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CERTAIN FOURIER OPERATORS AND THEIR ASSOCIATED POISSON SUMMATION FORMULAE ON GL_1

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We explore the possibility of using harmonic analysis on GL_1 to understand Langlands automorphic L-functions in general, as a vast generalization of the PhD Thesis of J. Tate in 1950. For a split reductive group G over a number field k, let $G^\vee(\mathbb{C})$ be its complex dual group and ρ be an n-dimensional complex representation of $G^\vee(\mathbb{C})$. For any irreducible cuspidal automorphic representation σ of $G(\mathbb{A})$, where $\mathbb{A}$ is the ring of adeles of k, we introduce the space $\mathcal{S}_{\sigma,\rho}(\mathbb{A}^\times)$ of (σ,ρ)-Schwartz functions on $\mathbb{A}^\times$ and (σ,ρ)-Fourier operator $\mathcal{F}_{\sigma,\rho,\psi}$ that takes $\mathcal{S}_{\sigma,\rho}(\mathbb{A}^\times)$ to $\mathcal{S}_{\widetilde{\sigma},\rho}(\mathbb{A}^\times)$, where $\widetilde{\sigma}$ is the contragredient of σ. By assuming the local Langlands functoriality for the pair (G,ρ), we show that the (σ,ρ)-theta functions $\Theta_{\sigma,\rho}(x,\phi) := \sum_{\alpha\in k^\times}\phi(\alpha x)$ converge absolutely for all $\phi\in\mathcal{S}_{\sigma,\rho}(\mathbb{A}^\times)$. We state conjectures on the (σ,ρ)-Poisson summation formula on GL_1, and prove them in the case where $G=\mathrm{GL}_n$ and ρ is the standard representation of $\mathrm{GL}_n(\mathbb{C})$. This is done with the help of results of Godement and Jacquet (1972). As an application, we provide a spectral interpretation of the critical zeros of the standard L-functions $L(s,\pi\times\chi)$ for any irreducible cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbb{A})$ and idele class character χ of k, extending theorems of C. Soulé (2001) and A. Connes (1999). Other applications are in the introduction.

1.	Introduction	302
2.	Godement–Jacquet theory and reformulation	309
3.	π-Schwartz functions and Fourier operators	320
4.	π-Poisson summation formula on GL_1	330
5.	Convergence of generalized theta functions	339
6.	(σ,ρ)-theta functions on GL_1	351
7.	Variants of Conjecture 1.5	355
8.	Critical zeros of $L(s,\pi\times\chi)$	364
Acknowledgement		370
References		370

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1. Introduction

Let k be a number field and $\mathbb{A}$ be the ring of adeles of k. It is well known that $\mathbb{A}$ is a locally compact abelian group and the diagonal embedding of k into $\mathbb{A}$ is a lattice, i.e., the image, which is still denoted by k, is discrete and the quotient $k\backslash\mathbb{A}$ is compact. The classical theory of harmonic analysis on the quotient $k\backslash\mathbb{A}$ — in particular, the famous 1950 Princeton thesis of J. Tate [44] — has had a great impact on the modern development of number theory, especially on the theory of automorphic L-functions.

In Tate's thesis, the classical Fourier transform and the associated Poisson summation formula are responsible for the meromorphic continuation and global functional equation of the Hecke L-function $L(s, \chi)$ attached to an automorphic character χ of $k^\times\backslash\mathbb{A}^\times$.

In their pioneering work in 1972, R. Godement and H. Jacquet extended the work of Tate on $L(s, \chi)$ (and also the work of T. Tamagawa in [43]) to the standard automorphic L-function $L(s, \pi)$ attached to any irreducible cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbb{A})$ [16]. In their work, the Fourier transform and the associated Poisson summation formula for $M_n(k)\backslash M_n(\mathbb{A})$ are responsible for the meromorphic continuation and global functional equation of $L(s, \pi)$. Here M_n denotes the space of all $n\times n$ matrices.

In 2000, A. Braverman and D. Kazhdan [6] proposed that there should exist a generalized Fourier transform $\mathcal{F}_{\rho,\psi}$ on $G(\mathbb{A})$ for any reductive group G defined over k and any finite-dimensional complex representation ρ of the L-group ${}^LG$; and if the associated Poisson summation formula could be established, then there is a hope to prove the Langlands conjecture [29] on meromorphic continuation and global functional equation for automorphic L-function $L(s, \pi, \rho)$ attached to the pair (π, ρ), where π is any irreducible cuspidal automorphic representation of $G(\mathbb{A})$. In [33; 34], one may find careful discussions on the spherical case of and a helpful introduction to the proposal. In his 2020 paper [37], B. C. Ngô suggests that such generalized Fourier transforms could be put in a framework that generalizes the classical Hankel transform for harmonic analysis on GL_1 and might be more useful in the trace formula approach to establish the Langlands conjecture of functoriality in general.

1A. GL_1*-theory.* We develop GL_1-theory to explore a possibility of using harmonic analysis on GL_1 to understand Langlands automorphic L-functions in general, which would be a vast generalization of the classical work of Tate in [44] or of the more systematical treatment by A. Weil in [48]. The development goes in two steps. The first step is to establish it for the standard automorphic L-function $L(s, \pi)$ associated with an irreducible cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbb{A})$. When $n = 1$ and π is an automorphic character χ, it is the theory developed in Tate's thesis. The second step is to formulate the framework for the general automorphic L-function $L(s,\pi,\rho)$ associated with a pair (π,ρ) as introduced above.

The GL_1-theory for a standard L-function $L(s,\pi)$ is a reformulation and refinement of the Godement–Jacquet theory [16] for $L(s,\pi)$ of GL_n. It is based on the determinant morphism

$$\det : M_n \to \mathbb{G}_a; \quad \mathrm{GL}_n \to \mathbb{G}_m, \tag{1-1}$$

where $\mathbb{G}_a(k) = k$ and $\mathbb{G}_m(k) = \mathrm{GL}_1(k) = k^\times$. We write $\pi = \bigotimes_{\nu\in|k|}\pi_\nu$ where $|k|$ is the set of local places of k and π_ν is an irreducible admissible representation of $\mathrm{GL}_n(k_\nu)$, which is of Casselman–Wallach type if k_ν is an Archimedean local field. For each π_ν, by taking the fiber integration along det as defined in (3-6), we define in Definition 3.3 the π_ν-Schwartz space $\mathcal{S}_{\pi_\nu}(k_\nu^\times)$. It is important to understand the structure of the space $\mathcal{S}_{\pi_\nu}(k_\nu^\times)$ of π_ν-Schwartz functions on $k_\nu^\times$, whose properties are discussed intensively in Section 3. In particular, by Proposition 3.2 and Corollary 3.8, we have that

$$\mathcal{C}_c^\infty(k_\nu^\times) \subset \mathcal{S}_{\pi_\nu}(k_\nu^\times) \subset \mathcal{C}^\infty(k_\nu^\times).$$

It is important to mention that Theorem 7.1 provides a new characterization of $\mathcal{C}_c^\infty(k_\nu^\times)$ as a subspace of $\mathcal{S}_{\pi_\nu}(k_\nu^\times)$ by means of the fiber integration along det in (3-6). Through diagram (3-16), we define the π_ν-Fourier operator (or transform) $\mathcal{F}_{\pi_\nu,\psi_\nu}$, where ψ_ν is the ν-component of a fixed nontrivial character ψ of $k\backslash\mathbb{A}$. By the local GL_1-theory (Theorems 3.4 and 3.10), there exists a so-called *basic function* $\mathbb{L}_{\pi_\nu} \in \mathcal{S}_{\pi_\nu}(k_\nu^\times)$ when $\nu < \infty$ and π_ν is unramified, and the π_ν-Fourier operator maps the π_ν-Schwartz space $\mathcal{S}_{\pi_\nu}(k_\nu^\times)$ to the $\widetilde{\pi}_\nu$-Schwartz space $\mathcal{S}_{\widetilde{\pi}_\nu}(k_\nu^\times)$ with $\mathcal{F}_{\pi_\nu,\psi_\nu}(\mathbb{L}_{\pi_\nu}) = \mathbb{L}_{\widetilde{\pi}_\nu}$. The global π-Schwartz space $\mathcal{S}_\pi(\mathbb{A}^\times)$ is defined to be the restricted tensor product

$$\mathcal{S}_\pi(\mathbb{A}^\times) := \bigotimes_{\nu\in|k|}\mathcal{S}_{\pi_\nu}(k_\nu^\times)$$

with respect to the basic functions $\mathbb{L}_{\pi_\nu}$ for almost all finite local places, and the global π-Fourier operator $\mathcal{F}_{\pi,\psi}$ is defined by

$$\mathcal{F}_{\pi,\psi}(\phi) := \bigotimes_{\nu\in|k|}\mathcal{F}_{\pi_\nu,\psi_\nu}(\phi_\nu)$$

for any factorizable functions $\phi = \bigotimes_{\nu\in|k|}\phi_\nu \in \mathcal{S}_\pi(\mathbb{A}^\times)$. One of the main results in the global GL_1-theory is the π-Poisson summation formula on GL_1.

Theorem 1.1 (π-Poisson summation formula, Theorem 4.7). *Let π be an irreducible cuspidal automorphic representation of* $\mathrm{GL}_n(\mathbb{A})$. *For any $\phi \in \mathcal{S}_\pi(\mathbb{A}^\times)$, the π-theta function*

$$\Theta_\pi(x,\phi) := \sum_{\alpha\in k^\times}\phi(\alpha x)$$

converges absolutely and locally uniformly as a function in $x \in \mathbb{A}^\times$, *and we have the identity*

$$\Theta_\pi(x, \phi) = \Theta_{\widetilde{\pi}}(x^{-1}, \mathcal{F}_{\pi,\psi}(\phi)) \quad \textit{for } x \in \mathbb{A}^\times. \tag{1-2}$$

According to the tradition in literature, the π-Poisson summation formula in (1-2) may also be called the π-theta inversion formula. Our proof of Theorem 1.1 (Theorem 4.7) is based on the work of Godement–Jacquet in [16].

The GL_1-theory for general L-functions $L(s, \sigma, \rho)$ is formulated by means of the local Langlands functorial conjecture associated with ρ, which is the major conjecture in the local theory of the Langlands program.

For a k-split reductive group G, let $G^\vee(\mathbb{C})$ be its complex dual group and ρ be an n-dimensional complex representation of $G^\vee(\mathbb{C})$. For any irreducible cuspidal automorphic representation $\sigma = \bigotimes_{\nu \in |k|} \sigma_\nu$ of $G(\mathbb{A})$, we assume that the local Langlands functorial transfer $\pi_\nu = \pi_\nu(\sigma_\nu, \rho)$ exists and is an irreducible admissible representation of $\mathrm{GL}_n(k_\nu)$, which is of the Casselman–Wallach type if k_ν is Archimedean. We define as in (6-5) the (σ_ν, ρ)-*Schwartz space* on $k_\nu^\times$ to be

$$\mathcal{S}_{\sigma_\nu,\rho}(k_\nu^\times) := \mathcal{S}_{\pi_\nu}(k_\nu^\times),$$

and at unramified local places, the (σ_ν, ρ)-basic function $\mathbb{L}_{\sigma_\nu,\rho}$ is taken to be the π_ν-basic function $\mathbb{L}_{\pi_\nu} \in \mathcal{S}_{\pi_\nu}(k_\nu^\times)$. Then we can define as in (6-6) the (σ, ρ)-*Schwartz space* on $\mathbb{A}^\times$ to be

$$\mathcal{S}_{\sigma,\rho}(\mathbb{A}^\times) := \bigotimes_\nu \mathcal{S}_{\sigma_\nu,\rho}(k_\nu^\times),$$

which is the restricted tensor product with respect to the basic function $\mathbb{L}_{\sigma_\nu,\rho}$ at almost all finite local places, and define, as in (6-8), the (σ, ρ)-Fourier operator (or transform) $\mathcal{F}_{\sigma,\rho,\psi}$ that takes $\mathcal{S}_{\sigma,\rho}(\mathbb{A}^\times)$ to $\mathcal{S}_{\widetilde{\sigma},\rho}(\mathbb{A}^\times)$, where $\widetilde{\sigma}$ is the contragredient of σ. The first result in the global GL_1-theory for $L(s, \sigma, \rho)$ is the following.

Theorem 1.2. *With notations as introduced above, for all* $\phi \in \mathcal{S}_{\sigma,\rho}(\mathbb{A}^\times)$, *the* (σ, ρ)-*theta function*

$$\Theta_{\sigma,\rho}(x, \phi) := \sum_{\alpha \in k^\times} \phi(\alpha x) \tag{1-3}$$

converges absolutely and locally uniformly as a function in $x \in \mathbb{A}^\times$.

It is clear that Theorem 1.2 is a special case of Theorem 6.2, which asserts the same result as in Theorem 1.2 for much more general σ. The proof of Theorem 6.2 is deduced from the technical result (Theorem 5.4), which can be stated as follows.

Theorem 1.3 (Theorem 5.4). *Let* $\pi = \bigotimes_{\nu \in |k|} \pi_\nu$ *be an irreducible admissible representation of* $\mathrm{GL}_n(\mathbb{A})$ *with Assumption 5.1. Then for any* $\phi \in \mathcal{S}_\pi(\mathbb{A}^\times) :=$

$\bigotimes_{\nu\in|k|}\mathcal{S}_{\pi_\nu}(k_\nu^\times)$, *the* π*-theta function*

$$\Theta_\pi(x,\phi):=\sum_{\alpha\in k^\times}\phi(\alpha x)$$

converges absolutely and locally uniformly as a function in $x\in\mathbb{A}^\times$.

We refer to Section 5 for notation not given here. Section 5 is devoted to develop the basic properties of such general theta functions. Then we show that for any irreducible admissible automorphic representation π of $\mathrm{GL}_n(\mathbb{A})$, Assumption 5.1 holds (Proposition 5.5). As a consequence, we obtain the following general assertion.

Corollary 1.4 (Corollary 5.6). *Let* π *be any irreducible admissible automorphic representation of* $\mathrm{GL}_n(\mathbb{A})$. *For any* $\phi\in\mathcal{S}_\pi(\mathbb{A}^\times)$, *the* π*-theta function*

$$\Theta_\pi(x,\phi)=\sum_{\alpha\in k^\times}\phi(\alpha x)$$

converges absolutely and locally uniformly as a function in $x\in\mathbb{A}^\times$.

It remains to be an interesting problem to establish the π-Poisson summation formula for such general π-theta functions as in Corollary 1.4, although Theorem 7.3 obtains the π-Poisson summation formula as in Theorem 1.1 for $\Theta_\pi(x,\phi)$ when π is any irreducible square-integrable automorphic representation of $\mathrm{GL}_n(\mathbb{A})$ and ϕ has restrictions at two local places (see Theorem 7.3 for details).

The following is the main statement in the global GL_1-theory for $L(s,\sigma,\rho)$.

Conjecture 1.5 ((σ,ρ)-Poisson summation formula). *Let* $\rho: G^\vee(\mathbb{C})\to\mathrm{GL}_n(\mathbb{C})$ *be any finite-dimensional representation of the complex dual group* $G^\vee(\mathbb{C})$ *and* σ *be an irreducible cuspidal automorphic representation of* $G(\mathbb{A})$. *Then there exist nontrivial* $k^\times$*-invariant linear functionals* $\mathcal{E}_{\sigma,\rho}$ *and* $\mathcal{E}_{\widetilde{\sigma},\rho}$ *on* $\mathcal{S}_{\sigma,\rho}(\mathbb{A}^\times)$ *and* $\mathcal{S}_{\widetilde{\sigma},\rho}(\mathbb{A}^\times)$, *respectively, such that the* (σ,ρ)*-Poisson summation formula*

$$\mathcal{E}_{\sigma,\rho}(\phi)=\mathcal{E}_{\widetilde{\sigma},\rho}(\mathcal{F}_{\sigma,\rho,\psi}(\phi))$$

holds for $\phi\in\mathcal{S}_{\sigma,\rho}(\mathbb{A}^\times)$, *where* $\mathcal{S}_{\sigma,\rho}(\mathbb{A}^\times)$ *and* $\mathcal{F}_{\sigma,\rho,\psi}$ *are defined in Section 6B.*

It is expected that such Poisson summation formulae on GL_1 should be responsible for the Langlands conjecture on the global functional equation of automorphic L-functions associated with the pairs (σ,ρ). Variants of Conjecture 1.5 will be discussed in Section 7C and see Conjecture 7.4 for details. It is clear that Theorem 1.1 proves Conjecture 1.5 for the case when σ is an irreducible cuspidal automorphic representation π of $G(\mathbb{A})=\mathrm{GL}_n(\mathbb{A})$ and ρ is the standard representation of $G^\vee(\mathbb{C})=\mathrm{GL}_n(\mathbb{C})$ (Theorem 4.7). A variant of Theorem 4.7 (Theorem 1.1) is established in Theorem 7.3 when π is an irreducible square-integrable automorphic representation of $\mathrm{GL}_n(\mathbb{A})$, based on the characterization in Theorem 7.1 of the subspace $\mathcal{C}_c^\infty(k_\nu^\times)$ in $\mathcal{S}_{\pi_\nu}(k_\nu^\times)$ through the fiber integration.

It is important to mention that according to the definition of $\mathcal{S}_{\sigma,\rho}(\mathbb{A}^\times)$ and $\mathcal{F}_{\sigma,\rho,\psi}$ in (6-6) and (6-8), respectively, if the image of σ under the Langlands functorial transfer associated with ρ (if it exists) is an irreducible cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbb{A})$, then the nontrivial $k^\times$-invariant linear functionals $\mathcal{E}_{\sigma,\rho}$ and $\mathcal{E}_{\widetilde{\sigma},\rho}$ in Conjecture 1.5 can be taken to be

$$\mathcal{E}_{\sigma,\rho}(\phi) = \Theta_{\sigma,\rho}(1,\phi) \quad \text{and} \quad \mathcal{E}_{\widetilde{\sigma},\rho}(\phi) = \Theta_{\widetilde{\sigma},\rho}(1,\phi)$$

for any $\phi \in \mathcal{S}_{\sigma,\rho}(\mathbb{A}^\times)$ (see Corollary 6.3 for details). In this case, Conjecture 1.5 follows from Theorem 1.1 (Theorem 4.7). Therefore, Conjecture 1.5 is supported by various known cases of the global Langlands functoriality conjecture associated with $\rho : G^\vee(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C})$.

From the point of view of the global Langlands functoriality conjecture, it is important to extend Theorem 1.1 (Theorem 4.7) to more general irreducible automorphic representations of $\mathrm{GL}_n(\mathbb{A})$, which may yield new understanding of the nature of the both nontrivial $k^\times$-invariant linear functionals $\mathcal{E}_{\sigma,\rho}$ and $\mathcal{E}_{\widetilde{\sigma},\rho}$ in Conjecture 1.5. At this point, we would also like to bring the attention of the reader to the work of L. Lafforgue [27; 28] on the relations between the global Langlands functoriality conjecture and a certain nonlinear Poisson formula conjecture.

The ultimate goal in the global theory for $L(s,\sigma,\rho)$ is to prove Conjecture 1.5 without using the global Langlands functoriality. It is expected that Conjecture 1.5 can be proved directly for a split classical group G and the standard representation ρ of the complex dual group $G^\vee(\mathbb{C})$, by using the doubling method of I. Piatetski-Shapiro and S. Rallis in [14] and the recent work of L. Zhang and the authors in [26] and of J. Getz and B. Liu in [15].

As applications of the GL_1-theory for automorphic L-functions and the π-Poisson summation formulas, we are able to provide in Theorem 8.1 a spectral interpretation of the critical zeros of the standard L-functions $L(s,\pi\times\chi)$ for any irreducible cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbb{A})$ and idele class character χ of k. Theorem 8.1 is a reformulation of [40, Theorem 2] in the adelic framework of A. Connes in [11] and is an extension of [11, Theorem III.1] from the Hecke L-functions $L(s,\chi)$ to the automorphic L-functions $L(s,\pi\times\chi)$. In [24], Zhaolin Li and Dihua Jiang provide a new proof of the Voronoi summation formula for GL_n [20, Theorem 1] by means of Theorem 4.7 (Theorem 1.1), in other words, by means of the GL_1-reformulation of the Godement–Jacquet theory for the standard L-functions of GL_n. This GL_1-theory also proves in [24] the (GL_n,π)-version with the Godement–Jacquet kernels of the Clozel theorem [10, Theorem 1.1], which was proved by L. Clozel for $n=1$ and with the Tate kernels. In their upcoming work [35], Ngô and Luo use the ideas and the methods of this paper and of [25] to treat the local theory of the Braverman–Kazhdan–Ngô proposal for the torus case.

1B. ***Brief explanation of each section.*** In Section 2, we reformulate the local theory of Godement–Jacquet [16] in terms of the framework of the Braverman–Kazhdan–Ngô proposal. We take $F = k_\nu$ for every $\nu \in |k|$ and recall the local theory of the Mellin transforms, mainly from [21, Chapter I]. In general, it could be highly nontrivial to reformulate the known Rankin–Selberg theory for certain automorphic L-functions in terms of the framework of the Braverman–Kazhdan–Ngô proposal as indicated in [26]. The key point is that one has to figure out the invariant distribution Φ_ν on $G(k_\nu)$, which controls the local theory proposed by Braverman–Kazhdan in [6] and by Ngô in [37]. Even in the case of Godement–Jacquet, the candidate of such an invariant distribution $\Phi_{\mathrm{GJ},\nu}$ is expected to the experts, but there is no written document available. We provide the details in Section 2C and the results are given in Proposition 2.8.

In Section 3, we fully develop the local theory of harmonic analysis on GL_1 for the Langlands local L-factors $L(s,\pi)$ and γ-factors $\gamma(s,\pi,\psi)$, attached to any irreducible admissible representations π of $\mathrm{GL}_n(F)$. When F is non-Archimedean, we take π to be irreducible smooth representations of $\mathrm{GL}_n(F)$; and when F is Archimedean, we take π to be irreducible Casselman–Wallach representations of $\mathrm{GL}_n(F)$ [4; 9; 41; 46]. The set of equivalence classes of all such representations of $\mathrm{GL}_n(F)$ is denoted by $\Pi_F(\mathrm{GL}_n)$.

By Theorem 2.3, via the Mellin inversion, the local Godement–Jacquet L-functions (or L-factors) (or even general local Langlands L-functions) could be a GL_1-object, i.e., there exists a subspace of smooth functions $\mathcal{C}^\infty(F^\times)$, whose Mellin transform sees the corresponding local L-functions. One of the goals in this section is to recover such a subspace associated to a local Godement–Jacquet L-function $L(s,\pi)$ by means of the matrix coefficients of π. More precisely, we introduce the space of π-Schwartz functions on $F^\times$ for any $\pi \in \Pi_F(\mathrm{GL}_n)$, which is denoted by $\mathcal{S}_\pi(F^\times)$ (Definition 3.3). By Proposition 3.2, we have that $\mathcal{S}_\pi(F^\times) \subset \mathcal{C}^\infty(F^\times)$. The first local result is Theorem 3.4, which establishes the local theory of zeta integrals on GL_1 for the Langlands local L-function $L(s,\pi)$ for any $\pi \in \Pi_F(\mathrm{GL}_n)$. The relevant local functional equation and the properties of the π-Fourier operator (transform) $\mathcal{F}_{\pi,\psi}$ as defined in (3-17) is established in Theorem 3.10, the second local result.

We note that in [25], a further local theory has been developed so that the π-Fourier operator $\mathcal{F}_{\pi,\psi}$ can be expressed as a convolution operator with kernel functions $k_{\pi,\psi}$ for any $\pi \in \Pi_F(\mathrm{GL}_n)$ [25, Theorem 5.1]. In [24], such kernel functions are proved to be the normalized Bessel functions associated with π and a certain Weyl group element of GL_n. Hence, the π-Fourier operator $\mathcal{F}_{\pi,\psi}$ is a natural generalization of the classical Hankel transform.

In Section 4, we develop the global theory of harmonic analysis on GL_1 for the standard automorphic L-functions $L(s,\pi)$ associated with any irreducible cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbb{A})$. To do this, we consider any irreducible admissible representation $\pi = \bigotimes_{\nu\in|k|} \pi_\nu$ of $\mathrm{GL}_n(\mathbb{A})$, with $\pi_\nu \in \Pi_{k_\nu}(\mathrm{GL}_n)$, and introduce, for more general π, the π-Schwartz space $\mathcal{S}_\pi(\mathbb{A}^\times) = \bigotimes_{\nu\in|k|} \mathcal{S}_{\pi_\nu}(k_\nu^\times)$ in (4-1), where the restricted tensor product with respect to the basic function $\mathbb{L}_{\pi_\nu}$ (as defined in Theorem 3.4) is taken at almost all finite local places ν. The π-Fourier operator $\mathcal{F}_{\pi,\psi}(\phi) = \bigotimes_{\nu\in|k|} \mathcal{F}_{\pi_\nu,\psi_\nu}(\phi_\nu)$ is defined in (4-3), with $\phi = \bigotimes_\nu \phi_\nu \in \mathcal{S}_\pi(\mathbb{A}^\times)$. The main global result in this section is Theorem 4.7, which is a restatement of Theorem 1.1 and establishes the π-Poisson summation formula on GL_1 for any irreducible cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbb{A})$.

To understand the Poisson summation formulae in Conjecture 1.5, it is desirable to explore variants of Theorem 4.7 when the automorphic representation π may not be cuspidal, from the point of view of the global Langlands functoriality. In Section 5, we first show that for any irreducible admissible representation π of $\mathrm{GL}_n(\mathbb{A})$, which may not be automorphic, but satisfies Assumption 5.1, the π-theta functions

$$\Theta_\pi(x,\phi) = \sum_{\gamma\in k^\times} \phi(\gamma x) \quad \text{for } \phi \in \mathcal{S}_\pi(\mathbb{A}^\times)$$

converge absolutely and locally uniformly as functions in $x \in \mathbb{A}^\times$ (Theorem 5.4). Then we show that Assumption 5.1 holds for any automorphic representation π of $\mathrm{GL}_n(\mathbb{A})$ (Proposition 5.5). With Theorem 5.4, we are ready to explore a more general situation in order to formulate Conjecture 1.5 and its variant (Conjecture 7.4).

In Section 6, we consider any k-split reductive group G. In Section 6B, for any finite-dimensional representation ρ of the complex dual group $G^\vee(\mathbb{C})$, we define the relevant Schwartz spaces $\mathcal{S}_{\sigma,\rho}(\mathbb{A}^\times)$, called the (σ,ρ)-Schwartz space, in (6-6), and (σ,ρ)-Fourier operators $\mathcal{F}_{\sigma,\rho,\psi}$ in (6-8) for any irreducible cuspidal automorphic representation σ of $G(\mathbb{A})$, under the assumption (Assumption 6.1) that the local Langlands reciprocity map exists for G over all finite local places ν of k. We prove in such a generality the convergence properties of the (σ,ρ)-theta function $\Theta_{\sigma,\rho}(x,\phi)$ as defined in (1-3) for any $\phi \in \mathcal{S}_{\sigma,\rho}(\mathbb{A}^\times)$ and any $x \in \mathbb{A}^\times$ (Theorem 6.2, which contains Theorem 1.2 as a special case).

In Section 7, after we establish a new characterization of $\mathcal{C}_c^\infty(k_\nu^\times)$ as a subspace of $\mathcal{S}_{\pi_\nu}(k_\nu^\times)$ in Theorem 7.1 at all local places of k, we prove a variant of Theorem 4.7 when π is an irreducible square-integrable automorphic representation of $\mathrm{GL}_n(\mathbb{A})$ (Theorem 7.3). Finally we write down a variant of Conjecture 1.5 with more details in Conjecture 7.4.

In order to understand the Poisson summation formulae in Conjectures 1.5 and 7.4, we have to explore and develop harmonic analysis on GL_1 initiated by the (σ,ρ)-Fourier operator $\mathcal{F}_{\sigma,\rho,\psi}$ and the (σ,ρ)-Schwartz space $\mathcal{S}_{\sigma,\rho}(\mathbb{A}^\times)$, both locally and globally. We refer to [24; 25] for a further discussion of the local theory, while a further global theory remains to be developed in our future work.

In Section 8, as an application of the GL_1-harmonic analysis we developed beforehand, we provide a spectral interpretation of the critical zeros of the automorphic L-functions $L(s, \pi \times \chi)$ (Theorem 8.1) for any irreducible cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbb{A})$ and any character χ of the idele class group of k. It can be viewed as a reformulation of [40, Theorem 2] in the adelic framework of A. Connes in [11] and an extension of [11, Theorem III.1] from Hecke L-functions $L(s, \chi)$ to automorphic L-functions $L(s, \pi \times \chi)$. The proof uses a combination of arguments in [40], and those in [11], together with the results developed before Section 8. Further results along the line of [11] will be written in our forthcoming work.

2. Godement–Jacquet theory and reformulation

2A. ***Mellin transforms.*** We recall the local theory of Mellin transforms from the book of Igusa [21, Chapter I] and state them in a slightly more general situation in order to treat the case that meromorphic functions may have poles that are not real numbers. Since the proofs are almost the same, we omit the details.

Let F be a local field of characteristic zero. This means that it is either the complex field $\mathbb{C}$, the real field $\mathbb{R}$, or a finite extension of the p-adic field $\mathbb{Q}_p$ for some prime p.

When F is non-Archimedean, let $\mathfrak{o}_F$ be the ring of integers with maximal ideal $\mathfrak{p}_F$ and fix a uniformizer ϖ_F of $\mathfrak{p}_F$. Let $\mathfrak{o}_F/\mathfrak{p}_F = \kappa_F \simeq \mathbb{F}_q$. Fix the norm $|x|_F = q^{-\mathrm{ord}_F(x)}$ where $\mathrm{ord}_F : F \to \mathbb{Z}$ is the valuation on F such that $\mathrm{ord}_F(\varpi_F) = 1$. Fix the Haar measure d^+x on F so that $\mathrm{vol}(\mathrm{d}^+x, \mathfrak{o}_F) = 1$. Let $\psi = \psi_F$ be an additive character of F which is trivial on $\mathfrak{o}_F$ but nontrivial on $\varpi_F^{-1} \cdot \mathfrak{o}_F$. In particular the standard Fourier transform defined via ψ_F is self-dual w.r.t. d^+x. Similarly, fix a multiplicative Haar measure $\mathrm{d}^\times x$ on $F^\times$, which is normalized so that $\mathrm{vol}(\mathrm{d}^\times x, \mathfrak{o}_F^\times) = 1$. In particular $\mathrm{d}^\times x = (1/\zeta_F(1)) \cdot (\mathrm{d}^+x/|x|_F)$, where $\zeta_F(s)$ is the local Dedekind zeta factor attached to F.

When F is Archimedean, define on F the norm

$$|z|_F = \begin{cases} \text{absolute value of } z, & F = \mathbb{R}, \\ z\bar{z}, & F = \mathbb{C}. \end{cases}$$

Take the Haar measure d^+x on F that is the usual Lebesgue measure on F, and set

$$\mathrm{d}^\times x = \begin{cases} \frac{\mathrm{d}^+x}{2|x|_F}, & F = \mathbb{R}, \\ \frac{\mathrm{d}^+x}{2\pi|x|_F}, & F = \mathbb{C}, \end{cases}$$

the multiplicative Haar measures on $F^\times$. The additive character $\psi = \psi_F$ of F is chosen as

$$\psi_F(x) = \begin{cases} \exp(2\pi i x), & F = \mathbb{R}, \\ \exp(2\pi i(x + \bar{x})), & F = \mathbb{C}. \end{cases}$$

For convenience, define on F the norm

$$|\cdot| = \begin{cases} |\cdot|_F, & F \neq \mathbb{C}, \\ |\cdot|_F^{1/2}, & F = \mathbb{C}. \end{cases}$$

We denote by $\mathfrak{X}(F^\times)$ the set of all quasicharacters of $F^\times$. Define the topological group Ω_F to be $\{\pm 1\}$ if $F = \mathbb{R}$, $\mathbb{C}_1^\times$ if $F = \mathbb{C}$, and the unit group $\mathfrak{o}_F^\times$ if F is non-Archimedean. It is clear that any $\chi \in \mathfrak{X}(F^\times)$ can be written as

$$\chi(x) = \chi_u(x) = \chi_{u,\omega}(x) = |x|_F^u\, \omega(\mathrm{ac}(x)), \tag{2-1}$$

for any $x \in F^\times$, with $u \in \mathbb{C}$ and $\omega \in \Omega_F^\wedge$, the Pontryagin dual of Ω_F. Here $\mathrm{ac}(x) = x/|x|_F \in \mathfrak{o}_F^\times$ if F is non-Archimedean, and

$$\mathrm{ac}(x) = \begin{cases} \frac{x}{|x|_F} \in \{\pm 1\}, & F = \mathbb{R}, \\ \frac{x}{|x|} = \frac{x}{|x|_F^{1/2}} \in \mathbb{C}_1^\times, & F = \mathbb{C}. \end{cases} \tag{2-2}$$

It is clear that the unitary character ω of Ω_F is uniquely determined by $\chi \in \mathfrak{X}(F^\times)$, in particular, we have

$$\omega(\mathrm{ac}(x)) = \mathrm{ac}(x)^p, \tag{2-3}$$

with $p \in \{0, 1\}$ if $F = \mathbb{R}$ and $p \in \mathbb{Z}$ if $F = \mathbb{C}$. Hence, we may sometimes write $\chi = (u, \omega)$ and $\omega(x) = \omega(\mathrm{ac}(x))$ for $x \in F^\times$.

For any local field F of characteristic zero, following [21, Sections I.4 and I.5], we define the following two spaces of functions associated to the local field F.

Definition 2.1. Let $\mathfrak{F}(F^\times)$ be the space of complex-valued functions $\mathfrak{f}$ such that:

(1) $\mathfrak{f} \in \mathcal{C}^\infty(F^\times)$, the space of all smooth functions on $F^\times$.

(2) When F is non-Archimedean, $\mathfrak{f}(x) = 0$ for $|x|_F$ sufficiently large. When F is Archimedean, we define $\mathfrak{f}^{(n)} := \mathrm{d}^n\mathfrak{f}/\mathrm{d}x^n$ if $F = \mathbb{R}$, and $\mathfrak{f}^{(n)} = \mathfrak{f}^{(a+b)} := \partial^{a+b}\mathfrak{f}/(\partial^a x \partial^b \bar{x})$ if $F = \mathbb{C}$ and $n = a + b$. Then we have

$$\mathfrak{f}^{(n)}(x) = \mathrm{o}(|x|_F^\rho)$$

as $|x|_F \to \infty$ for any ρ and any $n = a + b \in \mathbb{Z}_{\geq 0}$ with $a, b \in \mathbb{Z}_{\geq 0}$.

(3) When F is Archimedean, there exists

- a sequence $\{m_k\}_{k=0}^\infty$ of positive integers,
- a sequence of smooth functions $\{a_{k,m}\}$ on $\{\pm 1\}$ if $F = \mathbb{R}$ and on $\mathbb{C}_1^\times$ if $F = \mathbb{C}$, parameterized by $m = 1, 2, \ldots, m_k$ and $k \in \mathbb{Z}_{\geq 0}$,
- a sequence $\{\lambda_k\}_{k=0}^\infty$ of complex numbers with $\{\mathrm{Re}(\lambda_k)\}_{k=0}^\infty$ a strictly increasing sequence of real numbers with no finite accumulation point and $\mathrm{Re}(\lambda_0) \geq \lambda \in \mathbb{R}$,

such that

$$\lim_{|x|_F\to 0}\left\{\mathfrak{f}(x)-\sum_{k=0}^{\infty}\sum_{m=1}^{m_k} a_{k,m}(\mathrm{ac}(x))|x|_F^{\lambda_k}(\ln|x|_F)^{m-1}\right\}=0.$$

The limit is termwise differentiable and uniform (even after termwise differentiation) in $\mathrm{ac}(x)$.

When F is non-Archimedean, one can take the sequence $\{\lambda_k\}$ to be a finite set Λ and the sequence $\{m_k\}$ to be a finite subset of $\mathbb{Z}_{\geq 0}$. The smooth functions $\{a_{k,m}(\mathrm{ac}(x))\}$ are on the unit group $\mathfrak{o}_F^\times$.

Since the topological group Ω is compact and abelian, we have the following Fourier expansion for the smooth functions $\{a_{k,m}(\mathrm{ac}(x))\}$ on Ω:

$$a_{k,m}(\mathrm{ac}(x))=\sum_{\omega\in\Omega^\wedge} a_{k,m,\omega}\,\omega(\mathrm{ac}(x)).$$

In the Archimedean case, we may write $a_{k,m,\omega}=a_{k,m,p}$ with $p\in\{0,1\}$ if $F=\mathbb{R}$ and $p\in\mathbb{Z}$ if $F=\mathbb{C}$.

Definition 2.2. With the same notation as in Definition 2.1, let $\mathcal{Z}(\mathfrak{X}(F^\times))$ be the space of complex-valued functions $\mathfrak{z}(\chi_{s,\omega})=\mathfrak{z}\big(|\cdot|_F^s\,\omega(\mathrm{ac}(\cdot))\big)$ on $\mathfrak{X}(F^\times)$ such that:

(1) $\mathfrak{z}(\chi_{s,\omega})$ is meromorphic on $\mathfrak{X}(F^\times)$ with poles at most for $s=-\lambda_j$ with λ_j belonging to the given set $\{\lambda_k\}_{k=0}^{\infty}$ if F is Archimedean; and belonging to the given finite set Λ if F is non-Archimedean.

(2) For any $k\geq 0$, the difference

$$\mathfrak{z}(\chi_{s,\omega})-\sum_{m=1}^{m_k}\frac{b_{k,m,\omega}}{(s+\lambda_k)^m}$$

is holomorphic for s in a small neighborhood of $-\lambda_k$ if F is Archimedean; and is a polynomial in $\mathbb{C}[q^s,q^{-s}]$ if F is non-Archimedean.

(3) When F is non-Archimedean, the function $\mathfrak{z}(\chi_{s,\omega})$ is identically zero for almost all characters $\omega\in\Omega^\wedge$ with $\Omega=\mathfrak{o}_F^\times$. When F is Archimedean, for every polynomial $P(s,p)$ in s, p with coefficients in $\mathbb{C}$, and every pair of real numbers $a<b$, the function $P(s,p)\mathfrak{z}(\chi_{s,\omega})$ is bounded when s belongs to the vertical strip

$$S_{a,b}=\{s\in\mathbb{C}\mid a\leq\mathrm{Re}(s)\leq b\}, \tag{2-4}$$

with neighborhoods of $-\lambda_0,-\lambda_1,\ldots$ removed therefrom. More precisely, there exists a constant c depending only on P, $\mathfrak{z}$, a, b, but neither on s nor on p, such that

$$|P(s,p)\mathfrak{z}(\chi_{s,\omega})|\leq c$$

when s runs in the vertical strip $S_{a,b}$ with small neighborhoods of $-\lambda_0, -\lambda_1, \dots$ removed.

The main results on the local theory of Mellin transforms established in [21, Chapter I] are as follows.

Theorem 2.3 (Mellin transforms). *There is a bijective linear correspondence $\mathcal{M} = \mathcal{M}_F$ between the space $\mathfrak{F}(F^\times)$ and the space $\mathcal{Z}(\mathfrak{X}(F^\times))$. More precisely, for $\mathfrak{f} \in \mathfrak{F}(F^\times)$,*

$$\mathcal{M}(\mathfrak{f})(\chi_{s,\omega}) = \int_{F^\times} \mathfrak{f}(x)\chi_{s,\omega}(x)\,\mathrm{d}^\times x$$

defines a holomorphic function on

$$\mathfrak{X}_{-\sigma_0}(F^\times) = \{\chi_{s,\omega}(\,\cdot\,) = |\cdot|_F^s\omega(\mathrm{ac}(\,\cdot\,)) \in \mathfrak{X}(F^\times) \mid \mathrm{Re}(s) > -\sigma_0\}$$

for some $\sigma_0 \in \mathbb{R}$, which has a meromorphic continuation to all characters $\chi_{s,\omega} \in \mathfrak{X}(F^\times)$ and belongs to $\mathcal{Z}(\mathfrak{X}(F^\times))$ after meromorphic continuation. Conversely, for $\mathfrak{z} \in \mathcal{Z}(\mathfrak{X}(F^\times))$ and $x \in F^\times$, the Mellin inverse transform $\mathcal{M}_F^{-1}(\mathfrak{z})(x)$ belongs to the space $\mathfrak{F}(F^\times)$. We have the identities

$$\mathcal{M}(\mathcal{M}^{-1}(\mathfrak{z})) = \mathfrak{z} \quad \text{and} \quad \mathcal{M}^{-1}(\mathcal{M}(\mathfrak{f})) = \mathfrak{f}$$

for any $\mathfrak{f} \in \mathfrak{F}(F^\times)$ and $\mathfrak{z} \in \mathcal{Z}(\mathfrak{X}(F^\times))$. Here the Mellin inverse transform is explicitly given as follows.

When F is Archimedean, the Mellin inverse transform $\mathcal{M}_F^{-1}(\mathfrak{z})(x)$ is given by

$$\mathcal{M}^{-1}(\mathfrak{z})(x) := \sum_{\omega\in\Omega_F^\wedge} \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \mathfrak{z}(\chi_{s,\omega})\chi_{s,\omega}(x)^{-1}\,\mathrm{d}s \tag{2-5}$$

with $\omega(\mathrm{ac}(x)) = \mathrm{ac}(x)^p$, which defines a function $\mathfrak{f}$ in $\mathfrak{F}(F^\times)$ independent of $\sigma > -\sigma_0$, and the coefficients $a_{k,m,p}$ and $b_{k,m,p}$ satisfy the relations

$$b_{k,m,p} = (-1)^{m-1}(m-1)!\cdot a_{k,m,-p}$$

for every $k \geq 0$, $m \geq 1$ with $p \in \{0,1\}$ if $F = \mathbb{R}$ and $p \in \mathbb{Z}$ if $F = \mathbb{C}$. The coefficients $a_{k,m,p}$ and $b_{k,m,p}$ satisfy the relations

$$b_{\lambda,m,\omega} = \sum_{j=m}^{m_\lambda} e_{j,m}(-\ln q)^{j-1}a_{\lambda,j,\omega^{-1}}$$

with $e_{j,m}$ defined by the following identity of polynomials in a formal unknown t:

$$t^{n-1} = \sum_{\ell=1}^{n} e_{n,\ell}\binom{t+\ell-1}{\ell-1}.$$

If F is non-Archimedean, the Mellin inverse transform $\mathcal{M}_F^{-1}(\mathfrak{z})(x)$ is given by

$$\mathcal{M}_F^{-1}(\mathfrak{z})(x) := \sum_{\omega\in\Omega^\wedge} \big(\mathrm{Res}_{z=0}(\mathfrak{z}(\chi_{s,\omega})|x|_F^{-s}q^s)\big)\omega(\mathrm{ac}(x))^{-1}, \tag{2-6}$$

which defines a function $\mathfrak{f}$ in $\mathfrak{F}(F^\times)$. Here $z = q^{-s}$ for abbreviation.

2B. ***Local theory of Godement–Jacquet.*** Let $G_n := \mathrm{GL}_n$ be the general linear group defined over F. Fix the following maximal (open if F is non-Archimedean) compact subgroup K of $G_n(F) = \mathrm{GL}_n(F)$:

$$K = \begin{cases} \mathrm{GL}_n(\mathfrak{o}_F), & F \text{ is non-Archimedean},\\ O(n), & F = \mathbb{R},\\ U(n), & F = \mathbb{C}. \end{cases} \tag{2-7}$$

Fix the Haar measure $\mathrm{d}g = \mathrm{d}^+g/|\det g|_F^n$ on $G_n(F)$ where d^+g is the measure induced from the standard additive measure on $M_n(F)$, the F-vector space of $n\times n$-matrices. In particular, $G_n(F)$ embeds into $M_n(F)$ in a standard way.

Let $\Pi_F(G_n)$ be the set of equivalence classes of irreducible smooth representations of $G_n(F)$ when F is non-Archimedean; and of irreducible Casselman–Wallach representations of $G_n(F)$ when F is Archimedean. Let $\mathcal{C}(\pi)$ be the space of smooth matrix coefficients attached to π.

Let $\mathcal{S}(M_n(F))$ be the space of the standard Schwartz–Bruhat functions on $M_n(F)$. The standard Fourier transform $\mathcal{F}_\psi$ acting on $\mathcal{S}(M_n(F))$ is defined as

$$\mathcal{F}_\psi(f)(x) = \int_{M_n(F)} \psi(\mathrm{tr}(xy))f(y)\,\mathrm{d}^+y, \tag{2-8}$$

where ψ is a nontrivial additive character of F. The standard Fourier transform $\mathcal{F}_\psi$ extends to a unitary operator on the space $L^2(M(F), \mathrm{d}^+x)$ and satisfies the identity

$$\mathcal{F}_\psi \circ \mathcal{F}_{\psi^{-1}} = \mathrm{Id}. \tag{2-9}$$

For any $\pi \in \Pi_F(G_n)$ and any quasicharacter $\chi \in \mathfrak{X}(F^\times)$, the local zeta integral of Godement and Jacquet is defined by

$$\mathcal{Z}(s, f, \varphi_\pi, \chi) = \int_{G_n(F)} f(g)\varphi_\pi(g)\chi(\det g)|\det g|_F^{s+(n-1)/2}\,\mathrm{d}g, \tag{2-10}$$

for any $f \in \mathcal{S}(M_n(F))$ and $\varphi_\pi \in \mathcal{C}(\pi)$. The following theorem contains the main results in the local theory of the Godement and Jacquet zeta integrals [16, Chapter I].

Theorem 2.4. *With the notation introduced above, the following statements hold for any $f \in \mathcal{S}(M_n(F))$ and $\varphi_\pi \in \mathcal{C}(\pi)$:*

(1) *The zeta integral* $\mathcal{Z}(s, f, \varphi_\pi, \chi)$ *defined in* (2-10) *is absolutely convergent for* $\mathrm{Re}(s)$ *sufficiently large and admits a meromorphic continuation to* $s \in \mathbb{C}$.

(2) $\mathcal{Z}(s, f, \varphi_\pi, \chi)$ *is a holomorphic multiple of the Langlands local* L*-function* $L(s, \pi \times \chi)$ *associated to* (π, χ) *and the standard embedding*

$$\mathrm{std} : \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C}).$$

When F *is non-Archimedean, the fractional ideal* $\mathcal{I}_{\pi,\chi}$ *that is generated by the local zeta integrals* $\mathcal{Z}(s, f, \varphi_\pi, \chi)$ *is of the form*

$$\mathcal{I}_{\pi,\chi} = \{\mathcal{Z}(s, f, \varphi_\pi, \chi) \mid f \in \mathcal{S}(M_n(F)),\ \varphi_\pi \in \mathcal{C}(\pi)\} = L(s, \pi \times \chi) \cdot \mathbb{C}[q^s, q^{-s}];$$

and when F *is Archimedean, the local zeta integrals* $\mathcal{Z}(s, f, \varphi_\pi, \chi)$, *with unitary characters* χ, *have the following property. Let* $S_{a,b}$ *be the vertical strip for any* $a < b$, *defined in* (2-4). *If* $P_\chi(s)$ *is a polynomial in* s *such that the product* $P_\chi(s)L(s, \pi \times \chi)$ *is bounded in the vertical strip* $S_{a,b}$, *then the product* $P_\chi(s)\mathcal{Z}(s, f, \varphi_\pi, \chi)$ *must be bounded in the same vertical strip* $S_{a,b}$.

(3) *The local functional equation*

$$\mathcal{Z}(1 - s, \mathcal{F}_\psi(f), \varphi_\pi^\vee, \chi^{-1}) = \gamma(s, \pi \times \chi, \psi) \cdot \mathcal{Z}(s, f, \varphi_\pi, \chi)$$

holds after meromorphic continuation, where the function $\varphi_\pi^\vee(g)$ *is defined as* $\varphi_\pi(g^{-1}) \in \mathcal{C}(\widetilde{\pi})$, *and* $\gamma(s, \pi \times \chi, \psi)$ *is the Langlands local gamma function associated to* (π, χ) *and* std.

(4) *When* F *is non-Archimedean and* π *is unramified, take* $f^\circ(g) = \mathbb{1}_{M_n(\mathfrak{o}_F)}(g)$ *to be the characteristic function of* $M_n(\mathfrak{o}_F)$ *and* $\varphi_\pi^\circ(g)$ *to be the zonal spherical function associated to* π. *Then the identity*

$$\mathcal{Z}(s, f^\circ, \varphi_\pi^\circ, \chi) = L(s, \pi \times \chi)$$

holds for any unramified characters χ *and all* $s \in \mathbb{C}$ *as meromorphic functions in* s.

For the statements of the current version of Theorem 2.4, we have some comments in order. When F is non-Archimedean, the theorem is [16, Theorem 3.3]. When F is Archimedean, the statements were established in [16] only for K-finite vectors f in $\mathcal{S}(M_n(F))$ and φ_π in $\mathcal{C}(\pi)$, and were extended to general smooth vectors in [23, Section 4.7] and also in [32, Theorem 3.10]. About the boundedness on vertical strips, we refer to [23, Section 4].

2C. ***Reformulation of Godement–Jacquet theory.*** The local theory of Godement–Jacquet zeta integrals can be reformulated within harmonic analysis and L^2-theory.

For $f \in \mathcal{S}(M_n(F))$, we define

$$\xi_f(g) := |\det g|_F^{n/2} \cdot f(g) \tag{2-11}$$

for $g \in G_n(F)$. Then we define the Schwartz space on $G_n(F)$ to be

$$\mathcal{S}_{\mathrm{std}}(G_n(F)) := \{\xi \in \mathcal{C}^\infty(G_n(F)) \mid |\det g|^{-n/2} \cdot \xi(g) \in \mathcal{S}(M_n(F))\}. \tag{2-12}$$

Proposition 2.5. *The Schwartz space $\mathcal{S}_{\mathrm{std}}(G_n(F))$ is a subspace of $L^2(G_n(F), \mathrm{d}g)$, which is the space of square-integrable functions on $G_n(F)$.*

Proof. For $\xi \in \mathcal{S}_{\mathrm{std}}(G_n(F))$, write $\xi(g) = |\det g|_F^{n/2} \cdot f(g)$ for some $f \in \mathcal{S}(M_n(F))$. We deduce the square-integrability of ξ by the computation

$$\int_{G_n(F)} \xi(g)\overline{\xi(g)}\,\mathrm{d}g = \int_{G_n(F)} f(g)\overline{f(g)}\,\mathrm{d}^+g = \int_{M_n(F)} f(g)\overline{f(g)}\,\mathrm{d}^+g < \infty. \qquad \square$$

Define the distribution kernel in the local theory of Godement–Jacquet to be

$$\Phi_{\mathrm{GJ}}(g) := \psi(\mathrm{tr}\, g) \cdot |\det g|_F^{n/2}, \tag{2-13}$$

where ψ is a nontrivial additive character of F. We compute the convolution $\Phi_{\mathrm{GJ}} * \xi^\vee$ for any $\xi \in \mathcal{S}_{\mathrm{std}}(G_n(F))$ with $\xi(g) = |\det g|_F^{n/2} \cdot f(g)$ for some $f \in \mathcal{S}(M_n(F))$:

$$\begin{aligned}
\Phi_{\mathrm{GJ}} * \xi^\vee(g) &= \int_{G_n(F)} \Phi_{\mathrm{GJ}}(h)\xi(g^{-1}h)\,\mathrm{d}h \\
&= \int_{G_n(F)} \psi(\mathrm{tr}\, h) \cdot |\det h|_F^{n/2} \cdot |\det g^{-1}h|_F^{n/2} \cdot f(g^{-1}h)\,\mathrm{d}h \\
&= \int_{G_n(F)} f(h)\psi(\mathrm{tr}\, gh) \cdot |\det gh|_F^{n/2} \cdot |\det h|_F^{n/2}\,\mathrm{d}h \\
&= |\det g|_F^{n/2} \int_{M_n(F)} f(h)\psi(\mathrm{tr}\, gh)\,\mathrm{d}^+h \\
&= |\det g|_F^{n/2} \cdot \mathcal{F}_\psi(f)(g).
\end{aligned}$$

Since $\mathcal{F}_\psi(f)(g)$ belongs to $\mathcal{S}(M_n(F))$, by definition, we must have that $\Phi_{\mathrm{GJ}} * \xi^\vee(g)$ belongs to $\mathcal{S}_{\mathrm{std}}(G_n(F))$. We define the Fourier operator $\mathcal{F}_{\mathrm{GJ}}$ in the Godement–Jacquet theory to be

$$\mathcal{F}_{\mathrm{GJ}}(\xi)(g) := (\Phi_{\mathrm{GJ}} * \xi^\vee)(g) \tag{2-14}$$

for any $\xi \in \mathcal{S}_{\mathrm{std}}(G_n(F))$.

Proposition 2.6. *For any $\xi \in \mathcal{S}_{\mathrm{std}}(G_n(F))$ with $\xi(g) = |\det g|_F^{n/2} \cdot f(g)$ for some $f \in \mathcal{S}(M_n(F))$, the Fourier operator $\mathcal{F}_{\mathrm{GJ}}$ on $\mathcal{S}_{\mathrm{std}}(G_n(F))$ and the classical Fourier transform $\mathcal{F}_\psi$ on $\mathcal{S}(M_n(F))$ are related by the identity*

$$\mathcal{F}_{\mathrm{GJ}}(\xi)(g) = (\Phi_{\mathrm{GJ}} * \xi^\vee)(g) = |\det g|_F^{n/2} \cdot \mathcal{F}_\psi(f)(g) = |\det g|_F^{n/2} \cdot \mathcal{F}_\psi\big(|\det(\cdot)|^{-n/2}\xi\big)(g).$$

For any $\xi \in \mathcal{S}_{\mathrm{std}}(G_n(F))$ with $\xi(g) = |\det g|_F^{n/2} \cdot f(g)$ for some $f \in \mathcal{S}(M_n(F))$, the zeta integral can be renormalized as

$$\begin{aligned} \mathcal{Z}(s, f, \varphi_\pi, \chi) &= \int_{G_n(F)} |\det g|_F^{n/2} f(g)\varphi_\pi(g)\chi(\det g)|\det g|_F^{s-1/2}\,\mathrm{d}g \\ &= \mathcal{Z}(s, \xi, \varphi_\pi, \chi). \end{aligned} \tag{2-15}$$

We compute the other side of the functional equation of the Godement–Jacquet zeta integrals:

$$\begin{aligned} \mathcal{Z}(1-s, \mathcal{F}_\psi(f), \varphi_\pi^\vee, \chi^{-1}) &= \int_{G_n(F)} |\det g|_F^{n/2} \mathcal{F}_\psi(f)(g)\varphi_\pi^\vee(g)\chi^{-1}(g)|\det g|_F^{1/2-s}\mathrm{d}g \\ &= \int_{G_n(F)} \mathcal{F}_{\mathrm{GJ}}(\xi)(g)\varphi_\pi^\vee(g)\chi^{-1}(g)|\det g|_F^{1/2-s}\mathrm{d}g \\ &= \mathcal{Z}(1-s, \mathcal{F}_{\mathrm{GJ}}(\xi), \varphi_\pi^\vee, \chi^{-1}). \end{aligned}$$

Proposition 2.7. *For any $\xi \in \mathcal{S}_{\mathrm{std}}(G_n(F))$, $\varphi_\pi \in \mathcal{C}(\pi)$, and $\chi \in \mathfrak{X}(F^\times)$, the zeta integral defined by*

$$\mathcal{Z}(s, \xi, \varphi_\pi, \chi) = \int_{G_n(F)} \xi(g)\varphi_\pi(g)\chi(\det g)|\det g|_F^{s-1/2}\,\mathrm{d}g$$

satisfies the functional equation

$$\mathcal{Z}(1-s, \mathcal{F}_{\mathrm{GJ}}(\xi), \varphi_\pi^\vee, \chi^{-1}) = \gamma(s, \pi \times \chi, \psi) \cdot \mathcal{Z}(s, \xi, \varphi_\pi, \chi),$$

which holds as meromorphic functions in s.

We are going to understand the Godement–Jacquet distribution Φ_{GJ} in terms of the Bernstein center of $G_n(F)$, when F is non-Archimedean. Recall from [3] that the Bernstein center $\mathfrak{Z}(G(F))$ of a reductive group $G(F)$ over a non-Archimedean local field F is defined to be the endomorphism ring of the identity functor on the category of smooth representations of $G(F)$. It turns out that the Bernstein center $\mathfrak{Z}(G(F))$ can be identified with the space of invariant and essentially compactly supported distributions on $G(F)$, where an invariant distribution Φ on $G(F)$ is called essentially compactly supported if $\Phi * \mathcal{C}_c^\infty(G(F)) \subset \mathcal{C}_c^\infty(G(F))$. It was proved in [3] that through the Plancherel transform, the Bernstein center $\mathfrak{Z}(G(F))$ can also be identified with the space of regular functions on the Bernstein variety $\Omega(G(F))$ attached to $G(F)$, where $\Omega(G(F))$ is an infinite disjoint union of finite-dimensional complex algebraic varieties.

Proposition 2.8. *Let F be a non-Archimedean local field of characteristic zero. For any $m \in \mathbb{Z}$, define*

$$G_n(F)_m = \{g \in G_n(F) \mid |\det g|_F = q_F^{-m}\}.$$

Let $\mathbb{1}_m := \mathbb{1}_{G_n(F)_m}$ be the characteristic function of $G_n(F)_m \subset G_n(F)$. Then the following statements hold:

(1) *The invariant distribution*

$$\Phi_{\mathrm{GJ},m}(g) := \Phi_{\mathrm{GJ}}(g)\mathbb{1}_{G_n(F)_m}(g) = \Phi_{\mathrm{GJ}}(g)\mathbb{1}_m(g) \tag{2-16}$$

lies in the Bernstein center $\mathfrak{Z}(G_n(F))$ *of* $G_n(F)$.

(2) *Let* $f_{\mathrm{GJ},m}$ *be the regular function on* $\Omega(G_n(F))$ *attached to* $\Phi_{\mathrm{GJ},m} \in \mathfrak{Z}(G_n(F))$. *For every* $\pi \in \Pi_F(G_n)$, $\chi \in \mathfrak{X}(F^\times)$, *and* $s \in \mathbb{C}$, *define*

$$\pi_{\chi_s} := \pi \otimes \chi_s = \pi \otimes \chi(\det)|\det|_F^s.$$

Then the Laurent series

$$f_{\mathrm{GJ}}(\pi_{\chi_s}) = \sum_{m\in\mathbb{Z}} f_{\mathrm{GJ},m}(\pi_{\chi_s})$$

is convergent for $\mathrm{Re}(s)$ *sufficiently large*, *with a meromorphic continuation to* $s \in \mathbb{C}$, *and*

$$f_{\mathrm{GJ}}(\pi_{\chi_s}) = \gamma\big(\tfrac{1}{2}, \widetilde{\pi_{\chi_s}}, \psi\big) = \gamma\big(\tfrac{1}{2} - s, \widetilde{\pi} \times \chi^{-1}, \psi\big).$$

Proof. For part (1), we have to show that the invariant distribution $\Phi_{\mathrm{GJ},m}(g)$ is essentially compact on $G_n(F)$. By a simple reduction, it suffices to show that, for any open compact subgroup $\mathcal{K}$ of $G_n(\mathfrak{o}_F)$, we have

$$\Phi_{\mathrm{GJ},m} * \mathbb{1}_{\mathcal{K}} \in \mathcal{C}_c^\infty(G_n(F)).$$

Since $\mathbb{1}_{\mathcal{K}}(g) = \mathbb{1}_{\mathcal{K}}(g^{-1}) = \mathbb{1}_{\mathcal{K}}^\vee(g)$, the convolution $\Phi_{\mathrm{GJ},m} * \mathbb{1}_{\mathcal{K}} = \Phi_{\mathrm{GJ},m} * \mathbb{1}_{\mathcal{K}}^\vee$ can be written as

$$\begin{aligned}
\Phi_{\mathrm{GJ},m} * \mathbb{1}_{\mathcal{K}}^\vee(g) &= \int_{G_n(F)} \Phi_{\mathrm{GJ},m}(h)\mathbb{1}_{\mathcal{K}}(g^{-1}h)\,\mathrm{d}h \\
&= \int_{G_n(F)} \Phi_{\mathrm{GJ},m}(gh)\mathbb{1}_{\mathcal{K}}(h)\,\mathrm{d}h \\
&= \int_{G_n(F)} \psi(\mathrm{tr}\, gh)|\det gh|^{n/2}\mathbb{1}_m(gh)\mathbb{1}_{\mathcal{K}}(h)\,\mathrm{d}h.
\end{aligned}$$

By definition, $\mathbb{1}_{\mathcal{K}}(h) \neq 0$ if and only if $|\det h|_F = 1$, and $\mathbb{1}_m(gh) \neq 0$ if and only if $|\det g|_F = q_F^{-m}$, i.e., $g \in G_n(F)_m$. This implies that $\mathbb{1}_m(gh) = \mathbb{1}_m(g)$. The last integral can be written as

$$q_F^{-(mn)/2}\mathbb{1}_m(g)\int_{G_n(F)} \psi(\mathrm{tr}(gh))\mathbb{1}_{\mathcal{K}}(h)\,\mathrm{d}h,$$

which can be written as

$$q_F^{-(mn)/2}\mathbb{1}_m(g)\int_{M_n(F)} \psi(\mathrm{tr}(gh))\mathbb{1}_{\mathcal{K}}(h)\,\mathrm{d}^+h = q_F^{-(mn)/2}\mathbb{1}_m(g)\mathcal{F}_\psi(\mathbb{1}_{\mathcal{K}})(g).$$

Hence, we obtain that

$$\Phi_{\mathrm{GJ},m} * \mathbb{1}_{\mathcal{K}}(g) = |\det g|_F^{n/2} \mathbb{1}_m(g) \mathcal{F}_\psi(\mathbb{1}_{\mathcal{K}})(g).$$

Since $\mathcal{F}_\psi(\mathbb{1}_{\mathcal{K}})(g) \in \mathcal{S}(M_n(F))$ and $|\det g|_F^{n/2}\mathbb{1}_m(g)$ is smooth on $M_n(F)$, we obtain that the convolution $\Phi_{m,\psi} * \mathbb{1}_{\mathcal{K}}(g)$ belongs to $\mathcal{C}_c^\infty(G_n(F))$ and the invariant distribution $\Phi_{\mathrm{GJ},m}(g)$ is essentially compact on $G_n(F)$.

For part (2), recall from [3] that the regular function $f_{\mathrm{GJ},m}$ attached to $\Phi_{\mathrm{GJ},m}$ is defined as follows. For any $\pi \in \Pi_F(G_n)$ and $v \in \pi$, there exists an open compact subgroup $\mathcal{K}$ of $G_n(F)$, such that $v \in \pi^{\mathcal{K}}$, the subspace of $\mathcal{K}$-fixed vectors in π. We may define an action of $\Phi_{\mathrm{GJ},m}$ on π via

$$\pi(\Phi_{\mathrm{GJ},m})(v) := \pi(\Phi_{\mathrm{GJ},m} * \mathfrak{c}_{\mathcal{K}})(v), \tag{2-17}$$

where $\mathfrak{c}_{\mathcal{K}} := \mathrm{vol}(\mathcal{K})^{-1}\mathbb{1}_{\mathcal{K}}$ is the normalized characteristic function of $\mathcal{K}$. Since $\Phi_{\mathrm{GJ},m} * \mathfrak{c}_{\mathcal{K}}$ lies in $\mathcal{C}_c^\infty(G_n(F))$, the right-hand side is well defined, and so is the left-hand side. It is clear that the action defined in (2-17) does not depend on the choice of such an open compact subgroup $\mathcal{K}$. By Schur's lemma, there exists a constant $f_{\mathrm{GJ},m}(\pi)$, depending on π, such that

$$\pi(\Phi_{\mathrm{GJ},m}) = f_{\mathrm{GJ},m}(\pi) \cdot \mathrm{Id}_\pi. \tag{2-18}$$

For each $m \in \mathbb{Z}$, we define, for any $\xi \in \mathcal{C}_c^\infty(G_n(F))$,

$$\mathcal{F}_{\mathrm{GJ},m}(\xi)(g) := (\Phi_{\mathrm{GJ},m} * \xi^\vee)(g) = \int_{G_n(F)} \Phi_{\mathrm{GJ},m}(h)\xi(g^{-1}h)\,\mathrm{d}h. \tag{2-19}$$

In order to include the quasicharacters $\chi \in \mathfrak{X}(F^\times)$ in the gamma function, we write

$$\varphi_{\pi[\chi]}(g) := \varphi_\pi[\chi](g) := \varphi_\pi(g)\chi(\det g) = (\chi(g)\pi(g)v, \tilde{v}), \tag{2-20}$$

with $v \in V_\pi$ and $\tilde{v} \in V_{\tilde{\pi}}$, which is a matrix coefficient of π twisted by χ. We may denote the space of such twisted matrix coefficients of π by $\mathcal{C}(\pi[\chi])$. It is clear that we have

$$\mathcal{Z}(s, \xi, \varphi_{\pi[\chi]}) = \mathcal{Z}(s, \xi, \varphi_\pi, \chi).$$

For each $m \in \mathbb{Z}$, $\varphi_{\pi[\chi]} \in \mathcal{C}(\pi[\chi])$, and $\chi \in \mathfrak{X}(F^\times)$, consider the zeta function of Godement–Jacquet, with $\mathcal{F}_{\mathrm{GJ},m}(\xi)$ defined as in (2-19),

$$\mathcal{Z}(1-s, \mathcal{F}_{\mathrm{GJ},m}(\xi), \varphi^\vee_{\pi[\chi]}) = \mathcal{Z}(1-s, \Phi_{\mathrm{GJ},m} * \xi^\vee, \varphi^\vee_{\pi[\chi]}). \tag{2-21}$$

By part (1) as proved above, we obtain that $\Phi_{\mathrm{GJ},m} * \xi^\vee \in \mathcal{C}_c^\infty(G_n(F))$ for any $\xi \in \mathcal{C}_c^\infty(G_n(F))$. Hence, the integral in (2-21) is absolutely convergent for any $s \in \mathbb{C}$ when $\xi \in \mathcal{C}_c^\infty(G_n(F))$.

We write the right-hand side of (2-21) as

$$\int_{G_n(F)} \Phi_{\mathrm{GJ},m} * \xi^\vee(g)\varphi_\pi(g^{-1})\chi^{-1}(\det g)|\det g|_F^{1/2-s}\,\mathrm{d}g, \tag{2-22}$$

which is equal to

$$\begin{aligned}\int_{G_n(F)} \Phi_{\mathrm{GJ},m} * \xi^\vee(g)(v, \widetilde{\pi}(g)\tilde{v})\chi_{s-1/2}(\det g)^{-1}\,\mathrm{d}g \\ = \Bigl(v, \int_{G_n(F)} \Phi_{\mathrm{GJ},m} * \xi^\vee(g)\widetilde{\pi_{\chi_{s-1/2}}}(g)\tilde{v}\,\mathrm{d}g\Bigr).\end{aligned} \tag{2-23}$$

It is clear that

$$\begin{aligned}\int_{G_n(F)} \Phi_{\mathrm{GJ},m} * \xi^\vee(g)\widetilde{\pi_{\chi_{s-1/2}}}(g)\tilde{v}\,\mathrm{d}g &= \widetilde{\pi_{\chi_{s-1/2}}}(\Phi_{\mathrm{GJ},m} * \xi^\vee)\tilde{v} \\ &= \widetilde{\pi_{\chi_{s-1/2}}}(\Phi_{\mathrm{GJ},m})(\widetilde{\pi_{\chi_{s-1/2}}}(\xi^\vee)\tilde{v}).\end{aligned}$$

Since $\xi^\vee$ belongs to $\mathcal{C}_c^\infty(G_n(F))$, the vector $\widetilde{\pi_{\chi_{s-1/2}}}(\xi^\vee)\tilde{v}$ belongs to the space of $\widetilde{\pi_{\chi_{s-1/2}}}$. By definition, we have

$$\widetilde{\pi_{\chi_{s-1/2}}}(\Phi_{\mathrm{GJ},m}) = f_{\mathrm{GJ},m}(\widetilde{\pi_{\chi_{s-1/2}}}) \cdot I_{\widetilde{\pi_{\chi_{s-1/2}}}}. \tag{2-24}$$

Hence, we can write the right-hand side of (2-23) as

$$\Bigl(v, \int_{G_n(F)} \Phi_{\mathrm{GJ},m} * \xi^\vee(g)\widetilde{\pi_{\chi_{s-1/2}}}(g)\tilde{v}\,\mathrm{d}g\Bigr) = f_{\mathrm{GJ},m}(\widetilde{\pi_{\chi_{s-1/2}}}) \cdot (v, \widetilde{\pi_{\chi_{s-1/2}}}(\xi^\vee)\tilde{v}).$$

Next we compute the twisted coefficient $(v, \widetilde{\pi_{\chi_{s-1/2}}}(\xi^\vee)\tilde{v})$ on the right-hand side of the above equation as

$$\begin{aligned}&(v, \widetilde{\pi_{\chi_{s-1/2}}}(\xi^\vee)\tilde{v}) \\ &\quad= \int_{G_n(F)} \xi^\vee(h)(v, \widetilde{\pi_{\chi_{s-1/2}}}(h)\tilde{v})\,\mathrm{d}h = \int_{G_n(F)} \xi(h^{-1})(\pi_{\chi_{s-1/2}}(h^{-1})v, \tilde{v})\,\mathrm{d}h \\ &\quad= \int_{G_n(F)} \xi(h)(\pi_{\chi_{s-1/2}}(h)v, \tilde{v})\,\mathrm{d}h = \int_{G_n(F)} \xi(h)\varphi_{\pi[\chi]}(h)|\det h|^{s-1/2}\,\mathrm{d}h \\ &\quad= \mathcal{Z}(s, \xi, \varphi_{\pi[\chi]}).\end{aligned}$$

Hence, we obtain the functional equation

$$\mathcal{Z}(1-s, \mathcal{F}_{\mathrm{GJ},m}(\xi), \varphi^\vee_{\pi[\chi]}) = f_{\mathrm{GJ},m}(\widetilde{\pi_{\chi_{s-1/2}}}) \cdot \mathcal{Z}(s, \xi, \varphi_{\pi[\chi]}) \tag{2-25}$$

for any $\xi \in \mathcal{C}_c^\infty(G_n(F))$, $\varphi_\pi \in \mathcal{C}(\pi)$ and $\chi \in \mathfrak{X}(F^\times)$.

Theorem 2.4 implies that, when $\mathrm{Re}(s)$ is sufficiently small, the zeta integral $\mathcal{Z}(1-s, \mathcal{F}_{\mathrm{GJ}}(\xi), \varphi^\vee_{\pi[\chi]})$ converges absolutely for any $\xi \in \mathcal{C}_c^\infty(G_n(F))$, any $\varphi_\pi \in$

$\mathcal{C}(\pi)$ and any unitary character $\chi \in \mathfrak{X}(F^\times)$. We write it as

$$\begin{aligned}\mathcal{Z}(1-s, \mathcal{F}_{\mathrm{GJ}}(\xi), \varphi^\vee_{\pi[\chi]}) \\ = \sum_{m\in\mathbb{Z}} \mathcal{Z}(1-s, \mathcal{F}_{\mathrm{GJ},m}(\xi), \varphi^\vee_{\pi[\chi]}) = \mathcal{Z}(s, \xi, \varphi_{\pi[\chi]}) \cdot \sum_{m\in\mathbb{Z}} f_{\mathrm{GJ},m}(\widetilde{\pi_{\chi_{s-1/2}}}).\end{aligned}$$

By comparing with the right-hand side of the functional equation in Theorem 2.4, we obtain that, whenever $\mathrm{Re}(s)$ is sufficiently small,

$$f_{\mathrm{GJ}}(\widetilde{\pi_{\chi_{s-1/2}}}) = \sum_{m\in\mathbb{Z}} f_{\mathrm{GJ},m}(\widetilde{\pi_{\chi_{s-1/2}}}) = \gamma(s, \pi\otimes\chi, \psi) = \gamma(s, \pi_\chi, \psi). \tag{2-26}$$

By changing $s \to s + \frac{1}{2}$, we get

$$f_{\mathrm{GJ}}(\widetilde{\pi_{\chi_s}}) = \gamma\big(s+\tfrac{1}{2}, \pi_\chi, \psi\big) = \gamma\big(\tfrac{1}{2}, \pi_{\chi_s}, \psi\big).$$

By taking the contragredient of π_{χ_s}, we obtain that

$$f_{\mathrm{GJ}}(\pi_{\chi_s}) = \gamma\big(\tfrac{1}{2}, \widetilde{\pi_{\chi_s}}, \psi\big) = \gamma\big(\tfrac{1}{2}-s, \widetilde{\pi}\times\chi^{-1}, \psi\big).$$

This finishes the proof of part (2). □

3. π-Schwartz functions and Fourier operators

3A. ***Two spaces associated to π.*** For any $\pi \in \Pi_F(G_n)$, we are going to define two spaces associated to π: $\mathcal{L}_\pi(\mathfrak{X}(F^\times))$ and $\mathcal{S}_\pi(F^\times)$.

The space $\mathcal{L}_\pi = \mathcal{L}_\pi(\mathfrak{X}(F^\times))$ consists of $\mathbb{C}$-valued meromorphic functions $\mathfrak{z}(\chi)$ on $\mathfrak{X}(F^\times)$ that satisfy the following conditions:

(1) $\mathfrak{z}(\chi_{s,\omega})$ is a holomorphic multiple of the standard local L-function $L(s, \pi\times\omega)$ with $\chi_{s,\omega}(x) = |x|_F^s\omega(\mathrm{ac}(x))$.

(2) If F is non-Archimedean, $\mathfrak{z}(\chi_{s,\omega})$ is nonzero for finitely many $\omega \in \Omega^\wedge$, and for each $\omega\in\Omega^\wedge$, $\mathfrak{z}(\chi_{s,\omega}) \in L(s,\pi\times\omega)\cdot\mathbb{C}[q^s, q^{-s}]$.

(3) If F is Archimedean, for any polynomial $P(\chi_{s,\omega}) = P_\omega(s)$, if the function $P(\chi_{s,\omega})L(s,\pi\times\omega)$ is holomorphic in any vertical strip $S_{a,b}$ as in (2-4), with small neighborhoods at the possible poles of the L-function $L(s,\pi\times\omega)$ removed, then for any $\mathfrak{z}(\chi_{s,\omega}) \in \mathcal{L}_\pi$, the product $P(\chi_{s,\omega})\mathfrak{z}(\chi_{s,\omega})$ is bounded in the same strip $S_{a,b}$, with small neighborhoods at the possible poles of the L-function $L(s,\pi\times\omega)$ removed.

From part (3), we define a seminorm to be

$$\mu_{a,b:P}(\mathfrak{z}) := \sup_{a\le\mathrm{Re}(s)\le b} |P(\chi_{s,\omega})\cdot\mathfrak{z}(\chi_{s,\omega})|.$$

Then the space $\mathcal{L}_\pi$ is complete under the topology that is defined by the family of seminorms $\mu_{a,b:P}$ for all possible choice of data a, b; P as in part (3) [23, Section 4].

Proposition 3.1. *For any $\pi \in \Pi_F(G_n)$, the space $\mathcal{L}_\pi$ is a subspace of $\mathcal{Z}(\mathfrak{X}(F^\times))$ as defined in Definition 2.2.*

Proof. When F is non-Archimedean, the statement is a consequence of Theorem 2.4. We would like to focus on the case when F is Archimedean. In this case, it suffices to estimate the boundedness condition. To do so, recall the classical Stirling formula (see [23, p. 81], for instance)

$$\Gamma(x+iy) \sim (2\pi)^{1/2}|y|^{x-1/2}e^{-(\pi/2)|y|} \tag{3-1}$$

for x fixed and $|y| \to \infty$.

Consider the Archimedean local L-functions $L(s, \pi \times \omega) = L(s, \pi \times \mathrm{ac}(\,\cdot\,)^p)$, which can be explicitly expressed in terms of classical Γ-functions with the local Langlands parameter of π. For instance, from [16, Section 8], there exists a finite family of pairs $\{(l_i, u_i)\}_{i=1}^t$ with

$$u_i \in \mathbb{C}, \quad l_i \in \begin{cases} \mathbb{Z}/2\mathbb{Z} \simeq \{0, 1\}, & F = \mathbb{R}, \\ \mathbb{Z}, & F = \mathbb{C}, \end{cases}$$

such that in the fixed bounded vertical strip

$$S_{a,b} = \{s \in \mathbb{C} \mid a \le \mathrm{Re}(s) \le b\},$$

up to a bounded factor in $S_{a,b}$, we have

$$L(s, \pi \times \mathrm{ac}(\,\cdot\,)^p) \sim \begin{cases} \prod_{i=1}^t \Gamma\big(\frac{s+u_i+l_i+p}{2}\big), & F = \mathbb{R}, \\ \prod_{i=1}^t \Gamma\big(s+u_i+\frac{|l_i+p|}{2}\big), & F = \mathbb{C}, \end{cases}$$

with $p \in \mathbb{Z}/2\mathbb{Z} \simeq \{0, 1\}$ if $F = \mathbb{R}$; and $p \in \mathbb{Z}$ if $F = \mathbb{C}$. Here $l_i + p$ is understood to be zero if both l_i and p are equal to 1 when $F = \mathbb{R}$.

It follows from the classical Stirling formula in (3-1), in particular the exponential decay of $\Gamma(x+iy)$ along the imaginary axis, for any polynomial $P_\omega(s) = P(s) \in \mathbb{C}[s]$ when $F = \mathbb{R}$, and $P_\omega(s) = P(s, p) \in \mathbb{C}[s, p]$, the product $P(s, p)L(s, \pi \times \mathrm{ac}(\,\cdot\,)^p)$ is bounded in vertical strip $S_{a,b}$ with small neighborhoods at the possible poles removed. Hence, from the definition of the space $\mathcal{L}_\pi(\mathfrak{X}(F^\times))$, for any $\mathfrak{z}(\chi_{s,\omega}) \in \mathcal{L}_\pi(\mathfrak{X}(F^\times))$, the product $P(s, p)\mathfrak{z}(\chi)$, with $\chi(x) = |x|_F^s \mathrm{ac}(x)^p$ is bounded in vertical strip $S_{a,b}$ with small neighborhoods at the possible poles of the L-function $L(s, \pi \times \mathrm{ac}(\,\cdot\,)^p)$ removed. Therefore, we obtain that the space $\mathcal{L}_\pi = \mathcal{L}_\pi(\mathfrak{X}(F^\times))$ is contained in the space $\mathcal{Z}(\mathfrak{X}(F^\times))$, as defined in Definition 2.2. □

For any $\pi \in \Pi_F(G_n)$, we define (Definition 3.3) the π-Schwartz space $\mathcal{S}_\pi(F^\times) \subset \mathcal{C}^\infty(F^\times)$ attached to π, by using the theory of local zeta integrals of Godement–Jacquet, and prove that

$$\mathcal{S}_\pi(F^\times) = \mathcal{M}^{-1}(\mathcal{L}_\pi) \subset \mathcal{C}^\infty(F^\times) \tag{3-2}$$

by Theorems 2.3 and 2.4.

Consider the determinant map

$$\det = \det_F : G_n(F) = \mathrm{GL}_n(F) \to F^\times. \tag{3-3}$$

It is clear that the kernel ker(det) equals $\mathrm{SL}_n(F)$. For each $x \in F^\times$, the fiber of the determinant map det is

$$G_n(F)_x := \{g \in G_n(F) \mid \det g = x\}. \tag{3-4}$$

It is clear that each fiber $G_n(F)_x$ is an $\mathrm{SL}_n(F)$-torsor. Hence, one has the $\mathrm{SL}_n(F)$-invariant measure $\mathrm{d}_x g$ that is induced from the (normalized) Haar measure $\mathrm{d}_1 g$ on $\mathrm{SL}_n(F)$.

For $\xi \in \mathcal{S}_{\mathrm{std}}(G_n(F))$ as defined in (2-12), $\varphi_\pi \in \mathcal{C}(\pi)$, and $\chi \in \mathfrak{X}(F^\times)$, the local zeta integral of Godement and Jacquet, as normalized in (2-15), can be written as

$$\mathcal{Z}(s, \xi, \varphi_\pi, \chi) = \int_{F^\times} \Big(\int_{G_n(F)_x} \xi(g)\varphi_\pi(g)\,\mathrm{d}_x g \Big) \chi(x)|x|_F^{s-1/2}\,\mathrm{d}^\times x. \tag{3-5}$$

By part (1) of Theorem 2.4, the local zeta integral converges absolutely for $\mathrm{Re}(s)$ large. Hence, the inner integral of (3-5) satisfies

$$\phi_{\xi,\varphi_\pi}(x) := \int_{G_n(F)_x} \xi(g)\varphi_\pi(g)\,\mathrm{d}_x g = |x|_F^{n/2} \int_{G_n(F)_x} f(g)\varphi_\pi(g)\,\mathrm{d}_x g, \tag{3-6}$$

if $\xi(g) = |\det g|^{n/2} \cdot f(g)$ for some $f \in \mathcal{S}(M_n(F))$, is absolutely convergent for almost all $x \in F^\times$ and defines the fiber integration along the fibration in (3-3).

Proposition 3.2. *For $\xi \in \mathcal{S}_{\mathrm{std}}(G_n(F))$ and $\varphi_\pi \in \mathcal{C}(\pi)$, the fiber integration in* (3-6) *that defines the function $\phi_{\xi,\varphi_\pi}(x)$ is absolutely convergent for all $x \in F^\times$, and the function $\phi_{\xi,\varphi_\pi}(x)$ is smooth over $F^\times$.*

Proof. It is enough to show the proposition for the integral

$$\int_{G_n(F)_x} f(g)\varphi_\pi(g)\,\mathrm{d}_x g \tag{3-7}$$

with any $f \in \mathcal{S}(M_n(F))$ and $\varphi_\pi \in \mathcal{C}(\pi)$. In this case, the product $f \cdot \varphi_\pi$ is smooth on $G_n(F)$. Since the fiber $G_n(F)_x$ for any $x \in F^\times$ is closed in $G_n(F)$ and in $M_n(F)$, the restriction of f to the fiber $G_n(F)_x$ is a Schwartz function on $G_n(F)_x$ (see [5] for F non-Archimedean and [1, Theorem 4.6.1] for F Archimedean).

When F is non-Archimedean, any $\varphi_\pi(g) \in \mathcal{C}(\pi)$ is locally constant (smooth) on $G_n(F)$, and hence is smooth on the fiber $G_n(F)_x$. This implies that the restriction of $f \cdot \varphi_\pi$ is locally constant and compactly supported on the fiber $G_n(F)_x$. Hence, the integral in (3-7) is absolutely convergent for all $x \in F^\times$, and defines a smooth function in x over $F^\times$.

When F is Archimedean, since π is a Casselman–Wallach representation of $G_n(F)$, the matrix coefficient φ_π has at most polynomial growth on $G_n(F)$ [45, Theorem 4.3.5], as well as on the fiber $G_n(F)_x$. This implies that the restriction of

$f\cdot\varphi_\pi$ is a Schwartz function on the fiber $G_n(F)_x$ ([1, Definition 4.1.1]). Thus the integral in (3-7) is absolutely convergent for all $x\in F^\times$. Now we write the integral in (3-7) as

$$\int_{G_n(F)_x} f(g)\varphi_\pi(g)\,\mathrm{d}_x g=\int_{\mathrm{SL}_n(F)} f(t_1(x)g)\varphi_\pi(t_1(x)g)\,\mathrm{d}_1 g, \tag{3-8}$$

where $t_1(x)=\mathrm{diag}(x,1,\dots,1)\in G_n(F)$ and $\mathrm{d}_1 g$ is the Haar measure of $\mathrm{SL}_n(F)$. It is clear that the absolute convergence of the integral in (3-8) is uniform when x runs in any compact subset of $F^\times$. Hence, the integral in (3-7) defines a smooth function in x over $F^\times$. □

For $\xi\in\mathcal{S}_{\mathrm{std}}(G_n(F))$ and $\varphi_\pi\in\mathcal{C}(\pi)$, the function $\phi_{\xi,\varphi_\pi}(x)$ given in Proposition 3.2 via the fiber integration (3-6) is called a *π-Schwartz function* on $F^\times$ associated to the pair (ξ,φ_π). Here is the definition of π-Schwartz space.

Definition 3.3 (π-Schwartz space). For any $\pi\in\Pi_F(G_n)$, the space of π-Schwartz functions is defined by

$$\mathcal{S}_\pi(F^\times)=\mathrm{Span}\{\phi_{\xi,\varphi_\pi}\in\mathcal{C}^\infty(F^\times)\mid \xi\in\mathcal{S}_{\mathrm{std}}(G_n(F)),\ \varphi_\pi\in\mathcal{C}(\pi)\},$$

where the π-Schwartz function ϕ_{ξ,φ_π} associated to a pair (ξ,φ_π) is defined in (3-6).

For any $\phi\in\mathcal{S}_\pi(F^\times)$ and a quasicharacter $\chi\in\mathfrak{X}(F^\times)$, define a GL_1 zeta integral $\mathcal{Z}(s,\phi,\chi)$ associated to the pair (ϕ,χ) to be

$$\mathcal{Z}(s,\phi,\chi)=\int_{F^\times}\phi(x)\chi(x)|x|_F^{s-1/2}\,\mathrm{d}^\times x. \tag{3-9}$$

When $\phi=\phi_{\xi,\varphi_\pi}$ for some $\xi\in\mathcal{S}_{\mathrm{std}}(G_n(F))$ and $\varphi_\pi\in\mathcal{C}(\pi)$, from Theorem 2.4, we have the identity of local zeta integrals

$$\mathcal{Z}(s,\phi,\chi)=\mathcal{Z}(s,\xi,\varphi_\pi,\chi), \tag{3-10}$$

which holds for $\mathrm{Re}(s)$ sufficiently large and then for all $s\in\mathbb{C}$ by meromorphic continuation. Therefore, Theorem 2.4 can be restated for the GL_1 zeta integrals $\mathcal{Z}(s,\phi,\chi)$.

Theorem 3.4 (GL_1 zeta integrals). *The* GL_1 *zeta integral* $\mathcal{Z}(s,\phi,\chi)$ *as defined in* (3-9) *for any* $\phi\in\mathcal{S}_\pi(F^\times)$ *and any quasicharacter* $\chi\in\mathfrak{X}(F^\times)$ *enjoys the properties*:

(1) *The zeta integral* $\mathcal{Z}(s,\phi,\chi)$ *is absolutely convergent for* $\mathrm{Re}(s)$ *sufficiently large, and admits a meromorphic continuation to* $s\in\mathbb{C}$.

(2) *The zeta integral* $\mathcal{Z}(s,\phi,\chi)$ *is a holomorphic multiple of the Langlands local* L*-function* $L(s,\pi\times\chi)$ *associated to* (π,χ) *and the standard embedding*

$$\mathrm{std}:\mathrm{GL}_n(\mathbb{C})\times\mathrm{GL}_1(\mathbb{C})\to\mathrm{GL}_n(\mathbb{C}).$$

When F is non-Archimedean, the fractional ideal generated by the local zeta integrals $\mathcal{Z}(s,\phi,\chi)$ is of the form

$$\{\mathcal{Z}(s,\phi,\chi)\mid \phi\in\mathcal{S}_\pi(F^\times)\}=L(s,\pi\times\chi)\cdot\mathbb{C}[q^s,q^{-s}];$$

and when F is Archimedean, the GL_1 *zeta integrals $\mathcal{Z}(s,\phi,\chi)$, with unitary characters χ, have the following property. Let $S_{a,b}$ be the vertical strip for any $a<b$, as defined in* (2-4)*. If $P_\chi(s)$ is a polynomial in s such that the product $P_\chi(s)L(s,\pi\times\chi)$ is bounded in the vertical strip $S_{a,b}$, with small neighborhoods at the possible poles of the L-function $L(s,\pi\times\chi)$ removed, then the product $P_\chi(s)\mathcal{Z}(s,\phi,\chi)$ must be bounded in the same vertical strip $S_{a,b}$, with small neighborhoods at the possible poles of the L-function $L(s,\pi\times\chi)$ removed.*

(3) *When F is non-Archimedean, and π is unramified, define*

$$\mathbb{L}_\pi(x):=\phi_{\xi^\circ,\varphi^\circ_\pi}(x),$$

where $\xi^\circ(g)=|\det g|^{n/2}\mathbb{1}_{M_n(\mathfrak{o}_F)}(g)$, with $\mathbb{1}_{M_n(\mathfrak{o}_F)}(g)$ being the characteristic function of $M_n(\mathfrak{o}_F)$, and $\varphi^\circ_\pi(g)$ is the zonal spherical function associated to π. Then the identity

$$\mathcal{Z}(s,\mathbb{L}_\pi,\chi)=L(s,\pi\times\chi)$$

holds for any unramified characters χ and all $s\in\mathbb{C}$ as meromorphic functions in s.

We are going to discuss the relation between the π-Schwartz functions and the square-integrable functions in $L^2(F^\times,\mathrm{d}^\times x)$.

Proposition 3.5. *For any $\pi\in\Pi_F(G_n)$, there exists a real number α_π such that for any $\phi\in\mathcal{S}_\pi(F^\times)$ and for any $\kappa\ge\alpha_\pi+n/2$, the function $|x|_F^\kappa\phi(x)$ belongs to the space $L^2(F^\times,\mathrm{d}^\times x)$ of square-integrable functions on $F^\times$.*

Proof. For any $\alpha_0\in\mathbb{R}$, we consider the following inner product of the function $|x|^{\alpha_0/2}\phi(x)$ for any $\phi(x)\in\mathcal{S}_\pi(F^\times)$. We write $\phi=\phi_{\xi,\varphi_\pi}$ for some $\xi\in\mathcal{S}_{\mathrm{std}}(G_n(F))$ and $\varphi_\pi\in\mathcal{C}(\pi)$ and write $\xi(g)=|\det g|_F^{n/2}f(g)$ with $f\in\mathcal{S}(M_n(F))$. Then

$$\begin{aligned}\text{(3-11)}\quad &\int_{F^\times}\phi(x)\overline{\phi(x)}|x|_F^{\alpha_0}\,\mathrm{d}^\times x\\ &=\int_{F^\times}|x|_F^{\alpha_0+n}\,\mathrm{d}^\times x\int_{\det g_1=\det g_2=x}f(g_1)\varphi_\pi(g_1)\overline{f(g_2)}\overline{\varphi_\pi(g_2)}\,\mathrm{d}_x g_1\,\mathrm{d}_x g_2\\ &=\int_{(G_n(F)\times G_n(F))^\circ}f(g_1)\varphi_\pi(g_1)\overline{f(g_2)}\overline{\varphi_\pi(g_2)}|\det g_1|_F^{\alpha_0+n}\,\mathrm{d}(g_1,g_2)^\circ,\end{aligned}$$

where $(G_n(F)\times G_n(F))^\circ:=\{(g_1,g_2)\in G_n(F)\times G_n(F)\mid \det g_1=\det g_2\}$ and $\mathrm{d}(g_1,g_2)^\circ$ is a Haar measure on $(G_n(F)\times G_n(F))^\circ$, which makes the above fiber integration factorization hold.

We consider the natural embedding

$$(G_n(F)\times G_n(F))^\circ \hookrightarrow (M_n(F)\times M_n(F))^\circ$$

with an open dense image, where

$$(M_n(F)\times M_n(F))^\circ := \{(X,Y)\in M_n(F)\times M_n(F)\mid \det X=\det Y\},$$

which is the fiber product with respect to the determinant map $X\mapsto \det X$, and is a closed subvariety of the affine space $M_n(F)\times M_n(F)$. The natural group action of $G_n\times G_n$ on $M_n\times M_n$ via

$$(g,h)((X,Y))=(gX,hY)$$

for $(g,h)\in G_n\times G_n$ and $(X,Y)\in M_n\times M_n$ yields the action of $(G_n(F)\times G_n(F))^\circ$ on $(M_n(F)\times M_n(F))^\circ$ by restriction. Take $\mathrm{d}^+X\wedge \mathrm{d}^+Y$ to be an additive Haar measure on $M_n(F)\times M_n(F)$ with $|\det gh|_F^n$ the modulus function of the action of $G_n\times G_n$ on $M_n\times M_n$. Take the measure $\mathrm{d}^+(X,Y)^\circ$ on $(M_n(F)\times M_n(F))^\circ$, which is the pullback of the measure $\mathrm{d}^+X\wedge\mathrm{d}^+Y$ through the fiber product embedding. Then the modulus function of the action of $(G_n(F)\times G_n(F))^\circ$ on $(M_n(F)\times M_n(F))^\circ$ is

$$|\det gh|_F^n=|\det g|_F^{2n}=|\det h|_F^{2n}$$

for any $(g,h)\in (G_n(F)\times G_n(F))^\circ$. It is easy to check that $\mathrm{d}^+(g,h)^\circ/|\det gh|_F^n$ is a Haar measure on $(G_n(F)\times G_n(F))^\circ$. Hence, there is a constant $c>0$, such that

$$\mathrm{d}(g,h)^\circ = c\cdot \frac{\mathrm{d}^+(g,h)^\circ}{|\det gh|_F^n}.$$

The integral in (3-11) can be written as

$$\int_{(M_n(F)\times M_n(F))^\circ} f(X)\varphi_\pi(X)\overline{f(Y)}\overline{\varphi_\pi(Y)}|\det X|_F^{\alpha_0-n}\,\mathrm{d}^+(X,Y)^\circ. \tag{3-12}$$

Here we assume that $\alpha_0\ge n$ and both $\varphi_\pi(g_1)$ and $\varphi_\pi(g_2)$ are viewed as measurable functions on $M_n(F)$ that extend by zero to the boundary $M_n(F)\backslash \mathrm{GL}_n(F)$.

Since the F-analytical manifold $(M_n(F)\times M_n(F))^\circ$ is closed in $M_n(F)\times M_n(F)$, the restriction of the Schwartz function $f(g_1)\times\overline{f(g_2)}$ to $(M_n(F)\times M_n(F))^\circ$ is still a Schwartz function, which is smooth and compactly supported when F is non-Archimedean, and is in the sense of [1] when F is Archimedean. By Theorem 2.4, the zeta integral of Godement–Jacquet $\mathcal{Z}(s,f,\varphi_\pi,\chi)$ converges absolutely for $\mathrm{Re}(s)$ sufficiently large. It follows that for any $\pi\in\Pi_F(G_n)$, there exists a real number α_π such that for any $\varphi_\pi\in\mathcal{C}(\pi)$ and any $\mathrm{Re}(s)\ge\alpha_\pi$, the product $|\det(g)|_F^s\varphi_\pi(g)$ is bounded when $\det g$ tends to zero.

We write the F-analytical closed submanifold $(M_n(F)\times M_n(F))^\circ$ as a union of two closed submanifolds:

$$(M_n(F)\times M_n(F))^\circ=(M_n(F)\times M_n(F))^\circ_{\geq\varepsilon}\cup(M_n(F)\times M_n(F))^\circ_{\leq\varepsilon},$$

where

$$(M_n(F)\times M_n(F))^\circ_{\geq\varepsilon}=\{(g_1,g_2)\in M_n(F)^{\Delta\det}\mid|\det g_1|_F\geq\varepsilon\}$$

and

$$(M_n(F)\times M_n(F))^\circ_{\leq\varepsilon}=\{(g_1,g_2)\in M_n(F)^{\Delta\det}\mid|\det g_1|_F\leq\varepsilon\}.$$

For any $\pi\in\Pi_F(G_n)$, the restriction of the product $\varphi_\pi(g_1)\overline{\varphi_\pi(g_2)}\cdot|\det g_1|_F^{s-n}$ to the closed submanifold $(M_n(F)\times M_n(F))^\circ_{\geq\varepsilon}$ is of moderate growth and its restriction to the closed submanifold $(M_n(F)\times M_n(F))^\circ_{\leq\varepsilon}$ is bounded whenever $\mathrm{Re}(s)\geq 2\alpha_\pi+n$. It is also clear the Schwartz function $f(g_1)\times\overline{f(g_2)}$ on $(M_n(F)\times M_n(F))^\circ$ remains a Schwartz function when restricted to either the closed submanifold $(M_n(F)\times M_n(F))^\circ_{\geq\varepsilon}$ or the closed submanifold $(M_n(F)\times M_n(F))^\circ_{\leq\varepsilon}$. Hence, for any $\alpha_0\in\mathbb{R}$ with $\alpha_0\geq 2\alpha_\pi+n$, the integral

$$\int_{(M_n(F)\times M_n(F))^\circ}f(X)\varphi_\pi(X)\overline{f(Y)}\overline{\varphi_\pi(Y)}|\det X|_F^{\alpha_0-n}\,\mathrm{d}^+(X,Y)^\circ$$

converges absolutely, and so does the integral

$$\int_{F^\times}\phi(x)\overline{\phi(x)}|x|_F^{\alpha_0}\,\mathrm{d}^\times x.$$

It follows that the product $\phi(x)|x|_F^\kappa$ is square integrable on $F^\times$ for $\kappa=\alpha_0/2\geq\alpha_\pi+n/2$. □

Corollary 3.6. *If $\pi\in\Pi_F(G_n)$ is unitarizable, then for any $\phi\in\mathcal{S}_\pi(F^\times)$, the function $|x|_F^{n/2}\cdot\phi(x)$ belongs to the space $L^2(F^\times,\mathrm{d}^\times x)$ of square-integrable functions on $F^\times$.*

Proof. If $\pi\in\Pi_F(G_n)$ is unitarizable, then the matrix coefficient $\varphi_\pi(g)$ is bounded above over $G_n(F)$. For $\phi\in\mathcal{S}_\pi(F^\times)$, we write $\phi=\phi_{\xi,\varphi_\pi}$ with $\xi\in\mathcal{S}_{\mathrm{std}}(G_n(F))$ and $\varphi_\pi\in\mathcal{C}(\pi)$, and write $\xi(g)=|\det g|_F^{n/2}\cdot f(g)$ with $f\in\mathcal{S}(M_n(F))$. We compute the inner product of $|x|_F^{n/2}\cdot\phi(x)$ as

$$\begin{aligned}(3\text{-}13)\quad &\int_{F^\times}\phi(x)\overline{\phi(x)}|x|_F^n\,\mathrm{d}^\times x\\ &\leq\int_{F^\times}|x|_F^{2n}\int_{G_n(F)_x}|f(g_1)\varphi_\pi(g_1)|\,\mathrm{d}_xg_1\int_{G_n(F)_x}|f(g_2)\varphi_\pi(g_2)|\,\mathrm{d}_xg_2\,\mathrm{d}^\times x\\ &\leq c(\varphi_\pi)\cdot\int_{F^\times}|x|_F^{2n}\int_{G_n(F)_x}|f(g_1)|\,\mathrm{d}_xg_1\int_{G_n(F)_x}|f(g_2)|\,\mathrm{d}_xg_2\,\mathrm{d}^\times x\end{aligned}$$

for some positive constant $c(\varphi_\pi)$ depending on φ_π. By following the proof of Proposition 3.5, we obtain that

$$\int_{F^\times} \phi(x)\overline{\phi(x)}|x|_F^n \,\mathrm{d}^\times x \le c\cdot c(\varphi_\pi)\int_{(M_n(F)\times M_n(F))^\circ} |f(X)|\cdot|\overline{f(Y)}|\,\mathrm{d}(X,Y)^\circ. \tag{3-14}$$

The integral on the right-hand side of (3-14) comes from the integral in (3-12) with $\alpha_0 = n$. As explained in the proof of Proposition 3.5, the product $f(X)\times\overline{f(Y)}$ is a Schwartz function on $(M_n(F)\times M_n(F))^\circ$. Hence, the integral on the right-hand side of (3-14) converges. $\square$

By using Proposition 3.5 and Theorem 3.4, together with Theorem 2.3, we are able to understand the π-Schwartz space $\mathcal{S}_\pi(F^\times)$ by means of the L-functions $L(s,\pi\times\chi)$ for any $\pi\in\Pi_F(n)$.

Proposition 3.7. *For any $\pi\in\Pi_F(G_n)$, the π-Schwartz space $\mathcal{S}_\pi(F^\times)$ is contained in the space $\mathfrak{F}(F^\times)$ as defined in Definition 2.1*

Proof. Note first that the GL_1 zeta integral attached to $\phi\in\mathcal{S}_\pi(F^\times)$ is the same as the Mellin transform of ϕ up to a shift in s by the unramified part of χ. By Theorem 3.4 and Proposition 3.1, the image of $\mathcal{S}_\pi(F^\times)$ under Mellin transform is contained in the space $\mathcal{L}_\pi(\mathfrak{X}(F^\times))$ and hence in the space $\mathcal{Z}(\mathfrak{X}(F^\times))$. By Theorem 2.3, for any $\phi\in\mathcal{S}_\pi(F^\times)$, there exists $\phi_0\in\mathfrak{F}(F^\times)$, such that

$$\mathcal{M}(\phi-\phi_0)(\chi)=0 \tag{3-15}$$

holds identically for any quasicharacter $\chi\in\mathfrak{X}(F^\times)$. It remains to show that $\phi-\phi_0=0$ holds identically. By smoothness of ϕ and ϕ_0, it suffices to show that after unramified twist, both ϕ and ϕ_0 are square integrable on $F^\times$.

For $\phi_0\in\mathfrak{F}(F^\times)$, there exists $s_0\in\mathbb{R}$ such that, for any $\mathrm{Re}(s)>s_0$,

$$\lim_{x\to 0}\phi_0(x)|x|_F^{s+1}=0,$$

and the limit is preserved by differentiation on both sides when F is Archimedean. It follows that $\phi_0(x)|x|_F^s$ is indeed square integrable on $F^\times$ for $\mathrm{Re}(s)>s_0$, via the asymptotic formula appearing in the definition of $\mathfrak{F}(F^\times)$.

For any $\phi\in\mathcal{S}_\pi(F^\times)$, by Proposition 3.5, there exists $\alpha_\pi\in\mathbb{R}_{>0}$ such that the function $|x|_F^s\phi(x)$ is square integrable if $\mathrm{Re}(s)\ge\alpha_\pi+n/2$. By taking $\kappa>\max\{s_0,\alpha_\pi+n/2\}$, we obtain that both $\phi_0(x)|x|_F^\kappa$ and $\phi(x)|x|_F^\kappa$ are square integrable over $F^\times$. From (3-15), we obtain that the Mellin transform

$$\mathcal{M}\big(\phi(x)|x|_F^\kappa-\phi_0(x)|x|_F^\kappa\big)(\chi)=0$$

for all quasicharacters $\chi\in\mathfrak{X}(F^\times)$, in particular, for all unitary characters χ of $F^\times$. Therefore, by the Mellin inversion formula (Theorem 2.3), we obtain that

$$\phi(x)|x|_F^\kappa-\phi_0(x)|x|_F^\kappa=0$$

as functions in the space $L^2(F^\times, \mathrm{d}^\times x)$. Since both $\phi(x)$ and $\phi_0(x)$ are smooth, we must have that $\phi(x) = \phi_0(x) \in \mathfrak{F}(F^\times)$. □

Finally we are ready to characterize the Mellin inversion $\mathcal{M}^{-1}(\mathcal{L}_\pi)$ in terms of the π-Schwartz space $\mathcal{S}_\pi(F^\times)$ as in (3-2).

Corollary 3.8. *For any $\pi \in \Pi_F(G_n)$, the Mellin inversion $\mathcal{M}^{-1}(\mathcal{L}_\pi)$ coincides with the space $\mathcal{S}_\pi(F^\times)$ defined by*

$$\mathcal{S}_\pi(F^\times) = \mathcal{M}^{-1}(\mathcal{L}_\pi) \subset \mathcal{C}^\infty(F^\times).$$

In particular, the space $\mathcal{C}_c^\infty(F^\times)$ of smooth compactly supported functions on $F^\times$ is contained in the π-Schwartz space $\mathcal{S}_\pi(F^\times)$.

Proof. By Proposition 3.7, we have that the space $\mathcal{S}_\pi(F^\times)$ is contained in the space $\mathfrak{F}(F^\times)$. By Theorem 3.4, the Mellin transform (GL_1 zeta integral) of the space $\mathcal{S}_\pi(F^\times)$ is equal to the space $\mathcal{L}_\pi = \mathcal{L}_\pi(\mathfrak{X}(F^\times))$. Hence, we obtain that $\mathcal{S}_\pi(F^\times) = \mathcal{M}^{-1}(\mathcal{L}_\pi)$, because the Mellin transform is a bijective correspondence between the space $\mathfrak{F}(F^\times)$ and the space $\mathcal{Z}(\mathfrak{X}(F^\times))$ (Theorem 2.3). Finally, since the space $\mathcal{L}_\pi$ contains the space of holomorphic functions on $\mathfrak{X}(F^\times)$ that are of Paley–Wiener type along the vertical strips, it is clear from Theorem 2.3 again that $\mathcal{C}_c^\infty(F^\times)$ is contained in the π-Schwartz space $\mathcal{S}_\pi(F^\times)$. □

The relevant functional equation for GL_1 zeta integrals will be discussed in the next section.

3B. ***Fourier operators.*** We define a Fourier operator $\mathcal{F}_{\pi,\psi}$ from the π-Schwartz space $\mathcal{S}_\pi(F^\times)$ to the $\widetilde{\pi}$-Schwartz space $\mathcal{S}_{\widetilde{\pi}}(F^\times)$ for any $\pi \in \Pi_F(G_n)$ with smooth contragredient $\widetilde{\pi}$ and prove the functional equation for GL_1 zeta integrals $\mathcal{Z}(s, \phi, \chi)$.

The Fourier operator (transform) $\mathcal{F}_{\pi,\psi}$ is defined by the diagram

(3-16)
$$\begin{tikzcd}
\mathcal{S}_{\mathrm{std}}(G_n(F)) \otimes \mathcal{C}(\pi) \arrow[r, "(\mathcal{F}_{\mathrm{GJ}},(\cdot)^\vee)"] \arrow[d] & \mathcal{S}_{\mathrm{std}}(G_n(F)) \otimes \mathcal{C}(\widetilde{\pi}) \arrow[d] \\
\mathcal{S}_\pi(F^\times) \arrow[r, "\mathcal{F}_{\pi,\psi}"] & \mathcal{S}_{\widetilde{\pi}}(F^\times)
\end{tikzcd}$$

where ψ is a nontrivial additive character of F. More precisely, for $\phi = \phi_{\xi,\varphi_\pi} \in \mathcal{S}_\pi(F^\times)$ with a $\xi \in \mathcal{S}_{\mathrm{std}}(G_n(F))$ and a $\varphi_\pi \in \mathcal{C}(\pi)$, we define

$$\mathcal{F}_{\pi,\psi}(\phi) = \mathcal{F}_{\pi,\psi}(\phi_{\xi,\varphi_\pi}) := \phi_{\mathcal{F}_{\mathrm{GJ}}(\xi),\varphi_\pi^\vee}, \tag{3-17}$$

where $\varphi_\pi^\vee(g) = \varphi_\pi(g^{-1}) \in \mathcal{C}(\widetilde{\pi})$. Hence, we obtain that

$$\mathcal{F}_{\pi,\psi}(\phi) = \mathcal{F}_{\pi,\psi}(\phi_{\xi,\varphi_\pi}) \in \mathcal{S}_{\widetilde{\pi}}(F^\times). \tag{3-18}$$

It remains to check that the definition of the Fourier operator in (3-17) is independent of the choice of $\xi \in \mathcal{S}_{\mathrm{std}}(G_n(F))$ and $\varphi_\pi \in \mathcal{C}(\pi)$.

Proposition 3.9. *The Fourier operator* $\mathcal{F}_{\pi,\psi}$ *as in* (3-17) *is independent of the choice of* $\xi \in \mathcal{S}_{\mathrm{std}}(G_n(F))$ *and* $\varphi_\pi \in \mathcal{C}(\pi)$.

Proof. Assume that $\phi_{\xi_1,\varphi_{\pi,1}} = \phi_{\xi_2,\varphi_{\pi,2}}$ for some $\xi_1, \xi_2 \in \mathcal{S}_{\mathrm{std}}(G_n(F))$ and $\varphi_{\pi,1}, \varphi_{\pi,2} \in \mathcal{C}(\pi)$. We want to show that $\mathcal{F}_{\pi,\psi}(\phi_{\xi_1,\varphi_{\pi,1}}) = \mathcal{F}_{\pi,\psi}(\phi_{\xi_2,\varphi_{\pi,2}})$.

From (3-10), we must have that

$$\mathcal{Z}(s, \xi_1, \varphi_{\pi,1}, \chi) = \mathcal{Z}(s, \xi_2, \varphi_{\pi,2}, \chi)$$

for all quasicharacters $\chi \in \mathfrak{X}(F^\times)$ and all $s \in \mathbb{C}$. Of course, the identity holds for $\mathrm{Re}(s)$ large and then for all $s \in \mathbb{C}$ by meromorphic continuation. By the functional equation in Proposition 2.7, we obtain the identity

$$\mathcal{Z}(1-s, \mathcal{F}_{\mathrm{GJ}}(\xi_1), \varphi^\vee_{\pi,1}, \chi^{-1}) = \mathcal{Z}(1-s, \mathcal{F}_{\mathrm{GJ}}(\xi_2), \varphi^\vee_{\pi,2}, \chi^{-1})$$

for all $\chi \in \mathfrak{X}(F^\times)$ with $\mathrm{Re}(s)$ sufficiently small first and then all $s \in \mathbb{C}$ by meromorphic continuation. It follows by the identity in (3-10) again that, for all $\chi \in \mathfrak{X}(F^\times)$ and for $\mathrm{Re}(s) + \mathrm{Re}(\chi)$ sufficiently large, the identity

$$\int_{F^\times} (\phi_{\mathcal{F}_{\mathrm{GJ}}(\xi_1),\varphi^\vee_{\pi,1}}(x) - \phi_{\mathcal{F}_{\mathrm{GJ}}(\xi_2),\varphi^\vee_{\pi,2}}(x))\chi(x)|x|_F^{s-1/2}\,\mathrm{d}^\times x = 0$$

holds. By Proposition 3.7, we have that $\phi_{\mathcal{F}_{\mathrm{GJ}}(\xi_1),\varphi^\vee_{\pi,1}}(x) - \phi_{\mathcal{F}_{\mathrm{GJ}}(\xi_2),\varphi^\vee_{\pi,2}}(x)$ belongs to $\mathfrak{F}(F^\times)$. Finally, by Theorem 2.3, we must have that

$$\phi_{\mathcal{F}_{\mathrm{GJ}}(\xi_1),\varphi^\vee_{\pi,1}}(x) - \phi_{\mathcal{F}_{\mathrm{GJ}}(\xi_2),\varphi^\vee_{\pi,2}}(x) = 0$$

as functions on $F^\times$. Therefore, we proved that

$$\phi_{\mathcal{F}_{\mathrm{GJ}}(\xi_1),\varphi^\vee_{\pi,1}}(x) = \phi_{\mathcal{F}_{\mathrm{GJ}}(\xi_2),\varphi^\vee_{\pi,2}}(x)$$

as functions on $F^\times$, and $\mathcal{F}_{\pi,\psi}(\phi_{\xi_1,\varphi_{\pi,1}}) = \mathcal{F}_{\pi,\psi}(\phi_{\xi_2,\varphi_{\pi,2}})$. □

The following theorem on the local functional equation for the GL_1 zeta integrals $\mathcal{Z}(s, \phi, \chi)$ is a direct consequence of Theorem 2.4 and Proposition 3.9.

Theorem 3.10 (GL_1 functional equation). *For any* $\pi \in \Pi_F(G_n)$ *and its contragredient* $\widetilde{\pi} \in \Pi_F(G_n)$, *there exists a Fourier operator* $\mathcal{F}_{\pi,\psi}$, *which takes* $\phi \in \mathcal{S}_\pi(F^\times)$ *to* $\mathcal{F}_{\pi,\psi}(\phi) \in \mathcal{S}_{\widetilde{\pi}}(F^\times)$, *such that, after meromorphic continuation, the functional equation*

$$\mathcal{Z}(1-s, \mathcal{F}_{\pi,\psi}(\phi), \chi^{-1}) = \gamma(s, \pi \times \chi, \psi) \cdot \mathcal{Z}(s, \phi, \chi),$$

holds for any $\phi \in \mathcal{S}_\pi(F^\times)$. *The identities*

$$\mathcal{F}_{\widetilde{\pi},\psi^{-1}} \circ \mathcal{F}_{\pi,\psi} = \mathrm{Id} \quad \text{and} \quad \mathcal{F}_{\pi,\psi} \circ \mathcal{F}_{\widetilde{\pi},\psi^{-1}} = \mathrm{Id}$$

hold. When F is non-Archimedean, and π is unramified, the Fourier operator $\mathcal{F}_{\pi,\psi}$ takes the basic function $\mathbb{L}_\pi \in \mathcal{S}_\pi(F^\times)$ to the basic function $\mathbb{L}_{\widetilde{\pi}} \in \mathcal{S}_{\widetilde{\pi}}(F^\times)$:

$$\mathcal{F}_{\pi,\psi}(\mathbb{L}_\pi) = \mathbb{L}_{\widetilde{\pi}},$$

where the basic function $\mathbb{L}_\pi$ is defined in Theorem 3.4.

4. π-Poisson summation formula on GL_1

Let k be a number field and $\mathbb{A}$ be the ring of adeles of k. Denote by $|k|$ the set of all local places of k and by $|k|_\infty$ the set of all Archimedean local places of k. We may write

$$|k| = |k|_\infty \cup |k|_f,$$

where $|k|_f$ is the set of non-Archimedean local places of k. For each $\nu \in |k|$, we write $F = k_\nu$. Let $\Pi_{\mathbb{A}}(G_n)$ be the set of equivalence classes of irreducible admissible representations of $G_n(\mathbb{A})$. If we write $\pi = \bigotimes_{\nu\in|k|} \pi_\nu$, then we assume that $\pi_\nu \in \Pi_{k_\nu}(G_n)$, where at almost all finite local places ν, the local representations π_ν are unramified. When ν is a finite local place, π_ν is an irreducible admissible representation of $G_n(k_\nu)$, and when ν is an infinite local place, we assume that π_ν is of Casselman–Wallach type as representation of $G_n(k_\nu)$. Let $\mathcal{A}(G_n) \subset \Pi_{\mathbb{A}}(G_n)$ be the subset consisting of equivalence classes of irreducible admissible automorphic representations of $\mathrm{GL}_n(\mathbb{A})$, and $\mathcal{A}_{\mathrm{cusp}}(G_n)$ be the subset of cuspidal members of $\mathcal{A}(G_n)$.

4A. *π-Schwartz space and Fourier operator.* Take any $\pi = \bigotimes_{\nu\in|k|} \pi_\nu \in \Pi_{\mathbb{A}}(G_n)$. At each $\nu \in |k|$, the π_ν-Schwartz space $\mathcal{S}_{\pi_\nu}(k_\nu^\times)$ is defined in Definition 3.3. Recall from Theorems 3.4 and 3.10 the basic function $\mathbb{L}_{\pi_\nu} \in \mathcal{S}_{\pi_\nu}(k_\nu^\times)$ of π_ν when the local component π_ν of π is unramified. It is clear from the definition that $\mathbb{L}_{\pi_\nu}(1) = 1$ (We have to normalize various local measures in the computations. Actually it follows from the fact that the Laurent expansion of the unramified local L-factor has constant term 1.)

For the given $\pi = \bigotimes_\nu \pi_\nu \in \Pi_{\mathbb{A}}(G_n)$, we define the π-Schwartz space on $\mathbb{A}^\times$ to be

$$\mathcal{S}_\pi(\mathbb{A}^\times) := \bigotimes_{\nu\in|k|} \mathcal{S}_{\pi_\nu}(k_\nu^\times), \tag{4-1}$$

which is the restricted tensor product of the local π_ν-Schwartz space $\mathcal{S}_{\pi_\nu}(k_\nu^\times)$ with respect to the family of the basic functions $\mathbb{L}_{\pi_\nu}$ for the local places ν at which π_ν are unramified. The factorizable vectors $\phi = \bigotimes_\nu \phi_\nu$ in $\mathcal{S}_\pi(\mathbb{A}^\times)$ can be written as

$$\phi(x) = \prod_{\nu\in|k|} \phi_\nu(x_\nu). \tag{4-2}$$

Here for almost all finite local places ν, $\phi_\nu(x_\nu) = \mathbb{L}_{\pi_\nu}(x_\nu)$. According to our normalization, we have $\mathbb{L}_{\pi_\nu}(x_\nu) = 1$ when $x_\nu \in \mathfrak{o}_\nu^\times$, the unit group of the ring $\mathfrak{o}_\nu$ of integers at ν. Hence, for any given $x \in \mathbb{A}^\times$, the product in (4-2) is a finite product over Archimedean local places and finitely many non-Archimedean local places containing all ramified local places of π.

For any factorizable vectors $\phi = \bigotimes_\nu \phi_\nu$ in $\mathcal{S}_\pi(\mathbb{A}^\times)$, we define the π-Fourier operator

$$\mathcal{F}_{\pi,\psi}(\phi) := \bigotimes_{\nu\in|k|} \mathcal{F}_{\pi_\nu,\psi_\nu}(\phi_\nu), \tag{4-3}$$

where for each $\nu \in |k|$, $\mathcal{F}_{\pi_\nu,\psi_\nu}$ is the local Fourier operator as defined in (3-16) and (3-17). It is clear that $\mathcal{F}_{\pi_\nu,\psi_\nu}$ takes the π_ν-Schwartz space $\mathcal{S}_{\pi_\nu}(k_\nu^\times)$ to the $\widetilde{\pi}_\nu$-Schwartz space $\mathcal{S}_{\widetilde{\pi}_\nu}(k_\nu^\times)$ and enjoys the property

$$\mathcal{F}_{\pi_\nu,\psi}(\mathbb{L}_{\pi_\nu}) = \mathbb{L}_{\widetilde{\pi}_\nu}$$

when the data are unramified at ν. Hence, the Fourier operator $\mathcal{F}_{\pi,\psi}$ as defined in (4-3) maps the π-Schwartz space $\mathcal{S}_\pi(\mathbb{A}^\times)$ to the $\widetilde{\pi}$-Schwartz space $\mathcal{S}_{\widetilde{\pi}}(\mathbb{A}^\times)$.

4B. ***Global zeta integral.*** For any $\pi = \bigotimes_\nu \pi_\nu \in \Pi_{\mathbb{A}}(G_n)$, we define the global zeta integrals to be

$$\mathcal{Z}(s,\phi,\chi) := \int_{\mathbb{A}^\times} \phi(x)\chi(x)|x|_{\mathbb{A}}^{s-1/2}\,\mathrm{d}^\times x \tag{4-4}$$

for any $\phi \in \mathcal{S}_\pi(\mathbb{A}^\times)$ and characters χ of $k^\times\backslash\mathbb{A}^\times$. When $\phi = \bigotimes_\nu \phi_\nu$, we have

$$\mathcal{Z}(s,\phi,\chi) = \prod_{\nu\in|k|} \mathcal{Z}(s,\phi_\nu,\chi_\nu).$$

Let S be a finite subset of $|k|$, which contains all Archimedean local places and all the finite local places ν at which π_ν or χ_ν is ramified. Then we write

$$\mathcal{Z}(s,\phi,\chi) = L^S(s,\pi\times\chi)\cdot\prod_{\nu\in S}\mathcal{Z}(s,\phi_\nu,\chi_\nu),$$

according to Theorem 3.4. If π is unitarizable, the partial L-function $L^S(s,\pi\times\chi)$ converges absolutely for $\mathrm{Re}(s)$ large. By Theorem 3.4 again, the finite Euler product $\prod_{\nu\in S}\mathcal{Z}(s,\phi_\nu,\chi_\nu)$ converges absolutely for $\mathrm{Re}(s)$ large. We deduce the following proposition.

Proposition 4.1. *Let $\pi \in \Pi_{\mathbb{A}}(G_n)$ be unitarizable. Then for any $\phi \in \mathcal{S}_\pi(\mathbb{A}^\times)$ and any character χ of $k^\times\backslash\mathbb{A}^\times$, the zeta integral $\mathcal{Z}(s,\phi,\chi)$ as defined in* (4-4) *converges absolutely for* $\mathrm{Re}(s)$ *sufficiently large.*

We apply Proposition 4.1 to the case that $\pi \in \mathcal{A}_{\mathrm{cusp}}(G_n) \subset \mathcal{A}(G_n) \subset \Pi_{\mathbb{A}}(G_n)$. If $\pi \in \mathcal{A}_{\mathrm{cusp}}(G_n)$, then it is unitary. In this case, the zeta integral $\mathcal{Z}(s,\phi,\chi)$ can be identified with the Godement–Jacquet global zeta integral. For any $f = \bigotimes_\nu f_\nu \in$

$\mathcal{S}(M_n(\mathbb{A}))$ and any $\varphi_\pi \in \mathcal{C}(\pi)$, the Godement–Jacquet global zeta integral is defined to be

$$\mathcal{Z}(s, f, \varphi_\pi, \chi) := \int_{\mathrm{GL}_n(\mathbb{A})} f(g)\varphi_\pi(g)\chi(\det g)|\det g|_F^{s+(n-1)/2}\,\mathrm{d}g, \tag{4-5}$$

Theorem 4.2 [16, Theorem 13.8]. *For $\pi \in \mathcal{A}_{\mathrm{cusp}}(G_n)$ and any unitary automorphic character χ of $k^\times\backslash\mathbb{A}^\times$, the global zeta integral $\mathcal{Z}(s, f, \varphi_\pi, \chi)$ converges absolutely for* $\mathrm{Re}(s) > (n+1)/2$, *admits analytic continuation to an entire function in $s \in \mathbb{C}$, and satisfies the global functional equation*

$$\mathcal{Z}(s, f, \varphi_\pi, \chi) = \mathcal{Z}(1-s, \mathcal{F}_\psi(f), \varphi_\pi^\vee, \chi^{-1}), \tag{4-6}$$

where $\mathcal{F}_\psi$ is the global Fourier transform from $\mathcal{S}(M_n(\mathbb{A}))$ to $\mathcal{S}(M_n(\mathbb{A}))$ associated to the additive character ψ of $k\backslash\mathbb{A}$.

For $\mathrm{Re}(s) > (n+1)/2$, we write

$$\mathcal{Z}(s, f, \varphi_\pi, \chi) = \int_{\mathbb{A}^\times}\left(|x|_{\mathbb{A}}^{n/2}\int_{G_n(\mathbb{A})_x} f(g)\varphi_\pi(g)\,\mathrm{d}_x g\right)\chi(x)|x|_{\mathbb{A}}^{s-1/2}\,\mathrm{d}^\times x, \tag{4-7}$$

where $G_n(\mathbb{A})_x := \{g \in G_n(\mathbb{A}) \mid \det g = x\}$ is an $\mathrm{SL}_n(\mathbb{A})$-torsor, and the measure $\mathrm{d}_x g$ is $\mathrm{SL}_n(\mathbb{A})$-invariant. As in the local situations, we define, for any $x \in \mathbb{A}^\times$,

$$\phi_{\xi,\varphi_\pi}(x) := \int_{G_n(\mathbb{A})_x} \xi(g)\varphi_\pi(g)\,\mathrm{d}_x g = |x|_{\mathbb{A}}^{n/2}\int_{G_n(\mathbb{A})_x} f(g)\varphi_\pi(g)\,\mathrm{d}_x g, \tag{4-8}$$

where $\xi(g) := |\det g|_{\mathbb{A}}^{n/2}\cdot f(g)$ belongs to the space

$$\mathcal{S}_{\mathrm{std}}(G_n(\mathbb{A})) = \{\xi \in \mathcal{C}^\infty(G_n(\mathbb{A})) \mid \xi(g)\cdot|\det g|_{\mathbb{A}}^{-n/2} \in \mathcal{S}(M_n(\mathbb{A}))\}. \tag{4-9}$$

It is clear that

$$\mathcal{S}_{\mathrm{std}}(G_n(\mathbb{A})) = \bigotimes_{\nu\in|k|}\mathcal{S}_{\mathrm{std}}(G_n(k_\nu)). \tag{4-10}$$

Write $G_n(\mathbb{A})$ as a direct product decomposition:

$$G_n(\mathbb{A}) = A_n(\mathbb{R})^+\cdot G_n(\mathbb{A})^1, \tag{4-11}$$

where $G_n(\mathbb{A})^1 := \{g \in G_n(\mathbb{A}) \mid |\det g|_{\mathbb{A}} = 1\}$ and $A_n(\mathbb{R})^+$ is the identity connected component of the center $Z_{G_n}(\mathbb{R})$ of $G_n(\mathbb{R})$. As in [16, Section 13], any matrix coefficient φ_π of $\pi \in \mathcal{A}_{\mathrm{cusp}}(G_n)$ can be written as

$$\varphi_\pi(g) = \int_{A_n(\mathbb{R})^+G_n(k)\backslash G_n(\mathbb{A})} \alpha_\pi(hg)\alpha_{\widetilde{\pi}}(h)\,\mathrm{d}h = \int_{G_n(k)\backslash G_n(\mathbb{A})^1} \alpha_\pi(hg)\alpha_{\widetilde{\pi}}(h)\,\mathrm{d}h \tag{4-12}$$

for some $\alpha_\pi \in V_\pi$ and $\alpha_{\widetilde{\pi}} \in V_{\widetilde{\pi}}$, where V_π is the cuspidal automorphic realization of π in $L^2(G_n(k)\backslash G_n(\mathbb{A}), \omega)$ with central character $\omega_\pi = \omega$. In this case, we have $\omega_{\widetilde{\pi}} = \omega^{-1}$. In the integral in (4-8), the coefficient $\varphi_\pi(g)$ is bounded over $G_n(\mathbb{A})$. Since $f \in \mathcal{S}(M_n(\mathbb{A}))$ and $G_n(\mathbb{A})_x$ is a closed submanifold in $M_n(\mathbb{A})$, the restriction to $G_n(\mathbb{A})_x$ of the Schwartz function f is still a Schwartz function on $G_n(\mathbb{A})_x$. Hence, the integral in (4-8) converges absolutely for any $x \in \mathbb{A}^\times$, and the convergence is uniform when x runs in any given compact neighborhood of $\mathbb{A}^\times$.

Proposition 4.3. *For $\pi \in \mathcal{A}_{\mathrm{cusp}}(G_n)$, the function $\phi_{\xi,\varphi_\pi}(x)$ as defined in* (4-8) *is smooth on $\mathbb{A}^\times$. If $\xi(g) = \bigotimes_\nu \xi_\nu = |\det g|^{n/2} \cdot f(g) \in \mathcal{S}_{\mathrm{std}}(G_n(\mathbb{A}))$ with $f = \bigotimes_\nu f_\nu \in \mathcal{S}(M_n(\mathbb{A}))$ and $\varphi_\pi = \bigotimes_\nu \varphi_{\pi_\nu}$, then the function defined by*

$$\phi_{\xi,\varphi_\pi}(x) = \prod_{\nu \in |k|} \phi_{\xi_\nu,\varphi_{\pi_\nu}}(x_\nu)$$

for any $x \in \mathbb{A}^\times$ belongs to $\mathcal{S}_\pi(\mathbb{A}^\times)$.

Proof. Since the integral in (4-8) converges absolutely for any $x \in \mathbb{A}^\times$, and the convergence is uniform when x runs in any given compact neighborhood of $\mathbb{A}^\times$, the function $\phi_{\xi,\varphi_\pi}(x)$ is smooth on $\mathbb{A}^\times$.

To prove the second statement, we take $f = \bigotimes_\nu f_\nu \in \mathcal{S}(M_n(\mathbb{A}))$. Since $\mathcal{C}(\pi) = \bigotimes_\nu \mathcal{C}(\pi_\nu)$, we take $\varphi_\pi = \bigotimes_\nu \varphi_{\pi_\nu}$ with $\varphi_{\pi_\nu} \in \mathcal{C}(\pi_\nu)$. Then there exists a finite subset S_0 which contains all Archimedean local places of k such that for any finite local place ν of k, if $\nu \notin S_0$, then $f_\nu = f_\nu^\circ = \mathbb{1}_{M_n(\mathfrak{o}_\nu)}$, π_ν is unramified and $\varphi_{\pi_\nu} = \varphi^\circ_{\pi_\nu}$, which is the zonal spherical function on $G_n(k_\nu)$ associated to π_ν. For any $x \in \mathbb{A}^\times$, and for any finite subset S of $|k|$ that contains S_0 and $x_\nu \in \mathfrak{o}_\nu^\times$ if $\nu \notin S$, we have

$$\phi_{\xi,\varphi_\pi}(x) = \int_{\det g = x} \xi(g)\varphi_\pi(g)\,\mathrm{d}_x g = \lim_S \prod_{\nu \in S} \int_{\det g_\nu = x_\nu} \xi_\nu(g_\nu)\varphi_{\pi_\nu}(g_\nu)\,\mathrm{d}_{x_\nu} g_\nu \tag{4-13}$$

with $\xi(g) = |\det g|_{\mathbb{A}}^{n/2} \cdot f(g)$ and $\xi = \bigotimes_\nu \xi_\nu$, where $\xi_\nu(g) = |\det g|_\nu^{n/2} \cdot f_\nu(g)$. At $\nu \notin S$, we have $|x_\nu|_\nu = 1$ and the local integral identity

$$\begin{aligned}\int_{\det g_\nu = x_\nu} &\xi_\nu(g_\nu)\varphi_{\pi_\nu}(g_\nu)\,\mathrm{d}_{x_\nu} g_\nu \\ &= \int_{\det g_\nu = x_\nu} \mathbb{1}_{M_n(\mathfrak{o}_\nu)}(g_\nu)\varphi^\circ_{\pi_\nu}(g_\nu)\,\mathrm{d}_{x_\nu} g_\nu = \mathrm{vol}(G_n(\mathfrak{o}_\nu)_{x_\nu}) = 1.\end{aligned}$$

Hence, we obtain that $\phi_{\xi,\varphi_\pi}(x) = \prod_\nu \phi_{\xi_\nu,\varphi_{\pi_\nu}}(x_\nu)$. □

Hence, we obtain the relation between the global GL_1 zeta integrals defined in (4-4) and the global Godement–Jacquet zeta integrals defined in (4-5).

Corollary 4.4. *If $\pi \in \mathcal{A}_{\mathrm{cusp}}(G_n)$, then for any $\phi = \phi_{\xi,\varphi_\pi} \in \mathcal{S}_\pi(\mathbb{A}^\times)$ with $\xi(g) = |\det g|_{\mathbb{A}}^{n/2} \cdot f(g) \in \mathcal{S}_{\mathrm{std}}(G_n(\mathbb{A}))$ for some $f \in \mathcal{S}(M_n(\mathbb{A}))$ and $\varphi_\pi \in \mathcal{C}(\pi)$, the identity*

$$\mathcal{Z}(s, \phi, \chi) = \mathcal{Z}(s, f, \varphi_\pi, \chi)$$

holds for any character χ of $k^\times \backslash \mathbb{A}^\times$ and $\mathrm{Re}(s)$ sufficiently large.

Proposition 4.5. *If $\pi\in\mathcal{A}_{\mathrm{cusp}}(G_n)$, then for any $\phi=\phi_{\xi,\varphi_\pi}\in\mathcal{S}_\pi(\mathbb{A}^\times)$ with $\xi(g)=|\det g|_{\mathbb{A}}^{n/2}\cdot f(g)\in\mathcal{S}_{\mathrm{std}}(G_n(\mathbb{A}))$ for some $f\in\mathcal{S}(M_n(\mathbb{A}))$ and $\varphi_\pi\in\mathcal{C}(\pi)$, the identity*

$$\mathcal{F}_{\pi,\psi}(\phi_{\xi,\varphi_\pi})(x)=\phi_{\mathcal{F}_{\mathrm{GJ}}(\xi),\varphi_\pi^\vee}(x)$$

holds for any $x\in\mathbb{A}^\times$. For any $x\in\mathbb{A}^\times$, the $\mathbb{A}^\times$-equivariant property

$$\mathcal{F}_{\pi,\psi}(\phi^x)(y)=\mathcal{F}_{\pi,\psi}(\phi)(yx^{-1})$$

holds, where $\phi^x(y):=\phi(yx)$.

Proof. Assume that $\phi=\phi_{\xi,\varphi_\pi}\in\mathcal{S}_\pi(\mathbb{A}^\times)$ with $\xi(g)=|\det g|_{\mathbb{A}}^{n/2}\cdot f(g)\in\mathcal{S}_{\mathrm{std}}(G_n(\mathbb{A}))$ for some $f\in\mathcal{S}(M_n(\mathbb{A}))$ and $\varphi_\pi\in\mathcal{C}(\pi)$ is factorizable: $\phi=\bigotimes_\nu\phi_\nu$. By definition (4-3), we have

$$\mathcal{F}_{\pi,\psi}(\phi)(x)=\prod_{\nu\in|k|}\mathcal{F}_{\pi_\nu,\psi_\nu}(\phi_\nu)(x_\nu).$$

Write $\phi_\nu(x_\nu)=\phi_{\xi_\nu,\varphi_{\pi_\nu}}(x_\nu)$. Then we have

$$\mathcal{F}_{\pi_\nu,\psi_\nu}(\phi_\nu)(x_\nu)=\phi_{\mathcal{F}_{\mathrm{GJ},\nu}(\xi_\nu),\varphi_{\pi_\nu}^\vee}(x_\nu).$$

When the data involved are unramified, we have from the simple calculation below (4-13) that $\mathcal{F}_{\pi_\nu,\psi_\nu}(\phi_\nu)(x_\nu)=1$. Hence, we obtain

$$\mathcal{F}_{\pi,\psi}(\phi)(x)=\prod_\nu\mathcal{F}_{\pi_\nu,\psi_\nu}(\phi_\nu)(x_\nu)=\prod_\nu\phi_{\mathcal{F}_{\mathrm{GJ},\nu}(\xi_\nu),\varphi_{\pi_\nu}^\vee}(x_\nu)=\phi_{\mathcal{F}_{\mathrm{GJ}}(\xi),\varphi_\pi^\vee}(x)$$

as in (4-13).

In order to verify the $\mathbb{A}^\times$-equivariant property $\mathcal{F}_{\pi,\psi}(\phi^x)(y)=\mathcal{F}_{\pi,\psi}(\phi)(yx^{-1})$ for any $x,y\in\mathbb{A}^\times$, it is enough to verify that the local Fourier operators $\mathcal{F}_{\pi_\nu,\psi_\nu}$ for all local place $\nu\in|k|$ enjoy the same equivariant property. This local equivariant property for the Fourier operators $\mathcal{F}_{\pi_\nu,\psi_\nu}$ can be deduced from the local functional equation for the zeta integral $\mathcal{Z}(s,\phi,\chi)$ in Theorem 3.10 through a simple computation. □

We can deduce the following result from Theorem 4.2.

Theorem 4.6. *Let π be an irreducible unitary cuspidal automorphic representation of $G_n(\mathbb{A})$ with the local component π_ν being of Casselman–Wallach type at all $\nu\in|k|_\infty$. For any $\phi\in\mathcal{S}_\pi(\mathbb{A}^\times)$ and any unitary character χ of $k^\times\backslash\mathbb{A}^\times$, the global zeta integral $\mathcal{Z}(s,\phi,\chi)$ converges absolutely for* $\mathrm{Re}(s)>(n+1)/2$, *admits analytic continuation to an entire function in $s\in\mathbb{C}$, and satisfies the functional equation*

$$\mathcal{Z}(s,\phi,\chi)=\mathcal{Z}(1-s,\mathcal{F}_{\pi,\psi}(\phi),\chi^{-1}),$$

where $\mathcal{F}_{\pi,\psi}$ is the Fourier operator as defined in (4-3) *that takes $\mathcal{S}_\pi(\mathbb{A}^\times)$ to $\mathcal{S}_{\widetilde{\pi}}(\mathbb{A}^\times)$.*

4C. ***π-Poisson summation formula.*** We establish here the Poisson summation formula on GL_1 for the Fourier operator $\mathcal{F}_{\pi,\psi}$, which is associated to any $\pi \in \mathcal{A}_{\mathrm{cusp}}(G_n)$, and takes $\mathcal{S}_\pi(\mathbb{A}^\times)$ to $\mathcal{S}_{\widetilde{\pi}}(\mathbb{A}^\times)$. Technically, it is possible to establish such a summation formula from the global functional equation in Theorem 4.6. However, we are going to take a slightly different way below.

Theorem 4.7 (π-Poisson summation formula). *For any $\pi \in \mathcal{A}_{\mathrm{cusp}}(G_n)$, take $\widetilde{\pi}$ to be the contragredient of π. For any $\phi \in \mathcal{S}_\pi(\mathbb{A}^\times)$, the π-theta function*

$$\Theta_\pi(x,\phi) := \sum_{\alpha \in k^\times} \phi(\alpha x)$$

converges absolutely and locally uniformly for any $x \in \mathbb{A}^\times$, and we have the identity

$$\Theta_\pi(x,\phi) = \Theta_{\widetilde{\pi}}(x^{-1}, \mathcal{F}_{\pi,\psi}(\phi)),$$

as functions in $x \in \mathbb{A}^\times$, where $\mathcal{F}_{\pi,\psi}$ is the Fourier operator as defined in (4-3) *that takes $\mathcal{S}_\pi(\mathbb{A}^\times)$ to $\mathcal{S}_{\widetilde{\pi}}(\mathbb{A}^\times)$.*

Proof. It is clear that $\Theta_\pi(x,\phi) = \Theta_\pi(1,\phi^x)$ with $\phi^x(y) = \phi(xy)$. By Proposition 4.5, we have $\Theta_{\widetilde{\pi}}(x^{-1}, \mathcal{F}_{\pi,\psi}(\phi)) = \Theta_{\widetilde{\pi}}(1, \mathcal{F}_{\pi,\psi}(\phi^x))$. Since $\phi \in \mathcal{S}_\pi(\mathbb{A}^\times)$ is arbitrary, it is enough to show that

$$\Theta_\pi(1,\phi) := \sum_{\alpha \in k^\times} \phi(\alpha)$$

converges absolutely and the identity

$$\Theta_\pi(1,\phi) = \Theta_{\widetilde{\pi}}(1, \mathcal{F}_{\pi,\psi}(\phi))$$

holds.

In order to prove that the summation $\Theta_\pi(1,\phi)$ is absolutely convergent, we write $\phi = \phi_{\xi,\varphi_\pi} \in \mathcal{S}_\pi(\mathbb{A}^\times)$ with $\xi(g) = |\det g|_{\mathbb{A}}^{n/2} \cdot f(g) \in \mathcal{S}_{\mathrm{std}}(G_n(\mathbb{A}))$ for some $f \in \mathcal{S}(M_n(\mathbb{A}))$ and $\varphi_\pi \in \mathcal{C}(\pi)$. From (4-12) we have

$$\varphi_\pi(g) = \int_{A_n(\mathbb{R})^+ G_n(k)\backslash G_n(\mathbb{A})} \beta_1(hg)\beta_2(h)\,\mathrm{d}h = \int_{G_n(k)\backslash G_n(\mathbb{A})^1} \beta_1(hg)\beta_2(h)\,\mathrm{d}h \tag{4-14}$$

for some $\beta_1 \in V_\pi$ and $\beta_2 \in V_{\widetilde{\pi}}$, where V_π is the cuspidal automorphic realization of π in $L^2(G_n(k)\backslash G_n(\mathbb{A}), \omega)$ and so is $V_{\widetilde{\pi}}$.

First, we have that

$$\begin{aligned} \Theta_\pi(1,\phi) &= \sum_{\alpha \in k^\times} \phi_{\xi,\varphi_\pi}(\alpha) \\ &= \sum_{\alpha \in k^\times} \int_{G_n(\mathbb{A})_\alpha} \xi(g) \int_{G_n(k)\backslash G_n(\mathbb{A})^1} \beta_1(hg)\beta_2(h)\,\mathrm{d}h\,\mathrm{d}_\alpha g. \end{aligned} \tag{4-15}$$

By changing variable $g \to h^{-1}g$, we have that $\det g = \alpha\cdot\det h$ and the last expression in (4-15) becomes

$$\int_{G_n(k)\backslash G_n(\mathbb{A})^1} \sum_{\alpha\in k^\times} \int_{G_n(\mathbb{A})_{\alpha\cdot\det h}} \xi(h^{-1}g)\beta_1(g)\beta_2(h)\,\mathrm{d}_{\alpha\cdot\det h}g\,\mathrm{d}h. \tag{4-16}$$

For $g \in G_n(\mathbb{A})_{\alpha\cdot\det h}$, we change g to $t_1(\alpha)\cdot y$ with $\det y = \det h$, where $t_1(\alpha) = \mathrm{diag}(\alpha, I_{n-1}) \in G_n(k)$. Then (4-16) can be written as

$$\int_{G_n(k)\backslash G_n(\mathbb{A})^1} \sum_{\alpha\in k^\times} \int_{\mathrm{GL}_n(\mathbb{A})_{\det h}} \xi(h^{-1}t_1(\alpha)g)\beta_1(g)\beta_2(h)\,\mathrm{d}_{\det h}g\,\mathrm{d}h, \tag{4-17}$$

since β_1 is automorphic. For any $h \in G_n(\mathbb{A})^1$, we have $|\det h|_{\mathbb{A}} = 1$. Hence, we must have that $G_n(\mathbb{A})_{\det h} \subset G_n(\mathbb{A})^1$. It is clear that $G_n(\mathbb{A})_{\det h}$ is an $\mathrm{SL}_n(\mathbb{A})$-torsor and the measure $\mathrm{d}_{\det h}g$ is left-$\mathrm{SL}_n(k)$-invariant. Hence, (4-17) can be written as

$$\int_{G_n(k)\backslash G_n(\mathbb{A})^1} \sum_{\alpha\in k^\times} \sum_{\epsilon\in \mathrm{SL}_n(k)} \int_{\mathrm{SL}_n(k)\backslash G_n(\mathbb{A})_{\det h}} \xi(h^{-1}t_1(\alpha)\epsilon g)\beta_1(g)\beta_2(h)\,\mathrm{d}_{\det h}g\,\mathrm{d}h.$$

Since any element $\gamma \in G_n(k)$ can be written as a product of $t_1(\alpha)$ and ϵ in a unique way, we obtain that the above expression is equal to

$$\int_{G_n(k)\backslash G_n(\mathbb{A})^1} \int_{\mathrm{SL}_n(k)\backslash G_n(\mathbb{A})_{\det h}} \Big(\sum_{\gamma\in G_n(k)} \xi(h^{-1}\gamma g)\Big)\beta_1(g)\beta_2(h)\,\mathrm{d}_{\det h}g\,\mathrm{d}h. \tag{4-18}$$

Since $\xi(g) = |\det g|_{\mathbb{A}}^{n/2}\cdot f(g) \in \mathcal{S}_{\mathrm{std}}(G_n(\mathbb{A}))$ for some $f \in \mathcal{S}(M_n(\mathbb{A}))$, and $h \in G_n(\mathbb{A})^1$ and $g \in G_n(\mathbb{A})_{\det h}$, we must have that

$$\xi(h^{-1}\gamma g) = |\det(h^{-1}\gamma g)|_{\mathbb{A}}^{n/2}\cdot f(h^{-1}\gamma g) = f(h^{-1}\gamma g). \tag{4-19}$$

Hence, we obtain that

$$\sum_{\gamma\in G_n(k)} \xi(h^{-1}\gamma g) = \sum_{\gamma\in G_n(k)} f(h^{-1}\gamma g). \tag{4-20}$$

By [16, Lemma 11.7], for any $f \in \mathcal{S}(M_n(\mathbb{A}))$, the summation $\sum_{\gamma\in G_n(k)} f(h^{-1}\gamma g)$ is of moderate growth in $g, h \in G_n(k)\backslash G_n(\mathbb{A})$ as an automorphic function on $G_n(k)\backslash G_n(\mathbb{A}) \times G_n(k)\backslash G_n(\mathbb{A})$, and so is the summation $\sum_{\gamma\in G_n(k)} \xi(h^{-1}\gamma g)$ as an automorphic function in $g, h \in G_n(k)\backslash G_n(\mathbb{A})^1$. Since both $\beta_1(g)$ and $\beta_2(h)$ are cuspidal, we obtain that the integral in (4-18) converges absolutely, and so does the π-theta function $\Theta_\pi(1, \phi)$ at $x = 1$.

Now we continue with the integral in (4-18) to prove the identity

$$\Theta_\pi(1, \phi) = \Theta_{\widetilde{\pi}}(1, \mathcal{F}_{\pi,\psi}(\phi)). \tag{4-21}$$

Recall from [16, Section 11; 34, Theorem 4.0.1] the classical Poisson summation formula

$$\sum_{\gamma\in M_n(k)} f(h^{-1}\gamma g)=\sum_{\gamma\in M_n(k)} |\det gh^{-1}|_{\mathbb{A}}^{-n}\mathcal{F}_\psi(f)(g^{-1}\gamma h) \tag{4-22}$$

for any $f\in\mathcal{S}(M_n(\mathbb{A}))$ and $h,g\in G_n(\mathbb{A})$. When $g,h\in G_n(\mathbb{A})^1$, it can be rewritten according to the rank of $\gamma\in M_n(k)$ as

$$\begin{aligned}\sum_{\gamma\in G_n(k)} & f(h^{-1}\gamma g)\\ &=\sum_{\gamma\in G_n(k)}\mathcal{F}_\psi(f)(g^{-1}\gamma h)+\sum_{\substack{\gamma\in M_n(k)\\ \mathrm{rank}(\gamma)<n}}\mathcal{F}_\psi(f)(g^{-1}\gamma h)-\sum_{\substack{\gamma\in M_n(k)\\ \mathrm{rank}(\gamma)<n}} f(h^{-1}\gamma g).\end{aligned}$$

We denote the boundary terms by

$$B_f(h,g):=\sum_{\substack{\gamma\in M_n(k)\\ \mathrm{rank}(\gamma)<n}}\mathcal{F}_\psi(f)(g^{-1}\gamma h)-\sum_{\substack{\gamma\in M_n(k)\\ \mathrm{rank}(\gamma)<n}} f(h^{-1}\gamma g). \tag{4-23}$$

Then (4-18) can be written as a sum of the two terms

$$\int_{G_n(k)\backslash G_n(\mathbb{A})^1}\int_{\mathrm{SL}_n(k)\backslash G_n(\mathbb{A})_{\det h}}\Big(\sum_{\gamma\in G_n(k)}\mathcal{F}_\psi(f)(g^{-1}\gamma h)\Big)\beta_1(g)\beta_2(h)\,\mathrm{d}_{\det h}g\,\mathrm{d}h, \tag{4-24}$$

and

$$\int_{G_n(k)\backslash G_n(\mathbb{A})^1}\int_{\mathrm{SL}_n(k)\backslash G_n(\mathbb{A})_{\det h}} B_f(h,g)\beta_1(g)\beta_2(h)\,\mathrm{d}_{\det h}g\,\mathrm{d}h. \tag{4-25}$$

From the proofs of [16, Lemma 12.13; 34, Lemma 4.1.4], we must have that the term in (4-25) is zero, because of the cuspidality of both $\beta_1(g)$ and $\beta_2(h)$. Hence, we obtain that $\Theta_\pi(1,\phi)=\Theta_\pi(1,\phi_{\xi,\varphi_\pi})$ is equal to the term in (4-24).

Now we write (4-24) as

$$\int_{G_n(k)\backslash G_n(\mathbb{A})^1}\int_{\mathrm{SL}_n(k)\backslash G_n(\mathbb{A})_{\det h}}\Big(\sum_{\gamma\in G_n(k)}\mathcal{F}_\psi(f)((\gamma g)^{-1}h)\Big)\beta_1(g)\beta_2(h)\,\mathrm{d}_{\det h}g\,\mathrm{d}h.$$

By writing back that $\gamma=t_1(\alpha)\cdot\epsilon$ with $\alpha\in k^\times$ and $\epsilon\in\mathrm{SL}_n(k)$, we obtain that the above expression is equal to

$$\int_{G_n(k)\backslash G_n(\mathbb{A})^1}\int_{G_n(\mathbb{A})_{\det h}}\Big(\sum_{\alpha\in k^\times}\mathcal{F}_\psi(f)((t_1(\alpha)g)^{-1}h)\Big)\beta_1(g)\beta_2(h)\,\mathrm{d}_{\det h}g\,\mathrm{d}h. \tag{4-26}$$

By changing $t_1(\alpha)g$ to g, we write (4-26) as

$$\sum_{\alpha\in k^\times}\int_{G_n(k)\backslash G_n(\mathbb{A})^1}\int_{G_n(\mathbb{A})_{\alpha\cdot\det h}}\mathcal{F}_\psi(f)(g^{-1}h)\beta_1(g)\beta_2(h)\,\mathrm{d}_{\alpha\cdot\det h}g\,\mathrm{d}h. \tag{4-27}$$

After changing variable $g\to hg$, (4-27) can be written as

$$\sum_{\alpha\in k^\times}\int_{G_n(\mathbb{A})_\alpha}\mathcal{F}_\psi(f)(g^{-1})\int_{G_n(k)\backslash G_n(\mathbb{A})^1}\beta_1(hg)\beta_2(h)\,\mathrm{d}h\,\mathrm{d}_\alpha g, \tag{4-28}$$

which is equal to

$$\sum_{\alpha\in k^\times}\int_{G_n(\mathbb{A})_\alpha}\mathcal{F}_\psi(f)(g^{-1})\varphi_\pi(g)\,\mathrm{d}_\alpha g. \tag{4-29}$$

Finally, by changing g to g^{-1}, we obtain that (4-18) is equal to

$$\sum_{\alpha\in k^\times}\int_{G_n(\mathbb{A})_\alpha}\mathcal{F}_\psi(f)(g)\varphi_\pi(g^{-1})\,\mathrm{d}_\alpha g. \tag{4-30}$$

By Proposition 2.6, when $\det g=\alpha\in k^\times$, we have

$$\mathcal{F}_\psi(f)(g)=\mathcal{F}_{\mathrm{GJ}}(\xi)(g)$$

for $\xi(g)=|\det g|^{n/2}\cdot f(g)$. Hence, the summation in (4-30) is equal to

$$\sum_{\alpha\in k^\times}\mathcal{F}_{\pi,\psi}(\phi_{\xi,\varphi_\pi})(1)=\Theta_{\widetilde{\pi}}(1,\mathcal{F}_{\pi,\psi}(\phi_{\xi,\varphi_\pi})).$$

This proves the π-Poisson summation formula

$$\Theta_\pi(1,\phi)=\Theta_{\widetilde{\pi}}(1,\mathcal{F}_{\pi,\psi}(\phi))$$

for all $\phi\in\mathcal{S}_\pi(\mathbb{A}^\times)$.

For the locally uniform convergence of the π-theta function $\Theta_\pi(x,\phi)$, since $\Theta_\pi(x,\phi)=\Theta_\pi(1,\phi^x)$, it is enough to prove the locally uniform convergence of the π-theta function $\Theta_\pi(x,\phi)$ around $x=1$. One may verify this directly from the discussion in the proof given above. It also follows directly from Proposition 4.8 below in this case. We are done. □

Similar to the work of [40], we obtain the following uniform estimate of the π-theta function $\Theta_\pi(x,\phi)$, which is important to the application in Section 8.

Proposition 4.8. *For any $\pi\in\mathcal{A}_{\mathrm{cusp}}(G_n)$, take any $\phi\in\mathcal{S}_\pi(\mathbb{A}^\times)$. For any $\kappa>0$, there exists a positive constant $c_{\kappa,\phi}$ such that the π-theta function $\Theta_\pi(x,\phi)$ enjoys the property*

$$|\Theta_\pi(x,\phi)|\le c_{\kappa,\phi}\cdot\min\{|x|_{\mathbb{A}},|x|_{\mathbb{A}}^{-1}\}^\kappa.$$

Proof. This is a reformulation of part (ii) of [40, Theorem 1] and can be proved accordingly. We omit the details. □

Remark 4.9. The proof of the π-Poisson summation formula in Theorem 4.7 uses the Poisson summation formula associated to the classical Fourier transform $\mathcal{F}_\psi$ over the affine space $M_n(\mathbb{A})$, without using the global functional equation for the global zeta integrals $\mathcal{Z}(s, \phi, \chi)$ in Theorem 4.6. Hence, we are able to obtain the global functional equation for the global zeta integrals $\mathcal{Z}(s, \phi, \chi)$ as in Theorem 4.6 by using the π-Poisson summation formula in Theorem 4.7. Of course, this is essentially the same proof as the one that uses the global functional equation of Godement–Jacquet zeta functions in Theorem 4.2. However, it seems still meaningful to point out the contribution of the π-Poisson summation formulae on GL_1 in the theory of the global functional equation for the standard automorphic L-function $L(s, \pi \times \chi)$ for any automorphic characters χ of $\mathbb{A}^\times$ and any irreducible cuspidal automorphic representations π of $\mathrm{GL}_n(\mathbb{A})$, as an extension in a different perspective of Tate's thesis to the study of higher degree automorphic L-functions.

5. Convergence of generalized theta functions

In order to prove Theorem 1.2 and explore other possible cases of Conjecture 1.5, beyond Theorem 4.7 (or Theorem 1.1), we study the convergence issue of general π-theta functions associated with $\pi \in \Pi_{\mathbb{A}}(G_n)$, which may not be automorphic.

5A. ***Convergence of π-theta functions.*** Recall from Section 4, if $\pi = \bigotimes_{\nu\in|k|} \pi_\nu \in \Pi_{\mathbb{A}}(G_n)$, then for every $\nu \in |k|$, $\pi_\nu \in \Pi_{k_\nu}(G_n)$, the set of equivalence classes of irreducible admissible representations of $G_n(k_\nu)$, where at almost all finite local places ν, π_ν is unramified and at any infinite local place ν, π_ν is of Casselman–Wallach type as representation of $G_n(k_\nu)$. As in (4-1), for any $\pi = \bigotimes_\nu \pi_\nu \in \Pi_{\mathbb{A}}(G_n)$, we have that

$$\mathcal{S}_\pi(\mathbb{A}^\times) = \bigotimes_{\nu\in|k|} \mathcal{S}_{\pi_\nu}(k_\nu^\times).$$

For $\phi \in \mathcal{S}_\pi(\mathbb{A}^\times)$, we are going to show that the π-theta function

$$\Theta_\pi(x, \phi) = \sum_{\alpha\in k^\times} \phi(\alpha x) \tag{5-1}$$

converges absolutely and locally uniformly as a function in $x \in \mathbb{A}^\times$, under an assumption (Assumption 5.1) on the unramified local components π_ν of π.

For any $\pi = \bigotimes_\nu \pi_\nu \in \Pi_{\mathbb{A}}(G_n)$, let S_π be a finite subset of local places of k containing $|k|_\infty$ such that for any finite local place $\nu \notin S_\pi$, the local component π_ν is unramified. For any π_ν with $\nu \notin S_\pi$, via the Satake isomorphism, one has the

Frobenius–Hecke conjugacy class $c(\pi_\nu)$ in $G_n^\vee(\mathbb{C})$ associated to π_ν. We write

$$c(\pi_\nu) := \mathrm{diag}(q_\nu^{s_1(\pi_\nu)}, \dots, q_\nu^{s_n(\pi_\nu)}) \in \mathrm{GL}_n(\mathbb{C}) = G_n^\vee(\mathbb{C}), \tag{5-2}$$

up to the adjoint action of $G_n^\vee(\mathbb{C})$, with $s_j(\pi_\nu) \in \mathbb{C}$ for $j = 1, 2, \dots, n$, where q_ν is the cardinality of the residue field $\kappa_\nu = \mathfrak{o}_\nu/\mathfrak{p}_\nu$. The following is the assumption we take on the unramified local components π_ν of π.

Assumption 5.1 (uniform bound). *Let $\pi = \bigotimes_\nu \pi_\nu \in \Pi_{\mathbb{A}}(G_n)$ be an irreducible admissible representation of $G_n(\mathbb{A})$. There exists a positive real number κ_π, which depends only on π, such that*

$$\max_{1 \le j \le n} \{\mathrm{Re}(s_j(\pi_\nu))\} < \kappa_\pi$$

for every $\nu \notin S_\pi$.

Then we need to prove some technical local results.

Lemma 5.2. *For any $\pi = \bigotimes_\nu \pi_\nu \in \Pi_{\mathbb{A}}(G_n)$ with Assumption 5.1, there exists a positive real number $s_\pi \ge \kappa_\pi$ such that, for any real number $a_0 > s_\pi$, the limit*

$$\lim_{|x|_\nu \to 0} \phi_\nu(x)|x|_\nu^{a_0} = 0$$

holds, as a function in $x \in k_\nu^\times$, for any $\phi_\nu \in \mathcal{S}_{\pi_\nu}(k_\nu^\times)$ and any local place $\nu \in |k|$. In particular, $\phi_\nu(x)|x|_\nu^{a_0}$ extends to a continuous function on k_ν, which is compactly supported on k_ν if $\nu \in |k|_f$ and is of Schwartz type at ∞ of k_ν if $\nu \in |k|_\infty$.

Proof. By Proposition 3.7, at any $\nu \in |k|$, we have that $\mathcal{S}_{\pi_\nu}(k_\nu^\times) \subset \mathfrak{F}(k_\nu^\times)$, which is defined in Definition 2.1. In the following we discuss separately for $\nu \in |k|_\infty$ and for $\nu \in |k|_f$.

When $\nu \in |k|_\infty$, the asymptotic of $\phi_\nu \in \mathcal{S}_{\pi_\nu}(k_\nu^\times)$ near $x = 0$ is characterized in Definition 2.1. In particular, following the notation in Definition 2.1, the fixed sequence $\{\lambda_k\}_{k=0}^\infty$ has strictly increasing real part $\{\mathrm{Re}(\lambda_k)\}_{k=0}^\infty$. Hence, for any positive real number $s_0 \in \mathbb{R}$ satisfying the inequality

$$s_0 + \mathrm{Re}(\lambda_0) > 0,$$

the limit

$$\lim_{|x|_\nu \to 0} \phi_\nu(x) \cdot |x|_\nu^{s_0} = 0 \tag{5-3}$$

holds, because the limit formula in Definition 2.1 is termwise differentiable and uniform (even after termwise differentiation). Hence, the function $\phi_\nu(x) \cdot |x|_\nu^{s_0}$ is continuous over k_ν for any positive real number s_0 satisfying $s_0 + \mathrm{Re}(\lambda_0) > 0$. It is clear that the function $\phi_\nu(x) \cdot |x|_\nu^{s_0}$ is still of Schwartz type at ∞. Since the set $|k|_\infty$ is finite, it is possible to choose a sufficiently positive $s_\infty \in \mathbb{R}$ such that the prescribed property holds for all functions $\phi_\nu(x)\cdot|x|_\nu^{a_0}$ with $\phi_\nu\in\mathcal{S}_{\pi_\nu}(k_\nu^\times)$ at all $\nu\in|k|_\infty$, as long as $a_0\geq s_\infty$.

It remains to treat the case when $\nu\in|k|_f$, the finite local places of k. We consider the local zeta integrals $\mathcal{Z}(s,\phi_\nu,\omega_\nu)$ for any $\phi_\nu\in\mathcal{S}_{\pi_\nu}(k_\nu^\times)$, and any unitary character $\omega_\nu\in\Omega_\nu^\wedge$. By Theorem 3.4, it converges absolutely for $\mathrm{Re}(s)$ sufficiently positive and admits a meromorphic continuation to $s\in\mathbb{C}$. For each $\nu\in|k|_f$, we take c_{π_ν} to be a sufficiently positive real number, such that $\mathcal{Z}(s,\phi_\nu,\omega_\nu)$ converges absolutely for $\mathrm{Re}(s)>c_{\pi_\nu}$. If $\nu\notin S_\pi$, then π_ν is unramified. In this case, the zeta integral $\mathcal{Z}(s,\phi_\nu,\omega_\nu)$ converges absolutely for $\mathrm{Re}(s)>\kappa_\pi$, where the positive real number κ_π depends on π only, according to Assumption 5.1. Hence, if we take a positive real number c_π with

$$c_\pi:=\max\{\kappa_\pi,c_{\pi_\nu}\mid\nu\in S_\pi\cap|k|_f\}, \tag{5-4}$$

then for any $\phi_\nu\in\mathcal{S}_{\pi_\nu}(k_\nu^\times)$, and any unitary character $\omega_\nu\in\Omega_\nu^\wedge$, the local zeta integral $\mathcal{Z}(s,\phi_\nu,\omega_\nu)$ converges absolutely for $\mathrm{Re}(s)>c_\pi$ at all finite local places $\nu\in|k|_f$.

By the Mellin inversion formula as displayed in (2-6), we have

$$\phi_\nu(x)\cdot|x|_\nu^d=\sum_{\omega_\nu\in\Omega_\nu^\wedge}\big(\mathrm{Res}_{z=0}(\mathcal{Z}(s+d,\phi_\nu,\omega_\nu)|x|_\nu^{-s}q_\nu^s)\big)\omega_\nu(\mathrm{ac}(x))^{-1}, \tag{5-5}$$

where $z=q_\nu^{-s}$ and $d>c_\pi$. Since the summation on the right-hand side is finite, it suffices to show that the limit formula

$$\lim_{|x|_\nu\to0}\mathrm{Res}_{z=0}(\mathcal{Z}(s+d,\phi_\nu,\omega_\nu)|x|_\nu^{-s}q_\nu^s)=0 \tag{5-6}$$

holds for each $\omega_\nu\in\Omega_\nu^\wedge$.

It is clear that $\mathcal{Z}(s+d,\phi_\nu,\omega_\nu)$ is holomorphic for $\mathrm{Re}(s)>-(d-c_\pi)$. By Theorem 3.4, we have

$$\mathcal{Z}(s+d,\phi_\nu,\omega_\nu)=p_\nu(s)\cdot L(s+d,\pi_\nu\times\omega_\nu),$$

where $p_\nu(s)\in\mathbb{C}[q_\nu^s,q_\nu^{-s}]$, depending on ϕ_ν. By the supercuspidal support of $\pi_\nu\otimes\omega_\nu$, we obtain that the representation $\pi_\nu\otimes\omega_\nu$ can be embedded, as an irreducible subrepresentation, into the induced representation

$$\pi_\nu\otimes\omega_\nu\hookrightarrow\Pi_\nu:=\mathrm{Ind}_{P(k_\nu)}^{G_n(k_\nu)}\tau_{\nu,1}\otimes\cdots\otimes\tau_{\nu,t_\nu},$$

where $\tau_{\nu,j}$ is an irreducible supercuspidal representation of $G_{a_{\nu,j}}(k_\nu)$ with $n=a_{\nu,1}+\cdots+a_{\nu,t_\nu}$ (see [22]). By [16, Theorem 3.4], we have

$$L(s,\Pi_\nu)=L(s,\tau_{\nu,1})\cdots L(s,\tau_{\nu,t_\nu}).$$

By [16, Corollary 3.6], we have

$$\frac{L(s,\pi_\nu\times\omega_\nu)}{L(s,\Pi_\nu)}$$

is a polynomial in q_ν^{-s}. Hence, we obtain that for the given $\phi_\nu\in\mathcal{S}_{\pi_\nu}(k_\nu^\times)$, there exists a polynomial $\mathcal{P}_\nu(s)$ in q_ν^s and q_ν^{-s}, depending on $\pi_\nu\otimes\omega_\nu$ and ϕ_ν, such that

$$\mathcal{Z}(s+d,\phi_\nu,\omega_\nu)=\mathcal{P}_\nu(s)L(s+d,\Pi_\nu). \tag{5-7}$$

By applying [16, Proposition 5.11] to the local L-functions $L(s,\tau_{\nu,j})$, we obtain that $L(s,\tau_{\nu,j})=1$ when $\tau_{\nu,j}$ is either supercuspidal ($a_{\nu,j}\geq 2$) or a ramified character ($a_{\nu,j}=1$). Hence, there exists an integer $1\leq r_\nu\leq t_\nu\leq n$, such that

$$\mathcal{Z}(s+d,\phi_\nu,\omega_\nu)=\mathcal{P}_\nu(s)\prod_{j=1}^{r_\nu}\frac{1}{1-q_\nu^{-s-d+s_{\nu,j}}}=\prod_{j=1}^{r_\nu}\bigg(\sum_{\ell_j=0}^{\infty}q_\nu^{-(s+d-s_{\nu,j})\ell_j}\bigg) \tag{5-8}$$

for some $s_{\nu,j}\in\mathbb{C}$, with $j=1,2,\ldots,r_\nu$.

Now we are ready to discuss the limit in (5-6). For $z=q_\nu^{-s}$, we have

$$\mathcal{Z}(s+d,\phi_\nu,\omega_\nu)|x|_\nu^{-s}q_\nu^s=\mathcal{P}_\nu(z)\cdot\frac{\prod_{j=1}^{r_\nu}\big(\sum_{\ell_j=0}^{\infty}q_\nu^{-\ell_j(d-s_{\nu,j})}\cdot z^{\ell_j}\big)}{z^{\mathrm{ord}_\nu(x)+1}}, \tag{5-9}$$

where $\mathcal{P}_\nu(z)$ is a polynomial function in z, z^{-1}. By taking the residue at $z=0$, we obtain that

$$\mathrm{Res}_{z=0}(\mathcal{Z}(s+d,\phi_\nu,\omega_\nu)|x|_\nu^{-s}q_\nu^s)=\mathfrak{C}_0(x), \tag{5-10}$$

where $\mathfrak{C}_0(x)$ is the coefficient of the constant term of

$$\mathcal{P}_\nu(z)\cdot\frac{\prod_{j=1}^{r_\nu}\big(\sum_{\ell_j=0}^{\infty}q_\nu^{-\ell_j(d-s_{\nu,j})}\cdot z^{\ell_j}\big)}{z^{\mathrm{ord}_\nu(x)}}. \tag{5-11}$$

Since $\mathcal{P}_\nu(z)$ is a polynomial function in z, z^{-1} with degree depending on π, without loss of generality, we may assume that $\mathcal{P}_\nu(z)\equiv 1$ when we compute $\mathfrak{C}_0(x)$. In this case, the constant term of (5-11) with $\mathcal{P}_\nu(z)\equiv 1$ is equal to

$$\sum_{\substack{\ell_1+\cdots+\ell_{r_\nu}=\mathrm{ord}_\nu(x)\\ \ell_1,\ldots,\ell_{r_\nu}\geq 0}} q_\nu^{-\ell_1(d-s_{\nu,j})-\cdots-\ell_{t_0}(d-s_{\nu,j})}. \tag{5-12}$$

When $\nu\notin S_\pi$, π_ν is unramified,

$$\mathrm{diag}(q_\nu^{s_{\nu,1}},\ldots,q_\nu^{s_{\nu,n}})=c(\pi_\nu)$$

is the Frobenius–Hecke conjugacy class associated to π_ν in $G_n^\vee(\mathbb{C})$ with $s_{\nu,j}=s_j(\pi_\nu)$ for $j=1,2,\ldots,n$. By Assumption 5.1 and the definition of the positive real number

c_π as in (5-4), we take $d_0 = 0$ and have

$$d - \mathrm{Re}(s_{\nu,j}) > c_\pi - \mathrm{Re}(s_{\nu,j}) \geq 0 \tag{5-13}$$

for all $j = 1, 2, \ldots, n$. For the remaining finite local places ν, we may choose a positive real number d_0 such that

$$d + d_0 - \mathrm{Re}(s_{\nu,j}) > c_\pi + d_0 - \mathrm{Re}(s_{\nu,j}) \geq 0 \tag{5-14}$$

for all $j = 1, 2, \ldots, r_\nu$ and all $\nu \in S_\pi \cap |k|_f$. Hence, with the choice of d_0, we have

$$\begin{aligned} &\left|\mathrm{Res}_{z=0}(\mathcal{Z}(s+d+d_0, \phi_\nu, \omega_\nu)|x|_\nu^{-s} q_\nu^s)\right| \\ &\qquad \leq \sum_{\substack{\ell_1+\cdots+\ell_{r_\nu}=\mathrm{ord}_\nu(x) \\ \ell_1,\ldots,\ell_{r_\nu}\geq 0}} q_\nu^{-\sum_{j=1}^{r_\nu} \ell_j (d+d_0-\mathrm{Re}(s_{\nu,j}))} \\ &\qquad \leq \sum_{\substack{\ell_1+\cdots+\ell_{r_\nu}=\mathrm{ord}_\nu(x) \\ \ell_1,\ldots,\ell_{r_\nu}\geq 0}} q_\nu^{-\mathrm{ord}_\nu(x)(d+d_0-\max_j\{\mathrm{Re}(s_{\nu,j})\})} \\ &\qquad = \binom{\mathrm{ord}_\nu(x)+r_\nu-1}{r_\nu-1} \cdot q_\nu^{-\mathrm{ord}_\nu(x)(d+d_0-\max_j\{\mathrm{Re}(s_{\nu,j})\})}. \end{aligned} \tag{5-15}$$

Since $d + d_0 - \max_j\{\mathrm{Re}(s_{\nu,j})\} > 0$, and the function $\binom{\mathrm{ord}_\nu(x)+r_\nu-1}{r_\nu-1}$ is a polynomial in $\mathrm{ord}_\nu(x)$, we must have that

$$\lim_{\mathrm{ord}_\nu(x)\to+\infty} \binom{\mathrm{ord}_\nu(x)+r_\nu-1}{r_\nu-1} \cdot q_\nu^{-\mathrm{ord}_\nu(x)(d+d_0-\max_j\{\mathrm{Re}(s_{\nu,j})\})} = 0.$$

By (5-5), if $d > c_\pi + d_0$, then we must have that

$$\lim_{x_\nu\to 0} \phi_\nu(x) \cdot |x|_\nu^d = 0$$

for all $\phi_\nu \in \mathcal{S}_{\pi_\nu}(k_\nu^\times)$ and at all $\nu \in |k|_f$. It is clear that the function $\phi_\nu(x) \cdot |x|_\nu^d$ is continuous over k_ν and has compact support.

Finally, by taking a positive real number $s_\pi = \max\{s_\infty, c_\pi + d_0\}$, we obtain that for any $a_0 > s_\pi$, the function $\phi_\nu(x)|x|_\nu^{a_0}$ is continuous over k_ν and has the limit

$$\lim_{|x|_\nu\to 0} \phi_\nu(x)|x|_\nu^{a_0} = 0,$$

for any $\phi_\nu \in \mathcal{S}_{\pi_\nu}(k_\nu^\times)$ and at any local place $\nu \in |k|$. We are done. □

Lemma 5.3. *Let $\pi = \bigotimes_\nu \pi_\nu \in \Pi_{\mathbb{A}}(G_n)$ satisfy Assumption 5.1. For any $\nu \notin S_\pi$, the basic function $\mathbb{L}_{\pi_\nu} \in \mathcal{S}_{\pi_\nu}(k_\nu^\times)$ is supported on $\mathfrak{o}_\nu - \{0\}$ with*

$$\mathbb{L}_{\pi_\nu}(\mathfrak{o}_\nu^\times) = 1.$$

There exists a positive real number $b_\pi \geq s_\pi$, which is independent of ν, such that, for any $b_0 > b_\pi$,

$$\left|\mathbb{L}_{\pi_\nu}(x)\cdot |x|_\nu^{b_0}\right| \leq 1$$

holds, as a function in $x \in k_\nu^\times$, for all $\nu \notin S_\pi$.

Proof. We continue with the proof of Lemma 5.2 for the non-Archimedean case, and specialize it to the unramified situation. Note that the basic function $\mathbb{L}_{\pi_\nu} \in \mathcal{S}_{\pi_\nu}(k_\nu^\times)$ is the Mellin inversion of the local unramified L-factor

$$\mathcal{Z}(s, \mathbb{L}_{\pi_\nu}) = L(s, \pi_\nu),$$

whose Mellin inversion can be calculated by (5-5) after setting $\mathcal{P}_\nu(s) = 1$. In other words, taking the constant s_π as in Lemma 5.2, we have, for any $a_0 > s_\pi$,

$$\mathbb{L}_{\pi_\nu}(x)\cdot |x|^{a_0} = \mathrm{Res}_{z=0}(\mathcal{Z}(s+a_0, \mathbb{L}_{\pi_\nu})|x|_\nu^{-s} q_\nu^s).$$

As in (5-9), we write

$$\text{(5-16)}\quad \mathcal{Z}(s+a_0, \mathbb{L}_{\pi_\nu}) = \frac{1}{\prod_{j=1}^n (1-q_\nu^{-s-a_0+s_j(\pi_\nu)})} = \prod_{j=1}^n \left(\sum_{\ell_j \geq 0} q_\nu^{\ell_j(s_j(\pi_\nu)-a_0)} z^{\ell_j}\right),$$

where we write $z = q_\nu^{-s}$ and $c_j(\pi_\nu) = q_\nu^{s_j(\pi_\nu)}$. From the Laurent expansion on the right-hand side, we obtain that the function

$$\mathcal{Z}(s+a_0, \mathbb{L}_{\pi_\nu})|x|_\nu^{-s} q_\nu^s$$

is holomorphic in $z = q_\nu^{-s}$ whenever $x \notin \mathfrak{o}_\nu$. By taking the residue at $z = 0$, we obtain that

$$\mathbb{L}_{\pi_\nu}(x)\cdot |x|^{a_0} = 0 \quad \text{for } x \notin \mathfrak{o}_\nu.$$

Hence, the basic function $\mathbb{L}_{\pi_\nu}(x)$ has support included in $\mathfrak{o}_\nu$. Similarly, we apply the Mellin inversion, as calculated by (5-5), to the case $x \in \mathfrak{o}^\times$, and obtain that the residue picks up the constant term of the right-hand side of (5-16) as a function of $z = q^{-s}$, which is equal to 1. Therefore, we obtain

$$\mathbb{L}_{\pi_\nu}(\mathfrak{o}_\nu^\times) = 1.$$

Finally, whenever $x \in \mathfrak{o}_F \smallsetminus \{0\}$, we apply (5-15) to the unramified case, and obtain that

$$\left|\mathbb{L}_{\pi_\nu}(x)\cdot |x|^b\right| \leq \binom{\mathrm{ord}_\nu(x)+n-1}{n-1}\cdot q_\nu^{-\mathrm{ord}_\nu(x)\cdot \min_j\{b-\mathrm{Re}(s_j(\pi_\nu))\}},$$

as long as $b > s_\pi$. By Assumption 5.1, we have

$$\min_{1\leq j\leq n}\{b - s_j(\pi_\nu)\} > \min_{1\leq j\leq n}\{\kappa_\pi - s_j(\pi_\nu)\} > 0.$$

Therefore, whenever $\operatorname{ord}_\nu(x) \geq 1$,

$$\binom{\operatorname{ord}_\nu(x)+n-1}{n-1} \cdot q_\nu^{-\operatorname{ord}_\nu(x)\cdot\min_j\{b-\operatorname{Re}(s_j(\pi_\nu))\}} \leq \binom{\operatorname{ord}_\nu(x)+n-1}{n-1} \cdot 2^{-\operatorname{ord}_\nu(x)\cdot\min_j\{b-\operatorname{Re}(s_j(\pi_\nu))\}}$$

since $q_\nu \geq 2$ for any $\nu \notin S_\pi$. It turns out that we only need to find a positive integer $b_\pi \geq s_\pi \in \mathbb{R}$ such that, for any $b > b_\pi$,

$$\binom{\operatorname{ord}_\nu(x)+n-1}{n-1} \cdot 2^{-\operatorname{ord}_\nu(x)\cdot\min_j\{b-\operatorname{Re}(s_j(\pi_\nu))\}} \leq 1$$

holds for any $\nu \notin S_\pi$ and $\operatorname{ord}_\nu(x) \geq 1$. Equivalently, after applying the function $\log_2$ on both sides, the above inequality becomes

$$\log_2\binom{\operatorname{ord}_\nu(x)+n-1}{n-1} - \operatorname{ord}_\nu(x) \cdot \min_j\{b - \operatorname{Re}(s_j(\pi_\nu))\} \leq 0.$$

Hence, it suffices to show the existence of $b_\pi \in \mathbb{R}$ so that

$$\min_j\{b - \operatorname{Re}(s_j(\pi_\nu))\} = b - \max_j\{\operatorname{Re}(s_j(\pi_\nu))\} > b_\pi - \kappa_\pi \geq \frac{\log_2\binom{\operatorname{ord}_\nu(x)+n-1}{n-1}}{\operatorname{ord}_\nu(x)}$$

for any $\operatorname{ord}_\nu(x) \geq 1$, i.e.,

$$b_\pi \geq \kappa_\pi + \frac{\log_2\binom{\operatorname{ord}_\nu(x)+n-1}{n-1}}{\operatorname{ord}_\nu(x)} \tag{5-17}$$

for any $\operatorname{ord}_\nu(x) \geq 1$. As a function of $t \geq 1$,

$$\log_2\binom{t+n-1}{n-1} = \log_2 \frac{\prod_{k=1}^{n-1}(t+k)}{(n-1)!} \geq \log_2 \frac{\prod_{k=1}^{n-1}(1+k)}{(n-1)!} \geq \log_2 n \geq 0.$$

Thus we obtain that

$$\frac{\log_2\binom{t+n-1}{n-1}}{t} \geq 0$$

for any $t \geq 1$. On the other hand, by L'Hôspital's rule, one must have that

$$\lim_{t\to\infty} \frac{\log_2\binom{t+n-1}{n-1}}{t} = 0.$$

It follows, as a continuous function in $t \geq 1$, there exists a constant $c_0 \in \mathbb{R}$ such that

$$\frac{\log_2\binom{t+n-1}{n-1}}{t} < c_0$$

for any $t \geq 1$. It is clear now that the inequality in (5-17) holds for any

$$b_\pi \geq \kappa_\pi + c_0.$$

Therefore it suffices to take $b_\pi = \max\{s_\pi, \kappa_\pi + c_0\}$. We are done. $\square$

We are ready to establish the first property for the π-theta functions $\Theta_\pi(x, \phi)$ in such generality.

Theorem 5.4 (convergence of π-theta functions). *Fix any $\pi = \bigotimes_\nu \pi_\nu \in \Pi_{\mathbb{A}}(G_n)$ with Assumption 5.1. Then, for any $\phi \in \mathcal{S}_\pi(\mathbb{A}^\times)$, the π-theta function*

$$\Theta_\pi(x, \phi) := \sum_{\alpha \in k^\times} \phi(\alpha x)$$

converges absolutely and locally uniformly as a function in $x \in \mathbb{A}^\times$.

Proof. For any $\pi = \bigotimes_\nu \pi_\nu \in \Pi_{\mathbb{A}}(G_n)$, let S_π be a finite subset of local places of k containing $|k|_\infty$ and for any finite local place $\nu \notin S_\pi$, the local component π_ν is unramified. We may assume that $\phi \in \mathcal{S}_\pi(\mathbb{A}^\times)$ is a pure restricted tensor of the form

$$\phi = \Big(\bigotimes_{\nu \notin S_\pi} \mathbb{L}_{\pi_\nu}\Big) \otimes \Big(\bigotimes_{\nu \in S_\pi} \phi_\nu\Big) = \phi_\infty \otimes \phi_f \tag{5-18}$$

with $\phi_\nu \in \mathcal{S}_{\pi_\nu}(k_\nu^\times)$ for all $\nu \in S_\pi$, $\phi_\infty = \bigotimes_{\nu \in |k|_\infty} \phi_\nu$ and $\phi_f = \bigotimes_{\nu \in |k|_f} \phi_\nu$.

Fix a positive real number $s_0 > b_\pi \geq s_\pi \geq \kappa_\pi$ where the constants κ_π, s_π, and b_π are as given in Assumption 5.1, Lemma 5.2, and Lemma 5.3, respectively. By Lemma 5.3, for any $\nu \notin S_\pi$, we have the function $\mathbb{L}_{\pi_\nu}(x)\,|x|_\nu^{s_0}$ is continuous on k_ν and supported on $\mathfrak{o}_\nu$. We have

$$\left|\mathbb{L}_{\pi_\nu}(x)\,|x|_\nu^{s_0}\right| \leq 1 \tag{5-19}$$

for every $\nu \notin S_\pi$. Similarly, for any finite $\nu \in S_\pi \cap |k|_f$, the function $\phi_\nu(x)\,|x|_\nu^{s_0}$ is continuous on k_ν with compact support. We may assume that the support of $\phi_\nu(x)\,|x|_\nu^{s_0}$ is contained in a fractional ideal $\mathfrak{p}_\nu^{m_\nu}$ for some integer $m_\nu \in \mathbb{Z}$. Write $\mathfrak{o}_\phi := \prod_{\nu \notin S_\pi} \mathfrak{o}_\nu$ and $\mathfrak{m}_\phi := \prod_{\nu \in S_\pi \cap |k|_f} \mathfrak{p}^{m_\nu}$. Then, by the weak approximation theorem [48], the product

$$\mathfrak{m}(\phi) := \mathfrak{o}_\phi \cdot \mathfrak{m}_\phi \tag{5-20}$$

is a fractional ideal of $\mathfrak{o} = \mathfrak{o}_k$, the ring of integers in k.

For any $\alpha \in k^\times$, the Artin product formula shows that $|\alpha|_{\mathbb{A}} = 1$ [48]. Hence, we obtain that

$$\Theta_\pi(1, \phi) = \sum_{\alpha \in k^\times} \phi(\alpha) = \sum_{\alpha \in k^\times} \phi(\alpha) \cdot |\alpha|_{\mathbb{A}}^{s_0}. \tag{5-21}$$

From the support of the functions $\phi_\nu \cdot |\cdot|^{s_0}$ for all $\nu \in |k|_f$, we write

$$\Theta_\pi(1,\phi) = \sum_{\alpha\in k^\times\cap\mathfrak{m}(\phi)} (\phi_\infty(\alpha)\cdot|\alpha|_\infty^{s_0})\cdot(\phi_f(\alpha)\cdot|\alpha|_f^{s_0}). \tag{5-22}$$

It is clear that for $\alpha \in k^\times \cap \mathfrak{m}(\phi)$, we have that

$$\begin{aligned}\left|\phi_f(\alpha)\cdot|\alpha|_f^{s_0}\right| &= \Big(\prod_{\nu\notin S_\pi}\left|\mathbb{L}_{\pi_\nu}(\alpha)\cdot|\alpha|_\nu^{s_0}\right|\Big)\cdot\Big(\prod_{\nu\in S_\pi\cap|k|_f}\left|\phi_\nu(\alpha)\cdot|\alpha|_\nu^{s_0}\right|\Big)\\ &\le \prod_{\nu\in S_\pi\cap|k|_f}\left|\phi_\nu(\alpha)\cdot|\alpha|_\nu^{s_0}\right|,\end{aligned}$$

because of (5-19). By Lemma 5.2, there exists a real constant c_ϕ, such that

$$\prod_{\nu\in S_\pi\cap|k|_f}\left|\phi_\nu(\alpha)\cdot|\alpha|_\nu^{s_0}\right| \le c_\phi. \tag{5-23}$$

Hence, we obtain that

$$\left|\Theta_\pi(1,\phi)\right| \le c_\phi\cdot\sum_{\alpha\in k^\times\cap\mathfrak{m}(\phi)}|\phi_\infty(\alpha)|\cdot|\alpha|_\infty^{s_0}. \tag{5-24}$$

Since the fractional ideal $\mathfrak{m}(\phi)$ of k is a lattice in $\mathbb{A}_\infty = \prod_{\nu\in|k|_\infty} k_\nu$, it suffices to show that the summation

$$\sum_{\alpha\in\mathfrak{m}(\phi)}|\phi_\infty(\alpha)|\cdot|\alpha|_\infty^{s_0} \tag{5-25}$$

is absolutely convergent.

Consider the compact set

$$\mathcal{B}_\infty(1) := \{(\alpha_\nu)\in\mathbb{A}_\infty \mid |\alpha_\nu|_\nu \le 1,\ \forall\nu\in|k|_\infty\}. \tag{5-26}$$

We write (5-25) as

$$\sum_{\alpha\in\mathfrak{m}(\phi)\cap\mathcal{B}_\infty(1)}|\phi_\infty(\alpha)|\cdot|\alpha|_\infty^{s_0} + \sum_{\alpha\in\mathfrak{m}(\phi)\smallsetminus(\mathfrak{m}(\phi)\cap\mathcal{B}_\infty(1))}|\phi_\infty(\alpha)|\cdot|\alpha|_\infty^{s_0}. \tag{5-27}$$

It is clear that the intersection of $\mathfrak{m}(\phi)$ with $\mathcal{B}_\infty(1)$ is a finite set. By Lemma 5.2, the function $\phi_\infty(x)|x|_\infty^{s_0}$ is continuous over $\mathbb{A}_\infty$, and hence is bounded over $\mathcal{B}_\infty(1)$. Thus, in (5-27), the first summation

$$\sum_{\alpha\in\mathfrak{m}(\phi)\cap\mathcal{B}_\infty(1)}|\phi_\infty(\alpha)|\cdot|\alpha|_\infty^{s_0}$$

is bounded. The second summation in (5-27), which is

$$\sum_{\alpha\in\mathfrak{m}(\phi)\smallsetminus(\mathfrak{m}(\phi)\cap\mathcal{B}_\infty(1))}|\phi_\infty(\alpha)|\cdot|\alpha|_\infty^{s_0},$$

where the function $\phi_\infty(x)\cdot|x|_\infty^{s_0}$ is of Schwartz type over $\mathbb{A}_\infty\smallsetminus\mathcal{B}_\infty(1)$, is bounded by the same proof for the absolute convergence of the classical Poisson summation formula [21, Chapter 4; 47]. This proves the absolute convergence of $\Theta_\pi(x,\phi)$ for any $\phi\in\mathcal{S}_\pi(\mathbb{A}^\times)$.

For any $x\in\mathbb{A}^\times$, we have $\Theta_\pi(x,\phi)=\Theta_\pi(1,\phi^x)$ with $\phi^x(y)=\phi(yx)$. Hence, $\Theta_\pi(x,\phi)$ converges absolutely for any $\phi\in\mathcal{S}_\pi(\mathbb{A}^\times)$.

For the locally uniform convergence of the π-theta function $\Theta_\pi(x,\phi)$ at any $x\in\mathbb{A}^\times$, by using $\Theta_\pi(x,\phi)=\Theta_\pi(1,\phi^x)$ again, it is enough to show the locally uniform convergence of $\Theta_\pi(x,\phi)$ at $x=1$ for any given factorizable function ϕ as in (5-18). As in (5-21), we may write

$$\Theta_\pi(x,\phi)=\sum_{\alpha\in k^\times}\phi(\alpha x)\cdot|\alpha|_{\mathbb{A}}^{s_0}. \tag{5-28}$$

Since $\phi=\left(\bigotimes_{\nu\notin S_\pi}\mathbb{L}_{\pi_\nu}\right)\otimes\left(\bigotimes_{\nu\in S_\pi}\phi_\nu\right)$ as in (5-18), we have $\mathfrak{m}(\phi)=\prod_{\nu\in|k|_f}\mathfrak{m}(\phi)_\nu$ as in (5-20), where $\mathfrak{m}(\phi)_\nu$ is a fractional ideal of k_ν containing the support of the function $\phi_\nu(x)\cdot|x|_\nu^{s_0}$. As in (5-22), we write

$$\Theta_\pi(x,\phi)=\sum_{\alpha\in k^\times\cap\mathfrak{m}(\phi)}(\phi_\infty(\alpha x_\infty)\cdot|\alpha|_\infty^{s_0})\cdot(\phi_f(\alpha x_f)\cdot|\alpha|_f^{s_0}). \tag{5-29}$$

Take a compact open neighborhood $\Omega_f(\phi)$ of $x_f=1$ in $\mathbb{A}_f^\times$ to be

$$\Omega_f(\phi)=\left(\prod_{\nu\notin S_\pi}\mathfrak{o}_\nu^\times\right)\cdot\left(\prod_{\nu\in|k|_f\cap S_\pi}(1+\mathfrak{p}_\nu^{d_\nu})\right),$$

where d_ν is a positive integer for $\nu\in|k|_f\cap S_\pi$. For any $x_f\in\Omega_f(\phi)$, if $\nu\notin S_\pi$, then $x_\nu\in\mathfrak{o}_\nu^\times$ and $\alpha\neq0$ and $\alpha\in\mathfrak{o}_\nu$. Hence, $\alpha x_\nu\neq0$ and $\alpha x_\nu\in\mathfrak{o}_\nu$. In this case, we have that

$$\left|\phi_\nu(\alpha x_\nu)\cdot|\alpha|_\nu^{s_0}\right|=\left|\mathbb{L}_{\pi_\nu}(\alpha x_\nu)\cdot|\alpha x_\nu|_\nu^{s_0}\right|\le1$$

by (5-19). If $\nu\in S_\pi\cap|k|_f$, then $\alpha\in\mathfrak{p}_\nu^{m_\nu}$ and $x_\nu\in1+\mathfrak{p}_\nu^{d_\nu}$, and hence we have that $\alpha x_\nu\in\mathfrak{p}_\nu^{m_\nu}$. In this case, we have that

$$\left|\phi_\nu(\alpha x_\nu)\cdot|\alpha|_\nu^{s_0}\right|=\left|\phi_\nu(\alpha x_\nu)\cdot|\alpha x_\nu|_\nu^{s_0}\right|.$$

As in (5-23), there exists a real constant c_ϕ, which is independent of $x_f\in\Omega_f(\phi)$, such that

$$\left|\phi_f(\alpha x_f)\cdot|\alpha|_f^{s_0}\right|\le\prod_{\nu\in S_\pi\cap|k|_f}\left|\phi_\nu(\alpha x_\nu)\cdot|\alpha|_\nu^{s_0}\right|\le c_\phi.$$

Hence, we obtain that

$$|\Theta_\pi(x,\phi)|\le c_\phi\cdot\sum_{\alpha\in k^\times\cap\mathfrak{m}(\phi)}\left|\phi_\infty(\alpha x_\infty)\cdot|\alpha|_\infty^{s_0}\right|\le c_\phi\cdot\sum_{\alpha\in\mathfrak{m}(\phi)}\left|\phi_\infty(\alpha x_\infty)\cdot|\alpha|_\infty^{s_0}\right|. \tag{5-30}$$

When x_∞ runs over a compact neighborhood Ω_∞ of 1 in $\mathbb{A}_\infty$, by the same argument, we are reduced to showing that

$$\sum_{\alpha\in\mathfrak{m}(\phi)\smallsetminus(\mathfrak{m}(\phi)\cap\mathcal{B}_\infty(1))} |\phi_\infty(\alpha x_\infty)|\cdot|\alpha|_\infty^{s_0} = |x_\infty|_\infty^{-s_0}\cdot\sum_{\alpha\in\mathfrak{m}(\phi)\smallsetminus(\mathfrak{m}(\phi)\cap\mathcal{B}_\infty(1))} |\phi_\infty(\alpha x_\infty)|\cdot|\alpha x_\infty|_\infty^{s_0}$$

converges uniformly. Since the function $\phi_\infty(x)\cdot|x|_\infty^{s_0}$ is of Schwartz type over $\mathbb{A}_\infty\smallsetminus\mathcal{B}_\infty(1)$, the uniform convergence of the last summation with $x_\infty\in\Omega_\infty$ follows from the same proof of that for the classical theta functions. We omit the details and finish the proof. □

5B. ***Justification of Assumption 5.1.*** We prove Assumption 5.1 when $\pi\in\mathcal{A}(G_n)$ is any irreducible admissible automorphic representation of $G_n(\mathbb{A})$, which is contained in $\Pi_{\mathbb{A}}(G_n)$.

Proposition 5.5. *For any $\pi\in\mathcal{A}(G_n)$, Assumption 5.1 holds.*

Proof. A cuspidal datum (P,ε) of G_n consists of a standard parabolic subgroup P of G_n with Levi decomposition $P=M\cdot N$ with the Levi subgroup M and the unipotent radical N, and an irreducible cuspidal automorphic representation ε of $M(\mathbb{A})$, which is square integrable up to a twist of automorphic character of $M(\mathbb{A})$. For any $\pi=\bigotimes_{\nu\in|k|}\pi_\nu\in\mathcal{A}(G_n)$, by [30], there exists a cuspidal datum (P,ε) of G_n, such that π can be realized as an irreducible subquotient of the induced representation $\mathrm{Ind}_{P(\mathbb{A})}^{G_n(\mathbb{A})}(\varepsilon)$ of $G_n(\mathbb{A})$. It follows that for any $\nu\in|k|$, the ν-component π_ν can be realized as an irreducible subquotient of the induced representation $\mathrm{Ind}_{P(k_\nu)}^{G_n(k_\nu)}(\varepsilon_\nu)$ of $G_n(k_\nu)$, where ε_ν is the ν-component of $\varepsilon=\bigotimes_\nu\varepsilon_\nu$.

Let T be the maximal torus of G_n, consisting of all diagonal matrices, and $B=T\cdot U$ be the Borel subgroup of G_n, consisting of all upper-triangular matrices. Take S to be a finite subset of $|k|$, such that S contains $|k|_\infty$ and for any $\nu\notin S$, π_ν and ε_ν are unramified. It is well known (see [7], for instance) that there exists an unramified character η_ν of the maximal torus $T(k_\nu)$, such that ε_ν embeds as a subrepresentation into the unramified induced representation $\mathrm{Ind}_{(M\cap B)(k_\nu)}^{M(k_nu)}(\eta_\nu)$. By induction in stages, we have $\mathrm{Ind}_{P(k_\nu)}^{G_n(k_\nu)}(\varepsilon_\nu)$ embeds as a subrepresentation into the spherical induced representation $\mathrm{Ind}_{B(k_\nu)}^{G_n(k_\nu)}(\eta_\nu)$ of $G_n(k_\nu)$. Hence, the irreducible spherical representation π_ν is the unique spherical subquotient of $\mathrm{Ind}_{B(k_\nu)}^{G_n(k_\nu)}(\eta_\nu)$. Via the Satake isomorphism, the Frobenius–Hecke conjugacy class of π_ν in $G_n(\mathbb{C})$ is

$$c(\pi_\nu)=\mathrm{diag}(\eta_\nu^1(\varpi_\nu),\dots,\eta_\nu^n(\varpi_\nu)).$$

Here ϖ_ν is the uniformizer of the prime ideal $\mathfrak{p}_\nu$, and for any $t=\mathrm{diag}(t_1,\dots,t_n)\in T(k_\nu)$, the unramified character η_ν is given by

$$\eta_\nu(t)=\eta_\nu^1(t_1)\cdots\eta_\nu^n(t_n).$$

It is clear that the conjugacy class of the semisimple element $c(\pi_\nu)$ in the complex dual group $M^\vee(\mathbb{C})$ of the Levi subgroup M is the Frobenius–Hecke conjugacy class $c(\varepsilon_\nu)$ of ε_ν. In other words, both π_ν and ε_ν share the same Satake parameter in $T^\vee(\mathbb{C})^{W_n}$, where W_n is the Weyl group of $G_n^\vee(\mathbb{C})$.

Take δ_ε to be an automorphic character of $M(\mathbb{A})$ such that $\delta_\varepsilon \otimes \varepsilon$ is square integrable modulo the center of M. Then for $\nu \notin S$, the ν-component $(\delta_\varepsilon \otimes \varepsilon)_\nu$ is spherical and unitary. By the classification of the spherical unitary dual of GL_n over a non-Archimedean local field k_ν [42], we obtain

$$\left|\log_{q_\nu} \max_{1\le j\le n}\left\{|(\delta_\varepsilon)_\nu^j(\varpi_\nu)\eta_\nu^j(\varpi_\nu)|\right\}\right| \le \frac{n-1}{2}.$$

Since the unramified part of the automorphic character δ_ε is completely determined by ε and the cuspidal datum (P,ε) of π is uniquely determined by π, up to conjugation, we obtain that there exists a positive real number κ_π, depending only on $\pi \in \mathcal{A}(G_n)$, such that

$$\left|\log_{q_\nu} \max_{1\le j\le n}\left\{|\eta_\nu^j(\varpi_\nu)|\right\}\right| < \kappa_\pi.$$

This justifies the assumption. □

By Theorem 5.4 and Proposition 5.5, we obtain the following absolute convergence.

Corollary 5.6. *For any $\pi \in \mathcal{A}(G_n)$ and for any $\phi \in \mathcal{S}_\pi(\mathbb{A}^\times)$, the π-theta function*

$$\Theta_\pi(x,\phi) = \sum_{\alpha\in k^\times} \phi(\alpha x)$$

converges absolutely and locally uniformly as a function in $x \in \mathbb{A}^\times$.

Another consequence of Proposition 5.5 is the absolute convergence of the global zeta integral of Godement–Jacquet type for any $\pi \in \mathcal{A}(G_n)$.

Corollary 5.7. *For any $\pi \in \mathcal{A}(G_n)$, there exists a positive real number $r_\pi \in \mathbb{R}$, such that the global zeta integral*

$$\mathcal{Z}(s, f, \varphi_\pi) = \int_{\mathrm{GL}_n(\mathbb{A})} f(g)\varphi_\pi(g)|\det g|_{\mathbb{A}}^{s+(n-1)/2}\,\mathrm{d}g, \quad f \in \mathcal{S}(M_n(\mathbb{A})),\ \varphi_\pi \in \mathcal{C}(\pi)$$

is absolutely convergent for any $\mathrm{Re}(s) > r_\pi$.

Proof. There is no harm to assume that $f = \bigotimes_\nu f_\nu$ is a pure restricted tensor. Similarly, one can write $\varphi_\pi = \prod_\nu \varphi_{\pi_\nu}$. For the given $\pi \in \mathcal{A}(G_n)$, take the finite subset S of $|k|$ as in the proof of Proposition 5.5. Then for $\nu \notin S$, the function f_ν

is the characteristic function of $M_n(\mathfrak{o}_\nu)$, and φ_{π_ν} is the zonal spherical function attached to the unramified representation π_ν. From [16, Chapter I, §7], we have

$$\mathcal{Z}(s, f_\nu, \varphi_{\pi_\nu}) = \frac{1}{\det(I_n - \alpha(\pi_\nu)q_\nu^{-s})} = L(s, \pi_\nu),$$

where the left-hand side is absolutely convergent whenever $\mathrm{Re}(s) > \kappa_\pi$, where κ_π is determined in the proof of Proposition 5.5. It follows that

$$\prod_{\nu \notin S} \mathcal{Z}(s, f_\nu, \varphi_{\pi_\nu}) = \prod_{\nu \notin S} \frac{1}{\det(I_n - \alpha(\pi_\nu)q_\nu^{-s})} = L^S(s, \pi)$$

is absolutely convergent for $\mathrm{Re}(s) > \kappa_\pi + 1$. As S is a finite set, it is clear that one can choose a real number r_π to be sufficiently positive (depending on π only) such that the global zeta integral

$$\mathcal{Z}(s, f, \varphi_\pi) = L^S(s, \pi) \cdot \prod_{\nu \in S} \mathcal{Z}(s, f_\nu, \varphi_{\pi_\nu})$$

converges absolutely for $\mathrm{Re}(s) > r_\pi$. We are done. □

6. (σ, ρ)-theta functions on GL_1

For any k-split reductive group G, as in Section 4, we denote by $\Pi_{\mathbb{A}}(G)$ the set of irreducible admissible representations of $G(\mathbb{A})$. If we write $\sigma = \bigotimes_{\nu \in |k|} \sigma_\nu$, then we assume that $\sigma_\nu \in \Pi_{k_\nu}(G)$, where at almost all finite local places ν, the local representations σ_ν are unramified. When ν is a finite local place, σ_ν is an irreducible admissible representation of $G(k_\nu)$, and when ν is an infinite local place, we assume that σ_ν is of Casselman–Wallach type as a representation of $G(k_\nu)$. Let $\mathcal{A}(G) \subset \Pi_{\mathbb{A}}(G_n)$ be the subset consisting of equivalence classes of irreducible admissible automorphic representations of $G(\mathbb{A})$, and $\mathcal{A}_{\mathrm{cusp}}(G)$ be the subset of cuspidal members of $\mathcal{A}(G)$.

For any $\sigma \in \Pi_{\mathbb{A}}(G)$ and $\rho : G^\vee \to \mathrm{GL}_n(\mathbb{C})$, we are going to introduce the (σ, ρ)-Schwartz space $\mathcal{S}_{\sigma,\rho}(\mathbb{A}^\times)$, the (σ, ρ)-Fourier operator $\mathcal{F}_{\sigma,\rho,\psi}$ and (σ, ρ)-theta functions $\Theta_{\sigma,\rho}(x, \phi)$ by means of the existence of the local Langlands reciprocity map as in the local Langlands conjecture for G. The idea is to use the local Langlands conjecture for the pair (G, ρ) as input and to formulate the global statements, such as the (σ, ρ)-Poisson summation formula, which is expected to be responsible for the global functional equation for the Langlands L-function $L(s, \sigma, \rho)$ as predicted by the Langlands conjecture, as output. The goal in this section is to prove Theorem 6.2, which contains Theorem 1.2 as a special case and serves a base for the discussion on Conjecture 1.5 and its refinement in Section 7.

6A. ***On the local Langlands conjecture.*** We briefly review the local Langlands conjecture for G over any local field $F = k_\nu$ for any local place $\nu \in |k|$.

For any Archimedean local field, the local Langlands conjecture for G is a theorem of Langlands, which follows from the Langlands classification theory [31]. At any non-Archimedean local places, for unramified representations, their local Langlands parameters are uniquely determined by the Satake isomorphism [7; 38]. In the following we review the local Langlands conjecture for an F-split reductive group G over a non-Archimedean local field F of characteristic zero.

Let $\mathcal{W}_F$ be the Weil group attached to F. The set of local Langlands parameters is denoted by $\Phi_F(G)$, which consists of continuous, Frobenius semisimple homomorphisms

$$\varsigma : \mathcal{W}_F \times \mathrm{SL}_2(\mathbb{C}) \to G^\vee, \tag{6-1}$$

up to conjugation by $G^\vee$. The local Langlands conjecture asserts that there exists a reciprocity map

$$\mathfrak{R}_{F,G} : \mathrm{Rep}(G(F)) \to \Phi_F(G), \tag{6-2}$$

where $\mathrm{Rep}(G(F))$ is the set of equivalence classes of smooth representations of $G(F)$ of finite length. $\mathfrak{R}_{F,G}$ is expected to be surjective with finite fibers, and to satisfy a series of compatibility conditions. Beyond the existence, one has to formulate and prove the uniqueness of such a local Langlands reciprocity map.

When $G = \mathrm{GL}_n$, it is a theorem of Harris–Taylor [17], of G. Henniart [19] and of P. Scholze [39] that the local Langlands reciprocity map exists and is unique with compatibility of local factors, plus other conditions. Note that in this case, the uniqueness of such a local Langlands reciprocity map is proved by Henniart using the special case of the local converse theorem [18]. However, such a uniqueness is not known in general. When G is an F-quasisplit classical group, then such a local Langlands reciprocity map exists due to the endoscopic classification of J. Arthur [2].

In their recent work [13], L. Fargues and P. Scholze use the geometrization method to understand the local Langlands conjecture. In particular, they establish a local Langlands reciprocity map for any F-split reductive groups considered in this paper. More precisely, Theorem I.9.6 of [13] asserts that for any F-split reductive group G, there exists a local Langlands reciprocity map $\mathfrak{R}_{F,G}$ from $\mathrm{Rep}(G(F))$ to $\Phi_F(G)$, satisfying nine compatibility conditions. In particular when $G = \mathrm{GL}_n$, the reciprocity map of Fargues and Scholze coincides with the unique one for GL_n. When G is an F-quasisplit classical group, the reciprocity map of Fargues and Scholze coincides with the one by Arthur. Although it is still not known (as far as the authors know) if the reciprocity map of Fargues and Scholze is unique, it is the most promising one towards the local Langlands conjecture in great generality.

From now on, we are going to take the following assumption.

Assumption 6.1. *Over any non-Archimedean local field F of characteristic zero, for any F-split reductive group G, the reciprocity map $\mathfrak{R}_{F,G}$ exists for the local Langlands conjecture for G.*

We may simply take the reciprocity map $\mathfrak{R}_{F,G}$ as defined in [13, Theorem I.9.6] for the local Langlands conjecture. In fact, the relevant discussions in the rest of this paper make no essential difference on which reciprocity map $\mathfrak{R}_{F,G}$ we are going to take. Of course, the difference may occur if one discuss the definition of local L-functions $L(s, \sigma, \rho)$ or γ-functions $\gamma(s, \sigma, \rho, \psi)$. but we are not going to discuss those objects in the rest of this paper.

6B. ***Convergence of (σ, ρ)-theta functions.*** Let G be a k-split reductive group. Take $\rho : G^\vee(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C})$ to be any finite-dimensional representation of the complex dual group $G^\vee(\mathbb{C})$. For any $\sigma \in \Pi_{\mathbb{A}}(G)$, we write $\sigma = \bigotimes_\nu \sigma_\nu$ with $\sigma_\nu \in \Pi_{k_\nu}(G)$. By Assumption 6.1, for any local place $\nu \in |k|$, there exists a local L-parameter $\varsigma_\nu = \varsigma_\nu(\sigma_\nu)$ such that the composition $\rho \circ \varsigma_\nu$ is a local L-parameter for $G_n(k_\nu) = \mathrm{GL}_n(k_\nu)$. By the local Langlands conjecture for GL_n [17; 19; 31; 39], there exists a unique irreducible admissible representation

$$\pi_\nu = \pi_\nu(\sigma, \rho, \mathfrak{R}_{k_\nu,G}) \tag{6-3}$$

belonging to $\Pi_F(G_n)$, which we may simply denote, if there is no confusion, by

$$\pi_\nu = \pi_\nu(\sigma_\nu, \rho). \tag{6-4}$$

According to the Langlands functoriality conjecture, it makes sense to define the (σ_ν, ρ)-*Schwartz space* on $k_\nu^\times$ to be

$$\mathcal{S}_{\sigma_\nu,\rho}(k_\nu^\times) := \mathcal{S}_{\pi_\nu}(k_\nu^\times). \tag{6-5}$$

At unramified local places, the (σ_ν, ρ)-basic function $\mathbb{L}_{\sigma_\nu,\rho}$ is taken to be the π_ν-basic function $\mathbb{L}_{\pi_\nu} \in \mathcal{S}_{\pi_\nu}(k_\nu^\times)$. Then we can define the (σ, ρ)-*Schwartz space* on $\mathbb{A}^\times$ to be the restricted tensor product

$$\mathcal{S}_{\sigma,\rho}(\mathbb{A}^\times) := \bigotimes_\nu \mathcal{S}_{\sigma_\nu,\rho}(k_\nu^\times) \tag{6-6}$$

with respect to the basic function $\mathbb{L}_{\sigma_\nu,\rho}$ at almost all finite local places. Note that the definition of the (σ, ρ)-Schwartz space $\mathcal{S}_{\sigma,\rho}(\mathbb{A}^\times)$ may rely on the assumption of the local Langlands reciprocity map (Assumption 6.1) at the ramified finite local places of σ, when G is a general k-split reductive group.

Let $\psi = \bigotimes_\nu \psi_\nu$ be a nontrivial additive character of $\mathbb{A}$ with $\psi(a) = 1$ for any $a \in k$. Define the (σ_ν, ρ)-*Fourier operator* $\mathcal{F}_{\sigma_\nu,\rho,\psi_\nu}$ on $k_\nu^\times$ to be

$$\mathcal{F}_{\sigma_\nu,\rho,\psi_\nu} := \mathcal{F}_{\pi_\nu,\psi_\nu}, \tag{6-7}$$

which is a linear transformation from the (σ_ν, ρ)-Schwartz space $\mathcal{S}_{\sigma_\nu,\rho}(k_\nu^\times)$ to the $(\widetilde{\sigma_\nu}, \rho)$-Schwartz space $\mathcal{S}_{\widetilde{\sigma_\nu},\rho}(k_\nu^\times)$. Then we define the (σ, ρ)-*Fourier operator*

$$\mathcal{F}_{\sigma,\rho,\psi} := \bigotimes_\nu \mathcal{F}_{\sigma_\nu,\rho,\psi_\nu}, \tag{6-8}$$

which is a linear transformation from the (σ, ρ)-Schwartz space $\mathcal{S}_{\sigma,\rho}(\mathbb{A}^\times)$ to the $(\widetilde{\sigma}, \rho)$-Schwartz space $\mathcal{S}_{\widetilde{\sigma},\rho}(\mathbb{A}^\times)$. Again, the definition of the (σ, ρ)-Fourier operator $\mathcal{F}_{\sigma,\rho,\psi}$ may rely on the assumption of the local Langlands reciprocity map (Assumption 6.1) at the ramified finite local places of σ, when G is a general k-split reductive group.

Theorem 6.2 (convergence of (σ, ρ)-theta functions). *Let $\rho : G^\vee(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C})$ be any finite-dimensional representation of the complex dual group $G^\vee(\mathbb{C})$. Take Assumption 6.1 for G. Then, for any given unitary $\sigma \in \Pi_{\mathbb{A}}(G)$, the (σ, ρ)-theta function*

$$\Theta_{\sigma,\rho}(x, \phi) := \sum_{\alpha \in k^\times} \phi(\alpha x)$$

converges absolutely and locally uniformly for any $\phi \in \mathcal{S}_{\sigma,\rho}(\mathbb{A}^\times)$ and $x \in \mathbb{A}^\times$.

Proof. As discussed above, under Assumption 6.1 for G, for any $\sigma = \bigotimes_\nu \sigma_\nu \in \Pi_{\mathbb{A}}(G)$, we obtain $\pi_\nu = \pi_\nu(\sigma_\nu, \rho)$ of $\mathrm{GL}_n(k_\nu)$ for all $\nu \in |k|$. Note that at $\nu \in |k|_\infty$, π_ν is taken to be of Casselman–Wallach type. Hence, $\pi := \bigotimes_\nu \pi_\nu$ is an irreducible admissible representation of $G_n(\mathbb{A})$ and belongs to $\Pi_{\mathbb{A}}(G_n)$. From (6-5) and (6-6), we have that

$$\Theta_{\sigma,\rho}(x, \phi) = \Theta_\pi(x, \phi)$$

for any $\phi \in \mathcal{S}_{\sigma,\rho}(\mathbb{A}^\times) = \mathcal{S}_\pi(\mathbb{A}^\times)$. By Theorem 5.4, it is sufficient to verify Assumption 5.1 for this π.

Since $\sigma = \bigotimes_\nu \sigma_\nu$ is unitary as a representation of $G(\mathbb{A})$, we must have that σ_ν is an irreducible admissible unitary representation of $G(k_\nu)$ at every $\nu \in |k|$, and is unramified for almost all $\nu \in |k|$. Since G is k-split, we can fix a Borel pair (B, T) of G defined over k, with a fixed maximal k-split torus T of G. Let ϱ be the half-sum of positive roots with respect to the given pair (B, T) and let δ_B be the modular character of $B(k_\nu)$. Then, for any coweight $\omega^\vee \in \mathrm{Hom}(\mathbb{G}_m, T)$,

$$\delta_B(\omega^\vee(\varpi_\nu))^{1/2} = q_\nu^{\langle \varrho, \omega^\vee \rangle},$$

where ϖ_ν is a fixed uniformizer in $\mathfrak{o}_\nu$ and $\omega^\vee$ is viewed as a cocharacter from $k_\nu^\times$ to $T(k_\nu)$.

Let S be a finite subset of $|k|$ containing $|k|_\infty$, such that for any $\nu \notin S$, both σ_ν and π_ν are unramified. For any $\nu \notin S$, σ_ν is unitary and unramified. Then the zonal spherical function attached to σ_ν, which is the normalized matrix coefficient of σ_ν

attached to spherical vectors in σ_ν, is bounded by 1 — see [7, p. 151, (40)], for instance. Now let

$$c(\sigma_\nu) = (q_\nu^{s_1(\sigma_\nu)}, \dots, q_\nu^{s_r(\sigma_\nu)})$$

be the Frobenius–Hecke conjugacy class of σ_ν inside $T^\vee(\mathbb{C}) \simeq (\mathbb{C}^\times)^r$, where r is the k-rational rank of G. Then, by [36, Theorem 4.7.1],

$$\max_{1\le j\le r} \{|s_j(\sigma_\nu)|\} \le \max_{1\le j\le r} \{|\langle \varrho, \omega_j^\vee\rangle|\},$$

where $\{\omega_j^\vee\}_{j=1}^r$ is a fixed set of fundamental coweights. Note that the result of [36] assumes G to be simple-connected. But if we go over the proof of [36, Theorem 4.7.1], the only result used is the explicit formula for zonal spherical functions when the Frobenius–Hecke conjugacy class $c(\sigma_\nu)$ of σ_ν is nonsingular. Hence, it suffices to apply the general formula appearing in [8, Theorem 4.2] to the proof of [36, Theorem 4.7.1]. Therefore $\max_{1\le j\le r}\{|s_j(\sigma_\nu)|\}$ has an upper bound which is independent of the local places ν.

At unramified local places, we obtain the Frobenius–Hecke conjugacy class $c(\pi_\nu)$ of π_ν to be

$$c(\pi_\nu) = \rho(c(\sigma_\nu))$$

for all $\nu \notin S$. It is clear that for this $\pi = \bigotimes_\nu \pi_\nu \in \Pi_{\mathbb{A}}(G_n)$, the family of the Frobenius–Hecke conjugacy classes

$$\{c(\pi_\nu) \mid \forall \nu \notin S\}$$

associated to the irreducible admissible representation π satisfies Assumption 5.1. We are done. $\square$

Note that the definition of the (σ, ρ)-theta function $\Theta_{\sigma,\rho}(x, \phi)$ may depend on the existence of the local Langlands reciprocity map $\mathfrak{R}_{F,G}$ for general G (Assumption 6.1), However, the absolute convergence of $\Theta_{\sigma,\rho}(x, \phi)$ in Theorem 6.2 only depends on the unramified data, and hence is independent of Assumption 6.1. As a consequence of Theorem 4.7, we have:

Corollary 6.3. *Assume the global Langlands functoriality is valid for* (G, ρ). *For* $\sigma \in \mathcal{A}_{\mathrm{cusp}}(G)$, *if its functorial transfer* π *is cuspidal on* $G_n(\mathbb{A})$, *then Conjecture 1.5 holds with* $\mathcal{E}_{\sigma,\rho}(\phi) = \Theta_{\sigma,\rho}(1, \phi)$ *for any* $\phi \in \mathcal{S}_{\sigma,\rho}(\mathbb{A}^\times)$.

7. Variants of Conjecture 1.5

In Theorem 4.7, we established a π-Poisson summation formula (Conjecture 1.5) for any $\pi \in \mathcal{A}_{\mathrm{cusp}}(G_n)$ and $\rho = \mathrm{std}$. We explore the possibilities when π is not cuspidal.

7A. ***Certain special Schwartz functions.*** As before, we take F to be any local field of characteristic zero. For any $\pi \in \Pi_F(G_n)$, recall from Definition 3.3 the space of π-Schwartz functions

$$\mathcal{S}_\pi(F^\times) = \mathrm{Span}\{\phi_{\xi,\varphi_\pi} \in \mathcal{C}^\infty(F^\times) \mid \xi \in \mathcal{S}_{\mathrm{std}}(G_n(F)),\ \varphi_\pi \in \mathcal{C}(\pi)\},$$

where the π-Schwartz function ϕ_{ξ,φ_π} associated to a pair (ξ, φ_π) is defined in (3-6). We introduce here a subspace of $\mathcal{S}_\pi(F^\times)$:

$$\mathcal{S}^\circ_\pi(F^\times) := \mathrm{Span}\{\phi_{\xi,\varphi_\pi} \mid \xi \in \mathcal{C}^\infty_c(G_n(F)),\ \varphi_\pi \in \mathcal{C}(\pi)\}. \tag{7-1}$$

We prove the following result, which provides a new description of the test functions in $\mathcal{C}^\infty_c(F^\times)$, the space of all smooth, compactly supported functions on $F^\times$.

Theorem 7.1. *Let F be any local field of characteristic zero. For any $\pi \in \Pi_F(G_n)$, the subspace $\mathcal{S}^\circ_\pi(F^\times)$ of $\mathcal{S}_\pi(F^\times)$ as defined in* (7-1) *is equal to the space $\mathcal{C}^\infty_c(F^\times)$, i.e.*,

$$\mathcal{S}^\circ_\pi(F^\times) = \mathcal{C}^\infty_c(F^\times).$$

First of all, via the determinant morphism $\det : G_n \to \mathbb{G}_m$, it is not hard to verify that the fiber integration

$$\xi \mapsto \int_{\det g = x} \xi(g)\,\mathrm{d}_x g$$

yields a surjective homomorphism from $\mathcal{C}^\infty_c(G_n(F))$ to $\mathcal{C}^\infty_c(F^\times)$. For any $\xi \in \mathcal{C}^\infty_c(G_n(F))$ and $\varphi_\pi \in \mathcal{C}(\pi)$, the product $\xi(g)\varphi_\pi(g)$ belongs to $\mathcal{C}^\infty_c(G_n(F))$. With the fiber integration through det, the function $\phi_{\xi,\varphi_\pi}(x)$ belongs to $\mathcal{C}^\infty_c(F^\times)$. Hence, we obtain that

$$\mathcal{S}^\circ_\pi(F^\times) \subset \mathcal{C}^\infty_c(F^\times) \tag{7-2}$$

for any $\pi \in \Pi_F(G_n)$. To prove the converse of (7-2), we are going to use different arguments for the non-Archimedean case and the Archimedean case.

We first consider the *non-Archimedean case*. For any quasicharacter $\chi \in \mathfrak{X}(F^\times)$, it can be written in a unique way as $\chi(x) = |x|^s_F \cdot \omega(x)$ with $s \in \mathbb{C}$ and $\omega \in \Omega^\wedge$. We may write $\chi = \chi_{s,\omega}$ and $\mathfrak{X}(F^\times) = \mathbb{C} \times \Omega^\wedge$. Furthermore, we write the space $\mathcal{Z}(\mathfrak{X}(F^\times))$ defined in Definition 2.2 as $\mathcal{Z}(\mathbb{C} \times \Omega^\wedge)$. We denote by $\mathcal{L}_{\mathrm{cpt}}$ the subspace of $\mathcal{Z}(\mathbb{C} \times \Omega^\wedge)$ consisting of functions $\mathfrak{z}(\chi_{s,\omega}) = \mathfrak{z}(s, \omega) \in \mathcal{Z}(\mathbb{C} \times \Omega^\wedge)$ with the two properties

(1) $\mathfrak{z}(s, \omega)$ is nonzero at finitely many $\omega \in \Omega^\wedge$;

(2) for any $\omega \in \Omega^\wedge$, $\mathfrak{z}(s, \omega) \in \mathbb{C}[q^s, q^{-s}]$.

By Theorem 2.3, the subspace $\mathcal{L}_{\mathrm{cpt}}$ is in one-to-one correspondence with $\mathcal{C}^\infty_c(F^\times)$ via the Mellin transform and its inversion. Denote by $\mathcal{L}^\circ_\pi$ the subspace of $\mathcal{L}_{\mathrm{cpt}}$ that is in one-to-one correspondence with the subspace $\mathcal{S}^\circ_\pi(F^\times)$. From the discussion right after [34, Theorem 3.1.1], for any given $\omega \in \Omega^\wedge$, the subspace

$$\mathcal{I}^\circ_{\pi,\omega} := \{\mathcal{Z}(s, \xi, \varphi_\pi, \omega) \mid \xi \in \mathcal{C}^\infty_c(G_n(F)),\ \varphi_\pi \in \mathcal{C}(\pi)\}$$

of the fractional ideal $\mathcal{I}_{\pi,\omega}$ as in Theorem 2.4 is equal to $\mathbb{C}[q^s, q^{-s}]$. For the fixed $\omega \in \Omega^\wedge$, the space $\mathcal{I}^\circ_{\pi,\omega}$ consists of the restriction of functions in $\mathcal{L}_{\mathrm{cpt}}$ to the slice $\mathbb{C} \times \{\omega\}$, according to the definition of the space $\mathcal{L}_{\mathrm{cpt}}$. In other words, for any fixed $\omega \in \Omega^\wedge$ and $\mathfrak{z}(s, \omega) \in \mathcal{L}_{\mathrm{cpt}}$, there exists finitely many $\xi^j_\omega \in \mathcal{C}^\infty_c(G_n(F))$ and $\varphi^j_{\pi,\omega} \in \mathcal{C}(\pi)$, such that

$$\mathfrak{z}(s, \omega) = \sum_j \mathcal{Z}(s, \xi^j_\omega, \varphi^j_{\pi,\omega}, \omega) = \sum_j \mathcal{Z}(s, \phi_{\xi^j_\omega, \varphi^j_{\pi,\omega}}, \omega)$$

for any $s \in \mathbb{C}$. Hence, with any fixed $\omega \in \Omega^\wedge$, for any $\mathfrak{z}(s, \omega) \in \mathcal{L}_{\mathrm{cpt}}$, there exists $\mathfrak{z}^\circ(s, \omega) \in \mathcal{L}^\circ_\pi$ such that

$$\mathfrak{z}(s, \omega) = \mathfrak{z}^\circ(s, \omega) \tag{7-3}$$

as functions in $s \in \mathbb{C}$.

Define, for each $\omega_0 \in \Omega^\wedge$, a function $\mathfrak{z}_{\omega_0}(s, \omega)$ with the property

$$\mathfrak{z}_{\omega_0}(s, \omega) = \begin{cases} 1, & \text{if } \omega = \omega_0, \\ 0, & \text{if } \omega \neq \omega_0. \end{cases}$$

By definition, the function $\mathfrak{z}_{\omega_0}(s, \omega)$ belongs to $\mathcal{L}_{\mathrm{cpt}}$ for each $\omega_0 \in \Omega^\wedge$. Hence, from (7-3), we have

$$\mathfrak{z}(s, \omega) = \sum_{\omega_0 \in \Omega^\wedge} \mathfrak{z}_{\omega_0}(s, \omega) \cdot \mathfrak{z}(s, \omega) = \sum_{\omega_0 \in \Omega^\wedge} \mathfrak{z}_{\omega_0}(s, \omega) \cdot \mathfrak{z}^\circ(s, \omega_0), \tag{7-4}$$

for any $\mathfrak{z}(s, \omega) \in \mathcal{L}_{\mathrm{cpt}}$. Note here that the summations only take finitely many $\omega_0 \in \Omega^\wedge$. Hence, to prove the converse of (7-2), it is enough to show that

$$\mathfrak{z}_{\omega_0}(s, \omega) \cdot \mathfrak{z}^\circ(s, \omega_0) \in \mathcal{L}^\circ_\pi \tag{7-5}$$

for every $\omega_0 \in \Omega^\wedge$. It is clear that (7-5) is an easy consequence of the following proposition.

Proposition 7.2. *The space $\mathcal{L}_{\mathrm{cpt}}$ is an associative algebra without identity, and the space $\mathcal{L}^\circ_\pi$ is an $\mathcal{L}_{\mathrm{cpt}}$-module under multiplication.*

Proof. From the definition of $\mathcal{L}_{\mathrm{cpt}}$, it is clear that $\mathcal{L}_{\mathrm{cpt}}$ is an associative algebra under the multiplication and has no identity.

To prove that $\mathcal{L}^\circ_\pi$ is an $\mathcal{L}_{\mathrm{cpt}}$-module, we take $\mathfrak{z}(s, \omega) \in \mathcal{L}_{\mathrm{cpt}}$ and write ϕ as the Mellin inversion of $\mathfrak{z}(s, \omega)$. Via the determinant morphism $\det : G_n(F) \to F^\times$, we write

$$\phi(x)=\int_{\det g=x} f(g)\,\mathrm{d}_x g$$

for some $f\in\mathcal{C}_c^\infty(G_n(F))$. For any $\xi\in\mathcal{C}_c^\infty(G_n(F))$ and $\varphi_\pi\in\mathcal{C}(\pi)$, we write $\mathfrak{z}^\circ(s,\omega)\in\mathcal{L}_\pi^\circ$ to be the Mellin transform of the function $\phi_{\xi,\varphi_\pi}\in\mathcal{S}_\pi^\circ(F^\times)$. It is enough to show that

$$\mathfrak{z}(s,\omega)\cdot\mathfrak{z}^\circ(s,\omega)\in\mathcal{L}_\pi^\circ. \tag{7-6}$$

It is clear that

$$\mathfrak{z}(s,\omega)\cdot\mathfrak{z}^\circ(s,\omega)=\mathcal{Z}(s,\phi*\phi_{\xi,\varphi_\pi},\omega). \tag{7-7}$$

Now we compute the right-hand side of (7-7) with a fixed $\omega\in\Omega^\wedge$:

$$\begin{aligned}\mathcal{Z}(s,\phi*\phi_{\xi,\varphi_\pi},\omega)&=\int_{x\in F^\times}\omega(x)|x|_F^{s-1/2}\,\mathrm{d}^\times x\int_{y\in F^\times}\phi(y)\phi_{\xi,\varphi_\pi}(y^{-1}x)\,\mathrm{d}^\times y\\&=\int_{F^\times}\omega(x)|x|_F^{s-1/2}\,\mathrm{d}^\times x\int_{F^\times}\mathrm{d}^\times y\int_{\det g=y} f(g)\,\mathrm{d}_y g\\&\qquad\cdot\int_{\det h=y^{-1}x}\xi(h)\varphi_\pi(h)\,\mathrm{d}_{y^{-1}x}h.\end{aligned} \tag{7-8}$$

After changing variable $g\to gh^{-1}$, the right-hand side of (7-8) is equal to

$$\begin{aligned}&\int_{F^\times}\omega(x)|x|_F^{s-1/2}\,\mathrm{d}^\times x\int_{F^\times}\mathrm{d}^\times y\int_{\det g=x} f(gh^{-1})\,\mathrm{d}_x g\\&\qquad\cdot\int_{\det h=y^{-1}x}\xi(h)\varphi_\pi(h)\,\mathrm{d}_{y^{-1}x}h.\end{aligned} \tag{7-9}$$

In (7-9), the integration in $y\in F^\times$ yields the identity

$$\begin{aligned}&\int_{y\in F^\times}\mathrm{d}^\times y\int_{\det h=y^{-1}x} f(gh^{-1})\xi(h)\varphi_\pi(h)\,\mathrm{d}_{y^{-1}x}h\\&\qquad=\int_{G_n(F)} f(gh^{-1})\xi(h)\varphi_\pi(h)\,\mathrm{d}h.\end{aligned} \tag{7-10}$$

By applying (7-10) to (7-9), we can write (7-9) as

$$\int_{F^\times}\omega(x)|x|_F^{s-1/2}\,\mathrm{d}^\times x\int_{\det g=x}\int_{G_n(F)} f(gh^{-1})\xi(h)\varphi_\pi(h)\,\mathrm{d}h\,\mathrm{d}_x g,$$

which is equal to

$$\int_{g\in G_n(F)}\int_{h\in G_n(F)} f(gh^{-1})\xi(h)\varphi_\pi(h)\omega(\det g)|\det g|_F^{s-1/2}\,\mathrm{d}h\,\mathrm{d}g. \tag{7-11}$$

By taking a change of variable $h\to h^{-1}g$, (7-11) can be written as

$$\int_{g\in G_n(F)}\int_{h\in G_n(F)} f(h)\xi(h^{-1}g)\varphi_\pi(h^{-1}g)\omega(\det g)|\det g|_F^{s-1/2}\,\mathrm{d}h\,\mathrm{d}g. \tag{7-12}$$

Since $f, \xi \in \mathcal{C}_c^\infty(G_n(F))$, the function

$$(g, h) \mapsto f(h)\xi(h^{-1}g)$$

belongs to the space $\mathcal{C}_c^\infty(G_n(k_\nu) \times G_n(k_\nu))$. By [5, 1.22], we have

$$\mathcal{C}_c^\infty(G_n(k_\nu) \times G_n(k_\nu)) \simeq \mathcal{C}_c^\infty(G_n(k_\nu)) \otimes \mathcal{C}_c^\infty(G_n(k_\nu)).$$

We may write

$$f(h)\xi(h^{-1}g) = \sum_{j=1}^{r} \xi_j(g)\xi^j(h)$$

for some $\xi_j(g)$ and $\xi^j(h)$ in $\mathcal{C}_c^\infty(G_n(F))$. Meanwhile, we write

$$\varphi_\pi(h^{-1}g) = \langle \pi(h^{-1}g)v, \tilde{v}\rangle = \langle \pi(g)v, \widetilde{\pi}(h)\tilde{v}\rangle, \quad v \in \pi,\ \tilde{v} \in \widetilde{\pi}. \tag{7-13}$$

By applying those explicit expressions to the integral in (7-12), we obtain that (7-12) is equal to

$$\begin{aligned}
&\sum_{j=1}^{r} \int_{g \in G_n(F)} \int_{h \in G_n(F)} \xi_i(g)\xi^i(h)\langle \pi(g)v, \widetilde{\pi}(h)\tilde{v}\rangle \omega(\det g)|\det g|_F^{s-1/2}\, \mathrm{d}h\, \mathrm{d}g \\
&\quad = \sum_{j=1}^{r} \int_{g \in G_n(F)} \xi_i(g)\omega(\det g)|\det g|_F^{s-1/2}\, \mathrm{d}g \int_{h \in G_n(F)} \xi^i(h)\langle \pi(g)v, \widetilde{\pi}(h)\tilde{v}\rangle\, \mathrm{d}h \\
&\quad = \sum_{j=1}^{r} \int_{G_n(F)} \xi_i(g)\langle \pi(g)v, \widetilde{\pi}(\xi^j)\tilde{v}\rangle \omega(\det g)|\det g|_F^{s-1/2}\, \mathrm{d}g.
\end{aligned}$$

By writing $\varphi_{\pi,j}(g) := \langle \pi(g)v, \widetilde{\pi}(\xi^j)\tilde{v}\rangle$, we obtain that

$$\begin{aligned}
\mathcal{Z}(s, \phi * \phi_{\xi,\varphi_\pi}, \omega) &= \sum_{j=1}^{r} \int_{G_n(F)} \xi_i(g)\varphi_{\pi,j}(g)\omega(\det g)|\det g|_F^{s-1/2}\, \mathrm{d}g \\
&= \sum_{j=1}^{r} \mathcal{Z}(s, \phi_{\xi_j,\varphi_{\pi,j}}, \omega).
\end{aligned} \tag{7-14}$$

By definition of $\mathcal{L}_\pi^\circ$, we obtain that the right-hand side of (7-14) belongs to the space $\mathcal{L}_\pi^\circ$, and so does the function $\mathcal{Z}(s, \phi*\phi_{\xi,\varphi_\pi}, \omega)$. Therefore we have established (7-6). We are done. $\square$

We have finished the proof of Theorem 7.1 for the non-Archimedean case. Now we turn to the proof the converse of (7-2), and hence Theorem 7.1 for the Archimedean case.

We first treat the case when $F \simeq \mathbb{C}$. It is clear that the multiplication map

$$\begin{aligned} \mathfrak{m} : \mathbb{C}^\times \times \mathrm{SL}_n(\mathbb{C}) &\to G_n(\mathbb{C}) \\ (a, h) &\mapsto a \cdot h \end{aligned} \tag{7-15}$$

provides a surjective group homomorphism with finite kernel, which in particular is a smooth (covering) map. The push-forward map along $\mathfrak{m}$, which we denote by

$$\mathfrak{m}_* : \mathcal{C}_c^\infty(\mathbb{C}^\times \times \mathrm{SL}_n(\mathbb{C})) \to \mathcal{C}_c^\infty(G_n(\mathbb{C})) \tag{7-16}$$

is surjective. In fact, the surjectivity can be easily verified as follows. For any $f \in \mathcal{C}_c^\infty(G_n(\mathbb{C}))$, let $\mathfrak{m}^*(f)$ be the pull-back of f along $\mathfrak{m}$, which is given by

$$\mathfrak{m}^*(f)(a, h) = f(a \cdot h), \quad (a, h) \in \mathbb{C}^\times \times \mathrm{SL}_n(\mathbb{C}).$$

Then we obtain that

$$\mathfrak{m}_*(\mathfrak{m}^*(f))(h) = \sum_{\substack{(a,h) \in \mathbb{C}^\times \times \mathrm{SL}_n(\mathbb{C}) \\ a \cdot h = g}} f(a \cdot h) = |\ker(\mathfrak{m})| \cdot f(g), \quad g \in G_n(\mathbb{C}).$$

It is clear now that the subspace $\mathcal{S}_\pi^\circ(\mathbb{C}^\times)$ of $\mathcal{S}_\pi(\mathbb{C}^\times)$ is equal to the space spanned by the functions

$$\begin{aligned} \phi_{\mathfrak{m}_*(f), \varphi_\pi}(x) &= \int_{\det g = x} \mathfrak{m}_*(f)(g) \varphi_\pi(g) \, \mathrm{d}_x g \\ &= \int_{\det g = x} \sum_{\substack{(a,h) \in \mathbb{C}^\times \times \mathrm{SL}_n(\mathbb{C}) \\ a \cdot h = g}} f(a, h) \varphi_\pi(g) \, \mathrm{d}_x g \end{aligned} \tag{7-17}$$

with all $f \in \mathcal{C}_c^\infty(\mathbb{C}^\times \times \mathrm{SL}_n(\mathbb{C}))$ and $\varphi_\pi \in \mathcal{C}(\pi)$. Thus, in order to show the converse of (7-2), it suffices to show that any function in $\mathcal{C}_c^\infty(\mathbb{C}^\times)$ is of the form as in the last line of (7-17).

Let χ_π be the central character of π. By a change of variable, we write (7-17) as

$$\phi_{\mathfrak{m}_*(f), \varphi_\pi}(x) = \int_{\mathrm{SL}_n(\mathbb{C})} \sum_{a^n = x} f(a, h) \cdot \chi_\pi(a) \cdot \varphi_\pi(h) \, \mathrm{d}_1 h. \tag{7-18}$$

Assume that $f \in \mathcal{C}_c^\infty(\mathbb{C}^\times \times \mathrm{SL}_n(\mathbb{C}))$ is given by a pure tensor

$$f(a, h) = f_1(a) \cdot f_2(h)$$

with $f_1 \in \mathcal{C}_c^\infty(\mathbb{C}^\times)$ and $f_2 \in \mathcal{C}_c^\infty(\mathrm{SL}_n(\mathbb{C}))$. Then (7-18) can be written as

$$\phi_{\mathfrak{m}_*(f), \varphi_\pi}(x) = \left(\sum_{a^n = x} f_1(a) \chi_\pi(a) \right) \cdot \int_{\mathrm{SL}_n(\mathbb{C})} f_2(h) \varphi_\pi(h) \, \mathrm{d}_1 h. \tag{7-19}$$

It is clear that multiplying by the character $\chi_{\pi_{\mathbb{C}}}$ stabilizes the space $\mathcal{C}_c^\infty(\mathbb{C}^\times)$, which means that $f_1(a)\chi_\pi(a) \in \mathcal{C}_c^\infty(\mathbb{C}^\times)$ for any $f_1 \in \mathcal{C}_c^\infty(\mathbb{C}^\times)$. The map

$$\mathcal{C}_c^\infty(\mathbb{C}^\times) \to \mathcal{C}_c^\infty(\mathbb{C}^\times) \quad \text{with } f \mapsto \left(x \mapsto \sum_{a^n = x} f(a)\right)$$

is surjective, since it is the push-forward map along the surjective covering map

$$\mathbb{C}^\times \to \mathbb{C}^\times \quad \text{with } a \mapsto a^n.$$

Therefore, any function in $\mathcal{C}_c^\infty(\mathbb{C}^\times)$ can be written as $\phi_{\mathfrak{m}_*(f),\varphi_\pi}(x)$ for some $\varphi_\pi \in \mathcal{C}(\pi)$ and $f \in \mathcal{C}_c^\infty(\mathbb{C}^\times \times \mathrm{SL}_n(\mathbb{C}))$. This finishes the proof of the converse of (7-2).

We now turn to the case when $F = \mathbb{R}$ and treat the cases of n odd and of n even separately.

When n is odd, the multiplication map

$$\begin{aligned} \mathfrak{m} : \mathbb{R}^\times \times \mathrm{SL}_n(\mathbb{R}) &\to G_n(\mathbb{R}) \\ (a, g) &\mapsto a \cdot g \end{aligned}$$

is surjective, the proof in the complex case is applicable and yields a proof for this case. We omit the details here.

When n is even, we write $G_n(\mathbb{R})$ as a disjoint union two real smooth manifolds:

$$G_n(\mathbb{R}) = G_n^+(\mathbb{R}) \sqcup G_n^-(\mathbb{R}),$$

where $G_n^+(\mathbb{R})$ (resp. $G_n^-(\mathbb{R})$) consists of elements in $G_n(\mathbb{R})$ with positive (resp. negative) determinant.

By the surjectivity of the map

$$\mathbb{R}_{>0} \times \mathrm{SL}_n(\mathbb{R}) \to G_n^+(\mathbb{R}) \quad \text{with } (a, g) \mapsto a \cdot g,$$

the proof in the complex case shows that the space $\mathcal{S}_\pi^\circ(\mathbb{R}^\times)$ contains the space $\mathcal{C}_c^\infty(\mathbb{R}_{>0})$. Since $\mathbb{R}^\times = \mathbb{R}_{>0} \sqcup \mathbb{R}_{<0}$, we have that

$$\mathcal{C}_c^\infty(\mathbb{R}^\times) = \mathcal{C}_c^\infty(\mathbb{R}_{>0}) \oplus \mathcal{C}_c^\infty(\mathbb{R}_{<0}).$$

It remains to show that

$$\mathcal{C}_c^\infty(\mathbb{R}_{<0}) \subset \mathcal{S}_\pi^\circ(\mathbb{R}^\times). \tag{7-20}$$

Take $\theta = \mathrm{diag}(-1, 1, \ldots, 1) \in G_n(\mathbb{R})$ and consider the map

$$\mathfrak{m}^- : \mathbb{R}_{>0} \times \mathrm{SL}_n(\mathbb{R}) \to G_n^-(\mathbb{R}) \quad \text{with } (a, h) \mapsto a \cdot h \cdot \theta.$$

As the complex situation, for any $f \in \mathcal{C}_c^\infty(\mathbb{R}_{>0} \times \mathrm{SL}_n(\mathbb{R}))$, we set

$$\phi_{\mathfrak{m}_*^-(f),\varphi_\pi}(x) = \int_{\det g = x} \sum_{\substack{(a,h)\in\mathbb{R}_{>0}\times\mathrm{SL}_n(\mathbb{R})\\ a\cdot h\cdot\theta = g}} f(a,h)\cdot\varphi_\pi(g)\,\mathrm{d}_x g, \tag{7-21}$$

for $x \in \mathbb{R}_{<0}$. We only need to show the space spanned by the functions of the form

$$\{x \mapsto \phi_{\mathfrak{m}_*^-(f),\varphi_\pi}(x) \mid f \in \mathcal{C}_c^\infty(\mathbb{R}_{>0} \times \mathrm{SL}_n(\mathbb{R})),\ \varphi_\pi \in \mathcal{C}(\pi)\} \tag{7-22}$$

with $x \in \mathbb{R}_{<0}$ contains (and hence is equal to) the space $\mathcal{C}_c^\infty(\mathbb{R}_{<0})$.

By a simple change of variable, we obtain that

$$\phi_{\mathfrak{m}_*^-(f),\varphi_\pi}(x) = \int_{\mathrm{SL}_n(\mathbb{R})} \sum_{a^n=-x} f(a,h)\cdot\chi_\pi(a)\varphi_\pi(h\cdot\theta)\,\mathrm{d}_1 h, \tag{7-23}$$

where χ_π is the central character of $\pi \in \Pi_{\mathbb{R}}(n)$. Assume that $f(a,h) = f_1(a)\cdot f_2(h)$ is a pure tensor with $f_1 \in \mathcal{C}_c^\infty(\mathbb{R}_{>0})$ and $f_2 \in \mathcal{C}_c^\infty(\mathrm{SL}_n(\mathbb{R}))$. Then (7-23) can be written as

$$\phi_{\mathfrak{m}_*^-(f),\varphi_\pi}(x) = \sum_{a^n=-x} f_1(a)\chi_{\pi_{\mathbb{R}}}(a)\cdot\int_{\mathrm{SL}_n(\mathbb{R})} f_2(h)\varphi_\pi(h\cdot\theta)\,\mathrm{d}_1 h, \tag{7-24}$$

with $x \in \mathbb{R}_{<0}$. For $y = -x > 0$, the functions of the form

$$\sum_{a^n=y} f_1(a)\chi_{\pi_{\mathbb{R}}}(a)\cdot\int_{\mathrm{SL}_n(\mathbb{R})} f_2(h)\varphi_\pi(h\cdot\theta)\,\mathrm{d}_1 h$$

recover the space $\mathcal{C}_c^\infty(\mathbb{R}_{>0})$, as treated in the previous case. Thus, the functions of the form in (7-24) recover the space $\mathcal{C}_c^\infty(\mathbb{R}_{<0})$. This completes the proof for the converse of (7-2) for the Archimedean case. Therefore, we finish the proof of Theorem 7.1.

7B. ***A variant of π-Poisson summation formulae.*** For any $\pi = \bigotimes_{\nu\in|k|} \in \Pi_{\mathbb{A}}(G_n)$, we define in (4-1) the space of π-Schwartz functions on $\mathbb{A}^\times$:

$$\mathcal{S}_\pi(\mathbb{A}^\times) = \bigotimes_\nu \mathcal{S}_{\pi_\nu}(k_\nu^\times).$$

We define $\mathcal{S}_\pi^\circ(\mathbb{A}^\times)$ to be the subspace of $\mathcal{S}_\pi(\mathbb{A}^\times)$ that is spanned by the functions of the form $\phi = \bigotimes_\nu \phi_\nu \in \mathcal{S}_\pi(\mathbb{A}^\times)$ with at least one local component ϕ_ν belonging to $\mathcal{C}_c^\infty(k_\nu^\times)$. Note that for any $\phi = \bigotimes_\nu \phi_\nu \in \mathcal{S}_\pi(\mathbb{A}^\times)$, there are at most finitely many local components from $\mathcal{C}_c^\infty(k_\nu^\times)$. It is also easy to verify from the definition of the π-Fourier operator $\mathcal{F}_{\pi,\psi}$ as in (4-3) and Theorem 7.1 that there exist functions $\phi = \bigotimes_\nu \phi_\nu \in \mathcal{S}_\pi(\mathbb{A}^\times)$, such that

$$\mathcal{F}_{\pi,\psi}(\phi) = \bigotimes_{\nu\in|k|} \mathcal{F}_{\pi_\nu,\psi_\nu}(\phi_\nu) \in \mathcal{S}_{\widetilde{\pi}}^\circ(\mathbb{A}^\times).$$

We define $\mathcal{S}^{\circ\circ}_\pi(\mathbb{A}^\times)$ to be the subspace of $\mathcal{S}^{\circ}_\pi(\mathbb{A}^\times)$ that is spanned by the functions of the form $\phi = \bigotimes_\nu \phi_\nu \in \mathcal{S}^{\circ}_\pi(\mathbb{A}^\times)$ with the property that $\mathcal{F}_{\pi,\psi}(\phi) \in \mathcal{S}^{\circ}_{\widetilde{\pi}}(\mathbb{A}^\times)$.

Theorem 7.3. *Assume that $\pi \in \mathcal{A}(G_n)$ is square integrable. For any $\phi \in \mathcal{S}^{\circ\circ}_\pi(\mathbb{A}^\times)$, the π-Poisson summation formula*

$$\Theta_\pi(x, \phi) = \Theta_{\widetilde{\pi}}(x^{-1}, \mathcal{F}_{\pi,\psi}(\phi))$$

holds as functions in $x \in \mathbb{A}^\times$.

Proof. By Corollary 5.6, both $\Theta_\pi(x, \phi)$ and $\Theta_{\widetilde{\pi}}(x^{-1}, \mathcal{F}_{\pi,\psi}(\phi))$ are absolutely convergent. It suffices to show the equality. The proof goes in the same way as Theorem 4.7 when $\pi \in \mathcal{A}_{\mathrm{cusp}}(G_n)$. The first key point is that when π is square integrable, its matrix coefficients can also be realized as the integrals in (4-14), with $\beta_1, \beta_2 \in V_\pi$ being not necessarily cuspidal.

The second key point is to prove that the boundary terms defined in (4-23) vanish automatically by the local assumption on ϕ at the two local places ν_1 and ν_2. More precisely, take $\phi = \phi_{\xi,\varphi_\pi} \in \mathcal{S}_\pi(\mathbb{A}^\times)$ and assume that

$$\phi = \bigotimes_\nu \phi_\nu = \bigotimes_\nu \phi_{\xi_\nu,\varphi_{\pi_\nu}}$$

with $\xi_\nu(g) = |\det g|_\nu^{n/2} f_\nu(g)$ for some $f_\nu \in \mathcal{S}(M_n(k_\nu))$, and $\varphi_{\pi_\nu} \in \mathcal{C}(\pi_\nu)$. The assumption at the two local places ν_1 and ν_2 is the same as that $f_{\nu_1} \in \mathcal{C}^\infty_c(G_n(k_{\nu_1}))$ and $\mathcal{F}_{\psi_{\nu_2}}(f_{\nu_2}) \in \mathcal{C}^\infty_c(G_n(k_{\nu_2}))$. For $f = \bigotimes_\nu f_\nu$ and $\mathcal{F}_\psi(f) = \bigotimes_\nu \mathcal{F}_{\psi_\nu}(f_\nu)$ with the above f_{ν_1} at the given local place ν_1 and $\mathcal{F}_{\psi_{\nu_2}}(f_{\nu_2})$ at the given local place ν_2, the boundary terms $B_f(h, g)$ in (4-23) must vanish automatically. Therefore, the summation identity is established. We refer the other details of the proof to the proof of Theorem 4.7. $\square$

7C. *Refinement of Conjecture 1.5.* We are going to state our conjecture on (σ, ρ)-Poisson summation formula on GL_1 with more details, which refines Conjecture 1.5. We will continue with the discussions in Section 6B. By Assumption 6.1, for $\sigma \in \mathcal{A}_{\mathrm{cusp}}(G)$, there exists a $\pi = \bigotimes_\nu \pi_\nu \in \Pi_{\mathbb{A}}(G_n)$ with $\pi_\nu = \pi_\nu(\sigma_\nu, \rho)$ for all ν. We define the space $\mathcal{S}_{\sigma,\rho}(\mathbb{A}^\times)$ of (σ, ρ)-Schwartz functions as in (6-5) and (6-6); and the (σ, ρ)-Fourier operator $\mathcal{F}_{\sigma,\rho,\psi}$ as in (6-7) and (6-8). Finally we define the space $\mathcal{S}^{\circ\circ}_{\sigma,\rho}(\mathbb{A}^\times)$ to be equal to the space $\mathcal{S}^{\circ\circ}_\pi(\mathbb{A}^\times)$, which is defined in Section 7B.

Conjecture 7.4 (refinement of Conjecture 1.5). *Let G be a k-split reductive group, and $\rho : G^\vee(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C})$ be a representation of the complex dual group $G^\vee(\mathbb{C})$. With Assumption 6.1, for any $\sigma \in \mathcal{A}_{\mathrm{cusp}}(G)$, there exist $k^\times$-invariant linear functionals $\mathcal{E}_{\sigma,\rho}$ and $\mathcal{E}_{\widetilde{\sigma},\rho}$ on $\mathcal{S}_{\sigma,\rho}(\mathbb{A}^\times)$ and $\mathcal{S}_{\widetilde{\sigma},\rho}(\mathbb{A}^\times)$, respectively, such that the (σ, ρ)-Poisson summation formula*

$$\mathcal{E}_{\sigma,\rho}(\phi) = \mathcal{E}_{\widetilde{\sigma},\rho}(\mathcal{F}_{\sigma,\rho,\psi}(\phi)) \tag{7-25}$$

holds for $\phi \in \mathcal{S}_{\sigma,\rho}(\mathbb{A}^\times)$. If $\phi \in \mathcal{S}^{\circ\circ}_{\sigma,\rho}(\mathbb{A}^\times)$, then the identity in (7-25) *holds for*

$$\mathcal{E}_{\sigma,\rho}(\phi)(x) = \Theta_{\sigma,\rho}(x,\phi) = \sum_{\alpha\in k^\times} \phi(\alpha x)$$

with $x \in \mathbb{A}^\times$.

We make remarks on Conjecture 1.5 and its refinement Conjecture 7.4.

Remark 7.5. In Corollary 6.3, we have proved that if the global Langlands functoriality is valid for (G, ρ) and the image of σ under the functorial transfer is cuspidal on $G_n(\mathbb{A})$, then Conjectures 1.5 and 7.4 hold with

$$\mathcal{E}_{\sigma,\rho}(\phi)(x) = \Theta_{\sigma,\rho}(x,\phi) = \sum_{\alpha\in k^\times} \phi(\alpha x)$$

for any $\phi \in \mathcal{S}_{\sigma,\rho}(\mathbb{A}^\times)$ and any $x \in \mathbb{A}^\times$. If the global Langlands functoriality is valid for (G, ρ) and the image of σ under the functorial transfer is square integrable on $G_n(\mathbb{A})$, then by Theorem 7.3, a similar (σ, ρ)-Poisson summation formula in Conjecture 7.4 holds for $\phi \in \mathcal{S}^{\circ\circ}_\pi(\mathbb{A}^\times)$.

8. Critical zeros of $L(s, \pi \times \chi)$

We provide a spectral interpretation of critical zeros of the automorphic $L(s, \pi \times \chi)$ for any $\pi \in \mathcal{A}_{\text{cusp}}(G_n)$ and character χ of the idele class group $\mathcal{C}_k = k^\times\backslash\mathbb{A}^\times$ for a number field k. This can be viewed as a reformulation of [40, Theorem 2] (see also [12]) in the adelic formulation of A. Connes [11], and is a extension of [11, Theorem III.1] from the Hecke L-functions $L(s, \chi)$ to the standard automorphic L-functions $L(s, \pi \times \chi)$.

8A. ***Pólya–Hilbert–Connes pairs.*** For a number field k, denote by $\mathbb{A}^1 = \mathbb{A}^1_k :=$ $\ker(|\cdot|_{\mathbb{A}})$ the norm one ideles of k. Denote by $\mathcal{C}_k := k^\times\backslash\mathbb{A}^\times$ the idele class group of k, and define $\mathcal{C}^1_k := k^\times\backslash\mathbb{A}^1$. Then $\mathbb{A}^\times$ has a noncanonical decomposition

$$\mathbb{A}^\times = \mathbb{A}^1 \times \mathbb{R}^\times_+, \tag{8-1}$$

where $\mathbb{R}^\times_+ = |\mathbb{A}^\times|_{\mathbb{A}}$ is the connected component of 1. In the following, we choose and fix a section of the short exact sequence

$$1 \to \mathbb{A}^1 \to \mathbb{A}^\times \to \mathbb{R}^\times_+ \to 1.$$

This gives a fixed noncanonical decomposition

$$\mathcal{C}_k = \mathcal{C}^1_k \times \mathbb{R}^\times_+. \tag{8-2}$$

For any $\delta > 0$, define $L^2_\delta(\mathcal{C}_k)$ to the space consisting of measurable functions

$$\theta : \mathcal{C}_k \to \mathbb{C}$$

with a finite Sobolev norm $\|\cdot\|_\delta$ as defined by

$$\|\theta\|_\delta^2 := \int_{\mathcal{C}_k} |\theta(x)|^2 (1 + (\log|x|_{\mathbb{A}})^2)^{\delta/2} \, \mathrm{d}^\times x. \tag{8-3}$$

It is clear that the space $L^2_\delta(\mathcal{C}_k)$ is a $\mathcal{C}_k$-module via the right translation $\mathfrak{r}_\delta$ defined by

$$\mathfrak{r}_\delta(a)(\theta)(x) := \theta(xa) \tag{8-4}$$

for any $\theta \in L^2_\delta(\mathcal{C}_k)$ and $a, x \in \mathcal{C}_k$. Note that the $\mathcal{C}_k$-module $L^2_\delta(\mathcal{C}_k)$ is not unitary, but has the property

$$\|\mathfrak{r}_\delta(x)\| = o(\log|x|_{\mathbb{A}})^{\delta/2}, \quad |x|_{\mathbb{A}} \to \infty. \tag{8-5}$$

For any $\pi \in \mathcal{A}_{\mathrm{cusp}}(G_n)$, take any $\phi \in \mathcal{S}_\pi(\mathbb{A}^\times)$. By Proposition 4.8, for any $\kappa > 0$, there exists a positive constant $c_{\kappa,\phi}$ such that the π-theta function $\Theta_\pi(x, \phi)$ enjoys the property

$$|\Theta_\pi(x, \phi)| \le c_{\kappa,\phi} \cdot \min\{|x|_{\mathbb{A}}, |x|_{\mathbb{A}}^{-1}\}^\kappa,$$

in particular, $\Theta_\pi(x, \phi)$ decays rapidly when $|x|_{\mathbb{A}} \to 0$ or $|x|_{\mathbb{A}} \to \infty$, and hence belongs to $L^2_\delta(\mathcal{C}_k)$. Define

$$\|\phi\|_\delta^2 := \int_{\mathcal{C}_k} |\Theta_\pi(x, \phi)|^2 (1 + (\log|x|_{\mathbb{A}})^2)^{\delta/2} \, \mathrm{d}^\times x \tag{8-6}$$

for any $\phi \in \mathcal{S}_\pi(\mathbb{A}^\times)$. Then we have the embedding

$$\Theta_\pi : \phi \in \mathcal{S}_\pi(\mathbb{A}^\times) \mapsto \Theta_\pi(\,\cdot\,, \phi) \in L^2_\delta(\mathcal{C}_k) \tag{8-7}$$

with respect to the Sobolev norms defined in (8-3) and (8-6), respectively.

Denote by $\overline{\Theta_\pi}$ the completion of the image $\Theta_\pi(\mathcal{S}_\pi(\mathbb{A}^\times))$ in $L^2_\delta(\mathcal{C}_k)$. Since

$$\mathfrak{r}_\delta(y)(\Theta_\pi(\,\cdot\,, \phi))(x) = \Theta_\pi(x, \mathfrak{r}_\delta(y)\phi)$$

for any $\phi \in \mathcal{S}_\pi(\mathbb{A}^\times)$, with $x, y \in \mathcal{C}_k$, the closed subspace $\overline{\Theta_\pi}$ is also a $\mathcal{C}_k$-module. Define the quotient space

$$\mathcal{H}_{\pi,\delta} := L^2_\delta(\mathcal{C}_k)/\overline{\Theta_\pi}, \tag{8-8}$$

which is also a $\mathcal{C}_k$-module. The associated representation is denoted by $\mathfrak{r}_{\pi,\delta}$. It is clear that the restriction of the $\mathcal{C}_k$-module to $\mathcal{C}_k^1$ is unitary and has the decomposition

$$\mathcal{H}_{\pi,\delta}|_{\mathcal{C}_k^1} = \bigoplus_{\chi \in \widehat{\mathcal{C}_k^1}} \mathcal{H}_{\pi,\delta,\chi}. \tag{8-9}$$

By the fixed (noncanonical) decomposition in (8-2), each eigenspace $\mathcal{H}_{\pi,\delta,\chi}$ is a module of $\mathbb{R}_+^\times$. The associated representation is denoted by $\mathfrak{r}_{\pi,\delta,\chi}$. Note that $\mathfrak{r}_{\pi,\delta,\chi}$

is also a representation of $\mathcal{C}_k = \mathcal{C}_k^1 \times \mathbb{R}_+^\times$ on $\mathcal{H}_{\pi,\delta,\chi}$. The action of $\mathbb{R}_+^\times$ on $\mathcal{H}_{\pi,\delta,\chi}$ generates a flow with the infinitesimal generator

$$\mathfrak{D}_{\pi,\delta,\chi}(\theta) := \lim_{\epsilon\to 0} \frac{1}{\epsilon}\big(\mathfrak{r}_{\pi,\delta,\chi}(\exp(\epsilon) - 1)\big)\theta \tag{8-10}$$

for any $\theta \in \mathcal{H}_{\pi,\delta,\chi}$. As in [11], one should take the pair

$$(\mathcal{H}_{\pi,\delta,\chi}, \mathfrak{D}_{\pi,\delta,\chi}) \tag{8-11}$$

to be a candidate of the Pólya–Hilbert space. We call it a *Pólya–Hilbert–Connes pair*.

For any $\chi \in \widehat{\mathcal{C}_k^1}$, by the fixed noncanonical decomposition $\mathcal{C}_k = \mathcal{C}_k^1 \times \mathbb{R}_+^\times$ as in (8-2), the character χ has a unique extension to $\mathcal{C}_k$ by defining that it is trivial on $\mathbb{R}_+^\times$. We may still denote the extended character by χ.

Theorem 8.1 (critical zeros of $L(s, \pi \times \chi)$). *Given any $\pi \in \mathcal{A}_{\mathrm{cusp}}(G_n)$ and any character $\chi \in \widehat{\mathcal{C}_k^1}$, take $\mathfrak{D}_{\pi,\delta,\chi}$ as in* (8-10) *with $\delta > 1$.*

(1) *The spectrum $\mathrm{Sp}(\mathfrak{D}_{\pi,\delta,\chi})$ is discrete and is contained in $i \cdot \mathbb{R}$ with $i = \sqrt{-1}$.*

(2) *$\mu \in \mathrm{Sp}(\mathfrak{D}_{\pi,\delta,\chi})$ if and only if $L\big(\frac{1}{2} + \mu, \pi \times \chi\big) = 0$.*

(3) *The multiplicity $m_{\mathrm{Sp}(\mathfrak{D}_{\pi,\delta,\chi})}(\mu)$ is equal to the largest integer $m < \frac{1}{2}(1+\delta)$ with $m \le m_{L(s,\pi\times\chi)}\big(\frac{1}{2}+\mu\big)$, the multiplicity of $\frac{1}{2}+\mu$ as a zero of the automorphic L-function $L(s, \pi \times \chi)$.*

Note Theorem 8.1 can be viewed as a reformulation of [40, Theorem 2] in the adelic framework of [11] and is an extension of [11, Theorem III.1] from the Hecke L-functions $L(s, \chi)$ to the standard automorphic L-functions $L(s, \pi \times \chi)$. See also [12] for relevant discussion.

8B. *Proof of Theorem 8.1.* We are going to prove Theorem 8.1 by using an argument that combines the approach of [11] and that of [40].

Consider the pairing

$$L^2_\delta(\mathcal{C}_k) \times L^2_{-\delta}(\mathcal{C}_k) \to \mathbb{C} \quad \text{with } (\theta, \eta) \mapsto \langle\theta, \eta\rangle, \tag{8-12}$$

where the pairing is defined by the integral

$$\langle\theta, \eta\rangle := \int_{\mathcal{C}_k} \theta(x)\eta(x)\, \mathrm{d}^\times x.$$

For any $y \in \mathcal{C}_k$, we have

$$\langle\mathfrak{r}_\delta(y)\theta, \eta\rangle = \langle\theta, \mathfrak{r}_{-\delta}(y^{-1})\eta\rangle$$

for any $\theta \in L^2_\delta(\mathcal{C}_k)$ and $\eta \in L^2_{-\delta}(\mathcal{C}_k)$.

Consider a function $\eta \in L^2_{-\delta}(\mathcal{C}_k)$ as a distribution on the eigenspace $\mathcal{H}_{\pi,\delta,\chi}$. Then

$$\langle \theta, \eta \rangle = 0 \tag{8-13}$$

for any $\theta \in \overline{\Theta_\pi}$, and, for any $t \in \mathcal{C}^1_k$,

$$\mathfrak{r}_{-\delta}(t)\eta = \chi^{-1}(t)\eta$$

as a distribution on $\mathcal{H}_{\pi,\delta,\chi}$. Hence, we may write, for $x = ta \in \mathcal{C}_k = \mathcal{C}^1_k \times \mathbb{R}^\times_+$, the fixed noncanonical decomposition, that

$$\eta(x) = \chi^{-1}(t)\beta(a), \tag{8-14}$$

where $\beta(a)$ is a measurable function on $\mathbb{R}^\times_+$ with

$$\|\beta\|_\delta = \int_{\mathbb{R}^\times_+} |\beta(a)|^2 (1 + (\log |a|)^2)^{-\delta/2} \, \mathrm{d}^\times a < \infty.$$

The orthogonality in (8-13) can be written as

$$\int_{\mathcal{C}_k} \Theta_\pi(x, \phi)\eta(x) \, \mathrm{d}^\times x = 0 \tag{8-15}$$

for any $\phi \in \mathcal{S}_\pi(\mathbb{A}^\times)$. As in [40], we prove the following lemma, which is a reformulation of Lemma 1 of [40].

Lemma 8.2. *The subspace of $\overline{\Theta_\pi}$ generated by functions of type*

$$(b * \Theta_\pi(\,\cdot\,, \phi))(t) = \int_{\mathcal{C}_k} b(x)\Theta_\pi(x^{-1}t, \phi) \, \mathrm{d}^\times x$$

with all $b(x) \in \mathcal{C}^\infty_c(\mathcal{C}_k)$ is dense in $\overline{\Theta_\pi}$.

Proof. We reformulate the proof of [40, Lemma 1]. For any $\theta \in \overline{\Theta_\pi}$, we have

$$(b * \theta)(t) = \int_{\mathcal{C}_k} b(x)\theta(x^{-1}t) \, \mathrm{d}^\times x = \int_{\mathcal{C}_k} b(x)\theta^\vee(t^{-1}x) \, \mathrm{d}^\times x = \mathfrak{r}_\delta(b)(\theta^\vee)(t^{-1})$$

for any $b(x) \in \mathcal{C}^\infty_c(\mathcal{C}_k)$. Since $\overline{\Theta_\pi}$ is a closed subspace of $L^2_\delta(\mathcal{C}_k)$ and is a $\mathcal{C}_k$-module, it is clear that $b * \theta$ belongs to $\overline{\Theta_\pi}$. In particular, we have that $b * \Theta_\pi(\,\cdot\,, \phi)$ belongs to $\overline{\Theta_\pi}$ for all $b(x) \in \mathcal{C}^\infty_c(\mathcal{C}_k)$ and all $\phi \in \mathcal{S}_\pi(\mathbb{A}^\times)$.

Next, by [11, Lemma 5], there exists a sequence of functions $\{f_n\}$ with f_n belonging to the space $\mathcal{S}(\mathcal{C}_k)$ of the Bruhat–Schwartz functions on $\mathcal{C}_k$, such that $\mathfrak{r}_\delta(f_n)$ tends strongly to 1 in $L^2_\delta(\mathcal{C}_k)$ and the norm of $\mathfrak{r}_\delta(f_n)$ are bounded. Now following the same argument as in the proof of [40, Lemma 1], we obtain that there exists a sequence of functions $b_n \in \mathcal{C}^\infty_c(\mathcal{C}_k)$ with the properties

(1) $\mathfrak{r}_\delta(b_n)$ converges strongly to 1;

(2) the norm of $\mathfrak{r}_\delta(b_n)$ is bounded;

(3) $b_n * \Theta_\pi(\,\cdot\,, \phi)$ converges to $\Theta_\pi(\,\cdot\,, \phi)$ for any $\phi \in \mathcal{S}_\pi(\mathbb{A}^\times)$.

Therefore the linear span of $b * \Theta_\pi(\,\cdot\,,\phi)$ with $b(x) \in \mathcal{C}_c^\infty(\mathcal{C}_k)$ and $\phi \in \mathcal{S}_\pi(\mathbb{A}^\times)$ is dense in $\overline{\Theta_\pi}$. We are done. $\square$

By Lemma 8.2, it is enough to consider the orthogonality

$$\int_{\mathcal{C}_k} (b * \Theta_\pi(\,\cdot\,,\phi))(x)\eta(x)\,\mathrm{d}^\times x = 0 \tag{8-16}$$

for any $\phi \in \mathcal{S}_\pi(\mathbb{A}^\times)$ and $b(x) \in \mathcal{C}_c^\infty(\mathcal{C}_k)$.

Lemma 8.3. *For any $\eta \in L^2_{-\delta}(\mathcal{C}_k)$, the integral*

$$\int_{\mathcal{C}_k} (b * \Theta_\pi(\,\cdot\,,\phi))(x)\eta(x)\,\mathrm{d}^\times x$$

is zero for any $b \in \mathcal{C}_c^\infty(\mathcal{C}_k)$ and any $\phi \in \mathcal{S}_\pi(\mathbb{A}^\times)$ if and only if

$$L\big(\tfrac{1}{2} + i\mu, \pi \times \chi\big) \cdot \mathcal{M}(\eta)(\chi_{i\mu})$$

is zero as a function in $\chi_{i\mu}$, where $\chi_{i\mu}$ is any unitary character of $\mathcal{C}_k$ that can be written as $\chi_{i\mu}(x) = \chi(t)a^{i\mu}$ for $x = ta \in \mathcal{C}_k = \mathcal{C}_k^1 \times \mathbb{R}_+^\times$, the fixed noncanonical decomposition.

Proof. We are going to apply the Parseval formula for the Fourier transform from $\mathcal{C}_k$ to its unitary dual $\widehat{\mathcal{C}_k}$ to (8-16). Since $\chi_{i\mu}(x) = \chi(t)a^{i\mu}$, the Fourier transform for $\mathcal{C}_k$ is

$$\mathcal{M}(\theta)(\chi_{i\mu}) = \int_{\mathcal{C}_k} \theta(x)\chi_{i\mu}^{-1}(x)\,\mathrm{d}^\times x.$$

By applying the Parseval formula to the integral

$$\int_{\mathcal{C}_k} (b * \Theta_\pi(\,\cdot\,,\phi))(x)\eta(x)\,\mathrm{d}^\times x,$$

we obtain that (8-16) is equivalent to

$$\int_{\widehat{\mathcal{C}_k}} \mathcal{M}(b)(\chi_{i\mu})\mathcal{M}(\Theta_\pi(\,\cdot\,,\phi))(\chi_{i\mu})\mathcal{M}(\eta)(\chi_{i\mu})\,\mathrm{d}\chi_{i\mu} = 0 \tag{8-17}$$

for any $\phi \in \mathcal{S}_\pi(\mathbb{A}^\times)$ and $b(x) \in \mathcal{C}_c^\infty(\mathcal{C}_k)$. It is easy to verify from definition that

$$\mathcal{M}(\Theta_\pi(\,\cdot\,,\phi))(\chi_{i\mu}) = \mathcal{Z}\big(\tfrac{1}{2} + i\mu, \phi, \chi\big),$$

where the right-hand side is the global (GL_1) zeta integral as defined in (4-4). From Corollary 4.4 and [16, Proposition 13.9], the global zeta integral $\mathcal{Z}\big(\tfrac{1}{2} + i\mu, \phi, \chi\big)$ is a bounded function in μ. Hence, the product

$$\mathcal{T}_{\phi,\eta}(\chi_{i\mu}) := \mathcal{Z}\big(\tfrac{1}{2} + i\mu, \phi, \chi\big) \cdot \mathcal{M}(\eta)(\chi_{i\mu})$$

is a tempered distribution on $\widehat{\mathcal{C}_k}$. It follows that (8-17) is the same as

$$\int_{\widehat{\mathcal{C}_k}} \mathcal{M}(b)(\chi_{i\mu})\mathcal{T}_{\phi,\eta}(\chi_{i\mu})\,\mathrm{d}\chi_{i\mu} = 0 \tag{8-18}$$

for any $\phi \in \mathcal{S}_\pi(\mathbb{A}^\times)$ and $b(x) \in \mathcal{C}_c^\infty(\mathcal{C}_k)$. Denote by $\widehat{\mathcal{T}_{\phi,\eta}}(x)$ the (inverse) Fourier transform of $\mathcal{T}_{\phi,\eta}(\chi_{i\mu})$. By using the Parseval formula for the (inverse) Fourier transform, we obtain that (8-18) is equivalent to

$$\int_{\mathcal{C}_k} b(x)\widehat{\mathcal{T}_{\phi,\eta}}(x)\,\mathrm{d}^\times x = 0 \tag{8-19}$$

for all $\phi \in \mathcal{S}_\pi(\mathbb{A}^\times)$ and $b(x) \in \mathcal{C}_c^\infty(\mathcal{C}_k)$. Hence, we must have that (8-19) holds if and only if $\widehat{\mathcal{T}_{\phi,\eta}}(x) = 0$ as distribution on $\mathcal{C}_k$, which is equivalent to $\mathcal{T}_{\phi,\eta}(\chi_{i\mu}) = 0$ as distribution on $\widehat{\mathcal{C}_k}$. In other words, we obtain that for any $\eta \in L^2_{-\delta}(\mathcal{C}_k)$, the integral

$$\int_{\mathcal{C}_k} (b * \Theta_\pi(\,\cdot\,, \phi))(x)\eta(x)\,\mathrm{d}^\times x$$

is zero for any $b \in \mathcal{C}_c^\infty(\mathcal{C}_k)$ and any $\phi \in \mathcal{S}_\pi(\mathbb{A}^\times)$ if and only if

$$\mathcal{Z}\big(\tfrac{1}{2} + i\mu, \phi, \chi\big) \cdot \mathcal{M}(\eta)(\chi_{i\mu}) = 0 \tag{8-20}$$

for all $\phi \in \mathcal{S}_\pi(\mathbb{A}^\times)$. By Corollary 4.4 and [16, Theorem 13.8], there exist finitely many $\phi_1, \dots, \phi_\ell \in \mathcal{S}_\pi(\mathbb{A}^\times)$ such that

$$\mathcal{Z}\big(\tfrac{1}{2} + i\mu, \phi_1, \chi\big) + \cdots + \mathcal{Z}\big(\tfrac{1}{2} + i\mu, \phi_\ell, \chi\big) = L\big(\tfrac{1}{2} + i\mu, \pi \times \chi\big).$$

Thus we obtain that (8-20) implies

$$L\big(\tfrac{1}{2} + i\mu, \pi \times \chi\big) \cdot \mathcal{M}(\eta)(\chi_{i\mu}) = 0 \tag{8-21}$$

as a function in $\chi_{i\mu}$.

To prove the converse, we consider factorizable data $\phi = \bigotimes_\nu \phi_\nu \in \mathcal{S}_\pi(\mathbb{A}^\times)$ and $\chi = \bigotimes_\nu \chi_\nu$. The global zeta integral factorizes into an Euler product

$$\mathcal{Z}(s, \phi, \chi) = \prod_\nu \mathcal{Z}(s, \phi_\nu, \chi_\nu).$$

By Theorem 3.4, we obtain that

$$\mathcal{Z}(s, \phi, \chi) = L(s, \pi \times \chi) \cdot \prod_{\nu \in S} \frac{\mathcal{Z}(s, \phi_\nu, \chi_\nu)}{L(s, \pi_\nu \times \chi_\nu)},$$

where S is the finite set of local places, including all Archimedean local places of k, such that for any $\nu \notin S$, the data π_ν and χ_ν are unramified, and the quotient $\mathcal{Z}(s, \phi_\nu, \chi_\nu)/L(s, \pi_\nu \times \chi_\nu)$ is holomorphic in $s \in \mathbb{C}$. Hence, if $\eta \in L^2_{-\delta}(\mathbb{A}^\times)$ satisfies

$$L\big(\tfrac{1}{2} + i\mu, \pi \times \chi\big) \cdot \mathcal{M}(\eta)(\chi_{i\mu}) = 0$$

as a function in $\chi_{i\mu}$, i.e., (8-21) holds, then (8-20) holds for factorizable data $\phi = \bigotimes_\nu \phi_\nu \in \mathcal{S}_\pi(\mathbb{A}^\times)$ and $\chi = \bigotimes_\nu \chi_\nu$. Hence, it holds for all $\phi \in \mathcal{S}_\pi(\mathbb{A}^\times)$ and all χ. We are done. $\square$

The rest of the proof of Theorem 8.1 is exactly the same as the proof of [40, Theorem 2, page 178], which follows from the same argument of Connes (in the proof of [11, Theorem III.1, pp. 86–87]). We omit the details.

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Volume 326 No. 2 October 2023

Smooth local solutions to Schrödinger flows with damping term for maps into symplectic manifolds 187
BO CHEN and YOUDE WANG

Modules over the planar Galilean conformal algebra arising from free modules of rank one 227
JIN CHENG, DONGFANG GAO and ZITING ZENG

F-algebroids and deformation quantization via pre-Lie algebroids 251
JOHN ALEXANDER CRUZ MORALES, JIEFENG LIU and YUNHE SHENG

Existence of principal values of some singular integrals on Cantor sets, and Hausdorff dimension 285
JULIÀ CUFÍ, JUAN JESÚS DONAIRE, PERTTI MATTILA and JOAN VERDERA

Certain Fourier operators and their associated Poisson summation formulae on GL_1 301
DIHUA JIANG and ZHILIN LUO