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GROUP TOPOLOGIES ON AUTOMORPHISM GROUPS OF HOMOGENEOUS STRUCTURES

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We provide sufficient conditions for the standard topology (generated by stabilizers of finite sets) on the automorphism group of a countable homogeneous structure to be minimal among all Hausdorff group topologies on the group. Under certain assumptions, such as when the structure is the Fraïssé limit of a relational class with the free amalgamation property, we are able to classify all the group topologies on the automorphism group coarser than the standard topology even when the latter is not minimal.

1. Introduction

Minimality. A topological group (G, τ) consists of a group $(G, \cdot)$ and a topology τ on G such that the map $\rho : G\times G \to G$, where $\rho(g, h) = gh^{-1}$, is jointly continuous.

Definition 1.1. A Hausdorff topological group (G, τ) is called *minimal* if G does not admit a Hausdorff group topology strictly coarser than τ or, equivalently, if every bijective continuous homomorphism from G to another Hausdorff topological group is a homeomorphism. The topological group (G, τ) is *totally minimal* if every continuous surjective homomorphism to a Hausdorff topological group is open.

Clearly, every totally minimal group is minimal. Also, for a topological group (G, τ), if the only strictly coarser topology is $\{\varnothing, G\}$ then (G, τ) is totally minimal. Indeed, in that case for any continuous surjective homomorphism $\phi : (G, \tau)\to(H, \sigma)$ the pullback $\phi^*(\sigma)$ of σ by ϕ satisfies $\phi^*(\sigma) \subseteq \tau$ and thus $\phi^*(\sigma) \in \{\tau, \{\varnothing, G\}\}$, so the map σ is either a homeomorphism or the trivial map. For a group topology $\tau' \subsetneq \tau$, by considering the closure of the identity in τ', one easily sees that this applies, in particular, to the case in which (G, τ) is minimal and has no nontrivial normal closed subgroups.

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The notion of minimality for topological groups was introduced as early as 1971 as a generalization of compactness. In fact it is easy to see that any compact Hausdorff topological group is minimal. For more information about minimality, we refer the reader to the survey by Dikranjan and Megrelishvili [2014].

Given a group G of permutations of some set Ω and $A \subseteq \Omega$, let

$$G_A = \{g \in G \mid ga = a \text{ for all } a \in A\}.$$

Let $[\Omega]^{<\omega}$ be the set of all finite subsets of Ω. The collection $\{G_A \mid A \in [\Omega]^{<\omega}\}$ is a base of neighbourhoods at the identity of a group topology which we call the *standard topology* and denote by τ_{st}. More generally for each G-invariant $X \subseteq \Omega$ there is an associated group topology τ_{st}^X generated by $\{G_A \mid A \in [X]^{<\omega}\}$.

One of the earliest results on minimality due to Gaughan [1967] states that $(S_\infty, \tau_{\text{st}})$ is totally minimal, where S_∞ denotes the group of all permutations of a countable set Ω.

Given a countable first-order structure $\mathcal{M}$ with universe M, the automorphism group of $\mathcal{M}$ is a τ_{st}-closed subgroup of $S_\infty = S(M)$ and vice versa: any closed subgroup of $S(M)$ is the automorphism group of some countable structure on M. The interplay between the dynamical properties of $\text{Aut}(\mathcal{M})$ and the logical and combinatorial properties of $\mathcal{M}$ has been widely studied in the literature, beginning with the characterization due to Engeler, Ryll-Nardzewski, Svenonius and others of oligomorphic subgroups of S_∞ as the automorphism groups of ω-categorical countable structures. Recall that an oligomorphic group is a closed subgroup of S_∞ whose diagonal action on M^n has finitely many orbits, for each $n \in \mathbb{N}$.

In this context τ_{st} is often referred to in the literature as the pointwise convergence topology.

In light of the above the following is thus a natural question, already asked in [Dikranjan and Megrelishvili 2014].

Problem 1. *Let $\mathcal{M}$ be a countable ω-categorical* (ω*-saturated*, *sufficiently nice*) *first-order structure and $G = \text{Aut}(\mathcal{M})$. When is* (G, τ_{st}) (*totally*) *minimal*?

A deep result in this direction appeared in recent work by Ben Yaacov and Tsankov [2016], where the authors show that automorphism groups of countable ω-categorical, stable continuous structures are totally minimal with respect to the pointwise convergence topology. This specializes to the result that the automorphism groups of classical ω-categorical stable structures are totally minimal with respect to τ_{st}.

Not all oligomorphic groups are minimal with respect to τ_{st}. As pointed out in [Ben Yaacov and Tsankov 2016], an example of this is $\text{Aut}(\mathbb{Q}, <)$ (see Corollary C for a generalization). However even in those cases it is possible to formulate the following more general question:

Problem 2. *Let $\mathcal{M}$ be a countable ω-categorical (or sufficiently nice) first-order structure and $G = \mathrm{Aut}(\mathcal{M})$. Describe the lattice of all Hausdorff group topologies on G coarser than τ_{st}.*

This work was mainly motivated by [Ben Yaacov and Tsankov 2016] and is meant as a preliminary exploration of Problems 1 and 2 in the classical setting outside the stability constraint.

In its broadest lines the strategy followed by [Ben Yaacov and Tsankov 2016] goes back to [Uspenskij 2008], where the author shows that the isometry group of the Urysohn sphere is totally minimal with the pointwise convergence topology. Both proofs rely on the assumption that the group in question is Roelcke precompact and use a well-behaved independence relation among (small) subsets of the structure to endow the Roelcke precompletion of the group with a topological semigroup structure. Information on the topological quotients of the original group is then recovered from the latter via the functoriality of Roelcke compactification and the Ellis lemma. Recall that a topological group (G, τ) is *Roelcke precompact* if for any neighbourhood W of 1 there exists a finite $F \subset G$ such that $WFW = G$. For closed subgroups of S_∞ this is equivalent to being oligomorphic.

In contrast, our methods for obtaining (partial) minimality results are completely elementary. There are drawbacks to this lack of sophistication: for instance, we are not able to recover the result in [Ben Yaacov and Tsankov 2016] for classical structures. On the other hand we do not rely on assumptions of Roelcke precompactness (except for certain residual assumptions in some cases). Although we are not discussing metric structures or Urysohn spaces in this paper, we would like to mention that a refinement of the approached presented here has enabled us to answer in the positive the question about the minimality of the isometry group of the (unbounded) Urysohn space posed in [Uspenskij 2008].

Problems 1 and 2 could be also formulated for semigroup topologies on the endomorphism monoid of a countable relational structure. Some general techniques for characterising minimal and maximal semigroup topologies on the endomorphism monoid of a countable relational structure have been recently introduced in [Elliott et al. 2023].

Main results. Generally speaking, an *independence relation* is a ternary relation $\mathop{\smile\!\!\!\!\!|}$ defined on some collection of sets of elements of the structure such that $A \mathop{\smile\!\!\!\!\!|}_C B$ is meant to capture the intuitive idea that B does not contain any information about A not already contained in C. The paradigmatic example is that of forking independence in model theory. The study of the connections between the existence of a well-behaved independence relation on a homogeneous structure (see Definition 2.1) and the properties of the automorphism group goes back to [Tent and Ziegler 2013] (see also [Evans et al. 2016]).

We provide a simple technical criterion (Proposition 2.12) for (relative) minimality for τ_{st} in a relatively general setting. We derive from this general minimality results stated in terms of the existence of an independence relation satisfying certain axioms and in turn derive from this two main theorems. The first applies to Fraïssé limits of free amalgamation classes, i.e., Fraïssé classes closed under free amalgamation (more details in Section 3). Some well-known examples of Fraïssé limits of free amalgamation classes are the random graph, random hypergraph, homogeneous K_n-free graphs for $n \geqslant 3$, etc.

Theorem A. *Let $\mathcal{M}$ be the Fraïssé limit of a free amalgamation class in a countable relational language. Let $G = \mathrm{Aut}(\mathcal{M})$. Then any group topology $\tau \subseteq \tau_{\mathrm{st}}$ on G is of the form τ_{st}^X, where $X \subseteq M$ is some G-invariant set. In particular, if the action of G on M is transitive, then there are no nontrivial group topologies on G strictly coarser than τ_{st} and thus (G, τ_{st}) is totally minimal.*

Rather than the free amalgamation property directly, the proof of Theorem A uses the freedom axiom, a more abstract property introduced in [Conant 2017].

The second application of the Proposition 2.12 is in the context of *simple* theories. Simple structures (i.e., theories) occupy an important place in classification theory. We refer the reader to [Tent and Ziegler 2012], [Wagner 2000] and [Kim 2014] for the definition of simple theories, forking and canonical bases.

A simple theory T is called *one-based* if $\mathrm{Cb}(a/A) \subseteq \mathrm{bdd}(a)$ for any hyperimaginary element a and a small subset A of the monster model. Our second main result is the following:

Theorem B. *Let $\mathcal{M}$ be a simple, ω-saturated countable structure with locally finite algebraic closure and weak elimination of imaginaries. Assume furthermore that* $\mathrm{Th}(\mathcal{M})$ *is one-based. Let $G = \mathrm{Aut}(\mathcal{M})$. Then:*

(1) *If G acts transitively on M, then (G, τ_{st}) is minimal.*

(2) *If all singletons are algebraically closed, then any group topology τ on G coarser than τ_{st} is of the form τ_{st}^X for some G-invariant $X \subseteq M$.*

Technically speaking, the use of the freedom axiom and stationarity in Theorem A is replaced in Theorem B by that of one-basedness and the independence property for forking independence in simple theories.

One important class of structures that fall under the assumptions of Theorem B are Lie geometries and their affine spaces as described in [Cherlin and Hrushovski 2003] and [Kantor et al. 1989]. Another class of examples of structures to which Theorem B applies can be obtained using the general techniques in [Chatzidakis and Pillay 1998].

Finally, we present a natural variant of ideas of [Uspenskij 2008] and [Ben Yaacov and Tsankov 2016] in the context of automorphism groups of first-order structures.

Given a structure $\mathcal{M}$ with group of automorphisms G, we describe a semigroup of partial types $R^{\mathrm{pa}}(\mathcal{M})$ containing G consisting of partial infinitary types encoding the relationships between two copies of $\mathcal{M}$, and show that any idempotent in $R^{\mathrm{pa}}(\mathcal{M})$ which is invariant under the involution given by exchanging the blocks of coordinates corresponding to the two models and the action of G can be associated to a group topology on G coarser than τ_{st}.

We show that under certain mild conditions, the topology τ_{st} on the automorphism group of any distal Fraïssé limit is not minimal.

Corollary C. *Let $\mathcal{M}$ be any distal Fraïssé limit in a finite relational language with trivial algebraic closure. Then the type q_{inf} defines a group topology on $G = \mathrm{Aut}(\mathcal{M})$ strictly coarser than τ_{st}.*

Layout. The paper is organized as follows. In Section 2 we prove our main technical criterion, Proposition 2.12, of (relative) minimality for τ_{st}.

Section 3 contains some preliminary discussion on independence relations and Fraïssé constructions, along with the proofs of Theorems A and B. In Section 3D we have provided an example where we show total minimality is not preserved under taking open finite-index subgroups. Finally in Section 3E we have shown that τ_{st} in certain simple ω-categorical structures built using the Hrushovski construction method are minimal (Corollary 3.11). Structures that are built using this method and predimension functions are not one-based.

Section 4 is dedicated to the systematic connection between group topologies below the standard topology and types described above as well as the proof of Corollary C.

2. A relative minimality criterion for τ_{st}

Given a topological group (G, τ) and $g \in G$ we denote by $\mathcal{N}_\tau(g)$ the filter of (not necessarily open) neighbourhoods of g in τ. Since $\mathcal{N}_\tau(g) = g\mathcal{N}_\tau(1_G) = \mathcal{N}_\tau(1_G)g$ for any $g \in G$, any group topology τ is uniquely determined by $\mathcal{N}_\tau(1_G)$. Given a filter $\mathcal{V}$ on G at 1_G such that

- for every $U \in \mathcal{V}$ there is $V \in \mathcal{V}$ such that $V^{-1} \subseteq U$,
- for every $U \in \mathcal{V}$ there is $V \in \mathcal{V}$ such that $VV \subseteq U$, and
- $U^g \in \mathcal{V}$ for every $U \in \mathcal{V}$ and $g \in G$,

there is a unique group topology τ on G such that $\mathcal{V} = \mathcal{N}_\tau(1_G)$. Given a family $\mathcal{Y}$ of subsets of G containing 1_G, we say that $\mathcal{Y}$ generates a group topology τ at the identity if $\mathcal{Y}$ generates $\mathcal{N}_\tau(1_G)$ as a filter.

Given a set X we let $[X]^{<\omega}$ stand for the collection of all finite subsets of X. Our setting consists of an infinite set Ω and some $G \leqslant S(\Omega)$, where $S(\Omega)$ is the group of permutations of Ω. It is easy to see using the criterion above that the collection $\{G_A \mid A \in [\Omega]^{<\omega}\}$ is a base of neighbourhoods of the identity of a unique group topology τ_{st}, which we will refer to as the standard topology. We are mainly interested in the case in which Ω is countable, in which case $S(\Omega)$, abbreviated as S_∞, is a Polish group.

By a closure operator on $[\Omega]^{<\omega}$ we mean a map $\mathrm{cl} : [\Omega]^{<\omega} \to [\Omega]^{<\omega}$ that preserves inclusion and satisfies $A \subseteq \mathrm{cl}(A) = \mathrm{cl}(\mathrm{cl}(A))$, for each $A \in [\Omega]^{<\omega}$. There is a bijective correspondence between (G-equivariant) closure operators cl and (G-invariant) families $\mathfrak{X} \subseteq [\Omega]^{<\omega}$ closed under intersections. Each $\mathfrak{X}$ gives a closure operator $\mathrm{cl}(-)$ by taking as $\mathrm{cl}(A)$ for any finite A the smallest set in $\mathfrak{X}$ containing A. In the opposite direction we associate cl with the class of cl-closed sets: $\mathfrak{X} = \{A \in [\Omega]^{<\omega} \mid \mathrm{cl}(A) = A\}$.

Given a family $\mathfrak{X}$ of subsets of a set Ω, denote by $(\mathfrak{X})$ the collection of all (finite) tuples of elements whose coordinates enumerate some member of $\mathfrak{X}$. As is customary, the same letter will be used to refer to either a tuple or the corresponding set depending on the context. In particular we might use an expression such as BC to denote the union of the ranges of B and C.

Given tuples A, B, C of elements from Ω we write $A \cong^G B$ if there exists some $g \in G$ such that $gA = B$ and given an additional C we write $A \cong^G_C B$ if there is $g \in G_C$ such that $gA = B$. Given $A \subset \Omega$ we let $\mathrm{acl}^G(A)$ stand for the union of all elements of Ω whose orbit under G_{A_0} is finite for some finite subset A_0 of A. We say $\mathrm{acl}^G(-)$ is *locally finite* if $\mathrm{acl}^G(A)$ is finite whenever A is. In that case the restriction of acl^G to $[\Omega]^{<\omega}$ is a closure operator on $[\Omega]^{<\omega}$. We write $\mathfrak{X}^G = \{A \in [\Omega]^{<\omega} \mid \mathrm{acl}^G(A) = A\}$ and we say that acl^G is *trivial* if $\mathfrak{X}^G = [\Omega]^{<\omega}$.

Definition 2.1. Let $\mathcal{M}$ be a structure with universe M.

- The structure $\mathcal{M}$ is called *homogeneous* if for every $A, B \subseteq M$ such that $|A| = |B| < |M|$ and $\mathrm{tp}(A) = \mathrm{tp}(B)$ there is an automorphism of $\mathcal{M}$ which sends A to B.
- The structure $\mathcal{M}$ is called *ω-saturated* if for every $A \in [M]^{<\omega}$ any type over A is realised in $\mathcal{M}$.
- A relational structure $\mathcal{M}$ is called *ultrahomogeneous* if any isomorphism between finite substructures of $\mathcal{M}$ extends to an automorphism of $\mathcal{M}$.

Let G be the group of automorphisms of some structure $\mathcal{M}$ with universe M. Recall that if $\mathcal{M}$ is countable and ω-saturated, then for finite A we have that $\mathrm{acl}^G(A)$ coincides with the algebraic closure of A. If $\mathcal{M}$ is ω-saturated and countable, then it is homogeneous. In particular a relational structure $\mathcal{M}$ if ω-saturated and countable, then it is ultrahomogeneous. Typical examples of countable ultrahomogeneous structures are structures obtained from the Fraïssé construction method in a relational language (see Section 3B).

The proof of the following statements contains two auxiliary observations. As usual in such cases we mark the end of the proof of the subordinate results with a shaded (as opposed to white) square.

Proposition 2.2. *Let G be a group of permutations of a set Ω for which $\mathrm{acl}^G(-)$ is locally finite. Suppose we are given some G-invariant $X \subseteq \Omega$ and another group topology $\tau^* \subset \tau_{\mathrm{st}}^X$ such that for some constant $K \in \mathbb{N}$ the following property holds*:

($\diamond$) *For any $A, B \in \mathfrak{X}^G$ and $U \in \mathcal{N}_{\tau^*}(1_G)$ there exists $U' \in \mathcal{N}_{\tau^*}(1_G)$ such that $((G_A \cap U)G_B)^K = G_{A\cap B} \cap U'$.*

Then any group topology $\tau \subseteq \tau_{\mathrm{st}}^X$ must satisfy at least one of the following two conditions:

(1) *Given $x \in X$ there exists $W \in \mathcal{N}_\tau(1_G)$ such that $gx \in \mathrm{acl}^G(x)$ for each $g \in W$.*

(2) *There exists some G-invariant $X' \subsetneq X$ such that for all $W \in \mathcal{N}_\tau(1_G)$ there is $U' \in \mathcal{N}_{\tau^*}(1_G)$ and $U'' \in \mathcal{N}_{\tau_{\mathrm{st}}^{X'}}(1_G)$ such that $U' \cap U'' \subseteq W$.*

Proof. Assume the first alternative does not hold. Then there is $x_0 \in X$ such that for any $W \in \mathcal{N}_\tau(1_G)$ there exists $g \in W$ such that $g(x_0) \notin \mathrm{acl}^G(x_0)$. Let $X' = X \setminus G \cdot x_0$. Our goal is to show point (2), that is, that any neighbourhood W of 1_G in τ is also a neighbourhood of the identity in any topology containing τ^* and $\tau_{\mathrm{st}}^{X'}$. We prove this via two observations.

Observation 2.3. *For any $a \in G \cdot x_0$, any finite $B \subset \Omega$ and any $W \in \mathcal{N}_\tau(1_G)$ there exists some $g \in W$ such that $ga \notin B$.*

Proof. Suppose the condition above fails for some a, B, and W. By Neumann's lemma there exists some $h \in G_a$ such that $h(B) \cap B \subseteq \mathrm{acl}^G(a)$. This means that any g in $W \cap W^{h^{-1}} \in \mathcal{N}_\tau(1_G)$ must take a to a point in $\mathrm{acl}^G(a)$. This contradicts the choice of x_0 and the fact that any $a \in G \cdot x_0$ must have the same property, by invariance of $\mathcal{N}_\tau(1_G)$ under conjugation. ■

The following observation follows from ($\diamond$) by an induction argument and we leave the proof to the reader.

Observation 2.4. *There is a function $\mu : \mathbb{N} \to \mathbb{N}$ such that given any finite collection $\{B_j\}_{j=1}^r \subset \mathfrak{X}^G$, $U \in \mathcal{N}_{\tau^*}(1_G)$ and $W \subseteq G$ containing $U \cap \bigcup_{j=1}^r G_{B_j}$ there exists $U' \in \mathcal{N}_{\tau^*}(1_G)$ such that $G_{\bigcap_{j=1}^r B_j} \cap U' \subseteq W^{\mu(r)}$.*

Fix some arbitrary $W \in \mathcal{N}_\tau(1_G)$. Pick $W_0 = W_0^{-1} \in \mathcal{N}_\tau(1_G)$ such that $W_0^{2K} \subseteq W$. Since $\tau \subseteq \tau_{\mathrm{st}}^X$, there exists some finite $A \subset X$ such that $G_A \subseteq W_0$. By local finiteness we may assume $A = \mathrm{acl}^G(A)$. Let $\{a_j\}_{j=1}^r := A \cap (G \cdot x_0)$.

Pick $W_1 = W_1^{-1} \in \mathcal{N}_\tau(1_G)$ such that $W_1^{3\mu(r)} \subseteq W_0$, where μ is the function given by Observation 2.4. Let $B \subset \Omega$ be a finite subset such that $G_B \subset W_1$. We may assume again $B \in \mathfrak{X}^G$. By Observation 2.3 for any $1 \leqslant j \leqslant r$ there exists some $g_j \in W_1$ such that $g_j a_j \notin B$ or, equivalently, $a_j \notin B_j := g_j^{-1} B$. Notice that $G_{B_j} = G_B^{g_j} \subseteq W_1^3$.

Let $C = \bigcap_{j=1}^r B_j$. According to Observation 2.4 (for $U = G$) there is $U' \in \mathcal{N}_{\tau^*}(1_G)$ such that $G_C \cap U' \subset (W_1^3)^{\mu(r)} \subseteq W_0$. A final direct application of ($\diamond$) (again with $U = G$) yields some $U_0' \in \mathcal{N}_{\tau^*}(1_G)$ such that

$$U_0' \cap G_{C\cap A} \subseteq (G_C G_A)^K \subseteq W_0^{2K} \subseteq W.$$

By construction $C \cap A \subseteq X'$ and thus $U_0'' := G_{A\cap C} \in \tau_{\mathrm{st}}^{X'}$. As W is an arbitrary neighbourhood of 1_G in τ we conclude that case (2) of the statement holds and so we are done. □

We elaborate further on the same idea:

Lemma 2.5. *Let G be a group of permutations of a set Ω, $\{X_j\}_{j\in J}$ some collection of G-invariant subsets of Ω and $Z = \bigcap_{j\in J} X_j$. Assume that $\mathrm{acl}^G(x) = x$ for any $x \in \Omega$ and that there exists $K > 0$ such that for any finite $A, B \subset \Omega$ we have $(G_A G_B)^K = G_{A\cap B}$. Then $\tau_{\mathrm{st}}^Z = \bigcap_{j\in J} \tau_{\mathrm{st}}^{X_j}$.*

Proof. We begin by noting that just as in Observation 2.4 one can show by induction:

Observation 2.6. *There exists a function $\mu : \mathbb{N} \to \mathbb{N}$ such that for any finite collection $\{B_l\}_{l=1}^r \subseteq [\Omega]^{<\omega}$ and any $V \subseteq G$ containing G_{B_l} for all $1 \leqslant l \leqslant r$ we have $G_{\bigcap_{l=1}^r B_l} \subseteq V^{\mu(r)}$.*

Let $\tau_0 = \bigcap_{j\in J} \tau_{\mathrm{st}}^{X_j}$. The inclusion $\tau_{\mathrm{st}}^Z \subseteq \tau_0$ is clear. Take now any $W \in \mathcal{N}_{\tau_0}(1_G)$. Fix $j_0 \in J$. Since $W \in \tau_{\mathrm{st}}^{X_{j_0}}$, there exists some finite $A \subseteq X_{j_0}$ such that $G_A \subseteq W$. Let $\{a_j\}_{j=1}^r := A \setminus Z$. Pick $W_0 = W_0^{-1} \in \mathcal{N}_{\tau_0}(1_G)$ such that $W_0^{\mu(r+1)} \subseteq W$. For each $1 \leqslant l \leqslant r$ choose some $j_l \in J$ such that $a_l \notin X_{j_l}$ and then some finite $B_l \subseteq X_{j_l}$ such that $G_{B_l} \subseteq W_0$. Observation 2.6 and the choice of W_0 implies $G_C \subseteq W$, where $C = A \cap \bigcap_{l=1}^r B_l$. Since $C \subseteq Z$ we have shown $U \subseteq W$ for some $U \in \tau_{\mathrm{st}}^Z$. Since $W \in \mathcal{N}_{\tau_0}(1_G)$ was arbitrary we have $\tau_0 \subseteq \tau_{\mathrm{st}}^Z$ and we are done. □

Lemma 2.7. *Let G be the automorphism group of some structure $\mathcal{M}$ endowed with a G-invariant locally finite closure operator $\mathrm{cl}(-)$ on M and a group topology τ coarser than τ_{st}. Assume that the action of G is transitive and there is some $W \in \mathcal{N}_\tau(1_G)$ and $a \in M$ such that $ga \in \mathrm{cl}(a)$, for each $g \in W$. Then either τ is not Hausdorff or $\tau = \tau_{\mathrm{st}}$.*

Proof. Notice that by the transitivity of the action of G on M and continuity of the inverse operation for every $a \in M$ there are $U_a, W_a \in \mathcal{N}_\tau(1_G)$ such that $f(a) \in \mathrm{cl}(a)$ for any $f \in W_a$ and $g^{-1}(a) \in \mathrm{cl}(a)$ for any $g \in U_a$. For a finite tuple A in M we write $W_A = \bigcap_{a\in A} W_a$. Given $a, b \in M$, we say that $a \sim b$ if $a \in \mathrm{cl}(b)$ and $b \in \mathrm{cl}(a)$. This is clearly an equivalence relation. If we let $W_a' = W_a \cap \bigcap_{z\in \mathrm{cl}(a)} U_z$, then any $f \in W_a'$ must preserve the class $[a] \in M/{\sim}$ setwise, that is, $W_a' \subset G_{[a]}$. Indeed, if $g \in W'_a$, then $ga \in \mathrm{cl}(a)$. On the other hand, since $g \in U_{ga}$ we must have $a = g^{-1}ga \in \mathrm{cl}(ga)$, so $a \sim ga$.

For any $V \in \mathcal{N}_\tau(1_G)$ and any finite $\sim$-closed $A \subset M$ consider the set

$$Y_V^A = \{f : A \to A \mid \exists g \in V \text{ such that } g\restriction_A = f \text{ and } g([a]) = [a] \text{ for all } a \in A\}.$$

Notice that this set is finite, and that given $\sim$-closed $A \subset B \subset M$ and $f \in Y_V^B$ we have $f\restriction_A \in Y_V^A$. Invariance should be clear from the fact that A is $\sim$-closed and the definition of Y_V^A.

Lemma 2.8. *Either $Y_V^A = \{\mathrm{id}_A\}$ for some $V \in \mathcal{N}_\tau(1_G)$ and finite $\sim$-closed A or there exists $f \in G \setminus \{1_G\}$ such that for all $\sim$-closed $A \subset M$ and all $V \in \mathcal{N}_\tau(1_G)$ we have $f\restriction_A \in Y_V^A$.*

Proof. Recall that according to the assumption the closure is locally finite. If the first alternative is not the case, then from Observation 2.4 and König's lemma it follows that there is a function $f : M \to M$ such that $f\restriction_A \in Y_V^A$ for any $\sim$-closed A and $V \in \mathcal{N}_\tau(1_G)$. The fact that $f\restriction_A$ is a type-preserving bijection of A for any such A implies $f \in G$. ■

If the first possibility in Lemma 2.8 holds, then G_A contains $W'_A \cap V$ and is thus a neighbourhood of the identity in τ, which implies that $\tau = \tau_{\mathrm{st}}$. We claim that if the second possibility is satisfied the resulting $f \in G \setminus \{1_G\}$ satisfies $f \in \bigcap_{V \in \mathcal{N}_\tau(1_G)} V$, and therefore τ is not Hausdorff. Given any $V \in \mathcal{N}_\tau(1_G)$, the closure in τ of any $W \in \mathcal{N}_\tau(1_G) \cap \tau_{\mathrm{st}}$ satisfying $W = W^{-1}$ and $W^2 \subset V$ is itself contained in V. Indeed, if $h \in \overline{W}$ then there is $h' \in hW \cap W$ and thus $h = h'(h'')^{-1} \in W^2$ for some $h'' \in W$. Hence, $\mathcal{N}_\tau(1_G)$ admits a basis consisting entirely of τ_{st}-closed neighbourhoods of the identity. It is thus enough to show that f belongs to the closure of V in τ_{st} for any $V \in \mathcal{N}_\tau(1_G)$, which is immediate from the definition of Y_V^A. □

The following ubiquitous observation is crucial for the application of the results above. We provide a proof for the sake of completeness.

Lemma 2.9. *Let G be a group of permutations of a set Ω and A, B tuples of elements from Ω for which there is a chain $A = A_0, B_0, \ldots, B_{n-1}, A_n = g(A)$ such that $A_i B_i \cong^G A_{i+1} B_i \cong^G AB$ for $0 \leqslant i < n$. Then $g \in (G_A G_B)^n G_A$.*

Proof. The proof is by induction on n. In the base case $n = 0$ we have $A = g(A)$, that is, $g \in G_A$. Assume now $n > 0$. Since $AB_0 \cong^G AB$, there exists $h \in G_A$ such that $h(B_0) = B$. Now $A_1 B_0 \cong^G AB$ implies $h(A_1)B = h(A_1)h(B_0) \cong^G A_1 B_0 \cong^G AB$, which implies that there exists $h' \in G_B$ such that $h'(h(A_1)) = A$. Applying induction to the sequence $(A'_i, B'_i)_{i=0}^{n-1}$ given by $A'_i = h'h(A_{i+1})$, $B'_i = h'h(B_{i+1})$ yields that $h'hg \in (G_A G_B)^{n-1} G_A$, from which it follows that $g \in (G_A G_B)^n G_A$, as desired. □

Definition 2.10. Suppose we are given a group G of permutations of a set Ω, and $\mathfrak{X}$ a G-invariant family of subsets of Ω closed under intersection. We say $\mathfrak{X}$ has the *n-zigzag* property (with respect to the action of G) if for every $A, B \in (\mathfrak{X})$ and any A' with $A \cong^G_{A\cap B} A'$ there are $A_0, \ldots, A_n$ and $B_0, \ldots, B_{n-1}$ such that

(1) $A_0 := A$, and $A_n = A'$;

(2) $A_i B_i \cong^G A_{i+1} B_i \cong^G AB$ for $0 \leqslant i \leqslant n-1$.

We will refer to the sequence $A_0, B_0, A_1, \ldots, A_n$ above as an *(n, B)-zigzag path* from A to A'.

Observation 2.11. *Given an n-zigzag path as above if we write $C = A \cap B$ then $C \subseteq A_i B_i \cong^G_C A_{i+1} B_i \cong^G_C AB$ for all $0 \leqslant i \leqslant n-1$. In particular, $A_i \cap B_i = A_{i+1} \cap B_i = C$.*

Notice that for fixed A, B and n, the existence of a (n, B)-zigzag path from A to A' depends only on the orbit of A' under G_A.

Proposition 2.12. *Suppose $\mathcal{M}$ is a countable first-order structure and $G = \mathrm{Aut}(\mathcal{M})$. Assume $\mathrm{acl}^G(-)$ is locally finite and $\mathfrak{X}^G$ corresponding to acl^G has the n-zigzag property for some n. Then*:

(1) *If the action of G on M is transitive, then (G, τ_{st}) is minimal.*

(2) *If $\mathrm{acl}^G(x) = x$ for any $x \in M$, then any group topology $\tau \subseteq \tau_{\mathrm{st}}$ is of the form τ^X_{st} for some G-invariant $X \subseteq M$.*

Proof. For any $A, B \in \mathfrak{X}$ and any $g \in G_{A\cap B}$ the n-zigzag property applied to A, B and $A' = gA$, together with Lemma 2.9, implies that $g \in (G_A G_B)^n G_A$. Therefore $G_{A\cap B} = (G_A G_B)^n G_A$ and we can apply Proposition 2.2 with $\tau^* = \{\varnothing, G\}$ under the common assumptions of (1) and (2). By the same reason we can also apply Lemma 2.5 under the assumptions of (2).

Let us show (1) first. Let τ be a group topology on G coarser than τ_{st}. If the first alternative in Proposition 2.2 holds, then by Lemma 2.7 either τ is not Hausdorff or $\tau = \tau_{\mathrm{st}}$. Since by assumption the only invariant subsets of M are $\varnothing$ and M, the second alternative implies that $\tau = \{\varnothing, G\}$.

Let us now show (2). Let τ be a group topology on G coarser than τ_{st}. By Lemma 2.5 (see the discussion in the first paragraph) there exists some unique minimal G-invariant set X such that $\tau \subseteq \tau^X_{\mathrm{st}}$. Apply Proposition 2.2 with $\tau^* = \{\varnothing, G\}$. The second alternative produces some G-invariant $X' \subsetneq X$ such that $\tau \subseteq \tau^{X'}_{\mathrm{st}}$, in contradiction with the choice of X. Since we assume acl^G to be trivial, the first alternative implies $\tau = \tau^X_{\mathrm{st}}$. □

3. Minimality and independence

3A. ***Independence.*** Throughout this section we work in the following setting: Ω is a set, G is a permutation group of Ω, $\mathrm{cl}(-)$ a G-equivariant closure operator on $[\Omega]^{<\omega}$ and $\mathfrak{X} = \{\mathrm{cl}(A) \mid A \in [\Omega]^{<\omega}\}$ the associated family of closed sets. Our goal is to derive concrete applications from the results of the previous section to the case where Ω is the underlying set of a first-order structure $\mathcal{M}$ and $G = \mathrm{Aut}(\mathcal{M})$.

Definition 3.1. Given $\mathrm{cl}(-)$ and $\mathfrak{X}$ as above and a ternary relation $\mathop{\smile\!\!\!\!\!\!|}\,$ between members of $[\Omega]^{<\omega}$ we say that $(\mathrm{cl}, \perp)$ (alternatively, $(\mathfrak{X}, \perp)$) is a *compatible pair* if for all $A, B, C, D \in [\Omega]^{<\omega}$ the following properties are satisfied:

- (compatibility) $A \perp_C B$ if and only if $A \perp_{\mathrm{cl}(C)} B$ if and only if $\mathrm{cl}(AC) \perp_C \mathrm{cl}(BC)$.
- (invariance) If $g \in G$ and $A \perp_B C$ then $gA \perp_{gB} gC$.
- (weak monotonicity) If $A \perp_B CD$ or $AD \perp_B C$ then $A \perp_B C$.
- (antireflexivity) If $A \perp_C B$, then $A \cap B \subseteq \mathrm{cl}(C)$.

We write $A \perp B$ as an abbreviation of $A \perp_{\varnothing} B$.

Definition 3.2. We define some additional properties for a compatible pair $(\mathfrak{X}, \perp)$:

- (transitivity) If $A \perp_B C$ and $A \perp_{BC} D$, then $A \perp_B CD$.
- (symmetry) If $A \perp_B C$ then $C \perp_B A$.
- (existence) For any A, B, C there is $g \in G_B$ such that $gA \perp_B C$.
- (independence) Suppose we are given $A, B_1, B_2, C_1, C_2 \in (\mathfrak{X})$ such that $B_1 \perp_A B_2$, $A \subseteq B_i$ and $C_i \perp_A B_i$ for $i = 1, 2$ and $C_1 \cong^G_A C_2$. Then there exists $D \in \mathfrak{X}$ such that $D \cong^G_{B_i} C_i$ for $i = 1, 2$ and $D \perp_A B_1 B_2$.
- (stationarity) If $B \in \mathfrak{X}$ and $A_i \perp_B C$ for $i = 1, 2$, then $A_1 \cong^G_B A_2$ implies $A_1 \cong^G_{BC} A_2$.

We also consider these properties:

- (freedom) $\mathfrak{X} = [\Omega]^{<\omega}$ and if $A \perp_C B$ and $C \cap AB \subseteq D \subseteq C$, then $A \perp_D B$.
- (one-basedness) $A \perp_{A \cap B} B$ for every $A, B \in \mathfrak{X}$.

The one-basedness property admits the following generalization:

Definition 3.3. Given $k \geqslant 1$, we say that $(\mathfrak{X}, \perp)$ satisfies *k-narrowness* if, for any $C, A_0, A_1, \ldots, A_k$ in $\mathfrak{X}$, the conditions

- $A_i \cap A_{i+1} = C$ for each $0 \leqslant i \leqslant k-1$,
- $A_{i+1} \perp_{A_i} A_{i-1} \cdots A_0$ for each $1 \leqslant i \leqslant k-1$

imply that $A_0 \perp_C A_k$ (notice that for $k = 1$ we recover the one-basedness property).

Lemma 3.4. *Let* $(\mathfrak{X}, \perp)$ *be a compatible pair that satisfies existence. Then*:

(1) *If it satisfies freedom or one-basedness, then for any* $A, B \in \mathfrak{X}$ *there is* $A' \in \mathfrak{X}$ *such that* $A' \cong^G_B A$, $A' \cap A = A \cap B$ *and* $A \perp_{A \cap B} A'$.

(2) *If it satisfies transitivity, symmetry and $2m$-narrowness, then for any $A, B \in \mathfrak{X}$ there is $A' \in \mathfrak{X}$ such that an (m, B)-zigzag path from A to A' exists, $A' \cap A = A' \cap B$ and $A \mathop{\smile\!\!\!\!|}_{A \cap B} A'$.*

Proof. Existence yields $A' \in \mathfrak{X}$ such that $A' \cong^G_B A$ and $A' \mathop{\smile\!\!\!\!|}_B A$. Antireflexivity implies that $A' \cap A \subseteq B$, i.e., $A' \cap A \subseteq A \cap B$. On the other hand $A' \cong^G_B A$ implies $A \cap B = A' \cap B$.

If we assume the freedom axiom, then $A' \mathop{\smile\!\!\!\!|}_{A \cap B} A$ follows from $A' \mathop{\smile\!\!\!\!|}_B A$ and $B \cap (A' \cup A) = (B \cap A') \cup (B \cap A) = B \cap A$. Alternatively, the same conclusion follows directly from one-basedness.

Let $C = A \cap B$. For (2) construct sequences $B_0 = B, B_1, \dots, B_{m-1}$ and $A_0 = A, A_1, \dots, A_m$ as follows. Assuming we have already taken $(A_i, B_i)_{i=0}^k$, existence provides $A_{k+1} \cong^G_{B_k} A_k$ with $A_{k+1} \mathop{\smile\!\!\!\!|}_{B_k} A_0 B_0 \cdots A_k B_k$. By the same token, for $k \leqslant m$ we can choose $B_{k+1} \cong^G_{A_{k+1}} B_k$ with $B_{k+1} \mathop{\smile\!\!\!\!|}_{A_k} A_0 B_0 \cdots A_{k+1}$. It is clear that this yields an (m, B)-zigzag path from A to A_m.

By transitivity, $A_j \mathop{\smile\!\!\!\!|}_{B_{j-1}} A_l$ for any $0 \leqslant l \leqslant j-1$, so that $A_j \cap A_l \subseteq A_j \cap B_{j-1}$ by antireflexivity. Since $A_j \cap B_{j-1} = C$ and $C \subset A_j \cap A_l$ by Observation 2.11 we conclude that $A_j \cap A_l = C$. Arguing in a similar manner one can show that $A_j \cap B_l = C$ for any $0 \leqslant j \leqslant m$ and $0 \leqslant l \leqslant m-1$. This establishes that the sequence $A_0, B_0, \dots, B_{m-1}, A_m$ satisfies the first property of the condition in the definition of $2m$-narrowness, while the second follows by transitivity and construction. If we let $A' = A_m$ we then get $A' \mathop{\smile\!\!\!\!|}_C A$ and $A \mathop{\smile\!\!\!\!|}_C A'$ by symmetry, while the sequence above is an (m, B)-zigzag path from A to A'. □

Lemma 3.5. *Let $(\mathfrak{X}, \mathop{\smile\!\!\!\!|})$ be a compatible pair satisfying symmetry, existence and transitivity and assume that for any $A, B \in \mathfrak{X}$ there exists an (m, B)-zigzag path from A to some A_1 such that $A_1 \mathop{\smile\!\!\!\!|}_{A \cap B} A$. Then:*

(1) *If stationarity holds, then $\mathfrak{X}$ has the $2m$-zigzag property.*

(2) *If independence holds, then $\mathfrak{X}$ has the $4m$-zigzag property.*

Proof. Let $A, A', B \in \mathfrak{X}$ with $A' \cong^G_{A \cap B} A$. Let $C := A \cap B$. In both cases using the assumption we start by choosing $A_1 \in \mathfrak{X}$ for which there is an m-zigzag path from A to A_1 and $A_1 \mathop{\smile\!\!\!\!|}_C A$.

Consider (1) first. By extension there is A_2 such that $A_2 \cong^G_A A_1$ and $A_2 \mathop{\smile\!\!\!\!|}_A A'A$. The first implies the existence of an (m, B)-zigzag path from A to A_2. The second, together with $A_2 \mathop{\smile\!\!\!\!|}_C A$, implies $A_2 \mathop{\smile\!\!\!\!|}_C A'A$ by right transitivity. By weak monotonicity we get $A_2 \mathop{\smile\!\!\!\!|}_C A'$ and by symmetry $A \mathop{\smile\!\!\!\!|}_C A_2$ and $A' \mathop{\smile\!\!\!\!|}_C A_2$. Stationarity yields $A \cong^G_{A_2} A'$. Thus, there is also an (m, B')-zigzag path from A_2 to A', where $A'B' \cong^G AB$ and combining both paths we get a $(2m, B)$-zigzag path from A to A'.

We move on to case (2). By invariance and existence there is A'_1 such that $A'_1 A' \cong^G A_1 A$ (so that by invariance $A'_1 \mathop{\smile\!\!\!\!|}_C A'$) and $A'_1 \mathop{\smile\!\!\!\!|}_{A'} A' A_1$. Transitivity and monotonicity then imply $A'_1 \mathop{\smile\!\!\!\!|}_C A_1$.

Independence applied to the tuple C, A_1, A'_1, A, A' in place of the A, B_1, B_2, C_1, C_2 of the definition implies the existence of some D such that $DA_1 \cong^G AA_1$ and $DA'_1 \cong^G AA_1$. This witnesses the existence of a $(4m, B)$-zigzag path from A to A'. Notice that symmetry is required in order to get $A' \mathop{\smile\!\!\!\!\!\mid}_C A'_1$. □

3B. ***Review of Fraïssé construction.*** Let us briefly review the Fraïssé construction method in a relational language. For a more detailed discussion we refer the reader to the survey by Macpherson [2011].

Let $\mathfrak{L}$ be a relational signature and $\mathcal{K}$ be a countable class of finite $\mathfrak{L}$-structures closed under isomorphism. Suppose $A, B \in \mathcal{K}$. By $A \subseteq B$ we mean A is an $\mathfrak{L}$-substructure of B. We say $\mathcal{K}$ is a *Fraïssé class* if it satisfies the following properties:

- (HP) $\mathcal{K}$ is closed under substructures.
- (JEP) For any $A, B \in \mathcal{K}$ there is C in $\mathcal{K}$ such that $A, B \subseteq C$.
- (AP) Given $A_1, A_2, B \in \mathcal{K}$ and isometric embeddings $g_i : B \to A_i$ for $i = 1, 2$ there exists $C \in \mathcal{K}$ and isometric embeddings $h_i : A_i \to C$ such that $h_1 \circ g_1 = h_2 \circ g_2$.

According to a theorem of Fraïssé, for any Fraïssé class $\mathcal{K}$ there is a unique countable structure $\mathcal{M}$ called the *Fraïssé limit* of $\mathcal{K}$, denoted by $\mathrm{Flim}(\mathcal{K})$, such that

- $\mathcal{M}$ is ultrahomogeneous (see Definition 2.1);
- $\mathrm{Age}(\mathcal{M})$, the collection of all finite substructures of $\mathcal{M}$, coincides with $\mathcal{K}$.

Classical examples of Fraïssé limit structures are $(\mathbb{Q}, <)$ and the random graph. If $\mathfrak{L}$ is empty, then $\mathcal{K}$ is the class of finite sets and $\mathrm{Flim}(\mathcal{K})$ an infinite countable set. More generally, we say $\mathcal{K}$ is *trivial* if the equality type of a finite tuple of elements from M determines its type (equivalently, if $\mathrm{Aut}(\mathcal{M})$ is the full permutation group of M).

Suppose A, B and C are structures in some relational language $\mathfrak{L}$ with $A \subseteq B, C$. By the *free-amalgam* of B and C over A, denoted by $B \otimes_A C$, we mean the structure with domain $B \amalg_A C$ in which a relation holds for a tuple a if and only if it already did in either B or C.

By a *free amalgamation class* we mean a class $\mathcal{K}$ of finite structures in a relational language satisfying (HP) and such that $B \otimes_A C \in \mathcal{K}$ for any $A, B, C \in \mathcal{K}$ such that $A \subseteq B, C$. We write $B \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{fr}}_A C$ if and only if the structure generated by ABC is isomorphic (with the right identifications) with the free amalgam $B \otimes_A C$. If $B \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{fr}} C$ we say B and C are *free* from each other.

Theorem A. *Let $\mathcal{M}$ be the Fraïssé limit of a free amalgamation class in a countable relational language. Let $G = \mathrm{Aut}(\mathcal{M})$. Then any group topology $\tau \subseteq \tau_{\mathrm{st}}$ on G is of the form τ^X_{st}, where $X \subseteq M$ is some G-invariant set. In particular, if the action of G on M is transitive, then there are no nontrivial group topologies on G strictly coarser than τ_{st} and thus (G, τ_{st}) is totally minimal.*

Proof. First note that the algebraic closure in any Fraïssé limit of a free amalgamation class is trivial (follows from Lemma 2.1.4 in [Macpherson 2011]). If we let $\mathfrak{X} = [M]^{<\omega}$, where M is the underlying set of $\mathcal{M}$ and $\perp = \perp^{\mathrm{fr}}$, then part (1) of Lemma 3.4 and part (1) of Lemma 3.5 apply to the pair $(\mathfrak{X}, \perp)$. Together, they imply $\mathfrak{X}$ has the 2-zigzag property with respect to the action of G. The result then follows from an application of Proposition 2.12. □

3C. *Small, one-based simple theories.* Recall that given an $\mathfrak{L}$-structure $\mathcal{M}$ and $A \subseteq \mathcal{M}$, a subset X of $\mathcal{M}^n$ is called *definable over* A it is the solution set of some $\mathfrak{L}$-formula with parameters in A. For a model $\mathcal{M}$ of a complete theory and any definable equivalence relation E over $\varnothing$ on n-tuples one can consider the equivalence classes of $\mathcal{M}^n/E$ as elements of a new sort in an extended multisorted language. These classes are referred to as *imaginary* elements. A theory T is said to have *weak elimination of imaginaries* if for any $n \geqslant 1$ and any imaginary element $e = a/E$, where E is a definable equivalence relation on $\mathcal{M}^n$ over the empty set, there is a finite tuple c in $\mathcal{M}$ such that e is definable over c (i.e., the single solution of some formula over c) and c is algebraic over e (i.e., every element of c is a solution of some formula over e which has only finitely many solutions); see [Tent and Ziegler 2012]. Within a saturated model of the theory an element a is definable (algebraic) over B if its orbit under the stabilizer of B is a singleton (finite). Roughly speaking in theories with weak elimination of imaginaries, the imaginary elements are coded (in a weak sense) in the original structure.

Understanding simple theories requires dealing with hyperimaginaries. A *hyperimaginary* is an equivalence class of a type definable equivalence relation of a possibly infinite tuple over the empty set, where a type is an infinite conjunction of finitely consistent formulas. Recall that a theory *eliminates hyperimaginaries* if any hyperimaginary element is interdefinable with a sequence of imaginaries. See [Wagner 2000] or [Kim 2014] for details on these concepts.

Theorem B. *Let $\mathcal{M}$ be a simple, ω-saturated countable structure with locally finite algebraic closure and weak elimination of imaginaries. Assume furthermore that* $\mathrm{Th}(\mathcal{M})$ *is one-based. Let* $G = \mathrm{Aut}(\mathcal{M})$. *Then:*

(1) *If G acts transitively on M, then (G, τ_{st}) is minimal.*

(2) *If all singletons are algebraically closed, then any group topology τ on G coarser than τ_{st} is of the form τ_{st}^X for some G-invariant $X \subseteq M$.*

Proof. As cl we take the algebraic closure acl and $\perp$ the forking independence. We claim part (1) of Lemma 3.4 and part (2) of Lemma 3.5 both apply to $(\mathfrak{X}, \perp)$.

The pair clearly satisfies invariance, weak monotonicity, transitivity and symmetry. Existence follows from the fact that M is ω-saturated, so it is left to check one-basedness and independence in the sense of Definition 3.2.

It is known that *small* simple theories which admit *finite coding* have elimination of hyperimaginaries (for definitions and details, see [Wagner 2000, Section 6 and Proposition 6.1.21]). Furthermore, one-based simple theories admit the finite coding property. These all imply in our setting that we have elimination of hyperimaginaries.

Take $A, B \in \mathfrak{X}$. The fact that the theory is one-based and has elimination of hyperimaginaries implies $A \mathop{\smile\!\!\!\!\!\!|}_{\mathrm{acl}^{\mathrm{eq}}(A) \cap \mathrm{acl}^{\mathrm{eq}}(B)} B$. The relation $A \mathop{\smile\!\!\!\!\!\!|}_{A \cap B} B$ follows then from weak elimination of imaginaries.

Lastly, elimination of hyperimaginaries and weak elimination of imaginaries imply that the type of a tuple over a finite acl-closed set determines its Lascar strong type over that same set. Hence, Kim and Pillay's independence theorem [1998] (see also Chapter 2.3 and Theorem 2.3.1 in [Kim 2014]) translates into abstract independence (amalgamation of types) for (acl, $\mathop{\smile\!\!\!\!\!\!|}$) in that case. □

For stable theories the notion of being k-ample (for some $k \geqslant 1$) generalizes the negation of one-basedness. See [Evans 2003] for details. When algebraic closure is trivial, not k-ampleness translates into (acl, $\mathop{\smile\!\!\!\!\!\!|}^f$) being k-narrow, where $\mathop{\smile\!\!\!\!\!\!|}^f$ is the forking independence. From an argument similar to the one in the two theorems above we can deduce the following result:

Theorem 3.6. *Let $\mathcal{M}$ be a countable ω-saturated stable structure such that* $\mathrm{Th}(\mathcal{M})$ *has trivial algebraic closure, has weak elimination of imaginaries, and is not k-ample for some $k \geqslant 1$. Then any group topology on $G = \mathrm{Aut}(\mathcal{M})$ coarser than τ_{st} is of the form τ_{st}^X for some G-invariant $X \subseteq M$.*

3D. ***An example that shows total minimality is not preserved under taking open finite-index subgroups.*** Consider the relational language $\mathfrak{L}_1 = \{E^{(2)}, P^{(1)}\}$, and let $\mathcal{K}_1$ be the class of all finite $\mathfrak{L}_1$-structures in which E is interpreted as the edge relation of a bipartite graph with edges only between the domain of the unary predicate P and its complement. Consider also the class $\mathcal{K}_2$ in the language $\mathfrak{L}_2 = \{E^{(2)}, F^{(2)}\}$ consisting of all finite $\mathfrak{L}$-structures in which F is interpreted as an equivalence relation with at most 2 classes and E as the edge relation of a bipartite graph with edges only among vertices that belong to distinct F-classes.

Let $\mathcal{M}_i = \mathrm{Flim}(\mathcal{K}_i)$ and $G_i = \mathrm{Aut}(\mathcal{M}_i)$. Clearly $\mathcal{M}_2$ is a reduct of $\mathcal{M}_1$, so that $G_1 \lhd G_2$ and in fact $[G_2 : G_1] = 2$. It is easy to check that $\mathcal{K}_1$ has free amalgamation and then by Theorem A there are exactly two group topologies on G_1 strictly coarser than τ_{st}, namely $\tau_{\mathrm{st}}^{P(\mathcal{M}_1)}$ and $\tau_{\mathrm{st}}^{\neg P(\mathcal{M}_1)}$. Notice that both are Hausdorff, since no automorphism of $\mathcal{M}_1$ can fix $P(\mathcal{M}_1)$ or its complement (given any two points a, b, there exists c in P (resp. $\neg P$) such that $\mathrm{tp}(c, a) \neq \mathrm{tp}(c, b)$), so $(G_1, \tau_{\mathrm{st}})$ is not minimal.

In this case we have an additional non-Hausdorff group topology, $\tau^* = \{\varnothing, G_1\}$. Apply Proposition 2.2 to conclude that any group topology on G_1 strictly contained in τ_{st} is contained in τ^*.

On the other hand, it follows from Theorem B that $(G_2, \tau_{\mathrm{st}})$ is minimal.

3E. ***Simple nonmodular predimension Hrushovski construction.*** Hrushovski's predimension construction was introduced as a means of producing countable structures with a certain combinatorial property of the algebraic closure. This method was used by Hrushovski to build strongly minimal structures which are not field-like or vector space-like, as well as a stable ω-categorical pseudoplane. There are many variants of the method, but to fix notation, we consider the following basic case and later focus on a version that produces ω-categorical structures. We refer readers to [Wagner 1994; Baldwin and Shi 1996; Evans et al. 2016] for most of the properties that are mentioned here about Hrushovski constructions and some of their variations.

Suppose $s \geqslant 2$ and $\eta \in (0, 1]$. We work with the class $\mathcal{C}$ of finite s-uniform hypergraphs, that is, structures in a language with a single s-ary relation symbol $R(x_1, \ldots, x_s)$ whose interpretation is invariant under permutation of coordinates and satisfies $R(x_1, \ldots, x_s) \to \bigwedge_{i<j}(x_i \neq x_j)$.

To each $B \in \mathcal{C}$ we assign the predimension

$$\delta(B) = |B| - \eta|R[B]|,$$

where $R[B]$ denotes the set of hyperedges on B. For $A \subseteq B$, we define $A \leqslant B$ if and only if for all B' with $A \subseteq B' \subseteq B$ we have $\delta(A) \leqslant \delta(B')$, and let $\mathcal{C}_\eta := \{B \in \mathcal{C} \mid \varnothing \leqslant B\}$. The following is standard.

Lemma 3.7. *Suppose $A, B \subseteq C \in \mathcal{C}_\eta$. Then*:

(1) $\delta(AB) \leqslant \delta(A) + \delta(B) - \delta(A \cap B)$.

(2) *If $A \leqslant B$ and $X \subseteq B$, then $A \cap X \leqslant X$.*

(3) *If $A \leqslant B \leqslant C$, then $A \leqslant C$.*

If $A, B \subseteq C \in \mathcal{C}_\eta$ then we define $\delta(A/B) = \delta(AB) - \delta(B)$. Note that this is equal to $|A \setminus B| - \eta|R[AB] \setminus R[B]|$. Then $B \leqslant AB$ if and only if $\delta(A'/B) \geqslant 0$ for all $A' \subseteq A$. Moreover, if N is an infinite $\mathfrak{L}$-structure such that $A \subseteq N$, we write $A \leqslant N$ whenever $A \leqslant B$ for every finite substructure B of N that contains A. For $\mathfrak{L}$-structures A and X, where A is finite and X is of any cardinality, if $A \leqslant X$ then we say A is $\leqslant$-*closed* in X. One can show $\mathcal{C}_\eta$ has the $\leqslant$-free amalgamation property (see Lemma 4.8 in [Baldwin and Shi 1996]), by which we mean free amalgamation with respect to $\leqslant$ inclusions. An analogue of Fraïssé's theorem holds in this situation:

Proposition 3.8. *There is a unique countable structure $\mathcal{M}^\eta$, up to isomorphism, satisfying*:

(1) *The set of all finite substructures of $\mathcal{M}^\eta$, up to isomorphism, is precisely $\mathcal{C}_\eta$.*

(2) $\mathcal{M}^\eta = \bigcup_{i\in\omega} A_i$, *where $(A_i : i \in \omega)$ is a chain of $\leqslant$-closed finite sets.*

(3) *If $A \leqslant \mathcal{M}^\eta$ and $A \leqslant B \in \mathcal{C}_\eta$, then there is an embedding $f : B \to \mathcal{M}^\eta$ with $f\restriction_A = \mathrm{id}_A$ and $f(B) \leqslant \mathcal{M}^\eta$.*

The structure $\mathcal{M}^\eta$ that is obtained in the above proposition is called the *Hrushovski generic* structure.

Here we briefly discuss a variation on the Hrushovski's predimension construction method as a way to generate ω-categorical structures. The original version of this is used to provide a counterexample to Lachlan's conjecture, where it is used to construct a stable ω-categorical pseudoplane (see Section 5 in [Wagner 1994]). Here we follow a similar setting to that used in Section 5.2 of [Evans et al. 2016].

Suppose $\eta = m/n \in (0, 1]$, where $\gcd(m, n) = 1$. Consider the same setting of the previous subsection for $\mathfrak{L}$ and $\mathcal{C}_\eta$. For $A, B \in \mathcal{C}_\eta$, where $A \subset B$, define $A \leqslant_d B$ when $\delta(A'/A) > 0$, for all A' with $A \subsetneq A' \subseteq B$. For a suitable choice of an unbounded convex increasing function $f : \mathbb{R}^{\geqslant 0} \to \mathbb{R}^{\geqslant 0}$ and restricting $\mathcal{C}_\eta$ to

$$\mathcal{C}_\eta^f := \{A \in \mathcal{C}_\eta \mid \delta(X) \geqslant f(|X|) \text{ for all } X \subseteq A\},$$

one can show $(\mathcal{C}_\eta^f, \leqslant_d)$ has the $\leqslant_d$-free amalgamation property. We call these f *good* and denote the associated countable *generic structure* by $\mathcal{M}_\eta^f$, which is going to be ω-categorical.

Remark 3.9. To obtain a good function, we can take some piecewise smooth f whose right derivative f' satisfies $f'(x) \leqslant 1/x$ and is nonincreasing for $x \geqslant 1$. The latter condition implies that $f(x+y) \leqslant f(x) + yf'(x)$ (for $y \geqslant 0$). It can be shown that, under these conditions, $\mathcal{C}_\eta^f$ has the free $\leqslant_d$-amalgamation property. Details can be found in Section 6.2 and Example 6.2.27 in [Wagner 2000].

We assume that f is a good function. We will assume that $f(0) = 0$ and $f(1) > 0$, and in this case the $\leqslant$-closure of the empty set is empty. We shall also assume that $f(1) = n$ and one can show $\mathrm{Aut}(\mathcal{M}_\eta^f)$ acts transitively on M_η^f. See Examples 5.11 and 5.12 in Section 5.2 of [Evans et al. 2016] for details.

Given any finite subset X of $\mathcal{M}_\eta^f$, one can show there is a smallest finite subset Y with $X \subseteq Y \leqslant_d \mathcal{M}_\eta^f$, for which we use the notation $\mathrm{cl}^d(X)$. Let $\mathfrak{X}^d := \{\mathrm{cl}^d(A) \mid A \in [\mathcal{M}_\eta^f]^{<\omega}\}$. Given $A, B, C \in \mathfrak{X}^d$ one can define $A \mathop{\smile\!\!\!\!\!\mid}^d_B C$ if and only if $\mathrm{cl}^d(AB) \cup \mathrm{cl}^d(BC) = \mathrm{cl}^d(ABC)$ and $\mathrm{cl}^d(AB) \cap \mathrm{cl}^d(BC) = B$. Note that in this case, $\mathrm{cl}^d(ABC)$ is the free amalgam of $\mathrm{cl}^d(AB)$ and $\mathrm{cl}^d(BC)$ over B.

Lemma 3.10. $(\mathfrak{X}^d, \mathop{\smile\!\!\!\!\!\mid}^d)$ *satisfies* 3*-narrowness.*

Proof. Suppose C, A_0, A_1, A_2, A_3 are d-closed sets in $\mathfrak{X}^d$ with $A_i \cap A_{i+1} = C$ for $0 \leqslant i \leqslant 2$, where $A_3 \mathop{\smile\!\!\!\!\!\mid}^d_{A_2} A_1 A_0$ and $A_2 \mathop{\smile\!\!\!\!\!\mid}^d_{A_1} A_0$. We want to show $A_3 \mathop{\smile\!\!\!\!\!\mid}^d_C A_0$. First we claim $A_3 \cap A_0 = C$. By the assumption $C \subseteq A_0 \cap A_3$. From $A_3 \mathop{\smile\!\!\!\!\!\mid}^d_{A_2} A_1 A_0$ we know $\mathrm{cl}^d(A_3 A_2) \cap \mathrm{cl}^d(A_2 A_1 A_0) = A_2$, which implies $A_3 \cap A_0 \subseteq A_3 \cap A_2 = C$.

It remains to show $A_0 A_3$ is d-closed. If not, then there is $e \in \mathrm{cl}^d(A_0 A_3) \setminus A_0 A_3$ such that e is R-related to some elements in $\hat{A}_3 \subseteq A_3 \setminus C$ and to some elements in $\hat{A}_0 \subseteq A_0 \setminus C$, where $\delta(E/\hat{A}_0 \hat{A}_3 C) \leqslant 0$ for some $E \subseteq \mathrm{cl}^d(A_0 A_3)$, where $e \in E$ (see Section 4.2 in [Evans et al. 2016] for details of properties of minimally simply algebraic extensions). From $A_3 \mathop{\smile\!\!\!\!\!\!|}^{d}_{A_2} A_1A_0$ we know $\mathrm{cl}^d(A_0A_1A_2A_3)$ is the free amalgam of $\mathrm{cl}^d(A_3A_2)$ and $\mathrm{cl}^d(A_2A_1A_0)$ over A_2. Since $e \in \mathrm{cl}^d(A_0A_1A_2A_3)$, then this implies $e \in A_2$. Because $A_2 \mathop{\smile\!\!\!\!\!\!|}^{d}_{A_1} A_0$, we have $\hat{A}_0 \subseteq A_1$. This contradicts the fact that $A_0 \cap A_1 = C$. □

Then, combining Lemma 5.7 in [Evans et al. 2016] with Lemma 3.4(2), by Lemma 3.10, one can see $(\mathfrak{X}^d, \mathop{\smile\!\!\!\!\!\!|}^{d})$ satisfies all the properties of Lemma 3.5(1). Then using Proposition 2.12 we conclude the following.

Corollary 3.11. *Suppose that f is a good function and let $\mathcal{M}^f_\eta$ be an ω-categorical Hrushovski generic structure such that $G = \mathrm{Aut}(\mathcal{M}^f_\eta)$ acts transitively on M^f_η. Then (G, τ_{st}) is a minimal topological group.*

4. Topologies and types

In this section we describe a general way of constructing group topologies below the standard topology on the automorphism group of a first-order structure. Our ideas are inspired by [Ben Yaacov and Tsankov 2016] and [Uspenskij 2008]. In fact, when $\mathcal{M}$ is an ω-categorical structure the space $R^{\mathrm{pa}}(\mathcal{M})$ as defined below consisting of complete types can be identified with the Roelcke compactification of $\mathrm{Aut}(\mathcal{M})$ as described in [Ben Yaacov and Tsankov 2016]. However the goal here is to establish a way of parametrizing topologies that does not depend on the existence of a well-behaved independence relation. We prove Corollary C at the end of the section as an application.

Let $\mathcal{M}$ be a first-order structure and $T = \mathrm{Th}(\mathcal{M})$. Consider two tuples of variables $x = (x_m)_{m\in M}$ and $y = (y_m)_{m\in M}$ indexed by the elements of M. Given some finite tuple $a = (a_1, a_2, \ldots, a_k) \subset M$ we write x_a in lieu of $(x_{a_1}, x_{a_2}, \ldots, x_{a_k})$. Let $p_M(x) = \mathrm{tp}(M)$, where the variable x_m is made to correspond with $m \in M$. Let $R(\mathcal{M})$ stand for the collection of all T-complete types in variables x, y containing $p_M(x) \cup p_M(y)$ and write $R^{\mathrm{pa}}(\mathcal{M})$ for the collection of partial types in variables x, y in T containing $p_M(x) \cup p_M(y)$. Here we assume types are deduction closed. Given any partial type $p(x, y)$ we will denote the deduction closure of $p(x, y) \cup p_M(x) \cup p_M(y)$ in T as $\langle p\rangle$. The set $R^{\mathrm{pa}}(\mathcal{M})$ can be endowed with the so-called *logic topology*, which we denote by τ_L, generated by neighbourhoods of the form $[\phi] = \{p \in R^{\mathrm{pa}}(\mathcal{M}) \mid \phi \in p\}$, where ϕ is any formula in (x, y). The result is a Stone space.

Given $p_1, p_2 \in R^{\mathrm{pa}}(\mathcal{M})$ we let $(p_1 * p_2)(x, y) \in R^{\mathrm{pa}}(\mathcal{M})$ denote the collection of all formulas $\psi(x, y)$ such that there exist $\phi_i(x, y) \in p_i(x, y)$ for $i = 1, 2$ such that

$$\phi_1(x, z) \wedge \phi_2(z, y) \vdash \psi(x, y).$$

Given $p \in R^{\mathrm{pa}}$, let $\bar{p} \in R^{\mathrm{pa}}$ be defined by $\theta(x, y) \in \bar{p} \leftrightarrow \theta(y, x) \in p$. It can be checked that $*$ endows $R^{\mathrm{pa}}(\mathcal{M})$ with a semigroup structure. Furthermore, one can show that $*$ is a continuous map $R^{\mathrm{pa}}(\mathcal{M}) \times R^{\mathrm{pa}}(\mathcal{M}) \to R^{\mathrm{pa}}(\mathcal{M})$ and $p \mapsto \bar{p}$ is also continuous with respect to τ_L. For the first, assume $p_1, p_2 \in R^{\mathrm{pa}}(\mathcal{M})$ and $\psi(x, y)$ is a formula with $p_1 * p_2 \in [\psi(x, y)]$. Then the definition of $*$, together with compactness, implies the existence of $\phi_1(x, z) \in p_1$ and $\phi_2(z, y) \in p_2$ such that $T \cup \{\phi_1(x, z), \phi_2(z, y)\} \vdash \psi(x, y)$, which implies that $[\phi_1] * [\phi_2] \subseteq [\psi]$. If we let $0 = \langle \varnothing \rangle \in R^{\mathrm{pa}}$ then clearly $p * 0 = 0$ for any $p \in R^{\mathrm{pa}}$. We write $p \leqslant q$ for $p \vdash q$.

Every $g \in \mathrm{Aut}(\mathcal{M})$ is associated to some type $\iota(g) = \langle x_{gm} = y_m \rangle_{m \in M} \in R^{\mathrm{pa}}$. It can be easily checked that ι is a continuous homomorphic embedding of (G, τ_{st}) into $(R^{\mathrm{pa}}(\mathcal{M}), \tau_L)$ whose image is contained in $R(\mathcal{M})$. We will write simply g instead of $\iota(g)$. Notice that $p^g := g^{-1} * p * g = \{\phi(x_a, y_b) \mid \phi(x_{g \cdot a}, y_{g \cdot b}) \in p\}$ for any $p \in R^{\mathrm{pa}}$ and $g \in G$.

Definition 4.1. Suppose $\mathcal{M}$ is an $\mathfrak{L}$-structure and $G = \mathrm{Aut}(\mathcal{M})$. We say that $q \in R^{\mathrm{pa}}$ is an *invariant idempotent* if the following conditions are satisfied:

(1) $1_G \leqslant q$;

(2) $q = \bar{q}$;

(3) $q * q = q$; and

(4) $q = q^g$ for any $g \in G$.

Notice that (1) implies $q = 1_G * q \leqslant q * q$, so that item (3) could be replaced by the a priori weaker condition $q * q \leqslant q$.

Given a formula $\phi(x, y)$, let $N_\phi := \iota^{-1}([\phi]) = \{g \in G \mid \mathcal{M} \models \phi(ga, b)\}$. Given an invariant idempotent $q \in R^{\mathrm{pa}}(\mathcal{M})$, let $\mathcal{N}_q = \{N_\phi \mid \phi(x, y) \in q\}$.

Lemma 4.2. *Given any structure $\mathcal{M}$ the following statements hold, where $G =$* $\mathrm{Aut}(\mathcal{M})$:

(1) *Given any invariant idempotent $q \in R^{\mathrm{pa}}(\mathcal{M})$ the family $\mathcal{N}_q$ forms a basis of neighbourhoods of a group topology τ_q on G (necessarily unique by invariance under translations).*

(2) *The closure of 1_G in τ_q coincides with the collection of all $g \in G$ such that $g \leqslant q$.*

(3) *Given invariant idempotents $p, q \in R^{\mathrm{pa}}(\mathcal{M})$ such that $p \leqslant q$ we have $\tau_p \supseteq \tau_q$. Conversely, if $\mathcal{M}$ is countable and ω-saturated then $\tau_p \supseteq \tau_q$ implies $p \leqslant q$.*

Proof. On the one hand, for any $\phi(x_A, y_B) \in q$, we have

$$\begin{aligned} N^{-1}_{\phi(x,y)} &= \{g \in G \mid \mathcal{M} \models \phi(g^{-1}a, b)\} \\ &= \{g \in G \mid \mathcal{M} \models \phi(a, gb)\} = N_{\phi(y,x)} \in \mathcal{N}_{\bar{q}} = \mathcal{N}_q. \end{aligned}$$

On the other hand, the condition $q * q = q$ is equivalent to the following: for any ϕ and finite A and B there is $C \subset M$ and formulas $\psi(x_A, z_C), \psi'(z_C, y_B) \in q$ such that modulo T we have

(1) $$p_M(x) \cup p_M(y) \cup p_M(z) \cup \{\psi(x_A, z_C) \wedge \psi'(z_C, y_B)\} \vdash \phi(x_A, y_B).$$

Let $N = N_{\psi(x_A, y_C) \wedge \psi(x_C, y_B)}$. Given $h, g \in N$ we have $\mathcal{M} \models \psi(gA, C) \wedge \psi'(hC, B)$. The formulas are of course h-invariant, and hence $\mathcal{M} \models \psi(hgA, hC)$. Likewise, $hgA \models p_A$ and $hC \models p_C$ and thus by (1) we conclude that $\mathcal{M} \models \phi(hgA, B)$ and therefore $hg \in N_\phi$. This settles part (1). Part (2) follows easily from the fact that $\iota(g)$ is a complete type for $g \in G$ and is left to the reader. As for (3), the implication from left to right is trivial. Assume now $\mathcal{M}$ is countable and ω-saturated and we are given p, q such that $p \not\leq q$. Then there exists some $\phi(x_a, y_a) \in q$ for $a \in [M]^{<\omega}$ such that $p \notin [\phi]$.

Consider the type $r(x) \in S^{|a|}(a)$ given by

$$r(x) = \mathrm{tp}^x(a) \cup \{\neg\phi(x, a)\} \cup \{\psi(x, a) \mid \psi(x, y) \in p\}.$$

It follows from the discussion above that $r(x)$ is consistent and thus, by our assumption on $\mathcal{M}$, realized by some $a' \in M^{<\omega}$. Since $\mathcal{M}$ is homogeneous, there is $g \in G$ such that $a' = ga$. Since $\mathcal{M} \models \psi(ga, a)$, for each $\psi(x, y) \in p$ but $\mathcal{M} \models \neg\phi(ga, a)$ we conclude that $g \in N_\psi \setminus N_\phi$ for any $\psi \in p$ and thus that $\tau_p \not\supseteq \tau_q$. □

Remark 4.3. The element $1_G \in G$ seen as an element in R^{pa} is an invariant idempotent. The associated topology τ_{1_G} is just the standard topology. It can be checked by inspection that all topologies on automorphism groups that feature in this paper are of the form τ_q for some invariant idempotent q. In particular, any topology of the form τ^X_{st} for some $\mathrm{Aut}(\mathcal{M})$-invariant set X is of the form τ_p, where p is the type generated by all formulas of the form $x_a = y_a$, $a \in X$.

The following question arises naturally.

Question 3. *Let $\mathcal{M}$ be a countable w-categorical (homogeneous) structure. Is it true that any group topology on* $\mathrm{Aut}(\mathcal{M})$ *is of the form τ_q for some invariant idempotent $q \in R^{\mathrm{pa}}$?*

4A. *Nonminimality in the trivial* acl *case.* To conclude in this final subsection we show minimality fails for the automorphism groups of certain Fraïssé limits. Fix some structure $\mathcal{M}$ in a finite relational language in which acl is *trivial*, i.e., $\mathrm{acl}(A) = A$ for any finite $A \subset M$. Consider the type $q_{\inf} \in R^{\mathrm{pa}}(\mathcal{M})$ generated by all the formulas of the form $\phi(x_A, y_B)$, where $\phi \in \mathrm{tp}(A, B)$, for finite $A, B \subseteq M$ with $A \cap B = \varnothing$. Notice that $q_{\inf}$ is clearly invariant under the action of $\mathrm{Aut}(\mathcal{M})$ on x_M and y_M.

Definition 4.4. We say that $\mathcal{M}$ has the *separation property* if for any two disjoint finite tuples $a, b \in [M]^{<\omega}$ there exists $c \in [M]^{<\omega}$ disjoint from both a and b such that $\mathrm{tp}^{x,z}(a, c) \cup \mathrm{tp}^{z,y}(c, b) \vdash \mathrm{tp}^{x,y}(a, b)$.

Lemma 4.5. *Assuming* acl *is trivial in $\mathcal{M}$, the type q_{inf} is an invariant idempotent in $R^{\mathrm{pa}}(\mathcal{M})$ if and only if $\mathcal{M}$ has the separation property. If in addition to this $\mathcal{M}$ is countable and ω-saturated, then $q_{\mathrm{inf}} \not\leqslant 1_G$ and thus $\tau_{q_{\mathrm{inf}}}$ is strictly coarser than $\tau_{\mathrm{st}} = \tau_{1_G}$.*

Proof. Properties (1), (2) and (4) of Definition 4.1 are immediate from the definition of q_{inf}. For property (3) all we need to check is that $q * q \leqslant q$, as remarked after Definition 4.1, but this is precisely the content of the separation property, as in its definition, $\mathrm{tp}^{x,z}(a, c) \cup \mathrm{tp}^{z,y}(c, b) \vdash \mathrm{tp}^{x,y}(a, b)$, we have $\mathrm{tp}^{x,y}(a, c) \cup \mathrm{tp}^{x,y}(c, b) \subseteq q_{\mathrm{inf}}$ and thus $\mathrm{tp}^{x,y}(a, b) \subseteq q_{\mathrm{inf}} * q_{\mathrm{inf}}$ for the arbitrary fragment $\mathrm{tp}^{x,y}(a, b) \subseteq q_{\mathrm{inf}}$ we started with.

If $q_{\mathrm{inf}} = 1_G$, then for any $b \in M$ there must be some finite $A \subseteq M \setminus \{b\}$ such that $\mathrm{tp}^{x_A, y_b}(A, b) \vdash y_b = x_b$, which can only be the case if $b \in \mathrm{dcl}(A)$. The final claim then follows from last point of Lemma 4.2. Namely, from (3) of Lemma 4.2, if $1_G \leqslant q_{\mathrm{inf}}$ then $\tau_{1_G} = \tau_{\mathrm{st}} \supseteq \tau_{q_{\mathrm{inf}}}$. Using the second part of (3), if $\tau_{\mathrm{st}} = \tau_{q_{\mathrm{inf}}}$ then $q \leqslant 1_G$, which contradicts the fact that $q_{\mathrm{inf}} \not\leqslant 1_G$. □

Distal theories are a particular class of NIP theories introduced in [Simon 2013]. One main feature is the following fact [Chernikov and Simon 2015, Theorem 21]:

Fact 4.6. *Let T be distal. Then for any formula $\phi(x, y)$ there is a formula $\theta(x, z)$ such that for any $\mathrm{tp}^{\phi}(a/C)$ over a finite set of parameters C there is a tuple $d \subset C$ such that $\theta(a, d)$ holds, and $\theta(x, d) \vdash \mathrm{tp}^{\phi}(a/C)$, i.e., $\theta(x, y) \cup \mathrm{tp}^{y}(d, C) \vdash \mathrm{tp}^{\phi}(x/C)$, where $|y| = |d|$.*

Lemma 4.7. *Let $\mathcal{M}$ be any distal Fraïssé limit in a finite relational language with trivial algebraic closure. Then $\mathcal{M}$ has the separation property.*

Proof. Consider any two disjoint finite tuples $a, b \in M$. Since $\mathcal{M}$ has quantifier elimination, there exists some formula $\phi(x, y)$ such that for any $C \subset M$ the full type $\mathrm{tp}(a/C)$ is equivalent to the ϕ-type $\mathrm{tp}^{\phi}(a/C)$ ($|a| = |x|$). Let $\theta(x, z)$ be the formula provided by Fact 4.6 and let $s = |z|$. Take a sequence $b_{-s}, b_{-s+1}, \ldots, b_0 = b$, $b_1, \ldots, b_s$ of instances of $\mathrm{tp}(b/a)$ indiscernible over a, where b_i and b_j are disjoint for $i \neq j$. Let $C = b_{-s} b_{-s+1} \cdots b_s$, and let d be the tuple obtained from applying Fact 4.6 to $\mathrm{tp}(a/C)$. Let J be the set of indices $j \in \{-s, -s+1, \ldots, s\}$ such that $d \cap b_j \neq \varnothing$. Now, there must be some $j_0 \in \{-s, -s+1, \ldots, s\} \setminus J$ and some order-preserving bijection $\phi : J \cup \{j_0\} \to J' \subseteq \mathbb{Z}$ sending j_0 to 0. Since $(b_i)_i$ is indiscernible, the fact that $\mathrm{tp}(a/b_l)_{l \in J}$ isolates $\mathrm{tp}(a/b_l)_{l=-s}^{s}$ implies that $\mathrm{tp}(a/b_l)_{l \in J' \setminus \{0\}}$ isolates $\mathrm{tp}(a/b_l)_{l \in J'}$, so that the tuple $C = (b_l)_{l \in J' \setminus \{0\}}$ witnesses the separation property for the pair (a, b). □

Corollary C. *Let $\mathcal{M}$ be any distal Fraïssé limit in a finite relational language with trivial algebraic closure. Then the type q_{inf} defines a group topology on $G = \mathrm{Aut}(\mathcal{M})$ strictly coarser than τ_{st}.*

Many Fraïssé structures, such as nontrivial reducts of $(\mathbb{Q}, \leqslant)$ and ω-categorical finitely ramified ordered trees, satisfy the assumptions of Corollary C.

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