued holomorphic function and e^* a local frame for $\mathcal{O}_{P(E^*)}(1)$. So, $g^k(\Phi_{k,z}(\tilde{u}), \Phi_{k,z}(\tilde{u})) = |f|^2 e^{-k\phi}$. Meanwhile, recall the curvature $\Theta(g) = \partial \bar{\partial} \phi$. By Stokes' theorem and a count on degrees, we have

$$\begin{aligned} \sum_m \int_{V_m} \partial \bar{\partial} (g^k(\Phi_{k,z}(\tilde{u}), \Phi_{k,z}(\tilde{u})) \Theta(g)^{r-1}) \\ = \sum_m \int_{V_m} \sum_{\alpha, \beta} \frac{\partial^2 |f|^2 e^{-k\phi} \det(\phi_{i\bar{j}})}{\partial z_\alpha \partial \bar{z}_\beta} dz_\alpha \wedge d\bar{z}_\beta \bigwedge_j dw_j \wedge d\bar{w}_j. \end{aligned}$$

So, if the tangent vector $\eta = \sum_\alpha \eta_\alpha \frac{\partial}{\partial z_\alpha}$ in the coordinate neighborhood U_l , then

$$(3-6) \quad -H_k(\Theta_k u, u)(\eta, \bar{\eta}) = \sum_m \int_{V_m} \sum_{\alpha, \beta} \frac{\partial^2 |f|^2 e^{-k\phi} \det(\phi_{i\bar{j}})}{\partial z_\alpha \partial \bar{z}_\beta} \eta_\alpha \bar{\eta}_\beta \bigwedge_j dw_j \wedge d\bar{w}_j.$$

Note that the integrands in (3-6) are written in the local coordinates of corresponding U_l . A direct computation shows

$$\begin{aligned} \sum_{\alpha, \beta} \frac{\partial^2 |f|^2 e^{-k\phi} \det(\phi_{i\bar{j}})}{\partial z_\alpha \partial \bar{z}_\beta} \eta_\alpha \bar{\eta}_\beta \\ = e^{-k\phi} \det(\phi_{i\bar{j}}) \left| \sum_\alpha \frac{\partial f}{\partial z_\alpha} \eta_\alpha - f \sum_\alpha (k\phi_\alpha - (\log \det \phi_{i\bar{j}})_\alpha) \eta_\alpha \right|^2 \\ - |f|^2 e^{-k\phi} \det(\phi_{i\bar{j}}) \sum_{\alpha, \beta} (k\phi_{\alpha\bar{\beta}} - (\log \det \phi_{i\bar{j}})_{\alpha\bar{\beta}}) \eta_\alpha \bar{\eta}_\beta \\ \geq -|f|^2 e^{-k\phi} \det(\phi_{i\bar{j}}) \sum_{\alpha, \beta} (B_k)_{\alpha\beta} \eta_\alpha \bar{\eta}_\beta. \end{aligned}$$

By (3-3),

$$(3-7) \quad \frac{1}{k} \sum_{\alpha, \beta} (B_k)_{\alpha\beta} \eta_\alpha \bar{\eta}_\beta \leq \sum_{\alpha, \beta} \phi_{\alpha\bar{\beta}}|_{(z_0, [\zeta_l])} \eta_\alpha \bar{\eta}_\beta + \frac{r-M}{2} \sum_\alpha |\eta_\alpha|^2.$$

Using the coordinate system of U_l , the tangent vector $\eta = \sum_\alpha \eta_\alpha \frac{\partial}{\partial z_\alpha}$ at z induces a tangent vector $\eta_l = \sum_\alpha \eta_\alpha \frac{\partial}{\partial z_\alpha}|_{z_0}$ at z_0 . Denote the lifts of η_l to $T_{(z_0, [\zeta_l])}^{1,0} P(E^*)$

by $\tilde{\eta}_l$. According to (1-2), (1-3), and (2-3), we see

$$(3-8) \quad M \sum_{\alpha} |\eta_{\alpha}|^2 \geq -\theta(g)(\tilde{\eta}_l, \tilde{\eta}_l) = \inf_{q_*(\tilde{\eta}_l)=\eta_l} \Theta(g)(\tilde{\eta}_l, \tilde{\eta}_l) = \sum_{\alpha, \beta} \phi_{\alpha\bar{\beta}}|_{(z_0, [\zeta_l])} \eta_{\alpha} \bar{\eta}_{\beta}.$$

Therefore, (3-7) becomes

$$(3-9) \quad \frac{1}{k} \sum_{\alpha, \beta} (B_k)_{\alpha\beta} \eta_{\alpha} \bar{\eta}_{\beta} \leq \left(M + \frac{r-M}{2} \right) \sum_{\alpha} |\eta_{\alpha}|^2 \leq \left(M + \frac{r-M}{2} \right) \frac{\Omega(\eta, \bar{\eta})}{(1-\varepsilon)},$$

where we use (3-4) in the second inequality. So, (3-6) becomes

$$(3-10) \quad \begin{aligned} & -H_k(\Theta_k u, u)(\eta, \bar{\eta}) \\ & \geq \sum_m \int_{V_m} -|f|^2 e^{-k\phi} \det(\phi_{i\bar{j}}) \bigwedge_j dw_j \wedge d\bar{w}_j \left(M + \frac{r-M}{2} \right) k \frac{\Omega(\eta, \bar{\eta})}{(1-\varepsilon)} \\ & = - \left(M + \frac{r-M}{2} \right) k \frac{\Omega(\eta, \bar{\eta})}{(1-\varepsilon)} \end{aligned}$$

since $H_k(u, u) = 1$. □

We turn now to the metric h . The argument about h is similar to that about g , and it will be used in Theorems 2, 3, and 4. Given a point $(z_0, [\zeta_0]) \in P(E^*)$, we have the normal coordinate system from Section 2A with respect to the metric h . In this coordinate system, let us introduce the n -by- n matrix-valued function

$$A = (A_{\alpha\beta}) := \left(\phi_{\alpha\bar{\beta}} - \sum_{i,j} \phi_{\alpha\bar{j}} \phi^{i\bar{j}} \phi_{i\bar{\beta}} \right),$$

where $h(e^*, e^*) = e^{-\phi}$ and $(\phi^{i\bar{j}})$ is the inverse matrix of $(\phi_{i\bar{j}})$. By continuity, there is a neighborhood U of $(z_0, [\zeta_0])$ in $P(E^*)$ such that in U

$$(3-11) \quad rA + \frac{r-M}{4} \text{Id}_{n \times n} \geq rA|_{(z_0, [\zeta_0])}.$$

In summary:

Lemma 8. *Given a point $(z_0, [\zeta_0]) \in P(E^*)$, there exists a coordinate neighborhood U of $(z_0, [\zeta_0])$ in $P(E^*)$ such that in U*

$$(3-12) \quad rA + \frac{r-M}{4} \text{Id}_{n \times n} \geq r(\phi_{\alpha\bar{\beta}})|_{(z_0, [\zeta_0])}.$$

By Lemma 8, since $P(E^*)$ is compact, we can find on $P(E^*)$ finitely many points $\{(z_0, [\zeta_l])\}_l$ each of which corresponds to a coordinate neighborhood U_l in $P(E^*)$ such that the corresponding (3-12) holds, and $P(E^*) \subset \bigcup_l U_l$. The point z_0 has a neighborhood W' in X such that for $z \in W'$, the fiber $P(E^*_z)$ can

be partitioned as $\bigcup_m V_m$ with each V_m in U_l for some l . By shrinking W' , we can assume that for each U_l the corresponding $\Omega\left(\frac{\partial}{\partial z_\alpha}, \frac{\partial}{\partial \bar{z}_\beta}\right) := \Omega_{\alpha\bar{\beta}}$ satisfies

$$(3-13) \quad -\varepsilon\delta_{\alpha\beta} < \Omega_{\alpha\bar{\beta}}(z) - \delta_{\alpha\beta} < \varepsilon\delta_{\alpha\beta} \quad \text{for } z \in W',$$

where $\varepsilon := \frac{r-M}{5(r+M)}$.

Recall the Hermitian metric H on $\det E$ in (2-14) constructed using the metric h . Denote by Θ the curvature of H . We claim:

Lemma 9. *For $z \in W'$ and $\eta \in T_z^{1,0}X$, we have*

$$(3-14) \quad -\Theta(\eta, \bar{\eta}) \leq -\left(r - \frac{r-M}{4}\right) \frac{\Omega(\eta, \bar{\eta})}{(1+\varepsilon)}.$$

Proof. Using (2-15) and assuming $H(v, v) = 1$, we get

$$(3-15) \quad -\Theta(\eta, \bar{\eta}) \leq \sum_m \int_{V_m} \rho(\Psi_z(v), \Psi_z(v)) r \sum_{\alpha, \beta} \left(\sum_{i, j} \phi_{\alpha\bar{j}} \phi^{i\bar{j}} \phi_{i\bar{\beta}} - \phi_{\alpha\bar{\beta}} \right) \eta_\alpha \bar{\eta}_\beta \omega_z^{r-1},$$

where we again partition $P(E_z^*)$ as $\bigcup_m V_m$ with each V_m in U_l for some l . Note that the integrands in (3-15) are written in the local coordinates of corresponding U_l . In a fixed $V_m \subset U_l$, we have $\eta = \sum_\alpha \eta_\alpha \frac{\partial}{\partial z_\alpha}$, and by (3-12) we see

$$(3-16) \quad r \sum_{\alpha, \beta} A_{\alpha\beta} \eta_\alpha \bar{\eta}_\beta + \frac{r-M}{4} \sum_\alpha |\eta_\alpha|^2 \geq r \sum_{\alpha, \beta} \phi_{\alpha\bar{\beta}}|_{(z_0, [z_l])} \eta_\alpha \bar{\eta}_\beta.$$

In U_l , the tangent vector

$$\eta = \sum_\alpha \eta_\alpha \frac{\partial}{\partial z_\alpha} \quad \text{at } z$$

induces a tangent vector

$$\eta_l = \sum_\alpha \eta_\alpha \frac{\partial}{\partial z_\alpha} \Big|_{z_0} \quad \text{at } z_0.$$

Denote the lifts of η_l to $T_{(z_0, [z_l])}^{1,0}P(E^*)$ by $\tilde{\eta}_l$. By (1-2), (1-4), and (2-3), we see

$$(3-17) \quad \sum_\alpha |\eta_\alpha|^2 \leq -\theta(h)(\tilde{\eta}_l, \bar{\tilde{\eta}}_l) = \inf_{q_*(\tilde{\eta}_l) = \eta_l} \Theta(h)(\tilde{\eta}_l, \bar{\tilde{\eta}}_l) = \sum_{\alpha, \beta} \phi_{\alpha\bar{\beta}}|_{(z_0, [z_l])} \eta_\alpha \bar{\eta}_\beta.$$

Therefore, (3-16) becomes

$$r \sum_{\alpha, \beta} A_{\alpha\beta} \eta_\alpha \bar{\eta}_\beta \geq \left(r - \frac{r-M}{4}\right) \sum_\alpha |\eta_\alpha|^2 \geq \left(r - \frac{r-M}{4}\right) \frac{\Omega(\eta, \bar{\eta})}{1+\varepsilon},$$

where we use (3-13) in the second inequality. So, (3-15) becomes

$$\begin{aligned}
 -\Theta(\eta, \bar{\eta}) &\leq -\sum_m \int_{V_m} \rho(\Psi_z(v), \Psi_z(v)) \omega_z^{r-1} \left(r - \frac{r-M}{4} \right) \frac{\Omega(\eta, \bar{\eta})}{1+\varepsilon} \\
 &= -\left(r - \frac{r-M}{4} \right) \frac{\Omega(\eta, \bar{\eta})}{(1+\varepsilon)}
 \end{aligned}$$

because $H(v, v) = 1$. □

Now we put together the L^2 -metrics H_k on $S^k E$ in (2-11), and H on $\det E$ in (2-14). Since $(\det E^*)^k$ is a line bundle, we can identify $\text{End}(S^k E \otimes (\det E^*)^k)$ with $\text{End}(S^k E)$, and the curvature of the metric $H_k \otimes (H^*)^k$ on $S^k E \otimes (\det E^*)^k$ can be written as

$$\Theta_k - k\Theta \otimes \text{Id}_{S^k E},$$

where Θ_k and Θ are the curvature of H_k and H respectively. We claim that for $k \geq k_{\max}$ and in $W \cap W'$ a neighborhood of z_0 , the metric $H_k \otimes (H^*)^k$ is Griffiths negative. Indeed, as a result of Lemmas 7 and 9, for $k \geq k_{\max}$, $z \in W \cap W'$, $0 \neq \eta \in T_z^{1,0} X$, and $u \in S^k E_z$ with $H_k(u, u) = 1$, we see

$$H_k(\Theta_k u, u)(\eta, \bar{\eta}) - k\Theta(\eta, \bar{\eta}) \leq k \left(M + \frac{r-M}{2} \right) \frac{\Omega(\eta, \bar{\eta})}{(1-\varepsilon)} - k \left(r - \frac{r-M}{4} \right) \frac{\Omega(\eta, \bar{\eta})}{(1+\varepsilon)}.$$

The term on the right is negative after some computation using $\varepsilon = \frac{r-M}{5(r+M)}$. So, we have proved the claim that for $k \geq k_{\max}$ and in $W \cap W' \subset X$, the metric $H_k \otimes (H^*)^k$ is Griffiths negative. Since X is compact, $H_k \otimes (H^*)^k$ is Griffiths negative on the entire X for k large enough.

Now we fix k such that the Hermitian metric $H_k \otimes (H^*)^k$ on the bundle

$$S^k E \otimes (\det E^*)^k$$

is Griffiths negative on X . The Hermitian metric H_k by construction is an L^2 -integral, so its k -th root is a convex Finsler metric on E (see [Wu 2022, proof of Theorem 1] for details). By Lemma 5, the k -th root of $H_k \otimes (H^*)^k$ is a convex Finsler metric on $E \otimes \det E^*$ which we denote by F . Moreover, this Finsler metric F is strongly plurisubharmonic on $E \otimes \det E^* \setminus \{\text{zero section}\}$ due to Griffiths negativity of $H_k \otimes (H^*)^k$. By adding a small Hermitian metric, we can assume F is strongly convex and strongly plurisubharmonic.

In general, the Kobayashi curvature of Finsler metrics do not behave well under duality [Demailly 1999, Remark 2.7]. But since our Finsler metric F is strongly convex, the dual Finsler metric of F is in fact strongly pseudoconvex and Kobayashi positive (this duality result is originally due to Sommese [1978] and Demailly [1999, Theorem 2.5]. See also [Wu 2022, proof of Theorem 1 and Lemma 6]). In summary,

the dual Finsler metric of F is a convex, strongly pseudoconvex, and Kobayashi positive Finsler metric on $E^* \otimes \det E$. Hence the proof of [Theorem 1](#) is complete.

With slight modification on the proof, one has the following variant of [Theorem 1](#).

Theorem 10. *Assume $r \geq 2$ and the line bundle $O_{P(E^*)}(1)$ has a positively curved metric h and a metric g with $\Theta(g)|_{P(E_z^*)} > 0$ for all $z \in X$. If there exist a Hermitian metric Ω on X and a constant $M \geq 1$ such that the following inequalities of $(1, 1)$ -forms hold*

$$(3-18) \quad Mq^* \Omega \geq -\theta(g),$$

$$(3-19) \quad q^* \Omega \leq -\theta(h),$$

then for any positive integer $l > M/r$, the bundle $E^* \otimes (\det E)^l$ is Kobayashi positive.

4. Proof of [Theorem 2](#)

The proof is similar to what we do in [Section 3](#) except that we are dealing with not only $P(E^*)$ but $P(E)$ here. The metric h is used to define a Hermitian metric H on $\det E$ as in (2-14). The metric g is used to define Hermitian metrics H_k on $S^k E^* \otimes (\det E)^k$ as in (2-13).

Fix z_0 in X . For the metric h on $O_{P(E^*)}(1)$, we follow the path that leads to [Lemma 9](#) in [Section 3](#) to deduce a neighborhood W' of z_0 in X such that for $z \in W'$ and $\eta \in T_z^{1,0} X$, the curvature Θ of H satisfies

$$(4-1) \quad -\Theta(\eta, \bar{\eta}) \leq -\left(r - \frac{r-M}{4}\right) \frac{\Omega(\eta, \bar{\eta})}{(1+\varepsilon)},$$

with $\varepsilon = \frac{r-M}{5(r+M)}$.

For the metric g on $O_{P(E)}(1) \otimes p^* \det E$, we replace $O_{P(E^*)}(1) \rightarrow P(E^*)$ in [Section 3](#) with $O_{P(E \otimes \det E^*)}(1) \rightarrow P(E \otimes \det E^*)$ and use the canonical isomorphism between $O_{P(E \otimes \det E^*)}(1) \rightarrow P(E \otimes \det E^*)$ and $O_{P(E)}(1) \otimes p^* \det E \rightarrow P(E)$. Then following the argument leading to [Lemma 7](#), we obtain a positive integer k_{\max} and a neighborhood W of z_0 in X such that for $k \geq k_{\max}$, $z \in W$, $\eta \in T_z^{1,0} X$, and $u \in S^k E_z^* \otimes (\det E_z)^k$ with $H_k(u, u) = 1$, the curvature Θ_k of H_k satisfies

$$(4-2) \quad H_k(\Theta_k u, u)(\eta, \bar{\eta}) \leq \left(M + \frac{r-M}{2}\right) k \frac{\Omega(\eta, \bar{\eta})}{(1-\varepsilon)}.$$

On the bundle $[S^k E^* \otimes (\det E)^k] \otimes (\det E^*)^k$, there is a Hermitian metric $H_k \otimes (H^*)^k$ with curvature $\Theta_k - k\Theta \otimes \text{Id}_{S^k E^* \otimes (\det E)^k}$. As a result of (4-1) and (4-2), we deduce that, for $k \geq k_{\max}$, $z \in W \cap W'$, $\eta \in T_z^{1,0} X$, and $u \in S^k E_z^* \otimes (\det E_z)^k$ with $H_k(u, u) = 1$,

$$H_k(\Theta_k u, u)(\eta, \bar{\eta}) - k\Theta(\eta, \bar{\eta}) \leq k \left(M + \frac{r-M}{2}\right) \frac{\Omega(\eta, \bar{\eta})}{(1-\varepsilon)} - k \left(r - \frac{r-M}{4}\right) \frac{\Omega(\eta, \bar{\eta})}{(1+\varepsilon)}.$$

Again, the term on the right is negative using $\varepsilon = \frac{r-M}{5(r+M)}$. So we have proved that for $k \geq k_{\max}$ and in $W \cap W'$, the metric $\mathbf{H}_k \otimes (H^*)^k$ is Griffiths negative. Since X is compact, $\mathbf{H}_k \otimes (H^*)^k$ is Griffiths negative on X for k large.

Now we fix k such that $\mathbf{H}_k \otimes (H^*)^k$ on the bundle

$$[S^k E^* \otimes (\det E)^k] \otimes (\det E^*)^k \simeq S^k E^*$$

is Griffiths negative. Using the same argument as those at the end of [Section 3](#), we obtain a convex, strongly pseudoconvex, Kobayashi positive Finsler metric on E .

5. Proof of [Theorem 3](#)

We use the metric h to construct a Hermitian metric H on $\det E$ as in (2-14), and the metric g to construct a Hermitian metric H_1 on $S^1 E = E$ as in (2-11).

We start with the metric g . For $(z_0, [\zeta_0])$ in $P(E^*)$, there is a special coordinate system given in [Section 2B](#). In this coordinate system, we define the following n -by- n matrix-valued function:

$$B = (B_{\alpha\beta}) := (\phi_{\alpha\bar{\beta}} - (\log \det(\phi_{i\bar{j}}))_{\alpha\bar{\beta}}),$$

where $g(e^*, e^*) = e^{-\phi}$. By continuity, there is a neighborhood U of $(z_0, [\zeta_0])$ in $P(E^*)$ such that in U

$$B|_{(z_0, [\zeta_0])} + \frac{r-M}{4} \text{Id}_{n \times n} \geq B.$$

In summary:

Lemma 11. *Given a point $(z_0, [\zeta_0]) \in P(E^*)$, there exists a coordinate neighborhood U of $(z_0, [\zeta_0])$ in $P(E^*)$ such that in U*

$$(5-1) \quad B|_{(z_0, [\zeta_0])} + \frac{r-M}{4} \text{Id}_{n \times n} \geq B.$$

By [Lemma 11](#), since $P(E_{z_0}^*)$ is compact, we can find finitely many points $\{(z_0, [\zeta_l])\}_l$ on $P(E_{z_0}^*)$ each of which corresponds to a coordinate neighborhood U_l in $P(E^*)$ such that the corresponding (5-1) holds, and $P(E_{z_0}^*) \subset \bigcup_l U_l$. The fiber $P(E_{z_0}^*)$ can be partitioned as $\bigcup_m V_m$ with each V_m in U_l for some l .

Recall the Hermitian metric H_1 on E in (2-11) constructed using the metric g . Denote by Θ_1 the curvature of H_1 . We claim:

Lemma 12. *For $0 \neq \eta \in T_{z_0}^{1,0} X$ and $u \in E_{z_0}$ with $H_1(u, u) = 1$, we have*

$$(5-2) \quad H_1(\Theta_1 u, u)(\eta, \bar{\eta}) \leq \left(M + \frac{r-M}{4} \right) \Omega(\eta, \bar{\eta}).$$

Proof. As in [Section 2D](#), we extend the vector $u \in E_{z_0}$ to a local holomorphic section \tilde{u} whose covariant derivative at z_0 equals zero, and we have

$$\begin{aligned}
-H_1(\Theta_1 u, u) &= \partial \bar{\partial} H_1(\tilde{u}, \tilde{u})|_{z_0} = \int_{P(E_{z_0}^*)} \partial \bar{\partial} (g(\Phi_{1,z}(\tilde{u}), \Phi_{1,z}(\tilde{u})) \Theta(g)^{r-1}) \\
&= \sum_m \int_{V_m} \partial \bar{\partial} (g(\Phi_{1,z}(\tilde{u}), \Phi_{1,z}(\tilde{u})) \Theta(g)^{r-1}).
\end{aligned}$$

In a fixed $V_m \subset U_l$, we can write $\Phi_{1,z}(\tilde{u})$ as $f e^*$ with f a scalar-valued holomorphic function and e^* a local frame for $\mathcal{O}_{P(E^*)}(1)$. So,

$$g(\Phi_{1,z}(\tilde{u}), \Phi_{1,z}(\tilde{u})) = |f|^2 e^{-\phi}.$$

Meanwhile, recall the curvature $\Theta(g) = \partial \bar{\partial} \phi$. By Stokes' theorem and a count on degrees, we have

$$\begin{aligned}
\sum_m \int_{V_m} \partial \bar{\partial} (g(\Phi_{1,z}(\tilde{u}), \Phi_{1,z}(\tilde{u})) \Theta(g)^{r-1}) \\
= \sum_m \int_{V_m} \sum_{\alpha, \beta} \frac{\partial^2 |f|^2 e^{-\phi} \det(\phi_{i\bar{j}})}{\partial z_\alpha \partial \bar{z}_\beta} dz_\alpha \wedge d\bar{z}_\beta \bigwedge_j dw_j \wedge d\bar{w}_j.
\end{aligned}$$

So,

$$(5-3) \quad -H_1(\Theta_1 u, u)(\eta, \bar{\eta}) = \sum_m \int_{V_m} \sum_{\alpha, \beta} \frac{\partial^2 |f|^2 e^{-\phi} \det(\phi_{i\bar{j}})}{\partial z_\alpha \partial \bar{z}_\beta} \eta_\alpha \bar{\eta}_\beta \bigwedge_j dw_j \wedge d\bar{w}_j$$

for $T_{z_0}^{1,0} X \ni \eta = \sum_\alpha \eta_\alpha \frac{\partial}{\partial z_\alpha}$. A direct computation shows

$$\begin{aligned}
\sum_{\alpha, \beta} \frac{\partial^2 |f|^2 e^{-\phi} \det(\phi_{i\bar{j}})}{\partial z_\alpha \partial \bar{z}_\beta} \eta_\alpha \bar{\eta}_\beta \\
= e^{-\phi} \det(\phi_{i\bar{j}}) \left| \sum_\alpha \frac{\partial f}{\partial z_\alpha} \eta_\alpha - f \sum_\alpha (\phi_\alpha - (\log \det \phi_{i\bar{j}})_\alpha) \eta_\alpha \right|^2 \\
- |f|^2 e^{-\phi} \det(\phi_{i\bar{j}}) \sum_{\alpha, \beta} (\phi_{\alpha\bar{\beta}} - (\log \det \phi_{i\bar{j}})_{\alpha\bar{\beta}}) \eta_\alpha \bar{\eta}_\beta \\
\geq -|f|^2 e^{-\phi} \det(\phi_{i\bar{j}}) \sum_{\alpha, \beta} B_{\alpha\beta} \eta_\alpha \bar{\eta}_\beta.
\end{aligned}$$

By (1-8), (1-9), (2-6), and (2-9), we see

$$\begin{aligned}
M \sum_\alpha |\eta_\alpha|^2 &\geq -(r+1) \theta(g)(\tilde{\eta}, \bar{\tilde{\eta}}) + q^* \Theta(\det G)(\tilde{\eta}, \bar{\tilde{\eta}}) \\
&= \inf_{q_*(\tilde{\eta})=\eta} \Theta(g)(\tilde{\eta}, \bar{\tilde{\eta}}) - \sup_{q_*(\tilde{\eta})=\eta} \gamma_g(\tilde{\eta}, \bar{\tilde{\eta}}) \\
&= \sum_{\alpha, \beta} \phi_{\alpha\bar{\beta}}|_{(z_0, [\zeta_l])} \eta_\alpha \bar{\eta}_\beta - \sum_{\alpha, \beta} (\log \det(\phi_{i\bar{j}}))_{\alpha\bar{\beta}}|_{(z_0, [\zeta_l])} \eta_\alpha \bar{\eta}_\beta \\
&= \sum_{\alpha, \beta} B_{\alpha\beta}|_{(z_0, [\zeta_l])} \eta_\alpha \bar{\eta}_\beta.
\end{aligned}$$

Therefore, (5-1) becomes

$$(5-4) \quad \sum_{\alpha, \beta} B_{\alpha\beta} \eta_\alpha \bar{\eta}_\beta \leq \left(M + \frac{r-M}{4} \right) \sum_{\alpha} |\eta_\alpha|^2 = \left(M + \frac{r-M}{4} \right) \Omega(\eta, \bar{\eta}).$$

So, (5-3) becomes

$$(5-5) \quad \begin{aligned} & -H_1(\Theta_1 u, u)(\eta, \bar{\eta}) \\ & \geq \sum_m \int_{V_m} -|f|^2 e^{-\phi} \det(\phi_{i\bar{j}}) \bigwedge_j dw_j \wedge d\bar{w}_j \left(M + \frac{r-M}{4} \right) \Omega(\eta, \bar{\eta}) \\ & = - \left(M + \frac{r-M}{4} \right) \Omega(\eta, \bar{\eta}) \end{aligned}$$

since $H_1(u, u) = 1$. □

For the metric h on $O_{P(E^*)}(1)$, as in Lemma 9 from Section 3 with slight modification, we deduce that for $\eta \in T_{z_0}^{1,0}X$, the curvature Θ of H satisfies

$$(5-6) \quad -\Theta(\eta, \bar{\eta}) \leq - \left(r - \frac{r-M}{4} \right) \Omega(\eta, \bar{\eta}).$$

Finally, we consider the metric $H_1 \otimes H^*$ on $E \otimes \det E^*$. Since $\det E^*$ is a line bundle, we can identify $\text{End}(E \otimes \det E^*)$ with $\text{End } E$, and the curvature of the metric $H_1 \otimes H^*$ can be written as $\Theta_1 - \Theta \otimes \text{Id}_E$, where Θ_1 and Θ are the curvature of H_1 and H respectively. As a result of Lemma 12 and (5-6), we see for $0 \neq \eta \in T_{z_0}^{1,0}X$ and $u \in E_{z_0}$ with $H_1(u, u) = 1$,

$$H_1(\Theta_1 u, u)(\eta, \bar{\eta}) - \Theta(\eta, \bar{\eta}) \leq \left(M + \frac{r-M}{4} \right) \Omega(\eta, \bar{\eta}) - \left(r - \frac{r-M}{4} \right) \Omega(\eta, \bar{\eta}),$$

the term on the right is negative. Hence we have proved that at z_0 the metric $H_1 \otimes H^*$ is Griffiths negative. The point z_0 is arbitrary, so $H_1 \otimes H^*$ is Griffiths negative on X . As a result, the dual bundle $E^* \otimes \det E$ is Griffiths positive.

6. Proof of Theorem 4

The metric h is used to define a Hermitian metric H on $\det E$ as in (2-14). The metric g is used to define Hermitian metric H_1 on $E^* \otimes \det E$ as in (2-13).

Given z_0 in X . For the metric h on $O_{P(E^*)}(1)$, as in the formula (5-6) from Section 5, for $\eta \in T_{z_0}^{1,0}X$ we have

$$(6-1) \quad -\Theta(\eta, \bar{\eta}) \leq - \left(r - \frac{r-M}{4} \right) \Omega(\eta, \bar{\eta}).$$

For the metric g on $O_{P(E)}(1) \otimes p^* \det E$, we replace $O_{P(E^*)}(1) \rightarrow P(E^*)$ in Section 5 with $O_{P(E \otimes \det E^*)}(1) \rightarrow P(E \otimes \det E^*)$ and use the canonical isomorphism between $O_{P(E \otimes \det E^*)}(1) \rightarrow P(E \otimes \det E^*)$ and $O_{P(E)}(1) \otimes p^* \det E \rightarrow P(E)$. Then

as in Lemma 12, we get for $\eta \in T_{z_0}^{1,0}X$, and $u \in E_{z_0}^* \otimes (\det E_{z_0})$ with $\mathbf{H}_1(u, u) = 1$, the curvature Θ_1 of \mathbf{H}_1 satisfies

$$(6-2) \quad \mathbf{H}_1(\Theta_1 u, u)(\eta, \bar{\eta}) \leq \left(M + \frac{r-M}{4}\right) \Omega(\eta, \bar{\eta}).$$

On the bundle $(E^* \otimes \det E) \otimes \det E^*$, there is a Hermitian metric $\mathbf{H}_1 \otimes H^*$ with curvature $\Theta_1 - \Theta \otimes \text{Id}_{E^* \otimes \det E}$. As a result of (6-1) and (6-2), we deduce that for $\eta \in T_{z_0}^{1,0}X$, and $u \in E_{z_0}^* \otimes (\det E_{z_0})$ with $\mathbf{H}_1(u, u) = 1$,

$$(6-3) \quad \mathbf{H}_1(\Theta_1 u, u)(\eta, \bar{\eta}) - \Theta(\eta, \bar{\eta}) \leq \left(M + \frac{r-M}{4}\right) \Omega(\eta, \bar{\eta}) - \left(r - \frac{r-M}{4}\right) \Omega(\eta, \bar{\eta}),$$

the term on the right is negative. So the Hermitian metric $\mathbf{H}_1 \otimes H^*$ is Griffiths negative at z_0 an arbitrary point. Hence $\mathbf{H}_1 \otimes H^*$ is Griffiths negative on X , and the bundle E is Griffiths positive.

7. Examples

Example 13. We provide here an example where the assumptions in Theorems 1, 2, 3, and 4 are satisfied. Let L be a line bundle with a metric H whose curvature $\Theta > 0$. Let $E = L^9 \oplus L^8 \oplus L^7$ a vector bundle of rank $r = 3$. The induced metric $(H^*)^9 \oplus (H^*)^8 \oplus (H^*)^7$ on the dual bundle E^* has curvature

$$\Theta(E^*) = (-9\Theta) \oplus (-8\Theta) \oplus (-7\Theta),$$

which is Griffiths negative, so the corresponding metric h on $O_{P(E^*)}(1)$ is positively curved. According to (2-7), we see

$$-\theta(h) = -q^* \Theta(E^*)|_{O_{P(E^*)}(-1)}.$$

Hence we have

$$(7-1) \quad 7q^* \Theta \leq -\theta(h) \leq 9q^* \Theta.$$

For all four theorems, we will use this metric h on $O_{P(E^*)}(1)$ and take Ω to be 7Θ . So $q^* \Omega \leq -\theta(h)$ always holds. The choice of g will be different from case to case.

For Theorem 1, we choose g to be h , and hence by (7-1) and $\Omega = 7\Theta$ we get

$$(7-2) \quad q^* \Omega \leq -\theta(h) = -\theta(g) \leq \frac{9}{7} q^* \Omega.$$

To fulfill the assumption of Theorem 1, we can choose $M = \frac{9}{7}$ which is in the interval $[1, 3)$.

For Theorem 2, since $E \otimes \det E^* = (L^*)^{15} \oplus (L^*)^{16} \oplus (L^*)^{17}$ has induced curvature $(-15\Theta) \oplus (-16\Theta) \oplus (-17\Theta)$ which is Griffiths negative, the corresponding metric g on $O_{P(E)}(1) \otimes p^* \det E$ is positively curved and satisfies

$$(7-3) \quad 15p^* \Theta \leq -\theta(g) \leq 17p^* \Theta.$$

Together with (7-1) and $\Omega = 7\Theta$, we have

$$(7-4) \quad \frac{17}{7} p^* \Omega \geq -\theta(g) \quad \text{and} \quad q^* \Omega \leq -\theta(h).$$

We can choose $M = \frac{17}{7}$ which is in $[1, 3)$.

For Theorem 3, notice that h is induced from $(H^*)^9 \oplus (H^*)^8 \oplus (H^*)^7$ on E^* , so if we use $(H^*)^9 \oplus (H^*)^8 \oplus (H^*)^7$ for the Hermitian metric G , then the corresponding g is actually h . Since $\Theta(\det G) = -24\Theta$, by using (7-1) we have

$$(7-5) \quad -(r+1)\theta(g) + q^* \Theta(\det G) = -4\theta(h) - 24q^* \Theta \leq 12q^* \Theta = \frac{12}{7} q^* \Omega.$$

We choose $M = \frac{12}{7}$ which is in $[1, 3)$.

Finally for Theorem 4, on $E \otimes \det E^* = (L^*)^{15} \oplus (L^*)^{16} \oplus (L^*)^{17}$, we will use the metric $(H^*)^{15} \oplus (H^*)^{16} \oplus (H^*)^{17}$ for G , so $\Theta(\det G) = -48\Theta$. Moreover, the corresponding metric g on $O_{P(E)}(1) \otimes p^* \det E$ satisfies

$$(7-6) \quad 15p^* \Theta \leq -\theta(g) \leq 17p^* \Theta,$$

so we get

$$(7-7) \quad -(r+1)\theta(g) + p^* \Theta(\det G) \leq 20p^* \Theta = \frac{20}{7} p^* \Omega.$$

We choose $M = \frac{20}{7}$ which is in $[1, 3)$.

Example 14. Let X be a compact Riemann surface with a Hermitian metric ω . Let E be an ω -semistable ample vector bundle of rank r over X . The assumptions in Theorems 1, 2, 3, and 4 are all satisfied in this case. We will explain for only Theorems 2 and 4. Theorems 1 and 3 can be verified similarly. By [Li et al. 2021, Theorem 1.7, Remark 1.8, and Theorem 1.11], there exists a constant $c > 0$ such that for any $\delta > 0$, there exists a Hermitian metric H_δ on E satisfying

$$(7-8) \quad (c - \delta) \text{Id}_E \leq \sqrt{-1} \Lambda_\omega \Theta(H_\delta) \leq (c + \delta) \text{Id}_E,$$

where Λ_ω is the contraction with respect to ω . Since X is a Riemann surface, Λ_ω locally is multiplication by a positive function.

For Theorem 2, we choose $\delta = \frac{c}{5r}$. The Hermitian metric H_δ^* on E^* induces a metric h on $O_{P(E^*)}(1)$. Due to (2-7), we see

$$(7-9) \quad -\theta(h) = -q^* \Theta(H_\delta^*)|_{O_{P(E^*)}(-1)};$$

combining with (7-8), we have

$$(7-10) \quad (c - \delta) q^* \omega \leq -\theta(h) \leq (c + \delta) q^* \omega.$$

The Hermitian metric $H_\delta \otimes \det H_\delta^*$ on $E \otimes \det E^*$ induces on $O_{P(E)}(1) \otimes p^* \det E$ a metric g . Similar to (7-10), we have

$$(7-11) \quad -\theta(g) \leq [-(c - \delta) + r(c + \delta)] p^* \omega.$$

If we choose $\Omega = (c - \delta) \omega$ and $M = r - \frac{1}{2}$, then

$$[-(c - \delta) + r(c + \delta)] p^* \omega \leq M p^* \Omega.$$

As a result, we achieve the assumption in [Theorem 2](#):

$$q^* \Omega \leq -\theta(h) \quad \text{and} \quad -\theta(g) \leq M p^* \Omega.$$

For [Theorem 4](#), we choose $\delta = \frac{c}{9r}$. We still have (7-10). The Hermitian metric G on $E \otimes \det E^*$ is taken to be $H_\delta \otimes \det H_\delta^*$, so we get

$$\begin{aligned} & -(r+1)\theta(g) + p^* \Theta(\det G) \\ &= -(r+1)[p^* \Theta(H_\delta)|_{\mathcal{O}_{P(E)}(-1)} - p^* \Theta(\det H_\delta)] - (r-1)p^* \Theta(\det H_\delta) \\ &\leq [-(r+1)(c-\delta) + 2r(c+\delta)] p^* \omega. \end{aligned}$$

If we choose $\Omega = (c - \delta) \omega$ and $M = r - \frac{1}{2}$, then

$$[-(r+1)(c-\delta) + 2r(c+\delta)] p^* \omega \leq M p^* \Omega.$$

So the assumption of [Theorem 4](#) is satisfied.

In light of [[Li et al. 2021](#), Theorem 1.7], it is possible to modify our theorems so that semistability is not needed in this example.

Appendix

Here we prove the isomorphism $\bigwedge^{r-1} E \simeq E^* \otimes \det E$ where r is the rank of E .

Proof. Let $\{e_1, \dots, e_r\}$ and $\{f_1, \dots, f_r\}$ be two sets of local frames for E with the transition matrix $g = (g_{ij})$; namely, on the intersection of the two frames, we have $f_i = \sum_j g_{ij} e_j$. On the bundle $\bigwedge^{r-1} E$, we have the induced local frame $\{\hat{e}_1, \dots, \hat{e}_r\}$ where \hat{e}_k is $e_1 \wedge \dots \wedge e_r$ with e_k removed. Similarly, we have another frame $\{\hat{f}_1, \dots, \hat{f}_r\}$. Let $\hat{g} = (\hat{g}_{ij})$ be the corresponding transition matrix for the bundle $\bigwedge^{r-1} E$, namely, $\hat{f}_i = \sum_j \hat{g}_{ij} \hat{e}_j$. It is not hard to verify that \hat{g}_{ij} is the determinant of the matrix g with the i -th row and j -th column removed.

For the dual bundle E^* , the corresponding transition matrix for the dual frames $\{e_1^*, \dots, e_r^*\}$ and $\{f_1^*, \dots, f_r^*\}$ is the transpose of g^{-1} . Therefore, the transition matrix for the bundle $E^* \otimes \det E$ is $c = (c_{ij})$ where $c_{ij} = (-1)^{i+j} \hat{g}_{ij}$.

Now, let us denote by A the diagonal matrix whose i -th diagonal entry is $(-1)^i$. Notice that the inverse of A is still A . Also, after a straightforward computation, we have $AcA^{-1} = \hat{g}$. So, the two bundles $\bigwedge^{r-1} E$ and $E^* \otimes \det E$ are isomorphic. \square

Acknowledgements

I am grateful to László Lempert for his helpful comments. I would like to thank Academia Sinica, National Center for Theoretical Sciences, and National Taiwan

University for their support. I want to thank the referees for their careful reading of this paper and for their useful feedback.

References

- [Aikou 2004] T. Aikou, “Finsler geometry on complex vector bundles”, pp. 83–105 in *A sampler of Riemann–Finsler geometry*, Math. Sci. Res. Inst. Publ. **50**, Cambridge Univ. Press, Cambridge, 2004. [MR](#) [Zbl](#)
- [Berndtsson 2009a] B. Berndtsson, “Curvature of vector bundles associated to holomorphic fibrations”, *Ann. of Math. (2)* **169**:2 (2009), 531–560. [MR](#) [Zbl](#)
- [Berndtsson 2009b] B. Berndtsson, “Positivity of direct image bundles and convexity on the space of Kähler metrics”, *J. Differential Geom.* **81**:3 (2009), 457–482. [MR](#) [Zbl](#)
- [Berndtsson and Lempert 2016] B. Berndtsson and L. Lempert, “A proof of the Ohsawa–Takegoshi theorem with sharp estimates”, *J. Math. Soc. Japan* **68**:4 (2016), 1461–1472. [MR](#) [Zbl](#)
- [Campana and Flenner 1990] F. Campana and H. Flenner, “A characterization of ample vector bundles on a curve”, *Math. Ann.* **287**:4 (1990), 571–575. [MR](#) [Zbl](#)
- [Cao and Wong 2003] J.-G. Cao and P.-M. Wong, “Finsler geometry of projectivized vector bundles”, *J. Math. Kyoto Univ.* **43**:2 (2003), 369–410. [MR](#) [Zbl](#)
- [Demailly 1999] J.-P. Demailly, “Pseudoconvex-concave duality and regularization of currents”, pp. 233–271 in *Several complex variables* (Berkeley, CA, 1995–1996), Math. Sci. Res. Inst. Publ. **37**, Cambridge Univ. Press, Cambridge, 1999. [MR](#) [Zbl](#)
- [Demailly 2012] J.-P. Demailly, “Complex analytic and differential geometry”, preprint, Université de Grenoble, 2012, available at <https://www-fourier.ujf-grenoble.fr/~demailly/manuscripts/agbook.pdf>. [Zbl](#)
- [Demailly 2021] J.-P. Demailly, “Hermitian–Yang–Mills approach to the conjecture of Griffiths on the positivity of ample vector bundles”, *Sb. Math.* **212**:3 (2021), 305–318. [Zbl](#)
- [Feng et al. 2020] H. Feng, K. Liu, and X. Wan, “Complex Finsler vector bundles with positive Kobayashi curvature”, *Math. Res. Lett.* **27**:5 (2020), 1325–1339. [MR](#) [Zbl](#)
- [Finski 2022] S. Finski, “On Monge–Ampère volumes of direct images”, *Int. Math. Res. Not.* **2022**:16 (2022), 12072–12095. [MR](#) [Zbl](#)
- [Griffiths 1969] P. A. Griffiths, “Hermitian differential geometry, Chern classes, and positive vector bundles”, pp. 185–251 in *Global analysis (papers in honor of K. Kodaira)*, Univ. Tokyo Press, Tokyo, 1969. [MR](#) [Zbl](#)
- [Guan and Zhou 2015] Q. Guan and X. Zhou, “A solution of an L^2 extension problem with an optimal estimate and applications”, *Ann. of Math. (2)* **181**:3 (2015), 1139–1208. [MR](#) [Zbl](#)
- [Hacon et al. 2018] C. Hacon, M. Popa, and C. Schnell, “Algebraic fiber spaces over abelian varieties: around a recent theorem by Cao and Păun”, pp. 143–195 in *Local and global methods in algebraic geometry*, Contemp. Math. **712**, Amer. Math. Soc., Providence, RI, 2018. [MR](#) [Zbl](#)
- [Hartshorne 1966] R. Hartshorne, “Ample vector bundles”, *Inst. Hautes Études Sci. Publ. Math.* **29** (1966), 63–94. [MR](#) [Zbl](#)
- [Hering et al. 2010] M. Hering, M. Mustață, and S. Payne, “Positivity properties of toric vector bundles”, *Ann. Inst. Fourier (Grenoble)* **60**:2 (2010), 607–640. [MR](#) [Zbl](#)
- [Jacob 2014] A. Jacob, “Existence of approximate Hermitian–Einstein structures on semi-stable bundles”, *Asian J. Math.* **18**:5 (2014), 859–883. [MR](#) [Zbl](#)

- [Kobayashi 1975] S. Kobayashi, “Negative vector bundles and complex Finsler structures”, *Nagoya Math. J.* **57** (1975), 153–166. [MR](#) [Zbl](#)
- [Kobayashi 1987] S. Kobayashi, *Differential geometry of complex vector bundles*, Publications of the Mathematical Society of Japan **15**, Princeton University Press, Princeton, NJ, 1987. [MR](#) [Zbl](#)
- [Kobayashi 1996] S. Kobayashi, “Complex Finsler vector bundles”, pp. 145–153 in *Finsler geometry* (Seattle, WA, 1995), Contemp. Math. **196**, Amer. Math. Soc., Providence, RI, 1996. [MR](#) [Zbl](#)
- [Lempert 2017] L. Lempert, “Extrapolation, a technique to estimate”, pp. 271–281 in *Functional analysis, harmonic analysis, and image processing: a collection of papers in honor of Björn Jawerth*, Contemp. Math. **693**, Amer. Math. Soc., Providence, RI, 2017. [MR](#) [Zbl](#)
- [Li et al. 2021] C. Li, C. Zhang, and X. Zhang, “Mean curvature negativity and HN-negativity of holomorphic vector bundles”, preprint, 2021. [arXiv 2112.00488](#)
- [Liu and Yang 2015] K. Liu and X. Yang, “Effective vanishing theorems for ample and globally generated vector bundles”, *Comm. Anal. Geom.* **23**:4 (2015), 797–818. [MR](#) [Zbl](#)
- [Liu et al. 2013] K. Liu, X. Sun, and X. Yang, “Positivity and vanishing theorems for ample vector bundles”, *J. Algebraic Geom.* **22**:2 (2013), 303–331. [MR](#) [Zbl](#)
- [Ma and Zhang 2023] X. Ma and W. Zhang, “Superconnection and family Bergman kernels”, *Math. Ann.* **386**:3-4 (2023), 2207–2253. [MR](#) [Zbl](#)
- [Misra and Ray 2021] S. Misra and N. Ray, “On ampleness of vector bundles”, *C. R. Math. Acad. Sci. Paris* **359** (2021), 763–772. [MR](#) [Zbl](#)
- [Mourougane and Takayama 2007] C. Mourougane and S. Takayama, “Hodge metrics and positivity of direct images”, *J. Reine Angew. Math.* **606** (2007), 167–178. [MR](#) [Zbl](#)
- [Naumann 2021] P. Naumann, “An approach to the Griffiths conjecture”, *Math. Res. Lett.* **28**:5 (2021), 1505–1523. [MR](#) [Zbl](#)
- [Pingali 2021] V. P. Pingali, “A note on Demailly’s approach towards a conjecture of Griffiths”, *C. R. Math. Acad. Sci. Paris* **359** (2021), 501–503. [MR](#) [Zbl](#)
- [Sommese 1978] A. J. Sommese, “Concavity theorems”, *Math. Ann.* **235**:1 (1978), 37–53. [MR](#) [Zbl](#)
- [Umemura 1973] H. Umemura, “Some results in the theory of vector bundles”, *Nagoya Math. J.* **52** (1973), 97–128. [MR](#) [Zbl](#)
- [Ustinovskiy 2019] Y. Ustinovskiy, “The Hermitian curvature flow on manifolds with non-negative Griffiths curvature”, *Amer. J. Math.* **141**:6 (2019), 1751–1775. [MR](#) [Zbl](#)
- [Wan 2022] X. Wan, “Positivity preserving along a flow over projective bundles”, *Comm. Anal. Geom.* **30**:7 (2022), 1541–1574. [MR](#) [Zbl](#)
- [Wu 2022] K.-R. Wu, “Positively curved Finsler metrics on vector bundles”, *Nagoya Math. J.* **248** (2022), 766–778. [MR](#) [Zbl](#)
- [Zhou and Zhu 2018] X. Zhou and L. Zhu, “An optimal L^2 extension theorem on weakly pseudoconvex Kähler manifolds”, *J. Differential Geom.* **110**:1 (2018), 135–186. [MR](#) [Zbl](#)

Received March 24, 2023. Revised August 3, 2023.

KUANG-RU WU
INSTITUTE OF MATHEMATICS
ACADEMIA SINICA
TAIPEI
TAIWAN

krwu@gate.sinica.edu.tw

PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

msp.org/pjm

EDITORS

Don Blasius (Managing Editor)
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
blasius@math.ucla.edu

Matthias Aschenbrenner
Fakultät für Mathematik
Universität Wien
Vienna, Austria
matthias.aschenbrenner@univie.ac.at

Paul Balmer
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
balmer@math.ucla.edu

Vyjayanthi Chari
Department of Mathematics
University of California
Riverside, CA 92521-0135
chari@math.ucr.edu

Atsushi Ichino
Department of Mathematics
Kyoto University
Kyoto 606-8502, Japan
atsushi.ichino@gmail.com

Robert Lipshitz
Department of Mathematics
University of Oregon
Eugene, OR 97403
lipshitz@uoregon.edu

Kefeng Liu
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
liu@math.ucla.edu

Dimitri Shlyakhtenko
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
shlyakht@ipam.ucla.edu

Paul Yang
Department of Mathematics
Princeton University
Princeton NJ 08544-1000
yang@math.princeton.edu

Ruixiang Zhang
Department of Mathematics
University of California
Berkeley, CA 94720-3840
ruixiang@berkeley.edu

PRODUCTION

Silvio Levy, Scientific Editor, production@msp.org


See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2023 is US \$605/year for the electronic version, and \$820/year for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by [Mathematical Reviews](#), [Zentralblatt MATH](#), [PASCAL CNRS Index](#), [Referativnyi Zhurnal](#), [Current Mathematical Publications](#) and [Web of Knowledge \(Science Citation Index\)](#).

The Pacific Journal of Mathematics (ISSN 1945-5844 electronic, 0030-8730 printed) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW[®] from Mathematical Sciences Publishers.

PUBLISHED BY

 **mathematical sciences publishers**
nonprofit scientific publishing

<http://msp.org/>

© 2023 Mathematical Sciences Publishers

PACIFIC JOURNAL OF MATHEMATICS

Volume 326 No. 1 September 2023

Spin Lefschetz fibrations are abundant	1
MIHAIL ARABADJI and R. İNANÇ BAYKUR	
Some arithmetical properties of convergents to algebraic numbers	17
YANN BUGEAUD and KHOA D. NGUYEN	
Local Galois representations of Swan conductor one	37
NAOKI IMAI and TAKAHIRO TSUSHIMA	
Divisors of Fourier coefficients of two newforms	85
ARVIND KUMAR and MONI KUMARI	
Desingularizations of quiver Grassmannians for the equioriented cycle quiver	109
ALEXANDER PÜTZ and MARKUS REINEKE	
Varieties of chord diagrams, braid group cohomology and degeneration of equality conditions	135
VICTOR A. VASSILIEV	
Positively curved Finsler metrics on vector bundles, II	161
KUANG-RU WU	