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FOURIER BASES OF A CLASS OF PLANAR SELF-AFFINE MEASURES

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Let $\mu_{M,D}$ be the planar self-affine measure generated by an expansive integer matrix $M \in M_2(\mathbb{Z})$ and a noncollinear integer digit set

$$D = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}, \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}, \begin{pmatrix} -\alpha_1-\beta_1 \\ -\alpha_2-\beta_2 \end{pmatrix} \right\}.$$

We show that $\mu_{M,D}$ is a spectral measure if and only if there exists a matrix $Q \in M_2(\mathbb{R})$ such that $(\tilde{M}, \tilde{D})$ is admissible, where $\tilde{M} = QMQ^{-1}$ and $\tilde{D} = QD$. In particular, when $\alpha_1\beta_2 - \alpha_2\beta_1 \notin 2\mathbb{Z}$, $\mu_{M,D}$ is a spectral measure if and only if $M \in M_2(2\mathbb{Z})$. This completely settles the spectrality of the self-affine measure $\mu_{M,D}$.

1. Introduction

Let μ be a Borel probability measure with compact support on $\mathbb{R}^n$, and let $\langle \cdot, \cdot \rangle$ denote the standard inner product on $\mathbb{R}^n$. We say that μ is a *spectral measure* if there exists a countable set $\Lambda \subset \mathbb{R}^n$ such that the exponential function system $E_\Lambda := \{e^{2\pi i \langle \lambda, x \rangle} : \lambda \in \Lambda\}$ forms an orthonormal basis for the Hilbert space $L^2(\mu)$. In this case, we call Λ a *spectrum* of μ and (μ, Λ) a *spectral pair*. In particular, if μ is the normalized Lebesgue measure supported on a Borel set Ω, then Ω is called a *spectral set*.

Spectral measure is a natural generalization of spectral set introduced by Fuglede [20], who proposed the famous conjecture that Ω is a spectral set if and only if Ω is a translational tile. It is known [22] that a spectral measure μ must be of pure type: μ is either discrete, or absolutely continuous or singularly continuous. The first singularly continuous spectral measure was constructed by Jorgensen and

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Pedersen in 1998 [24]. They proved that the middle-fourth Cantor measure is a spectral measure with a spectrum

$$\Lambda = \left\{ \sum_{k=0}^{n} 4^k \ell_k : \ell_k \in \{0, 1\},\ n \in \mathbb{N} \right\}.$$

Following this discovery, there is a considerable number of papers on the spectrality of self-affine measures and the construction of their spectra; see [2; 3; 5; 6; 7; 8; 12; 13; 16; 18; 29]. These results are generalized further to some classes of Moran measures (see, e.g., [1; 9; 19]), and some surprising convergence properties of the associated Fourier series were discovered in [38; 39]. These fractal measures also have very close connections with the theory of multiresolution analysis in wavelet analysis; see [11].

In [14], Dutkay and Jorgensen summarized some known results regarding iterated function systems (IFS); see [23] for details. Two approaches to harmonic analysis on IFS have been popular: one based on a discrete version of the more familiar and classical second-order Laplace differential operator of potential theory; see [27; 28; 30]; and the other is based on Fourier series. The first model in turn is motivated by infinite discrete network of resistors, and the harmonic functions are defined by minimizing a global measure of resistance, but this approach does not rely on Fourier series. In contrast, the second approach begins with Fourier series, and it has its classical origins in lacunary Fourier series [26].

For an expansive real matrix $M \in M_n(\mathbb{R})$ and a finite digit set $D \subset \mathbb{R}^n$ with cardinality $\#D$, the *iterated function system* (IFS) $\{\phi_d(x)\}_{d\in D}$ is defined by $\phi_d(x) = M^{-1}(x + d)$ $(x \in \mathbb{R}^n,\ d \in D)$. By [23], there exists a unique probability measure $\mu_{M,D}$ satisfying

$$\mu_{M,D} = \frac{1}{\#D} \sum_{d\in D} \mu_{M,D} \circ \phi_d^{-1}. \tag{1-1}$$

It is supported on the unique nonempty compact set $T(M, D) = \bigcup_{d\in D} \phi_d(T(M, D))$. Hence

$$T(M, D) = \left\{ \sum_{k=1}^{\infty} M^{-k} d_k : d_k \in D \right\} := \sum_{k=1}^{\infty} M^{-k} D.$$

The measure $\mu_{M,D}$ and the set $T(M, D)$ are called *self-affine measure* and *self-affine set*, respectively. It is known that a self-affine measure $\mu_{M,D}$ can be expressed by the infinite convolution of discrete measures as

$$\mu_{M,D} = \delta_{M^{-1}D} * \delta_{M^{-2}D} * \delta_{M^{-3}D} * \cdots ,$$

where $*$ is the convolution sign, $\delta_E = \frac{1}{\#E} \sum_{e\in E} \delta_e$ for a finite set E and δ_e is the Dirac measure at the point e.

Self-affine measures have the advantage that their Fourier transforms (see (2-1)) can be explicitly written down as an infinite product, which allows us to compute their zeros. The previous research on self-affine measures $\mu_{M,D}$ and their Fourier transform have revealed some surprising connections with a number of areas in mathematics such as harmonic analysis, dynamical systems, number theory and others (see, e.g., [21; 25; 37]).

In the previous works, the spectral self-affine measures are usually generated by compatible pairs (known also as Hadamard triples). The appearance of compatible pairs stems from the terminology of [38].

Definition 1.1. Let $M \in M_n(\mathbb{Z})$ be an expansive integer matrix, and let $D, S \subset \mathbb{Z}^n$ be two finite digit sets with $\#D = \#S = N$. We say that (M, D) is *admissible* (or $(M^{-1}D, S)$ forms a *compatible pair* or (M, D, S) forms a *Hadamard triple*) if the matrix

$$H = \frac{1}{\sqrt{N}} \left(e^{2\pi i \langle M^{-1}d, s\rangle} \right)_{d \in D, s \in S}$$

is unitary, i.e., $H^*H = I$, where I is a $n \times n$ identity matrix.

The well-known result of Jorgensen and Pedersen [24] shows that if (M, D) is admissible, then there are infinite families of orthogonal exponential functions in $L^2(\mu_{M,D})$. Dutkay and Jorgensen [13; 15] formulated the famous conjecture that if (M, D) is admissible, then $\mu_{M,D}$ is a spectral measure. It was first proved in one dimension by Łaba and Wang [29]. The conjecture is true in higher dimensions under some additional assumptions, introduced by Strichartz [38]. There are many other papers that investigated it in higher dimensional cases; see [12; 32]. In the end, Dutkay, Haussermann and Lai [16] proved that:

Theorem 1.2. *Let $M \in M_n(\mathbb{Z})$ be an expansive integer matrix, and let $D \subset \mathbb{Z}^n$ be a finite digit set. If (M, D) is admissible, then $\mu_{M,D}$ is a spectral measure.*

In [18], Fu, He and Lau gave an example to illustrate that the sufficient condition in Theorem 1.2 is not necessary in one dimension. For an expansive integer matrix $M \in M_2(\mathbb{Z})$ and the classic digit set $D = \left\{\binom{0}{0}, \binom{1}{0}, \binom{0}{1}\right\}$, the spectrality and nonspectrality of the corresponding self-affine measure $\mu_{M,D}$ has been widely investigated by many researchers; see [12; 31; 32]. Eventually, An, He and Tao [2] completely settled the spectrality of $\mu_{M,D}$. More precisely, they showed that $\mu_{M,D}$ is a spectral measure if and only if (M, D) is admissible. For a more general integer digit set D with $0 \in D$ and $\#D = 3$, there is also a complete spectral characterization; see [4; 35; 36]. In addition to these, another important integer digit set is

$$D = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}, \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}, \begin{pmatrix} -\alpha_1-\beta_1 \\ -\alpha_2-\beta_2 \end{pmatrix} \right\}, \tag{1-2}$$

where $\alpha_1\beta_2 - \alpha_2\beta_1 \neq 0$. The existence of infinitely many orthogonal exponentials in $L^2(\mu_{M,D})$ has been fully studied in [33; 40; 41]. Recently, Fu and Tang [17] considered the special case where $\alpha_1 = 1$, $\alpha_2 = 0$, $\beta_1 = 0$ and $\beta_2 = 1$. They fully characterized the spectrality of the corresponding self-affine measures. However, to the best of our knowledge, the complete description of spectral properties of the general case (1-2) is not known yet. A natural subsequent question is:

Question 1. For an expansive integer matrix $M \in M_2(\mathbb{Z})$ and the digit set D given by (1-2), what is the sufficient and necessary condition for $\mu_{M,D}$ to be a spectral measure?

In the study of the spectrality of self-affine measures $\mu_{M,D}$ on $\mathbb{R}^n$, the finiteness and rationality of the set $\mathcal{Z}_D^n := \{x \in [0,1)^n : \sum_{d\in D} e^{2\pi i\langle d,x\rangle} = 0\}$ are pivotal. Many classic digit sets, such as $\{0, 1, \dots, N-1\}$, $\{(0,0)^t, (1,0)^t, (0,1)^t\}$ and the digit set D given by (1-2), exhibit the desired property. This has attracted a large number of researchers to study their spectrality of the corresponding self-affine measures. However, if $\mathcal{Z}_D^n$ is infinite or irrational, resolving the spectrality of the corresponding self-affine measure becomes a formidable challenge. For instance, consider $M \in M_2(\mathbb{Z})$ and $D = \{(0,0)^t, (1,0)^t, (0,1)^t, (1,1)^t\}$. It is easy to get that

$$\mathcal{Z}_D^2 = \left\{ \begin{pmatrix} \frac{1}{2} \\ a \end{pmatrix} \cup \begin{pmatrix} a \\ \frac{1}{2} \end{pmatrix} : a \in [0,1) \right\}.$$

This means that $\mathcal{Z}_D^2$ encompasses a submanifold characterized by the free variable $a \in [0,1)$. For the more general digit set $D = \{0, u, v, u+v\} \subset \mathbb{Z}^2$, the set $\mathcal{Z}_D^2$ is infinite and includes free variables. The spectral properties of these self-affine measures have not been resolved.

The cardinality $\#D$ of a digit set D significantly influences the properties of $\mathcal{Z}_D^n$. In [3], An, He and Lai extensively classified four-element digit spectral self-similar measures on $\mathbb{R}$. They showed that if $\#D = 4$ and the corresponding self-similar measure is a spectral measure, then D is rational and $\mathcal{Z}_D^1$ is finite and rational. However, if D does not have any special structures and $\#D \geq 5$, the set $\mathcal{Z}_D^n$ is hard to calculate and may be irrational. For example, let $D = \{0, 1, 3, 5, 6\}$. Then $\mathcal{Z}_D^1 \subset \mathbb{R} \setminus \mathbb{Q}$ by [3, Example 5.2]. This makes it very difficult to study the spectrality of the corresponding self-similar measure.

Inspired by the above researches and due to the finiteness and rationality of the set $\mathcal{Z}_D^2$ corresponding to the digit set D given by (1-2), we can give an answer to Question 1. Before presenting our results, a reasonable assumption for the digit set D is necessary. Without loss of generality, we can assume that $\gcd(\alpha_1, \alpha_2, \beta_1, \beta_2) = 1$ by Lemma 2.2.

Our first main result is as follows:

Theorem 1.3. *Let $\mu_{M,D}$ be defined by* (1-1), *where $M \in M_2(\mathbb{Z})$ is an expansive integer matrix and D is given by* (1-2). *Then $\mu_{M,D}$ is a spectral measure if and only if there exists a matrix $Q \in M_2(\mathbb{R})$ such that $(\tilde{M}, \tilde{D})$ is admissible, where $\tilde{M} = QMQ^{-1}$ and $\tilde{D} = QD$.*

We remark that Theorem 1.3 gives a complete answer to the spectral Question 1. We now outline the strategy of the proof of Theorem 1.3. The sufficiency of Theorem 1.3 follows directly from Theorem 1.2 and Lemma 2.2. The more challenging part of the proof is the necessity. The key point is to construct a self-affine measure $\mu_{\tilde{M},\tilde{D}}$ so that it has the same spectrality as the measure $\mu_{M,D}$, and then the necessity follows immediately from Theorems 1.5 and 1.6. What is exciting is that the proof method of the necessity is new and completely different from the previous work proving spectral self-affine measures.

It is worth noting that if D satisfies $\alpha_1\beta_2 - \alpha_2\beta_1 \notin 2\mathbb{Z}$, we can give more explicit sufficient and necessary conditions for $\mu_{M,D}$ to be a spectral measure. Before presenting them, some notation is needed. For any integer $p \geq 2$, we define

$$\mathcal{F}_p^2 := \frac{1}{p}\left\{\begin{pmatrix} l_1 \\ l_2 \end{pmatrix} : 0 \leq l_1, l_2 \leq p-1,\ l_i \in \mathbb{Z}\right\} \quad \text{and} \quad \mathring{\mathcal{F}}_p^2 := \mathcal{F}_p^2 \setminus \{\mathbf{0}\}. \tag{1-3}$$

Under the above notation and the assumption of $\alpha_1\beta_2 - \alpha_2\beta_1 \notin 2\mathbb{Z}$, we give the second main result:

Theorem 1.4. *Let $\mu_{M,D}$ and $\mathring{\mathcal{F}}_p^2$ be defined by* (1-1) *and* (1-3), *respectively*, *where $M \in M_2(\mathbb{Z})$ is an expansive integer matrix and D is given by* (1-2). *If $\alpha_1\beta_2 - \alpha_2\beta_1 \notin 2\mathbb{Z}$, then the following statements are equivalent*:

(i) *$\mu_{M,D}$ is a spectral measure.*

(ii) $M \in M_2(2\mathbb{Z})$.

(iii) $M\mathring{\mathcal{F}}_2^2 \subset \mathbb{Z}^2$.

(iv) *(M, D) is admissible.*

We point out that the proofs of Theorems 1.3 and 1.4 are based on the precise form of the matrix $\tilde{M}$ in Theorem 1.3. Before giving the form, some technical work needs to be done. For an expansive integer matrix $M \in M_2(\mathbb{Z})$ and the digit set D given by (1-2), we can let $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $\alpha_1\beta_2 - \alpha_2\beta_1 = 2^\eta\gamma$ with $\eta \geq 0$ and $\gamma \notin 2\mathbb{Z}$. Without loss of generality, we assume $\gcd(\alpha_1, \alpha_2) = \alpha$ with $\alpha \notin 2\mathbb{Z}$ (otherwise, we can choose $\alpha = \gcd(\beta_1, \beta_2)$ with $\alpha \notin 2\mathbb{Z}$ since $\gcd(\alpha_1, \alpha_2, \beta_1, \beta_2) = 1$). Let $\alpha_1 = \alpha t_1$ and $\alpha_2 = \alpha t_2$ with $\gcd(t_1, t_2) = 1$. Then there exist $p, q \in \mathbb{Z}$ such that $pt_1 + qt_2 = 1$. Clearly, $\alpha = p\alpha_1 + q\alpha_2$ and $\alpha \mid \gamma$. For convenience, we define $\omega = p\beta_1 + q\beta_2$ and $\beta = \gamma/\alpha$. It is easy to check that $t_1\alpha_2 = t_2\alpha_1$ and $t_1\beta_2 - t_2\beta_1 = 2^\eta\beta$ with $\beta \notin 2\mathbb{Z}$.

Define $Q = \left(\begin{smallmatrix} p & q \\ -t_2 & t_1 \end{smallmatrix}\right)$. Then one has

$$\tilde{M} := QMQ^{-1} = \begin{pmatrix} (pa+qc)t_1+(pb+qd)t_2 & (pb+qd)p-(pa+qc)q \\ (ct_1-at_2)t_1+(dt_1-bt_2)t_2 & (dt_1-bt_2)p-(ct_1-at_2)q \end{pmatrix} \tag{1-4}$$

and

$$\tilde{D} := QD = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \alpha \\ 0 \end{pmatrix}, \begin{pmatrix} \omega \\ 2^\eta \beta \end{pmatrix}, \begin{pmatrix} -\alpha-\omega \\ -2^\eta \beta \end{pmatrix} \right\} \subset \mathbb{Z}^2. \tag{1-5}$$

Obviously, $\tilde{M}$ is an expansive integer matrix with $\det(\tilde{M}) = \det(M)$. Also, $\eta = 0$ and $\eta > 0$ are equivalent to $\alpha_1\beta_2 - \alpha_2\beta_1 \notin 2\mathbb{Z}$ and $\alpha_1\beta_2 - \alpha_2\beta_1 \in 2\mathbb{Z}$, respectively.

For $\eta = 0$ in $\tilde{D}$, we have the following conclusion, which is equivalent to Theorem 1.4 by using the property of similarity transformation.

Theorem 1.5. *Let $\mu_{\tilde{M},\tilde{D}}$ and $\mathring{\mathcal{F}}_p^2$ be defined by* (1-1) *and* (1-3), *respectively, where $\tilde{M}$ and $\tilde{D}$ are given by* (1-4) *and* (1-5), *respectively. If $\eta = 0$, then the following statements are equivalent*:

(i) $\mu_{\tilde{M},\tilde{D}}$ *is a spectral measure.*

(ii) $\tilde{M} \in M_2(2\mathbb{Z})$.

(iii) $\tilde{M}\mathring{\mathcal{F}}_2^2 \subset \mathbb{Z}^2$.

(iv) $(\tilde{M}, \tilde{D})$ *is admissible.*

On the other hand, if $\eta > 0$ in $\tilde{D}$, the form of $\tilde{M}$ is different from that in the case $\eta = 0$.

Theorem 1.6. *Let $\mu_{\tilde{M},\tilde{D}}$ be defined by* (1-1), *where $\tilde{M}$ and $\tilde{D}$ are given by* (1-4) *and* (1-5), *respectively. If $\eta > 0$, then $\mu_{\tilde{M},\tilde{D}}$ is a spectral measure if and only if the matrix $\tilde{M} = \left(\begin{smallmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{smallmatrix}\right)$ satisfies $\tilde{a}, \tilde{d} \in 2\mathbb{Z}$ and $2^{\eta+1} \mid \tilde{c}$.*

We now give a brief explanation of the proofs of Theorems 1.5 and 1.6. The main technical difficulty in the proofs lies in "(i) $\Longrightarrow$ (ii)" of Theorem 1.5 and the necessity of Theorem 1.6. More precisely, the key point is to construct a Moran measure $\mu_{A,\tilde{M},\tilde{D}}$ (see (3-1)) so that it has the same spectrality as $\mu_{\tilde{M},\tilde{D}}$. For the matrix A, we need to cleverly describe its complete residue system (Proposition 3.3). We carefully investigate the structure of the spectrum of $\mu_{A,\tilde{M},\tilde{D}}$ (see (3-11)). And then we get a property of decomposition on the spectrum of $\mu_{\tilde{M},\tilde{D}}$ under the assumption that $\mu_{A,\tilde{M},\tilde{D}}$ is a spectral measure (Lemma 3.5). With their help, the proof becomes within reach.

The paper is organized as follows. In Section 2, we introduce some basic definitions and lemmas. In Section 3, we focus on proving Theorems 1.5 and 1.6. Finally, we prove Theorems 1.3 and 1.4, and give some concluding remarks in Section 4.

2. Preliminaries

For the self-affine measure $\mu_{M,D}$ defined by (1-1), the Fourier transform of $\mu_{M,D}$ is defined by

$$\hat{\mu}_{M,D}(\xi) = \int e^{2\pi i \langle x, \xi \rangle}\, d\mu_{M,D}(x) = \prod_{j=1}^{\infty} m_D(M^{*-j}\xi), \quad \xi \in \mathbb{R}^n, \tag{2-1}$$

where M^* denotes the transpose of M and $m_D(\cdot) = \frac{1}{\#D}\sum_{d\in D} e^{2\pi i \langle d, \cdot\rangle}$ is the *mask polynomial* of D. We denote the set of all the roots of $f(x)$ by $\mathcal{Z}(f)$, i.e., $\mathcal{Z}(f) = \{x : f(x) = 0\}$. Using (2-1), one has

$$\mathcal{Z}(\hat{\mu}_{M,D}) = \bigcup_{j=1}^{\infty} M^{*j}(\mathcal{Z}(m_D)). \tag{2-2}$$

For a countable set $\Lambda \subset \mathbb{R}^n$, $E_\Lambda = \{e^{2\pi i \langle \lambda, x\rangle} : \lambda \in \Lambda\}$ is an orthogonal family of $L^2(\mu_{M,D})$ if and only if $\hat{\mu}_{M,D}(\lambda_1 - \lambda_2) = 0$ for any $\lambda_1 \neq \lambda_2$, which is equivalent to

$$(\Lambda - \Lambda) \setminus \{\mathbf{0}\} \subset \mathcal{Z}(\hat{\mu}_{M,D}). \tag{2-3}$$

If E_Λ forms an orthogonal family of $L^2(\mu_{M,D})$, then Λ is called an *orthogonal set* of $\mu_{M,D}$. Note that the properties of spectra are invariant under a translation, so we can always assume that $\mathbf{0} \in \Lambda$.

In a number of applications, one encounters a measure μ and a subset Λ such that the functions $e^{2\pi i \langle \lambda, x\rangle}$ indexed by Λ are orthogonal in $L^2(\mu)$, but a separate argument is needed in order to show that the family is complete. Let

$$Q_{\mu,\Lambda}(\xi) = \sum_{\lambda\in\Lambda} |\hat{\mu}(\xi + \lambda)|^2, \quad \xi \in \mathbb{R}^n. \tag{2-4}$$

The following result is a basic criterion for the spectrality of μ.

Theorem 2.1 [24]. *Let μ be a Borel probability measure with compact support on $\mathbb{R}^n$, and let $\Lambda \subset \mathbb{R}^n$ be a countable set. Then*:

(i) Λ *is an orthogonal set of μ if and only if $Q_{\mu,\Lambda}(\xi) \le 1$ for $\xi \in \mathbb{R}^n$.*

(ii) Λ *is a spectrum of μ if and only if $Q_{\mu,\Lambda}(\xi) \equiv 1$ for $\xi \in \mathbb{R}^n$.*

The following lemma indicates that the spectrality of $\mu_{M,D}$ is invariant under a similarity transformation.

Lemma 2.2 [12]. *Let $D_1, D_2 \subset \mathbb{R}^n$ be two finite digit sets with the same cardinality, and let $M_1, M_2 \in M_n(\mathbb{R})$ be two expansive real matrices. If there exists a matrix $Q \in M_n(\mathbb{R})$ such that $M_2 = QM_1Q^{-1}$ and $D_2 = QD_1$, then μ_{M_1,D_1} is a spectral measure with spectrum Λ if and only if μ_{M_2,D_2} is a spectral measure with spectrum $Q^{*-1}\Lambda$.*

The following result is a known fact, which was proved in [16] and will be used in the proof of Proposition 3.3.

Lemma 2.3. *Let $M \in M_n(\mathbb{Z})$ be an expansive integer matrix, and let $D, S \subset \mathbb{Z}^n$ be two finite digit sets with the same cardinality. Then the following three statements are equivalent*:

(i) (M, D, S) *is a Hadamard triple.*

(ii) $m_D(M^{*-1}(s_1 - s_2)) = 0$ *for any distinct* $s_1, s_2 \in S$.

(iii) $(\delta_{M^{-1}D}, S)$ *is a spectral pair.*

Recalling that $\mu_{M,D}$ is defined by (1-1), we let A be a nonsingular matrix and define the Moran measure

$$\mu_{A,M,D} = \delta_{A^{-1}D} * \delta_{A^{-1}M^{-1}D} * \delta_{A^{-1}M^{-2}D} * \cdots . \tag{2-5}$$

It is clear that $\mu_{A,M,D} = \mu_{M,D}$ if $A = M$. The following lemma indicates the spectrality of $\mu_{M,D}$ is independent of A. The proof is the same as that of [9, Lemma 3.1; 10, Lemma 2.6]. For the convenience of readers, we include the proof here.

Lemma 2.4. *Let A be a nonsingular matrix, and let $\mu_{A,M,D}$ be defined by* (2-5). *Then*

$$\mu_{M,D} = \mu_{A,M,D} \circ (A^{-1}M).$$

Also, $(\mu_{M,D}, \Lambda)$ *is a spectral pair if and only if* $(\mu_{A,M,D}, A^*M^{*-1}\Lambda)$ *is a spectral pair.*

Proof. Applying (2-1) and (2-5), we have

$$\begin{aligned}\hat{\mu}_{A,M,D}(A^*M^{*-1}\xi) &= m_D(A^{*-1}A^*M^{*-1}\xi)\prod_{j=1}^{\infty} m_D(M^{*-j}A^{*-1}A^*M^{*-1}\xi) \\ &= \prod_{j=1}^{\infty} m_D(M^{*-j}\xi) = \hat{\mu}_{M,D}(\xi).\end{aligned} \tag{2-6}$$

Then $\mu_{M,D} = \mu_{A,M,D} \circ (A^{-1}M)$ by the uniqueness of Fourier transform.

Recall $Q_{\mu,\Lambda}(\xi)$ is defined by (2-4). Then, for $\xi \in \mathbb{R}^2$, it follows from (2-6) that

$$\begin{aligned}Q_{\mu_{M,D},\Lambda}(\xi) &= \sum_{\lambda\in\Lambda} |\hat{\mu}_{M,D}(\xi+\lambda)|^2 \\ &= \sum_{\lambda\in\Lambda} |\hat{\mu}_{A,M,D}(A^*M^{*-1}(\xi+\lambda))|^2 \\ &= \sum_{\lambda\in\Lambda} |\hat{\mu}_{A,M,D}(A^*M^{*-1}\xi + A^*M^{*-1}\lambda)|^2 \\ &= Q_{\mu_{A,M,D},A^*M^{*-1}\Lambda}(A^*M^{*-1}\xi).\end{aligned}$$

Hence the second assertion follows by Theorem 2.1. □

We conclude this section by recalling a useful lemma in our investigation, which was proved by Deng et al. in [9, Lemma 2.5].

Lemma 2.5. *Let $p_{i,j}$ be positive numbers such that $\sum_{j=1}^{n} p_{i,j} = 1$, and let $q_{i,j}$ be nonnegative numbers such that $\sum_{i=1}^{m} \max_{1\le j\le n} q_{i,j} \le 1$. Then*

$$\sum_{i=1}^{m}\sum_{j=1}^{n} p_{i,j}q_{i,j} = 1$$

if and only if $q_{i,1} = \cdots = q_{i,n}$ for $1 \le i \le m$ and $\sum_{i=1}^{m} q_{i,1} = 1$.

3. Proofs of Theorems 1.5 and 1.6

We focus on proving Theorems 1.5 and 1.6, that is, studying the spectrality of the measure $\mu_{\tilde{M},\tilde{D}}$, where $\tilde{M}$ and $\tilde{D}$ are given by (1-4) and (1-5), respectively. For this purpose, we first give some properties of $\mathcal{Z}(m_{\tilde{D}})$, and then investigate the structure of the spectrum of $\mu_{\tilde{M},\tilde{D}}$ under the assumption that $\mu_{A,\tilde{M},\tilde{D}}$ is a spectral measure, where $\mu_{A,\tilde{M},\tilde{D}}$ is defined by (2-5). With these preparations, we will achieve our goal.

By Lemma 2.4, without loss of generality, we assume in the rest of the paper that

$$A = \begin{pmatrix} 2^{\eta+1}\alpha\beta & 0 \\ 0 & 2^{\eta+1}\alpha\beta \end{pmatrix}.$$

The matrix A will be pivotal in constructing the spectrum of $\mu_{\tilde{M},\tilde{D}}$. Consequently,

$$\begin{aligned} \mu_{A,\tilde{M},\tilde{D}} &= \delta_{\frac{1}{2^{\eta+1}\alpha\beta}\tilde{D}} * (\mu_{\tilde{M},\tilde{D}} \circ 2^{\eta+1}\alpha\beta), \\ \hat{\mu}_{A,\tilde{M},\tilde{D}}(\xi) &= m_{\tilde{D}}\left(\frac{\xi}{2^{\eta+1}\alpha\beta}\right)\hat{\mu}_{\tilde{M},\tilde{D}}\left(\frac{\xi}{2^{\eta+1}\alpha\beta}\right). \end{aligned} \tag{3-1}$$

It is known that $m_{\tilde{D}}(x) = 0$ if and only if

$$\begin{cases} \alpha x_1 = \frac{1}{2}+k_1, \\ \omega x_1 + 2^\eta\beta x_2 = k_1', \end{cases} \quad \begin{cases} \alpha x_1 = k_2, \\ \omega x_1 + 2^\eta\beta x_2 = \frac{1}{2}+k_2', \end{cases} \quad \text{or} \quad \begin{cases} \alpha x_1 = \frac{1}{2}+k_3, \\ \omega x_1 + 2^\eta\beta x_2 = \frac{1}{2}+k_3', \end{cases} \tag{3-2}$$

where $k_1, k_2, k_3, k_1', k_2', k_3' \in \mathbb{Z}$. By a direct calculation, we have that

$$\mathcal{Z}(m_{\tilde{D}}) = \Theta_1 \cup \Theta_2 \cup \Theta_3, \tag{3-3}$$

where

$$\Theta_1 = \left\{ \frac{1}{2^{\eta+1}\alpha\beta} \begin{pmatrix} 2^\eta(2k_1\beta+\beta) \\ 2k_1'\alpha - 2k_1\omega - \omega \end{pmatrix} : k_1, k_1' \in \mathbb{Z} \right\},$$

$$\Theta_2 = \left\{ \frac{1}{2^{\eta+1}\alpha\beta} \begin{pmatrix} 2^{\eta+1}k_2\beta \\ 2k_2'\alpha - 2k_2\omega + \alpha \end{pmatrix} : k_2, k_2' \in \mathbb{Z} \right\},$$

$$\Theta_3 = \left\{ \frac{1}{2^{\eta+1}\alpha\beta} \begin{pmatrix} 2^\eta(2k_3\beta+\beta) \\ 2k_3'\alpha - 2k_3\omega + \alpha - \omega \end{pmatrix} : k_3, k_3' \in \mathbb{Z} \right\}.$$

Define

$$\Theta_0 = \left\{ \frac{1}{2^{\eta+1}\alpha\beta} \begin{pmatrix} 2^{\eta+1}k_0\beta \\ 2k_0'\alpha - 2k_0\omega \end{pmatrix} : k_0, k_0' \in \mathbb{Z} \right\}.$$

We now make a detailed analysis on the zero set $\mathcal{Z}(m_{\tilde{D}})$ of $m_{\tilde{D}}$.

Proposition 3.1. *With the above notation, the following statements hold:*

(i) $(\Theta_i - \Theta_i) \cap \mathcal{Z}(m_{\tilde{D}}) = \varnothing$ *for any* $i \in \{0, 1, 2, 3\}$.

(ii) $\Theta_i - \Theta_j \subset \mathcal{Z}(m_{\tilde{D}})$ *for any distinct* $i, j \in \{0, 1, 2, 3\}$.

(iii) *If* $\eta = 0$, *then* $\mathring{\mathcal{F}}_2^2 \subset \mathcal{Z}(m_{\tilde{D}})$, *where* $\mathring{\mathcal{F}}_2^2$ *is defined by* (1-3).

Proof. (i) Since $\alpha, \beta \in 2\mathbb{Z}+1$, from the definitions of $\mathcal{Z}(m_{\tilde{D}})$ and Θ_0, it can easily be seen that $\Theta_i - \Theta_i \subset \Theta_0$ for any $i \in \{0, 1, 2, 3\}$ and $\Theta_i \cap \Theta_0 = \varnothing$ for any $i \in \{1, 2, 3\}$. This yields $(\Theta_i - \Theta_i) \cap \mathcal{Z}(m_{\tilde{D}}) = \varnothing$ for all i, which proves (i).

(ii) For any $\theta_i \in \Theta_i$, it is easy to verify that

$$\pm(\theta_i - \theta_0) \in \Theta_i \quad (i \in \{1, 2, 3\}),$$
$$\pm(\theta_1 - \theta_2) \in \Theta_3, \quad \pm(\theta_1 - \theta_3) \in \Theta_2 \quad \text{and} \quad \pm(\theta_2 - \theta_3) \in \Theta_1.$$

Hence the assertion follows by using (3-3).

(iii) As $\eta = 0$ and $\alpha, \beta \in 2\mathbb{Z}+1$, it follows from (3-2) and (3-3) that

$$\left(\tfrac{1}{2}, 0\right)^t \in \Theta_1, \quad \left(0, \tfrac{1}{2}\right)^t \in \Theta_2 \quad \text{and} \quad \left(\tfrac{1}{2}, \tfrac{1}{2}\right)^t \in \Theta_3$$

if $\omega \in 2\mathbb{Z}$ and

$$\left(\tfrac{1}{2}, 0\right)^t \in \Theta_3, \quad \left(0, \tfrac{1}{2}\right)^t \in \Theta_2 \quad \text{and} \quad \left(\tfrac{1}{2}, \tfrac{1}{2}\right)^t \in \Theta_1$$

if $\omega \in 2\mathbb{Z}+1$. Therefore, $\mathring{\mathcal{F}}_2^2 \subset \Theta_1 \cup \Theta_2 \cup \Theta_3 = \mathcal{Z}(m_{\tilde{D}})$. □

Remark 3.2. Observing that $\alpha, \beta \in 2\mathbb{Z}+1$ in $\tilde{D}$, without loss of generality, we can further assume that $\alpha, \beta \geq 1$. In fact, if $\alpha < 0$ or $\beta < 0$, we take

$$Q = \begin{cases} \operatorname{diag}(-1, 1), & \text{if } \alpha < 0, \beta > 0, \\ \operatorname{diag}(1, -1), & \text{if } \alpha > 0, \beta < 0, \\ \operatorname{diag}(-1, -1), & \text{if } \alpha, \beta < 0. \end{cases}$$

Let $\bar{M} = Q\tilde{M}Q^{-1}$ and $\bar{D} = Q\tilde{D}$. By Lemma 2.2, we only need to consider the spectrality of $\mu_{\bar{M},\bar{D}}$. This implies that the assumption is reasonable.

To investigate the spectrality of $\mu_{\tilde{M},\tilde{D}}$, we need to construct a complete residue system of matrix A. In view of (3-1) and (3-3), one may easily get that

$$\text{(3-4)} \quad \mathcal{Z}(\hat{\mu}_{A,\tilde{M},\tilde{D}}) = \bigcup_{j=0}^{\infty} A^* \tilde{M}^{*j}(\mathcal{Z}(m_{\tilde{D}})) = \bigcup_{j=0}^{\infty} \tilde{M}^{*j}(2^{\eta+1}\alpha\beta(\Theta_1 \cup \Theta_2 \cup \Theta_3)) \subset \mathbb{Z}^2.$$

Throughout this paper, we set $\hbar_p = \{0, 1, \dots, p-1\}$ for an integer $p \geq 1$, and let

$$\mathcal{S}_q = \left\{ \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} : s_1 \in \hbar_{2^q\beta},\ s_2 \in \hbar_\alpha \right\} \quad \text{and} \quad \mathcal{T}_q = \bigcup_{i=0}^{3} \mathcal{T}_{q,i}, \tag{3-5}$$

where q is a nonnegative integer and

$$\mathcal{T}_{q,0} = \left\{ \frac{1}{2^{q+1}\alpha\beta} \begin{pmatrix} 2^{q+1}k_0\beta \\ 2k_0'\alpha - 2k_0\omega \end{pmatrix} : k_0 \in \hbar_\alpha,\ k_0' \in \hbar_{2^q\beta} \right\},$$

$$\mathcal{T}_{q,1} = \left\{ \frac{1}{2^{q+1}\alpha\beta} \begin{pmatrix} 2^q(2k_1\beta + \beta) \\ 2k_1'\alpha - 2k_1\omega - \omega \end{pmatrix} : k_1 \in \hbar_\alpha,\ k_1' \in \hbar_{2^q\beta} \right\},$$

$$\mathcal{T}_{q,2} = \left\{ \frac{1}{2^{q+1}\alpha\beta} \begin{pmatrix} 2^{q+1}k_2\beta \\ 2k_2'\alpha - 2k_2\omega + \alpha \end{pmatrix} : k_2 \in \hbar_\alpha,\ k_2' \in \hbar_{2^q\beta} \right\},$$

$$\mathcal{T}_{q,3} = \left\{ \frac{1}{2^{q+1}\alpha\beta} \begin{pmatrix} 2^q(2k_3\beta + \beta) \\ 2k_3'\alpha - 2k_3\omega + \alpha - \omega \end{pmatrix} : k_3 \in \hbar_\alpha,\ k_3' \in \hbar_{2^q\beta} \right\}.$$

Proposition 3.3. *With the above notation, the following statements hold*:

(i) $\mathcal{T}_{\eta,i} \subset \Theta_i$ *for any* $i \in \{0, 1, 2, 3\}$.

(ii) $(\delta_{A^{-1}\tilde{D}}, \mathcal{C})$ *is a spectral pair, where* $A = \operatorname{diag}(2^{\eta+1}\alpha\beta, 2^{\eta+1}\alpha\beta)$ *and* $\mathcal{C} = 2^{\eta+1}\alpha\beta\{\ell_0, \ell_1, \ell_2, \ell_3\}$ *for any* $\ell_i \in \mathcal{T}_{\eta,i}$.

(iii) $\mathcal{S}_\eta \oplus 2^{\eta+1}\alpha\beta\mathcal{T}_\eta$ *is a complete residue system of matrix* A *in* (ii).

Proof. According to the definitions of $\mathcal{T}_{\eta,i}$ and Θ_i, (i) is obvious. We now prove (ii). In view of Lemma 2.3, it suffices to prove that $m_{\tilde{D}}(A^{*-1}(c-c')) = 0$ for all distinct $c, c' \in \mathcal{C}$. Since $A = \operatorname{diag}(2^{\eta+1}\alpha\beta, 2^{\eta+1}\alpha\beta)$, it follows from Proposition 3.1(ii) and Proposition 3.3(i) that $A^{*-1}(c-c') \in \mathcal{Z}(m_{\tilde{D}})$. This implies $m_{\tilde{D}}(A^{*-1}(c-c')) = 0$, and the assertion (ii) follows.

Finally, we prove (iii). It is clear that the set $\mathcal{S}_\eta \oplus 2^{\eta+1}\alpha\beta\mathcal{T}_\eta$ can be written as

$$\begin{aligned} &\mathcal{S}_\eta \oplus 2^{\eta+1}\alpha\beta\mathcal{T}_\eta \\ &= \left\{ \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} : s_1 \in \hbar_{2^\eta\beta},\ s_2 \in \hbar_\alpha \right\} \oplus \begin{pmatrix} 2^\eta\beta & 0 \\ -\omega & \alpha \end{pmatrix} \left\{ \begin{pmatrix} k \\ k' \end{pmatrix} : k \in \hbar_{2\alpha},\ k' \in \hbar_{2^{\eta+1}\beta} \right\} \\ &:= \mathcal{S}_\eta \oplus \begin{pmatrix} 2^\eta\beta & 0 \\ -\omega & \alpha \end{pmatrix} \mathcal{Q}. \end{aligned} \tag{3-6}$$

To prove $\mathcal{S}_\eta \oplus 2^{\eta+1}\alpha\beta\mathcal{T}_\eta$ is a complete residue system of $A = \operatorname{diag}(2^{\eta+1}\alpha\beta, 2^{\eta+1}\alpha\beta)$, by using (3-6), it suffices to show that for any $(x, y)^t \in \mathbb{Z}^2$, there exist $(s_1, s_2)^t \in \mathcal{S}_\eta$, $(k, k')^t \in \mathcal{Q}$ and $(x', y')^t \in \mathbb{Z}^2$ such that

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} + \begin{pmatrix} 2^\eta\beta & 0 \\ -\omega & \alpha \end{pmatrix} \begin{pmatrix} k \\ k' \end{pmatrix} + 2^{\eta+1}\alpha\beta \begin{pmatrix} x' \\ y' \end{pmatrix}. \tag{3-7}$$

Since $\{0, 1, \dots, 2^\eta\beta - 1\} \oplus 2^\eta\beta\{0, 1, \dots, 2\alpha - 1\}$ is a complete residue system of $2^{\eta+1}\alpha\beta$, it follows that there exist $s_1 \in \{0, 1, \dots, 2^\eta\beta - 1\}$, $k \in \{0, 1, \dots, 2\alpha - 1\}$ and $x' \in \mathbb{Z}$ such that

$$x = s_1 + 2^\eta\beta k + 2^{\eta+1}\alpha\beta x'. \tag{3-8}$$

Also note that $\{0, 1, \dots, \alpha-1\}\oplus\alpha\{0, 1, \dots, 2^{\eta+1}\beta-1\}$ is another complete residue system of $2^{\eta+1}\alpha\beta$; thus there exist $s_2 \in \{0, 1, \dots, \alpha-1\}$, $k' \in \{0, 1, \dots, 2^{\eta+1}\beta-1\}$ and $y' \in \mathbb{Z}$ such that

$$y + \omega k = s_2 + \alpha k' + 2^{\eta+1}\alpha\beta y'. \tag{3-9}$$

The above equations (3-8) and (3-9) imply that (3-7) holds. □

Let Λ be a spectrum of $\mu_{A,\tilde{M},\tilde{D}}$ with $0 \in \Lambda$. By (2-3) and (3-4), we have $\Lambda \subset \mathbb{Z}^2$. This together with Proposition 3.3(iii) implies that for any $\lambda \in \Lambda$, there exist some $s \in \mathcal{S}_\eta$ and $\ell \in \mathcal{T}_\eta$ such that $\lambda = s + 2^{\eta+1}\alpha\beta\ell + 2^{\eta+1}\alpha\beta\gamma$ for some $\gamma \in \mathbb{Z}^2$. Then for $s \in \mathcal{S}_\eta$ and $\ell \in \mathcal{T}_\eta$, define

$$\Lambda_{s,\ell} = \{\gamma \in \mathbb{Z}^2 : s + 2^{\eta+1}\alpha\beta\ell + 2^{\eta+1}\alpha\beta\gamma \in \Lambda\}. \tag{3-10}$$

Then using (3-5), we have the decomposition

$$\Lambda = \bigcup_{s\in\mathcal{S}_\eta} \bigcup_{i\in\{0,1,2,3\}} \bigcup_{\ell\in\mathcal{T}_{\eta,i}} (s + 2^{\eta+1}\alpha\beta\ell + 2^{\eta+1}\alpha\beta\Lambda_{s,\ell}), \tag{3-11}$$

where $s + 2^{\eta+1}\alpha\beta\ell + 2^{\eta+1}\alpha\beta\Lambda_{s,\ell} = \varnothing$ if $\Lambda_{s,\ell} = \varnothing$. As $0 \in \Lambda$, it follows that

$$\Lambda_{0,0} \neq \varnothing. \tag{3-12}$$

Lemma 3.4. *Let Λ be a spectrum of $\mu_{A,\tilde{M},\tilde{D}}$ with $0 \in \Lambda$. If $\Lambda_{s,\ell}$ is a nonempty set, then $\Lambda_{s,\ell}$ is an orthogonal set of $\mu_{\tilde{M},\tilde{D}}$ for each $s \in \mathcal{S}_\eta$ and $\ell \in \mathcal{T}_\eta$.*

Proof. Suppose that $\Lambda_{s,\ell}$ is a nonempty set for $s \in \mathcal{S}_\eta$ and $\ell \in \mathcal{T}_\eta$. Then for any distinct $\lambda_1, \lambda_2 \in \Lambda_{s,\ell}$, it follows from (3-11) that

$$s + 2^{\eta+1}\alpha\beta\ell + 2^{\eta+1}\alpha\beta\lambda_1,\ s + 2^{\eta+1}\alpha\beta\ell + 2^{\eta+1}\alpha\beta\lambda_2 \in \Lambda.$$

Applying (2-3), we have $2^{\eta+1}\alpha\beta(\lambda_1 - \lambda_2) \in \mathcal{Z}(\hat{\mu}_{A,\tilde{M},\tilde{D}})$. Together with (3-1), $\lambda_1, \lambda_2 \in \mathbb{Z}^2$ and $m_{\tilde{D}}(\lambda_1 - \lambda_2) = 1$, we have

$$0 = \hat{\mu}_{A,\tilde{M},\tilde{D}}(2^{\eta+1}\alpha\beta(\lambda_1 - \lambda_2)) = m_{\tilde{D}}(\lambda_1 - \lambda_2)\hat{\mu}_{\tilde{M},\tilde{D}}(\lambda_1 - \lambda_2) = \hat{\mu}_{\tilde{M},\tilde{D}}(\lambda_1 - \lambda_2).$$

Thus $\lambda_1 - \lambda_2 \in \mathcal{Z}(\hat{\mu}_{\tilde{M},\tilde{D}})$, which means that $\Lambda_{s,\ell}$ is an orthogonal set of $\mu_{\tilde{M},\tilde{D}}$. □

The following lemma gives the structure of the spectrum of $\mu_{\tilde{M},\tilde{D}}$ under the assumption that $\mu_{A,\tilde{M},\tilde{D}}$ is a spectral measure.

Lemma 3.5. *Let Λ be a spectrum of $\mu_{A,\tilde{M},\tilde{D}}$ with $0 \in \Lambda$. For any $s \in \mathcal{S}_\eta$, choose a $i_s \in \{0, 1, 2, 3\}$ and write*

$$\Gamma = \bigcup_{s\in\mathcal{S}_\eta} \bigcup_{\ell\in\mathcal{T}_{\eta,i_s}} \left(\frac{s + 2^{\eta+1}\alpha\beta\ell}{2^{\eta+1}\alpha\beta} + \Lambda_{s,\ell} \right),$$

where $\Lambda_{s,\ell}$ is defined by (3-10). *Then Γ is a spectrum of $\mu_{\tilde{M},\tilde{D}}$ or an empty set.*

Proof. If Γ is a nonempty set, we will complete the proof in the following two steps.

Step 1. We prove that Γ is an orthogonal set of $\mu_{\tilde{M},\tilde{D}}$.

For any distinct $\varsigma_1, \varsigma_2 \in \Gamma$, we can write

$$\varsigma_k = \frac{s_k + 2^{\eta+1}\alpha\beta\ell_k}{2^{\eta+1}\alpha\beta} + \lambda_k,$$

where $s_k \in \mathcal{S}_\eta$, $\ell_k \in \mathcal{T}_{\eta,i_{s_k}}$, $\lambda_k \in \Lambda_{s_k,\ell_k}$ and $i_{s_k} \in \{0, 1, 2, 3\}$, $k = 1, 2$. Applying (3-1), the fact $\lambda_1, \lambda_2 \in \mathbb{Z}^2$ and the $\mathbb{Z}^2$-periodicity of $m_{\tilde{D}}$, one has

$$\begin{aligned} 0 &= \hat{\mu}_{A,\tilde{M},\tilde{D}}(2^{\eta+1}\alpha\beta(\varsigma_1 - \varsigma_2)) \\ &= m_{\tilde{D}}(\varsigma_1 - \varsigma_2)\hat{\mu}_{\tilde{M},\tilde{D}}(\varsigma_1 - \varsigma_2) \\ &= m_{\tilde{D}}\left(\frac{s_1 - s_2}{2^{\eta+1}\alpha\beta} + \ell_1 - \ell_2 + \lambda_1 - \lambda_2\right)\hat{\mu}_{\tilde{M},\tilde{D}}(\varsigma_1 - \varsigma_2) \\ &= m_{\tilde{D}}\left(\frac{s_1 - s_2}{2^{\eta+1}\alpha\beta} + \ell_1 - \ell_2\right)\hat{\mu}_{\tilde{M},\tilde{D}}(\varsigma_1 - \varsigma_2). \end{aligned} \tag{3-13}$$

We now claim that $m_{\tilde{D}}((s_1 - s_2)/(2^{\eta+1}\alpha\beta) + \ell_1 - \ell_2) \neq 0$. The proof will be divided into the following two cases.

Case 1: $s_1 = s_2$. In this case, it is clear that $\ell_1, \ell_2 \in \mathcal{T}_{\eta,i_{s_1}}$ by the definition of Γ. With Proposition 3.1(i) and Proposition 3.3(i), we derive that $\ell_1 - \ell_2 \notin \mathcal{Z}(m_{\tilde{D}})$. Thus the claim follows.

Case 2: $s_1 \neq s_2$. For this case, we prove the claim by contradiction. Suppose, on the contrary, that

$$\frac{s_1 - s_2}{2^{\eta+1}\alpha\beta} + \ell_1 - \ell_2 \in \mathcal{Z}(m_{\tilde{D}}). \tag{3-14}$$

By Proposition 3.1 and Proposition 3.3(i), one has $\ell_1 - \ell_2 \in \Theta_0 \cup \mathcal{Z}(m_{\tilde{D}})$. Combining this with (3-14), we conclude that

$$\frac{s_1 - s_2}{2^{\eta+1}\alpha\beta} \in \Theta_0 \cup \mathcal{Z}(m_{\tilde{D}}). \tag{3-15}$$

Using (3-5) and $s_1 \neq s_2$, it is easy to check that $s_1 - s_2 \in \mathfrak{B}$, where

$$\mathfrak{B} = \left\{ \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} : t_1 \in \{1 - 2^\eta \beta, \ldots, 2^\eta \beta - 1\},\ t_2 \in \{1 - \alpha, \ldots, \alpha - 1\} \right\} \setminus \{\mathbf{0}\}.$$

Write $s_1 - s_2 = (t_1, t_2)^t \in \mathfrak{B}$. We first prove $t_1 = 0$. If $t_1 \neq 0$, it follows $t_1 \notin 2^\eta \beta \mathbb{Z}$. Then from the definitions of $\mathcal{Z}(m_{\tilde{D}})$ and Θ_0, it can easily be seen that

$$\frac{s_1 - s_2}{2^{\eta+1}\alpha\beta} = \frac{1}{2^{\eta+1}\alpha\beta} \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} \notin \Theta_0 \cup \mathcal{Z}(m_{\tilde{D}}).$$

This contradicts (3-15), which proves $t_1 = 0$.

Since $t_1 = 0$, it follows from $\beta \in 2\mathbb{Z} + 1$ that

$$\frac{s_1 - s_2}{2^{\eta+1}\alpha\beta} \notin \Theta_1 \cup \Theta_3.$$

Together with (3-15) and $t_1 = 0$, we have

$$\frac{s_1 - s_2}{2^{\eta+1}\alpha\beta} = \frac{1}{2^{\eta+1}\alpha\beta} \begin{pmatrix} 0 \\ t_2 \end{pmatrix} \in \Theta_0 \cup \Theta_2.$$

By a simple calculation, we deduce from $\beta \in 2\mathbb{Z} + 1$ that $t_2 \in \alpha\mathbb{Z}$. However, $(t_1, t_2)^t = (0, t_2)^t \in \mathfrak{B}$ means that $t_2 \notin \alpha\mathbb{Z}$, a contradiction. Hence the claim follows.

Applying the claim and (3-13), we obtain that

$$\hat{\mu}_{\tilde{M},\tilde{D}}(\varsigma_1 - \varsigma_2) = 0.$$

This implies that Γ is an orthogonal set of $\mu_{\tilde{M},\tilde{D}}$.

Step 2. We prove the completeness of the exponential function system $E_\Gamma = \{e^{2\pi i \langle \lambda, x\rangle} : \lambda \in \Gamma\}$.

Fix $s \in \mathcal{S}_\eta$. In view of Proposition 3.3(ii) and Theorem 2.1, one may get that, for any $\ell_{i_s} \in \mathcal{T}_{\eta,i_s}$,

$$\sum_{i_s=0}^{3} \left| m_{\tilde{D}}\left(\frac{s + 2^{\eta+1}\alpha\beta\ell_{i_s} + \xi}{2^{\eta+1}\alpha\beta} \right) \right|^2 \equiv 1. \tag{3-16}$$

In (3-16), let three of ℓ_0, ℓ_1, ℓ_2 and ℓ_3 be fixed, and the other be altered in $\mathcal{T}_{\eta,i_s}$. We can easily verify that, for all distinct $\ell, \ell' \in \mathcal{T}_{\eta,i_s}$,

$$\left| m_{\tilde{D}}\left(\frac{s + 2^{\eta+1}\alpha\beta\ell + \xi}{2^{\eta+1}\alpha\beta} \right) \right| = \left| m_{\tilde{D}}\left(\frac{s + 2^{\eta+1}\alpha\beta\ell' + \xi}{2^{\eta+1}\alpha\beta} \right) \right|. \tag{3-17}$$

Since $\Lambda_{s,\ell} \subset \mathbb{Z}^2$ and Λ is a spectrum of $\mu_{A,\tilde{M},\tilde{D}}$, it follows from the $\mathbb{Z}^2$-periodicity of $m_{\tilde{D}}(x)$ that

$$
\begin{aligned}
(3\text{-}18)\quad 1 &\equiv \sum_{\lambda\in\Lambda} |\hat{\mu}_{A,\tilde{M},\tilde{D}}(\xi+\lambda)|^2 \\
&= \sum_{s\in\mathcal{S}_\eta} \sum_{i_s=0}^{3} \sum_{\ell\in\mathcal{T}_{\eta,i_s}} \sum_{\lambda'\in\Lambda_{s,\ell}} |\hat{\mu}_{A,\tilde{M},\tilde{D}}(\xi+s+2^{\eta+1}\alpha\beta\ell+2^{\eta+1}\alpha\beta\lambda')|^2 \\
&= \sum_{s\in\mathcal{S}_\eta} \sum_{i_s=0}^{3} \sum_{\ell\in\mathcal{T}_{\eta,i_s}} \Big| m_{\tilde{D}}\Big(\frac{s+2^{\eta+1}\alpha\beta\ell+\xi}{2^{\eta+1}\alpha\beta}\Big)\Big|^2 \\
&\qquad\qquad \sum_{\lambda'\in\Lambda_{s,\ell}} \Big|\hat{\mu}_{\tilde{M},\tilde{D}}\Big(\frac{s+2^{\eta+1}\alpha\beta\ell+\xi}{2^{\eta+1}\alpha\beta}+\lambda'\Big)\Big|^2 \\
&= \sum_{s\in\mathcal{S}_\eta} \sum_{i_s=0}^{3} \Big| m_{\tilde{D}}\Big(\frac{s+2^{\eta+1}\alpha\beta\ell_{i_s}+\xi}{2^{\eta+1}\alpha\beta}\Big)\Big|^2 \\
&\qquad\qquad \sum_{\ell\in\mathcal{T}_{\eta,i_s}} \sum_{\lambda'\in\Lambda_{s,\ell}} \Big|\hat{\mu}_{\tilde{M},\tilde{D}}\Big(\frac{s+2^{\eta+1}\alpha\beta\ell+\xi}{2^{\eta+1}\alpha\beta}+\lambda'\Big)\Big|^2,
\end{aligned}
$$

where $\ell_{i_s} \in \mathcal{T}_{\eta,i_s}$, the first line follows from Theorem 2.1 and the second, third and fourth line follow from (3-11), (3-1) and (3-17), respectively.

We now choose $\xi \in \mathbb{R}^2 \setminus \mathbb{Q}^2$, and, for simplicity, write

$$
\begin{aligned}
p_{s,i_s} &= \Big| m_{\tilde{D}}\Big(\frac{s+2^{\eta+1}\alpha\beta\ell_{i_s}+\xi}{2^{\eta+1}\alpha\beta}\Big)\Big|^2, \\
q_{s,i_s} &= \sum_{\ell\in\mathcal{T}_{\eta,i_s}} \sum_{\lambda'\in\Lambda_{s,\ell}} \Big|\hat{\mu}_{\tilde{M},\tilde{D}}\Big(\frac{s+2^{\eta+1}\alpha\beta\ell+\xi}{2^{\eta+1}\alpha\beta}+\lambda'\Big)\Big|^2.
\end{aligned}
$$

Then one may derive from (3-3) that $p_{s,i_s} > 0$, and (3-18) becomes

$$
\sum_{s\in\mathcal{S}_\eta} \sum_{i_s=0}^{3} p_{s,i_s} q_{s,i_s} = 1. \tag{3-19}
$$

Note that Γ is an orthogonal set of $\mu_{\tilde{M},\tilde{D}}$; thus Theorem 2.1 implies that

$$
\sum_{s\in\mathcal{S}_\eta} \max\{q_{s,0}, q_{s,1}, q_{s,2}, q_{s,3}\} \le 1.
$$

Together with (3-16), (3-19) and Lemma 2.5, we conclude that

$$
\sum_{s\in\mathcal{S}_\eta} \sum_{\ell\in\mathcal{T}_{\eta,i_s}} \sum_{\lambda'\in\Lambda_{s,\ell}} \Big|\hat{\mu}_{\tilde{M},\tilde{D}}\Big(\frac{s+2^{\eta+1}\alpha\beta\ell+\xi}{2^{\eta+1}\alpha\beta}+\lambda'\Big)\Big|^2 = 1, \quad i_s = 0, 1, 2, 3, \tag{3-20}
$$

and

$$
\begin{aligned}
\text{(3-21)}\quad \sum_{\ell\in\mathcal{T}_{\eta,0}}\sum_{\lambda'\in\Lambda_{s,\ell}}\left|\hat{\mu}_{\tilde{M},\tilde{D}}\Big(\frac{s+2^{\eta+1}\alpha\beta\ell+\xi}{2^{\eta+1}\alpha\beta}+\lambda'\Big)\right|^2
&=\sum_{\ell\in\mathcal{T}_{\eta,1}}\sum_{\lambda'\in\Lambda_{s,\ell}}\left|\hat{\mu}_{\tilde{M},\tilde{D}}\Big(\frac{s+2^{\eta+1}\alpha\beta\ell+\xi}{2^{\eta+1}\alpha\beta}+\lambda'\Big)\right|^2\\
&=\sum_{\ell\in\mathcal{T}_{\eta,2}}\sum_{\lambda'\in\Lambda_{s,\ell}}\left|\hat{\mu}_{\tilde{M},\tilde{D}}\Big(\frac{s+2^{\eta+1}\alpha\beta\ell+\xi}{2^{\eta+1}\alpha\beta}+\lambda'\Big)\right|^2\\
&=\sum_{\ell\in\mathcal{T}_{\eta,3}}\sum_{\lambda'\in\Lambda_{s,\ell}}\left|\hat{\mu}_{\tilde{M},\tilde{D}}\Big(\frac{s+2^{\eta+1}\alpha\beta\ell+\xi}{2^{\eta+1}\alpha\beta}+\lambda'\Big)\right|^2
\end{aligned}
$$

for any $s\in\mathcal{S}_\eta$.

By continuity, we conclude that (3-20) and (3-21) hold for all $\xi\in\mathbb{R}^2$. Therefore, Theorem 2.1 shows that Γ is a spectrum of $\mu_{\tilde{M},\tilde{D}}$ for any group $\{i_s\}_{s\in\mathcal{S}_\eta}$ with $i_s\in\{0,1,2,3\}$. The proof is complete. □

Remark 3.6. Suppose $\Lambda=\bigcup_{s\in\mathcal{S}_\eta}\bigcup_{i\in\{0,1,2,3\}}\bigcup_{\ell\in\mathcal{T}_{\eta,i}}(s+2^{\eta+1}\alpha\beta\ell+2^{\eta+1}\alpha\beta\Lambda_{s,\ell})$ is a spectrum of $\mu_{A,\tilde{M},\tilde{D}}$ with $0\in\Lambda$. Then we can conclude from (3-21) that for any $s\in\mathcal{S}_\eta$, one of the following two statements holds:

(i) There exist some $\ell_{i_s}\in\mathcal{T}_{\eta,i_s}$ such that $\Lambda_{s,\ell_{i_s}}\neq\varnothing$ for all $0\le i_s\le 3$.

(ii) $\Lambda_{s,\ell}=\varnothing$ for any $\ell\in\mathcal{T}_\eta=\bigcup_{i=0}^{3}\mathcal{T}_{\eta,i}$.

In particular, the assumption $0\in\Lambda$ implies $\Lambda_{0,0}\neq\varnothing$. Therefore, (i) always holds for $s=0$, which illustrates that there must exist $\ell_{i_0}\in\mathcal{T}_{\eta,i_0}$ such that $\Lambda_{0,\ell_{i_0}}\neq\varnothing$ for all $1\le i_0\le 3$.

In order to prove Theorems 1.5 and 1.6 more conveniently, we define

$$
\begin{aligned}
\Phi_0&=\{\upsilon\in\mathbb{Z}^2:\upsilon=(0,0)^t \pmod{2\mathbb{Z}^2}\},\\
\Phi_1&=\{\upsilon\in\mathbb{Z}^2:\upsilon=(1,0)^t \pmod{2\mathbb{Z}^2}\},\\
\Phi_2&=\{\upsilon\in\mathbb{Z}^2:\upsilon=(0,1)^t \pmod{2\mathbb{Z}^2}\},\\
\Phi_3&=\{\upsilon\in\mathbb{Z}^2:\upsilon=(1,1)^t \pmod{2\mathbb{Z}^2}\}.
\end{aligned}
$$

Then

$$
\mathbb{Z}^2=\bigcup_{i=0}^{3}\Phi_i. \tag{3-22}
$$

We have all ingredients for the proof of Theorem 1.5.

Proof of Theorem 1.5. We will prove this theorem by the circle (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv) $\Longrightarrow$ (i) $\Longrightarrow$ (ii).

(ii) $\Longrightarrow$ (iii): If $\tilde{M} \in M_2(2\mathbb{Z})$, we can write $\tilde{M} = \left(\begin{smallmatrix} 2\tilde{a} & 2\tilde{b} \\ 2\tilde{c} & 2\tilde{d} \end{smallmatrix}\right)$ with $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d} \in \mathbb{Z}$. Then with (1-3), it is easy to verify that

$$\tilde{M}\mathring{\mathcal{F}}_2^2 = \left\{ \begin{pmatrix} \tilde{a} \\ \tilde{c} \end{pmatrix}, \begin{pmatrix} \tilde{b} \\ \tilde{d} \end{pmatrix}, \begin{pmatrix} \tilde{a}+\tilde{b} \\ \tilde{c}+\tilde{d} \end{pmatrix} \right\} \subset \mathbb{Z}^2.$$

Hence the assertion follows.

(iii) $\Longrightarrow$ (iv): Suppose $\tilde{M}\mathring{\mathcal{F}}_2^2 \subset \mathbb{Z}^2$, which implies $\tilde{C} := \tilde{M}^*\mathcal{F}_2^2 \subset \mathbb{Z}^2$. Then using Lemma 2.3 and Proposition 3.1(iii), we obtain that $(\tilde{M}, \tilde{D}, \tilde{C})$ is a Hadamard triple. Therefore, $(\tilde{M}, \tilde{D})$ is admissible.

(iv) $\Longrightarrow$ (i): If $(\tilde{M}, \tilde{D})$ is admissible, $\mu_{\tilde{M},\tilde{D}}$ is a spectral measure by Theorem 1.2.

(i) $\Longrightarrow$ (ii): Suppose that $\mu_{\tilde{M},\tilde{D}}$ is a spectral measure, and let $A = \mathrm{diag}(2\alpha\beta, 2\alpha\beta)$. In view of Lemma 2.4, one may derive that $\mu_{A,\tilde{M},\tilde{D}}$ is also a spectral measure. Let Λ be a spectrum of $\mu_{A,\tilde{M},\tilde{D}}$ with $0 \in \Lambda$. First, we construct a spectrum of $\mu_{\tilde{M},\tilde{D}}$. Recall that $\mathcal{T}_{\eta,i}$ and Φ_i are defined by (3-5) and (3-22), respectively. By $\eta = 0$ and a simple calculation, one has $2\alpha\beta\tilde{M}^*\mathcal{T}_{\eta,0} \subset \Phi_0$. For $i \in \{1, 2, 3\}$, we can suppose that $2\alpha\beta\tilde{M}^*\mathcal{T}_{\eta,i} \subset \Phi_{j_i}$ for some $j_i \in \{0, 1, 2, 3\}$. Consequently,

$$\bigcup_{i=1}^{3} 2\alpha\beta\tilde{M}^*\mathcal{T}_{\eta,i} \subset \bigcup_{i=1}^{3} \Phi_{j_i}.$$

This means that for any $s \in \mathcal{S}_\eta \setminus \{0\}$, there exists $i_s \in \{0, 1, 2, 3\}$ such that $s + 2\alpha\beta\ell_s \notin \bigcup_{j=1}^{3} 2\alpha\beta\tilde{M}^*\mathcal{T}_{\eta,j} + 2\mathbb{Z}^2$ for any $\ell_s \in \mathcal{T}_{\eta,i_s}$. Define

$$\Gamma = \Delta_{0,0} \cup \bigcup_{s\in\mathcal{S}_\eta\setminus\{0\}} \Delta_{s,i_s}, \tag{3-23}$$

where $\Delta_{0,0} = \bigcup_{\ell_0\in\mathcal{T}_{\eta,0}} (\ell_0 + \Lambda_{0,\ell_0})$, $\Delta_{s,i_s} = \bigcup_{\ell_s\in\mathcal{T}_{\eta,i_s}} ((s + 2\alpha\beta\ell_s)/(2\alpha\beta) + \Lambda_{s,\ell_s})$ with

$$(s + 2\alpha\beta\mathcal{T}_{\eta,i_s}) \cap \left(\bigcup_{j=1}^{3} 2\alpha\beta\tilde{M}^*\mathcal{T}_{\eta,j} + 2\mathbb{Z}^2 \right) = \varnothing, \tag{3-24}$$

and Λ_{s,ℓ_s} is defined by (3-10). In view of Lemma 3.5, we get that Γ is a spectrum of $\mu_{\tilde{M},\tilde{D}}$. Moreover, it follows from $0 \in \Lambda$ and Lemma 2.4 that $0 \in \Gamma$.

Second, we prove that for any $i \in \{1, 2, 3\}$, there must exist $\ell_i \in \mathcal{T}_{\eta,i}$ such that $2\alpha\beta\tilde{M}^*\ell_i \in 2\mathbb{Z}^2$. Since Γ is a spectrum of $\mu_{\tilde{M},\tilde{D}}$ with $0 \in \Gamma$, it follows from Lemma 2.4 that $2\alpha\beta\tilde{M}^{*-1}\Gamma$ is a spectrum of $\mu_{A,\tilde{M},\tilde{D}}$ with $0 \in 2\alpha\beta\tilde{M}^{*-1}\Gamma$. Using (3-11), one has

$$2\alpha\beta\tilde{M}^{*-1}\Gamma = \bigcup_{s'\in\mathcal{S}_\eta} \bigcup_{i\in\{0,1,2,3\}} \bigcup_{\ell'_i\in\mathcal{T}_{\eta,i}} (s' + 2\alpha\beta\ell'_i + 2\alpha\beta\Lambda'_{s',\ell'_i}), \tag{3-25}$$

where

$$\Lambda'_{s',\ell'_i} = \{\gamma \in \mathbb{Z}^2 : s' + 2\alpha\beta\ell'_i + 2\alpha\beta\gamma \in 2\alpha\beta\tilde{M}^{*-1}\Gamma\}.$$

For $s'=0$ and $\ell'_i = 0 \in \mathcal{T}_{\eta,0}$, we have $\Lambda'_{0,0} \neq \varnothing$ since $0 \in 2\alpha\beta\tilde{M}^{*-1}\Gamma$. By Remark 3.6, there must exist $\ell'_i \in \mathcal{T}_{\eta,i}$ such that $\Lambda'_{0,\ell'_i} \neq \varnothing$ for all $1 \le i \le 3$. Let $\lambda'_i \in \Lambda'_{0,\ell'_i}$, where $i = 1, 2, 3$. Therefore, (3-23) and (3-25) imply that there exist $s_i \in \mathcal{S}_\eta$, $\ell_i \in \bigcup_{j=0}^{3} \mathcal{T}_{\eta,j}$ and $\lambda_i \in \Lambda_{s_i,\ell_i}$ such that $(s_i + 2\alpha\beta\ell_i)/(2\alpha\beta) + \lambda_i \in \Gamma$ and

$$2\alpha\beta\tilde{M}^*\ell'_i + 2\alpha\beta\tilde{M}^*\lambda'_i = s_i + 2\alpha\beta\ell_i + 2\alpha\beta\lambda_i \quad \text{for } i = 1, 2, 3. \tag{3-26}$$

Moreover, it follows from (3-24) that $s_i + 2\alpha\beta\ell_i \notin \bigcup_{j=1}^{3} 2\alpha\beta\tilde{M}^*\mathcal{T}_{\eta,j} + 2\mathbb{Z}^2$ if $s_i \neq 0$ for $i = 1, 2, 3$. However, by noting that $\lambda_i, \lambda'_i \in \mathbb{Z}^2$, (3-26) implies that

$$s_i + 2\alpha\beta\ell_i \in 2\alpha\beta\tilde{M}^*\ell'_i + 2\mathbb{Z}^2 \subset 2\alpha\beta\tilde{M}^*\mathcal{T}_{\eta,i} + 2\mathbb{Z}^2 \subset \bigcup_{j=1}^{3} 2\alpha\beta\tilde{M}^*\mathcal{T}_{\eta,j} + 2\mathbb{Z}^2$$

for $i = 1, 2, 3$. Therefore, the above discussion shows that $s_i = 0$ for $i = 1, 2, 3$, and hence $\ell_i \in \mathcal{T}_{\eta,0}$ by the definition of Γ. This implies $2\alpha\beta\ell_i \in 2\mathbb{Z}^2$ for $i = 1, 2, 3$. Combining this with $\tilde{M} \in M_2(\mathbb{Z})$, $s_i = 0$ and $\lambda_i, \lambda'_i \in \mathbb{Z}^2$, one may infer from (3-26) that

$$2\alpha\beta\tilde{M}^*\ell'_i = 2\alpha\beta\ell_i + 2\alpha\beta(\lambda_i - \tilde{M}^*\lambda'_i) \in 2\mathbb{Z}^2 \quad \text{for } i = 1, 2, 3.$$

Therefore, $2\alpha\beta\tilde{M}^*\ell'_i \in 2\mathbb{Z}^2$ for some $\ell'_i \in \mathcal{T}_{\eta,i}$, where $i = 1, 2, 3$.

It remains to prove $\tilde{M} \in M_2(2\mathbb{Z})$. For any $i \in \{1, 2, 3\}$, the above conclusion shows that there must exist $\ell_i \in \mathcal{T}_{\eta,i}$ such that $2\alpha\beta\tilde{M}^*\ell_i \in 2\mathbb{Z}^2$. For these $\ell_i \in \mathcal{T}_{\eta,i}$, $i = 1, 2, 3$, by the definition of $\mathcal{T}_{\eta,i}$ and the fact $\alpha, \beta \in 2\mathbb{Z}+1$, it can easily be checked that

$$\{2\alpha\beta\ell_i : i = 1, 2, 3\} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} \pmod{2\mathbb{Z}^2}.$$

This together with $2\alpha\beta\tilde{M}^*\ell_i \in 2\mathbb{Z}^2$ and a simple calculation gives that $\tilde{M}^* \in M_2(2\mathbb{Z})$, which is equivalent to $\tilde{M} \in M_2(2\mathbb{Z})$. This finishes the proof of Theorem 1.5. □

The following lemma plays an important role in the proof of Theorem 1.6.

Lemma 3.7. *Let $\mu_{\tilde{M},\tilde{D}}$ be a spectral measure, where $\tilde{M}$ and $\tilde{D}$ are given by* (1-4) *and* (1-5), *respectively. If $\eta > 0$ in $\tilde{D}$, then $\tilde{M} = \left(\begin{smallmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{smallmatrix}\right)$ satisfies $2^{\eta+1} \mid \tilde{c}$.*

Proof. Suppose, on the contrary, that $2^{\eta+1} \nmid \tilde{c}$. Then one may write $\tilde{c} = 2^\tau c'$ for some integer $\tau \le \eta$ and $c' \in 2\mathbb{Z}+1$. Let $Q_1 = \operatorname{diag}(1, 1/2^\tau)$. A simple calculation gives

$$M_1 := Q_1\tilde{M}Q_1^{-1} = \begin{pmatrix} \tilde{a} & 2^\tau\tilde{b} \\ c' & \tilde{d} \end{pmatrix} \in M_2(\mathbb{Z})$$

and

$$D_1 := Q_1\tilde{D} = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \alpha \\ 0 \end{pmatrix}, \begin{pmatrix} \omega \\ 2^{\eta-\tau}\beta \end{pmatrix}, \begin{pmatrix} -\alpha-\omega \\ -2^{\eta-\tau}\beta \end{pmatrix} \right\} \subset \mathbb{Z}^2,$$

where $\alpha, \beta \in 2\mathbb{Z}+1$. Since $\mu_{\tilde{M},\tilde{D}}$ is a spectral measure, it follows from Lemmas 2.2 and 2.4 that μ_{M_1,D_1} and μ_{A_1,M_1,D_1} are also spectral measures, where $A_1 = \mathrm{diag}(2^{\eta-\tau+1}\alpha\beta, 2^{\eta-\tau+1}\alpha\beta)$ and μ_{A_1,M_1,D_1} is defined by (2-5).

If $\tau = \eta$, it follows from Theorem 1.5 that $M_1 \in M_2(2\mathbb{Z})$. This means that $c' \in 2\mathbb{Z}$, a contradiction. Hence the assertion follows.

If $\tau < \eta$, we derive the contradiction by constructing a spectrum of μ_{M_1,D_1}. Recall that $\mathcal{S}_{\eta-\tau}$ and $\mathcal{T}_{\eta-\tau} = \bigcup_{i=0}^{3} \mathcal{T}_{\eta-\tau,i}$ are defined by (3-5). We first prove the following two claims.

Claim 1. *Let Φ_1 and Φ_3 be given by* (3-22). *Then*

$$2^{\eta-\tau+1}\alpha\beta M_1^* \mathcal{T}_{\eta-\tau,2} \subset \begin{cases} \Phi_1, & \text{if } \tilde{d} \in 2\mathbb{Z}, \\ \Phi_3, & \text{if } \tilde{d} \in 2\mathbb{Z}+1. \end{cases}$$

Proof of Claim 1. For any $\ell \in \mathcal{T}_{\eta-\tau,2}$, there exist $k \in \hbar_\alpha$ and $k' \in \hbar_{2^{\eta-\tau}\beta}$ such that

$$\ell = \frac{1}{2^{\eta-\tau+1}\alpha\beta} \begin{pmatrix} 2^{\eta-\tau+1}k\beta \\ 2k'\alpha - 2k\omega + \alpha \end{pmatrix}. \tag{3-27}$$

Since $M_1 = \left(\begin{smallmatrix} \tilde{a} & 2^\tau \tilde{b} \\ c' & \tilde{d} \end{smallmatrix}\right)$, $\tau < \eta$ and $\alpha, c' \in 2\mathbb{Z}+1$, it follows from (3-27) that

$$2^{\eta-\tau+1}\alpha\beta M_1^* \ell = \begin{pmatrix} 2(2^{\eta-\tau}k\tilde{a}\beta + (k'\alpha - k\omega)c') + c'\alpha \\ 2(2^{\eta}k\tilde{b}\beta + (k'\alpha - k\omega)\tilde{d}) + \tilde{d}\alpha \end{pmatrix} = \begin{pmatrix} 1 \\ \tilde{d} \end{pmatrix} \pmod{2\mathbb{Z}^2}.$$

Consequently, $2^{\eta-\tau+1}\alpha\beta M_1^*\ell \in \Phi_1$ if $\tilde{d} \in 2\mathbb{Z}$, and $2^{\eta-\tau+1}\alpha\beta M_1^*\ell \in \Phi_3$ if $\tilde{d} \in 2\mathbb{Z}+1$. So the claims follows. □

Claim 2. *Let Φ_1 and Φ_3 be given by* (3-22). *Then for any $s \in \mathcal{S}_{\eta-\tau} \setminus \{0\}$, the following two statements hold*:

(i) *There exist some $i_s \in \{0, 1, 2, 3\}$ such that $s + 2^{\eta-\tau+1}\alpha\beta\ell_s \notin \Phi_1$ for any $\ell_s \in \mathcal{T}_{\eta-\tau,i_s}$.*

(ii) *There exist some $i_s \in \{0, 1, 2, 3\}$ such that $s + 2^{\eta-\tau+1}\alpha\beta\ell_s \notin \Phi_3$ for any $\ell_s \in \mathcal{T}_{\eta-\tau,i_s}$.*

Proof of Claim 2. Begin by observing that if $\alpha \in 2\mathbb{Z}+1$ and $\tau < \eta$, then for any $\ell_i \in \mathcal{T}_{\eta-\tau,i}$, $i = 0, 1, 2, 3$, we have

$$2^{\eta-\tau+1}\alpha\beta\ell_0 = \begin{pmatrix} 2^{\eta-\tau+1}k_0\beta \\ 2k_0'\alpha - 2k_0\omega \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \pmod{2\mathbb{Z}^2},$$

$$2^{\eta-\tau+1}\alpha\beta\ell_1 = \begin{pmatrix} 2^{\eta-\tau}(2k_1\beta + \beta) \\ 2k_1'\alpha - 2k_1\omega - \omega \end{pmatrix} = \begin{pmatrix} 0 \\ \omega \end{pmatrix} \pmod{2\mathbb{Z}^2},$$

$$2^{\eta-\tau+1}\alpha\beta\ell_2 = \begin{pmatrix} 2^{\eta-\tau+1}k_2\beta \\ 2k_2'\alpha - 2k_2\omega + \alpha \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \pmod{2\mathbb{Z}^2}$$

and

$$2^{\eta-\tau+1}\alpha\beta\ell_3=\begin{pmatrix}2^{\eta-\tau}(2k_3\beta+\beta)\\ 2k_3'\alpha-2k_3\omega+\alpha-\omega\end{pmatrix}=\begin{pmatrix}0\\ \omega-1\end{pmatrix}\pmod{2\mathbb{Z}^2}$$

for some $k_i\in\hbar_\alpha$ and $k_i'\in\hbar_{2^{\eta-\tau}\beta}$. Without loss of generality, we assume that $\omega\in 2\mathbb{Z}$ (the case $\omega\in 2\mathbb{Z}+1$ can be similarly proved). Then a simple calculation gives

$$2^{\eta-\tau+1}\alpha\beta\ell_0,\,2^{\eta-\tau+1}\alpha\beta\ell_1\in\Phi_0\quad\text{and}\quad 2^{\eta-\tau+1}\alpha\beta\ell_2,\,2^{\eta-\tau+1}\alpha\beta\ell_3\in\Phi_2. \tag{3-28}$$

Recall that $\mathcal{T}_{\eta-\tau}=\bigcup_{i=0}^{3}\mathcal{T}_{\eta-\tau,i}$. Then for any $s=(s_1,s_2)^t\in\mathcal{S}_{\eta-\tau}\setminus\{\mathbf{0}\}$, we take

$$\ell_s\in\begin{cases}\mathcal{T}_{\eta-\tau}, & \text{if } s_1\in 2\mathbb{Z},\\ \mathcal{T}_{\eta-\tau,2}\cup\mathcal{T}_{\eta-\tau,3}, & \text{if } s_1\in 2\mathbb{Z}+1,\ s_2\in 2\mathbb{Z},\\ \mathcal{T}_{\eta-\tau,0}\cup\mathcal{T}_{\eta-\tau,1}, & \text{if } s_1,s_2\in 2\mathbb{Z}+1.\end{cases}$$

This together with (3-28) yields that $s+2^{\eta-\tau+1}\alpha\beta\ell_s\notin\Phi_1$, which proves (i). For (ii), we take

$$\ell_s\in\begin{cases}\mathcal{T}_{\eta-\tau}, & \text{if } s_1\in 2\mathbb{Z},\\ \mathcal{T}_{\eta-\tau,0}\cup\mathcal{T}_{\eta-\tau,1}, & \text{if } s_1\in 2\mathbb{Z}+1,\ s_2\in 2\mathbb{Z},\\ \mathcal{T}_{\eta-\tau,2}\cup\mathcal{T}_{\eta-\tau,3}, & \text{if } s_1,s_2\in 2\mathbb{Z}+1.\end{cases}$$

Consequently, $s+2^{\eta-\tau+1}\alpha\beta\ell_s\notin\Phi_3$ by (3-28). Thus Claim 2 follows. □

We now continue with the proof of the case $\tau<\eta$. In the following proof, we might as well assume $\tilde{d}\in 2\mathbb{Z}$ in M_1. If $\tilde{d}\in 2\mathbb{Z}+1$, we only need to replace Claim 2(i) with Claim 2(ii).

Since $\tau<\eta$ and $\tilde{d}\in 2\mathbb{Z}$, it follows from Claim 2(i) that for any $s\in\mathcal{S}_{\eta-\tau}\setminus\{\mathbf{0}\}$, there must exist some $i_s\in\{0,1,2,3\}$ such that $s+2^{\eta-\tau+1}\alpha\beta\ell_s\notin\Phi_1$ for any $\ell_s\in\mathcal{T}_{\eta-\tau,i_s}$. Let $\tilde{\Lambda}$ be a spectrum of μ_{A_1,M_1,D_1} with $0\in\tilde{\Lambda}$. Define

$$\tilde{\Gamma}=\tilde{\Delta}_{0,0}\cup\bigcup_{s\in\mathcal{S}_{\eta-\tau}\setminus\{0\}}\tilde{\Delta}_{s,i_s},$$

where

$$\tilde{\Delta}_{0,0}=\bigcup_{\ell_0\in\mathcal{T}_{\eta-\tau,0}}(\ell_0+\tilde{\Lambda}_{0,\ell_0}),\qquad \tilde{\Delta}_{s,i_s}=\bigcup_{\ell_s\in\mathcal{T}_{\eta-\tau,i_s}}\left(\frac{s+2^{\eta-\tau+1}\alpha\beta\ell_s}{2^{\eta-\tau+1}\alpha\beta}+\tilde{\Lambda}_{s,\ell_s}\right)$$

with

$$(s+2^{\eta-\tau+1}\alpha\beta\mathcal{T}_{\eta-\tau,i_s})\cap\Phi_1=\varnothing,$$

and

$$\tilde{\Lambda}_{s,\ell_s}=\{\gamma\in\mathbb{Z}^2: s+2^{\eta-\tau+1}\alpha\beta\ell_s+2^{\eta-\tau+1}\alpha\beta\gamma\in\tilde{\Lambda}\}.$$

Using the similar argument as in the proof of Lemma 3.5, we can show that $\tilde{\Gamma}$ is a spectrum of μ_{M_1,D_1} with $0\in\tilde{\Gamma}$.

Next, we prove that there must exist $\ell \in \mathcal{T}_{\eta-\tau,2}$ such that $2^{\eta-\tau+1}\alpha\beta M_1^*\ell \in 2\mathbb{Z}^2$. Since $\tilde{\Gamma}$ is a spectrum of μ_{M_1,D_1} with $0 \in \tilde{\Gamma}$, it follows from Lemma 2.4 that $2^{\eta-\tau+1}\alpha\beta M_1^{*-1}\tilde{\Gamma}$ is a spectrum of μ_{A_1,M_1,D_1} with $0 \in 2^{\eta-\tau+1}\alpha\beta M_1^{*-1}\tilde{\Gamma}$. Similar to (3-25), we have that

$$2^{\eta-\tau+1}\alpha\beta M_1^{*-1}\tilde{\Gamma} = \bigcup_{s'\in\mathcal{S}_{\eta-\tau}} \bigcup_{i\in\{0,1,2,3\}} \bigcup_{\ell_i'\in\mathcal{T}_{\eta-\tau,i}} (s' + 2^{\eta-\tau+1}\alpha\beta\ell_i' + 2^{\eta-\tau+1}\alpha\beta\tilde{\Lambda}'_{s',\ell_i'}),$$

where

$$\tilde{\Lambda}'_{s',\ell_i'} = \{\gamma \in \mathbb{Z}^2 : s' + 2^{\eta-\tau+1}\alpha\beta\ell_i' + 2^{\eta-\tau+1}\alpha\beta\gamma \in 2^{\eta-\tau+1}\alpha\beta M_1^{*-1}\tilde{\Gamma}\}.$$

For $s' = 0$ and $\ell_i' = 0 \in \mathcal{T}_{\eta-\tau,0}$, it follows from $0 \in 2^{\eta-\tau+1}\alpha\beta M_1^{*-1}\tilde{\Gamma}$ that $\tilde{\Lambda}'_{0,0} \neq \varnothing$. Similar to Remark 3.6, one may infer that there exists $\ell_2' \in \mathcal{T}_{\eta-\tau,2}$ such that $\tilde{\Lambda}'_{0,\ell_2'} \neq \varnothing$. Therefore, applying Claim 1 and the similar argument as in the proof of Theorem 1.5, we can easily conclude that $2^{\eta-\tau+1}\alpha\beta M_1^*\ell_2' \in 2\mathbb{Z}^2$. Thus the assertion follows.

Finally, we prove $2^{\eta+1} \mid \tilde{c}$. The above discussion means that there exist some $\ell \in \mathcal{T}_{\eta-\tau,2}$ such that $2^{\eta-\tau+1}\alpha\beta M_1^*\ell \in 2\mathbb{Z}^2$. For these $\ell \in \mathcal{T}_{\eta-\tau,2}$, it follows from (3-27) that

$$2^{\eta-\tau+1}\alpha\beta M_1^*\ell = \begin{pmatrix} 2(2^{\eta-\tau}k\tilde{a}\beta+(k'\alpha-k\omega)c')+c'\alpha \\ 2(2^{\eta}k\tilde{b}\beta+(k'\alpha-k\omega)\tilde{d})+\tilde{d}\alpha \end{pmatrix}$$

for some $k \in \hbar_\alpha$ and $k' \in \hbar_{2^{\eta-\tau}\beta}$. Together with $2^{\eta-\tau+1}\alpha\beta M_1^*\ell \in 2\mathbb{Z}^2$, it yields that $c'\alpha \in 2\mathbb{Z}$. This contradicts the fact $c', \alpha \in 2\mathbb{Z}+1$, and hence the assumption $2^{\eta+1} \nmid \tilde{c}$ does not hold. Therefore, we obtain $2^{\eta+1} \mid \tilde{c}$, and complete the proof. □

Having established the above preparation, now we are in a position to prove Theorem 1.6.

Proof of Theorem 1.6. We first prove the necessity. Suppose $\mu_{\tilde{M},\tilde{D}}$ is a spectral measure. In view of Lemma 3.7, we have that $\tilde{M} = \left(\begin{smallmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{smallmatrix}\right)$ satisfies $2^{\eta+1} \mid \tilde{c}$. Thus one may write $\tilde{c} = 2^{\eta+1}\kappa$ with $\kappa \in \mathbb{Z}$. Let $\tilde{Q} = \mathrm{diag}(1, 1/2^\eta)$. By a simple calculation, we get

$$\bar{M} := \tilde{Q}\tilde{M}\tilde{Q}^{-1} = \begin{pmatrix} \tilde{a} & 2^\eta\tilde{b} \\ 2\kappa & \tilde{d} \end{pmatrix} \tag{3-29}$$

and

$$\bar{D} := \tilde{Q}\tilde{D} = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \alpha \\ 0 \end{pmatrix}, \begin{pmatrix} \omega \\ \beta \end{pmatrix}, \begin{pmatrix} -\alpha-\omega \\ -\beta \end{pmatrix} \right\}. \tag{3-30}$$

Since $\mu_{\tilde{M},\tilde{D}}$ is a spectral measure, it follows from Lemma 2.2 that $\mu_{\bar{M},\bar{D}}$ is also a spectral measure. Then with Theorem 1.5, we have $\bar{M} \in M_2(2\mathbb{Z})$. This together with (3-29) gives that $\tilde{a}, \tilde{d} \in 2\mathbb{Z}$. Hence the necessity follows.

Now we are devoted to proving the sufficiency. Suppose $\tilde{M} = \left(\begin{smallmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{smallmatrix}\right)$, where $\tilde{a}, \tilde{d} \in 2\mathbb{Z}$ and $2^{\eta+1} \mid \tilde{c}$. Then there exist $a^*, c^*, d^* \in \mathbb{Z}$ such that $\tilde{a} = 2a^*$, $\tilde{c} = 2^{\eta+1}c^*$ and $\tilde{d} = 2d^*$. Let $\tilde{Q} = \operatorname{diag}(1, 1/2^\eta)$. A simple calculation gives

$$M' := \tilde{Q}\tilde{M}\tilde{Q}^{-1} = \begin{pmatrix} 2a^* & 2^\eta \tilde{b} \\ 2c^* & 2d^* \end{pmatrix},$$

and $\bar{D} = \tilde{Q}\tilde{D}$ is given by (3-30). Since $\eta > 0$, it follows from Theorem 1.5 that $\mu_{M',\bar{D}}$ is a spectral measure. Therefore, $\mu_{\tilde{M},\tilde{D}}$ is a spectral measure by Lemma 2.2.

This completes the proof of Theorem 1.6. □

4. Proofs of Theorems 1.3 and 1.4

We are committed to investigating the spectrality of the measure $\mu_{M,D}$, where $M \in M_2(\mathbb{Z})$ is an expansive integer matrix and D is given by (1-2). We first prove Theorem 1.3 by using Theorems 1.5 and 1.6, and then prove Theorem 1.4. Finally, we provide some concluding remarks.

Proof of Theorem 1.3. The sufficiency follows directly from Theorem 1.2 and Lemma 2.2. Now we are devoted to proving the necessity. Suppose that $\mu_{M,D}$ is a spectral measure. Let $\eta = \max\{r : 2^r \mid (\alpha_1\beta_2 - \alpha_2\beta_1)\}$, and let $\tilde{M}$ and $\tilde{D}$ be given by (1-4) and (1-5), respectively. That is, $\tilde{M} = QMQ^{-1}$ and $\tilde{D} = QD$. In view of Lemma 2.2, $\mu_{\tilde{M},\tilde{D}}$ is a spectral measure. It suffices to prove that there exists a matrix $\tilde{Q} \in M_2(\mathbb{R})$ such that $(\bar{M}, \bar{D})$ is admissible, where $\bar{M} = \tilde{Q}\tilde{M}\tilde{Q}^{-1}$ and $\bar{D} = \tilde{Q}\tilde{D}$. The proof will be divided into the following two cases.

Case 1: $\eta = 0$. Since $\mu_{\tilde{M},\tilde{D}}$ is a spectral measure, it follows from $\eta = 0$ and Theorem 1.5 that $(\tilde{M}, \tilde{D})$ is admissible. Thus the assertion follows by taking $\tilde{Q} = \operatorname{diag}(1, 1)$.

Case 2: $\eta > 0$. Since $\mu_{\tilde{M},\tilde{D}}$ is a spectral measure, Theorem 1.6 implies that one may write $\tilde{M} = \left(\begin{smallmatrix} 2a' & b' \\ 2^{\eta+1}c' & 2d' \end{smallmatrix}\right)$, where $a', b', c', d' \in \mathbb{Z}$. We take $\tilde{Q} = \operatorname{diag}(1, 1/2^\eta)$. Then

$$\bar{M} = \tilde{Q}\tilde{M}\tilde{Q}^{-1} = \begin{pmatrix} 2a' & 2^\eta b' \\ 2c' & 2d' \end{pmatrix} \quad \text{and} \quad \bar{D} = \tilde{Q}\tilde{D} = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \alpha \\ 0 \end{pmatrix}, \begin{pmatrix} \omega \\ \beta \end{pmatrix}, \begin{pmatrix} -\alpha-\omega \\ -\beta \end{pmatrix} \right\}.$$

Using $\eta > 0$, it is clear that $\bar{M} \in M_2(2\mathbb{Z})$. Hence $(\bar{M}, \bar{D})$ is admissible by Theorem 1.5.

This completes the proof of Theorem 1.3. □

Next, we focus on proving Theorem 1.4.

Proof of Theorem 1.4. Let $\tilde{M}$ and $\tilde{D}$ be given by (1-4) and (1-5), respectively. That is,

$$\tilde{M} = QMQ^{-1} \quad \text{and} \quad \tilde{D} = QD, \tag{4-1}$$

where the matrix $Q \in M_2(\mathbb{Z})$ satisfies $\det(Q) = 1$. In view of Lemma 2.2, $\mu_{M,D}$ is a spectral measure if and only if $\mu_{\tilde{M},\tilde{D}}$ is a spectral measure. This implies that Theorem 1.4(i) is equivalent to Theorem 1.5(i). Note that $\det(Q) = 1$; hence, by a simple calculation, one has that

$$M \in M_2(2\mathbb{Z}) \iff \tilde{M} \in M_2(2\mathbb{Z}).$$

Thus Theorem 1.4(ii) and (iii) are equivalent to Theorem 1.5(ii) and (iii), respectively. Finally, from the Definition 1.1 and (4-1), it is easy to see that $(\tilde{M}, \tilde{D})$ is admissible $\iff$ there exists a set $\tilde{C} \subset \mathbb{Z}^2$ such that $(\tilde{M}, \tilde{D}, \tilde{C})$ is a Hadamard triple $\iff$ $(M, D, Q^*\tilde{C})$ is a Hadamard triple $\iff$ (M, D) is admissible. Consequently, Theorem 1.4(iv) is equivalent to Theorem 1.5(iv).

Therefore, the desired result now is obtained by appeal to Theorem 1.5. □

At the end of this paper, we give some further remarks and list an open question which is related to our main results. The following example is specifically used to display our results, which are convenient to judge whether the measure $\mu_{M,D}$ in Question 1 is a spectral measure.

Example 4.1. Let $M_1 = \begin{pmatrix} 2 & b \\ 2 & 2 \end{pmatrix}$ and $M_2 = \begin{pmatrix} 2 & b \\ 4 & 2 \end{pmatrix}$ be two expansive integer matrices, and let

$$D_1 = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix} \right\} \quad \text{and} \quad D_2 = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ -2 \end{pmatrix} \right\}.$$

Then the following statements hold:

(i) μ_{M_1,D_1} and μ_{M_2,D_1} are spectral measures if and only if $b \in 2\mathbb{Z}$.

(ii) μ_{M_1,D_2} is a nonspectral measure, while μ_{M_2,D_2} is a spectral measure.

Proof. By a simple calculation, this follows directly from Theorems 1.5 and 1.6. □

It is worth noting that if $\alpha_1\beta_2 - \alpha_2\beta_1 \in 2\mathbb{Z}$ in Theorem 1.3, we cannot give the specific form of matrix M. However, if $\alpha_1, \alpha_2, \beta_1$ and β_2 are fixed, we can describe the specific form by applying Theorem 1.6. The following simple but interesting example is devoted to illustrating this fact.

Example 4.2. Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an expansive integer matrix, and let

$$D = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 8 \end{pmatrix}, \begin{pmatrix} -4 \\ -10 \end{pmatrix} \right\}.$$

Then $\mu_{M,D}$ is a spectral measure if and only if $a, d \in 2\mathbb{Z}$ and $c \in 4\mathbb{Z}$.

Proof. Write $Q = \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix}$. Then it is direct to compute that

$$\tilde{M} := QMQ^{-1} = \begin{pmatrix} 3a-c+2(3b-d) & 3a-c+3(3b-d) \\ c-2a+2(d-2b) & c-2a+3(d-2b) \end{pmatrix}$$

and

$$\tilde{D} := QD = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -2 \\ -2 \end{pmatrix} \right\}.$$

By Lemma 2.2, $\mu_{M,D}$ is a spectral measure if and only if $\mu_{\tilde{M},\tilde{D}}$ is a spectral measure.

For the sufficiency, it follows from $a, d \in 2\mathbb{Z}$ and $c \in 4\mathbb{Z}$ that there exist $\tilde{a}, \tilde{c}, \tilde{d} \in \mathbb{Z}$ such that $a = 2\tilde{a}$, $d = 2\tilde{d}$ and $c = 4\tilde{c}$. Thus $\tilde{M}$ becomes

$$\tilde{M} = \begin{pmatrix} 2(3\tilde{a}-2\tilde{c}+3b-d) & 3a-c+3(3b-d) \\ 4(\tilde{c}-\tilde{a}+\tilde{d}-b) & 2(2\tilde{c}-a+3\tilde{d}-b) \end{pmatrix}.$$

This together with Theorem 1.6 yields that $\mu_{\tilde{M},\tilde{D}}$ is a spectral measure, and hence the sufficiency follows.

Conversely, suppose $\mu_{\tilde{M},\tilde{D}}$ is a spectral measure. Applying Theorem 1.6, we have

$$\begin{aligned} &3a - c + 2(3b - d) \in 2\mathbb{Z}, \\ &c - 2a + 2(d - 2b) \in 4\mathbb{Z}, \\ &c - 2a + 3(d - 2b) \in 2\mathbb{Z}. \end{aligned}$$

Consequently, $3a - c, c + 3d \in 2\mathbb{Z}$ and $c - 2a + 2d \in 4\mathbb{Z}$. By a simple calculation, we infer that $a, d \in 2\mathbb{Z}$ and $c \in 4\mathbb{Z}$. This proves the necessity. □

We remark here that the digit set D in (1-2) satisfies $\alpha_1\beta_2 - \alpha_2\beta_1 \neq 0$, and so it is of interest to consider the following question:

Question 2. For an expansive matrix $M \in M_2(\mathbb{Z})$ and the digit set

$$D = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}, \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}, \begin{pmatrix} -\alpha_1-\beta_1 \\ -\alpha_2-\beta_2 \end{pmatrix} \right\}$$

with $\alpha_1\beta_2 - \alpha_2\beta_1 = 0$, what is the sufficient and necessary condition for $\mu_{M,D}$ to be a spectral measure?

In fact, for the matrix M and the digit set D given in the above question, using the methods of [34], we can find an integer matrix Q such that $\bar{M} := QMQ^{-1}$ and

$$\bar{D} := QD = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \alpha \\ 0 \end{pmatrix}, \begin{pmatrix} \beta \\ 0 \end{pmatrix}, \begin{pmatrix} -\alpha-\beta \\ 0 \end{pmatrix} \right\},$$

where $\alpha, \beta \in \mathbb{Z}$ and $\bar{M}$ is an expansive integer matrix with $\det(\bar{M}) = \det(M)$. Lemma 2.2 indicates that to consider the spectrality of $\mu_{M,D}$, we only need to consider the measure $\mu_{\bar{M},\bar{D}}$. However, it is apparent that the set $\mathcal{Z}(m_{\bar{D}})$ includes free variables since the root of

$$m_{\bar{D}}(\xi) = \frac{1}{\#\bar{D}} \sum_{d \in \bar{D}} e^{2\pi i \langle d, \xi \rangle} = \frac{1}{\#\bar{D}} (1 + e^{2\pi\alpha\xi_1} + e^{2\pi\beta\xi_1} + e^{2\pi(-\alpha-\beta)\xi_1}) = 0$$

is independent of ξ_2, where $\xi = (\xi_1, \xi_2)^t$. We have not yet discovered an effective method to address this situation. An answer to Question 2 may provide insights into the study of the spectrality of fractal measures.

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