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SMOOTH LOCAL SOLUTIONS TO SCHRÖDINGER FLOWS WITH DAMPING TERM FOR MAPS INTO SYMPLECTIC MANIFOLDS

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We show the existence of short-time very regular solutions to the initial Neumann boundary value problem of Schrödinger flows with damping term (or Landau–Lifshitz–Gilbert flows) for maps from a 3-dimensional compact Riemannian manifold with smooth boundary into a compact symplectic manifold.

1. Introduction

Let (M, g) be a compact Riemannian manifold with smooth boundary and (N, J, ω) be a symplectic manifold, where ω is the symplectic form and $J : TN \to TN$ with $J^2 = -\,\mathrm{id}$ is an ω-tamed almost complex structure. For a smooth map $u \in C^2(M, N)$, the tension field is defined by

$$\tau(u) = \mathrm{tr}_g(\nabla du),$$

where ∇ denotes the induced connection on the pullback bundle u^*TN.

Recently, in [Chen and Wang 2023b; 2023a] we have addressed the local existence of strong or even smooth solutions to the initial Neumann boundary value problems to the Schrödinger flows from a smooth bounded domain Ω^m $(m = 2, 3)$ into a standard sphere $\mathbb{S}^2$. A natural problem is whether or not one can extend the local existence of smooth solutions to the initial Neumann boundary value problem to the following Schrödinger flow from a compact Riemannian manifold with boundary (M, g) into a general symplectic manifold (N, J, ω):

$$\begin{cases} \partial_t u = J(u)\tau(u), & (x,t) \in M \times \mathbb{R}^+, \\ \partial u/\partial \nu = 0, & (x,t) \in \partial M \times \mathbb{R}^+, \\ u(x,0) = u_0 : M \to N. \end{cases}$$

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In this paper, we are concerned with a geometric flow for maps between (M, g) and (N, J, ω), which is a close relative of the Schrödinger flow. If u is a time-dependent map from (M, g) into N satisfying

$$\partial_t u + \gamma \nabla_v u = \alpha \tau(u) - \beta J(u)\tau(u),$$

we call this geometric flow a Schrödinger flow with damping term $\alpha\tau(u)$ (or a Landau–Lifshitz–Gilbert (LLG) geometric flow) for maps from (M, g) into (N, J, ω), where $\alpha > 0$, β and γ are fixed real numbers, $v : M \times \mathbb{R}^+ \to TM$ is a vector field satisfying $\mathrm{div}(v) = 0$ inside M for any $t \in \mathbb{R}^+$, and $\nabla_v u$ is defined by

$$\nabla_v u = du(v).$$

We are interested in the well-posedness to the initial Neumann boundary value problem of the above geometric flow

$$\begin{cases} \partial_t u + \gamma \nabla_v u = \alpha \tau(u) - \beta J(u)\tau(u), & (x, t) \in M \times \mathbb{R}^+, \\ \partial u/\partial \nu = 0, & (x, t) \in \partial M \times \mathbb{R}^+, \\ u(x, 0) = u_0 : M \to N. \end{cases} \tag{1-1}$$

In fact, the study of system (1-1) above can be regarded as the first step to approach the previous initial Neumann boundary value problem on the Schrödinger map flow. This is also the main motivation of this paper.

On the other hand, system (1-1) is of strong physical background. Now, let us recall some background materials and related equations of this flow.

1A. ***Background: Landau–Lifshitz–Gilbert equation and the Schrödinger map flow.*** Let Ω be a bounded domain in $\mathbb{R}^3$. In physics, for a map u from Ω into a standard sphere $\mathbb{S}^2$, the Landau–Lifshitz (LL) equation

$$\partial_t u = -u \times \Delta u \tag{1-2}$$

is a fundamental evolution equation for the ferromagnetic spin chain and was proposed on the phenomenological ground in studying the dispersive theory of magnetization of ferromagnets. It was first derived by Landau and Lifshitz [1935], and then proposed by Gilbert [1955] with dissipation as the form

$$\partial_t u = -\alpha u \times (u \times \Delta u) - \beta u \times \Delta u, \tag{1-3}$$

where β is a real number and $\alpha \geq 0$ is called the Gilbert damping coefficient. Hence, equation (1-3) above is also called the Landau–Lifshitz–Gilbert (LLG) equation if $\alpha > 0$. Here "$\times$" denotes the cross product in $\mathbb{R}^3$ and Δ is the Laplace operator in $\mathbb{R}^3$.

Let $i : \mathbb{S}^2 \to \mathbb{R}^3$ be the canonical inclusion map, which induces an embedding $i_* : T\mathbb{S}^2 \to \mathbb{S}^2 \times \mathbb{R}^3$, namely $i_*(p, v) = (p, di_p(v))$ for any $p \in \mathbb{S}^2$ and $v \in T_p\mathbb{S}^2$.

Let $\iota : \mathbb{R}^3 \setminus \{0\} \to \mathbb{S}^2$ be the projection defined by $\iota(y) = y/|y|$. Then a direct calculation shows

$$d\iota|_y(w) = \pi_y(w) = w - \langle w, y\rangle y$$

for $y \in \mathbb{S}^2$ and $w \in \mathbb{R}^3$, where π is the orthogonal projection from $\mathbb{R}^3$ to $T_y\mathbb{S}^2$. Moreover, it satisfies

$$i_{*y} \circ \pi_y = \pi_y, \quad \pi_y \circ i_{*y} = \mathrm{id}.$$

Then $u\times$ has the intrinsic form

$$u\times = i_{*u} \circ J(u) \circ \pi_u.$$

Here J is the complex structure on $\mathbb{S}^2$, i.e., $J(u) : T_u\mathbb{S}^2 \to T_u\mathbb{S}^2$ rotates vectors $\frac{\pi}{2}$ radians counterclockwise in the tangent space of $\mathbb{S}^2$. Therefore, (1-3) can be written as

$$\partial_t u = \alpha\pi_u \Delta u - \beta i_*(u) \circ J(u) \circ \pi_u \Delta u.$$

Since $\tau(u) = \pi_u \Delta u \in T_u\mathbb{S}^2$ (i.e., the tension field) and $\pi_u \partial_t u = \partial_t u$, we get the intrinsic version of (1-3) as

$$\partial_t u = \alpha\tau(u) - \beta J(u)\tau(u). \tag{1-4}$$

In the case $\alpha = 0$, it is just the Schrödinger flow into $\mathbb{S}^2$, which is introduced independently in [Ding and Wang 2001] and [Terng and Uhlenbeck 2006] as a geometric Hamiltonian flow of maps between manifolds. The intrinsic equation (1-4) can be defined between general manifolds and gives a natural generalization of the LLG equation, which is a parabolic perturbation of the Schrödinger flow. Namely, suppose that (M, g) is a Riemannian manifold and (N, J, ω) is a symplectic manifold, the LLG geometric flow for map $u : M \times \mathbb{R}^+ \to N \hookrightarrow \mathbb{R}^K$ is defined by

$$\partial_t u = \alpha\tau(u) - \beta J(u)\tau(u), \tag{1-5}$$

where

$$\tau(u) = \Delta u + A(u)(\nabla u, \nabla u)$$

is the tension field, $A(u)(\cdot\,,\cdot)$ is the second fundamental form of N in $\mathbb{R}^K$. Here we have embedded isometrically N into $\mathbb{R}^K$ by applying the well-known Nash embedding theorem. In the following, we always assume that $N \subset \mathbb{R}^K$ is just a submanifold in $\mathbb{R}^K$ for the sake of convenience and without loss of generality.

Let $v : M \times \mathbb{R}^+ \to TM$ be a vector field with $\mathrm{div}(v) \equiv 0$ inside M. The equation

$$\partial_t u + \gamma \nabla_v u = \alpha\tau(u) - \beta J(u)\tau(u) \tag{1-6}$$

appears in magnetoelastic theory, where $\gamma \in \mathbb{R}$ is a constant. One can refer to [Benešová et al. 2018; Kalousek et al. 2021] for more details.

In the special case of $\alpha = 0$, the equation

$$\partial_t u + \gamma \nabla_v u = -\beta J(u)\tau(u)$$

is called an incompressible Schrödinger flow, which was derived for the purely Eulerian simulation of incompressible fluids by Chern et al. [2016].

We should mention that (1-5) and (1-6) are gauge equivalent. Let $\phi_t : M \to M$ be a family of diffeomorphisms of M generated by γv, which preserves the volume element. Namely, ϕ_t is the solution to the ODE

$$\begin{cases} \dfrac{\partial \phi}{\partial t} = \gamma v(\phi_t(x), t), \\ \phi(\,\cdot\,, 0) = \phi_0, \end{cases} \tag{1-7}$$

where $\phi_0 : M \to M$ is a given diffeomorphism. If $\partial M \neq \varnothing$, we additionally assume $\gamma \langle v, \nu \rangle |_{\partial M} = 0$, where ν is the outer normal vector of ∂M. Let u solve (1-6), and set $\tilde{u}(x, t) = u(\phi_t(x), t)$. Then we have

$$\partial_t \tilde{u} = (\partial_t u + \gamma \nabla_v u) \circ \phi_t(x) = \phi_t^*(\alpha\tau(u) - \beta J(u)\tau(u)) = \alpha\tau(\tilde{u}) - \beta J(\tilde{u})\tau(\tilde{u}).$$

This is the standard LLG equation

$$\partial_t \tilde{u} = \alpha\tau(\tilde{u}) - \beta J(\tilde{u})\tau(\tilde{u})$$

with respect to the pullback metric $g_t = \phi_t^* g$.

It is worthy to point out that if the vector field v is the velocity field in magnetic fluid, which satisfies a Navier–Stokes equation involving a magnetic term, we can derive the so-called magnetic elasticity system (see [Benešová et al. 2018] for more details)

$$\begin{cases} \partial_t v + \nabla_v v + \nabla P = \mu\Delta v - \nabla \cdot (\nabla u \odot \nabla u - W'(F)F), \\ \operatorname{div}(v) = 0, \\ \partial_t F + (v \cdot \nabla)F - \nabla v F = \kappa \Delta F, \\ \partial_t u + \gamma \nabla_v u = \alpha\tau(u) - \beta u \times \Delta u, \end{cases} \tag{1-8}$$

accompanied by some suitable initial-boundary value conditions. Here μ, κ are two positive constants, $u : \Omega^m \times \mathbb{R}^+ \to \mathbb{S}^2$ is the magnetization field, $v : \Omega^m \times \mathbb{R}^+ \to \mathbb{R}^m$ is the velocity field of the fluid, P is the pressure function, and $F : \Omega^m \to \mathbb{R}^{m\times m}$ is the deformation gradient, where Ω^m is a domain in $\mathbb{R}^m$ with $m = 2, 3$. The term $\nabla u \odot \nabla u$ is an $m \times m$ matrix with (i, j)-th entry

$$(\nabla u \odot \nabla u)_{ij} = \langle \nabla_i u, \nabla_j u \rangle,$$

W is the elastic energy which satisfies $W(RS) = W(S)$ for all $R \in \mathbb{SO}(m)$ (and thus $W'(RS) = RW'(S)$) for all matrices $S \in \mathbb{R}^{m\times m}$, and

$$\tau(u) = \Delta u + |\nabla u|^2 u.$$

In the special case $\alpha = 0$ and $F \equiv 0$, equation (1-8) is the Navier–Stokes–Schrödinger flow, which can be used to describe the dispersive theory of magnetization of ferromagnets with quantum effects.

Next, we briefly recall a few results that are closely related to our work in the present paper. In 1985, the existence of global weak solutions to the LLG equation (i.e., (1-3) with $\alpha > 0$) was established by Visintin [1985]. P.L. Sulem, C. Sulem, and C. Bardos [Sulem et al. 1986] employed a difference method to prove that the LL equation (1-2) without a dissipation term defined on $\mathbb{R}^n$ admits a global weak solution and a smooth local solution. Later, Alouges and Soyeur [1992] showed the nonuniqueness of weak solutions to the LLG equation defined on a bounded domain $\Omega \subset \mathbb{R}^3$. Y.D. Wang [1998] adopted a more geometric approximation method (i.e., the complex structure approximation method) than the Ginzburg–Landau penalized method used for the LLG equation in [Alouges and Soyeur 1992; Bonithon 2007; Tilioua 2011] to obtain the global existence of weak solutions to the Schrödinger flow for maps from a closed Riemannian manifold or a bounded domain in $\mathbb{R}^n$ into $\mathbb{S}^2$. For recent developments of weak solutions to a class of generalized LL equations and related flows, we refer to [Jia and Wang 2019; 2020; Chen and Wang 2021] for various results.

The global well-posedness result for the LL equation on $\mathbb{R}^n$ with $n \geq 2$ was well studied by Ionescu, Kenig, and Bejanaru et al., we refer to [Bejenaru 2008; Bejenaru et al. 2007; 2011; Ionescu and Kenig 2007] for more details. For the Schrödinger flow from a closed manifold or $\mathbb{R}^n$ onto a compact Kähler manifold (i.e., (1-9) with $\alpha = 0$), the existence of local smooth solutions was obtained by Ding and Wang et al., one can refer to [Ding and Wang 1998; 2001; Sulem et al. 1986; Pang et al. 2000; 2001; 2002; Zhou et al. 1991].

In the case the domain manifold is a smooth bounded domain in $\mathbb{R}^3$, Carbou and Fabrie [2001] proved the local existence and uniqueness of regular solutions of the initial Neumann boundary value problem to the LLG equation. Recently, the local existence of very regular solutions to the LLG equation with $\alpha > 0$ was addressed by applying the delicate Galerkin approximation method and adding initial Neumann boundary compatibility conditions on the initial map [Carbou and Jizzini 2018]. Inspired by this method, which essentially stems from [Sulem et al. 1986], we obtained local-in-time very regular solutions to the LLG equation with spin-polarized transport in [Chen and Wang 2023c].

Very recently, the authors of this paper studied the most challenging LL equation (i.e., the Schrödinger flow into $\mathbb{S}^2$) on a smooth bounded domain in $\mathbb{R}^3$, and proved the existence and uniqueness of local-in-time strong solutions and local very regular solutions to its initial Neumann boundary value problem (see [Chen and Wang 2023b; 2023a]).

1B. ***Motivations and main results.*** Although we have proved the existence and uniqueness of local-in-time strong solutions and local very regular solutions to the initial Neumann boundary value problem of the Schrödinger flow from a smooth bounded domain in $\mathbb{R}^3$ into $\mathbb{S}^2$ (see [Chen and Wang 2023b; 2023a]), the existence of the initial Neumann boundary value problem of the Schrödinger flow from a smooth bounded domain M in $\mathbb{R}^3$ into a compact Kähler manifold N is still an open problem:

$$\begin{cases} \partial_t u = J(u)\tau(u), & (x,t)\in M\times\mathbb{R}^+, \\ \partial u/\partial\nu = 0, & (x,t)\in \partial M\times\mathbb{R}^+, \\ u(x,0)=u_0 : M\to N. \end{cases}$$

To this end, the first step is to extend Carbou's work [Carbou and Jizzini 2018] on the LLG equation for maps from a smooth bounded domain in $\mathbb{R}^3$ into $\mathbb{S}^2$ to the case from a compact Riemannian manifold with smooth boundary into a symplectic manifold. So, in this paper we consider the existence of regular solutions to the initial Neumann boundary value problem of (1-5) with $\alpha > 0$.

Because the geometry of the domain manifold M does not affect our analysis and the main results, for simplicity, we assume that Ω is a smooth bounded domain in $\mathbb{R}^m$. Let u be a time-dependent map from Ω to N. We consider the initial Neumann boundary value problem of the general LLG flow (equation)

$$\begin{cases} \partial_t u + \gamma\nabla_v u = \alpha\tau(u) - \beta J(u)\tau(u), & (x,t)\in\Omega\times\mathbb{R}^+, \\ \partial u/\partial\nu = 0, & (x,t)\in\partial\Omega\times\mathbb{R}^+, \\ u(x,0) = u_0 : \Omega\to N \hookrightarrow \mathbb{R}^K, \end{cases} \tag{1-9}$$

where $\alpha > 0$, γ and β are fixed real numbers. Here $v : \Omega\times\mathbb{R}^+\to\mathbb{R}^m$ is a vector field satisfying $\operatorname{div}(v) = 0$, and $\nabla_v u$ is defined by

$$\nabla_v u = du(v).$$

No doubt, the initial Neumann boundary value problem of the corresponding incompressible Schrödinger flow

$$\begin{cases} \partial_t u + \gamma\nabla_v u = -\beta J(u)\tau(u), & (x,t)\in\Omega\times\mathbb{R}^+, \\ \partial u/\partial\nu = 0, & (x,t)\in\partial\Omega\times\mathbb{R}^+, \\ u(x,0) = u_0 : \Omega\to N, \end{cases} \tag{1-10}$$

and related problems are more challenging and will be carried out in our forthcoming papers.

Our main results are the following two theorems:

Theorem 1.1. *Let Ω be a smooth bounded domain in $\mathbb{R}^3$ and N be a compact symplectic manifold. Let $u_0 \in H^2(\Omega, N)$ satisfy the compatibility condition*

$$\frac{\partial u_0}{\partial \nu}|_{\partial\Omega} = 0.$$

Suppose $v \in L^\infty(\mathbb{R}^+, W^{1,3}(\Omega))$, $\operatorname{div}(v) = 0$ for any $t \in \mathbb{R}^+$, and $\langle v, \nu\rangle|_{\partial\Omega\times\mathbb{R}^+} = 0$. Then there exists a constant $T_0 > 0$ depending only on γ, α, β, $\|u_0\|_{H^2(\Omega)}$, and $\|v\|_{L^\infty(\mathbb{R}^+, W^{1,3}(\Omega))}$ such that (1-9) *admits a unique local solution u for any $T < T_0$ which satisfies*

$$u \in C^0\big([0,T], H^2(\Omega,N)\big) \cap L^2\big([0,T], H^3(\Omega,N)\big). \tag{1-11}$$

Furthermore, if $u_0 \in H^3(\Omega, N)$, $v \in C^0(\mathbb{R}^+, H^1(\Omega))$, and $\partial_t v \in L^2(\mathbb{R}^+, H^1(\Omega))$, then this solution u satisfies

$$\partial_t^i u \in C^0\big([0,T], H^{3-2i}(\Omega)\big) \cap L^2\big([0,T], H^{4-2i}(\Omega)\big) \tag{1-12}$$

for $T < T_0$ and $i = 0, 1$.

Moreover, we can obtain a very regular solution to (1-9) by adding higher order compatibility conditions on an initial map:

Theorem 1.2. *Let Ω be a smooth bounded domain in $\mathbb{R}^3$ and N be a compact symplectic manifold. Let $k \geq 4$, $u_0 \in H^k(\Omega, N)$ satisfy the compatibility condition at $\left[\frac{k}{2}\right] - 1$ order, which is given in the Definition 5.1. Suppose that $\operatorname{div}(v) = 0$ for any $t \in \mathbb{R}^+$ and $\langle v, \nu\rangle|_{\partial\Omega\times\mathbb{R}^+} = 0$, and for any $i \leq \left[\frac{k}{2}\right] - 1$,*

$$\partial_t^i v \in C^0\big(\mathbb{R}^+, H^{k-2(i+1)}(\Omega, \mathbb{R}^3)\big) \cap L^2\big(\mathbb{R}^+, H^{2[k/2]-2i}(\Omega, \mathbb{R}^3)\big);$$

moreover, if k is odd, we additionally assume that $\partial_t^{[k/2]} v \in L^2(\mathbb{R}^+, L^2(\Omega))$. Then, for u and $T_0 > 0$ which are given in Theorem 1.1, we have that for any $T < T_0$ and $0 \leq i \leq \left[\frac{k}{2}\right] - 1$,

$$\partial_t^i u \in C^0\big([0,T], H^{k-2i}(\Omega, N)\big) \cap L^2\big([0,T], H^{k+1-2i}(\Omega, N)\big).$$

Remark 1.3. (1) Theorems 1.1 and 1.2 still hold true when Ω is a compact 3-dimensional Riemannian manifold with smooth boundary.

(2) By almost the same arguments as in the proofs of Theorem 1.1 and Theorem 1.2, we can also get a short-time very regular solution to the equation

$$\begin{cases} \partial_t u + \gamma \nabla_v u = \alpha(\tau(u) + \gamma J(u)\nabla_v u) + J(u)\tau(u), & (x,t) \in \Omega \times \mathbb{R}^+, \\ \quad \partial u/\partial\nu = 0, & (x,t) \in \partial\Omega \times \mathbb{R}^+, \\ \quad u(x,0) = u_0 : \Omega \to N \hookrightarrow \mathbb{R}^K, \end{cases} \tag{1-13}$$

on $\Omega \times \mathbb{R}^+$, provided that u_0 satisfies some suitable compatibility conditions on the boundary. Here $\alpha > 0$ and $\gamma \in \mathbb{R}$.

To prove Theorem 1.1, we need to consider an extrinsic version (see (3-1)) of (1-9) and then use the solution of the auxiliary equation

$$\begin{cases} \partial_t u + \gamma \nabla_v u = \alpha(\Delta u + \mathcal{P}(u)(\nabla u, \nabla u)) - \beta \mathfrak{J}(u)\Delta u, & (x,t) \in \Omega \times \mathbb{R}^+, \\ \quad \partial u / \partial \nu = 0, & (x,t) \in \partial\Omega \times \mathbb{R}^+, \\ \quad u(x,0) = u_0 : \Omega \to N \hookrightarrow \mathbb{R}^K, \end{cases} \tag{1-14}$$

which preserves the original geometric structures of (1-9), to approximate a solution of (1-9). Here $\mathcal{P}(\cdot,\cdot)$ and $\mathfrak{J}(u)$ are the extensions of $A(\cdot,\cdot)$ and J defined in Section 3A, respectively. We then prove the main result Theorem 1.1 by the following process $\mathscr{T}(1)$:

(1) We apply Galerkin approximation to (1-14), and then estimate some suitable energies directly to get a unique solution u to (1-14) satisfying

$$u \in C^0\big([0,T], H^2(\Omega, \mathbb{R}^K)\big) \cap L^2\big([0,T], H^3(\Omega, \mathbb{R}^K)\big).$$

Since $u_0 \in H^2(\Omega, N)$, the geometric structures of the above auxiliary equation (1-14) guarantee $u(x,t) \in N$ for a.e. $(x,t) \in \Omega \times [0, T_0)$. Therefore, u is also a solution to (1-9) satisfying (1-11).

(2) Since the space of test functions associated to (1-14) is small, we cannot get higher energy estimates directly to improve the regularity of u. We then consider the differential of Galerkin approximation to (1-14) with respect to time and then apply an energy method to show (1-12).

Next, with higher order compatibility conditions on initial data at hand we can prove Theorem 1.2 by following the ideas in [Carbou and Jizzini 2018; Chen and Wang 2023c]. More precisely, we consider the equation satisfied by $\partial_t^k u$ (i.e., (5-9)) with $k \geq 1$ and repeat the process $\mathscr{T}(1)$ in the proof of Theorem 1.1 with $\partial_t^k u$ in place of u. Namely, we prove the main result Theorem 1.2 by showing the so-called property $\mathscr{T}(k)$ which is defined in Section 5.

Our proof of Theorems 1.1 and 1.2 is similar to that of [Carbou and Jizzini 2018; Chen and Wang 2023c], but is more complicated. There are two technical issues we need to address in our presentation. The first one is that we obtain the extensions of $A(\cdot,\cdot)$ and J in a tubular neighborhood $U_{2\delta}(N)$ of N by using the canonical projections $\iota : U_{2\delta}(N) \to N$ and $\pi : N \times \mathbb{R}^K \to TN$, which satisfy the original geometric structures of $A(\cdot,\cdot)$ and J, respectively. Then by multiplying a truncation function involving the distance function $\operatorname{dist}(\cdot, N)$, we get the desired extensions (i.e., $\mathcal{P}(\cdot,\cdot)$ and $\mathfrak{J}(u)$) on $\mathbb{R}^K$ (see Section 3A). In particular, the extension $\mathfrak{J}$ of J is still antisymmetric, which plays an essential role in our proof. The second one is that the property $\operatorname{div}(v) = 0$ can be applied to eliminate some terms involving v in the process of the energy estimate. This makes the assumptions on regularity for v in Theorems 1.1 and 1.2 weaker than those for the electric current in [Carbou and Jizzini 2018], one can refer to [Carbou and Jizzini 2018] for more details.

The rest of our paper is organized as follows: In Section 2, we introduce basic notations on Sobolev spaces and some preliminary lemmas. In Section 3 and Section 4, we give the proof of Theorem 1.1. Finally, the proof of Theorem 1.2 is given in Section 5.

2. Preliminary

2A. ***Notations.*** In this subsection, we fix some notations on manifolds and Sobolev spaces which will be used in the following context:

Let (N, J, ω) be an n-dimensional symplectic manifold, where ω is the symplectic form and $J : TN \to TN$ with $J^2 = -\,\mathrm{id}$ is an ω-tamed almost complex structure, that is, for any $X, Y \in \Gamma(TN)$,

$$\omega(JX, JY) = \omega(X, Y).$$

Then ω and J induce a canonical Riemannian metric g on N as

$$g(X, Y) = \omega(X, JY),$$

which also satisfies

$$g(JX, JY) = g(X, Y).$$

By the Nash embedding theorem, we always embed isometrically (N, g) into $\mathbb{R}^K$ hence without loss of generality we assume $N \subset \mathbb{R}^K$ is an embedded submanifold of $\mathbb{R}^K$ with the induced metric. Let Ω be a smooth bounded domain in $\mathbb{R}^m$ with $m \geq 1$. Let $u = (u^1, \dots, u^K) : \Omega \to N \hookrightarrow \mathbb{R}^K$ be a map. We set

$$H^k(\Omega) = W^{k,2}(\Omega, \mathbb{R}^K)$$

and

$$H^k(\Omega, N) = \{u \in H^k(\Omega) : u(x) \in N \text{ for a.e. } x \in \Omega\}.$$

Moreover, let $(B, \|\cdot\|_B)$ be a Banach space and $f : [0, T] \to B$ be a map. For any $p > 0$ and $T > 0$, we define

$$\|f\|_{L^p([0,T],B)} := \left(\int_0^T \|f\|_B^p \, dt\right)^{1/p},$$

and set

$$L^p([0, T], B) := \{f : [0, T] \to B : \|f\|_{L^p([0,T],B)} < \infty\}.$$

In particular, we set

$$L^p([0, T], H^k(\Omega, N)) = \{u \in L^p([0, T], H^k(\Omega)) : u(x, t) \in N \text{ for a.e. } (x, t) \in \Omega \times [0, T]\},$$

where $k \in \mathbb{N}$ and $p \geq 1$.

2B. ***Some basic lemmas.*** Next, we recall some crucial lemmas which will be used later. The following lemma of equivalent norms for Sobolev functions with Neumann boundary condition can be found in [Wehrheim 2004]:

Lemma 2.1. *Let Ω be a smooth bounded domain in $\mathbb{R}^m$ and $k \in \mathbb{N}$. There exists a constant $C_{k,m}$ such that, for all $u \in H^{k+2}(\Omega)$ with $\frac{\partial u}{\partial \nu}|_{\partial\Omega} = 0$,*

$$\|u\|_{H^{k+2}(\Omega)} \le C_{k,m}(\|u\|_{L^2(\Omega)} + \|\Delta u\|_{H^k(\Omega)}). \tag{2-1}$$

Here, for simplicity we define $H^0(\Omega) := L^2(\Omega)$.

In particular, the above lemma implies that we can define the H^{k+2}-norm of u as follows:

$$\|u\|_{H^{k+2}(\Omega)} := \|u\|_{L^2(\Omega)} + \|\Delta u\|_{H^k(\Omega)}.$$

We also need to use the following ODE comparison theorem and the classical compactness results in [Boyer and Fabrie 2013; Simon 1987] to show the uniform estimates and the convergence of solutions to the approximate equation constructed in the coming sections:

Lemma 2.2. *Let $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ be a continuous function, which is locally Lipschitz in the second variable. Let $z : [0, T^*) \to \mathbb{R}$ be the maximal solution of the Cauchy problem*

$$\begin{cases} z' = f(t, z), \\ z(0) = z_0. \end{cases}$$

Let $y : \mathbb{R}^+ \to \mathbb{R}$ be a C^1 function such that

$$\begin{cases} y' \le f(t, y), \\ y(0) \le z_0. \end{cases}$$

Then, we have

$$y(t) \le z(t), \quad t \in [0, T^*).$$

Lemma 2.3 (Aubin–Lions–Simon compactness lemma, see [Simon 1987]). *Let $X \subset B \subset Y$ be Banach spaces with compact embedding $X \hookrightarrow B$. Let $1 \le p, q, r \le \infty$. For $T > 0$, we define*

$$E_{p,r} = \left\{ f : f \in L^p((0, T), X) \quad \text{and} \quad \frac{df}{dt} \in L^r((0, T), Y) \right\},$$

which is equipped with a norm $\|f\| := \|f\|_{L^p((0,T),X)} + \|df/dt\|_{L^r((0,T),Y)}$. Then the following properties hold true:

1. *If $p < \infty$ and $p < q$, the embedding $E_{p,r} \cap L^q((0, T), B)$ in $L^s((0, T), B)$ is compact for all $1 \le s < q$.*
2. *If $p = \infty$ and $r > 1$, the embedding of $E_{p,r}$ in $C^0([0, T], B)$ is compact.*

Lemma 2.4 [Boyer and Fabrie 2013, Theorem II.5.14]. *Let $k \in \mathbb{N}$, then the space*

$$E_{2,2} = \left\{ f : f \in L^2((0,T), H^{k+2}(\Omega)),\ \frac{\partial f}{\partial t} \in L^2((0,T), H^k(\Omega)) \right\}$$

is continuously embedded in $C^0([0,T], H^{k+1}(\Omega))$.

2C. ***Galerkin basis and Galerkin projection.*** Let Ω be a smooth bounded domain in $\mathbb{R}^m$, λ_i be the i-th eigenvalue of the operator $\Delta - I$ with Neumann boundary condition. We denote the corresponding eigenfunction of λ_i by f_i, that is,

$$(\Delta - I) f_i = -\lambda_i f_i \quad \text{with } \frac{\partial f_i}{\partial \nu}\Big|_{\partial\Omega} = 0.$$

Without loss of generality, we assume that $\{f_i\}_{i=1}^{\infty}$ is a complete, standard orthonormal basis of $L^2(\Omega, \mathbb{R}^1)$. Let $H_n = \operatorname{span}\{f_1, \dots f_n\}$ be a finite subspace of L^2, $P_n : L^2 \to H_n$ be the Galerkin projection such that for any $f \in L^2$, $f^n = P_n f = \sum_1^n \langle f, f_i \rangle_{L^2} f_i$. Then the following result is proved in [Carbou and Jizzini 2018]:

Lemma 2.5. *There exists a constant C such that for all n, the projection P_n satisfies the following properties*:

(1) *For $f \in H^1(\Omega, \mathbb{R}^1)$, $\|P_n(f)\|_{H^1(\Omega)} \le \|f\|_{H^1(\Omega)}$,*

(2) *For $f \in H^2(\Omega, \mathbb{R}^1)$ with $\frac{\partial f}{\partial \nu}|_{\partial\Omega} = 0$, $\|P_n(f)\|_{H^2(\Omega)} \le C\|f\|_{H^2(\Omega)}$,*

(3) *For $f \in H^3(\Omega, \mathbb{R}^1)$ with $\frac{\partial f}{\partial \nu}|_{\partial\Omega} = 0$, $\|P_n(f)\|_{H^3(\Omega)} \le C\|f\|_{H^3(\Omega)}$.*

Here we set $H^k(\Omega, \mathbb{R}^1) = W^{k,2}(\Omega, \mathbb{R}^1)$ for $k \in \mathbb{N}$.

3. Local strong solution

3A. ***Approximation equation.*** We start with constructing the approximation equation of (1-9). Let N be a complete compact Riemannian manifold, and $N \subset \mathbb{R}^K$. Let $\pi : N \times \mathbb{R}^K \to TN$ be the canonical orthonormal projection induced by the inclusion map $i : N \hookrightarrow \mathbb{R}^K$. Then there exists a positive constant δ such that there exists a canonical well-defined projection

$$\iota : U_{2\delta}(N) \to N, \quad x \longmapsto \iota(x),$$

satisfying $\operatorname{dist}(x, N) = |x - \iota(x)|$, where

$$U_{2\delta}(N) := \{x \in \mathbb{R}^K \mid \operatorname{dist}(x, N) < 2\delta\}.$$

Moreover, we have the following theorem (refer to [Simon 1996] for a proof):

Theorem 3.1. *Let N be a compact n-dimensional C^∞-submanifold embedded in $\mathbb{R}^K$. Then there exists a positive number $\delta(N) > 0$ and a smooth projection map*

$$\iota : U_{2\delta}(N) \to N \subset \mathbb{R}^K$$

such that the following properties hold:

(1) *For any* $y \in U_{2\delta}(N)$, *we have* $y - \iota(y) \in T^{\perp}_{\iota(y)}N$ *with* $|y - \iota(y)| = \operatorname{dist}(y, N)$. *Moreover*, *if* $z \in N \setminus \{\iota(y)\}$, *we have* $|y - z| > |y - \iota(y)|$.

(2) *For any* $y \in N$ *and* $z \in T^{\perp}_{y}N$ *with* $|z| < 2\delta$, *we have*

$$\iota(y + z) = y.$$

(3) *For* $v \in \mathbb{R}^K$ *and* $y \in N$, *we have*

$$d\iota|_y(v) = \pi_y(v) \in T_yN.$$

(4) *For* $y \in N$ *and* $v_1, v_2 \in T_yN$, *we have*

$$\operatorname{Hess} \iota|_y(v_1, v_2) = \nabla\pi_y(v_1, v_2) = -A(y)(v_1, v_2).$$

We next restrict to the case where (N, J, ω) is a compact symplectic manifold. The almost complex structure is a map $J : TN \to TN$ such that $J^2 = -\operatorname{id}$. Then we can define an extension $\bar{J}$ of J on $U \times \mathbb{R}^K$ by

$$\begin{array}{ccc} U \times \mathbb{R}^K & \xrightarrow{\ \bar{J}\ } & U \times \mathbb{R}^K \\ \Big\downarrow{\scriptstyle (\iota, \pi\circ\iota)} & & \Big\uparrow{\scriptstyle i_*} \\ TN & \xrightarrow{\ J\ } & TN \end{array}$$

where we define $U := U_{2\delta}(N)$. That is $\bar{J}(u) = (\iota(u), i_* \circ J(\iota(u)) \circ \pi_{\iota(u)} w)$ for any $(u, w) \in U \times \mathbb{R}^K$. If we restrict $\bar{J}$ to $\mathbb{R}^K$, the second component of $\bar{J}$ can be interpreted as a map

$$\hat{J} = i_* \circ J(\iota(u)) \circ \pi_{\iota(u)} : U \to \mathbb{R}^K \otimes \mathbb{R}^K, \quad \hat{J}(u) = (\hat{J}_{\alpha,\beta}(u))_{K \times K}.$$

To proceed, the following property on $\hat{J}$ will be used:

Lemma 3.2. *Let* $\hat{J} : U \to \mathbb{R}^K \otimes \mathbb{R}^K$ *be the smooth map defined as above. Then* $\hat{J}$ *is antisymmetric. Namely, for any* $u \in U$ *and* $X, Y \in \mathbb{R}^K$,

$$\langle \hat{J}(u)X, Y\rangle = -\langle X, \hat{J}(u)Y\rangle.$$

Proof. For any $u \in U$ and $X, Y \in \mathbb{R}^K$,

$$\begin{aligned} \langle \hat{J}(u)X, Y\rangle &= \langle i_* \circ J(\iota(u)) \circ \pi_{\iota(u)}(X), i_* \circ \pi_{\iota(u)}Y\rangle \\ &= \langle J(\iota(u)) \circ \pi_{\iota(u)}(X), \pi_{\iota(u)}(Y)\rangle_{T_{\iota(u)}N} \\ &= -\langle \pi_{\iota(u)}(X), J(\iota(u)) \circ \pi_{\iota(u)}(Y)\rangle_{T_{\iota(u)}N} = -\langle X, \hat{J}(u)Y\rangle. \end{aligned}$$

Hence, the proof is completed. □

Therefore, (1-9) has the following extrinsic version:

$$\begin{cases} \partial_t u + \gamma \nabla_v u = \alpha(\Delta u + P(u)(\nabla u, \nabla u)) - \beta \hat{J}(u)\Delta u, & (x,t) \in \Omega \times \mathbb{R}^+, \\ \partial u/\partial \nu = 0, & (x,t) \in \partial\Omega \times \mathbb{R}^+, \\ u(x,0) = u_0 : \Omega \to N \hookrightarrow \mathbb{R}^K. \end{cases} \tag{3-1}$$

Here we set $P(u) = -\operatorname{Hess}\iota(u)$, and have used the facts

$$\pi \circ \Delta u = \tau(u) = \Delta u + A(u)(\nabla u, \nabla u)$$

and

$$\operatorname{Hess}\iota|_u(\nabla u, \nabla u) = -A(u)(\nabla u, \nabla u)$$

for $u : \Omega \to N$ (see Theorem 3.1).

Let ζ be a cut-off function such that $\zeta = 1$ on $[0, \delta^2]$ and $\zeta = 0$ on $[2\delta^2, \infty)$. Then the definition domains of $\hat{J}$ and P can be naturally extended to $\mathbb{R}^K$ in the following way:

$$\mathfrak{J}(u) = \begin{cases} \zeta(\operatorname{dist}(u, N)^2) i_* \circ J(\iota(u)) \circ \pi_{\iota(u)}, & \operatorname{dist}(u, N) \le \sqrt{2}\delta, \\ 0, & \operatorname{dist}(u, N) > \sqrt{2}\delta, \end{cases}$$

and

$$\mathcal{P}(u) = \begin{cases} -\zeta(\operatorname{dist}(u, N)^2) \operatorname{Hess}\iota(u), & \operatorname{dist}(u, N) \le \sqrt{2}\delta, \\ 0, & \operatorname{dist}(u, N) > \sqrt{2}\delta, \end{cases}$$

where $\mathfrak{J}(u)$ is still a smooth antisymmetric matrix-valued function with compact support set. Then we consider the following approximation equation of (1-9):

$$\begin{cases} \partial_t u + \gamma \nabla_v u = \alpha(\Delta u + \mathcal{P}(u)(\nabla u, \nabla u)) - \beta \mathfrak{J}(u)\Delta u, & (x,t) \in \Omega \times \mathbb{R}^+, \\ \partial u/\partial \nu = 0, & (x,t) \in \partial\Omega \times \mathbb{R}^+, \\ u(x,0) = u_0 : \Omega \to N \hookrightarrow \mathbb{R}^K. \end{cases} \tag{3-2}$$

3B. ***Galerkin approximation of* (3-2) *and a priori estimates.*** Next, we seek a solution u^n in H_n to the Galerkin approximation equation associated to (3-2), i.e.,

$$\begin{cases} \partial_t u^n - \alpha \Delta u^n = P_n(-\gamma \nabla_v u^n + \alpha \mathcal{P}(u^n)(\nabla u^n, \nabla u^n)) - \beta P_n(\mathfrak{J}(u^n)\Delta u^n), \\ u^n(x,0) = u_0^n : \Omega \to \mathbb{R}^K. \end{cases} \tag{3-3}$$

Here $u^n(x,t) = \sum_{i=1}^n g_i^n(t) f_i(x)$, $g^n(t) = \{g_1^n(t), \dots, g_n^n(t)\}$ is a vector-valued function. One can refer to Section 2C for the notions of H_n and f_i. A direct calculation shows that g^n satisfies the ODE

$$\begin{cases} \dfrac{\partial g^n}{\partial t} = F(g^n(t)), \\ g^n(0) = (\langle u_0, f_1\rangle, \dots, \langle u_0, f_n\rangle), \end{cases}$$

where $F(y)$ is a smooth function of y because of the smoothness of $\mathcal{P}$ and $\mathfrak{J}$. Then there exists a regular solution $g^n(t)$ on $[0, T^n)$, where T^n is the maximal time of existence. So, we get a regular solution u^n to (3-3) on $[0, T^n)$.

Next, by taking u^n as a test function of (3-3), we can see that

$$\frac{1}{2}\frac{\partial}{\partial t}\int_\Omega |u^n|^2\,dx+\alpha\int_\Omega |\nabla u^n|^2\,dx$$
$$=-\gamma\int_\Omega\langle\nabla_v u^n,u^n\rangle\,dx+\alpha\int_\Omega\langle\mathcal{P}(u^n)(\nabla u^n,\nabla u^n),u^n\rangle\,dx-\beta\int_\Omega\langle\mathfrak{J}(u^n)\Delta u^n,u^n\rangle\,dx.$$

First of all, we use the fact that $\operatorname{div}(v)=0$ with $\langle v,\nu\rangle|_{\partial\Omega}=0$ for all t to eliminate the term

$$\int_\Omega\langle\nabla_v u^n,u^n\rangle\,dx=\frac{1}{2}\int_\Omega \operatorname{div}(v|u^n|^2)\,dx=0.$$

On the other hand, since $\mathfrak{J}(u^n)$ is antisymmetric and $u^n\in H_n$, we have

$$\int_\Omega\langle\mathfrak{J}(u^n)\Delta u^n,u^n\rangle\,dx=-\int_\Omega\langle\nabla(\mathfrak{J}(u^n))\cdot\nabla u^n,u^n\rangle\,dx.$$

It follows that

$$\frac{1}{2}\frac{\partial}{\partial t}\int_\Omega |u^n|^2\,dx+\alpha\int_\Omega |\nabla u^n|^2\,dx\le C_{\alpha,\beta}\int_\Omega|\nabla u^n|^2\,dx \tag{3-4}$$

since $\mathcal{P}$ and $\mathfrak{J}$ are smooth maps with compact supports.

Next, taking Δ^2u^n as another test function of (3-3), we can show

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}\int_\Omega |\Delta u^n|^2\,dx&+\alpha\int_\Omega |\nabla\Delta u^n|^2\,dx\\&=\gamma\int_\Omega\langle\nabla(v\cdot\nabla u^n),\nabla\Delta u^n\rangle\,dx+\beta\int_\Omega\langle\nabla(\mathfrak{J}(u^n))\Delta u^n,\nabla\Delta u^n\rangle\,dx\\&\qquad-\alpha\int_\Omega\langle\nabla(\mathcal{P}(u^n)(\nabla u^n,\nabla u^n)),\nabla\Delta u^n\rangle\,dx\\&=I+II+III.\end{aligned}\tag{3-5}$$

To proceed, we estimate the above three terms as follows:

$$\begin{aligned}|I|&\le|\gamma|\Big(\int_\Omega|\nabla v||\nabla u^n||\nabla\Delta u^n|\,dx+\int_\Omega|v||\nabla^2u^n||\nabla\Delta u^n|\,dx\Big)\\&\le C|\gamma|\|\nabla v\|_{L^3}\|\nabla u^n\|_{L^6}\|\nabla\Delta u^n\|_{L^2}+C|\gamma|\|v\|_{L^6}\|\nabla^2u^n\|_{L^3}\|\nabla\Delta u^n\|_{L^2}\\&\le C_\alpha|\gamma|^2\|v\|^2_{W^{1,3}}\|u^n\|^2_{H^2}+\tfrac{1}{16}\alpha\|\nabla\Delta u^n\|^2_{L^2}\\&\qquad+C|\gamma|\|v\|_{H^1}(\|u^n\|_{H^2}\|\nabla\Delta u^n\|_{L^2}+\|u^n\|^{1/2}_{H^2}\|\nabla\Delta u^n\|^{3/2}_{L^2})\\&\le C_\alpha(|\gamma|^2\|v\|^2_{H^1}\|u^n\|^2_{H^2}+|\gamma|^4\|v\|^4_{H^1}\|u^n\|^2_{H^2})\\&\qquad+C_\alpha|\gamma|^2\|v\|^2_{W^{1,3}}\|u^n\|^2_{H^2}+\tfrac{1}{8}\alpha\|\nabla\Delta u^n\|^2_{L^2},\end{aligned}$$

where we have used the interpolation inequality

$$\|\nabla^2u^n\|_{L^3}\le\|\nabla^2u^n\|^{1/2}_{L^2}\|\nabla^2u^n\|^{1/2}_{L^6},$$

and the Sobolev embedding inequality

$$\|f\|_{L^6}\le C\|f\|_{H^1}$$

for any $f\in H^1(\Omega)$.

The second term II can be estimated as follows:

$$\begin{aligned}|II| &\le C|\beta|\int_\Omega |\nabla u^n||\Delta u^n||\nabla\Delta u^n|\,dx\\ &\le C|\beta|\|\nabla u^n\|_{L^6}\|\Delta u^n\|_{L^3}\|\nabla\Delta u^n\|_{L^2}\\ &\le C|\beta|\|u^n\|_{H^2}\|\Delta u^n\|_{L^2}^{1/2}\|\Delta u^n\|_{L^6}^{1/2}\|\nabla\Delta u^n\|_{L^2}\\ &\le C|\beta|\|u^n\|_{H^2}^{3/2}(\|u^n\|_{H^2}^{1/2}+\|\nabla\Delta u^n\|_{L^2}^{1/2})\|\nabla\Delta u^n\|_{L^2}\\ &\le C_{\alpha,\beta}(\|u^n\|_{H^2}^4+\|u^n\|_{H^2}^6)+\tfrac{1}{8}\alpha\|\nabla\Delta u^n\|_{L^2}^2.\end{aligned}$$

Similarly, for the last term III, we have

$$\begin{aligned}|III| &\le C_\alpha\int_\Omega(|\nabla u^n|^3+|\nabla u^n||\nabla^2u^n|)|\nabla\Delta u^n|\,dx\\ &\le C_\alpha\|\nabla u^n\|_{L^6}^6+\tfrac{1}{16}\alpha\|\nabla\Delta u^n\|_{L^2}^2+C\|\nabla u^n\|_{L^6}\|\nabla^2u^n\|_{L^3}\|\nabla\Delta u^n\|_{L^2}\\ &\le C_\alpha\|u^n\|_{H^2}^6+C_\alpha(\|u^n\|_{H^2}^4+\|u^n\|_{H^2}^6)+\tfrac{1}{8}\alpha\|\nabla\Delta u^n\|_{L^2}^2.\end{aligned}$$

In view of the above estimates of terms I, II, and III, we have

$$\frac{\partial}{\partial t}\int_\Omega|\Delta u^n|^2\,dx+\alpha\int_\Omega|\nabla\Delta u^n|^2\,dx\le C_{\alpha,\gamma,\beta}(\|v\|_{W^{1,3}(\Omega)}^2+1)^2(\|u^n\|_{H^2(\Omega)}^2+1)^3. \tag{3-6}$$

Therefore, by combining (3-4) with (3-6), we conclude

$$\frac{\partial}{\partial t}\|u^n\|_{H^2(\Omega)}^2+\alpha\int_\Omega|\nabla\Delta u^n|^2\,dx\le C_{\alpha,\gamma,\beta}(\|v\|_{W^{1,3}(\Omega)}^2+1)^2(\|u^n\|_{H^2(\Omega)}^2+1)^3. \tag{3-7}$$

Proposition 3.3. *Let Ω be a smooth bounded domain in $\mathbb{R}^3$. Suppose that u_0 is in $H^2(\Omega)$ and*

$$\frac{\partial u_0}{\partial\nu}\Big|_{\partial\Omega}=0,$$

$v\in L^\infty(\mathbb{R}^+, W^{1,3}(\Omega))$, *and* $\operatorname{div}(v)=0$ *with* $\langle v,\nu\rangle|_{\partial\Omega\times\mathbb{R}^+}=0$. *Then, there exists a positive constant T_0 depending only on α, γ, β, and $\|u_0\|_{H^2}$, such that the above approximate solutions u^n satisfy*

$$\sup_{0\le t\le T}(\|u^n\|_{H^2(\Omega)}^2+\|\partial_tu^n\|_{L^2(\Omega)}^2)+\alpha\int_0^T(\|u^n\|_{H^3(\Omega)}^2+\|\partial_tu^n\|_{H^1(\Omega)}^2)\,dt\le C(T) \tag{3-8}$$

for $0<T<T_0$, where $C(T)$ is a constant depending on T.

Proof. Let $f(t)=\|u^n\|_{H^2}^2+1$. Since $v\in L^\infty(\mathbb{R}^+, W^{1,3}(\Omega))$, inequality (3-7) implies $f(t)$ satisfies

$$\begin{cases} f'(t)\le C(f(t)+1)^3,\\ f(0)=\|u_0^n\|_{H^2}^2+1\le C\|u_0\|_{H^2}^2+1.\end{cases}$$

Here we have used the inequality

$$\|u_0^n\|_{H^2}^2\le C\|u_0\|_{H^2}^2,$$

since $\frac{\partial u_0}{\partial\nu}|_{\partial\Omega}=0$.

Then, by Lemma 2.1 and the classical comparison theorem of ODE, Lemma 2.2, we can show that there exists a positive constant T_0 depending only on α, γ, β, and $\|u_0\|_{H^2}$, such that for any $0 < T < T_0$,

$$\sup_{0\le t\le T} \|u^n\|^2_{H^2(\Omega)} + \alpha \int_0^T \|u^n\|^2_{H^3(\Omega)}\, dt \le C(T).$$

By (3-3), it is not difficult to show

$$\sup_{0\le t\le T} \|\partial_t u^n\|^2_{L^2(\Omega)} + \alpha \int_0^T \|\partial_t u^n\|^2_{H^1(\Omega)}\, dt \le C(T).$$

Therefore, the proof is completed. □

With the above uniform estimate (3-8) of u^n at hand, we can show that there exists a local strong solution to (3-2) by applying the compactness Lemma 2.3 and letting $n \to \infty$. Therefore, we conclude:

Theorem 3.4. *Let Ω be a smooth bounded domain in $\mathbb{R}^3$ and $u_0 \in H^2(\Omega)$ with*

$$\frac{\partial u_0}{\partial \nu}\Big|_{\partial\Omega} = 0.$$

Suppose that $v \in L^\infty(\mathbb{R}^+, W^{1,3}(\Omega))$ and $\operatorname{div}(v) = 0$ with $\langle v, \nu\rangle|_{\partial\Omega\times\mathbb{R}^+} = 0$. Then there exists a positive constant T_0 depending only on α, γ, β, and $\|u_0\|_{H^2}$, such that the initial Neumann boundary value problem (3-2) admits a local strong solution $u \in C^0\big([0, T], H^2(\Omega)\big) \cap L^2\big([0, T], H^3(\Omega)\big)$, which satisfies

$$\sup_{0\le t\le T} (\|u\|^2_{H^2(\Omega)} + \|\partial_t u\|^2_{L^2(\Omega)}) + \alpha \int_0^T (\|u\|^2_{H^3(\Omega)} + \|\partial_t u\|^2_{H^1(\Omega)})\, dt \le C(T) \tag{3-9}$$

for $0 < T < T_0$, where $C(T)$ is a constant depending on T.

Since the proof of the above theorem is almost the same as that in [Chen and Wang 2023c], we omit it. To show that u is a strong solution to (1-9) or (3-1), we need to prove $u(x, t) \in N$ for almost all $(x, t) \in \Omega \times [0, T_0)$.

Proposition 3.5. *The solution u constructed in Theorem 3.4 satisfies $u(x, t) \in N$ for almost every $(x, t) \in \Omega \times [0, T_0)$, and hence u is a local strong solution to (1-9).*

Proof. Since $u \in L^\infty([0, T], H^2(\Omega))$ and $\partial u/\partial t \in L^2([0, T], L^2(\Omega))$ for $T < T_0$, Lemma 2.3 implies

$$u \in C^0([0, T], W^{1,4}(\Omega)).$$

It follows that

$$\sup_{x\in\Omega} |u(x, t) - u(x, 0)| \le C\|u(\cdot, t) - u(\cdot, 0)\|_{W^{1,4}} \to 0$$

as $t \to 0$. Then there exists a positive number $t_1 \le T$ such that for $t \le t_1$, we have

$$\sup_{x\in\Omega} |u(x, t) - u(x, 0)| \le \delta,$$

namely $u(x,t) \in U_\delta(N)$ for $(x,t) \in \Omega \times [0,t_1]$. Therefore, by the definition of the cut-off function ζ, u satisfies

$$\frac{\partial u}{\partial t} + \gamma \nabla_v u = \alpha(\Delta u - \operatorname{Hess}\iota(\nabla u, \nabla u)) - \beta i_* \circ J(\iota(u)) \circ \pi_{\iota(u)} \Delta u.$$

Let $\rho(u) = u - \iota(u)$, then we have

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}\int_\Omega |\rho(u)|^2\,dx &= \int_\Omega \Big\langle \rho(u), \frac{\partial u}{\partial t}\Big\rangle\,dx \\ &= \int_\Omega \langle \rho(u), \alpha(\Delta\rho(u) + d\iota(\Delta u))\rangle\,dx \\ &\qquad - \beta\int_\Omega \langle \rho(u), \hat{J}(u)\Delta u\rangle\,dx - \gamma\int_\Omega v\cdot\langle \rho(u), \nabla u\rangle\,dx \\ &= \alpha\int_\Omega \langle \rho(u), \Delta\rho(u)\rangle\,dx - \frac{\gamma}{2}\int_\Omega v\cdot\nabla|\rho(u)|^2\,dx \\ &= -\alpha\int_\Omega |\nabla\rho(u)|^2\,dx.\end{aligned}$$

Here we have used the following facts:

(1) Since $\Delta\iota(u) = d\iota(\Delta u) + \operatorname{Hess}\iota(\nabla u, \nabla u)$, we have

$$\Delta u - \operatorname{Hess}\iota(\nabla u, \nabla u) = \Delta\rho(u) + d\iota(\Delta u).$$

(2) Since $\rho(u) \in T^\perp_{\iota(u)}N$ and $\hat{J}(u)\Delta u \in T_{\iota(u)}N$,

$$\langle \rho(u), \hat{J}(u)\Delta u\rangle = 0 \quad\text{and}\quad \langle \rho(u), d\iota(\Delta u)\rangle = 0.$$

(3) Since $\operatorname{div}(v) = 0$ and $\langle v, \nu\rangle|_{\partial\Omega} = 0$,

$$\int_\Omega v\cdot\nabla|\rho(u)|^2\,dx = 0.$$

Then the Gronwall inequality implies $\rho(u) = 0$ for almost all $(x,t) \in \Omega\times[0,t_1]$. Finally, we can prove this proposition by repeating the above argument. □

To end this section, we show the uniqueness of the solution u constructed above:

Proposition 3.6. *The solution to* (3-1) *in* $L^\infty([0,T], H^2(\Omega)) \cap L^2([0,T], H^3(\Omega))$ *is unique.*

Proof. Assume u_1 and u_2 are two solutions in $L^\infty([0,T], H^2)\cap L^2([0,T], H^3(\Omega))$, then $\bar{u} = u_1 - u_2$ satisfies

$$\begin{cases} \partial_t\bar{u} = -\gamma\nabla_v\bar{u} + \alpha\Delta\bar{u} + \alpha(P(u_1)(\nabla u_1, \nabla u_1) - P(u_2)(\nabla u_2, \nabla u_2)) \\ \qquad\qquad - \beta(\hat{J}(u_1) - \hat{J}(u_2))\Delta u_1 - \beta\hat{J}(u_2)\Delta\bar{u}, \\ \bar{u}(x,0) = 0 \\ \partial\bar{u}/\partial\nu = 0. \end{cases} \tag{3-10}$$

By taking $\bar{u}$ as a test function to (3-10), we can show

$$\frac{1}{2}\frac{\partial}{\partial t}\int_\Omega |\bar{u}|^2\,dx + \alpha\int_\Omega |\nabla\bar{u}|^2\,dx = -\gamma\int_\Omega \langle \nabla_v\bar{u}, \bar{u}\rangle\,dx + I + II + III.$$

Here

$$|\gamma|\cdot\left|\int_\Omega\langle\nabla_v\bar{u},\bar{u}\rangle\,dx\right|=\left|\frac{\gamma}{2}\right|\cdot\left|\int_\Omega\operatorname{div}(v|\bar{u}|^2)\,dx\right|=0,$$

since $\langle v,\nu\rangle|_{\partial\Omega}=0$ and $\operatorname{div}(v)=0$.

$$\begin{aligned}
|I|&=\alpha\left|\int_\Omega\langle P(u_1)(\nabla u_1,\nabla u_1)-P(u_2)(\nabla u_2,\nabla u_2),\bar{u}\rangle\,dx\right|\\
&\le C\alpha\Big(\int_\Omega|\bar{u}|^2|\nabla u_1|^2\,dx+\int_\Omega|\nabla\bar{u}|(|\nabla u_1|+|\nabla u_2|)|\bar{u}|\,dx\Big)\\
&\le C\alpha(\|u_1\|^2_{H^3}+\|u_2\|^2_{H^3})\int_\Omega|\bar{u}|^2\,dx+\frac{\alpha}{8}\int_\Omega|\nabla\bar{u}|^2\,dx,\\
|II|&=|\beta|\left|\int_\Omega\langle(\hat{J}(u_1)-\hat{J}(u_2))\Delta u_1,\bar{u}\rangle\,dx\right|\\
&\le C|\beta|\left|\int_\Omega\langle\operatorname{div}((\hat{J}(u_1)-\hat{J}(u_2))\nabla u_1),\bar{u}\rangle\,dx\right|\\
&\qquad\qquad+C|\beta|\left|\int_\Omega\langle\nabla((\hat{J}(u_1)-\hat{J}(u_2))\cdot\nabla u_1),\bar{u}\rangle\,dx\right|\\
&\le C|\beta|\int_\Omega|\hat{J}(u_1)-\hat{J}(u_2)||\nabla u_1||\nabla\bar{u}|\,dx\\
&\qquad\qquad+C|\beta|\left|\int_\Omega\langle\nabla((\hat{J}(u_1)-\hat{J}(u_2))\cdot\nabla u_1),\bar{u}\rangle\,dx\right|\\
&\le C|\beta|\|\nabla u_1\|_{L^\infty}\int_\Omega|\nabla\bar{u}||\bar{u}|\,dx+C|\beta|(|\nabla u_1|^2_\infty+|\nabla u_2|^2_\infty)\int_\Omega|\bar{u}|^2\,dx\\
&\le C_\alpha|\beta|(\|u_1\|^2_{H^3}+\|u_2\|^2_{H^3})\int_\Omega|\bar{u}|^2\,dx+\frac{\alpha}{8}\int_\Omega|\nabla\bar{u}|^2\,dx,
\end{aligned}$$

$$\begin{aligned}
|III|&=|\beta|\left|\int_\Omega\langle\hat{J}(u_2)\Delta\bar{u},\bar{u}\rangle\,dx\right|\\
&\le|\beta|\left|\int_\Omega\langle\nabla(\hat{J}(u_2))\cdot\nabla\bar{u},\bar{u}\rangle\,dx\right|+|\beta|\left|\int_\Omega|\langle\operatorname{div}(\hat{J}(u_2)\nabla\bar{u}),\bar{u}\rangle\,dx\right|\\
&\le C|\beta||\int_\Omega|\nabla u_2||\nabla\bar{u}||\bar{u}|\,dx|\\
&\le C_\alpha|\beta|\|u_2\|^2_{H^3}\int_\Omega|\bar{u}|^2\,dx+\frac{\alpha}{8}\int_\Omega|\nabla\bar{u}|^2\,dx.
\end{aligned}$$

In view of the above estimates of terms I, II and III, we get

$$\frac{\partial}{\partial t}\int_\Omega|\bar{u}|^2\,dx+\alpha\int_\Omega|\nabla\bar{u}|^2\,dx\le C_{\alpha,\beta,\gamma}(\|u_1\|^2_{H^3(\Omega)}+\|u_2\|^2_{H^3(\Omega)}+1)\int_\Omega|\bar{u}|^2\,dx.$$

Then, since $\|u_1\|^2_{H^3(\Omega)}+\|u_2\|^2_{H^3(\Omega)}\in L^1[0,T]$, the Gronwall inequality implies $u_1\equiv u_2$. □

4. Local regular solution

In the previous section, we obtained a strong solution u to the equation

$$(4\text{-}1)\quad \begin{cases} \partial_t u+\gamma\nabla_v u=\alpha(\Delta u+P(u)(\nabla u,\nabla u))-\beta\hat{J}(u)\Delta u, & (x,t)\in\Omega\times[0,T_0),\\ \partial u/\partial\nu=0, & (x,t)\in\partial\Omega\times[0,T_0),\\ u(x,0)=u_0:\Omega\to N\hookrightarrow\mathbb{R}^K. \end{cases}$$

Here $u:\Omega\times[0,T_0)\to N$,

$$P(u)=-\operatorname{Hess}\iota(u):\mathbb{R}^K\otimes\mathbb{R}^K\to\mathbb{R}^K$$

is a bilinear functional, and

$$\hat{J}(u)=i_*\circ J\circ d\iota(u):\mathbb{R}^K\to\mathbb{R}^K$$

is an antisymmetric matrix, since $d\iota(u)=\pi_u$.

Suppose $u_0\in H^3(\Omega,N)$ and $\frac{\partial u_0}{\partial\nu}|_{\partial\Omega}=0$, we can improve the regularity of u by applying the differential of Galerkin approximation to (3-2) with respect to the time variable t.

Theorem 4.1. *Let Ω be a smooth bounded domain in $\mathbb{R}^3$ and $u_0\in H^3(\Omega,N)$ with*

$$\frac{\partial u_0}{\partial\nu}\Big|_{\partial\Omega}=0.$$

Suppose that $v\in L^\infty(\mathbb{R}^+,W^{1,3}(\Omega))\cap C^0(\mathbb{R}^+,H^1(\Omega))$, $\partial_t v\in L^2(\mathbb{R}^+,H^1(\Omega))$, and $\operatorname{div}(v)=0$ *with $\langle v,\nu\rangle|_{\partial\Omega\times\mathbb{R}^+}=0$. Then, the solution u given in Theorem 3.4 satisfies*

$$\partial_t^i u\in C^0([0,T],H^{3-2i}(\Omega))\cap L^2([0,T],H^{4-2i}(\Omega))$$

for $T<T_0$ and $i=0,1$.

Proof. We divide the proof into two steps.

Step 1: *H^2-estimate of $\partial_t u$.*

To get H^2-estimates of the solution $\partial_t u$, we consider the equation of $w^n=\partial_t u^n$ as follows, where u^n is the Galerkin approximation of u:

$$(4\text{-}2)\quad \begin{aligned}\partial_t w^n=\alpha\Delta w^n&+P_n(-\gamma\nabla_v w^n-\gamma\nabla_{\partial_t v}u^n)\\ &+\alpha P_n(\mathcal{P}(u^n)(\nabla w^n,\nabla u^n)+\partial_t\mathcal{P}(u^n)(\nabla u^n,\nabla u^n))\\ &-\beta P_n(\partial_t\mathfrak{J}(u^n)\Delta u^n-\mathfrak{J}(u^n)\Delta w^n).\end{aligned}$$

Then we take Δw^n as a test function for (4-2) to give

$$\begin{aligned}\frac{\partial}{\partial t}\int_\Omega |\nabla w^n|^2\,dx + 2\alpha\int_\Omega |\Delta w^n|^2\,dx
&\le C_{\alpha,\beta}\Big(\int_\Omega |w^n|^2|\Delta u^n|^2\,dx + \int_\Omega |w^n|^2|\nabla u^n|^4\,dx\Big)\\
&\quad + C_{\alpha,\gamma}\Big(\int_\Omega |\nabla w^n|^2|\nabla u^n|^2\,dx + |\partial_t v|^2|\nabla u^n|^2\,dx\Big)\\
&\quad + |\gamma|\int_\Omega |\nabla w^n|^2|\nabla v|\,dx + \frac{\alpha}{2}\int_\Omega |\Delta w^n|^2\,dx\\
&\le C_{\alpha,\beta,\gamma}(\|u^n\|_{H^3}^2 + \|u^n\|_{H^2}^4 + 1)\|w^n\|_{H^1}^2 + C_{\alpha,\gamma}\|u^n\|_{H^2}^2\|\partial_t v\|_{H^1}^2\\
&\quad + |\gamma|\int_\Omega |\nabla w^n|^2|\nabla v|\,dx + \frac{\alpha}{2}\int_\Omega |\Delta w^n|^2\,dx.\end{aligned}$$

Here we have used the fact $\operatorname{div}(v)=0$ and $\langle v, \nu\rangle|_{\partial\Omega\times\mathbb{R}^+}=0$ to show

$$\int_\Omega \langle v\cdot\nabla w^n, \Delta w^n\rangle\,dx = -\int_\Omega \langle \nabla v\cdot\nabla w^n, \nabla w^n\rangle\,dx.$$

On the other hand, we have

$$\begin{aligned}\int_\Omega |\nabla w^n|^2|\nabla v|\,dx &\le \|\nabla w^n\|_{L^3}^2\|\nabla v\|_{L^3} \le \|\nabla w^n\|_{L^2}\|\nabla w^n\|_{L^6}\|\nabla v\|_{L^3}\\
&\le C\|v\|_{W^{1,3}}\|\nabla w^n\|_{L^2}(\|w^n\|_{L^2} + \|\Delta w^n\|_{L^2})\\
&\le C_\alpha\|v\|_{W^{1,3}}^2\|w^n\|_{H^1}^2 + \frac{\alpha}{2}\int_\Omega |\Delta w^n|^2\,dx.\end{aligned}$$

It follows that

$$\begin{aligned}&\frac{\partial}{\partial t}\int_\Omega |\nabla w^n|^2\,dx + \alpha\int_\Omega |\Delta w^n|^2\,dx\\
&\le C(\alpha,\beta,\gamma)(\|u^n\|_{H^3}^2 + \|u^n\|_{H^2}^4 + \|v\|_{W^{1,3}}^2 + 1)\|w^n\|_{H^1}^2 + C(\alpha,\gamma)\|u^n\|_{H^2}^2\|\partial_t v\|_{H^1}^2.\end{aligned}$$

By assumption we know that $v\in L^2(\mathbb{R}^+, W^{1,3}(\Omega))$ and $\partial_t v\in L^2(\mathbb{R}^+, H^1(\Omega))$. Hence, by applying the Gronwall inequality we can derive from (3-8) that

$$\sup_{0<t\le T}\|w^n\|_{H^1(\Omega)} + \alpha\int_0^T \|w^n\|_{H^2(\Omega)}^2\,dt \le C(\alpha,\beta,\gamma,T,\|w^n|_{t=0}\|_{H^1(\Omega)}). \tag{4-3}$$

Now, it remains to give a bound of $\|w^n|_{t=0}\|_{H^1}$. Since

$$w^n(\,\cdot\,,0) = \alpha\Delta u_0^n + P_n(-\gamma\nabla_v u_0^n + \alpha\mathcal{P}(u_0^n)(\nabla u_0^n, \nabla u_0^n) - \beta\mathfrak{J}(u_0^n)\Delta u_0^n),$$

it is not difficult to show

$$\|w^n|_{t=0}\|_{H^1(\Omega)} \le C(\|u_0\|_{H^3(\Omega)}^2, \|v(\,\cdot\,,0)\|_{H^1}^2).$$

Here we have used the fact

$$\|u_0^n\|_{H^3(\Omega)}^2 \le C\|u_0\|_{H^3(\Omega)}^2$$

by providing $\frac{\partial u_0}{\partial\nu}|_{\partial\Omega}=0$.

Without loss of generality, the estimate (4-3) implies w^n weakly converges to $\partial_t u$, which satisfies

$$\partial_t u \in L^\infty\big([0, T], H^1(\Omega)\big) \cap L^2\big([0, T], H^2(\Omega)\big)$$

for any $0 < T < T_0$.

Step 2: *H^4-estimate of u.*

Equation (1-5) is equivalent to

$$\Delta u = -A(u)(\nabla u, \nabla u) + \frac{1}{\alpha^2+\beta^2}(\alpha\partial_t u + \beta J(u)\partial_t u) + \frac{\gamma}{\alpha^2+\beta^2}(\alpha\nabla_v u + \beta J(u)\nabla_v u).$$

Under the assumption that $v \in L^\infty(\mathbb{R}^+, H^1(\Omega))$, one can easily show

$$\Delta u \in L^\infty([0, T], L^3(\Omega)),$$

the classical L^p-theory of elliptic equations gives

$$u \in L^\infty([0, T], W^{2,3}(\Omega)).$$

Hence, by using the assumption $v \in L^\infty\big(\mathbb{R}^+, W^{1,3}(\Omega)\big) \cap L^2\big(\mathbb{R}^+, H^2(\Omega)\big)$, we can take almost the same argument as in [Chen and Wang 2023c] to show

$$\Delta u \in L^\infty\big([0, T], H^1(\Omega)\big) \cap L^2\big([0, T], H^2(\Omega)\big),$$

hence the classical L^2-theory of elliptic equations again gives

$$u \in L^\infty\big([0, T], H^3(\Omega)\big) \cap L^2\big([0, T], H^4(\Omega)\big).$$

Moreover, since $u \in L^2([0, T], H^4(\Omega))$ and $\partial_t u \in L^2([0, T], H^2(\Omega))$, Lemma 2.4 tells us that $u \in C^0([0, T], H^3(\Omega))$. It follows that

$$\partial_t u \in C^0([0, T], H^1(\Omega))$$

by using (4-1) and the fact $v \in C^0(\mathbb{R}^+, H^1(\Omega))$. □

The proof of Theorem 1.1. By combining Theorem 3.4, Propositions 3.5, 3.6, and Theorem 4.1, we can obtain the results in Theorem 1.1 and finish its proof. □

5. Local very regular solution

In this section, we prove Theorem 1.2.

5A. ***Compatibility conditions of the initial data.*** In order to make the LLG equation (4-1) (an extrinsic version of the LLG equation (1-9)) admit a regular or smooth solution, we need to pose some compatibility conditions of the initial data. We start with a brief description of the compatibility conditions of the initial data. For the sake of convenience, we assume v is a smooth vector field and u is a smooth solution to the initial Neumann boundary value problem of the LLG equation (4-1). Then, for any $k \in \mathbb{N}$, $u_k = \partial_t^k u$ satisfies the linear equation

$$\partial_t u_k = \alpha \Delta u_k - \beta \hat{J}(u) \Delta u_k - \gamma \nabla_v u_k + L_k(u_k, u) + F_k(u) \tag{5-1}$$

with the initial data

$$V_k(u_0) := \partial_t^k u|_{t=0}.$$

In particular, $V_0 = u_0$ and

$$V_1(u_0) = -\gamma \nabla_{v(x,0)} u_0 + \alpha \tau(u_0) - \beta J(u_0) \tau(u_0).$$

Here we set

$$L_k(u_k, u) = 2\alpha P(u)(\nabla u_k, \nabla u) + \alpha d P(u)(u_k, \nabla u, \nabla u) - \beta d \hat{J}(u)(u_k) \Delta u,$$

and

$$F_k(u) = -\gamma \sum_{\substack{i+j=k \\ i \geq 1}} C_k^i \nabla_{v_i} u_j + \alpha \sum_{\substack{i_1+\cdots+i_s+m+l=k \\ 1 \leq i_j < k}} \nabla^s P(u) \# u_{i_1} \# \cdots \# u_{i_s} \# \nabla u_m \# \nabla u_l$$
$$+ \beta \sum_{\substack{i_1+\cdots+i_s+m=k \\ 1 \leq i_j < k}} \nabla^s \hat{J}(u) \# u_{i_1} \# \cdots \# u_{i_s} \# \Delta u_m,$$

where $v_i = \partial_t^i v$ and # denotes the linear contraction.

Then the compatibility conditions of the initial data is defined as follows:

Definition 5.1. Let $k \in \mathbb{N}$, $u_0 \in H^{2k+2}(\Omega, N)$. Suppose that for any $0 \leq i \leq k$, v satisfies

$$\partial_t^i v \in C^0(\mathbb{R}^+, H^{2k-2i}(\Omega)).$$

We say u_0 satisfies the compatibility condition at order k, if for any $j \in \{0, 1, \ldots, k\}$

$$\frac{\partial V_j}{\partial \nu}\Big|_{\partial\Omega} = 0. \tag{5-2}$$

Intrinsically, if we set

$$\widetilde{V}_k(u_0) = \nabla_t^k u(x, 0) \in \Gamma(u_0^*(TN))$$

where $\nabla_t = \nabla^N_{\partial u/\partial t}$, then the compatibility conditions defined in Definition 5.1 has the following equivalent characterization:

Proposition 5.2. *Let $k \in \mathbb{N}$, $u_0 \in H^{2k+2}(\Omega, N)$. Suppose that for any $0 \leq i \leq k$, v satisfies*

$$\partial_t^i v \in C^0(\mathbb{R}^+, H^{2k-2i}(\Omega)).$$

Then u_0 satisfies the compatibility condition of order k if and only if for any $j \in \{0, 1, \ldots, k\}$,

$$\nabla_\nu \widetilde{V}_j|_{\partial\Omega} = 0. \tag{5-3}$$

Proof. The necessity is proved by induction on k. Since $V_1 = \widetilde{V}_1$, if we assume $\frac{\partial V_1}{\partial \nu}|_{\partial\Omega} = 0$, then we have

$$\nabla_\nu \widetilde{V}_1|_{\partial\Omega} = \frac{\partial \widetilde{V}_1}{\partial \nu}\Big|_{\partial\Omega} + A(u_0)\bigg(\frac{\partial u_0}{\partial \nu}\Big|_{\partial\Omega}, \widetilde{V}_1\bigg) = 0.$$

Then, we assume that the result is true for $1 \le l \le k-1$. For the case $l = k \le 2$, by definition of $\widetilde{V}_k$, a simple calculations gives

$$\widetilde{V}_k = V_k + \sum_\sigma B_{\sigma(k)}(u_0)(V_{a_1}, \dots, V_{a_s})$$

where the sum is over all multi-indices $a_1, \dots, a_s$ such that $1 \le a_i \le k-1$ and $a_1 + \cdots + a_s = k$ for all i,

$$(a_1, \dots, a_s) = \sigma(k)$$

is a partition of k, and B is a multilinear functional on $u_0^*(TN)$.

Hence, by using the assumption of induction, we have

$$\begin{aligned}\nabla_\nu \widetilde{V}_k|_{\partial\Omega} =& \frac{\partial \widetilde{V}_k}{\partial \nu}\Big|_{\partial\Omega} + A(u_0)\bigg(\frac{\partial u_0}{\partial \nu}\Big|_{\partial\Omega}, \widetilde{V}_k\bigg)\\ =& \frac{\partial V_k}{\partial \nu}\Big|_{\partial\Omega} + \sum_\sigma \nabla B_{\sigma(k)}(u_0)\bigg(\frac{\partial u_0}{\partial \nu}\Big|_{\partial\Omega}, V_{a_1}, \dots, V_{a_s}\bigg) = 0.\end{aligned}$$

For the converse the proof is almost the same as above, so we omit it. □

Remark 5.3. If $\gamma = 0$ in (1-9) and $\nabla^N J = 0$, we set

$$W_k = \nabla_t^{k-1}\tau(u)(x,0) \quad \text{and} \quad \widetilde{W}_k = \partial_t^{k-1}\tau(u)(x,0)$$

for any $k \ge 1$, and set $W_0 = \widetilde{W}_0 = u_0$. Then, by taking a similar argument to that in the proof of Proposition 5.2 or Proposition 3.2 in [Chen and Wang 2023b], we can show the k-order compatibility condition defined in Definition 5.1 is equivalent to one of the following:

(1) For $1 \le j \le k$,

$$\nabla_\nu W_j|_{\partial\Omega} = 0.$$

(2) For $1 \le j \le k$,

$$\frac{\partial \widetilde{W}_j}{\partial \nu}\Big|_{\partial\Omega} = 0.$$

Next, we apply the method of induction to show the existence of very regular solutions to (4-1) by considering the initial Neumann boundary value problem of equation of $\partial_t^k u$ for $k \ge 1$ with corresponding initial data V_k. For this purpose, we intend to prove the main result Theorem 1.2 by showing the following process $\mathscr{T}(k)$ with $k \ge 2$:

(1) Assume that $u_0 \in H^{2k}(\Omega)$ satisfies the $(k-1)$-order compatibility conditions. Suppose moreover

$$\partial_t^i v \in C^0\big([0,T], H^{2k-2(i+1)}(\Omega)\big) \cap L^2\big([0,T], H^{2k-2i}(\Omega)\big)$$

for any $0 < T < T_0$ and $i \in \{0, 1, \dots, k-1\}$. Then for any $0 \le i \le k-1$, we have

$$\partial_t^i u \in C^0\big([0,T], H^{2k-2i}(\Omega)\big) \cap L^2\big([0,T], H^{2k-2i+1}(\Omega)\big),$$

and

$$\partial_t^k u \in L^\infty\big([0,T], L^2(\Omega)\big) \cap L^2\big([0,T], H^1(\Omega)\big).$$

(2) Additionally, if $u_0 \in H^{2k+1}(\Omega)$,

$$\partial_t^i v \in C^0\big([0,T], H^{2k+1-2(i+1)}(\Omega)\big) \cap L^2\big([0,T], H^{2k-2i}(\Omega)\big)$$

for $i \in \{0, 1, \dots, k-1\}$ and $\partial_t^k v \in L^2([0,T], L^2(\Omega))$, then for any $0 \le i \le k$ we have

$$\partial_t^i u \in C^0\big([0,T], H^{2k-2i+1}(\Omega)\big) \cap L^2\big([0,T], H^{2k-2i+2}(\Omega)\big).$$

5B. ***H^5-regularity of u (i.e., the proof of property $\mathscr{T}(2)$).*** For any $T < T_0$, Theorem 4.1 implies that $\partial_t u \in C^0\big([0,T], H^1(\Omega)\big) \cap L^2\big([0,T], H^2(\Omega)\big)$ is a strong solution to

$$\text{(5-4)}\quad \begin{cases} \partial_t u_1 + \gamma \nabla_v u_1 \\ \qquad = \alpha \Delta u_1 - \beta \hat{J}(u) \Delta u_1 + L_1(u_1, u) + F_1(u), & (x,t) \in \Omega \times [0,T], \\ \partial u_1 / \partial \nu = 0, & (x,t) \in \partial\Omega \times [0,T], \\ u_1(x,0) = V_1 \end{cases}$$

where

$$L_1(u_1, u) = 2\alpha P(u)(\nabla u_1, \nabla u) + \alpha d P(u)(u_1, \nabla u, \nabla u) - \beta d \hat{J}(u)(u_1) \Delta u,$$

and

$$F_1(u) = -\gamma \nabla_{\partial_t v} u.$$

To improve the regularity of $\partial_t u$, we solve the initial Neumann boundary value problem (5-4) with compatibility condition

$$\frac{\partial V_1}{\partial \nu}\Big|_{\partial\Omega} = 0.$$

As before, we consider the Galerkin approximation equation of (5-4):

$$\text{(5-5)}\quad \begin{cases} \partial_t u_1^n + \gamma P_n(\nabla_v u_1^n) = \alpha \Delta u_1^n - \beta P_n(\hat{J}(u) \Delta u_1^n) + P_n(L_1(u_1^n, u) + F_1(u)), \\ \qquad u_1(x,0) = V_1^n. \end{cases}$$

Since the operators P and $\hat{J}$ satisfy

(1) $|P(u)| + |\hat{J}(u)| \leq C$,

(2) $|\nabla(P(u))| + |\nabla(\hat{J}(u))| \leq C|\nabla u|$,

(3) $\hat{J}$ is antisymmetric,

we can apply almost the same argument as that in [Chen and Wang 2023c] to give the estimate

$$\sup_{0\leq t\leq T}(\|u_1^n\|_{H^2}^2 + \|\partial_t u_1^n\|_{L^2}^2) + \alpha\int_0^T \|\Delta\nabla u_1^n\|_{L^2}^2 + \|\nabla\partial_t u_1^n\|_{L^2}^2\,dt \leq C(\|u_0\|_{H^3}, \|P_n(V_1)\|_{H^2}),$$

by providing $v \in C^0(\mathbb{R}^+, H^2(\Omega))$ and $\partial_t v \in L^\infty(\mathbb{R}^+, L^2(\Omega)) \cap L^2(\mathbb{R}^+, H^1(\Omega))$, then taking u_1^n and $\Delta^2 u_1^n$ as test functions to (5-5).

On the other hand, since

$$V_1 = -\gamma\nabla_{v(x,0)}u_0 + \alpha\tau(u_0) - \beta J(u_0)\tau(u_0)$$

and

$$\frac{\partial V_1}{\partial \nu}\Big|_{\partial\Omega} = 0,$$

Lemma 2.5 implies that

$$\|P_n(V_1)\|_{H^2(\Omega)} \leq C\|V_1\|_{H^2} \leq C(\|u_0\|_{H^4(\Omega)}, \|v(.,0)\|_{H^2(\Omega)}).$$

Without loss of generality, by using the compactness in Lemma 2.3, we can infer that u_1^n converges to a map $u_1 \in L^\infty([0,T], H^2(\Omega)) \cap L^2([0,T], H^3(\Omega))$ which solves (5-4). To show that $\partial_t u = u_1$ on $\Omega \times [0, T_0)$, we need to use the following result:

Proposition 5.4. *The solution to* (5-4) *in* $C^0([0,T], H^1) \cap L^2([0,T], H^2(\Omega))$ *is unique.*

Proof. Let v_1 and v_2 be two solutions of (5-4), which belong to the space

$$C^0([0,T], H^1(\Omega)) \cap L^2([0,T], H^2(\Omega)).$$

Then, $\omega = v_1 - v_2$ satisfies

$$\begin{cases} \partial_t\omega + \gamma\nabla_v\omega = \alpha\Delta\omega - \beta\hat{J}(u)\Delta\omega + L_1(\omega, u), & (x,t) \in \Omega\times[0,T], \\ \partial\omega/\partial\nu = 0, & (x,t) \in \partial\Omega\times[0,T], \\ \omega(x,0) = 0. \end{cases}$$

By choosing ω as a test function of this equation and taking a simple calculation we obtain

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}\int_\Omega|\omega|^2\,dx+\alpha\int_\Omega|\nabla\omega|^2\,dx&\\ &\le C_\alpha\int_\Omega(|\nabla u||\nabla\omega||\omega|+|\nabla u|^2|\omega|^2)\,dx-\beta\int_\Omega\langle d\hat{J}(u)(\omega)\Delta u,\omega\rangle\,dx\\ &\le C_\alpha\int_\Omega(|\nabla u||\nabla\omega||\omega|+|\nabla u|^2|\omega|^2)\,dx-\beta\int_\Omega\langle\nabla u,\nabla(d\hat{J}(u)(\omega)\omega)\rangle\,dx\\ &\le C_{\alpha,\beta}\int_\Omega(|\nabla u||\nabla\omega||\omega|+|\nabla u|^2|\omega|^2)\,dx\\ &\le C_{\alpha,\beta}\|u\|^2_{L^\infty([0,T],H^3(\Omega))}\int_\omega|\omega|^2\,dx+\frac{\alpha}{2}\int_\Omega|\nabla\omega|^2\,dx.\end{aligned}$$

Consequently, the Gronwall inequality implies $\omega\equiv 0$, completing the proof. □

It follows from Proposition 5.4 that

$$\partial_t u\in L^\infty\big([0,T],H^2(\Omega)\big)\cap L^2\big([0,T],H^3(\Omega)\big). \tag{5-6}$$

Additionally, if we provide $u_0\in H^5(\Omega)$, $\partial_t v\in C^0([0,T],H^1(\Omega))$, and $\partial_t^2 v$ in $L^2(\mathbb{R}^+,L^2(\Omega))$, we can apply a similar argument to Step 1 of the proof of Theorem 4.1 to show

$$\partial_t^2 u\in L^\infty\big([0,T],H^1(\Omega)\big)\cap L^2\big([0,T],H^2(\Omega)\big) \tag{5-7}$$

by considering the equation of $\partial u_1^n/\partial t$.

To enhance the regularity of u, we need to use the following technical lemmas:

Lemma 5.5. *Let Ω be a smooth bounded domain in $\mathbb{R}^3$, $n\ge 0$, and $m\ge 2$. If $f\in H^n(\Omega)$ (we set $H^0(\Omega)=L^2(\Omega)$) and $g\in H^m(\Omega)$, then $fg\in H^l(\Omega)$ with $l=\min\{n,m\}$. Moreover, there exists a constant $C(\|f\|_{H^n},\|g\|_{H^m})$ such that we have*

$$\|fg\|_{H^l(\Omega)}\le C(\|f\|_{H^n},\|g\|_{H^m}).$$

One can consult [Carbou and Jizzini 2018] for a proof. As a direct corollary, we have:

Corollary 5.6. *Let Ω be a smooth bounded domain in $\mathbb{R}^3$ and N be a compact Riemannian submanifold of $\mathbb{R}^K$. If*

$$u\in L^\infty\big([0,T],H^k(\Omega,N)\big)\cap L^2\big([0,T],H^{k+1}(\Omega,N)\big)$$

with $k\ge 2$ and $L:N\to\mathbb{R}^K\otimes\mathbb{R}^K$ is a smooth map, then $L(u)$ belongs to $L^\infty\big([0,T],H^k(\Omega)\big)\cap L^2\big([0,T],H^{k+1}(\Omega)\big)$.

Proof. It is not difficult to show that the result holds true for $k=2$. Hence, without loss of generality we can assume that $k\ge 3$.

Since $\nabla(L(u)) = \nabla L(u)\#\nabla u$, the fact that $u \in L^\infty([0,T], H^k(\Omega, N))$ with $k \geq 3$ implies

$$\nabla(L(u)) \in L^\infty([0,T], L^2(\Omega, N)).$$

On the other hand, a simple calculation gives

$$\begin{aligned}\nabla^l(L(u)) =& \sum_{\substack{i_1+\cdots+i_s=l\\ 1\le s\le l,\, i_j\ge 1}} \nabla^s L(u)\#\nabla^{i_1}u\#\cdots\#\nabla^{i_s}u\\ =& \nabla L(u)\#\nabla^l u + \nabla^2 L(u)\#\nabla^{l-1}u\#\nabla u + \sum_{\substack{i_1+\cdots+i_s=l\\ 2\le s\le l\\ 1\le i_j\le l-2}} \nabla^s L(u)\#\nabla^{i_1}u\#\cdots\#\nabla^{i_s}u\end{aligned}$$

for $2 \leq l \leq k+1$. Since $u \in L^\infty([0,T], H^k(\Omega, N))$ and $\sup_{y\in N} |\nabla^s L|(y) \leq C(s)$, Lemma 5.5 above implies

$$\nabla^l(L(u)) \in L^\infty([0,T], L^2(\Omega))$$

for $2 \leq l \leq k$.

To show $\nabla^{k+1}(L(u)) \in L^2([0,T], L^2(\Omega))$, we need only to deal with the following term of $\nabla^{k+1}(L(u))$:

$$I = \nabla^2 L(u)\#\nabla^{k-1}u\#\nabla^2 u,$$

since the other terms can be bounded directly by applying Lemma 5.5.

By using the facts $\nabla^{k-1}u \in L^2([0,T], H^2(\Omega))$ and $\nabla^2 u \in L^\infty([0,T], H^1(\Omega))$, we have

$$\begin{aligned}\int_0^T\int_\Omega |I|^2\,dx dt \leq& C\int_0^T \|\nabla^{k-1}u\|^2_{L^\infty(\Omega)}\,dt \sup_{t\in[0,T]}\int_\Omega |\nabla^2 u|^2\,dx\\ \leq& C\int_0^T \|\nabla^{k-1}u\|^2_{H^2}\,dt \sup_{t\in[0,T]}\int_\Omega |\nabla^2 u|^2\,dx < \infty.\end{aligned}$$

Therefore, we finish the proof. □

We are now in position to prove the main result (i.e., $\mathscr{T}(2)$) of this subsection:

Proposition 5.7. *Suppose that $u_0 \in H^4(\Omega, N)$ satisfies the 1-order compatibility condition defined in Definition 5.1, $v \in C^0(\mathbb{R}^+, H^2(\Omega)) \cap L^2(\mathbb{R}^+, H^3(\Omega))$, and $\partial_t v \in L^\infty(\mathbb{R}^+, L^2(\Omega)) \cap L^2(\mathbb{R}^+, H^1(\Omega))$. Then for any $0 < T < T_0$ we have*

$$\partial_t^i u \in C^0([0,T], H^{4-2i}(\Omega)) \cap L^2([0,T], H^{5-2i}(\Omega))$$

for $i \in \{0, 1\}$, and

$$\partial_t^2 u \in L^\infty([0,T], L^2(\Omega)) \cap L^2([0,T], H^1(\Omega)).$$

Additionally, if $u_0 \in H^5(\Omega, N)$, $\partial_t^i v \in C^0\big(\mathbb{R}^+, H^{3-2i}(\Omega)\big) \cap L^2\big(\mathbb{R}^+, H^{4-2i}(\Omega)\big)$ *with* $i = 0, 1$, *and* $\partial_t^2 v \in L^2(\mathbb{R}^+, L^2(\Omega))$, *we obtain*

$$\partial_t^i u \in C^0\big([0, T], H^{5-2i}(\Omega)\big) \cap L^2\big([0, T], H^{6-2i}(\Omega)\big)$$

for $i \in \{0, 1, 2\}$.

Proof. Our proof is divided into two steps:

Step 1: H^5*-estimate of* u.

By using (4-1) and taking a simple computation we can show

$$\Delta u = -P(u)(\nabla u, \nabla u) + \frac{1}{\alpha^2+\beta^2}(\alpha \partial_t u + \beta \hat{J}(u)\partial_t u) + \frac{\gamma}{\alpha^2+\beta^2}(\alpha \nabla_v u + \beta \hat{J}(u)\nabla_v u). \tag{5-8}$$

In the case $u_0 \in H^4(\Omega, N)$, Lemma 5.5 and Corollary 5.6 tell us that

$$\Delta u \in L^\infty([0, T], H^2(\Omega)),$$

since $u \in L^\infty([0, T], H^3(\Omega))$, $v \in C^0(\mathbb{R}^+, H^2(\Omega))$ and by estimate (5-6). Hence, by the L^2-theory of elliptic equations we know that

$$u \in L^\infty([0, T], H^4(\Omega)).$$

Moreover, if we assume $v \in L^2([0, T], H^3(\Omega))$, we can apply Lemma 5.5 and Corollary 5.6 again to show

$$\Delta u \in L^2([0, T], H^3(\Omega)),$$

and hence we have $u \in L^2([0, T], H^5(\Omega))$. Consequently, Lemma 2.4 implies

$$\partial_t^i u \in C^0([0, T], H^{4-2i}(\Omega))$$

for $i = 0, 1$.

Step 2: H^6*-estimate of* u.

On the other hand, it follows from (5-8) that

$$\Delta \partial_t u = \frac{1}{\alpha^2+\beta^2}(\alpha \partial_t^2 u + \beta \hat{J}(u)\partial_t^2 u) + \frac{\beta}{\alpha^2+\beta^2} d\hat{J}(u)\#\partial_t u\#\partial_t u + \frac{\gamma}{\alpha^2+\beta^2}\partial_t(\alpha \nabla_v u + \beta \hat{J}(u)\nabla_v u) - \partial_t(P(u)(\nabla u, \nabla u)).$$

Then, by using estimate (5-7) and taking the same argument as above, we can show

$$\Delta \partial_t u \in L^\infty\big([0, T], H^1(\Omega)\big) \cap L^2\big([0, T], H^2(\Omega)\big);$$

hence the L^2-theory of the Laplace operator again implies

$$\partial_t u \in L^\infty\big([0, T], H^3(\Omega)\big) \cap L^2\big([0, T], H^4(\Omega)\big).$$

Finally, we can show

$$u \in L^\infty\big([0,T], H^5(\Omega)\big) \cap L^2\big([0,T], H^6(\Omega)\big),$$

by providing $v \in L^\infty\big([0,T], H^3(\Omega)\big) \cap L^2\big([0,T], H^4(\Omega)\big)$.

Now, by Lemma 2.4 we can also derive that

$$\partial_t^i u \in C^0([0,T], H^{5-2i}(\Omega))$$

for $i \in 0, 1$. Hence, it follows that $\partial_t^2 u \in C^0([0,T], H^1(\Omega))$ by using the equation of $\partial_t u$ and the fact $\partial_t^i v \in C^0(\mathbb{R}^+, H^{3-2i}(\Omega))$ with $i = 0, 1$. □

5C. ***Higher order regularity of u (i.e., the proof of $\mathscr{T}(k)$ with $k \geq 2$).*** In Section 5B, we have proved property $\mathscr{T}(k)$ in the case $k = 2$. Next, we assume that $\mathscr{T}(k)$ has been established for $k \geq 2$, then we intend to show $\mathscr{T}(k+1)$ is true. To this end, we assume that $u_0 \in H^{2(k+1)}(\Omega)$ satisfies the k-order compatibility conditions, and v satisfies

$$\partial_t^i v \in C^0\big([0,T], H^{2(k+1)-2(i+1)}(\Omega)\big) \cap L^2\big([0,T], H^{2(k+1)-2i}(\Omega)\big)$$

for any $0 < T < T_0$ and any $i \in \{0, 1, \dots, k\}$. Moreover, property $\mathscr{T}(k)$ implies

$$\partial_t^i u \in C^0\big([0,T], H^{2k-2i+1}(\Omega)\big) \cap L^2\big([0,T], H^{2k-2i+2}(\Omega)\big)$$

for any $0 \leq i \leq k$.

In particular, $u_k = \partial_t^k u \in C^0\big([0,T], H^1(\Omega)\big) \cap L^2\big([0,T], H^2(\Omega)\big)$ is a strong solution to the equation

$$\begin{cases} \partial_t w + \gamma \nabla_v w = \alpha \Delta w - \beta \hat{J}(u) \Delta w + L_k(w,u) + F_k(u), & (x,t) \in \Omega \times [0,T], \\ \partial w / \partial \nu = 0, & (x,t) \in \partial\Omega \times [0,T], \\ w(x,0) = V_k(u_0) : \Omega \to \mathbb{R}^K. \end{cases} \tag{5-9}$$

In the following context, we improve the regularity of u by proving the following three claims:

(1) If $u_0 \in H^{2(k+1)}(\Omega)$ satisfies the k-order compatibility conditions, then we get a regular solution to (5-9):

$$w \in C^0\big([0,T], H^2(\Omega)\big) \cap L^2\big([0,T], H^3(\Omega)\big).$$

(2) It follows from an argument on uniqueness that $w = u_k$. Hence we can show

$$u_i \in C^0\big([0,T], H^{2(k+1)-2i}(\Omega)\big) \cap L^2\big([0,T], H^{2(k+1)+1-2i}(\Omega)\big)$$

for any $0 \leq i \leq k+1$, by using (4-1).

(3) Additionally if $u_0 \in H^{2(k+1)+1}(\Omega)$, we can further prove

$$u_{k+1} \in C^0\big([0,T], H^1(\Omega)\big) \cap L^2\big([0,T], H^2(\Omega)\big)$$

by considering differentiation of the Galerkin approximation equation to (5-9) in the time direction. This implies

$$u_i \in C^0\big([0,T], H^{2(k+1)+1-2i}(\Omega)\big) \cap L^2\big([0,T], H^{2(k+1)+2-2i}(\Omega)\big)$$

for any $0 \le i \le k+1$.

5D. ***Regular solution to*** **(5-9).** To show the existence of local regular solutions to (5-9) by applying a similar argument to that in Section 3, first of all, we estimate the nonhomogeneous term F_k by using the estimates given in Lemmas 5.5 and 5.6.

Lemma 5.8. *Assume that, for* $0 \le i \le k$, *the field* v *satisfies*

$$\partial_t^i v \in C^0\big([0,T], H^{2(k+1)-2(i+1)}(\Omega)\big) \cap L^2\big([0,T], H^{2(k+1)-2i}(\Omega)\big)$$

and property $\mathscr{T}(k)$ *holds true. Then, we have*

$$F_i \in L^\infty\big([0,T], H^{2k-2i}(\Omega)\big) \cap L^2\big([0,T], H^{2k-2i+2}(\Omega)\big)$$

for $0 \le i \le k$.

Proof. For any $0 \le i \le k$, by setting $v_i = \partial_t^i v$, we have

$$\begin{aligned} F_i(u) = -\gamma \sum_{\substack{m+j=i\\ m\ge 1}} v_m \# \nabla u_j + \alpha \sum_{\substack{i_1+\cdots+i_s+m+l=i\\ 1\le i_j<i}} \nabla^s P(u) \# u_{i_1} \# \cdots \# u_{i_s} \# \nabla u_m \# \nabla u_l \\ + \beta \sum_{\substack{i_1+\cdots+i_s+m=i\\ 1\le i_j<i}} \nabla^s \hat{J}(u) \# u_{i_1} \# \cdots \# u_{i_s} \# \Delta u_m \\ = I + II + III. \end{aligned}$$

Next we estimate the above three terms step by step. For term I: since $1 \le m \le i$ and $0 \le j = i - m \le i-1$, then we have

$$v_m \in L^\infty\big([0,T], H^{2k-2i}(\Omega)\big) \cap L^2\big([0,T], H^{2k-2i+2}(\Omega)\big)$$

and

$$u_j \in L^\infty([0,T], H^{2k-2i+3}(\Omega)).$$

Hence, Lemma 5.5 claims

$$I \in L^\infty\big([0,T], H^{2k-2i}(\Omega)\big) \cap L^2\big([0,T], H^{2k-2i+2}(\Omega)\big).$$

For term II: since $1 \le i_j \le i-1$ and $0 \le m \le i-1$, we have

$$u_{i_j} \in L^\infty([0,T], H^{2k-2i+3}(\Omega))$$

and

$$\nabla u_m \in L^\infty\big([0,T], H^{2k-2i+2}(\Omega)\big) \cap L^2\big([0,T], H^{2k-2i+3}(\Omega)\big).$$

It follows from Lemma 5.5 that

$$II \in L^\infty([0,T], H^{2k-2i+2}(\Omega)).$$

Similarly, by applying Corollary 5.6 with $\nabla^s \hat{J}$ in place of L, we can also show

$$III \in L^\infty\big([0,T], H^{2k-2i+1}(\Omega)\big) \cap L^2\big([0,T], H^{2k-2i+2}(\Omega)\big).$$

Therefore, the desired results are proved. □

Now we turn to considering the Galerkin approximation of (5-9):

$$\text{(5-10)} \quad \begin{cases} \partial_t w^n + \gamma P_n(\nabla_v w^n) = \alpha \Delta w^n - \beta P_n(\hat{J}(u)\Delta w^n) + P_n(L_k(w^n, u) + F_k(u)), \\ \qquad w^n(x,0) = P_n(V_k(u_0)) : \Omega \to \mathbb{R}^{n+k}. \end{cases}$$

It is not difficult to show that there exists a unique solution $w^n \in H^n$ to (5-10) on a maximal interval $[0, T_*^n)$, and we will show $T_0 \le T_*^n$.

Next, we choose w^n and $\Delta^2 w^n$ as test functions of (5-10) and take a simple calculation to show

$$\frac{\partial}{\partial t}\int_\Omega |w^n|^2\,dx + \alpha\int_\Omega |\nabla w^n|^2\,dx \le C_\alpha(1+|\beta|^2)(\|u\|_{H^3}^2+1)\|w^n\|_{H^1}^2 + \int_\Omega |F_k|^2\,dx,$$

$$\frac{\partial}{\partial t}\int_\Omega |\Delta w^n|^2\,dx + \alpha\int_\Omega |\nabla\Delta w^n|^2\,dx$$
$$\le C_\alpha(1+|\beta|^2+|\gamma|^2)(\|u\|_{H^3}^6+\|v\|_{H^2}^2+1)\|w^n\|_{H^2}^2 + C_\alpha\int_\Omega |\nabla F_k|^2\,dx.$$

It follows that

$$\frac{\partial}{\partial t}\|w^n\|_{H^2}^2 + \alpha\int_\Omega |\nabla\Delta w^n|^2\,dx \le C_{\alpha,\beta,\gamma}\, p(t)\|w^n\|_{H^2}^2 + C_\alpha q(t),$$

where

$$p(t) := \|u\|_{H^3}^6 + \|v\|_{H^2}^2 + 1 \le C(T)$$

and

$$q(t) := \|F_k\|_{H^1}^2 \in L^1([0,T])$$

for any $T < T_0$.

On the other hand, since $u_0 \in H^{2k+2}(\Omega)$ and $v_i \in C^0([0,T], H^{2k-2i}(\Omega))$ with $0 \le i \le k$, it is not difficult to show

$$\|V_k^n\|_{H^2(\Omega)}^2 \le C\|V_k\|_{H^2(\Omega)}^2 \le C(T, \|u_0\|_{H^{2(k+1)}(\Omega)}^2).$$

Here we have used Lemma 2.5 in the first inequality.

Thus, by the Gronwall inequality we can infer from the above

$$\sup_{0<t\le T}(\|w^n\|_{H^2}^2 + \|\partial_t w^n\|_{L^2}^2) + \alpha\int_0^T (\|w^n\|_{H^3}^2 + \|\partial_t w^n\|_{H^1}^2)\,dt \le C(T).$$

Hence without loss of generality, we assume that w^n converges to a regular solution $w \in L^\infty\big([0,T], H^2\big) \cap L^2\big([0,T], H^3(\Omega)\big)$ to (5-9). Moreover, $\partial_t w$ is in $L^\infty\big([0,T], L^2(\Omega)\big) \cap L^2\big([0,T], H^1(\Omega)\big)$. By Lemma 2.4 we know that

$$w \in C^0([0,T], H^2(\Omega)).$$

5E. ***Uniqueness of strong solutions to equation*** **(5-9).**

Proposition 5.9. *There exists a unique solution to equation* (5-9) *in the space* $L^\infty([0,T],H^1(\Omega))\cap L^2([0,T],H^2(\Omega))$.

Proof. Suppose w_1 and w_2 are two solutions to (5-9) belonging to the space $L^\infty([0,T],H^1)\cap L^2([0,T],H^2(\Omega))$. Then, the difference $\bar{w}=w_1-w_2$ satisfies

$$\begin{cases} \partial_t\bar{w}+\gamma\nabla_v\bar{w}=\alpha\Delta\bar{w}-\beta\hat{J}(u)\Delta\bar{w},u)+F_k(u), & (x,t)\in\Omega\times[0,T],\\ \partial\bar{w}/\partial\nu=0, & (x,t)\in\partial\Omega\times[0,T],\\ \bar{w}(x,0)=0. \end{cases}$$

Taking $\bar{w}$ as a test function to the above equation, we can show

$$\begin{aligned}\text{(5-11)}\quad \frac{1}{2}\frac{\partial}{\partial t}\int_\Omega|\bar{w}|^2\,dx+\alpha\int_\Omega|\nabla\bar{w}|^2\,dx &\\ =-\gamma\int_\Omega\langle v\cdot\nabla\bar{w},\bar{w}\rangle\,dx-\beta\int_\Omega\langle\hat{J}(u)\Delta\bar{w},\bar{w}\rangle\,dx+\int_\Omega\langle L_k(\bar{w},u),\bar{w}\rangle\,dx &\\ =I+II+III.&\end{aligned}$$

We estimates the above three terms as follows:

$$I=-\frac{\gamma}{2}\int_\Omega v\cdot\nabla|\bar{w}|^2\,dx=-\frac{\gamma}{2}\int_\Omega\operatorname{div}(v|\bar{w}|^2)\,dx=0,$$

since $\operatorname{div}(v)=0$ and $\langle v,\nu\rangle|_{\partial\Omega}=0$.

$$\begin{aligned}|II|&=|\beta|\left|\int_\Omega\langle\hat{J}(u)\Delta\bar{w},\bar{w}\rangle\,dx\right|\\ &\le C|\beta|\int_\Omega|\nabla\bar{w}||\nabla u||\bar{w}|\,dx\le C_\alpha\beta^2\|u\|^2_{H^3}\int_\Omega|\bar{w}|^2\,dx+\frac{\alpha}{4}\int_\Omega|\nabla\bar{w}|^2\,dx.\\ |III|&\le C\alpha\int_\Omega(|\bar{w}||\nabla\bar{w}||\nabla u|+|\bar{w}|^2|\nabla u|)\,dx+C|\beta|\left|\int_\Omega\langle d\hat{J}(\bar{w})\Delta u,\bar{w}\rangle\,dx\right|\\ &\le C_\alpha(1+\beta^2)\|u\|^2_{H^3}\int_\Omega|\bar{w}|^2\,dx+\frac{\alpha}{4}\int_\Omega|\nabla\bar{w}|^2\,dx.\end{aligned}$$

Here we have used the fact

$$\left|\int_\Omega\langle\nabla\hat{J}(\bar{w})\Delta u,\bar{w}\rangle\,dx\right|\le\left|\int_\Omega\langle\nabla(d\hat{J}(\bar{w}))\cdot\nabla u,\bar{w}\rangle\,dx\right|+\left|\int_\Omega\langle(d\hat{J}(\bar{w}))\cdot\nabla u,\nabla\bar{w}\rangle\,dx\right|$$

since $\frac{\partial u}{\partial\nu}|_{\partial\Omega}=0$.

By combining the estimates of I–III with (5-11), we get

$$\frac{\partial}{\partial t}\int_\Omega|\bar{w}|^2\,dx+\alpha\int_\Omega|\nabla\bar{w}|^2\,dx\le C_{\alpha,\beta}\int_\Omega|\bar{w}|^2\,dx.$$

It follows from the Gronwall inequality that $\bar{w}\equiv 0$. Therefore, the proof is completed. □

As a direct conclusion of the above proposition, we have $u_k\equiv w$ and hence

$$u_k\in C^0([0,T],H^2(\Omega))\cap L^2([0,T],H^3(\Omega)).$$

5F. ***The proof of item (1) of property $\mathscr{T}(k+1)$.*** Now we are in position to prove item (1) of property $\mathscr{T}(k+1)$ as follows:

Proposition 5.10. *Assume that $u_0 \in H^{2(k+1)}(\Omega)$ satisfies the k-order compatibility condition, for $i \in \{0, 1, \dots, k\}$,*

$$v_i = \partial_t^i v \in C^0\big([0,T], H^{2(k+1)-2(i+1)}(\Omega)\big) \cap L^2\big([0,T], H^{2(k+1)-2i}(\Omega)\big),$$

and property $\mathscr{T}(k)$ holds true. Then, for any $i \in \{0, 1, \dots, k+1\}$,

$$u_i \in L^\infty\big([0,T], H^{2(k+1)-2i}(\Omega)\big) \cap L^2\big([0,T], H^{2(k+1)+1}(\Omega)\big).$$

It follows that, for any $i \in \{0, 1, \dots, k\}$,

$$u_i \in C^0\big([0,T], H^{2(k+1)-2i}(\Omega)\big) \cap L^2\big([0,T], H^{2(k+1)+1}(\Omega)\big).$$

Proof. Since

$$u_{k+1} = \alpha \Delta u_k - \beta \hat{J}(u) \Delta u_k - \gamma \nabla_v u_k + L_k(u_k, u) + F_k(u)$$

and $u_k \in L^\infty\big([0,T], H^2\big) \cap L^2\big([0,T], H^3(\Omega)\big)$, a direct calculation shows

$$u_{k+1} \in L^\infty\big([0,T], L^2(\Omega)\big) \cap L^2\big([0,T], H^1(\Omega)\big).$$

Next we prove this proposition by inducting on $k+1-l$. We have shown the result is true for $l=0$ and $l=1$. Now, we assume that for $l=i\geq 1$ the result has been proved. Then, we need to establish it for $l=i+1$, where $i \leq k-1$. Thus, we consider the following equation of u_{k-i}:

$$\begin{aligned}
\text{(5-12)}\quad \Delta u_{k-i} &= \frac{1}{\sigma}(\alpha u_{k-i+1} + \beta \hat{J}(u) u_{k-i+1}) + \frac{\alpha\gamma}{\sigma} \sum_{q+m=k-i} v_q \# u_m \\
&\quad + \sum_{i_1+\cdots+i_q+s+m=k-i} \nabla^q P \# u_{i_1} \# \cdots \# u_{i_q} \# \nabla u_s \# \nabla u_m \\
&\quad + \frac{\beta\gamma}{\sigma} \sum_{i_1+\cdots+i_q+s+m=k-i} \nabla^q \hat{J} \# u_{i_1} \# \cdots \# u_{i_q} \# v_s \# \nabla u_m \\
&\quad + \frac{\beta}{\sigma} \sum_{\substack{i_1+\cdots+i_q+m=k-i \\ m<k-i}} \nabla^q \hat{J} \# u_{i_1} \# \cdots \# u_{i_q} \# u_{m+1} \\
&= J_1 + J_2 + J_3 + J_4 + J_5,
\end{aligned}$$

where σ denotes $\alpha^2+\beta^2$.

Next we estimate the above five terms step by step. First of all, by the assumptions of induction, we have the following:

(1) For $i+1 \leq l \leq k+1$, $u_{k+1-l} \in L^\infty\big([0,T], H^{2l-1}(\Omega)\big) \cap L^2\big([0,T], H^{2l}(\Omega)\big)$.

(2) For $0 \leq l \leq i < k$, $u_{k+1-l} \in L^\infty\big([0,T], H^{2l}(\Omega)\big) \cap L^2\big([0,T], H^{2l+1}(\Omega)\big)$.

(3) For $0 \le s \le k$,

$$v_s \in L^\infty\big([0,T], H^{2k-2s}(\Omega)\big) \cap L^2\big([0,T], H^{2(k+1)-2l}(\Omega)\big).$$

The estimate of term J_1: since

$$u_{k-i+1} \in L^\infty\big([0,T], H^{2i}(\Omega)\big) \cap L^2\big([0,T], H^{2i+1}(\Omega)\big)$$

and

$$u \in L^\infty\big([0,T], H^{2k+1}(\Omega)\big) \cap L^2\big([0,T], H^{2k+2}(\Omega)\big),$$

by applying Corollary 5.6 with L replaced by $\hat{J}$ and Lemma 5.5, we have

$$J_1 \in L^\infty\big([0,T], H^{2i}(\Omega)\big) \cap L^2\big([0,T], H^{2i+1}(\Omega)\big).$$

The estimate of term J_3: A simple computation shows that J_3 satisfies

$$\begin{aligned} J_3 &= \nabla P \# u_{k-i} \# \nabla u \# \nabla u + P(u) \# \nabla u_{k-i} \# \nabla u \\ &\quad + \sum_{\substack{i_1+\cdots+i_q+s+m=k-i \\ i_j,m,s \le k-i-1}} \nabla^q P \# u_{i_1} \# \cdots \# u_{i_q} \# \nabla u_s \# \nabla u_m \\ &= a+b+c. \end{aligned}$$

Since $u_{k-i} \in L^\infty\big([0,T], H^{2i+1}\big) \cap L^2\big([0,T], H^{2i+2}\big)$ with $i \le k-1$ and

$$\nabla u \in L^\infty\big([0,T], H^{2k}(\Omega)\big) \cap L^2\big([0,T], H^{2k+1}(\Omega)\big),$$

Lemma 5.5 implies

$$a+b \in L^\infty\big([0,T], H^{2i}(\Omega)\big) \cap L^2\big([0,T], H^{2i+1}(\Omega)\big).$$

On the other hand, by using the fact $i_j, m, s \le k-i-1$, we have

$$u_{i_j} \in L^\infty([0,T], H^{2(i+1)+1}(\Omega)) \quad \text{and} \quad \nabla u_m \in L^\infty([0,T], H^{2(i+1)}(\Omega)).$$

It follows that $c \in L^\infty([0,T], H^{2(i+1)})$. Consequently, we obtain

$$J_3 \in L^\infty\big([0,T], H^{2i}(\Omega)\big) \cap L^2\big([0,T], H^{2i+1}(\Omega)\big).$$

Taking almost the same argument as in estimating J_3, we obtain

$$J_2 + J_4 \in L^\infty\big([0,T], H^{2i}(\Omega)\big) \cap L^2\big([0,T], H^{2i+1}(\Omega)\big).$$

Then we show the last term:

$$\begin{aligned} J_5 &= \frac{\beta}{\sigma} \nabla \hat{J}(u) \# u_{k-i} \# u_1 + \frac{\beta}{\sigma} \sum_{\substack{i_1+\cdots+i_q+m=k-i \\ i_j,m<k-i}} \nabla^q \hat{J}(u) \# u_{i_1} \# \cdots \# u_{i_q} \# u_{m+1} \\ &= d+e. \end{aligned}$$

Since $u_{k-i} \in L^\infty([0,T], H^{2i+1}(\Omega))$ with $i \le k-1$ and

$$u_1 \in L^\infty([0,T], H^{2k-1}(\Omega)) \subset L^\infty([0,T], H^{2i+1}(\Omega)),$$

we have

$$d \in L^\infty([0,T], H^{2i+1}(\Omega)).$$

Since $m, i_j \le k-i-1$, by Lemma 5.5 and Corollary 5.6, it is not difficult to show

$$e \in L^\infty([0,T], H^{2i+1}(\Omega)).$$

Combining the above estimates of J_1–J_5 with formula (5-12), we conclude that

$$\Delta u_{k-i} \in L^\infty\big([0,T], H^{2i}(\Omega)\big) \cap L^2\big([0,T], H^{2i+1}(\Omega)\big).$$

Then, by the L^2-theory of Laplace operator we have

$$u_{k-i} \in L^\infty\big([0,T], H^{2(i+1)}(\Omega)\big) \cap L^2\big([0,T], H^{2(i+1)+1}(\Omega)\big)$$

for $1 \le i \le k-1$.

It remains to show the result in the case of $l = k+1$. Since

$$\Delta u = -P(u)(\nabla u, \nabla u) + \frac{1}{\sigma}(\alpha\partial_t u + \beta \hat{J}(u)\partial_t u) + \frac{\gamma}{\sigma}(\alpha\nabla_v u + \beta\hat{J}(u)\nabla_v u) \tag{5-13}$$

and

- $u \in L^\infty([0,T], H^{2k+1}(\Omega))$,
- $\partial_t u \in L^\infty\big([0,T], H^{2k}(\Omega)\big) \cap L^2\big([0,T], H^{2k+1}(\Omega)\big)$,
- $v \in L^\infty\big([0,T], H^{2k}(\Omega)\big) \cap L^2\big([0,T], H^{2k+2}(\Omega)\big)$,

we can apply Lemma 5.5 to show

$$\Delta u \in L^\infty([0,T], H^{2k}(\Omega)),$$

which gives $u \in L^\infty([0,T], H^{2k+2}(\Omega))$.

And again it follows that

$$\Delta u \in L^2([0,T], H^{2k+1}(\Omega)),$$

then the L^2-theory of the Laplace operator yields

$$u \in L^2([0,T], H^{2(k+1)+1}(\Omega)).$$

Therefore, the proof is completed. □

5G. *The proof of item (2) in property $\mathscr{T}(k+1)$.* In the last part, we assume that $u_0 \in H^{2(k+1)+1}(\Omega)$ satisfies the k-order compatibility conditions. Furthermore, suppose that there hold true the following properties $C(k)$:

- for any $i \in \{0, 1, \dots, k\}$,

  $$v_i \in C^0\big([0,T], H^{2(k+1)+1-2(i+1)}(\Omega)\big) \cap L^2\big([0,T], H^{2(k+1)-2i}(\Omega)\big)$$

  and $\partial_t^{k+1} v \in L^2([0,T], L^2(\Omega))$;

- for any $i \in \{0, 1, \dots, k+1\}$, we have

$$u_i \in C^0\big([0,T], H^{2(k+1)-2i}(\Omega)\big) \cap L^2\big([0,T], H^{2(k+1)+1-2i}(\Omega)\big).$$

Next, we turn to proving item (2) of property $\mathscr{T}(k+1)$.

First of all, taking almost the same argument as in Lemma 5.8, we can show:

Proposition 5.11. *For any* $i \in \{0, 1, \dots, k\}$,

$$\partial_t F_i \in L^2([0,T], H^{2k-2i}(\Omega)).$$

Next, we can also prove the following proposition, which is analogous to the main theorem in Section 4:

Proposition 5.12. *Assume that* $u_0 \in H^{2(k+1)+1}(\Omega)$ *satisfies the k-order compatibility conditions. If the properties* $C(k)$ *hold true, then we have*

$$u_{k+1} \in C^0\big([0,T], H^1(\Omega)\big) \cap L^2\big([0,T], H^2(\Omega)\big).$$

Proof. It follows from the Galerkin approximation equation (5-10) that $w_t^n := \partial_t w^n$ satisfies

$$\partial_t w_t^n - \alpha \Delta w_t^n = P_n \partial_t(-\gamma \nabla_v w^n - \beta \hat{J}(u)\Delta w^n + L_k(w^n, u) + F_k(u)).$$

Then, taking $-\Delta w_t^n$ as a test function to this equation, we obtain

$$\begin{aligned}
\frac{1}{2}\frac{\partial}{\partial t}\int_\Omega |\nabla w_t^n|^2\,dx + \alpha\int_\Omega |\Delta w_t^n|\,dx
&= \gamma\int_\Omega \langle \partial_t(v\cdot\nabla w^n), \Delta w_t^n\rangle\,dx + \beta\int_\Omega \langle \partial_t(\hat{J}(u)\Delta w^n), \Delta w_t^n\rangle\,dx \\
&\qquad - \int_\Omega \langle \partial_t L_k(w^n, u), \Delta w_t^n\rangle\,dx - \int_\Omega \langle \partial_t F_k(u), \Delta w_t^n\rangle\,dx \\
&= M_1 + M_2 + M_3 + M_4.
\end{aligned}$$

By direct calculations, we show the estimates of M_1–M_4 as follows:

$$\begin{aligned}
|M_1| &\le C|\gamma|\int_\Omega (|\partial_t v||\nabla w^n| + |v||\nabla\partial_t w^n|)|\Delta w_t^n|\,dx \\
&\le C_\alpha|\gamma|^2\|\partial_t v\|_{H^1}^2\|w^n\|_{H^2}^2 + C_\alpha\|v\|_{H^2}^2|\gamma|^2\int_\Omega |\nabla w_t^n|^2\,dx + \frac{\alpha}{8}\int_\Omega |\Delta w_t^n|^2\,dx, \\
|M_2| &= |\beta|\left|\int_\Omega \langle \partial_t(\hat{J}(u))\Delta w^n, \Delta w_t^n\rangle\,dx\right| \\
&\le C_\alpha|\beta|^2\|\partial_t u\|_{H^2}^2\|w^n\|_{H^2}^2 + \frac{\alpha}{8}\int_\Omega |\Delta w_t^n|^2\,dx,
\end{aligned}$$

$$
\begin{aligned}
|M_3| &= \left| \int_\Omega \langle \partial_t L_k(w^n, u), \Delta w_t^n \rangle \, dx \right| \\
&\le C\alpha \int_\Omega (|\nabla w_t^n|^2 |\nabla u|^2 + |u_t|^2 |\nabla w^n|^2 |\nabla u|^2 + |\nabla u_t|^2 |\nabla w^n|^2) \, dx \\
&\qquad + C\alpha \int_\Omega (|w_t^n|^2 |\nabla u|^4 + |u_t|^2 |w^n|^2 |\nabla u|^4 + |\nabla u_t|^2 |\nabla u|^2 |w^n|^2) \, dx \\
&\qquad\qquad + C_\alpha |\beta|^2 \int_\Omega (|w_t^n|^2 |\Delta u|^2 + |u_t|^2 |w^n|^2 |\Delta u|^2 + |\Delta u_t|^2 |w^n|^2) \, dx \\
&\qquad\qquad\qquad + \frac{\alpha}{8} \int_\Omega |\Delta w_t^n|^2 \, dx \\
&\le C_\alpha (1+\beta^2) f(t) \Big(\int_\Omega |w_t^n|^2 \, dx \Big) + C\alpha \|u\|_{H^3}^2 \int_\Omega |\nabla w_t^n|^2 \, dx + \frac{\alpha}{8} \int_\Omega |\Delta w_t^n|^2 \, dx,
\end{aligned}
$$

where

$$
f(t) := \|u_t\|_{H^2}^2 \|w^n\|_{H^2}^2 (\|u\|_{H^2}^2 + 1)^2 \le C(T).
$$

The last term satisfies the estimate

$$
|M_4| \le C(\alpha) \|\partial_t F_k\|_{L^2}^2 + \frac{\alpha}{8} \int_\Omega |\Delta w_t^n|^2 \, dx.
$$

Hence, we conclude that

$$
\frac{\partial}{\partial t} \int_\Omega |\nabla w_t^n|^2 \, dx + \alpha \int_\Omega |\Delta w_t^n| \, dx \le C_{\gamma,\alpha,\beta,T} \int_\Omega |\nabla w_t^n|^2 \, dx + C_\alpha \|\partial_t F_k\|_{L^2}^2.
$$

It follows

$$
\sup_{0 \le t \le T} \|\partial_t w^n\|_{H^1}^2 + \alpha \int_0^T \int_\Omega |\Delta w_t^n|^2 \, dx dt \le C(T, \|V_k^n\|_{H^3}^2),
$$

since $\|\partial_t F_k\|_{L^2}^2 \in L^1([0, T])$.

Now, it remains to show there exists a uniform bound of $\|V_k^n\|_{H^3}^2$. By using the fact $v_i \in C^0([0, T], H^{2k-2i+1})$ with $0 \le i \le k$, we can show

$$
\|V_k^n\|_{H^3(\Omega)}^2 \le C \|V_k\|_{H^3(\Omega)}^2 \le C(\|u_0\|_{H^{2(k+1)+1}(\Omega)}^2).
$$

Hence, without loss of generality we can assume that w_t^n converges weakly to

$$
u_{k+1} \in L^\infty\big([0, T], H^1(\Omega)\big) \cap L^2\big([0, T], H^2(\Omega)\big).
$$

It follows that

$$
\partial_t u_{k+1} \in L^2([0, T], L^2(\Omega))
$$

by applying the equation of u_{k+1} and the fact $\partial_t F_k \in L^2([0, T], L^2(\Omega))$. Then, Lemma 2.4 gives

$$
u_{k+1} \in C^0([0, T], H^1(\Omega)). \qquad \square
$$

Consequently, taking the estimates in Propositions 5.10–5.12 into consideration, and adopting almost the same argument as in the proof of Proposition 5.10, we can see that it is not difficult to show

$$u_i \in L^\infty\big([0, T], H^{2(k+1)-2i+1}(\Omega)\big) \cap L^2\big([0, T], H^{2(k+1)-2i+2}(\Omega)\big)$$

for any $0 \le i \le k+1$. Hence, Lemma 2.4 implies that for any $i \in \{0, \dots, k\}$,

$$u_i \in C^0([0, T], H^{2(k+1)-2i+1}(\Omega)).$$

Therefore, the second term (2) in property $\mathscr{T}(k+1)$ is proved.

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