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A multisection of a 4-manifold is a decomposition into 1-handlebodies intersecting pairwise along 3-dimensional handlebodies or along a central closed surface; this generalizes the Gay–Kirby trisections. We show how to compute the twisted absolute and relative homology, the torsion and the equivariant intersection form of a 4-manifold from a multisection diagram. The homology and torsion are given by a complex of free modules defined by the diagram and the intersection form is expressed in terms of the intersection form on the central surface. We give efficient proofs, with very few computations, thanks to a retraction of the (possibly punctured) 4-manifold onto a CW-complex determined by the multisection diagram. Further, a multisection induces an open book decomposition on the boundary of the 4-manifold; we describe the action of the monodromy on the homology of the page from the multisection diagram.

1. Introduction and main results

A trisection is a type of combinatorial structure on 4-manifolds which was discovered by Gay and Kirby [2016] via Morse 2-functions. They proved that any smooth 4-manifold, possibly with boundary, can be decomposed as the union of three 4-dimensional 1-handlebodies, with 3-dimensional 1-handlebodies as pairwise intersections and a compact surface as global intersection. Such a trisection can be described by a diagram, namely the central surface with collections of curves that define the 3-dimensional pieces. A trisection diagram determines a smooth 4-manifold up to diffeomorphism, so that one should be able to read topological invariants of the manifold on the diagram. In the setting of closed 4-manifolds, Feller, Klug, Schirmer and Zemke [Feller et al. 2018] provided a computation of the homology and intersection form of the manifold from a trisection diagram, and Florens and Moussard [2022] derived the twisted homology and torsion, and the equivariant intersection form. Following these papers, Tanimoto [2023] computed the homology of 4-manifolds with connected boundary. Here we recover and

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generalize these results, computing from a diagram the twisted absolute and relative homology and torsion and the equivariant intersection form for any trisected 4-manifold with boundary. Moreover, we work with "multisections" in the sense of Islambouli and Naylor [2024], namely a cyclic decomposition of the manifold into any number of 4-dimensional 1-handlebodies, where successive pieces meet along 3-dimensional 1-handlebodies while nonsuccessive ones meet along the central surface. We propose a more efficient approach. While Feller, Klug, Schirmer and Zemke worked with a handle decomposition of the manifold underlying the trisection, Florens and Moussard directly used the datum of the trisection. This last method reduced the homological computations, but the computation of torsion was quite intricate. Here we consider a deformation-retraction of the (possibly punctured) manifold onto a CW-complex associated with the multisection diagram. This simplifies the computations and provides the torsion "for free". This retraction could be useful for further computations of homological or homotopical invariants.

A multisection of a 4-manifold X with boundary induces an open book decomposition on the boundary. The monodromy of this open book has been described algorithmically by Castro, Gay and Pinzón-Caicedo [Castro et al. 2018a] from a diagram. Here we derive the action of the monodromy on the homology of the page from which can be derived a computation of the homology of ∂X as well as the Alexander module of the binding determined by the monodromy.

For 4-manifolds with boundary, the handlebodies of a multisection inherit (hyper) compression bodies structures related to the way they intersect the boundary of the manifold.

Definition 1.1. A *compression body* C is a cobordism from a compact orientable surface $\partial_- C$ to a connected compact orientable surface $\partial_+ C$ which is constructed using only 1-handles. Likewise a *hyper compression body* V is a cobordism from a compact orientable 3-manifold $\partial_- V$ to a connected compact orientable 3-manifold $\partial_+ V$ constructed using only 1-handles. A *lensed* (hyper) compression body is then obtained by collapsing the vertical boundary of the cobordism so that the boundary of $\partial_+ C$ ($\partial_+ V$) becomes identified with the boundary of $\partial_- C$ ($\partial_- V$).

In the case that $\partial_- C = \varnothing$, it is understood at C is built using only 1-handles attached to a single 0-handle. A (lensed) compression body is *trivial* if $\partial_- C \cong \partial_+ C$. This means it is just a thickened surface $S \times I$, or if lensed, it is obtained from $S \times I$ by collapsing the I-fibers of $\partial S \times I$.

Definition 1.2. A *multisection* of a compact orientable 4-manifold X is a decomposition $X = X_1 \cup \cdots \cup X_n$ into 4-dimensional 1-handlebodies X_i with the following properties (all arithmetic involving indices is mod n):

(1) Each X_i has a lensed hyper compression body structure such that $\partial_- X_i = X_i \cap \partial X$, and if $\partial X \neq \varnothing$, there is a fixed surface Σ_∂ such that, for all $1 \leq i \leq n$,

$\partial_- X_i$ is diffeomorphic to the trivial lensed compression body obtained by pinching the vertical boundary of $\Sigma_\partial \times I$.

(2) $\Sigma = \bigcap_{i=1}^{n} X_i$ is a compact connected orientable surface.

(3) $C_i = X_i \cap X_{i+1}$ is a 3-dimensional 1-handlebody with a lensed compression body structure satisfying $\partial_+ C_i = \Sigma$ and $\partial_- C_i = C_i \cap \partial X \cong \Sigma_\partial$ for all i.

(4) $X_i \cap X_j = \Sigma$ when $|i - j| > 1$.

A multisection is called a *trisection* when $n = 3$.

The condition that the C_i are 1-handlebodies implies that Σ is closed if and only if Σ_∂ is closed. We shall consider the case when Σ_∂ contains no closed components, and within this context Σ will be closed if and only if X itself is closed. This is the framework of most of the literature on trisections and within this framework a unified calculation of the algebraic topology is possible. The specific case where more general compression bodies are allowed, i.e., the case in which Σ_∂ has closed components, was considered in the original paper of Gay and Kirby [2016]; however, the calculations become more delicate and require special treatment. Moreover in the case that Σ_∂ contains components that are spheres, diagrams no longer determine a unique 4-manifolds up to diffeomorphism. We postpone the homology computations in this case to a forthcoming publication in order to avoid the extra complications here.

In the case that $\partial X \neq \varnothing$, it is also to be understood that for all $i \mod n$, $\partial_- X_i$ is parametrized as $\Sigma_\partial \times I/\!\sim$ in such a way that $\partial_- C_{i-1} = \Sigma_\partial \times \{0\}$ and $\partial_- C_i = \Sigma_\partial \times \{1\}$. Thus, the multisection induces an open book decomposition on ∂X with page Σ_∂.

We fix once and for all a multisected manifold $X = \bigcup_{1 \le i \le n} X_i$, and set $C_i = X_i \cap X_{i+1}$ and $\Sigma = \bigcap_i X_i$.

Definition 1.3. Let C be a compression body. A *defining collection of disks* for C is a collection $\mathcal{D}$ of disks properly embedded in C such that $C \setminus \eta(\mathcal{D})$ is a thickening of $\partial_- C$ (for instance the cocore disks of the 1-handles in the definition). The boundary $\partial \mathcal{D} \subset \partial_+ C$ is a *defining collection of curves* for C.

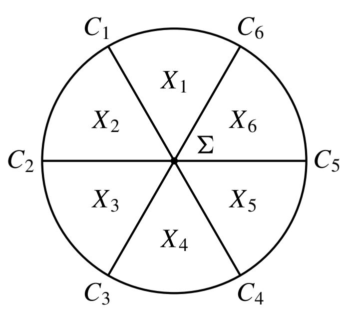

Figure 1. Schematic of a multisection.

Definition 1.4. A *diagram* of the multisection $X=\bigcup_{1\leq i\leq n} X_i$ is a tuple $(\Sigma; c_1, \dots, c_n)$ where c_i is a defining collection of curves for C_i.

A multisection diagram determines a unique smooth 4-manifold [Castro et al. 2018b]. The structure of the X_i gives some constraints on the curves of a multisection diagram. For each i, X_i is obtained from a thickened $\partial_- X_i$ by attaching 1-handles, so that $\partial_+ X_i \cong (S^2 \times S^1)^{\sharp k} \# \big(\# \partial_- X_i\big)$, where k is the number of 1-handles in excess of the minimum required to connect $\partial_- X_i$, and $\# \partial_- X_i$ is the connected sum of all components of $\partial_- X_i$. Now Definition 1.2 implies that $C_{i-1} \cup_\Sigma C_i$ is a sutured Heegaard splitting of $\partial_+ X_i$, so that the Heegaard diagram $(\Sigma; c_{i-1}, c_i)$ is always handleslide-diffeomorphic to a standard diagram as represented in Figure 2.

Fix a homomorphism $\varphi : \mathbb{Z}[\pi_1(X)] \to R$, where R is a commutative ring. We shall express the absolute and relative homology of X, twisted by φ, in terms of the multisection diagram. Fix a point $* \in \mathrm{Int}(\Sigma)$ and let L_i^φ be the submodule of $H_1^\varphi(\Sigma, *)$ generated by the homology classes of the curves in c_i. In Section 3, we obtain the following result (Theorem 3.8, Remark 3.9 and Lemma 3.11).

Theorem 1.5. *The homology of X is given by the chain complex of free R-modules*

$$(\mathcal{C}) \qquad 0 \to \bigoplus_{i=1}^{n} (L_{i-1}^\varphi \cap L_i^\varphi) \xrightarrow{\partial_2} \bigoplus_{i=1}^{n} L_i^\varphi \xrightarrow{\partial_1} H_1^\varphi(\Sigma, *) \xrightarrow{\partial_0} H_0^\varphi(*),$$

where

$$\partial_2((x_i)_{1\leq i\leq n}) = (x_i - x_{i+1})_{1\leq i\leq n} \quad \textit{and} \quad \partial_1((x_i)_{1\leq i\leq n}) = \sum_{i=1}^{n} x_i .$$

Moreover, if R is a field, an explicit complex basis of $(\mathcal{C})$ can be given such that $\tau^\varphi(X; h) = \tau(\mathcal{C}; b, h)$.

Let Σ' be the surface Σ with a small open disk removed, such that the point $*$ lies on the boundary of the removed disk. For $1 \leq i \leq n$, let $\mathcal{J}_i^\varphi$ be the orthogonal complement in $H_1^\varphi(\Sigma', \partial\Sigma)$ of L_i^φ with respect to the equivariant intersection pairing on $H_1^\varphi(\Sigma, *) \times H_1^\varphi(\Sigma', \partial\Sigma)$. We prove the following in Section 4 (Theorem 4.9, Lemma 4.6 and Remark 4.10).

Theorem 1.6. *If $\partial X \neq \varnothing$, the twisted homology of $(X, \partial X)$ is given by the chain complex of free R-modules*

$$(\mathcal{C}_\partial) \qquad H_2^\varphi(\Sigma, \Sigma') \xrightarrow{\partial_3} \bigoplus_i (\mathcal{J}_{i-1}^\varphi \cap \mathcal{J}_i^\varphi) \xrightarrow{\partial_2} \bigoplus_i \mathcal{J}_i^\varphi \xrightarrow{\partial_1} H_1^\varphi(\Sigma', \partial\Sigma) \to 0,$$

where

$$\partial_3([\Sigma]) = [\partial(\Sigma\setminus\Sigma')], \quad \partial_2((x_i)_{1\leq i\leq n}) = ((x_i - x_{i+1})_{1\leq i\leq n}), \quad \partial_1((x_i)_{1\leq i\leq n}) = \sum_{i=1}^{n} x_i .$$

Moreover, if R is a field, an explicit complex basis of $(\mathcal{C}_\partial)$ can be given such that $\tau^\varphi(X, \partial X; h) = \tau(\mathcal{C}_\partial; b, h)$.

When Σ is closed, we define the L_i^φ in $H_1^\varphi(\Sigma', *)$. In this closed case, these are lagrangians, namely they are their own orthogonal complement with respect to the intersection form. The next result is obtained in Section 6 (Theorem 6.4, Remark 6.5 and Lemma 6.2).

Theorem 1.7. *If X is closed, the twisted homology of X is given by the chain complex of free R-modules*

$$(\bar{\mathcal{C}})\qquad H_2^\varphi(\Sigma,\Sigma') \xrightarrow{\partial_3} \bigoplus_i (L_{i-1}^\varphi \cap L_i^\varphi) \xrightarrow{\partial_2} \bigoplus_i L_i^\varphi \xrightarrow{\partial_1} H_1^\varphi(\Sigma', *) \to H_0^\varphi(*),$$

where

$$\partial_3([\Sigma]) = [\partial\Sigma'],\quad \partial_2((x_i)_{1\le i\le n}) = ((x_i - x_{i+1})_{1\le i\le n}),\quad \partial_1((x_i)_{1\le i\le n}) = \sum_{i=1}^n x_i.$$

Moreover, if R is a field, an explicit complex basis of $(\bar{\mathcal{C}})$ can be given such that $\tau^\varphi(X;h) = \tau(\bar{\mathcal{C}}; b, h)$.

These three theorems allow us to represent homology classes by mainly explicit chains in the multisected manifold which meet transversely along copies of the central surface Σ. This provides a simple description of the intersection form on X (Theorems 5.1 and 6.6).

Theorem 1.8. *Suppose $h_1 = [(x_i)_{1\le i\le n}]$ and $h_2 = [(y_i)_{1\le i\le n}]$ in $H_2^\varphi(X)$, where $(x_i)_{1\le i\le n}, (y_i)_{1\le i\le n} \in \bigoplus_i L_i^\varphi$. Then*

$$\langle h_1, h_2\rangle_X^\varphi = \sum_{1\le i<j\le n} \langle x_i, y_j\rangle_\Sigma^\varphi,$$

where $\langle\cdot,\cdot\rangle_X^\varphi$ and $\langle\cdot,\cdot\rangle_\Sigma^\varphi$ are the equivariant intersection forms on $H_2^\varphi(X)$ and $H_1^\varphi(\Sigma,*)$ respectively ($H_1^\varphi(\Sigma',*)$ if X is closed).

The intersection pairing on $H_2^\varphi(X)\times H_2^\varphi(X,\partial X)$ is similar (Theorem 5.3). In odd dimensions, the intersection pairings are especially simple (Theorems 5.4 and 6.6).

Theorem 1.9. *Suppose that either $h_1 \in H_1^\varphi(X)$ corresponds to the element $a \in H_1^\varphi(\Sigma,*)$ and $h_2 \in H_3^\varphi(X,\partial X)$ corresponds to the element $b \in \bigcap_i \mathcal{J}_i^\varphi$, or $h_1 \in H_1^\varphi(X,\partial X)$ corresponds to the element $a \in H_1^\varphi(\Sigma',\partial\Sigma)$ and $h_2 \in H_3^\varphi(X)$ corresponds to the element $b \in \bigcap_i L_i^\varphi$ ($a \in H_1^\varphi(\Sigma',\partial\Sigma)$ and $b \in \bigcap_i L_i^\varphi$ if X is closed). Then*

$$\langle h_1, h_2\rangle_X^\varphi = \langle a, b\rangle_\Sigma^\varphi.$$

Plan of the paper. In Section 2, we recall the definitions of twisted homology, torsion and equivariant intersection pairing. Our discussion is somewhat discursive to help readers build intuition. Sections 3 and 4 are devoted to the twisted homology and torsion of a 4-manifold with nonempty boundary, respectively absolute and relative. In Section 5, we describe the intersection forms. Section 6 treats the case of a closed 4-manifold. Section 7 deals with the boundary: action in homology

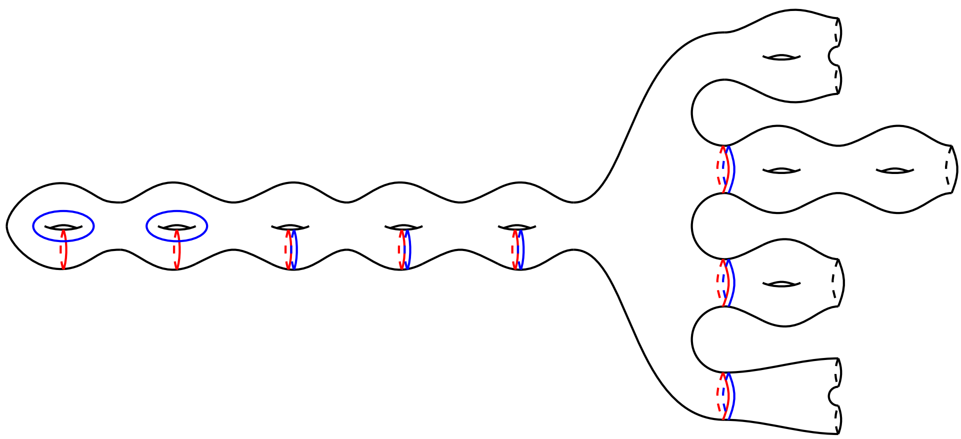

Figure 2. Heegaard diagram for $C_{i-1} \cup C_i$. In this example, C_{i-1} and C_i are constructed with eight 1-handles and X_i with six 1-handles. The manifold X has four boundary components. The components of the page Σ_∂ have a pair (genus, number of boundary components) equal to $(1, 2)$, $(2, 1)$, $(1, 1)$ and $(0, 2)$.

of the monodromy of the open book and homology of the boundary. Finally, in Section 8, we treat some examples.

Conventions. The notation we set above is assumed to be fixed for the remainder of the paper. That is, X is always multisected by n hyper compression bodies X_i which meet in compression bodies C_i, all of which are attached radially about the central fiber Σ. Additionally, $Y = C_1 \cup \cdots \cup C_n$ shall be referred to as the *spine* of the multisection. Also, $\varphi : \mathbb{Z}[\pi_1(X)] \to R$ is a homomorphism to a commutative ring R. Throughout the paper, if Z is a subset of a manifold M, $\eta(Z)$ denotes a regular neighborhood of Z in M.

2. Algebraic preliminaries

2A. ***Twisted homology.*** Let $\pi = \pi_1(X)$ and let R be a ring. A group homomorphism $\varphi : \pi \to R^*$ induces a ring homomorphism $\mathbb{Z}[\pi] \to R$. Throughout, both of these homomorphisms shall be denoted by φ and called the "twisting map."

Let $\widetilde{X}$ denote the universal cover of X, and for any $Z \subset X$, let $\widetilde{Z}$ denote the inverse image of Z under the covering map $\widetilde{X} \to X$ ($\widetilde{Z}$ will usually not be the universal cover of Z). Then π acts on both $\widetilde{X}$ and $\widetilde{Z}$ by deck transformations, which induces a left $\mathbb{Z}[\pi]$-module structure on $C_*(\widetilde{X}, \widetilde{Z})$. This allows us to define a chain complex of R-modules

$$C_i^\varphi(X, Z) = R \otimes_\varphi C_i(\widetilde{X}, \widetilde{Z}).$$

The usual boundary maps on $C_*(\widetilde{X}, \widetilde{Z})$ induce $\mathbb{Z}[\pi]$-module morphisms, and the boundary maps of $C^\varphi_*(X, Z)$ are then obtained by tensoring with the identity map on R. The resulting homology groups are denoted by $H^\varphi_*(X, Z)$.

To understand the structure of the twisted chain groups, observe first that by definition, for any $g \in \pi_1(X)$ and choice of lift $\tilde{e}$ of an i-cell e of X, $1 \otimes (g \cdot [\tilde{e}]) = \varphi(g) \otimes [\tilde{e}]$. It follows from the transitivity of the π action on $\widetilde{X}$ that a choice of lift for every i-cell in X determines an R-basis of $C^\varphi_i(X, Z)$, and thus $C^\varphi_i(X, Z)$ is always a freely generated R-module of the same rank as the $\mathbb{Z}$-rank of $C_i(X, Z)$, for any twisting map.

The effect of the twisting map is to be found in how the boundary maps are changed, and thereby also the resulting homology groups. Intuitively, one thinks of R as something like a tangent space to each point of X, and multiplication by $\varphi(g)$ corresponds to the monodromy action of g. For example, if a 1-chain e corresponds to the element $g \in \pi$ with its endpoints on the 0-chain v, then in untwisted homology we would have $\partial e = v - v = 0$, but with twisted homology we have $\partial e = \varphi(g)v - v = (\varphi(g) - 1)v$. The choice of lift does not affect the homology because different choices of lift amount to scalar multiplication of a basis element by a unit in R.

For example, if φ is the trivial map $\pi \to R^*$, so that $\varphi(g) = 1$ for every $g \in \pi$, then in this case for all lifts $\tilde{e}_1, \tilde{e}_2$ of a given cell e of X, we have

$$1 \otimes [\tilde{e}_1] = 1 \otimes (g \cdot [\tilde{e}_2]) = \varphi(g) \otimes [\tilde{e}_2] = 1 \otimes [\tilde{e}_2].$$

In other words, all the lifts of e determine the same chain in $C^\varphi_i(X, Z)$, and the projection map $\widetilde{X} \to X$ thus induces a chain isomorphism $C^\varphi_*(X, Z) \to C_*(X, Z; R)$, where $C_*(X, Z; R)$ is the usual chain complex for the (untwisted) homology with R coefficients.

On the other extreme, if φ is the inclusion $\pi \hookrightarrow \mathbb{Z}[\pi]^*$, then *all* distinct lifts of a cell e to $\widetilde{X}$ determine chains which differ by multiplication by a unit in $\pi \subset \mathbb{Z}[\pi]$. This example where $\varphi = \iota : \pi \to \mathbb{Z}[\pi]^*$ is in a sense universal. For if one can compute matrices which describe the boundary maps of $C^\iota_*(X, Z)$ in terms of some fixed cellular basis, then for any other map $\varphi : \pi \to R^*$, one simply substitutes $\varphi(g)$ for every g in the matrices of $C^\iota_*(X, Z)$ to obtain matrices of the boundary maps for $C^\varphi_*(X, Z)$ with respect to the same basis.

As a simple but instructive example, if $X = S^1$ and $Z = \varnothing$, then we may identify π with the cyclic group generated by t, and

$$C^\iota_0(S^1) \cong C^\iota_1(S^1) \cong \mathbb{Z}[t, t^{-1}].$$

All other chain groups are trivial as with the untwisted case, and the one nontrivial boundary map is multiplication by $t^n(t - 1)$ (where n depends only on the choices of lifts). Therefore $H^\iota_1(S^1) \cong 0$ and $H^\iota_0(S^1)$ is $\mathbb{Z}$, considered as a $\mathbb{Z}[t, t^{-1}]$-module whose action is given by $P(t) \cdot a = P(1)a$. More generally, given a homomorphism

$\varphi : \pi \to R^*$, where R is a ring,

$$C_0^{\varphi}(S^1) \cong C_1^{\varphi}(S^1) \cong R,$$

and the boundary map is multiplication by $\varphi(t)^n(\varphi(t)-1)$. If $\varphi(t)=1$ (the untwisted case), then

$$H_0^{\varphi}(S^1) \cong H_1^{\varphi}(S^1) = R,$$

but if $(\varphi(t)-1)$ is a unit then the entire twisted homology becomes trivial.

Returning to generalities, if we let $\widehat{X}$ denote the cover of X associated with a normal subgroup $N \subset \ker(\varphi)$, then φ factors through a map $\pi_1(X)/N \to R$, which we also call φ. If $\widehat{Z}$ is the inverse image of Z under the covering map $\widehat{X} \to X$, then the chain complex $C_*(\widehat{X}, \widehat{Z})$ becomes a $\mathbb{Z}[\pi]$-module and if we define the chain groups $R \otimes_{\varphi} C_*(\widehat{X}, \widehat{Z})$, the resulting homology will be the same as $H_*^{\varphi}(X, Z)$ through an isomorphism induced by the subcovering $\widetilde{X} \to \widehat{X}$, similar to how twisted homology reduces to ordinary homology when φ is the trivial map. This observation is important for making geometric sense of long exact sequences in the twisted context. Consider, for example, the long exact sequence associated to the pair (X, Z), which looks like this:

$$\cdots \to H_i^{\varphi}(X) \to H_i^{\varphi}(X, Z) \xrightarrow{f} H_{i-1}^{\varphi\circ\iota}(Z) \xrightarrow{g} H_{i-1}^{\varphi}(X) \to \cdots.$$

Considering $H_{i-1}^{\varphi\circ\iota}(Z)$ in isolation, observe the twisting map is $\varphi \circ \iota$, where $\iota : \pi_1(Z) \to \pi_1(X)$ is induced by inclusion. If the map ι is a surjection, then the inverse image $p^{-1}(Z)$ of Z under the universal cover $p : \widetilde{X} \to X$ will be a subcover of the universal cover $\widetilde{Z} \to Z$. We cannot easily give a geometric interpretation of f if we use the universal cover of Z to define its twisted homology, but if we instead use the cover $p^{-1}(Z)$ of Z to recover $H_{i-1}^{\varphi\circ\iota}(Z)$, then f will be induced on the chain level by taking the boundary of a relative i-cycle in $R \otimes_{\varphi} C_i(\widetilde{X}, \widehat{Z})$, and g will be induced by inclusion as usual. More generally, $p^{-1}(Z)$ may correspond to a disjoint union of copies of the subcover $\widehat{Z} \to Z$ associated with $\ker(\varphi \circ \iota)$. In this case, f first takes the boundary of a relative i-cycle in $(\widetilde{X}, p^{-1}(Z))$ and these, after multiplication by an appropriate element of $\mathbb{Z}[\pi]$, will all be identified with cells in one preferred component of $p^{-1}(Z)$, which we identify with $\widehat{Z}$.

For the reader who would like to build intuition with twisted coefficients, we strongly recommend working out the geometric details of the long exact sequence of the pairs (S^1, v), where v is a point in S^1, and (T^2, G), where T^2 is the 2-torus and G is its standard 1-skeleton with two edges and one vertex. It is similar to working with integral homology; the only complication is that one must understand the topology of appropriate covers to carry out calculations.

2B. ***Torsion.*** We recall the algebraic setup; see [Milnor 1966] and [Turaev 2001] for further details. Let $\mathbb{K}$ be a field. If V is a finite-dimensional $\mathbb{K}$-vector space and b and c are two bases of V, we denote by $[b/c]$ the determinant of the matrix expressing the basis change from b to c. The bases b and c are *equivalent* if $[b/c]=1$. Let $\mathcal{C}$ be a finite complex of finite-dimensional $\mathbb{K}$-vector spaces:

$$\mathcal{C} = (\mathcal{C}_m \xrightarrow{\partial_m} \mathcal{C}_{m-1} \longrightarrow \cdots \xrightarrow{\partial_1} \mathcal{C}_0).$$

A *complex basis* of $\mathcal{C}$ is a family $c = (c_m, \dots, c_0)$ where c_i is a basis of $\mathcal{C}_i$ for all $i \in \{0, \dots, m\}$. A *homology basis* of $\mathcal{C}$ is a family $h = (h_m, \dots, h_0)$ where h_i is a basis of the homology group $H_i(\mathcal{C})$ for all $i \in \{0, \dots, m\}$. If we have chosen a basis b_j of the space of j-dimensional boundaries $B_j(\mathcal{C}) = \operatorname{Im} \partial_{j+1}$ for all $j \in \{0, \dots, m-1\}$, and a homology basis h of $\mathcal{C}$, this induces a class of bases $(b_i h_i)\bar{b}_{i-1}$ which consists of the elements of b_i, a choice of representatives for h_i, and the image $\bar{b}_{i-1}$ of b_{i-1} under some section of ∂_i. Neither the choice of h_i-representatives nor the choice of section used to define $\bar{b}_{i-1}$ affects the equivalence class of the resulting basis of $\mathcal{C}_i$, because they differ from one another by linear combinations of b_i.

The *torsion* of the $\mathbb{K}$-complex $\mathcal{C}$, equipped with a complex basis c and a homology basis h, is the scalar

$$\tau(\mathcal{C}; c, h) = \prod_{i=0}^{m} [(b_i h_i)\bar{b}_{i-1}/c_i]^{(-1)^{i+1}} \in \mathbb{K}^*,$$

where $[(b_i h_i)\bar{b}_{i-1}/c_i]$ denotes the determinant of the change of basis matrix from c_i to $(b_i h_i)\bar{b}_{i-1}$.

This definition does not depend even on the choice of $b_0, \dots, b_m$, because of the alternating exponent. The value depends only on the choice of c and h. Of course, by making appropriate choices of c and h, we can make the torsion equal to whatever we want (indeed, just multiply one element of c or h by a scalar and you will multiply or divide the entire torsion by that scalar). In practice, $\mathcal{C}$ will be the twisted cellular chain complex associated with the CW-space X, and c_i will be a *geometric* base of the chain groups $R \otimes_\varphi C_i^\varphi(\widetilde{X})$ that is represented by the cells in a lift of the i-skeleton of X to $\widetilde{X}$. Different choices of lift can change the final value of the torsion by an element of $\pm\varphi(\pi)$, so we mod out by this ambiguity and obtain a torsion function $\tau : \mathcal{H}(X) \to \mathbb{K}^*/(\pm\varphi(\pi))$, where $\mathcal{H}(X)$ is the set of all homology bases of the associated (twisted) cellular chain complex of X.

Specifically, assume R is a field, and (X, Z) is a CW-pair. Let $\tilde{c}$ be a basis of the complex of free $\mathbb{Z}[\pi_1(X)]$-modules $C(\widetilde{X}, \widetilde{Z})$ obtained by lifting each relative cell of (X, Z) to $\widetilde{X}$. Then $c = \tilde{c} \otimes 1$ is a *geometric basis* of $C^\varphi(X, Z)$. We need such bases to define the torsion.

Definition 2.1. Given a homology basis h of $H^\varphi(X, Z)$ and a geometric basis c of $C^\varphi(X, Z)$, the torsion of $(X, Z; \varphi)$ is

$$\tau^\varphi(X, Z; h) = \tau(C^\varphi(X, Z); c, h) \in R/\pm\varphi(\pi_1(X)).$$

The remarkable fact is that τ^φ is a topological invariant when $Z = \varnothing$: any two choices of CW-decomposition for X will result in the same torsion. In the case of a true pair ($Z \neq \varnothing$) the torsion remains invariant under CW-subdivision.

Our results below will describe how to pick out geometric bases given the curves on a multisection diagram. In particular, we will describe the curves on the central surface Σ that explicitly correspond to 1-, 2- and 3-cells in the multisected 4-manifold.

2C. ***The equivariant intersection form.*** Let W be a compact oriented m-manifold, R be a commutative ring and $\varphi : \mathbb{Z}[\pi_1(W)] \to R$ be a morphism. Let A and B be disjoint subsets of ∂W. For $q \in \{0, \dots, m\}$, the *equivariant intersection pairing of W relative to A and B* with coefficient in R, introduced by Reidemeister [1939], is the sesquilinear map

$$\langle \cdot , \cdot \rangle_W^\varphi : H_q^\varphi(W, A) \times H_{m-q}^\varphi(W, B) \to R$$

defined by

$$\langle [x \otimes r], [x' \otimes r'] \rangle_W^\varphi = \sum_{h \in H_1(W)/\ker(\varphi)} \langle x, h.x' \rangle_{\overline{W}} \varphi(h) r r',$$

where we are abusing notation slightly by letting φ denote the group homomorphism from $H_1(W)$ into R^*, $\overline{W} \twoheadrightarrow W$ is the covering associated with $\ker\varphi$ and $\langle \cdot , \cdot \rangle_{\overline{W}}$ stands for the algebraic intersection in $\overline{W}$. By Blanchfield's duality theorem [1957, Theorem 2.6], if W is smooth, $\varphi(H_1(W))$ is a free multiplicative subgroup of R, and $\partial W = A \sqcup B$, this pairing is nondegenerate on $(H_q^\varphi(W, A)/\mathrm{Tor}) \times (H_{m-q}^\varphi(W, B)/\mathrm{Tor})$. The standard (i.e., nonequivariant) intersection pairing is recovered with a trivial twisting map (i.e., $R = \mathbb{Z}$ and $\varphi(\pi_1(W)) = \{1\}$).

When $A = B = \varnothing$ and $q = m - q$, the equivariant intersection pairing defines a nondegenerate *equivariant intersection form* on $H_q^\varphi(W)/\mathrm{Tor}$. (In general, if the modules H and K are identified by a canonical isomorphism, a pairing on the product $H \times K$ defines a form on $H \cong K$. A pairing may be considered up to isomorphism of either H or K, while for a form, one may restrict to applying the same isomorphism on both factors. Therefore a form carries more information.)

For a 4-manifold X, the intersection form is standardly defined as a bilinear form on $H^2(X, \partial X) \times H^2(X, \partial X)$ by applying the cup product of two cochains to the fundamental form $[X] \in H_4(X, \partial X)$. Via Poincaré duality, this form can be defined on $H_2(X) \times H_2(X)$ and, as such, it coincides with the above intersection form for a trivial twisting map.

3. The twisted absolute homology groups and torsion

In this section we derive chain complexes for the twisted homology groups $H_*^\varphi(X)$, assuming that $\partial X \neq \varnothing$. Recall that, in this case, $\partial\Sigma \neq \varnothing$.

Definition 3.1. Let V be a hyper compression body. A *defining collection of balls* for V is a collection $\mathcal{B}$ of 3-balls properly embedded in V such that $V \setminus \eta(\mathcal{B})$ is a thickening of $\partial_- V$.

We assume for the remainder of this section that a fixed choice of defining collections $\mathcal{D}_i$ and $\mathcal{B}_i$ of disks and balls has been made for all $1 \le i \le n$.

The following lemmas provide a 3-complex onto which X deformation retracts. Our calculations hinge on a careful understanding of how the cells of this complex are mirrored by simple closed curves on Σ. Recall that $Y = C_1 \cup \dots \cup C_n$ denotes the spine of the multisection.

Lemma 3.2. *The manifold X retracts onto $\Sigma \cup \bigcup_{i=1}^{n} (\mathcal{D}_i \cup \mathcal{B}_i)$. Further, the quad $(X, Y, \Sigma, *)$ deformation retracts on a CW-complex (Z_3, Z_2, Z_1, Z_0), where $Z_0 = *$, Z_1 is a bouquet of loops defining a basis of $H_1(\Sigma)$, $Z_2 = Z_1 \cup \mathcal{D}$, and $Z_3 = Z_2 \cup \mathcal{B}$.*

Proof. By definition, $X_i \setminus \eta(\mathcal{B}_i) \cong \partial_- X_i \times I$, so each X_i retracts onto $\partial_+ X_i \cup \mathcal{B}_i$, and hence X retracts onto $Y \cup \bigcup_i \mathcal{B}_i$. Likewise each C_i retracts onto $\partial_+ C_i \cup \mathcal{D}_i = \Sigma \cup \mathcal{D}_i$, so that Y further retracts down to $\Sigma \cup \bigcup_i \mathcal{D}_i$. This gives the first assertion. For the second assertion, we get $Z_1 = \Sigma$ and $Z_2 = \Sigma \cup \mathcal{D}$ and we conclude by further retracting Σ. □

Corollary 3.3. *The twisted homology of X is the homology of the following complex*:

$$(\mathcal{C}') \qquad 0 \to H_3^\varphi(X, Y) \to H_2^\varphi(Y, \Sigma) \to H_1^\varphi(\Sigma, *) \to H_0^\varphi(*).$$

Proof. Lemma 3.2 shows that the complex above is isomorphic to the cellular homology complex

$$0 \to H_3^\varphi(Z_3, Z_2) \to H_2^\varphi(Z_2, Z_1) \to H_1^\varphi(Z_1, Z_0) \to H_0^\varphi(Z_0)$$

via the map induced by the inclusion $Z_3 \hookrightarrow X$, which is a simple homotopy. □

Definition 3.4. Let L_i^φ denote the submodule of $H_1^\varphi(\Sigma, *)$ generated by the twisted homology classes of the components of c_i.

Following the approach of [Florens and Moussard 2022], we shall express the complex $(\mathcal{C}')$ in terms of these submodules. They have the following homological interpretation.

Lemma 3.5. *The module L_i^φ naturally identifies with the kernel of the inclusion map $\iota_* : H_1^\varphi(\Sigma) \to H_1^\varphi(C_i)$.*

Proof. Since the components of c_i bound disks in C_i, it is clear that $L_i^\varphi \subset \ker(\iota_*)$; since these disks cut C_i into a thickened $\partial_- C_i$, the reverse inclusion follows. □

Lemma 3.6. $$H_2^\varphi(C_i, \Sigma) \cong L_i^\varphi \quad \textit{for all } i.$$

Proof. $H_1^\varphi(C_i, \Sigma) = 0$ because C_i deformation-retracts onto $\Sigma \cup \mathcal{D}_i$, and thus the exact sequence of the pair (C_i, Σ) gives $H_2^\varphi(C_i, \Sigma) \cong \ker(H_1^\varphi(\Sigma) \to H_1^\varphi(C_i))$. □

Lemma 3.7. $H_3^\varphi(X_i, C_{i-1}\cup C_i) \cong L_{i-1}^\varphi \cap L_i^\varphi$ *for all* i.

Proof. Since X_i is a 4-dimensional 1-handlebody, its order 2 and 3 homology is trivial, and the exact sequence of the pair $(X_i, C_{i-1}\cup C_i)$ gives $H_3^\varphi(X_i, C_{i-1}\cup C_i)\cong H_2^\varphi(C_{i-1}\cup C_i)$. Now the exact sequence of the pair $(C_{i-1}\cup C_i, \Sigma)$ gives

$$0 \to H_2^\varphi(C_{i-1}\cup C_i) \xrightarrow{\iota} H_2^\varphi(C_{i-1},\Sigma)\oplus H_2^\varphi(C_i,\Sigma) \xrightarrow{\pi} H_1^\varphi(\Sigma),$$

where the identification $H_2^\varphi(C_{i-1}\cup C_i,\Sigma)\cong H_2^\varphi(C_{i-1},\Sigma)\oplus H_2^\varphi(C_i,\Sigma)$ follows from the Mayer–Vietoris sequence associated to the decomposition of the pair $(C_{i-1}\cup C_i,\Sigma)$ into (C_{i-1},Σ) and (C_i,Σ). Now, the map π is the difference of the maps $H_2^\varphi(C_{i-1},\Sigma)\to H_1^\varphi(\Sigma)$ and $H_2^\varphi(C_i,\Sigma)\to H_1^\varphi(\Sigma)$ given by the exact sequences of the pairs (C_{i-1},Σ) and (C_i,Σ), which give the identifications $H_2^\varphi(C_{i-1},\Sigma)\cong L_{i-1}^\varphi$ and $H_2^\varphi(C_i,\Sigma)\cong L_i^\varphi$ of Lemma 3.6. It follows that the kernel of π, and thus the image of ι, identifies with the intersection $L_{i-1}^\varphi\cap L_i^\varphi$. □

Theorem 3.8. *The homology of X is given by the chain complex*

$$(\mathcal{C}) \qquad 0 \to \bigoplus_{i=1}^{n}(L_{i-1}^\varphi\cap L_i^\varphi) \xrightarrow{\partial_2} \bigoplus_{i=1}^{n} L_i^\varphi \xrightarrow{\partial_1} H_1^\varphi(\Sigma,*) \xrightarrow{\partial_0} H_0^\varphi(*),$$

where $\partial_2((x_i)_{1\le i\le n}) = (x_i - x_{i+1})_{1\le i\le n}$ and $\partial_1((x_i)_{1\le i\le n}) = \sum_{i=1}^n x_i$. Moreover, if R is a field, the complex basis b of $(\mathcal{C})$ described in Remark 3.9 forms a geometric basis for the torsion of X, meaning that $\tau^\varphi(X;h)=\tau(\mathcal{C};b,h)$.

Proof. Since $H_2^\varphi(Y,\Sigma)\cong\bigoplus_i H_2^\varphi(C_i,\Sigma)$ and $H_3^\varphi(X,Y)\cong\bigoplus_i H_3^\varphi(X_i,C_{i-1}\cup C_i)$, we can conclude with Corollary 3.3 and Lemmas 3.6 and 3.7. See Remark 3.9 for the explication of the geometric bases. □

Remark 3.9. The geometric bases of $\mathcal{C}'$ are images of cellular bases under the map induced by inclusion $Z_3 \hookrightarrow X$. The maps described in Lemmas 3.6 and 3.7 then define an isomorphism from $\mathcal{C}'$ to $\mathcal{C}$, and the geometric bases of $\mathcal{C}$ are then the images of geometric bases of $\mathcal{C}'$ under this map. This yields the following more concrete description of what the geometric bases b look like for $\mathcal{C}$:

- b_0 is given by the basepoint $*$,
- b_1 is defined by any set of loops on which Σ retracts,
- b_2 is any basis corresponding to a tuple of defining curves $(c_i)_{1\le i\le n}$,
- b_3 is any basis corresponding to a tuple of "double curves" for the pairs (c_{i-1},c_i).

By a "double curve" for a pair (c_{i-1},c_i), we mean any curve on Σ which simultaneously bounds disks in C_{i-1} and C_i. The constraints on multisection diagrams imply that $L_{i-1}^\varphi\cap L_i^\varphi$ admits bases represented by double curves; see Figure 2.

It might not be easy to find a system of double curves from a diagram, since it implies some possibly unobvious handleslides. It is not necessary in this algebraic computation; see Remark 3.12.

Corollary 3.10. *We have the expressions*

$$H_1^{\varphi}(X) \cong H_1^{\varphi}(\Sigma)/\Big(\bigoplus_i L_i^{\varphi}\Big), \quad H_3^{\varphi}(X) \cong \bigcap_i L_i^{\varphi},$$

where we slightly abuse notation by viewing $L_i^{\varphi} \subset H_1^{\varphi}(\Sigma) \subset H_1^{\varphi}(\Sigma, *)$.

Proof. For H_1, the pair $(\Sigma, *)$ gives $H_1^{\varphi}(\Sigma) \cong \ker(H_1^{\varphi}(\Sigma, *) \to H_0^{\varphi}(*))$. □

A satisfying point in Theorem 3.8 is that the modules of the complex $(\mathcal{C})$ are free.

Lemma 3.11. *The modules $H_1^{\varphi}(\Sigma, *)$ and L_i^{φ} are free R-modules of respective ranks $2g+b-1$ and p, where g is the genus of Σ, b is its number of boundary components and p is the number of curves in each collection c_i. The modules $L_{i-1}^{\varphi} \cap L_i^{\varphi}$ are also free, and their rank does not depend on R and φ.*

Proof. Since $\partial\Sigma \neq \varnothing$, Σ deformation retracts onto a bouquet of $2g+b-1$ loops with central vertex $*$. Hence $C_1^{\varphi}(\Sigma, *) \cong R^{2g+b-1}$ is the only nontrivial twisted chain module of $(\Sigma, *)$ and $H_1^{\varphi}(\Sigma, *) \cong R^{2g+b-1}$. The retraction can be chosen so that the components of c_i are loops of the bouquet, and hence L_i^{φ} is a free submodule of $H_1^{\varphi}(\Sigma, *)$ with basis given by the classes of these components. Moreover, up to handleslide, we can assume the components of c_{i-1} and c_i are in standard position (see Figure 2), so that a basis of $L_{i-1}^{\varphi} \cap L_i^{\varphi}$ is given by the parallel curves in these collections. □

Remark 3.12. We can now explain how to simplify the computation of torsion, avoiding the explicit exhibition of systems of double curves. Consider the subring $R_0 = \varphi(\mathbb{Z}[\pi_1(X)])$ of the field R; note that $R_0 = \mathbb{Z}[\varphi(\pi_1(X))]$ and $R_0^* = \pm\varphi(\pi_1(X))$. To avoid confusion, we denote by $L_i^{R_0}$ the module associated to the map φ viewed with values in R_0. The natural map from $H_1^{\varphi}(\Sigma, *; R_0)$ to $H_1^{\varphi}(\Sigma, *; R)$ sends $L_i^{R_0}$ onto L_i^{φ}. The submodules of $H_1^{\varphi}(\Sigma, *; R)$ that appear in the complex $(\mathcal{C})$ of Theorem 3.8 are free and are images of the similar submodules of $H_1^{\varphi}(\Sigma, *; R_0)$, which have the same rank. An R_0-*basis* of such a submodule of $H_1^{\varphi}(\Sigma, *; R)$ is a basis that is the image of a basis of the corresponding submodule of $H_1^{\varphi}(\Sigma, *; R_0)$. A tuple of double curves defines such an R_0-basis. Any other R_0-basis can be used to compute the torsion. Actually, changing the basis of a homology module in the computation of the torsion multiplies the torsion by the determinant of the change of basis. For R_0-bases, this determinant is the same as the determinant of the corresponding change of basis of the corresponding submodule of $H_1^{\varphi}(\Sigma, *; R_0)$, so that it is an element of $R_0^* = \pm\varphi(\pi_1(X))$.

4. The twisted relative homology groups and torsion

In this section, we compute the twisted relative homology and torsion of X. The computation of the homology of $(X, \partial X)$ ends up being formally similar to that of X: it involves a retraction onto a 3-complex in X. However, in order to make the relative cellular structure clearly apparent, we leave ∂X fixed throughout the retraction. To carry out such a retraction, X has to be punctured, but once the homology of the punctured version of X is computed rel ∂X, it is easy to recover the homology of X itself rel ∂X.

Definition 4.1. Let C be a lensed compression body. An *r-defining collection of disks* for C is a disjoint union $\mathcal{D}^r$ of disks, with boundary in $\partial_+ C$ or made of an arc in $\partial_+ C$ and an arc in $\partial_- C$, such that $C \setminus \eta(\mathcal{D}^r)$ is a 3-ball. The intersection with $\partial_+ C$ of an r-defining collection of disks for C is a *complete collection of arcs and curves* for C.

Likewise if V is a hyper compression body then an *r-defining collection of balls* for V is a union of 3-balls $\mathcal{B}^r$ such that $V \setminus \eta(\mathcal{B}^r)$ is a 4-ball.

Remark 4.2. The r in these definitions stands for "relative". Note that an r-defining collection of disks can be chosen to contain a defining collection of disks, and similarly for collections of balls.

Such r-defining collections of disks do exist. First take a subcollection of a defining collection of disks for C, dropping those that do not carry homology relative to boundary. Then add the products with the interval in $\partial_- C \times I$ of arcs that cut $\partial_- C$ into a disjoint union of disks. A similar argument shows existence of r-defining collections of balls for the hyper compression bodies under consideration here.

Fix r-defining collections $\mathcal{D}_i^r$ and $\mathcal{B}_i^r$ of disks and balls for C_i and X_i respectively. Set $\mathcal{D}^r = \bigcup_{i=1}^n \mathcal{D}_i^r$ and $\mathcal{B}^r = \bigcup_{i=1}^n \mathcal{B}_i^r$. For all $Z \subset X$, let $Z' = Z \setminus \eta(*)$.

Lemma 4.3. *The manifold X' deformation retracts onto $\Sigma' \cup \mathcal{D}^r \cup \mathcal{B}^r \cup \partial X$. Further, the quad $(X', Y' \cup \partial X, \Sigma' \cup \partial X, \partial X)$ deformation retracts rel ∂X onto a CW-complex $(Z_3^\partial \cup \partial X, Z_2^\partial \cup \partial X, Z_1^\partial \cup \partial X, \partial X)$, where Z_1^∂ is made of arcs and loops on Σ', $Z_2^\partial = Z_1^\partial \cup \mathcal{D}^r$, $Z_3^\partial = Z_2^\partial \cup \mathcal{B}^r$.*

Proof. The proof is similar to that of Lemma 3.2, but instead of retracting from the boundary, we retract "inside out" from the puncture $*$. Because $X_i \setminus \eta(\mathcal{B}_i^r)$ is a ball and meets $\eta(*)$ in a small 4-ball that has been "scooped out" of the boundary, we obtain a retraction of X_i' onto $(\partial X_i)' \cup \mathcal{B}_i^r$. Carrying this retraction out for each i yields a retraction of X' onto $Y' \cup \mathcal{B}^r \cup \partial X$ — recall that $\partial X_i = (X_i \cap X_{i-1}) \cup (X_i \cap X_{i+1}) \cup (X_i \cap \partial X)$. Since each $C_i \setminus \eta(\mathcal{D}_i^r)$ is also a ball which intersects $\eta(*)$ along a scooped out 3-ball, Y' can further be retracted onto $\Sigma' \cup \mathcal{D}^r$. This gives the first assertion, and the second one follows. $\square$

Corollary 4.4. *The homology of* $(X, \partial X)$ *is given by the chain complex*

$$(\mathcal{C}'_\partial)\quad H_4^\varphi(X, X') \to H_3^\varphi(X', Y' \cup \partial X) \to H_2^\varphi(Y', \Sigma' \cup \partial Y) \to H_1^\varphi(\Sigma', \partial\Sigma) \to 0.$$

Proof. Lemma 4.3 immediately gives the following cellular chain complex for $(X', \partial X)$:

$$0 \to H_3^\varphi(X', Y' \cup \partial X) \to H_2^\varphi(Y' \cup \partial X, \Sigma' \cup \partial X) \to H_1^\varphi(\Sigma' \cup \partial X, \partial X) \to 0,$$

or equivalently,

$$0 \to H_3^\varphi(X', Y' \cup \partial X) \to H_2^\varphi(Y', \Sigma' \cup \partial Y) \to H_1^\varphi(\Sigma', \partial\Sigma) \to 0.$$

Now, the long exact sequence of the triple $(X, X', \partial X)$ shows that $H_k^\varphi(X, \partial X) \cong H_k^\varphi(X', \partial X')$ for $k = 1, 2$ and $H_3^\varphi(X, \partial X) \cong H_3^\varphi(X', \partial X)/\mathrm{Im}(H_4^\varphi(X, X'))$. Finally, the long exact sequence of the triple $(X, Y' \cup \partial X, \partial X)$ identifies $H_4^\varphi(X, \partial X)$ with $H_4^\varphi(X, Y' \cup \partial X)$ and the long exact sequence of the triple $(X, X', Y' \cup \partial X)$ identifies $H_4^\varphi(X, Y' \cup \partial X)$ with the kernel of $H_4^\varphi(X, X') \to H_3^\varphi(X', Y' \cup \partial X)$. □

Definition 4.5. Let $\mathcal{J}_i^\varphi$ denote the subgroup of $H_1^\varphi(\Sigma', \partial\Sigma)$ generated by any complete collection of arcs and curves for C_i on Σ'.

Lemma 4.6 gives an alternative interpretation of $\mathcal{J}_i^\varphi$. Identifying $H_1^\varphi(\Sigma, *)$ with $H_1(\Sigma', \partial\eta(*))$ via the excision equivalence, and using the decomposition $\partial\Sigma' = \partial\Sigma \cup \partial\eta(*)$, we have an equivariant intersection form on $H_1^\varphi(\Sigma, *) \times H_1^\varphi(\Sigma', \partial\Sigma)$.

Lemma 4.6. *The modules* $H_1^\varphi(\Sigma', \partial\Sigma)$ *and* $\mathcal{J}_i^\varphi$ *are free* R*-modules of respective ranks* $2g+b-1$ *and* $2g+b-1-n$*. The modules* $\mathcal{J}_{i-1}^\varphi \cap \mathcal{J}_i^\varphi$ *are also free. Moreover,* $\mathcal{J}_i^\varphi$ *is the orthogonal complement of* L_i^φ *with respect to the equivariant intersection pairing on* $H_1^\varphi(\Sigma, *) \times H_1^\varphi(\Sigma', \partial\Sigma)$.

Proof. Let Z_1^∂ be any collection of $2g + b - 1$ arcs properly embedded in Σ' which are pairwise disjoint and cut Σ' into a disk. Then Σ' retracts onto $Z_1^\partial \cup \partial\Sigma$, showing that $H_1^\varphi(\Sigma', \partial\Sigma) \cong R^{2g+b-1}$. The 1-complex Z_1^∂ can be chosen so that $2g - p + b - 1$ of the arcs form a complete collection of arcs and curves for C_i (start with a complete collection of arcs and curves, replace closed curves by arcs, and add as many arcs as needed) whose twisted homology classes generate $\mathcal{J}_i^\varphi$. A basis of $\mathcal{J}_{i-1}^\varphi \cap \mathcal{J}_i^\varphi$ is provided by a subcollection of these.

Now, a curve c_i^0 in the family c_i bounds a disk in C_i, while an arc γ in a complete collection of arcs and curves for C_i cobounds a disk in C_i with an arc in $\partial_- C_i$. Assuming transversality of the two disks, it follows that the intersection of c_i^0 and γ is the boundary of a union of embedded intervals and hence contains as many positive as negative intersection points. Hence $\mathcal{J}_i^\varphi$ is contained in the orthogonal complement of L_i^φ, and the equality follows by a dimension argument, using the nondegeneracy of the intersection form. □

Lemma 4.7. $H_2^{\varphi}(C_i', (\partial C_i')) \cong \mathcal{J}_i^{\varphi}$ *for all* i.

Proof. The long exact sequence of the triple $(C_i', (\partial C_i)', \partial_- C_i)$, together with the excision equivalence $((\partial C_i)', \partial_- C_i) \sim (\Sigma', \partial\Sigma)$ give the short exact sequence

$$0 \to H_2(C_i', (\partial C_i)') \to H_1(\Sigma', \partial\Sigma) \xrightarrow{\zeta} H_1(C_i', \partial_- C_i).$$

Now C_i' is obtained from a thickened $\partial_- C_i$ by adding only 1-handles, so that the kernel of ζ contains the homology classes of curves in Σ' that have trivial algebraic intersection with the cocores of these 1-handles, cocores whose boundaries generate L_i^{φ}. We conclude that $H_2(C_i', (\partial C_i)') \cong \ker(\zeta) \cong \mathcal{J}_i^{\varphi}$. □

Lemma 4.8. $H_3^{\varphi}(X_i', (\partial X_i)') \cong \mathcal{J}_{i-1}^{\varphi} \cap \mathcal{J}_i^{\varphi}$ *for all* i.

Proof. Since X_i' is obtained from a thickened $\partial_- X_i$ by adding 1-handles, the exact sequence of the triple $(X_i', (\partial X_i)', \partial_- X_i)$ gives an isomorphism $H_3^{\varphi}(X_i', (\partial X_i)') \cong H_2^{\varphi}((\partial X_i)', \partial_- X_i)$; this last module is isomorphic to $H_2(C_{i-1}' \cup C_i', \partial_- C_{i-1} \cup \partial_- C_i)$. The long exact sequence of the triple $\big(C_{i-1}' \cup C_i', (\partial C_{i-1})' \cup (\partial C_i)', \partial_- C_{i-1} \cup \partial_- C_i\big)$ and the excision equivalence $\big((\partial C_{i-1})' \cup (\partial C_i)', \partial_- C_{i-1} \cup \partial_- C_i\big) \sim (\Sigma', \partial\Sigma)$ give

$$0 \to H_2^{\varphi}(C_{i-1}' \cup C_i', \partial_- C_{i-1} \cup \partial_- C_i) \xrightarrow{\iota} H_2^{\varphi}(C_{i-1}', (\partial C_{i-1})') \oplus H_2^{\varphi}(C_i', (\partial C_i)') \xrightarrow{\pi} H_1^{\varphi}(\Sigma', \partial\Sigma).$$

The conclusion follows from Lemma 4.7 with an argument analogous to that of Lemma 3.7. □

Theorem 4.9. *If* $\partial X \neq \varnothing$, *the twisted homology of* $(X, \partial X)$ *is given by the chain complex*

$$(\mathcal{C}_\partial) \qquad H_2^{\varphi}(\Sigma, \Sigma') \xrightarrow{\partial_3} \bigoplus_i (\mathcal{J}_{i-1}^{\varphi} \cap \mathcal{J}_i^{\varphi}) \xrightarrow{\partial_2} \bigoplus_i \mathcal{J}_i^{\varphi} \xrightarrow{\partial_1} H_1^{\varphi}(\Sigma', \partial\Sigma) \to 0,$$

where $\partial_3([\Sigma]) = [\partial(\Sigma \setminus \Sigma')]$, $\partial_2((x_i)_{1\le i\le n}) = ((x_i - x_{i+1})_{1\le i\le n})$ *and* $\partial_1((x_i)_{1\le i\le n}) = \sum_{i=1}^n x_i$. *Moreover, if* R *is a field, the complex basis* b *of* $(\mathcal{C}_\partial)$ *described in Remark 4.10 forms a geometric basis for the relative torsion of* X, *meaning that* $\tau^{\varphi}(X, \partial X; h) = \tau(\mathcal{C}_\partial; b, h)$.

Proof. Start with the complex $(\mathcal{C}_\partial')$ of Corollary 4.4. For the order 2 and 3 terms, use Mayer–Vietoris sequences to get the identifications $H_2^{\varphi}(Y', \Sigma' \cup \partial Y) \cong \bigoplus_i H_2^{\varphi}(C_i', (\partial C_i)')$ and $H_3^{\varphi}(X', Y' \cup \partial X) \cong \bigoplus_i H_3^{\varphi}(X_i', (\partial X_i)')$ and conclude with Lemmas 4.7 and 4.8. A generator of $H_4^{\varphi}(X, X')$ is sent onto the class of $\partial\eta(*)$ in $H_3^{\varphi}(X', Y' \cup \partial X)$. Following the isomorphisms in Lemma 4.8, we see that this class corresponds to the class in $\bigoplus_i \mathcal{J}_{i-1}^{\varphi} \cap \mathcal{J}_i^{\varphi}$ of the curve $\partial\eta(*)$, where the neighborhood is now understood in Σ, which is the boundary of a generator of $H_2(\Sigma, \Sigma')$. □

Remark 4.10. As with the absolute case, we can obtain a concrete description of what the geometric bases b look like for $\mathcal{C}_\partial$:

- $b_1 = \{[e_1], [e_2], \dots, [e_n]\}$, where each e_i is an edge of Z_1^∂ (i.e., any set of arcs which cut Σ into a disk),
- $b_2 =$ any basis corresponding to a tuple of complete collections of arcs and curves for C_i,
- $b_3 =$ any basis corresponding to a tuple of "double arcs and curves" for the pairs (C_{i-1}, C_i), or any other R_0-basis with $R_0 = \varphi(\mathbb{Z}[\pi_1(X)])$ (see Remark 3.12),
- $b_4 =$ the fundamental class of $H_2(\Sigma, \Sigma')$.

Corollary 4.11. *We have the following expressions for the twisted homology of* $(X, \partial X)$:

$$H_1^\varphi(X, \partial X) \cong H_1^\varphi(\Sigma', \partial\Sigma)/\bigoplus_i \mathcal{J}_i^\varphi, \quad H_3^\varphi(X, \partial X) \cong \bigcap_i \overline{\mathcal{J}}_i^\varphi,$$

where $\overline{\mathcal{J}}_i^\varphi$ *denotes the image of* $\mathcal{J}_i^\varphi$ *under the inclusion map* $H_1^\varphi(\Sigma', \partial\Sigma) \to H_1^\varphi(\Sigma, \partial\Sigma)$.

Proof. For H_3, the long exact sequence of the triple $(\Sigma, \Sigma', \partial\Sigma)$ gives the exact sequence

$$H_2^\varphi(\Sigma, \Sigma') \xrightarrow{\zeta} H_1^\varphi(\Sigma', \partial\Sigma) \to H_1^\varphi(\Sigma, \partial\Sigma) \to 0,$$

and we have $H_3^\varphi(X, \partial X) \cong \left(\bigcap_i \mathcal{J}_i^\varphi\right)/\mathrm{Im}(\zeta)$. □

5. Intersection forms

We keep in this section the assumption that $\partial X \neq \varnothing$. The intersection forms are formally identical to the closed case treated in [Florens and Moussard 2022]. The upshot is that the intersections between various cycles in X can all be made to coincide with intersections in Σ. Below we assume that $\Sigma' = \Sigma \setminus \eta(*)$, so that there is a natural isomorphism $H_1^\varphi(\Sigma, *) \cong H_1^\varphi(\Sigma', \partial\eta(*))$, and we identify each L_i^φ with its image under this map below. Note that, in the nontwisted case, $H_1(\Sigma, *)$ naturally identifies with $H_1(\Sigma)$.

Theorem 5.1. *Suppose* $h_1 = [(x_i)_{1\le i\le n}]$ *and* $h_2 = [(y_i)_{1\le i\le n}]$ *in* $H_2^\varphi(X)$, *where* $(x_i)_{1\le i\le n}, (y_i)_{1\le i\le n} \in \bigoplus_i L_i^\varphi$. *Then*

$$\langle h_1, h_2\rangle_X^\varphi = \sum_{1\le i<j\le n} \langle x_i, y_j\rangle_\Sigma^\varphi,$$

where $\langle\cdot,\cdot\rangle_X^\varphi$ *and* $\langle\cdot,\cdot\rangle_\Sigma^\varphi$ *are the equivariant intersection forms on* $H_2^\varphi(X)$ *and* $H_1^\varphi(\Sigma, *)$ *respectively.*

Proof. It suffices to show that the analogous claim holds true in the untwisted integral homology groups of $\overline{X}$, which denotes the cover of X associated to $\ker(\varphi)$. For any $Z \subset X$ let $\overline{Z}$ denote the inverse image of Z under the cover $\overline{X} \to X$. Because $\pi_1(\Sigma)$, $\pi_1(C_i)$, and $\pi_1(X_i)$ all surject onto $\pi_1(X)$ via the inclusion map,

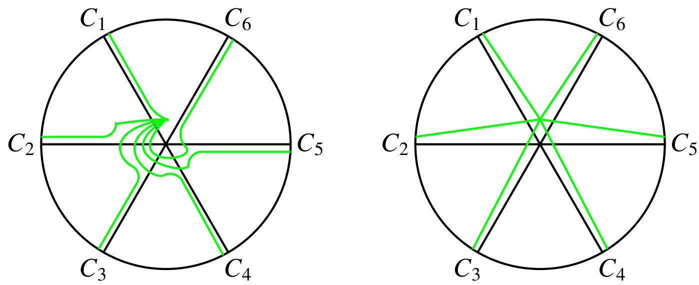

Figure 3. Pushing the relative 2-skeleton. The 2-skeleton appears in black, while the pushed relative 2-skeleton appears in green. At each intersection of the black lines and the green lines lies a copy of Σ.

$\bar{\Sigma}$, $\bar{C}_i$, and $\bar{X}_i$ are connected as well. In the finite case these lifts combine to form a multisection of $\bar{X}$, and in the case of an infinite-sheeted cover they form what is essentially a multisection, except the pieces involved have infinite genus. In particular, just as in the finite case, $\eta(\bar{\Sigma})$ is a trivial disk bundle and the lifted compression bodies $\bar{C}_i$ meet each disk in the bundle along rays which are disjoint except at the center point.

There is a cellular structure on $\bar{X}$ obtained by lifting the cell structures of X and $(X, \partial X)$ described in Lemmas 3.2 and 4.3 to $\bar{X}$. If Z_2 is the 2-skeleton of X described in Lemma 3.2, then $\bar{Z}_2$ is a 2-skeleton for $\bar{X}$ which lies in $\overline{\bigcup_i C_i}$. As observed in [Florens and Moussard 2022], we may push each $\bar{Z}_2 \cap \bar{C}_i$ slightly into its collar so that it is pushed into $\overline{\bigcup_{1 \le j \le i} X_j}$. This being done, the intersections between 2-chains in $\bar{Z}_2$ and 2-chains in the pushed $\bar{Z}_2$ will coincide with intersections between the boundaries of the subchains lying just in $\overline{Z_2 \cap C_i}$, and these intersections occur on diverse copies of Σ; see the left-hand side of Figure 3. □

Remark 5.2. Different expressions can be given for the intersection form by diversely pushing the relative 2-skeleton. The right-hand side of Figure 3 suggests another possibility with fewer terms.

Similarly one can compute the intersection pairings on $H_k^\varphi(X) \times H_{4-k}^\varphi(X, \partial X)$. For $k = 2$, the expression is analogous to that of Theorem 5.1.

Theorem 5.3. *If $h_1 = [(x_i)_{1 \le i \le n}] \in H_2^\varphi(X)$ and $h_2 = [(y_i)_{1 \le i \le n}] \in H_2^\varphi(X, \partial X)$, where $(x_i)_{1 \le i \le n} \in \bigoplus_i L_i^\varphi$, and $(y_i)_{1 \le i \le n} \in \bigoplus_i \mathcal{J}_i^\varphi$, then*

$$\langle h_1, h_2 \rangle_X^\varphi = \sum_{1 \le i < j \le n} \langle x_i, y_j \rangle_\Sigma^\varphi,$$

where $\langle\,\cdot\,,\cdot\,\rangle_X^\varphi$ and $\langle\,\cdot\,,\cdot\,\rangle_\Sigma^\varphi$ are the equivariant intersection pairings on $H_2^\varphi(X) \times H_2^\varphi(X, \partial X)$ and $H_1^\varphi(\Sigma, *) \times H_1^\varphi(\Sigma', \partial\Sigma)$ respectively.

Proof. The proof of Theorem 5.1 applies with the following adaptation: we consider the relative 2-skeleton Z_2^∂ of Lemma 4.3 and we look at intersections between 2-chains in $\bar{Z}_2$ and 2-chains in the pushed $\overline{Z_2^\partial}$. □

The intersection pairings on the odd-dimensional homology groups are described even more simply.

Theorem 5.4. *Suppose that either $h_1 \in H_1^\varphi(X)$ corresponds to the element $a \in H_1^\varphi(\Sigma, *)$ and $h_2 \in H_3^\varphi(X, \partial X)$ corresponds to the element $b \in \bigcap_i \mathcal{J}_i^\varphi$, or $h_1 \in H_1^\varphi(X, \partial X)$ corresponds to the element $a \in H_1^\varphi(\Sigma', \partial\Sigma)$ and $h_2 \in H_3^\varphi(X)$ corresponds to the element $b \in \bigcap_i L_i^\varphi$. Then*

$$\langle h_1, h_2\rangle_X^\varphi = \langle a, b\rangle_\Sigma^\varphi.$$

Proof. The proof is similar in structure to the proof of Theorem 5.1, except that now we observe that every chain in $H_1(\overline{X})$ or $H_1(\overline{X}, \partial\overline{X})$ is geometrically represented by linear combinations of curves $c \subset \overline{\Sigma} \subset \overline{X}$, and the chains in $H_1(\overline{X}, \partial\overline{X})$ or $H_3(\overline{X})$ can be geometrically represented by linear combinations of balls which meet $\overline{\Sigma}$ only in linear combinations of double curves. Thus no isotopy is needed, the intersections between the 1-chains and the 3-chains already correspond exactly to the intersections of their representatives in $H_1(\overline{\Sigma})$. □

6. The case of closed 4-manifolds

In this section, we compute the twisted homology, torsion and intersection forms when X is closed. It mainly follows the lines of the computation of relative homology, since we need again to puncture X. However, it mixes some features of the absolute and relative cases. For instance, when X is closed, r-defining collections of disks and balls are the same as ordinary defining collections. Since there is no additive difficulty with regards to the nonclosed case, we skip the details.

We fix $\star \in \Sigma$; for $Z \subset X$, we set $Z' = Z \setminus \eta(\star)$ and we fix $* \in \partial\Sigma'$. Let $\mathcal{D}$ and $\mathcal{B}$ be unions of defining collections of disks and balls for the C_i and the X_i respectively. Lemma 4.3 still holds, and provides the following corollary.

Lemma 6.1. *The quad $(X', Y', \Sigma', *)$ deformation retracts onto a CW-complex (Z_3, Z_2, Z_1, Z_0), where $Z_0 = *$, Z_1 is made of loops on Σ', $Z_2 = Z_1^\partial \cup \mathcal{D}$, $Z_3 = Z_2 \cup \mathcal{B}$. Subsequently, the homology of X is given by the chain complex*

$$(\bar{\mathcal{C}}') \qquad H_4^\varphi(X, X') \to H_3^\varphi(X', Y') \to H_2^\varphi(Y', \Sigma') \to H_1^\varphi(\Sigma', *) \to H_0^\varphi(*).$$

Now, L_i^φ denotes the submodule of $H_1^\varphi(\Sigma', *)$ generated by the homology classes of the curves in c_i.

Lemma 6.2. *The modules $H_1^\varphi(\Sigma', *)$ and L_i^φ are free R-modules of ranks $2g$ and g respectively. The modules $L_{i-1}^\varphi \cap L_i^\varphi$ are also free. Moreover, L_i^φ is a lagrangian for the equivariant intersection form on $H_1^\varphi(\Sigma', *)$.*

Lemma 6.3. *$H_2^\varphi(C_i', (\partial C_i)') \cong L_i^\varphi$ and $H_3^\varphi(X_i', (\partial X_i)') \cong L_{i-1}^\varphi \cap L_i^\varphi$ for all i.*

Theorem 6.4. *If X is closed, the twisted homology of X is given by the chain complex*

$$(\bar{\mathcal{C}}) \qquad H_2^\varphi(\Sigma, \Sigma') \xrightarrow{\partial_3} \bigoplus_i (L_{i-1}^\varphi \cap L_i^\varphi) \xrightarrow{\partial_2} \bigoplus_i L_i^\varphi \xrightarrow{\partial_1} H_1^\varphi(\Sigma', *) \to H_0^\varphi(*),$$

where

$$\partial_3([\Sigma]) = [\partial\Sigma'], \quad \partial_2((x_i)_{1\le i\le n}) = ((x_i - x_{i+1})_{1\le i\le n}), \quad \partial_1((x_i)_{1\le i\le n}) = \sum_{i=1}^n x_i.$$

Moreover, if R is a field, the complex basis b of $(\bar{\mathcal{C}})$ described in Remark 6.5 forms a geometric basis for the torsion of X, meaning that $\tau^\varphi(X; h) = \tau(\bar{\mathcal{C}}; b, h)$.

Remark 6.5. Once again, we can describe geometric torsion bases b for $\bar{\mathcal{C}}$:

- $b_0 = [*]$,
- $b_1 =$ any set of loops based at $*$ which cut Σ into a disk,
- $b_2 =$ any basis corresponding to a tuple of defining collections of curves for C_i,
- $b_3 =$ any basis corresponding to a tuple of "double curves" for the pairs (C_{i-1}, C_i), or any other R_0-basis with $R_0 = \varphi(\mathbb{Z}[\pi_1(X)])$ (see Remark 3.12),
- $b_4 =$ the fundamental class of $H_2(\Sigma, \Sigma')$.

Expressions for the intersection forms on $H_2^\varphi(X)$ and on $H_1^\varphi(X) \times H_3^\varphi(X)$ are again obtained in terms of the intersection form on $H_1^\varphi(\Sigma', *)$. Strictly speaking, this intersection form is defined on $H_1^\varphi(\Sigma', *_1) \times H_1^\varphi(\Sigma', *_2)$ for two distinct basepoints $*_1$ and $*_2$ on $\partial\Sigma'$. Again, in the nontwisted case, $H_1(\Sigma', *)$ identifies with $H_1(\Sigma)$.

Theorem 6.6. *Suppose that $h_1 = [(x_i)_{1\le i\le n}]$ and $h_2 = [(y_i)_{1\le i\le n}] \in H_2^\varphi(X)$, where $(x_i)_{1\le i\le n}, (y_i)_{1\le i\le n} \in \bigoplus_i L_i^\varphi$. Then*

$$\langle h_1, h_2\rangle_X^\varphi = \sum_{1\le i<j\le n} \langle x_i, y_j\rangle_\Sigma^\varphi,$$

where $\langle\cdot,\cdot\rangle_X^\varphi$ and $\langle\cdot,\cdot\rangle_\Sigma^\varphi$ are the equivariant intersection forms on $H_2^\varphi(X)$ and $H_1^\varphi(\Sigma', *)$.

Suppose that $h_1 \in H_1^\varphi(X)$ corresponds to the element $a \in H_1^\varphi(\Sigma', *)$ and that $h_2 \in H_3^\varphi(X)$ corresponds to the element $b \in \bigcap_i L_i^\varphi$. Then $\langle h_1, h_2\rangle_X^\varphi = \langle a, b\rangle_\Sigma^\varphi$.

7. The boundary: monodromy and homology

In this section, we assume $\partial X \neq \varnothing$ and we compute the action of the monodromy of the open book induced on ∂X on the homology of the page Σ_∂; we then deduce the homology of ∂X. All homology groups are considered with coefficients in $\mathbb{Z}$. We denote by Σ_i the result of compressing Σ along c_i, which is a copy of Σ_∂.

Given a compact surface S with no closed component, a *cut system for S* is a family of arcs on S that cuts each component of S into a disk.

Our main tool is the algorithm of Castro, Gay and Pinzón-Caicedo which describes the monodromy of the open book from a trisection diagram [Castro et al. 2018a]. Although they work with trisections in the case of a connected page, their result extends directly to the setting of multisections with multiple boundary components.

Proposition 7.1 (Castro, Gay and Pinzón-Caicedo). *Let e be any choice of arcs in Σ, disjoint from c_1, that forms a cut system for Σ_1. The monodromy $\phi : \Sigma_1 \to \Sigma_1$ which defines the open book decomposition of ∂X is encoded by its action on e, which in turn is described by the following algorithm. For i running from 1 to n, slide the curves c_{i+1} over one another and slide the arcs e over the curves c_i, until e is disjoint from the curves c_{i+1}. The family $c_1' \cup e'$ which results from these n steps will generally be distinct from the original family $c_1 \cup e$. Perform one final sequence of handleslides of the arcs and curves $c_1' \cup e'$ which sends c_1' to c_1. The resulting cut system e' for Σ_1 is $\phi(e)$.*

It is necessary to explicitly index the arcs e and keep track of this index throughout the algorithm, but the simple closed curves c_i need not be indexed.

We denote by L_i the subgroup of $H_1(\Sigma)$ generated by the homology classes of the curves in c_i, and we let $\mathcal{J}_i$ denote its orthogonal in $H_1(\Sigma, \partial\Sigma)$ with respect to the intersection pairing on $H_1(\Sigma) \times H_1(\Sigma, \partial\Sigma)$ (see Section 2C). Similarly, we denote by $\mathcal{L}_i$ the subgroup of $H_1(\Sigma, \partial\Sigma)$ generated by the homology classes of the curves in c_i, and we let J_i denote its orthogonal in $H_1(\Sigma)$.

Lemma 7.2. *The groups L_i, J_i, $\mathcal{L}_i$ and $\mathcal{J}_i$ are free abelian groups of ranks p, $g+h+b-1$, $g-h$ and $2g+b-p-1$ respectively, where g is the genus of Σ, h the genus of Σ_∂ and b the number of boundary components of both. Moreover, L_i and $\mathcal{L}_i$ are primitive subgroups of J_i and $\mathcal{J}_i$ respectively, so that the quotients J_i/L_i and $\mathcal{J}_i/\mathcal{L}_i$ both are free abelian groups of rank $g+h+b-p-1$.*

Proof. Up to diffeomorphism and handleslides, the curves of the collection c_i can be put in standard position; see Figure 2. From this standard position, one can draw curves providing bases for the different groups under study; see Figure 4.

The group L_i is generated by the homology classes of the curves in c_i, which are p homologically independent curves on Σ, and thus $\operatorname{rk} L_i = p$. The group $\mathcal{L}_i$ is generated by the same curves, but some of them are trivial in $H_1(\Sigma, \partial\Sigma)$. There are $g-h$ nontrivial ones, which are p homologically independent, so that $\operatorname{rk} \mathcal{L}_i = g-h$. Bases for J_i and $\mathcal{J}_i$ can be obtained by completing the given bases for L_i and $\mathcal{L}_i$ respectively, giving the remainder of the statement. Note that the ranks of J_i and $\mathcal{J}_i$ can be recovered from the fact that these groups are orthogonal complements of $\mathcal{L}_i$ and L_i respectively. □

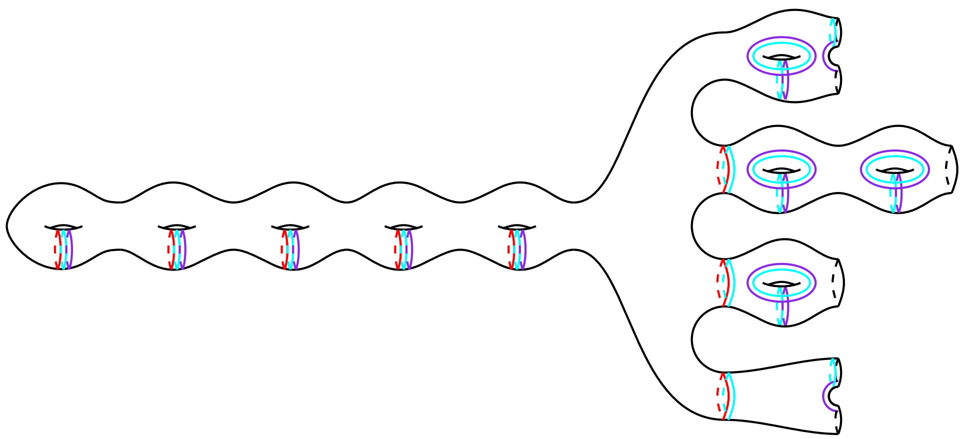

Figure 4. Curves on Σ for the compression body C_i. The curves of c_i are in red; their homology classes in $H_1(\Sigma)$ form a basis of L_i and the five leftmost ones provide a basis of $\mathcal{L}_i \subset H_1(\Sigma, \partial\Sigma)$. The homology classes of the blue and violet curves form bases of J_i and $\mathcal{J}_i$ respectively.

Lemma 7.3. *There are natural identifications $H_1(\Sigma_i, \partial\Sigma) \cong \mathcal{J}_i/\mathcal{L}_i$ and $H_1(\Sigma_i) \cong J_i/L_i$.*

Proof. Recall C_i is a lensed compression body, so that we can write its boundary as $\partial C_i = \Sigma \cup_{\partial\Sigma} \Sigma_i$. In particular, we have excision equivalences $(\partial C_i, \Sigma_i) \sim (\Sigma, \partial\Sigma)$ and $(\partial C_i, \Sigma) \sim (\Sigma_i, \partial\Sigma)$.

We first view C_i as a thickened Σ_i with 1-handles attached on the positive boundary whose cocores are the curves in c_i. This shows that $H_2(C_i, \Sigma_i) = 0$ and $H_1(C_i, \Sigma_i)$ is generated by the classes of the cores of the 1-handles. Hence the long exact sequence of the triple $(C_i, \partial C_i, \Sigma_i)$ gives

$$0 \to H_2(C_i, \partial C_i) \to H_1(\Sigma, \partial\Sigma) \to H_1(C_i, \Sigma_i) \to 0.$$

Now the image of an element of $H_1(\Sigma, \partial\Sigma)$ is determined by its algebraic intersection with the curves in c_i, and thus $H_2(C_i, \partial C_i) \cong \mathcal{J}_i$.

Likewise, viewing the compression body C_i as a thickened Σ with 2-handles glued along the curves in c_i on the negative boundary shows that $H_1(C_i, \Sigma) = 0$ and $H_2(C_i, \Sigma)$ is generated by the classes of the cores of the 2-handles. Now the long exact sequence of the triple $(C_i, \partial C_i, \Sigma)$ gives

$$H_2(C_i, \Sigma) \to H_2(C_i, \partial C_i) \to H_1(\Sigma_i, \partial\Sigma) \to 0.$$

Since the 2-handles are glued along the curves in c_i, the image of $H_2(C_i, \Sigma)$ corresponds to $\mathcal{L}_i$ in the above identification of $H_2(C_i, \partial C_i)$ with the subgroup $\mathcal{J}_i$ of $H_1(\Sigma, \partial\Sigma)$. This gives the identification $H_1(\Sigma_i, \partial\Sigma) \cong \mathcal{J}_i/\mathcal{L}_i$.

We now repeat the whole argument replacing ∂C_i by $\Sigma \sqcup \overline{\Sigma}_i$, where $\overline{\Sigma}_i$ is obtained from Σ_i by removing an open collar neighborhood of its boundary. The first step gives

$$0 \to H_2(C_i, \Sigma \sqcup \overline{\Sigma}_i) \to H_1(\Sigma) \to H_1(C_i, \Sigma_i)$$

and $H_2(C_i, \Sigma \sqcup \overline{\Sigma}_i) \cong J_i$, and the second step gives $H_2(C_i, \Sigma) \to H_2(C_i, \Sigma \sqcup \overline{\Sigma}_i) \to H_1(\Sigma_i) \to 0$ and $H_2(C_i, \Sigma) \cong L_i$ in $H_2(C_i, \Sigma \sqcup \overline{\Sigma}_i) \cong J_i$. □

Let e be a family of arcs in Σ, disjoint from c_1, that forms a cut system for Σ_1; note that it defines a basis of $H_1(\Sigma_1, \partial\Sigma)$. Let a_i be a family of simple closed curves on Σ that defines a basis of $\mathcal{L}_i/(\mathcal{L}_i \cap \mathcal{L}_{i+1})$ or $L_i/(L_i \cap L_{i+1})$ (in the sequel, we may consider their homology classes in $H_1(\Sigma)$ or $H_1(\Sigma, \partial\Sigma)$). For $\mu = (\mu_i)_{1\le i\le s}$ and $\mu' = (\mu'_i)_{1\le i\le t}$ families of $H_1(\Sigma, \partial\Sigma)$ and $H_1(\Sigma)$, define the matrix $\mu \cdot \mu' = (\langle \mu_i, \mu'_j\rangle_\Sigma)_{1\le i\le s, 1\le j\le t}$.

Proposition 7.4. *Let $\phi : \Sigma_1 \to \Sigma_1$ be the monodromy which defines the open book on ∂X. Define matrices R_i and families e_i in $H_1(\Sigma, \partial\Sigma)$ recursively as follows*:

- $R_0 = 0$ *and* $e_1 = e$,
- $R_i = -(e_i \cdot a_{i+1})(a_i \cdot a_{i+1})^{-1}$ *and* $e_{i+1} = e_i + R_i a_i$.

Fix a basis of the free $\mathbb{Z}$-module $\mathcal{J}_1$ which admits e as a subfamily and write the families e and e_{n+1} in this basis. Then the action of the monodromy of the open book of ∂X on $H_1(\Sigma_1, \partial\Sigma) \cong \mathcal{J}_1/(\mathcal{L}_1)$ is given in the basis e by the matrix of $R = e^t e_{n+1}$, where e^t is the transpose of e.

Proof. Following the algorithm of Proposition 7.1, we define families of arcs and curves e_i on Σ, disjoint from c_i, that define bases of $H_1(\Sigma_i, \partial\Sigma)$, by $e_1 = e$ and $e_{i+1} = e_i + r_i a_i$, where the r_i are matrices to compute. Since e_i is disjoint from c_i, we have $0 = e_{i+1} \cdot a_{i+1} = e_i \cdot a_{i+1} + r_i(a_i \cdot a_{i+1})$, so that $r_i = R_i$. Now e_{n+1} expresses $\phi(e)$ in the fixed basis of $\mathcal{J}_1$. Multiply by e^t to get it in the basis e of $H_1(\Sigma_1, \partial\Sigma)$. □

The following lemma gives the homology of a 3-manifold from an open book decomposition. A similar computation can be found in [Etnyre and Ozbagci 2008, Section 2.1].

Lemma 7.5. *Let M be a 3-manifold with an open book (S, ϕ). The homology of M is the homology of the complex*

$$0 \to \mathbb{Z}^s \xrightarrow{0} H_1(S, \partial S) \xrightarrow{\xi} H_1(S) \xrightarrow{0} \mathbb{Z}^s \to 0,$$

where $\xi([\mu]) = [-\mu \cup \phi(\mu)]$ and s is the number of components of S.

Proof. First note that S and M necessarily have the same number of connected components, so that s is also the number of components of M.

Consider the triple $\big(S\times[0,1], \partial(S\times[0,1]), S\times\{0\}\big)$. Since $S\times[0,1]$ deformation retracts on $S\times\{0\}$, the homology of the corresponding pair is trivial. Also, the open book structure gives a map $\phi : S\times[0,1]\to M$, injective on the interior, such that the $S_t=\phi(S\times\{t\})$ are the pages, with $S_0=S_1=S$. The map ϕ induces an isomorphism in homology: $H_*\big(S\times[0,1], \partial(S\times[0,1])\big)\cong H_*(M,S)$. Further, the inclusion of S as $S\times\{1\}$ in $\partial(S\times[0,1])$ induces an isomorphism $H_*\big(\partial(S\times[0,1]), S\times\{0\}\big)\cong H_*(S,\partial S)$. Finally $H_*(M,S)\cong H_{*-1}(S,\partial S)$. Hence the long exact sequence of the pair (M,S) gives

$$0\to H_2(M)\to H_1(S,\partial S)\xrightarrow{\xi} H_1(S)\to H_1(M)\to 0.$$

Finally, given an arc a properly embedded in $(S,\partial S)$, $a\times[0,1]$ is a relative 2-cycle for the pair

$$\big(S\times[0,1], \partial(S\times[0,1])\big)\sim(M,S),$$

whose boundary is $-a\cup\phi(a)$. □

To compute the homology of ∂X, we need to understand the homology classes $\phi(\mu)-\mu$ in $H_1(\Sigma_1)$. We keep the notations defined before and in Proposition 7.4.

Proposition 7.6. *Define families ε_i in $H_1(\Sigma)$ as follows: $\varepsilon_1=0$ and $\varepsilon_{i+1}=\varepsilon_i+R_i a_i$. Fix a basis b_L of $\mathcal{L}_1\cong L_1$ and complete it into a basis (b_L,b) of J_1. Write e in the basis (b_L,e) of $\mathcal{J}_1$ and ε_{n+1} in the basis (b_L,b) of J_1. The homology of ∂X is the homology of the complex*

$$0\to\mathbb{Z}^s\xrightarrow{0}\frac{\mathcal{J}_1}{\mathcal{L}_1}\xrightarrow{\xi}\frac{J_1}{L_1}\xrightarrow{0}\mathbb{Z}^s\to 0,$$

where s is the number of components of Σ and ξ is given in the bases e and b by the matrix $S=e^t\varepsilon_{n+1}$.

Proof. The ε_i represent the homology classes in $H_1(\Sigma)$ of the e_i-e. Throughout the algorithm of Proposition 7.1, they are added curves as in Proposition 7.4, but we now view the result in $H_1(\Sigma)$ at each step. □

8. Sample calculations

Example 1. The trisection diagram $(\Sigma;\alpha,\beta,\gamma)$ in Figure 5 is a diagram for a disk bundle X over S^2 with Euler number -2 obtained by Castro, Gay and Pinzón-Caicedo in [Castro et al. 2018a, Section 5.1]. In this example, all homology groups have coefficients in $\mathbb{Z}$. We first compute the (relative) homology and intersection form of X from this diagram.

In $H_1(\Sigma)=\langle\alpha_1,\beta_1,\alpha_2,\beta_2,\gamma_1\rangle$, we have $L_\alpha=\langle\alpha_1,\alpha_2\rangle$, $L_\beta=\langle\beta_1,\beta_2\rangle$, $L_\gamma=\langle\gamma_1,\alpha_2-2\beta_1+\beta_2\rangle$. All pairwise intersections of these subgroups are trivial. The

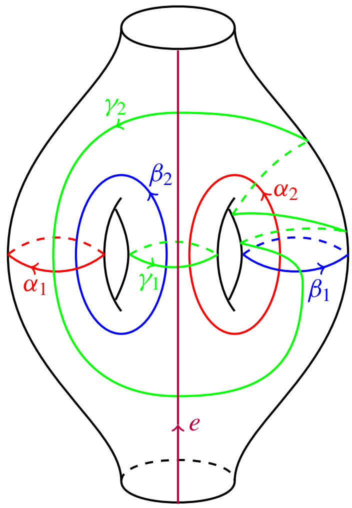

Figure 5. A trisection diagram of a disk bundle over S^2 with Euler number -2.

homology of X is the homology of the complex

$$0 \to L_\alpha \oplus L_\beta \oplus L_\gamma \to H_1(\Sigma) \xrightarrow{0} \mathbb{Z},$$

giving $H_1(X) = 0$, $H_2(X) \cong \mathbb{Z}$ and $H_3(X) = 0$. Note that the rightmost differential in always zero when working with coefficients in $\mathbb{Z}$.

In $H_1(\Sigma, \partial\Sigma) = \langle \alpha_1, \beta_1, \alpha_2, \beta_2, e\rangle$, we have $\mathcal{J}_\alpha = \langle \alpha_1, \alpha_2, e\rangle$, $\mathcal{J}_\beta = \langle \beta_1, \beta_2, e\rangle$, $\mathcal{J}_\gamma = \langle \alpha_1 - \beta_1, \alpha_2 - 2\beta_1 + \beta_2, e - \beta_2\rangle$; for $\mathcal{J}_\gamma$, we obtain these expressions by considering a complete collection of arcs and curves for C_γ made of γ_1, γ_2 and an arc joining the two boundary components avoiding the γ_i. Pairwise intersections are $\mathcal{J}_\alpha \cap \mathcal{J}_\beta = \langle e\rangle$, $\mathcal{J}_\beta \cap \mathcal{J}_\gamma = \langle e - \beta_2\rangle$, $\mathcal{J}_\gamma \cap \mathcal{J}_\alpha = \langle 2\alpha_1 - \alpha_2 - e\rangle$. The relative homology of X is the homology of the complex

$$\mathbb{Z} \xrightarrow{0} \bigoplus_{\nu \neq \nu'} \mathcal{J}_\nu \cap \mathcal{J}_{\nu'} \to \bigoplus_{\nu} \mathcal{J}_\nu \to H_1(\Sigma, \partial\Sigma) \to 0,$$

where $\nu, \nu' \in \{\alpha, \beta, \gamma\}$, giving $H_1(X, \partial X) = 0$, $H_2(X, \partial X) \cong \mathbb{Z}$ and $H_3(X, \partial X) = 0$. Note that the leftmost differential in always zero when working with coefficients in $\mathbb{Z}$.

A generator of $H_2(X)$ is given by $(\alpha_2, \beta_2 - 2\beta_1, -\gamma_2) \in L_\alpha \oplus L_\beta \oplus L_\gamma$. Using this generator, we can compute the intersection form of X:

$$\langle \alpha_2, \beta_2 - 2\beta_1\rangle_\Sigma + \langle \alpha_2, -\gamma_2\rangle_\Sigma + \langle \beta_2 - 2\beta_1, -\gamma_2\rangle_\Sigma = 2.$$

We now consider the monodromy of the open book on ∂X. We set $a_1 = (\alpha_1, \alpha_2)$, $a_2 = (\beta_1, \beta_2)$ and $a_3 = (\gamma_1, \gamma_2)$. Starting with $R_0 = 0$ and $e_1 = e$, we compute

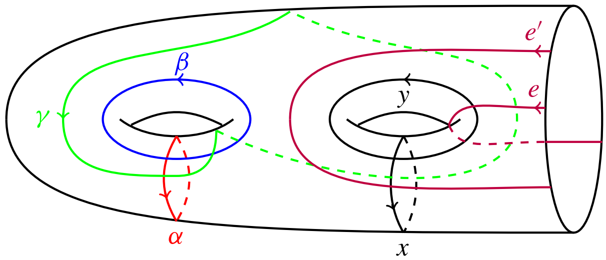

Figure 6. A trisection diagram.

$R_1 = (0\ \ 0)$, so that $e_2 = e$, then $R_2 = (0\ \ -1)$ and $e_3 = e - \beta_2$, and finally $R_3 = (-2\ \ 1)$ and $e_4 = e - 2\alpha_1 + \alpha_2$. Utilizing the basis (α_1, α_2, e) of $\mathcal{J}_\alpha$, this gives $R = (1)$ and shows that the action of the monodromy on $H_1(\Sigma_1, \partial\Sigma_1)$ is trivial. Now starting with $\varepsilon_1 = 0$, we get $\varepsilon_2 = 0$, $\varepsilon_3 = -\beta_2$ and $\varepsilon_4 = \alpha_2 - 2\alpha_1 - 2\zeta$, where $\zeta = \gamma_1 + \beta_1 - \alpha_1$ is a generator of J_α / L_α that we use as basis. Hence $S = (-2)$. Finally, the homology of ∂X is the homology of the complex

$$0 \to \mathbb{Z} \xrightarrow{0} \langle e \rangle \xrightarrow{-2} \langle \zeta \rangle \xrightarrow{0} \mathbb{Z} \to 0,$$

giving $H_1(\partial X) = \mathbb{Z}/2\mathbb{Z}$ and $H_2(\partial X) = 0$.

Example 2. Let X be the 4-manifold defined by the trisection diagram $(\Sigma; \alpha, \beta, \gamma)$ in Figure 6.

In $H_1(\Sigma) = \langle \alpha, \beta, x, y \rangle$, we have $L_\alpha = \langle \alpha \rangle$, $L_\beta = \langle \beta \rangle$, $L_\gamma = \langle -\alpha + \beta + y \rangle$. Pairwise intersections are trivial and we get $H_1(X; \mathbb{Z}) = \langle x \rangle \cong \mathbb{Z}$, $H_2(X; \mathbb{Z}) = 0$ and $H_3(X; \mathbb{Z}) = 0$.

In $H_1(\Sigma, \partial\Sigma) = \langle \alpha, \beta, e, e' \rangle$, we have $\mathcal{J}_\alpha = \langle \alpha, e, e' \rangle$, $\mathcal{J}_\beta = \langle \beta, e, e' \rangle$, $\mathcal{J}_\gamma = \langle \beta - \alpha, \alpha + e, e' \rangle$. This gives $H_1(X, \partial X) = 0$, $H_2(X, \partial X) = 0$ and $H_3(X, \partial X) = \mathcal{J}_\alpha \cap \mathcal{J}_\beta \cap \mathcal{J}_\gamma \cong \mathbb{Z}$.

We define $\varphi : \mathbb{Z}[\pi_1(X, *)] \to \mathbb{Z}[t^{\pm 1}]$ by $\varphi(x) = t$. Let us compute the associated twisted homology and torsion. Fix a lift $\tilde{*}$ of the basepoint $*$. For $\zeta \in \pi_1(\Sigma, *)$, we denote by $\tilde{\zeta}$ the lift of ζ starting at $\tilde{*}$. Since $\gamma = \alpha^{-1} x y x^{-1} \alpha \beta \alpha^{-1}$ in $\pi_1(\Sigma, *)$, we have $\tilde{\gamma} = -\tilde{\alpha} + \tilde{\beta} + t\tilde{y}$ in $H_1^\varphi(\Sigma, *)$. Hence, in $H_1^\varphi(\Sigma, *) = \langle \tilde{\alpha}, \tilde{\beta}, \tilde{x}, \tilde{y} \rangle$, we have $L_\alpha^\varphi = \langle \tilde{\alpha} \rangle$, $L_\beta^\varphi = \langle \tilde{\beta} \rangle$, $L_\gamma^\varphi = \langle -\tilde{\alpha} + \tilde{\beta} + t\tilde{y} \rangle$. From the complex

$$0 \to L_\alpha^\varphi \oplus L_\beta^\varphi \oplus L_\gamma^\varphi \to H_1^\varphi(\Sigma, *) \to H_0^\varphi(*) \to 0,$$

we get $H_0^\varphi(X; \mathbb{Z}[t^{\pm 1}]) \cong \mathbb{Z}[t^{\pm 1}]/(t-1) \cong \mathbb{Z}$ and $H_i^\varphi(X; \mathbb{Z}[t^{\pm 1}]) = 0$ for $i > 0$. This implies that the homology of X with coefficients in $\mathbb{Q}(t)$ is trivial, so that the torsion won't depend on the choice of a homology basis. Set $c_2 = (\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$, $c_1 = (\tilde{\alpha}, \tilde{\beta}, \tilde{x}, \tilde{y})$ and $c_0 = (\tilde{*})$ as complex bases for the above complex and $b_1 = (\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$ and $b_0 = ((t-1)\tilde{*})$ as bases of the images of the boundary map.

Then the torsion is given by

$$\tau^{\varphi}(X)=\left[\frac{b_1}{c_2}\right]^{-1}\left[\frac{b_1b_0}{c_1}\right]\left[\frac{b_0}{c_0}\right]^{-1}=-t(t-1)^{-1}\in\mathbb{Q}(t)/\mathbb{Z}[t^{\pm1}].$$

Finally, we consider the monodromy of the open book on ∂X. We set $a_1=\alpha$, $a_2=\beta$ and $a_3=\gamma$; note that $a_3=e'-\alpha+\beta$ in $H_1(\Sigma,\partial\Sigma)$. Starting with $R_0=0$ and $e_1=\binom{e}{e'}$, we get $R_1=\binom{0}{0}$ and $e_2=e_1$, then $R_2=\binom{1}{0}$ and $e_3=\binom{e+\beta}{e'}$, and finally $R_3=\binom{-1}{0}$ and $e_4=\binom{e-e'+\alpha}{e'}$. Utilizing the basis (e,e',α) of $\mathcal{J}_1$, we obtain

$$e^te_4=\begin{pmatrix}1&0&0\\0&1&0\end{pmatrix}\begin{pmatrix}1&0\\-1&1\\1&0\end{pmatrix}=\begin{pmatrix}1&0\\-1&1\end{pmatrix}$$

as the matrix giving the action of the monodromy in the basis (e,e') of $H_1(\Sigma_1,\partial\Sigma)$.

To get the homology of ∂X, we start with $\varepsilon_1=(0,0)$ and the computation gives $\varepsilon_2=(0,0)$, $\varepsilon_3=(\beta,0)$, $\varepsilon_4=(\alpha-y,0)$. It follows that the homology of ∂X is the homology of the complex

$$0\to\mathbb{Z}\xrightarrow{0}\langle e,e'\rangle\xrightarrow{\xi}\langle x,y\rangle\xrightarrow{0}\mathbb{Z}\to 0,$$

where $\xi(e)=-y$ and $\xi(e')=0$. Thus $H_1(\partial X)\cong H_2(\partial X)\cong\mathbb{Z}$.

Example 3. The quadrisection diagram $(\Sigma;(c_i)_{1\le i\le 4})$ represents the manifolds $S^2\times S^2$ (see for instance [Islambouli and Naylor 2024], or decompose each factor S^2 into two disks and recover this quadrisection). We shall use it to recover the homology with coefficients in $\mathbb{Z}$ and the intersection form of $S^2\times S^2$.

In $H_1(\Sigma)=\langle c_1,c_2\rangle$, we have $L_1=L_3=\langle c_1\rangle$ and $L_2=L_4=\langle c_2\rangle$. All pairwise intersections are trivial. The homology of $S^2\times S^2$ is the homology of the complex

$$\mathbb{Z}\to 0\to\bigoplus_{1\le i\le 4}L_i\to H_1(\Sigma)\xrightarrow{0}\mathbb{Z},$$

giving $H_1(X)=0$, $H_2(X)\cong\mathbb{Z}^2$ and $H_3(X)=0$.

A basis of $H_2(S^2\times S^2)$ is given by $(c_1,0,-c_1,0)$ and $(0,c_2,0,-c_2)$ in $\bigoplus_{1\le i\le 4}L_i$. In this basis, we obtain the intersection form as $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$.

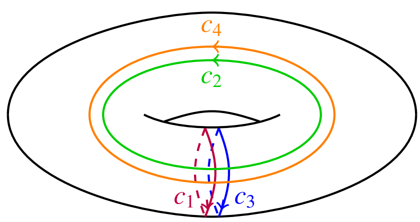

Figure 7. A quadrisection diagram of $S^2\times S^2$.

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