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It is shown that the fundamental group of the Griffiths double cone space is isomorphic to that of the triple cone. More generally if κ is a cardinal such that $2 \leq \kappa \leq 2^{\aleph_0}$ then the κ -fold cone has the same fundamental group as the double cone. The isomorphisms produced are nonconstructive, and no isomorphism between the fundamental group of the 2- and of the κ -fold cones, with $2 < \kappa$, can be realized via continuous mappings.

1. Introduction

The Griffiths double cone over the Hawaiian earring, which we denote $\mathbb{G}\mathbb{S}_2$, was introduced by H. B. Griffiths [1954] and has long stood as an interesting example in topology (Figure 1). Although $\mathbb{G}\mathbb{S}_2$ is a path connected, locally path connected compact metric space (a *Peano continuum*) which embeds as a subspace of \mathbb{R}^3 , it has some subtle properties. Despite being a wedge of two contractible spaces, $\mathbb{G}\mathbb{S}_2$ is not itself contractible, and more surprisingly the fundamental group of $\mathbb{G}\mathbb{S}_2$ is uncountable. The fundamental group is freely indecomposable and includes a copy of the additive group of the rationals and of the fundamental group of the Hawaiian earring. This group has found use in defining cotorsion-free groups in the nonabelian setting [Eda and Fischer 2016] and continues to serve as a counterexample [Zastrow 1994] and as a test model for notions of infinitary abelianization [Brazas and Gillespie 2022].

It is easy to see that analogous behavior is exhibited when one uses more cones in the wedge, as in the triple wedge $\mathbb{G}\mathbb{S}_3$ of cones over the Hawaiian earring or more generally in the κ -fold wedge $\mathbb{G}\mathbb{S}_\kappa$ (the one-point union of cones, indexed by κ , with the natural metric topology). A natural question is whether the isomorphism type of the fundamental group changes with this change in subscript. In light of the

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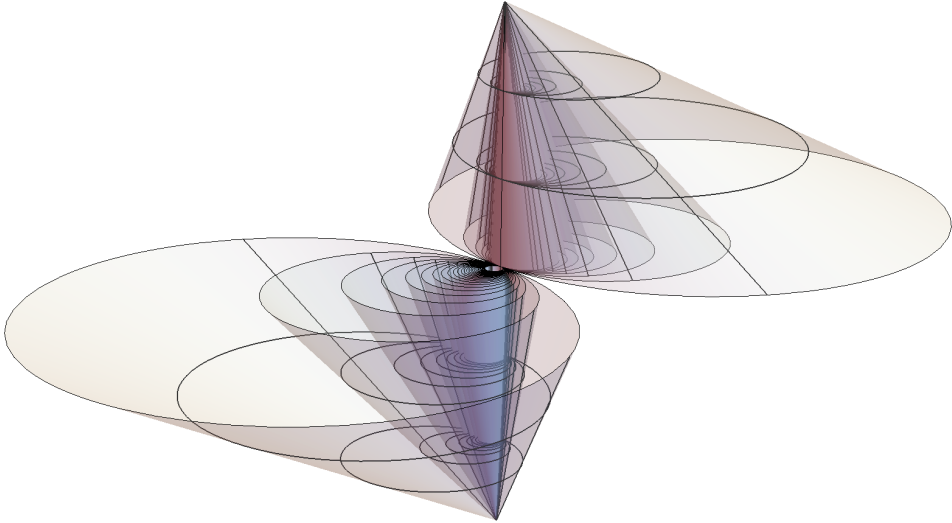


Figure 1. The Griffiths double cone $\mathbb{G}\mathbb{S}_2$.

intuitive fact that no spatial isomorphism can be defined the following answer is surprising.

Theorem A. *If κ is a cardinal such that $2 \leq \kappa \leq 2^{\aleph_0}$ then $\pi_1(\mathbb{G}\mathbb{S}_2) \simeq \pi_1(\mathbb{G}\mathbb{S}_\kappa)$.*

The bounds on κ in the statement of Theorem A are the best possible. The spaces $\mathbb{G}\mathbb{S}_0$ and $\mathbb{G}\mathbb{S}_1$ both strongly deformation retract to a point and therefore have trivial fundamental group, and when $\kappa > 2^{\aleph_0}$ one has $|\pi_1(\mathbb{G}\mathbb{S}_\kappa)| > 2^{\aleph_0} = |\pi_1(\mathbb{G}\mathbb{S}_2)|$ (Theorem 2.11). Using techniques of [Eda and Fischer 2016] or [Herfort and Hojka 2017] one can compute the abelianizations of $\pi_1(\mathbb{G}\mathbb{S}_2)$ and $\pi_1(\mathbb{G}\mathbb{S}_3)$ and see that these abelianizations are isomorphic.

A notable point of comparison is that the wedge of 2, 3, etc. Hawaiian earrings (without cones) is again homeomorphic to the Hawaiian earring, and so these spaces have isomorphic fundamental groups. However the fundamental group of a wedge of \aleph_0 Hawaiian earrings, under the topology that we are considering, will not have isomorphic fundamental group. This follows since the \aleph_0 -wedge of Hawaiian earrings retracts to a subspace which is the \aleph_0 -wedge of circles each having diameter 1, and this shows that the fundamental group of the \aleph_0 -wedge homomorphically surjects onto an infinite rank free group, which the fundamental group of the Hawaiian earring cannot do [Higman 1952].

The isomorphism given in Theorem A is produced combinatorially by a back-and-forth argument, using the axiom of choice. It is intuitively clear that there is no continuous function from $\mathbb{G}\mathbb{S}_2$ to $\mathbb{G}\mathbb{S}_3$ or vice versa which can yield an isomorphism of fundamental groups. A comparable situation in the setting of topological groups is that \mathbb{R} and \mathbb{R}^2 are isomorphic as abstract groups, since by picking a Hamel

basis over \mathbb{Q} one sees that both are isomorphic to $\bigoplus_{2^{\aleph_0}} \mathbb{Q}$. There is no continuous, or even Baire measurable, isomorphism between these topological groups. By contrast Theorem A does not seem to follow by producing isomorphisms to an easily understood third group like $\bigoplus_{2^{\aleph_0}} \mathbb{Q}$.

Another curiosity worth mentioning is that despite the necessary constraints on the cardinality of κ in Theorem A, the first-order logical theory of $\pi_1(\mathbb{GS}_2)$ and $\pi_1(\mathbb{GS}_\kappa)$ are the same whenever $\kappa \geq 2$.

Theorem B. *If $2 \leq \gamma \leq \kappa$ then $\pi_1(\mathbb{GS}_\gamma)$ elementarily embeds in $\pi_1(\mathbb{GS}_\kappa)$. Thus for $\kappa \geq 2$ the groups $\pi_1(\mathbb{GS}_2)$ and $\pi_1(\mathbb{GS}_\kappa)$ are elementarily equivalent.*

Of course when κ is 0 or 1 the fundamental group $\pi_1(\mathbb{GS}_\kappa)$ is trivial and therefore not elementarily equivalent to $\pi_1(\mathbb{GS}_2)$. The proof of Theorem B utilizes Theorem A and the action of the automorphism group, and no previous knowledge of first-order logic is required to understand the proof.

The ideas used in proving Theorem A seem to have very broad applications, and we state two now. Another space that is often mentioned along with the Griffiths space is the harmonic archipelago \mathbb{HA} of Bogley and Sieradski [2000]. The spaces \mathbb{GS}_2 and \mathbb{HA} share many common properties. Each embeds as a subspace of \mathbb{R}^3 , both contain a distinguished point at which every loop can be homotoped to have arbitrarily small image, and both have uncountable fundamental group. Cannon and Conner have conjectured that the two spaces share a further property, namely that they have isomorphic fundamental group [Conner 2011], and in a forthcoming paper we will show that this is indeed the case (the reader can see a proof of this fact in [Corson 2021, Theorem D]). By further reworking these ideas one can produce a correct proof of the main theorem of [Conner et al. 2015] (some errors have been pointed out by K. Eda) as well as answer many of the questions of that paper in the affirmative (see [Corson 2023]).

We describe the layout of this paper. In Section 2 we give the formal definition of the Griffiths space and its κ -fold analogues. We also present some combinatorially defined groups \mathcal{C}_κ and show them to be isomorphic to the fundamental groups $\pi_1(\mathbb{GS}_\kappa)$. In Section 3 we prove Theorems A and B.

2. The cone groups

We give a construction of \mathbb{GS}_2 and more generally of the κ -fold Griffiths space \mathbb{GS}_κ for any cardinal κ . We consider each cardinal number κ as being the set of all ordinals below it in the standard way. Thus $0 = \emptyset$, $n = \{0, \dots, n-1\}$ for each $n \in \omega$, $\omega + 2 = \{0, 1, \dots, \omega, \omega + 1\}$, etc. Let 2^{\aleph_0} denote the cardinal of the continuum. Given a point $p \in \mathbb{R}^2$ and $r \in [0, \infty)$ we let $C(p, r)$ denote the circle centered at p of radius r (in case $r = 0$ we obtain the degenerate circle consisting only of the point p). The *Hawaiian earring* is the subspace $\mathbb{E} = \bigcup_{n \in \omega} C((0, \frac{1}{n+3}), \frac{1}{n+3})$

of \mathbb{R}^2 . Let $\mathbb{G}\mathbb{S}_1 \subseteq \mathbb{R}^3$ be the subspace $\bigcup_{r \in [0,1]} (\bigcup_{n \in \omega} C((0, \frac{1-r}{n+3}), \frac{r}{n+3}) \times \{r\})$. The space $\mathbb{G}\mathbb{S}_1$ may also be viewed as the space obtained by first taking the Hawaiian earring sitting in the xy -plane $\mathbb{E} \times \{0\}$ and joining each point of $\mathbb{E} \times \{0\}$ to the point $(0, 0, 1)$ by a geodesic line segment. A third, topological way of viewing $\mathbb{G}\mathbb{S}_1$ is by simply taking the topological cone over the Hawaiian earring. In other words, $\mathbb{G}\mathbb{S}_1$ is homeomorphic to the quotient space obtained by beginning with $\mathbb{E} \times [0, 1]$ and identifying all points which have 1 in the last coordinate.

We define $\mathbb{G}\mathbb{S}_0$ to be the metric space consisting of the single point o_0 . Let $\kappa \geq 1$ be a cardinal. We take $\mathbb{G}\mathbb{S}_\kappa$ to be the set obtained by taking κ -many disjoint isometric copies $\bigsqcup_{\alpha < \kappa} X_\alpha$ of $\mathbb{G}\mathbb{S}_1$ and identifying all copies of $(0, 0, 0)$ to a single point o_κ . Thus we consider $o_\kappa \in X_\alpha$ for all $\alpha < \kappa$. Metrize $\mathbb{G}\mathbb{S}_\kappa$ by letting D_α be the metric on X_α (making X_α an isometric copy of $\mathbb{G}\mathbb{S}_1$) and

$$D(x, y) = \begin{cases} D_\alpha(x, y) & \text{if } x, y \in X_\alpha, \\ D_\alpha(x, o_\kappa) + d_{\alpha'}(o_\kappa, y) & \text{if } x \in X_\alpha \setminus \{o_\kappa\} \text{ and } y \in X_{\alpha'} \setminus \{o_\kappa\}, \alpha \neq \alpha'. \end{cases}$$

We note that this definition yields an isometric copy of $\mathbb{G}\mathbb{S}_1$ when $\kappa = 1$ and so the definition is consistent. When κ is finite, the space $\mathbb{G}\mathbb{S}_\kappa$ is a Peano continuum and $\mathbb{G}\mathbb{S}_\kappa$ is homeomorphic to the topological wedge of κ -many copies of $\mathbb{G}\mathbb{S}_1$ with the copies of the point $(0, 0, 0)$ identified. When $\kappa \geq \aleph_0$ the space $\mathbb{G}\mathbb{S}_\kappa$ is neither compact nor homeomorphic to the quotient space obtained by identifying all copies of $(0, 0, 0)$ in the topological disjoint union of κ -many copies of $\mathbb{G}\mathbb{S}_1$.

Next we give a description of what we call the *cone group* \mathcal{C}_κ for each cardinal κ . The description involves infinitary word combinatorics. Fix a cardinal κ . We start with a set $\mathcal{A}_\kappa = \{a_{\alpha,n}^{\pm 1} \mid \alpha < \kappa, n < \omega\}$ equipped with formal inverses. We call the elements of \mathcal{A}_κ *letters* and a letter is *positive* if it has superscript 1. For convenience we shall usually leave off the superscript 1 on positive letters. A letter which is not positive is *negative*. Let proj_0 and proj_1 be the functions defined on \mathcal{A}_κ which project respectively the first and second subscript of a letter. Thus $\text{proj}_0(a_{\alpha,n}^{-1}) = \alpha$ and $\text{proj}_1(a_{\alpha,n}^{-1}) = n$.

A *word* in \mathcal{A}_κ is a function $W : \overline{W} \rightarrow \mathcal{A}_\kappa$ such that \overline{W} is a totally ordered set and for each $N \in \omega$ the set $\{i \in \overline{W} \mid \text{proj}_1(W(i)) \leq N\}$ is finite. The domain of a word is necessarily countable. We write $W_0 \equiv W_1$ if there exists an order isomorphism $\iota : \overline{W}_0 \rightarrow \overline{W}_1$ such that $W_1(\iota(i)) = W_0(i)$ for all $i \in \overline{W}_0$, and write $\iota : W_0 \equiv W_1$ in this case. Let E denote the word with empty domain.

Let \mathcal{W}_κ denote the set of all \equiv classes of words in \mathcal{A}_κ . For $W \in \mathcal{W}_\kappa$ we let $d(W) = \min\{\text{proj}_1(W(i)) \mid i \in \overline{W}\}$ and $d(E) = \infty$. There is a natural associative binary operation on \mathcal{W}_κ given by word concatenation, defined by letting $W_0 W_1$ be the word W such that $\overline{W} = \overline{W}_0 \sqcup \overline{W}_1$ has the ordering that extends the orders of \overline{W}_0 and \overline{W}_1 , placing elements in \overline{W}_0 below those of \overline{W}_1 , and

$$W(i) = \begin{cases} W_0(i) & \text{if } i \in \overline{W}_0, \\ W_1(i) & \text{if } i \in \overline{W}_1. \end{cases}$$

There is similarly a notion of infinite concatenation. If Λ is a totally ordered set and $\{W_\lambda\}_{\lambda \in \Lambda}$ is a collection of words such that for every $N \in \omega$ the set $\{\lambda \in \Lambda : d(W_\lambda) \leq N\}$ is finite then we can take a concatenation $\prod_{\lambda \in \Lambda} W_\lambda$ whose domain is the disjoint union $\bigsqcup_{\lambda \in \Lambda} \overline{W}_\lambda$ ordered in the natural way and whose outputs are given by $(\prod_{\lambda \in \Lambda} W_\lambda)(i) = W_\lambda(i)$ where $i \in \overline{W}_\lambda$. We also use this notation for the concatenation of ordered sets. If $\{\Lambda_\lambda\}_{\lambda \in \Lambda}$ is a collection of ordered sets and Λ is itself ordered we let $\prod_{\lambda \in \Lambda} \Lambda_\lambda$ be the ordered set obtained by taking the disjoint union of the Λ_λ and ordering the elements in the obvious way. To further abuse notation we write $\Lambda \equiv \Theta$ if Λ is order isomorphic to Θ .

We also have an inversion operation on words given by letting W^{-1} have domain \overline{W} under the reverse order and letting $W^{-1}(i) = (W(i))^{-1}$. For each $N \in \omega$ and word W we let $p_N(W)$ be the restriction $W \upharpoonright \{i \in \overline{W} \mid \text{proj}_1(W(i)) \leq N\}$. Thus $p_N(W)$ is a finite word in the alphabet \mathcal{A}_κ . We write $W_0 \sim W_1$ if for every $N \in \omega$ the words $p_N(W_0)$ and $p_N(W_1)$ are equal when considered as elements in the free group on positive elements of \mathcal{A}_κ . As an example, the word $W \equiv a_{0,0}a_{0,0}^{-1}a_{0,1}a_{0,1}^{-1} \cdots$ satisfies $W \sim E$ since $p_N(W) \equiv a_{0,0}a_{0,0}^{-1}a_{0,1}a_{0,1}^{-1} \cdots a_{0,N}a_{0,N}^{-1}$ is freely equal to E for each $N \in \omega$. Let $[W]$ denote the \sim equivalence class of W . We obtain a group structure on \mathcal{W}_κ/\sim by letting $[W_0][W_1] = [W_0W_1]$, from which one gets inverses defined by $[W]^{-1} = [W^{-1}]$ and $[E]$ as the identity element. Let H_κ denote this group. Define a word W to be α -pure if $\text{proj}_0 \circ W(i) = \alpha$ for all $i \in \overline{W}$. More generally a word is pure if it is α -pure for some α . The empty word E is α -pure for every α . Define the group C_κ to be the quotient of H_κ by the smallest normal subgroup including the set of \sim equivalence classes of pure words.

We work towards the proof that $C_\kappa \simeq \pi_1(\mathbb{G}\mathbb{S}_\kappa, \circ_\kappa)$. Recall that the Hawaiian earring $\mathbb{E} \times \{0\}$ is a subspace of $\mathbb{G}\mathbb{S}_1$. Each copy X_α of $\mathbb{G}\mathbb{S}_1$ which appears in the wedge $\mathbb{G}\mathbb{S}_\kappa$ therefore has such a copy of the Hawaiian earring, which we denote E_α , at its “base”. Let \mathbb{E}_κ denote the union of all of these copies E_α of the Hawaiian earring.

In [Cannon and Conner 2000] is a description of an isomorphism of H_1 with the fundamental group of the Hawaiian earring $\pi_1(\mathbb{E}_1, \circ_1)$, which we give and generalize here. Let \mathcal{I} denote the set of maximal open intervals in the closed interval $[0, 1]$ minus the Cantor ternary set. The natural ordering on \mathcal{I} is order isomorphic to that of the rationals, and so every countable order type embeds in \mathcal{I} . For each $n \in \omega$ let L_n be a loop based at \circ_1 which passes exactly once around the circle $C((0, \frac{1}{n+3}), \frac{1}{n+3})$ and is injective except at 0 and 1. Given a word $W \in \mathcal{W}_1$ we let $\iota : \overline{W} \rightarrow \mathcal{I}$ be an order embedding. Let $R_\iota(W) : [0, 1] \rightarrow \mathbb{E}_1$ be the loop given by

$$R_\iota(W)(t) = \begin{cases} L_n\left(\frac{t - \inf I}{\sup I - \inf I}\right) & \text{if } W(i) = a_{0,n} \text{ and } t \in I = \iota(i), \\ L_n^{-1}\left(\frac{t - \inf I}{\sup I - \inf I}\right) & \text{if } W(i) = a_{0,n}^{-1} \text{ and } t \in I = \iota(i), \\ \circ_1 & \text{otherwise.} \end{cases}$$

If $\iota_0 : \overline{W} \rightarrow \mathcal{I}$ is a distinct order embedding, then $R_\iota(W)$ and $R_{\iota_0}(W)$ are homotopic via a straightforward homotopy whose image lies inside the common image $R_\iota(W)([0, 1]) = R_{\iota_0}(W)([0, 1])$. Thus we have a well-defined map $R : \mathcal{W} \rightarrow \pi_1(\mathbb{E}_1, \circ_1)$. Less obvious is the fact that $W \sim U$ implies $R(W) = R(U)$, so that R descends to a map, which we also name R , from H_1 to $\pi_1(\mathbb{E}_1, \circ_1)$ which is in fact an isomorphism. Each loop at \circ_1 , moreover, can be homotoped in its image to a loop which is precisely $R_\iota(W)$ for some ι and W .

We'll use these facts to produce such a map R for larger values of κ . To simplify the work we introduce the notion of reduced words. As is the case with finitary words, there is a notion of reducedness for words in \mathcal{W}_κ . We say $W \in \mathcal{W}_\kappa$ is *reduced* if $W \equiv W_0 W_1 W_2$ and $W_1 \sim E$ implies $W_1 \equiv E$. We state the following, whose proof would follow in precisely the same way as that of [Eda 1992, Theorem 1.4, Corollary 1.7].

Lemma 2.1. *Given $W \in \mathcal{W}_\kappa$ there exists a reduced word $W_0 \in \mathcal{W}_\kappa$ such that $[W] = [W_0]$ and this W_0 is unique up to \equiv . Moreover, letting W and U be reduced, there exist unique words W_0, W_1, U_0, U_1 such that:*

- (1) $W \equiv W_0 W_1$.
- (2) $U \equiv U_0 U_1$.
- (3) $W_1 \equiv U_0^{-1}$.
- (4) $W_0 U_1$ is reduced.

Let Red_κ denote the set of reduced words in \mathcal{W}_κ and for each $W \in \mathcal{W}_\kappa$ let $\text{Red}(W)$ be the reduced word such that $W \sim \text{Red}(W)$. The proof of the following is straightforward.

Lemma 2.2. *We have $\text{Red}(WU) \equiv \text{Red}(\text{Red}(W) \text{Red}(U))$ given $W \in \mathcal{W}_\kappa$ and $U \in \mathcal{W}_\kappa$. Similarly, given $W_0, W_1, W_2 \in \mathcal{W}_\kappa$ we have*

$$\text{Red}(W_0 W_1 W_2) \equiv \text{Red}(W_0 \text{Red}(W_1 W_2)) \equiv \text{Red}(\text{Red}(W_0 W_1) W_2).$$

Lemma 2.2 implies the group H_κ is isomorphic to the set Red_κ under the group operation $W * U = \text{Red}(WU)$. We give the following definition (see [Cannon and Conner 2000, Definition 3.4]):

Definition 2.3. Given a word $W \in \mathcal{W}_\kappa$ we say $\mathcal{S} \subseteq \overline{W} \times \overline{W}$ is a *cancellation* provided the following:

- (1) For $\langle i_0, i_1 \rangle \in \mathcal{S}$, we have $i_0 < i_1$.
- (2) If $\langle i_0, i_1 \rangle \in \mathcal{S}$ and $\langle i_0, i_2 \rangle \in \mathcal{S}$, then $i_2 = i_1$.
- (3) If $\langle i_0, i_1 \rangle \in \mathcal{S}$ and $\langle i_2, i_1 \rangle \in \mathcal{S}$, then $i_2 = i_0$.
- (4) If $\langle i_0, i_1 \rangle \in \mathcal{S}$ and $i_2 \in (i_0, i_1) \subseteq \overline{W}$, there exists $i_3 \in (i_0, i_1)$ such that either $\langle i_2, i_3 \rangle \in \mathcal{S}$ or $\langle i_3, i_2 \rangle \in \mathcal{S}$.

(5) If $\langle i_0, i_1 \rangle \in \mathcal{S}$, then $W(i_0) = (W(i_1))^{-1}$.

The $\langle \cdot, \cdot \rangle$ notation for ordered pairs is used here in order to avoid confusion with parenthetical notation (\cdot, \cdot) which can be interpreted as an open interval. We shall use $\langle \cdot \rangle$ to denote a generated subgroup, and the lack of a comma makes this use unambiguous.

A cancellation may be understood as a transfinite strategy for freely reducing a word. Conditions (2) and (3) imply that a cancellation is a pairing of elements in a subset of elements of \overline{W} . Condition (5) says that the pairing requires the associated letters in W to be inverses of each other. Condition (4) requires the pairing to be complete in the sense that each element between paired elements must also be paired by \mathcal{S} . Condition (4) also requires that the pairing is noncrossing in the sense that if an element i lies between two paired elements i_0 and i_1 , then the element with which i is paired must also be between i_0 and i_1 .

Zorn's lemma implies that each cancellation \mathcal{S} in a word W is included in a maximal cancellation \mathcal{S}' ; that is, $\mathcal{S} \subseteq \mathcal{S}'$ and \mathcal{S}' is not a proper subset of a cancellation in W . It turns out that a maximal cancellation reveals the reduced word representative, as happens with freely reducing a finitary word until free reductions are no longer possible. We omit the proof of the following, but it follows in precisely the same manner as [Cannon and Conner 2000, Theorem 3.9]:

Lemma 2.4. *If \mathcal{S} is a maximal cancellation for $W \in \mathcal{W}_\kappa$ then*

$$W \upharpoonright \{i \in \overline{W} \mid (\neg \exists i') (\langle i, i' \rangle \in \mathcal{S} \text{ or } \langle i', i \rangle \in \mathcal{S})\} \equiv \text{Red}(W).$$

Thus a word has only trivial cancellation if and only if that word is reduced. As a consequence, if $W \in \mathcal{W}_\kappa$ with $W \equiv \prod_{\lambda \in \Lambda} W_\lambda$ then $\text{Red}(W) \equiv \text{Red}(\prod_{\lambda \in \Lambda} \text{Red}(W_\lambda))$.

Now we define our homomorphism from Red_κ to $\pi_1(\mathbb{E}_\kappa, \circ_\kappa)$. For each $\alpha < \kappa$ and $n < \omega$ we let $L_{\alpha,n}$ be a loop based at \circ_κ which goes exactly once around the n -th circle of E_α and is injective except at 0, 1. One can use an isometry between \mathbb{E}_1 and E_α to define $L_{\alpha,n}$ from L_n if wished. Given a reduced word $W \in \text{Red}_\kappa$ and an order embedding $\iota : \overline{W} \rightarrow \mathcal{I}$ we get a loop $R_\iota(W)$ defined by

$$R_\iota(W)(t) = \begin{cases} L_{\alpha,n} \left(\frac{t - \inf I}{\sup I - \inf I} \right) & \text{if } W(i) = a_{0,n} \text{ and } t \in I = \iota(i), \\ L_{\alpha,n}^{-1} \left(\frac{t - \inf I}{\sup I - \inf I} \right) & \text{if } W(i) = a_{0,n}^{-1} \text{ and } t \in I = \iota(i), \\ \circ_\kappa & \text{otherwise.} \end{cases}$$

The check that this function on $[0, 1]$ is continuous is straightforward. Given some other order embedding $\iota_0 : \overline{W} \rightarrow \mathcal{I}$ we obtain a different loop R_{ι_0} which is homotopic to R_ι via a homotopy which is a reparametrization.

In particular we have a well-defined map $R : \text{Red}_\kappa \rightarrow \pi_1(\mathbb{E}_\kappa, \circ_\kappa)$. To see that this is a homomorphism, we let $W, U \in \text{Red}_\kappa$ and let W_0, W_1, U_0, U_1 be as in

Lemma 2.1. The loop $R(W_1)$ is readily seen to be the inverse of $R(U_0)$. The word W_0U_1 is reduced and therefore we have

$$\begin{aligned}
 R(W * U) &= R(\text{Red}(WU)) \\
 &= R(W_0U_1) \\
 &\simeq R(W_0)R(U_0)^{-1}R(U_0)R(U_1) \\
 &= R(W_0)R(W_1)R(U_0)R(U_1) \\
 &= R(W_0W_1)R(U_0U_1) \\
 &= R(W)R(U).
 \end{aligned}$$

Suppose now that $W \in \text{Red}_\kappa$ is in the kernel of R . Suppose for contradiction that $W \neq E$. We'll construct a cancellation \mathcal{S} of W to obtain a contradiction. Fix an order embedding $\iota : \overline{W} \rightarrow \mathcal{I}$. Let $H : [0, 1] \times [0, 1] \rightarrow \mathbb{E}_\kappa$ be a nullhomotopy of $R_i(W)$. That is, $H(t, 0) = R_i(W)(t)$ and $H(0, s) = H(1, s) = H(t, 1)$ for all $t, s \in [0, 1]$. For each $I \in \mathcal{I}$ we let $m(I)$ signify the midpoint $m(I) = \frac{1}{2}(\sup I + \inf I)$. Consider the set of points $M = \{(m(\iota(i)), 0)\}_{i \in \overline{W}} \subseteq [0, 1] \times [0, 1]$. For each point $p \in M$ we consider its path component P_p in $[0, 1] \times [0, 1] \setminus H^{-1}(\circ_\kappa)$. Each $p \in M$ is associated with a unique interval $\iota(i_p)$ and therefore with a unique element $i_p \in \overline{W}$, and each $i \in \overline{W}$ is in turn associated with a unique point $p \in M$. Moreover, the natural order on points in M is isomorphic with the elements of \overline{W} in this association.

Fixing $p \in M$ the set $P_p \cap M$ is necessarily finite, because each element of $P_p \cap M$ corresponds to exactly one occurrence of a loop $L_{\alpha, n}$ or of its inverse, for a fixed α and n , and there are only finitely many such occurrences since there are finitely many occurrences of $a_{\alpha, n}^{\pm 1}$ in W . Write $P_p \cap M = \{p_0, p_1, \dots, p_j\}$ listing elements in the natural order. By modifying H to have output \circ_κ outside of P_p , we see that H witnesses a nullhomotopy of the loop $R_i(W \upharpoonright \{i_{p_0}, \dots, i_{p_j}\})$, which lies entirely in the n -th circle of E_α . Then there are exactly as many i_{p_k} for which $W(i_{p_k}) = a_{\alpha, n}$ as there are for which $W(i_{p_k}) = a_{\alpha, n}^{-1}$. Select neighboring points p_k, p_{k+1} which are of opposite parity and let $\langle i_{p_k}, i_{p_{k+1}} \rangle \in \mathcal{S}$. Among the remaining points $P_p \cap M \setminus \{p_k, p_{k+1}\}$ select two which are neighboring under the new order and add this ordered pair to \mathcal{S} . Continue in this way until all elements of $P_p \cap M$ are used. Perform this procedure on all path components P_p for $p \in M$. It is straightforward to check that \mathcal{S} satisfies the rules of a cancellation. We have obtained our contradiction. Thus R is an injection.

We check that R is a surjection. Let $L : [0, 1] \rightarrow \mathbb{E}_\kappa$ be a loop at \circ_κ . Let \mathcal{J} be the set of maximal open intervals in $[0, 1] \setminus L^{-1}(\circ_\kappa)$. This set is countable and has a natural ordering. For each restriction $L \upharpoonright \bar{J}$, where $J \in \mathcal{J}$, there is a homotopy $H_J : \bar{J} \times [0, 1] \rightarrow L(\bar{J})$ to a loop $L_J : \bar{J} \rightarrow L(\bar{J})$ which is either constant, or

$L_{\alpha,n}(\frac{t-\inf J}{\sup J-\inf J})$ or $L_{\alpha,n}^{-1}(\frac{t-\inf J}{\sup J-\inf J})$. By gluing these homotopies together we get a homotopy of L to a loop whose restriction to each nonconstant interval \bar{J} is of the form $L_{\alpha,n}(\frac{t-\inf J}{\sup J-\inf J})$ or $L_{\alpha,n}^{-1}(\frac{t-\inf J}{\sup J-\inf J})$.

Thus assuming L is of this form, we define a word $W : \mathcal{J} \rightarrow \mathcal{A}_\kappa$ by letting $W(J) = a_{\alpha,n}^{\pm 1}$ where the α, n and superscript are determined in the straightforward way. That the mapping W is indeed a word (no n in the subscript occurs infinitely often) follows from the fact that L is continuous. Let \mathcal{S} be a maximal cancellation on W . This \mathcal{S} can be used to homotope L so that the new associated word is $\text{Red}(W)$. More explicitly, we define $H : [0, 1] \times [0, 1] \rightarrow \mathbb{E}_\kappa$ by having $H(t, s) = L(t)$ if t does not lie inside an interval $(\inf J_0, \sup J_1)$ where $\langle J_0, J_1 \rangle \in \mathcal{S}$. If a point $(t, s) \in [0, 1] \times [0, 1]$ lies on the semicircle determined by points $(t_0, 0)$ and $(t_1, 0)$ which is perpendicular to $[0, 1] \times \{0\}$ where $t_0 \in J_0, t_1 \in J_1$ and $\langle J_0, J_1 \rangle \in \mathcal{S}$ with $L(t_0) = L(t_1)$, we let $H(t, s) = L(t_0) = L(t_1)$. Give H output \circ_κ everywhere else. That H is continuous and produces a loop $H(t, 1)$ as described is intuitive but tedious to check. Thus we may now assume that the associated word W is reduced. By reparametrizing L we may make it so that all the intervals in \mathcal{J} are elements in \mathcal{I} , which immediately gives an order embedding ι of \bar{W} to \mathcal{I} for which $L = R_\iota(W)$. We have shown surjectivity and finished the proof of the following:

Lemma 2.5. *The function $R : \text{Red}_\kappa \rightarrow \pi_1(\mathbb{E}_\kappa, \circ_\kappa)$ is an isomorphism.*

We now approach the isomorphism $\mathcal{C}_\kappa \simeq \pi_1(\mathbb{G}\mathbb{S}_\kappa, \circ_\kappa)$. For finite values of κ this can be done by a straightforward argument in which van Kampen’s Theorem is iterated finitely many times, as is done in [Eda and Fischer 2016, Section 4]. We present an argument which works for every cardinal κ .

Lemma 2.6. *Given $\epsilon > 0$ and a loop $L : [0, 1] \rightarrow \mathbb{G}\mathbb{S}_\kappa$ based at \circ_κ , there is a loop homotopic to L whose image is of diameter at most ϵ .*

Proof. Let \mathcal{J} be the set of maximal open intervals in $[0, 1] \setminus L^{-1}(\circ_\kappa)$. There are only finitely many intervals $J \in \mathcal{J}$ for which the diameter of the image $\text{diam}(L \upharpoonright J)$ is at least $\frac{1}{2}\epsilon$. But for every $J \in \mathcal{J}$ the loop $L \upharpoonright \bar{J}$ lies entirely in a contractible space, a homeomorph of $\mathbb{G}\mathbb{S}_1$. In particular each restriction $L \upharpoonright \bar{J}$ is nullhomotopic. Thus letting $\mathcal{J}' \subseteq \mathcal{J}$ be the set of those intervals whose images are of diameter at least $\frac{1}{2}\epsilon$ we have L homotopic to the loop $L' : [0, 1] \rightarrow \mathbb{G}\mathbb{S}_\kappa$ given by

$$L'(t) = \begin{cases} L(t) & \text{if } t \notin \bigcup \mathcal{J}', \\ \circ_\kappa & \text{if } t \in \bigcup \mathcal{J}', \end{cases}$$

which has diameter at most ϵ . □

Let each copy of $(0, 0, 1)$ in the copies of $\mathbb{G}\mathbb{S}_1$ whose wedge forms $\mathbb{G}\mathbb{S}_\kappa$ be called a “cone tip”. Let $\mathbb{G}\mathbb{S}'_\kappa$ denote the space $\mathbb{G}\mathbb{S}_\kappa$ minus the set of cone tips. The following is easy to see.

Lemma 2.7. *The space $\mathbb{G}\mathbb{S}'_\kappa$ strongly deformation retracts to \mathbb{E}_κ .*

Now we let $U \subseteq \mathbb{G}\mathbb{S}_\kappa$ be the open set which is the union over all $\alpha < \kappa$ of images $E_\alpha \times [0, \frac{2}{3})$ in the cone over E_α . For each $\alpha < \kappa$ we let V_α be the image of $E_\alpha \times (\frac{1}{3}, 1]$ in the cone over E_α . An application of van Kampen’s theorem gives the following.

Theorem 2.8. *The isomorphism $R : \text{Red}_\kappa \rightarrow \pi_1(\mathbb{E}_\kappa, \circ_\kappa)$ descends to an isomorphism $R_{\mathcal{C}_\kappa} : \mathcal{C}_\kappa \rightarrow \pi_1(\mathbb{G}\mathbb{S}_\kappa, \circ_\kappa)$.*

We immediately obtain the following (cf. [Bogopolski and Zastrow 2012, Theorem 8.1]):

Corollary 2.9. *A reduced word W is in the kernel of the map $\text{Red}_\kappa \rightarrow \mathcal{C}_\kappa$ if and only if there exist finitely many intervals I_0, \dots, I_p such that $W \upharpoonright I_j$ is pure for each j and $\text{Red}(W \upharpoonright (\overline{W} \setminus \bigcup_{j=0}^p I_j)) = E$.*

Lemma 2.10. *Suppose that we have a word $V \equiv \prod_{n \in \omega} V_n$ with $V \in \text{Red}_\kappa$, and that the following properties are verified:*

- (1) *Any interval $I \subseteq \overline{V}$ such that $V \upharpoonright I$ is pure is a subinterval of $\overline{\prod_{n=0}^m V_n}$ for some $m \in \omega$.*
- (2) *For each $n \in \omega$ there exists $j_n \in \omega$ such that $|\{i \in \overline{V}_n \mid \text{proj}_1(V_n(i)) = j_n\}| > \sum_{m \neq n} |\{i \in \overline{V}_m \mid \text{proj}_1(V_m(i)) = j_n\}|$.*

Then $[[V]] \neq [[E]]$ in \mathcal{C}_κ .

Proof. Suppose for contradiction that $[[V]] = [[E]]$, so by Corollary 2.9 we obtain a finite collection of intervals I_0, \dots, I_p in \overline{V} such that $V \upharpoonright I_k$ is pure for each $0 \leq k \leq p$ and $\text{Red}(V \upharpoonright (\overline{V} \setminus \bigcup_{k=0}^p I_k)) = E$. Let \mathcal{S} be a maximal cancellation of $V \upharpoonright (\overline{V} \setminus \bigcup_{k=0}^p I_k)$. We know by (1) that $\bigcup_{k=0}^p I_k \subseteq \overline{\prod_{n=0}^m V_n}$ for some $m \in \omega$. All elements of $Z = \{i \in \overline{V}_{m+1} \mid \text{proj}_1(V_{m+1}(i)) = j_{m+1}\}$ must participate in \mathcal{S} since $\text{Red}(V \upharpoonright (\overline{V} \setminus \bigcup_{k=0}^p I_k)) = E$, but since V_{m+1} is reduced we know that the elements of Z are paired with elements of $\overline{V} \setminus (\overline{V}_{m+1} \cup \bigcup_{k=0}^p I_k)$, but this is impossible by condition (2). □

For a reduced word W we let $[[W]]$ denote the equivalence class of W in \mathcal{C}_κ and if $[[W]] = [[U]]$ we write $W \approx U$.

Theorem 2.11. *For each cardinal κ we have*

$$|\mathcal{C}_\kappa| = \begin{cases} 1 & \text{if } \kappa = 0, \\ \kappa^{\aleph_0} & \text{if } \kappa \geq 1. \end{cases}$$

Proof. We have already seen that the formula holds in case $\kappa = 0, 1$. Suppose $\kappa \geq 2$. Notice that the space $\mathbb{G}\mathbb{S}_\kappa$ has $2^{\aleph_0} \cdot \kappa = \max\{2^{\aleph_0}, \kappa\}$ points in it. Every continuous function from $[0, 1]$ to the metric space $\mathbb{G}\mathbb{S}_\kappa$ is totally determined by the restriction

to $[0, 1] \cap \mathbb{Q}$. Thus there are at most $(\max\{2^{\aleph_0}, \kappa\})^{\aleph_0} = \kappa^{\aleph_0}$ loops in the space, so in particular $|\mathcal{C}_\kappa| \leq \kappa^{\aleph_0}$. We must show $|\mathcal{C}_\kappa| \geq \kappa^{\aleph_0}$.

If $2 \leq \kappa \leq 2^{\aleph_0}$ then let Σ be a collection of infinite subsets of ω such that for distinct $X, Y \in \Sigma$ we have $X \cap Y$ finite and such that $|\Sigma| = 2^{\aleph_0}$. Such a construction is straightforward, see for example [Kunen 1980, Chapter II, Theorem 1.3]. For each $X \in \Sigma$ let $X = \{n_{0,X}, n_{1,X}, \dots\}$ be the enumeration of X in the natural order. Let

$$W_X \equiv a_{0,n_{0,X}} a_{1,n_{1,X}} a_{0,n_{2,X}} a_{1,n_{3,X}} \cdots$$

Since W_X uses only positive letters it is clear that W_X and also any deletion of finitely many letters of W_X is a reduced word. By the conditions on Σ it is clear that $[[W_X]] \neq [[W_Y]]$ if $X \neq Y$. Then $\kappa^{\aleph_0} \leq |\mathcal{C}_\kappa|$.

Suppose that $2^{\aleph_0} < \kappa$ and that $\kappa^{\aleph_0} = \kappa$. Let $f : \kappa \times \omega \rightarrow \kappa$ be an injection and for each $\alpha < \kappa$ we define $W_\alpha \equiv a_{f(\alpha,0),0} a_{f(\alpha,1),1} \cdots$. It is clear that $[[W_\alpha]] \neq [[W_\beta]]$ for distinct $\alpha, \beta < \kappa$.

Suppose finally that $2^{\aleph_0} < \kappa$ and that $\kappa^{\aleph_0} > \kappa$. Let X be the set of all functions from ω to κ and consider two functions $\sigma_0, \sigma_1 \in X$ to be equivalent if they are eventually identical: for some $m \in \omega$ we have $\sigma_0(m+n) = \sigma_1(m+n)$ for all $n \in \omega$. Each equivalence class is of cardinality κ , so there are exactly κ^{\aleph_0} distinct equivalence classes. Letting $Y \subset X$ be a selection from each equivalence class we define a map $Y \rightarrow \mathcal{C}_\kappa$ by letting $\sigma \mapsto W_\sigma$ where $W_\sigma \equiv a_{f(\sigma(0),0)} a_{f(\sigma(1),1)} \cdots$ and again $f : \kappa \times \omega \rightarrow \kappa$ is an injection. It is easy to see that for distinct elements of Y the assigned words are not equivalent in \mathcal{C}_κ . □

An interval I in a totally ordered set Λ is *initial* if it is a union of intervals of the form $(-\infty, i]$ and is *terminal* if a union of intervals of form $[i, \infty)$ (an initial or terminal interval may be empty). Given a nonempty word $W \in \text{Red}_\kappa$ there exists a unique maximal initial interval I_0 of \overline{W} for which there exists a terminal interval $I_1 \subseteq \overline{W}$ such that $W \upharpoonright I_0 \equiv (W \upharpoonright I_1)^{-1}$. By the proof of [Eda 1992, Corollary 1.6] the maximal such initial interval I_0 and the accompanying I_1 are disjoint and $\overline{W} \setminus (I_0 \cup I_1)$ is nonempty, and this set is clearly an interval, say I_2 . Thus $W \equiv (W \upharpoonright I_0)(W \upharpoonright I_2)(W \upharpoonright I_0)^{-1}$ and we call the word $W \upharpoonright I_2$ the *cyclic reduction* of W . Clearly if U is the cyclic reduction of W then the cyclic reduction of U is again U , so cyclic reduction is an idempotent operation. A word whose cyclic reduction is itself is called *cyclically reduced*. It is clear from Lemma 2.4 that a word U is cyclically reduced if and only if the word U^n is reduced for all $n \geq 1$, thus if and only if U^2 is reduced.

3. Theorem A

We begin with a description of the overall strategy and then describe the structure of this section. An isomorphism between two cone groups \mathcal{C}_{κ_0} and \mathcal{C}_{κ_1} will be

constructed by induction on specially defined subgroups. We cannot expect that such an isomorphism will be imposed by a homomorphism $\text{Red}_{\kappa_0} \rightarrow \text{Red}_{\kappa_1}$. However, the idea is that establishing careful correspondences between certain words in Red_{κ_0} and certain words in Red_{κ_1} will allow us to ultimately produce homomorphisms $\phi_0 : \text{Red}_{\kappa_0} \rightarrow \mathcal{C}_{\kappa_1}$ and $\phi_1 : \text{Red}_{\kappa_1} \rightarrow \mathcal{C}_{\kappa_0}$ which will descend to isomorphisms $\Phi_0 : \mathcal{C}_{\kappa_0} \rightarrow \mathcal{C}_{\kappa_1}$ and $\Phi_1 : \mathcal{C}_{\kappa_1} \rightarrow \mathcal{C}_{\kappa_0}$ with $\Phi_1 = \Phi_0^{-1}$.

What sort of correspondences between words should be produced? They should not be so rigid as to produce a homomorphism $\text{Red}_{\kappa_0} \rightarrow \text{Red}_{\kappa_1}$. Rather, they should be forgiving enough to produce the homomorphisms ϕ_0 and ϕ_1 described above. The correspondences should also agree with each other so that the ϕ_0 and ϕ_1 are well defined.

Each word in Red_{κ_0} and Red_{κ_1} may be decomposed in a natural way as a concatenation of maximal pure subwords (the index over which concatenation is written is unique up to order isomorphism and is called the *p-index*). Taking concatenations over subintervals of the p-index gives us words which are recognizable pieces of the original word (which we will call *p-chunks*). There is a natural way of comparing certain words $W \in \text{Red}_{\kappa_0}$ with other words $U \in \text{Red}_{\kappa_1}$ via an order isomorphism between a subset of the p-index of W and that of U . These subsets will be large enough to “capture” any interval of the p-index, up to deletion of finitely many elements, and there will be a correspondence between the p-chunks of W and those of U . The bijections between the subsets of the p-indices will honor word concatenation (up to finite deletion of pure subwords) and will allow us to define isomorphisms between the subgroups of \mathcal{C}_{κ_0} and \mathcal{C}_{κ_1} which are generated by the p-chunks of the words on which we have defined such bijections.

In order to have the isomorphisms be well defined, it is essential that the imposed correspondences between p-chunks are in agreement with each other. That is, suppose that $W_0, W_1 \in \text{Red}_{\kappa_0}$ and $U_0, U_1 \in \text{Red}_{\kappa_1}$ and W_i is made to correspond to U_i for $i = 0, 1$. If $W \in \text{Red}_{\kappa_0}$ is a p-chunk of each of W_0 and W_1 then we should be able to make W correspond to a word $U \in \text{Red}_{\kappa_0}$ in a way that honors the correspondences $W_i \leftrightarrow U_i$, so any choice of such a U should be independent of whether we are considering W as a p-chunk of W_0 or of W_1 , up to the equivalence \approx .

It will be necessary to be able to define many such correspondences between words, so as to make the isomorphism between subgroups of \mathcal{C}_{κ_0} and \mathcal{C}_{κ_1} have larger and larger domain and range. Keeping such new correspondences in agreement with the previously defined ones requires us to consider concatenations of words on which such bijections have already been defined, concatenations of order type ω and of order type \mathbb{Q} are of particular concern. If we can continue to do this for sufficiently many steps (2^{\aleph_0} steps will suffice) then we can succeed in the construction.

This section is organized into subsections for the sake of clarity. We introduce

and prove some basic properties of p -chunks in Section 3A. In Section 3B we will make precise the concept of a “sufficiently large” subset of an ordered set. In Section 3C we define what it means for bijections between sufficiently large subsets of p -indices to honor word concatenation (up to deletion of finitely many pure subwords). In Section 3D we give some baby steps towards defining such bijections on more words, and in Sections 3E and 3F we show how to extend such notions for ω - and \mathbb{Q} -type concatenations, respectively. Finally in Section 3G we combine all the previous ideas to prove Theorems A and B.

3A. P -chunks. Let κ be a cardinal. For each word $W \in \text{Red}_\kappa$ we have a decomposition of the domain $\overline{W} \equiv \prod_{\lambda \in \Lambda} \Lambda_\lambda$ such that each Λ_λ is a nonempty maximal interval with $W \upharpoonright \Lambda_\lambda$ pure. We’ll call this decomposition the *pure decomposition of the domain of W* . Write $W \equiv_p \prod_{\lambda \in \Lambda} W_\lambda$ to express that $\overline{W} \equiv \prod_{\lambda \in \Lambda} \overline{W}_\lambda$ is the p -decomposition of the domain of W , and call this writing $W \equiv_p \prod_{\lambda \in \Lambda} W_\lambda$ the *p -decomposition of W* and Λ the *p -index*, denoted $p^*(W)$. By definition we therefore have $E \equiv_p \prod_{\lambda \in \Lambda} W_\lambda$ with $\Lambda = \emptyset$. If $W \equiv_p \prod_{\lambda \in p^*(W)} W_\lambda$ and I is an interval in $p^*(W)$ then let $W \upharpoonright_p I$ denote the word $\prod_{\lambda \in I} W_\lambda$. Call a word W' a *p -chunk* of W if for some interval $I \subseteq p^*(W)$ we have $W' \equiv W \upharpoonright_p I$. For a given $W \in \text{Red}_\kappa$ we let $p\text{-chunk}(W)$ denote the set of p -chunks of W . A pure p -chunk of a word $W \equiv_p \prod_{\lambda \in \Lambda} W_\lambda$ will, of course, either be empty or one of the W_λ . Notice as well that an equivalence $W \equiv U$ immediately gives an order isomorphism from $p^*(W)$ to $p^*(U)$.

Lemma 3.1. *Suppose that $W \equiv_p \prod_{\lambda \in \Lambda} W_\lambda$ and $U \equiv_p \prod_{\lambda' \in \Lambda'} U_{\lambda'}$. Then there exists a (possibly empty) initial interval $I \subseteq \Lambda$ and a (possibly empty) terminal interval $I' \subseteq \Lambda'$ such that either:*

- (i) $\text{Red}(WU) \equiv_p \prod_{\lambda \in I} W_\lambda \prod_{\lambda' \in I'} U_{\lambda'}$; or
- (ii) *there exist $\lambda_0 \in \Lambda$ which is the least element strictly above all elements in I , $\lambda_1 \in \Lambda'$ which is the greatest element strictly below all elements of I' and*

$$\text{Red}(WU) \equiv_p \left(\prod_{\lambda \in I} W_\lambda \right) V \left(\prod_{\lambda' \in I'} U_{\lambda'} \right)$$

where $V \equiv \text{Red}(W_{\lambda_0} U_{\lambda_1}) \neq E$ is pure.

Proof. Since both W and U are reduced we have reduced words W_0, W_1, U_0, U_1 such that $W \equiv W_0 W_1, U \equiv U_0 U_1, W_1 \equiv U_0^{-1}$ and $W_0 U_1$ is reduced, by Lemma 2.1. Select $I_0 \subseteq \Lambda$ to be a maximal initial interval for which $\bigcup_{\lambda \in I_0} \overline{W}_\lambda \subseteq \overline{W}_0$. Select $I'_1 \subseteq \Lambda'$ to be a maximal terminal interval such that $\bigcup_{\lambda' \in I'_1} \overline{U}_{\lambda'} \subseteq \overline{U}_1$.

Suppose $\prod_{\lambda \in I_0} W_\lambda \equiv W_0$ and $\prod_{\lambda' \in I'_1} U_{\lambda'} \equiv U_1$. If I_0 has a maximal element λ_0 and I'_1 has a minimal element λ_1 such that the words W_{λ_0} and U_{λ_1} are both α -pure for some α , then we let $I = I_0 \setminus \{\lambda_0\}$ and $I' = I'_1 \setminus \{\lambda_1\}$ and $V \equiv W_{\lambda_0} U_{\lambda_1}$ and

obviously condition (ii) holds. If there are no such maximal and minimal elements then condition (i) holds.

Suppose that $\prod_{\lambda \in I_0} W_\lambda \not\equiv W_0$. Then there exists some λ_0 which is the least element strictly above all elements in I_0 and nonempty words $W_{\lambda_0,0}$ and $W_{\lambda_0,1}$ such that

$$W_{\lambda_0} \equiv W_{\lambda_0,0}W_{\lambda_0,1}; \quad W_0 \equiv_p \left(\prod_{\lambda \in I_0} W_\lambda \right) W_{\lambda_0,0}; \quad W_1 \equiv_p W_{\lambda_0,1} \left(\prod_{\lambda \in \Lambda \setminus (I_0 \cup \{\lambda_0\})} W_\lambda \right).$$

If in addition $\prod_{\lambda' \in I_1} U_{\lambda'} \equiv U_1$ then $\Lambda' \setminus I_1$ has a maximum element λ_1 which satisfies $U_{\lambda_1} \equiv W_{\lambda_0,1}^{-1}$. Thus we let $I = I_0 \setminus \{\lambda_0\}$ and $I' = I_1$ and $V \equiv W_{\lambda_0,0} \equiv \text{Red}(W_{\lambda_0} U_{\lambda_1})$ and we have condition (ii). On the other hand, if in addition we have $\prod_{\lambda' \in I_1} U_{\lambda'} \not\equiv U_1$ then $\Lambda' \setminus I_1$ has a maximum element λ_1 and there exist nonempty words $U_{\lambda_1,0}$ and $U_{\lambda_1,1}$ for which

$$U_{\lambda_1} \equiv U_{\lambda_1,0}U_{\lambda_1,1}; \quad U_0 \equiv_p \left(\prod_{\lambda' \in \Lambda' \setminus I_1} U_{\lambda'} \right) U_{\lambda_1,0}; \quad U_1 \equiv_p U_{\lambda_1,1} \left(\prod_{\lambda' \in I_1} U_{\lambda'} \right).$$

Then we let $V \equiv W_{\lambda_0,0}V_{\lambda_1,1} \equiv \text{Red}(W_{\lambda_0}U_{\lambda_1})$ and $I = I_0$ and $I' = I_1$ and condition (ii) holds.

The case where $\prod_{\lambda \in I_0} W_\lambda \equiv W_0$ and $\prod_{\lambda' \in I_1'} U_{\lambda'} \not\equiv U_1$ follows from dualizing the proof of an earlier case. \square

Lemma 3.2. *Suppose that $X \subseteq \text{Red}_\kappa$. For each nonempty element W of the subgroup $\langle \bigcup_{U \in X} \text{p-chunk}(U) \rangle \leq \text{Red}_\kappa$, if $W \equiv_p \prod_{\lambda \in \Lambda} W_\lambda$, then there exist nonempty intervals I_0, \dots, I_n in Λ such that:*

- (i) $\Lambda \equiv \prod_{i=0}^n I_i$.
- (ii) For each $0 \leq i \leq n$, at least one of the following holds:
 - (a) I_i is a singleton $\{\lambda\}$ such that W_λ is the reduction of a finite concatenation of pure p -chunks of elements in $X^{\pm 1}$.
 - (b) $\prod_{\lambda \in I_i} W_\lambda$ is a p -chunk of some element in $X^{\pm 1}$.

Proof. The elements of $\langle \bigcup_{U \in X} \text{p-chunk}(U) \rangle$ are of form $\text{Red}(U_0 \cdots U_l)$ where each U_i is a p -chunk of an element of $X^{\pm 1}$. The claim will follow by an induction on the number l . If $l = 0$ or $l = 1$ then we are already done. Supposing that the claim holds for l , we suppose $W \equiv \text{Red}(U_0 \cdots U_{l+1}) \equiv \text{Red}(\text{Red}(U_0 \cdots U_l)U_{l+1})$ and let $W' \equiv \text{Red}(U_0 \cdots U_l)$ and $U \equiv U_{l+1}$. Let $W' \equiv_p \prod_{\lambda \in \Lambda} W_\lambda$ and $U \equiv_p \prod_{\lambda' \in \Lambda'} U_{\lambda'}$. By induction we have for the word W' a decomposition $I_0, \dots, I_{n'}$ as in the conclusion of this lemma. We can select an initial interval $I \subseteq \Lambda$ and a terminal interval $I' \subseteq \Lambda'$ as in the conclusion of Lemma 3.1. Consider the two possible cases in Lemma 3.1 for the word $W \equiv \text{Red}(W'U)$. If case (i) of Lemma 3.1 holds then we can decompose the p -chunk total order for W into at most $n' + 1$

intervals as in (i) and (ii) of the statement of the lemma that we are proving. If case (ii) of Lemma 3.1 holds then we can decompose the p-chunk total order for W into at most $n' + 2$ intervals, at least one of which will be a singleton. \square

We say a subgroup G of Red_κ is *p-fine* if each p-chunk U of each $W \in G$ is also in G (cf. [Eda 1999, page 600]).

Lemma 3.3. *If $X \subseteq \text{Red}_\kappa$ then the subgroup $\langle \bigcup_{U \in X} \text{p-chunk}(U) \rangle \leq \text{Red}_\kappa$ is p-fine. This is the smallest p-fine subgroup including the set X .*

Proof. This follows immediately from the characterization in Lemma 3.2. \square

Given a set $X \subseteq \text{Red}_\kappa$ we'll denote the subgroup

$$\left\langle \bigcup_{U \in X} \text{p-chunk}(U) \right\rangle \leq \text{Red}_\kappa$$

by $\text{Pfine}(X)$.

Lemma 3.4. *If $X \subseteq \text{Red}_\kappa$ then there are at most $(|X| + 1) \cdot \aleph_0$ pure p-chunks of elements in $\text{Pfine}(X)$.*

Proof. If X is empty then $\text{Pfine}(X)$ has only the empty word and so there is one pure p-chunk of elements in $\text{Pfine}(X)$ and the claim is true. If X is not empty then there are there are at most $|X| \cdot \aleph_0$ pure p-chunks of elements in X (since a p-index is at most countable), and therefore we have at most $|X| \cdot \aleph_0 \cdot \aleph_0 = |X| \cdot \aleph_0$ finite products of p-chunks, or their inverses, of elements in X . By Lemma 3.2 we know all pure p-chunks of elements in $\text{Pfine}(X)$ arise in this way and so we are also done in this case. \square

3B. Close subsets. We take a diversion through a concept which will be useful in later subsections.

Definition 3.5. Let Λ be a totally ordered set. We say $\Lambda_0 \subseteq \Lambda$ is *close in Λ* , and write $\text{Close}(\Lambda_0, \Lambda)$, if every infinite interval in Λ has nonempty intersection with Λ_0 .

The idea of a close subset Λ_0 in Λ is that there are no infinite gaps in Λ which miss elements in Λ_0 . We give some elementary examples. If Λ_0 is cofinite in Λ then $\text{Close}(\Lambda_0, \Lambda)$. Any infinite subset of the ordered set ω of natural numbers is close. A subset of \mathbb{Z} is close precisely when it contains numbers of arbitrarily large positive numbers and arbitrarily large negative numbers. A subset of \mathbb{Q} is close when it is dense.

Lemma 3.6. *The following hold:*

- (i) *If $\text{Close}(\Lambda_0, \Lambda)$ then for any infinite interval $I \subseteq \Lambda$ the set $I \cap \Lambda_0$ is infinite.*
- (ii) *If $\Lambda_2 \subseteq \Lambda_1 \subseteq \Lambda_0$ with $\text{Close}(\Lambda_{i+1}, \Lambda_i)$ for $i = 0, 1$, then $\text{Close}(\Lambda_2, \Lambda_0)$.*

(iii) If we have that $\Lambda \equiv \prod_{\theta \in \Theta} \Lambda_\theta$, $\text{Close}(\Lambda_{\theta,0}, \Lambda_\theta)$ for each $\theta \in \Theta$, and also $\text{Close}(\{\theta \in \Theta \mid \Lambda_{\theta,0} \neq \emptyset\}, \Theta)$, then $\text{Close}(\bigcup_{\theta \in \Theta} \Lambda_{\theta,0}, \Lambda)$.

(iv) If I_0 is an interval in Λ and $\text{Close}(\Lambda_0, \Lambda)$, then $\text{Close}(\Lambda_0 \cap I_0, I_0)$.

Proof. (i) If instead $I \cap \Lambda_0 = \{\lambda_0, \lambda_1, \dots, \lambda_n\}$ with $\lambda_i < \lambda_{i+1}$ then at least one of the intervals $I \cap (-\infty, \lambda_0)$, (λ_0, λ_1) , \dots , $(\lambda_{n-1}, \lambda_n)$, $I \cap (\lambda_n, \infty)$ in Λ is infinite, but each has empty intersection with Λ_0 and this is a contradiction.

(ii) Let $I \subseteq \Lambda_0$ be an infinite interval. Notice that $I \cap \Lambda_1$ is infinite by (i) and so $I \cap \Lambda_1$ is an infinite interval in Λ_1 , so $I \cap \Lambda_2 = (I \cap \Lambda_1) \cap \Lambda_2 \neq \emptyset$.

(iii) Let $I \subseteq \Lambda$ be an infinite interval. The set $\underline{I} = \{\theta \in \Theta \mid I \cap \Lambda_\theta \neq \emptyset\}$ is an interval in Θ . If \underline{I} is finite then as $I = \bigsqcup_{\theta \in \underline{I}} (I \cap \Lambda_\theta)$ there is some $\theta_0 \in \underline{I}$ for which $|I \cap \Lambda_{\theta_0}| = \infty$, and as $I \cap \Lambda_{\theta_0}$ is an infinite interval in Λ_{θ_0} we see that $I \cap \Lambda_{\theta_0,0} \neq \emptyset$, so $I \cap \bigcup_{\theta \in \Theta} \Lambda_{\theta,0} \neq \emptyset$. If \underline{I} is infinite then $\underline{I} \cap \{\theta \in \Theta \mid \Lambda_{\theta,0} \neq \emptyset\}$ is infinite by (i), as we are assuming $\text{Close}(\{\theta \in \Theta \mid \Lambda_{\theta,0} \neq \emptyset\}, \Theta)$. Then there exists some $\theta_0 \in \underline{I} \cap \{\theta \in \Theta \mid \Lambda_{\theta,0} \neq \emptyset\}$ for which $I \supseteq \Lambda_{\theta_0}$. Thus $I \cap \Lambda_{\theta_0,0} \neq \emptyset$.

(iv) This is obvious. \square

If $\text{Close}(\Lambda_0, \Lambda)$ then for each interval $I \subseteq \Lambda$ we let $\alpha(I, \Lambda_0)$ denote the smallest interval in Λ which includes the set $I \cap \Lambda_0$. In other words $\alpha(I, \Lambda_0) = \bigcup_{\lambda_0, \lambda_1 \in I \cap \Lambda_0, \lambda_0 \leq \lambda_1} [\lambda_0, \lambda_1]$ where the intervals $[\lambda_0, \lambda_1]$ are being considered in Λ .

Lemma 3.7. Let $\text{Close}(\Lambda_0, \Lambda)$ and $I \subseteq \Lambda$ be an interval.

(i) The inclusion $I \supseteq \alpha(I, \Lambda_0)$ holds and $\alpha(I, \Lambda_0) = \alpha(\alpha(I, \Lambda_0), \Lambda_0)$.

(ii) The set $I \setminus \alpha(I, \Lambda_0)$ is the disjoint union of an initial and terminal subinterval $I_0, I_1 \subseteq I$ (either subinterval could be empty) with $|I_0|, |I_1| < \infty$.

Proof. (i) The claimed inclusion is obvious. For the claimed equality it is therefore sufficient to prove that $\alpha(I, \Lambda_0) \subseteq \alpha(\alpha(I, \Lambda_0), \Lambda_0)$. We let $\lambda \in \alpha(I, \Lambda_0)$ be given. Select $\lambda_0, \lambda_1 \in I \cap \Lambda_0$ such that $\lambda_0 \leq \lambda \leq \lambda_1$. Then $\lambda_0, \lambda_1 \in \alpha(I, \Lambda_0) \cap \Lambda_0$ and $\lambda_0 \leq \lambda \leq \lambda_1$, so $\lambda \in \alpha(\alpha(I, \Lambda_0), \Lambda_0)$.

(ii) If $I \cap \Lambda_0 = \emptyset$ then I is finite (since $\text{Close}(\Lambda_0, \Lambda)$) and we can let $I_0 = \emptyset$ and $I_1 = I$. If $I \cap \Lambda_0 \neq \emptyset$ then we let $I_0 = \{\lambda \in I \mid (\forall \lambda_0 \in I \cap \Lambda_0) \lambda < \lambda_0\}$ and $I_1 = \{\lambda \in I \mid (\forall \lambda_0 \in I \cap \Lambda_0) \lambda > \lambda_0\}$. Clearly $I \equiv I_0 \alpha(I, \Lambda_0) I_1$. Each of I_0 and I_1 is a subinterval of I and therefore a subinterval of Λ as well. If, say, I_0 is infinite then $I_0 \cap \Lambda_0 \neq \emptyset$ but this is an obvious contradiction. \square

We will say that two totally ordered sets Λ and Θ are *close-isomorphic* if there exist $\Lambda_0 \subseteq \Lambda$ and $\Theta_0 \subseteq \Theta$ with $\text{Close}(\Lambda_0, \Lambda)$, $\text{Close}(\Theta_0, \Theta)$ and Λ_0 order isomorphic to Θ_0 ; and if ι is an order isomorphism between such a Λ_0 and Θ_0 then we will call ι a *close order isomorphism from Λ to Θ* . It is obvious that the inverse of a close order isomorphism from Λ to Θ is a close order isomorphism from Θ to Λ .

From a close order isomorphism (abbreviated *coi*) between totally ordered sets one obtains a reasonable way of identifying intervals in one totally ordered set with intervals in the other, which we now describe. Given *coi* ι between Λ and Θ , with Λ_0 and Θ_0 being the respective domain and range of ι , and an interval $I \subseteq \Lambda$ we let $\alpha(I, \iota)$ denote the smallest interval in Θ which includes the set $\iota(I \cap \Lambda_0)$. Thus $\alpha(I, \iota) = \bigcup_{\theta_0, \theta_1 \in \iota(I \cap \Lambda_0), \theta_0 \leq \theta_1} [\theta_0, \theta_1]$, where each interval $[\theta_0, \theta_1]$ is being considered in Θ .

Lemma 3.8. *If $\iota : \Lambda_0 \rightarrow \Theta_0$ is a coi between Λ and Θ and $I \subseteq \Lambda$ is an interval then $\alpha(\alpha(I, \iota), \iota^{-1}) = \alpha(I, \Lambda_0)$. \square*

We point out that a coi ι between Λ and Θ also induces a coi between the reversed orders Λ^{-1} and Θ^{-1} in the obvious way.

Lemma 3.9. *Let $\Lambda \equiv I_0 \cdots I_n$ and $\iota : \Lambda_0 \rightarrow \Theta_0$ a coi from Λ to Θ . Then there exist (possibly empty) finite subintervals I'_0, \dots, I'_{n+1} of $\alpha(\Lambda, \iota)$ such that*

$$\alpha(\Lambda, \iota) \equiv I'_0 \alpha(I_0, \iota) I'_1 \alpha(I_1, \iota) I'_2 \cdots \alpha(I_n, \iota) I'_{n+1}.$$

Proof. Assume the hypotheses. Clearly each $\alpha(I_j, \iota)$ is a subinterval of $\alpha(\Lambda, \iota)$, and it is easy to see that all elements of $\alpha(I_j, \iota)$ are strictly below all elements of $\alpha(I_{j+1}, \iota)$ for $0 \leq j < n$. Thus we may indeed write

$$\alpha(\Lambda, \iota) \equiv I'_0 \alpha(I_0, \iota) I'_1 \alpha(I_1, \iota) I'_2 \cdots \alpha(I_n, \iota) I'_{n+1}$$

and we conclude by pointing out that $I'_l \cap \Theta_0 = I'_l \cap \iota(\Lambda_0) = I'_l \cap (\bigcup_{j=0}^n \iota(I_j \cap \Lambda_0)) \subseteq \bigcup_{j=0}^n (I'_l \cap \alpha(I_j, \iota)) = \emptyset$ for each $0 \leq l \leq n + 1$, and since $\text{Close}(\Theta_0, \Theta)$ we have I'_l finite. \square

Lemma 3.10. *Let $\iota : \Lambda_0 \rightarrow \Theta_0$ be a coi from Λ to Λ_0 . If $I \subseteq \Lambda$ is finite then $\alpha(I, \iota)$ is finite.*

Proof. Since I is finite, we know $I \cap \Lambda_0$ is finite. Clearly we have $\alpha(I, \iota) \cap \Theta_0 = \iota(I \cap \Lambda_0)$, so $\alpha(I, \iota)$ is an interval in Θ having finite intersection with Θ_0 . Thus $\alpha(I, \iota)$ is finite by Lemma 3.6 (i). \square

3C. Coherent coi triples. Suppose that κ_0 and κ_1 are cardinal numbers greater than or equal to 2. For words $W \in \text{Red}_{\kappa_0}$ and $U \in \text{Red}_{\kappa_1}$ we'll write $\text{coi}(W, \iota, U)$ to denote that ι is a coi between $\text{p}^*(W)$ and $\text{p}^*(U)$ and say that $\text{coi}(W, \iota, U)$ is a *coi triple from Red_{κ_0} to Red_{κ_1}* . We will often abuse language and say that ι is a coi from W to U when really ι is a coi from $\text{p}^*(W)$ to $\text{p}^*(U)$.

Definition 3.11. A collection $\{\text{coi}(W_x, \iota_x, U_x)\}_{x \in X}$ of coi triples from Red_{κ_0} to Red_{κ_1} is *coherent* if for any choice of $x_0, x_1 \in X$, intervals $I_0 \subseteq \text{p}^*(W_{x_0})$ and $I_1 \subseteq \text{p}^*(W_{x_1})$ and $i \in \{-1, 1\}$ such that $W_{x_0} \upharpoonright_p I_0 \equiv (W_{x_1} \upharpoonright_p I_1)^i$ we get

$$[[U_{x_0} \upharpoonright_p \alpha(I_0, \iota_{x_0})]] = [[(U_{x_1} \upharpoonright_p \alpha(I_1, \iota_{x_1}))^i]],$$

and similarly for any choice of $x_2, x_3 \in X$, intervals $I_2 \subseteq \text{p-chunk}(U_{x_2})$ and $I_3 \subseteq \text{p-chunk}(U_{x_3})$ and $j \in \{-1, 1\}$ such that $U_{x_2} \upharpoonright_p I_2 \equiv (U_{x_3} \upharpoonright_p I_3)^j$ we get

$$[[W_{x_2} \upharpoonright_p \alpha(I_2, \iota_{x_2}^{-1})]] = [[(W_{x_3} \upharpoonright_p \alpha(I_3, \iota_{x_3}^{-1}))^j]].$$

It is clear from the symmetric nature of this definition that if the collection of coi triples $\{\text{coi}(W_x, \iota_x, U_x)\}_{x \in X}$ from Red_{k_0} to Red_{k_1} is coherent then so is the collection of coi triples $\{\text{coi}(U_x, \iota_x^{-1} W_x)\}_{x \in X}$ from Red_{k_1} to Red_{k_0} . We emphasize that a word can appear multiple times in a coherent collection. For example, if each element of $\{W_x\}_{x \in X}$ is pure then the collection $\{(W_x, \iota_x, E)\}_{x \in X}$ is obviously coherent (each ι_x is the empty function).

Lemma 3.12. *Suppose that Θ is a totally ordered set and that $\{\mathcal{T}_\theta\}_{\theta \in \Theta}$ is a collection of coherent collections of coi triples from Red_{k_0} to Red_{k_1} such that $\theta \leq \theta'$ implies $\mathcal{T}_\theta \subseteq \mathcal{T}_{\theta'}$. Then $\bigcup_{\theta \in \Theta} \mathcal{T}_\theta$ is coherent.*

Proof. Supposing that $\text{coi}(W_{x_0}, \iota_{x_0}, U_{x_0}), \text{coi}(W_{x_1}, \iota_{x_1}, U_{x_1}) \in \bigcup_{\theta \in \Theta} \mathcal{T}_\theta$ and intervals $I_0 \subseteq \text{p}^*(W_{x_0})$ and $I_1 \subseteq \text{p}^*(W_{x_1})$ and $i \in \{-1, 1\}$ are such that $W_{x_0} \upharpoonright_p I_0 \equiv (W_{x_1} \upharpoonright_p I_1)^i$, we select $\theta \in \Theta$ such that $\text{coi}(W_{x_0}, \iota_{x_0}, U_{x_0}), \text{coi}(W_{x_1}, \iota_{x_1}, U_{x_1}) \in \mathcal{T}_\theta$. As \mathcal{T}_θ is coherent we get

$$[[U_{x_0} \upharpoonright_p \alpha(I_0, \iota_{x_0})]] = [[(U_{x_1} \upharpoonright_p \alpha(I_1, \iota_{x_1}))^i]].$$

The comparable check for words $U_{x_2}, U_{x_3} \in \text{Red}_{k_1}$ is analogous. \square

Lemma 3.13. *Suppose $\{\text{coi}(W_x, \iota_x, U_x)\}_{x \in X}$ is coherent, $x \in X$, $I \subseteq \text{p}^*(W_x)$ is an interval, $I \equiv I_0 I_1 \cdots I_n$. Suppose also that for each $0 \leq j \leq n$ we have an $x_j \in X$, an interval I'_j in $\text{p}^*(W_{x_j})$ and $i_j \in \{-1, 1\}$ such that $W_x \upharpoonright_p I_j \equiv (W_{x_j} \upharpoonright_p I'_j)^{i_j}$. Then*

$$[[U_x \upharpoonright_p \alpha(I, \iota_x)]] = \prod_{j=0}^n [[(U_{x_j} \upharpoonright_p \alpha(I'_j, \iota_{x_j}))^{i_j}]].$$

Furthermore, if $L = \{0 \leq j \leq n \mid |I_j| > 1\}$ we have

$$[[U_x \upharpoonright_p \alpha(I, \iota_x)]] = \prod_{j \in L} [[(U_{x_j} \upharpoonright_p \alpha(I'_j, \iota_{x_j}))^{i_j}]].$$

Proof. For each $0 \leq j \leq n$ we have $W_x \upharpoonright_p I_j \equiv (W_{x_j} \upharpoonright_p I'_j)^{i_j}$, so that by the fact that $\{\text{coi}(W_x, \iota_x, U_x)\}_{x \in X}$ is coherent we see that

$$[[U_x \upharpoonright_p \alpha(I_j, \iota_x)]] = [[(U_{x_j} \upharpoonright_p \alpha(I'_j, \iota_{x_j}))^{i_j}]]$$

for all $0 \leq j \leq n$. In particular we have

$$\prod_{j=0}^n [[U_x \upharpoonright_p \alpha(I'_j, \iota_x)]] = \prod_{j=0}^n [[(U_{x_j} \upharpoonright_p \alpha(I'_j, \iota_{x_j}))^{i_j}]]$$

and so we will be done with the first claim if we show that $[[U_x \downarrow_p \alpha(I, \iota_x)]] = \prod_{j=0}^n [[U_x \downarrow_p \alpha(I_j, \iota_x)]]$. But this is true since by Lemma 3.9 the (possibly unreduced) word $\prod_{j=0}^n U_x \downarrow_p \alpha(I_j, \iota_x)$ is obtained from $U_x \downarrow_p \alpha(I, \iota_x)$ by deleting finitely many pure subwords.

Next we let L be as in the statement of the lemma. Notice that for each $0 \leq j \leq n$ with $j \notin L$ we have $|I_j| = |I'_j| \leq 1$ and so $\alpha(I'_j, \iota_{x_j})$ is a finite interval, by Lemma 3.10. Thus for each such j we have $[(U_{x_j} \downarrow_p \alpha(I'_j, \iota_{x_j}))^{i_j}] = [[E]]$ since $U_{x_j} \downarrow_p \alpha(I'_j, \iota_{x_j})$ is a finite concatenation of pure words. Thus removing all such j from the multiplication expression $\prod_{j=0}^n [(U_{x_j} \downarrow_p \alpha(I'_j, \iota_{x_j}))^{i_j}]$ will not change the value in the group, and so we are done with the second claim. \square

What follows is a rather technical result that will allow us to conclude that certain natural maps are well defined despite certain choices that are made.

Lemma 3.14. *Let the collection $\{\text{coi}(W_x, \iota_x, U_x)\}_{x \in X}$ be coherent and let W be in $\text{Pfine}(\{W_x\}_{x \in X})$. Let I_0, \dots, I_n be a finite set of subintervals of $\mathfrak{p}^*(W)$ as in the conclusion of Lemma 3.2 and let $J = \{0 \leq j \leq n \mid |I_j| > 1\}$. For each $j \in J$ select $x_j \in X$, $i_j \in \{-1, 1\}$, and an interval $\Lambda_j \subseteq \mathfrak{p}^*(W_{x_j})$ such that $W \downarrow_p I_j \equiv (W_{x_j} \downarrow_p \Lambda_j)^{i_j}$. Again, let $I'_0, \dots, I'_{n'}$ be a finite set of subintervals of $\mathfrak{p}^*(W)$ as in the conclusion of Lemma 3.2 and let $J' = \{0 \leq j' \leq n' \mid |I'_{j'}| > 1\}$. For each $j' \in J'$ select $y_{j'} \in X$, $m_{j'} \in \{-1, 1\}$, and an interval $\Lambda'_{j'} \subseteq \mathfrak{p}^*(W_{y_{j'}})$ such that $W \downarrow_p I'_{j'} \equiv (W_{y_{j'}} \downarrow_p \Lambda'_{j'})^{m_{j'}}$. Then*

$$\prod_{j \in J} [[(U_{x_j} \downarrow_p \alpha(\Lambda_j, \iota_{x_j}))^{i_j}]] = \prod_{j' \in J'} [[(U_{y_{j'}} \downarrow_p \alpha(\Lambda'_{j'}, \iota_{y_{j'}}))^{m_{j'}}]].$$

Proof. Assume the hypotheses. Take \mathbb{I} to be the set of nonempty intervals obtained by intersecting an I_j with an $I'_{j'}$. For each $0 \leq j \leq n$ we can write $I_j \equiv I_{(j,0)} I_{(j,1)} \cdots I_{(j,n_j)}$ where each $I_{(j,q)}$ is an element of \mathbb{I} . Similarly for each $0 \leq j' \leq n'$ we write $I'_{j'} \equiv I'_{(j',0)} \cdots I'_{(j',n'_{j'})}$ where each $I'_{(j',r)}$ is an element of \mathbb{I} .

We have $\mathbb{I} = \{I_{(j,q)}\}_{0 \leq j \leq n, 0 \leq q \leq n_j} = \{I'_{(j',r)}\}_{0 \leq j' \leq n', 0 \leq r \leq n'_{j'}}$. Let $F : \mathbb{I} \rightarrow \{(j, q) \mid 0 \leq j \leq n, 0 \leq q \leq n_j\}$ be the unique order isomorphism between the domain and codomain where the codomain is given the lexicographic order, comparing the leftmost coordinate first and define $F' : \mathbb{I} \rightarrow \{(j', r) \mid 0 \leq j' \leq n', 0 \leq r \leq n'_{j'}\}$ similarly. Let $h : \{(j, q) \mid 0 \leq j \leq n, 0 \leq q \leq n_j\} \rightarrow \{0, \dots, n\}$ denote projection to the first coordinate, and similarly define $h' : \{(j', r) \mid 0 \leq j' \leq n', 0 \leq r \leq n'_{j'}\} \rightarrow \{0, \dots, n'\}$. Let $\mathbb{J} \subseteq \mathbb{I}$ denote the set of intervals in \mathbb{I} which are of cardinality at least 2; that is,

$$\mathbb{J} = \{I_{(j,q)} \mid 0 \leq j \leq n, 0 \leq q \leq n_j, |I_{(j,q)}| \geq 2\}.$$

For each $j \in J$ and each $I_{(j,q)} \in \mathbb{J}$ we know that $W \downarrow_p I_{(j,q)} \in \mathfrak{p}\text{-chunk}(W_{x_j}^{i_j})$, so select an interval $\Lambda_{(j,q)} \subseteq \mathfrak{p}^*(W_{x_j})$ such that $W \downarrow_p I_{(j,q)} \equiv (W_{x_j} \downarrow_p \Lambda_{(j,q)})^{i_j}$. Now

we have that

$$\begin{aligned}
 & \prod_{j \in J} [(U_{x_j} \downarrow_{\mathfrak{p}} \propto(\Lambda_j, t_{x_j}))^{i_j}] \\
 &= \prod_{j \in J} \prod_{\substack{0 \leq q \leq n_j \\ I_{(j,q)} \in \mathbb{J}}} [(U_{x_j} \downarrow_{\mathfrak{p}} \propto(\Lambda_{(j,q)}, t_{x_j}))^{i_j}] \\
 &= \prod_{I \in \mathbb{J}} [(U_{x_{\text{ho}F(I)}} \downarrow_{\mathfrak{p}} \propto(\Lambda_{F(I)}, t_{x_{\text{ho}F(I)}}))^{i_{\text{ho}F(I)}}] \\
 &= \prod_{j' \in J'} \prod_{\substack{0 \leq r \leq n'_{j'} \\ I_{(j',r)} \in \mathbb{J}}} [(U_{x_{\text{ho}F \circ (F')^{-1}(j',r)}} \downarrow_{\mathfrak{p}} \propto(\Lambda_{F \circ (F')^{-1}(j',r)}, t_{x_{\text{ho}F \circ (F')^{-1}(j',r)}}))^{i_{\text{ho}F \circ (F')^{-1}(j',r)}}] \\
 &= \prod_{j' \in J'} [(U_{y_{j'}} \downarrow_{\mathfrak{p}} \propto(\Lambda'_{j'}, t_{y_{j'}}))^{m'_{j'}}]
 \end{aligned}$$

where the first equality holds by Lemma 3.13, the second and third equalities are simply a rewriting of the order index, and the last equality holds by another application of Lemma 3.13. This completes the proof. \square

Now we may conclude that a coherent collection of cois produces well-defined homomorphisms. For each $i \in \{0, 1\}$ we let $\sqsupset_{\kappa_i} : \text{Red}_{\kappa_i} \rightarrow \mathcal{C}_{\kappa_i}$ denote the surjection given by $W \mapsto [[W]]$.

Proposition 3.15. *Let $\{\text{coi}(W_x, t_x, U_x)\}_{x \in X}$ be coherent. By selecting for each $W \in \text{Pfine}(\{W_x\}_{x \in X})$ a finite set of subintervals I_0, \dots, I_n of $\mathfrak{p}^*(W)$ as in the conclusion of Lemma 3.2, letting $J = \{0 \leq j \leq n \mid |I_j| > 1\}$, selecting for each $j \in J$ an element $x_j \in X$, $i_j \in \{-1, 1\}$, and an interval $\Lambda_j \subseteq \mathfrak{p}^*(W_{x_j})$ such that $W \downarrow_{\mathfrak{p}} I_j \equiv (W_{x_j} \downarrow_{\mathfrak{p}} \Lambda_j)^{i_j}$ we obtain a homomorphism*

$$\phi_0 : \text{Pfine}(\{W_x\}_{x \in X}) \rightarrow \sqsupset_{\kappa_1}(\text{Pfine}(\{U_x\}_{x \in X}))$$

given by $\phi_0(W) = \prod_{j \in J} [(U_{x_j} \downarrow_{\mathfrak{p}} \propto(\Lambda_j, t_{x_j}))^{i_j}]$, whose definition is independent of the choices made of the set of subintervals I_0, \dots, I_n , elements $x_j \in X$ and $i_j \in \{-1, 1\}$, and intervals $\Lambda_j \subseteq \mathfrak{p}^*(W_{x_j})$. The comparable map

$$\phi_1 : \text{Pfine}(\{U_x\}_{x \in X}) \rightarrow \sqsupset_{\kappa_0}(\text{Pfine}(\{W_x\}_{x \in X}))$$

similarly is a homomorphism whose definition is independent of the various selections made.

Proof. From Lemma 3.14 we see that the described function ϕ_0 is well defined and independent of the numerous choices made. We must check that ϕ_0 is a homomorphism.

We note first that if $W \in \text{Pfine}(\{W_x\}_{x \in X})$ and $\mathfrak{p}^*(W)$ has a first or last element, say $\lambda = \max(\mathfrak{p}^*(W))$, then $\phi_0(W) = \phi_0(W \downarrow_{\mathfrak{p}} \mathfrak{p}^*(W) \setminus \{\lambda\})$. This is easily seen by

selecting the set of intervals I_0, \dots, I_n for W to be such that $I_n = \{\lambda\}$. The fact that $|I_n| = 1$ and therefore $I_n \notin J$ completes the argument.

Suppose that $W \in \text{Pfine}(\{W_x\}_{x \in X})$ and $W \equiv W_0 W_1$ where also both $W_0, W_1 \in \text{Pfine}(\{W_x\}_{x \in X})$. Choose subintervals I_0, \dots, I_n in $p^*(W_0)$ as in Lemma 3.2, let $J = \{0 \leq j \leq n \mid |I_j| > 1\}$, select $x_j \in X$ and $i_j \in \{-1, 1\}$ and intervals $\Lambda_j \subseteq p^*(W_{x_j})$ with $W \upharpoonright_p I_j \equiv (W_{x_j} \upharpoonright_p \Lambda_j)^{i_j}$. Similarly choose intervals $I'_0, \dots, I'_{n'}$ in $p^*(W_1)$ and define J' and choose $y_{j'} \in X$, $m_{j'} \in \{-1, 1\}$ and $\Lambda'_{j'} \subseteq p^*(W_{y_{j'}})$ for each $j' \in J'$. Notice that $p^*(W) \equiv I_0 \cdots I_n I'_0 \cdots I'_{n'}$ is a decomposition as in Lemma 3.2 and $J \cup J'$ is precisely the set of indices whose accompanying interval is of cardinality at least two. Then

$$\begin{aligned} \phi_0(W) &= \left(\prod_{j \in J} [(U_{x_j} \upharpoonright_p \alpha(\Lambda_j, t_{x_j}))^{i_j}] \right) \left(\prod_{j' \in J'} [(U_{y_{j'}} \upharpoonright_p \alpha(\Lambda'_{j'}, t_{y_{j'}}))^{m_{j'}}] \right) \\ &= \phi_0(W_0) \phi_0(W_1). \end{aligned}$$

Next we suppose that $W \in \text{Pfine}(\{W_x\}_{x \in X})$ and let subintervals I_0, \dots, I_n in $p^*(W_0)$ be as in Lemma 3.2, let $J = \{0 \leq j \leq n \mid |I_j| > 1\}$, select $x_j \in X$ and $i_j \in \{-1, 1\}$ and intervals $\Lambda_j \subseteq p^*(W_{x_j})$ with $W \upharpoonright_p I_j \equiv (W_{x_j} \upharpoonright_p \Lambda_j)^{i_j}$. Notice that $p^*(W^{-1})$ may be written as $p^*(W^{-1}) \equiv I'_n \cdots I'_0$ as in Lemma 3.2, where I'_j is order isomorphic to the ordered set $(I_j)^{-1}$, and $W \upharpoonright_p I_j \equiv (W^{-1} \upharpoonright_p I'_j)^{-1}$. Also, $\{0 \leq j \leq n \mid |I'_j| > 1\}$ is equal to the set J . Then

$$\begin{aligned} \phi_0(W) &= \prod_{j \in J} [(U_{x_j} \upharpoonright_p \alpha(\Lambda_j, t_{x_j}))^{i_j}] = \left(\prod_{j \in J^{-1}} [(U_{x_j} \upharpoonright_p \alpha(\Lambda_j, t_{x_j}))^{-i_j}] \right)^{-1} \\ &= (\phi_0(W^{-1}))^{-1}, \end{aligned}$$

where J^{-1} denotes the set J under the reverse order. Thus $\phi_0(W^{-1}) = (\phi_0(W))^{-1}$.

Finally we let $W_0, W_1 \in \text{Pfine}(\{W_x\}_{x \in X})$ be given. As in Lemma 2.1 we write $W_0 \equiv W_{00} W_{01}$ and $W_1 \equiv W_{10} W_{11}$ with $W_{01} \equiv W_{10}^{-1}$ and the word $W_{00} W_{11}$ reduced. We will give the argument in the most difficult case and sketch how the argument goes in the less difficult ones. Suppose that W_{00} ends with a nonempty α -pure word and W_{11} begins with a nonempty α -pure word, and also that W_{01} begins with a nonempty α -pure word. From this last assumption we know that W_{10} ends with a nonempty α -pure word.

We have $W_{00} W_{11} \equiv W'_{00} W_a W'_{11}$ where we denote $\lambda_0 = \max(p^*(W_{00}))$, $\lambda_1 = \min(p^*(W_{11}))$ and

$$\begin{aligned} W'_{00} &\equiv W_{00} \upharpoonright_p \{\lambda \in p^*(W_{00}) \mid \lambda < \lambda_0\}, & W'_{11} &\equiv W_{11} \upharpoonright_p \{\lambda \in p^*(W_{11}) \mid \lambda > \lambda_1\}, \\ W_a &\equiv (W_{00} \upharpoonright_p \{\lambda_0\})(W_{11} \upharpoonright_p \{\lambda_1\}). \end{aligned}$$

Note that $W'_{00}, W_a, W'_{11} \in \text{Pfine}(\{W_x\}_{x \in X})$ since the concatenation $W_{00} W_{11}$ is in $\text{Pfine}(\{W_x\}_{x \in X})$ and each of W'_{00}, W_a , and W'_{11} are p-chunks of this word, whereas

for example $W_{00} \upharpoonright_p \{\lambda_0\}$ might not be in $\text{Pfine}(\{W_x\}_{x \in X})$. Furthermore suppose $\lambda_2 = \min(\mathfrak{p}^*(W_{01}))$ and $\lambda_3 = \max(\mathfrak{p}^*(W_{10}))$ and define

$$\begin{aligned} W'_{01} &\equiv W_{01} \upharpoonright_p (\mathfrak{p}^*(W_{01}) \setminus \{\lambda_2\}), & W_b &\equiv (W_{00} \upharpoonright_p \{\lambda_0\})(W_{01} \upharpoonright_p \{\lambda_2\}), \\ W'_{10} &\equiv W_{10} \upharpoonright_p (\mathfrak{p}^*(W_{10}) \setminus \{\lambda_3\}), & W_c &\equiv (W_{10} \upharpoonright_p \{\lambda_3\})(W_{11} \upharpoonright_p \{\lambda_1\}). \end{aligned}$$

Notice that $W'_{01} \equiv (W'_{10})^{-1}$ and that each of the words $W'_{01}, W_b, W'_{10}, W_c$ is in $\text{Pfine}(\{W_x\}_{x \in X})$.

By our work so far we get

$$\begin{aligned} \phi_0(W_{00}W_{11}) &= \phi_0(W'_{00}W_aW'_{11}) \\ &= \phi_0(W'_{00})\phi_0(W_a)\phi_0(W'_{11}) \\ &= \phi_0(W'_{00})\phi_0(W'_{11}) \\ &= \phi_0(W'_{00})\phi_0(W'_{01})\phi_0(W'_{10})\phi_0(W'_{11}) \\ &= \phi_0(W'_{00})\phi_0(W_b)\phi_0(W'_{01})\phi_0(W'_{10})\phi_0(W_c)\phi_0(W'_{11}) \\ &= \phi_0(W_0)\phi_0(W_1). \end{aligned}$$

In the simpler case where W_{01} does not begin with an α -pure word (hence W_{10} does not end with an α -pure word) we let $W'_{01} = W_{01}$, $W'_{10} = W_{10}$ and both W_b and W_c be the empty word and the equalities above will all hold. In the case there does not exist $\alpha < \kappa_0$ such that both W_{00} ends with a nonempty α -pure word and W_{11} begins with an α -pure word we let $W'_{00} \equiv W_{00}$, $W'_{11} \equiv W_{11}$ and $W_a \equiv E$. It may still be the case that W_{00} ends with a nonempty β -pure word and W_{01} begins with a nonempty β -pure word, $\beta < \kappa_0$, and for this we define

$$\begin{aligned} W'_{01} &\equiv W_{01} \upharpoonright_p (\mathfrak{p}^*(W_{01}) \setminus \{\lambda_2\}), & W_b &\equiv (W_{00} \upharpoonright_p \{\lambda_0\})(W_{01} \upharpoonright_p \{\lambda_2\}), \\ W'_{10} &\equiv W_{10} \upharpoonright_p (\mathfrak{p}^*(W_{10}) \setminus \{\lambda_3\}), \end{aligned}$$

and let W_c be given by

$$\begin{cases} (W_{10} \upharpoonright_p \{\lambda_3\})(W_{11} \upharpoonright_p \{\lambda_1\}) & \text{in case } W_{11} \text{ begins with a nonempty} \\ & \beta\text{-pure word and } \lambda_3 = \min \mathfrak{p}^*(W_{11}); \\ W_{10} \upharpoonright_p \{\lambda_3\} & \text{otherwise.} \end{cases}$$

The case where W_{11} and W_{10} respectively begin and end with a β -pure word, for some $\beta < \kappa_0$, is analogous. If none of these cases holds then we simply let $W'_{00} \equiv W_{00}$, $W'_{01} \equiv W_{01}$, $W'_{10} \equiv W_{10}$, $W'_{11} \equiv W_{11}$ and $W_a \equiv W_b \equiv W_c \equiv E$. This exhausts all possibilities and the proof is complete (the arguments for ϕ_1 are made in the analogous way). \square

Proposition 3.16. *The homomorphisms ϕ_0 and ϕ_1 descend respectively to isomorphisms*

$$\begin{aligned} \Phi_0 &: \beth_0(\text{Pfine}(\{W_x\}_{x \in X})) \rightarrow \beth_1(\text{Pfine}(\{U_x\}_{x \in X})), \\ \Phi_1 &: \beth_1(\text{Pfine}(\{U_x\}_{x \in X})) \rightarrow \beth_0(\text{Pfine}(\{W_x\}_{x \in X})), \end{aligned}$$

with $\Phi_0 = \Phi_1^{-1}$.

Proof. If $W \in \text{Pfine}(\{W_x\}_{x \in X})$ is a pure word the set $\text{p}^*(W)$ is a singleton and for any decomposition of $\text{p}^*(W)$ by Lemma 3.2 the accompanying set J will necessarily be empty. Thus all pure words in $\text{Pfine}(\{W_x\}_{x \in X})$ are in $\ker(\phi_0)$ and so we get the induced Φ_0 , and similarly we obtain an induced Φ_0 .

By Lemma 3.2 each element of the group $\sqsupset_0(\text{Pfine}(\{W_x\}_{x \in X}))$ may be written as a product $[[W_0]][[W_1]] \cdots [[W_n]]$ where each W_i is in $(\bigcup_{x \in X} \text{p-chunk}(W_x))^{\pm 1}$. For each $0 \leq j \leq n$ we select x_j and i_j and an interval $\Lambda_j \subseteq \text{p}^*(W_{x_j})$ such that $W_j \equiv (W_{x_j} \upharpoonright_{\text{p}} \Lambda_j)^{i_j}$. Now

$$\begin{aligned} \Phi_1 \circ \Phi_0([[W_0]] \cdots [[W_n]]) &= \prod_{j=0}^n \Phi_1[[(U_{x_j} \upharpoonright_{\text{p}} \alpha(\Lambda_j, \iota_{x_j}))^{i_j}]] \\ &= \prod_{j=0}^n (\Phi_1([[U_{x_j} \upharpoonright_{\text{p}} \alpha(\Lambda_j, \iota_{x_j})]])^{i_j}) \\ &= \prod_{j=0}^n [[W_{x_j} \upharpoonright_{\text{p}} \alpha(\alpha(\Lambda_j, \iota_{x_j}), \iota_{x_j}^{-1})]]^{i_j} \\ &= \prod_{j=0}^n [[W_{x_j} \upharpoonright_{\text{p}} \Lambda_j]]^{i_j} \\ &= \prod_{j=0}^n [[W_j]], \end{aligned}$$

where the fourth equality holds by Lemma 3.8 — the word $W_{x_j} \upharpoonright_{\text{p}} \alpha(\alpha(\Lambda_j, \iota_{x_j}), \iota_{x_j}^{-1})$ is obtained from the word $W_{x_j} \upharpoonright_{\text{p}} \Lambda_j$ by deleting finitely many pure subwords, namely those associated with the set $\Lambda_j \setminus \alpha(\alpha(\Lambda_j, \iota_{x_j}), \iota_{x_j}^{-1})$. Thus $\Phi_1 \circ \Phi_0$ is the identity map, and that $\Phi_0 \circ \Phi_1$ is also the identity map follows from the same reasoning. \square

3D. Extensions of coherent collections. By Proposition 3.16, the problem of finding an isomorphism between cone groups is reduced to that of finding a coherent collection of coi triples $\{\text{coi}(W_x, U_x, \iota_x)\}_{x \in X}$ such that $\sqsupset_0(\text{Pfine}(\{W_x\}_{x \in X})) = \mathcal{C}_{\kappa_0}$ and $\sqsupset_1(\text{Pfine}(\{U_x\}_{x \in X})) = \mathcal{C}_{\kappa_1}$. Thus, in this and all remaining subsections we approach the problem of extending collections of coi triples. We still assume that $\kappa_0, \kappa_1 \geq 2$ and that the coi collections are from Red_{κ_0} to Red_{κ_1} .

Lemma 3.17. *Let $\{\text{coi}(W_x, \iota_x, U_x)\}_{x \in X}$ be coherent. If W is in $\text{Pfine}(\{W_x\}_{x \in X})$ then there exists $U \in \text{Red}_{\kappa_1}$ and a coi ι from W to U such that $\{\text{coi}(W_x, \iota_x, U_x)\}_{x \in X} \cup \{(W, \iota, U)\}$ is coherent. Moreover if W is nonempty the domain (and range) of ι can be made to be nonempty.*

Proof. If W is empty then we let U and ι be empty. Else we choose subintervals I_0, \dots, I_n in $\text{p}^*(W)$ as in Lemma 3.2, let $J = \{0 \leq j \leq n \mid |I_j| > 1\}$, select $x_j \in X$ and

$i_j \in \{-1, 1\}$ and intervals $\Lambda_j \subseteq p^*(W_{x_j})$ with $W \upharpoonright_p I_j \equiv (W_{x_j} \upharpoonright_p \Lambda_j)^{i_j}$. Let $J' \subseteq J$ be given by

$$J' = \{j \in J \mid (U_{x_j} \upharpoonright_p \alpha(\Lambda_j, \iota_{x_j}))^{i_j} \neq E\}.$$

For each $j \in J'$ let $U'_j \equiv (U_{x_j} \upharpoonright_p \alpha(\Lambda_j, \iota_{x_j}))^{i_j}$. For every $0 \leq j \leq n$ with $j \notin J'$ we let $U'_j \equiv a_{0,0}$.

The word $\prod_{j=0}^n U'_j$ is probably not reduced, and so we will make slight modifications in order to obtain a reduced word. We know that each subword U'_j is reduced and nonempty. Let $U_n \equiv U'_n$. Let $0 \leq j < n$ be given. There are a few possibilities:

- $p^*(U'_j)$ has a maximal element and $p^*(U'_{j+1})$ has a minimal element and both $U'_j \upharpoonright_p \{\max p^*(U'_j)\}$ and $U'_{j+1} \upharpoonright_p \{\min p^*(U'_{j+1})\}$ are α -pure for some $\alpha < \kappa_1$.
- $p^*(U'_j)$ has a maximal element and $p^*(U'_{j+1})$ has a minimal element and both $U'_j \upharpoonright_p \{\max p^*(U'_j)\}$ and $U'_{j+1} \upharpoonright_p \{\min p^*(U'_{j+1})\}$ are not α -pure for some $\alpha < \kappa_1$.
- $p^*(U'_j)$ does not have a maximal element or $p^*(U'_{j+1})$ does not have a minimal element.

In the middle case we let $U_j \equiv U'_j$. In the first or last case we choose $\alpha'_j < \kappa_1$ such that U'_j does not end with an α'_j -pure word (here we are using the fact that $\kappa_1 \geq 2$) and let $U_j \equiv U'_j a_{\alpha',0}$. The word $U_j U'_{j+1}$ is reduced, and so the word $U_j U_{j+1}$ is reduced (since U_{j+1} is nonempty), and so the word $U \equiv \prod_{j=0}^n U_j$ is reduced.

We now define the coi ι from W to U in a very natural way. If $j \in J'$ then we let the domain of ι_{x_j} be Λ'_j , and so $\text{Close}(\Lambda'_j, p^*(W_{x_j}))$. Let $\Lambda''_j \subseteq I_j$ be the image of $\Lambda'_j \cap \Lambda_j$ under the order isomorphism given by $W \upharpoonright_p I_j \equiv (W_{x_j} \upharpoonright_p \Lambda_j)^{i_j}$. Similarly we let $\Theta''_j \subseteq p^*(U'_j) \subseteq p^*(U_j)$ be the image of $\iota_{x_j}(\Lambda_j \cap \Lambda'_j)$ under the order isomorphism given by $U'_j \equiv (U_{x_j} \upharpoonright_p \alpha(\Lambda_j, \iota_{x_j}))^{i_j}$. Define $\iota_j : \Lambda''_j \rightarrow \Theta''_j$ to be the order isomorphism given by the restriction to Λ''_j of the composition of the order isomorphism given by $W \upharpoonright_p I_j \equiv (W_{x_j} \upharpoonright_p \Lambda_j)^{i_j}$ with ι_{x_j} with the order isomorphism given by $(U_{x_j} \upharpoonright_p \alpha(\Lambda_j, \iota_{x_j}))^{i_j} \equiv U'_j$. It is easy to check that $\text{Close}(\Lambda''_j, I_j)$ and $\text{Close}(\Theta''_j, p^*(U_j))$, since for $0 \leq j \leq n$ either $U_j \equiv U'_j$ or U_j is obtained from U'_j by appending a word of length one on the right.

If $0 \leq j \leq n$ and $j \notin J'$ then I_j is finite and nonempty, as is $p^*(U_j)$, and we simply select elements $\lambda \in I_j$ and $\lambda' \in p^*(U_j)$ and let $\Lambda''_j = \{\lambda\}$, $\Theta''_j = \{\lambda'\}$ and $\iota_j : \Lambda''_j \rightarrow \Theta''_j$ be the unique function. Clearly $\text{Close}(\Lambda''_j, I_j)$ and $\text{Close}(\Theta''_j, p^*(U_j))$.

Let $\Lambda'' = \bigcup_{j=0}^n \Lambda''_j$ and $\Theta'' = \bigcup_{j=0}^n \Theta''_j$, and note that $\text{Close}(\Lambda'', p^*(W))$ and $\text{Close}(\Theta'', p^*(U))$ by Lemma 3.6 (iii). Let $\iota : \Lambda'' \rightarrow \Theta''$ be the unique extension of the ι_j . Now $\text{coi}(W, \iota, U)$.

We check that $\{\text{coi}(W_x, \iota_x, U_x)\}_{x \in X} \cup \{\text{coi}(W, \iota, U)\}$ is coherent. Suppose that $y \in X$ and intervals $I \subseteq p\text{-chunk}(W)$ and $I' \subseteq p\text{-chunk}(W_y)$ and $i \in \{-1, 1\}$ are such that $W \upharpoonright_p I \equiv (W_y \upharpoonright_p I')^i$. Let $L \subseteq \{0, \dots, n\}$ denote the set of those j such that $I_j \cap I \neq \emptyset$. For each $j \in L \cap J$ we have $W \upharpoonright_p (I_j \cap I) \equiv (W_{x_j} \upharpoonright_p \Lambda_j^*)^{i_j}$ for the obvious

choice of interval $\Lambda_j^* \subseteq \Lambda_j \subseteq \text{p-chunk}(W_{x_j})$. Thus $(W_{x_j} \downarrow_{\text{p}} \Lambda_j^*)^{i \cdot i_j} \equiv W_y \downarrow_{\text{p}} I'_j$ for the obvious choice of interval $I'_j \subseteq I'$. By the coherence of $\{\text{coi}(W_x, \iota_x, U_x)\}_{x \in X}$ we therefore have

$$\begin{aligned} [[U \downarrow_{\text{p}} \alpha(I, \iota)]] &= \prod_{j \in L} [[U \downarrow_{\text{p}} \alpha(I_j \cap I, \iota)]] \\ &= \prod_{j \in L \cap J'} [[U \downarrow_{\text{p}} \alpha(I_j \cap I, \iota)]] \\ &= \prod_{j \in L \cap J'} [[U_{x_j} \downarrow_{\text{p}} \alpha(\Lambda_j^*, \iota_{x_j})]]^{i_j} \\ &= \prod_{j \in (L \cap J')^i} [[U_y \downarrow_{\text{p}} \alpha(I'_j, \iota_y)]]^i \\ &= [[(U_y \downarrow_{\text{p}} \alpha(I', \iota_y))^i]]. \end{aligned}$$

If we select intervals $I, I' \subseteq \text{p}^*(W)$ and $i \in \{-1, 1\}$ such that $W \downarrow_{\text{p}} I \equiv (W \downarrow_{\text{p}} I')^i$ then a similar strategy of finitely decomposing I and I' is employed to show $[[U \downarrow_{\text{p}} \alpha(I, \iota)]] = [[(U \downarrow_{\text{p}} \alpha(I', \iota))^i]]$.

With slight modifications, we check in a similar way that if $U \downarrow_{\text{p}} Q \equiv (U_z \downarrow_{\text{p}} Q')^i$, with $z \in X$, then the appropriate elements of \mathcal{C}_{κ_0} are equal. Suppose $z \in X$, $i \in \{-1, 1\}$, and intervals $Q \subseteq \text{p}^*(U)$ and $Q' \subseteq \text{p}^*(U_z)$ are such that $U \downarrow_{\text{p}} Q \equiv (U_z \downarrow_{\text{p}} Q')^i$. By construction we know that $\text{p}^*(U'_j)$ is an initial interval in $\text{p}^*(U_j)$, with $\text{p}^*(U_j) \setminus \text{p}^*(U'_j)$ being of cardinality at most 1. Also, $\text{p}^*(U) \equiv \text{p}^*(U_0) \cdots \text{p}^*(U_n)$. Let $T \subseteq \{0, \dots, n\}$ be the set of those j such that $\text{p}^*(U_j) \cap Q \neq \emptyset$. For each $j \in T \cap J'$ we have $U \downarrow_{\text{p}} \text{p}^*(U'_j) \cap Q \equiv (U_{x_j} \downarrow_{\text{p}} \Theta_j^*)^{i_j}$ for the obvious interval $\Theta_j^* \subseteq \text{p-chunk}(U_{x_j})$, and $(U_{x_j} \downarrow_{\text{p}} \Theta_j^*)^{i \cdot i_j} \equiv U_z \downarrow_{\text{p}} Q'_j$ for an appropriate $Q'_j \subseteq \text{p}^*(U_z)$. We see that

$$\begin{aligned} [[W \downarrow_{\text{p}} \alpha(Q, \iota^{-1})]] &= \prod_{j \in T} [[W \downarrow_{\text{p}} \alpha(\text{p}^*(U_j) \cap Q, \iota^{-1})]] \\ &= \prod_{j \in L \cap J'} [[W \downarrow_{\text{p}} \alpha(\text{p}^*(U'_j) \cap Q, \iota)]] \\ &= \prod_{j \in L \cap J'} [[W_{x_j} \downarrow_{\text{p}} \alpha(\Theta_j^*, (\iota_{x_j})^{-1})]]^{i_j} \\ &= \prod_{j \in (L \cap J')^i} [[W_z \downarrow_{\text{p}} \alpha(Q'_j, \iota_y)]]^i \\ &= [[(W_z \downarrow_{\text{p}} \alpha(Q', \iota_y))^i]]. \end{aligned}$$

Similar modifications are enacted if $Q, Q' \subseteq \text{p}^*(U)$, and the proof is complete. \square

We introduce some extra notation for convenience. For a not necessarily reduced word W we let

$$\|W\| = \sup \left\{ \frac{1}{n+1} \mid n = \text{proj}_1(W(i)) \text{ for some } i \in \overline{W} \right\}$$

where the supremum is considered in the set of nonnegative reals. As examples we have $\|E\| = 0$ and $\|a_{\alpha,5}^{-1}a_{\alpha',10}\| = \frac{1}{6}$. By comparison to earlier notation, we have $d(W) = 1/\|W\| - 1$.

Lemma 3.18. *Suppose that κ_0 and κ_1 are cardinal numbers greater than or equal to 2. Suppose that $\{\text{coi}(W_x, \iota_x, U_x)\}_{x \in X}$ is coherent, $z \in X$ and that $\epsilon > 0$ is a real number. Then there exists $U \in \text{Red}_{\kappa_1}$ with $\|U\| < \epsilon$ and a coi ι from W_z to U such that $\{\text{coi}(W_x, \iota_x, U_x)\}_{x \in X} \cup \{\text{coi}(W_z, \iota, U)\}$ is coherent. Moreover the domain (and codomain) of ι may be chosen to be nonempty provided ι_z satisfies this property.*

Proof. If W_z is empty then let U be empty and $\iota = \emptyset$. Otherwise let $U_z \equiv_p \prod_{\lambda \in p^*(U_z)} U_\lambda$ and $J = \{\lambda \in p^*(U_z) \mid \|U_\lambda\| \geq \epsilon\}$. Since U_z is a word, we know that J is finite. Let $N \in \omega$ be large enough that $\frac{1}{N+1} < \epsilon$. For each $\lambda \in p^*(U_z)$ we let

$$U'_\lambda \equiv \begin{cases} U_\lambda & \text{if } \lambda \notin J, \\ a_{\alpha,N} & \text{if } \lambda \in J \text{ and } U_\lambda \text{ is } \alpha\text{-pure.} \end{cases}$$

We let $U \equiv \prod_{\lambda \in p^*(U_x)} U'_\lambda$. It is easy to see that U is reduced (a cancellation in U would necessarily include the pairing of a letter $a_{\alpha,N} \equiv U_\lambda$, with $\lambda \in J$, with a letter in $U'_{\lambda'}$ where λ' is the immediate successor or immediate predecessor of λ in $p^*(U_x)$, and thus U'_λ and $U'_{\lambda'}$ are both α -pure, so U_λ and $U_{\lambda'}$ are as well, a contradiction). Moreover $U \equiv_p \prod_{\lambda \in p^*(U_x)} U'_\lambda$ and clearly $\|U\| < \epsilon$. Letting $\iota = \iota_z$ it is immediate that ι is a coi from W_z to U . The rather intuitive fact that $\{\text{coi}(W_x, \iota_x, U_x)\}_{x \in X} \cup \{\text{coi}(W_z, \iota, U)\}$ is coherent is proved along similar lines used in earlier proofs. \square

Lemma 3.19. *Suppose that $\kappa_1 \geq 2$ and that $|X| < 2^{\aleph_0}$. Given $N \in \omega \setminus \{0\}$ and an ordinal $\alpha < \kappa_1$ there exists an α -pure word $U \in \text{Red}_{\kappa_1}$ using only positive letters such that $\|U\| = \frac{1}{N}$, and $U(\max(\bar{U})) = a_{\alpha,N-1} = U(\min(\bar{U}))$, and $U \notin \text{Pfine}(\{U_x\}_{x \in X})$.*

Proof. Assume the hypotheses. We will let $\bar{U} = [0, 1] \cap \mathbb{Q}$. It is easy to see that the set of all functions $f : ([0, 1] \cap \mathbb{Q}) \rightarrow \{a_{\alpha,n}\}_{n \geq N-1}$ such that $f(0) = f(1) = a_{\alpha,N-1}$ and the restriction $f \upharpoonright (0, 1) \cap \mathbb{Q}$ is injective is of cardinality 2^{\aleph_0} , and each such function is an element of Red_{κ_1} since there are no inverse letters with which to perform a cancellation. On the other hand we have by Lemma 3.4 that there are less than 2^{\aleph_0} pure elements in $\text{Pfine}(\{U_x\}_{x \in X})$. The lemma follows immediately. \square

3E. ω -type concatenations. In this subsection we prove the following:

Proposition 3.20. *Suppose that κ_0 and κ_1 are cardinal numbers greater than or equal to 2. Suppose that $\{\text{coi}(W_x, \iota_x, U_x)\}_{x \in X}$ is coherent, that W is reduced, that $p^*(W) \equiv \prod_{n \in \omega} I_n$ with each $I_n \neq \emptyset$, $W \upharpoonright_p I_n \in \text{Pfine}(\{W_x\}_{x \in X})$, and $W \notin \text{Pfine}(\{W_x\}_{x \in X})$. Suppose also that $|X| < 2^{\aleph_0}$. Then there exists $U \in \text{Red}_{\kappa_1}$ and a coi ι from W to U such that $\{\text{coi}(W_x, \iota_x, U_x)\}_{x \in X} \cup \{\text{coi}(W, \iota, U)\}$ is coherent.*

Proof. For each $n \in \omega$ write $W_n \equiv W \upharpoonright_p I_n$. As $W_0 \in \text{Pfine}(\{W_x\}_{x \in X})$ is nontrivial we select a word $U_0 \in \text{Red}_{\kappa_1}$ and $\text{coi } \iota_0$ from W_0 to U_0 such that the domain of ι_0 is nonempty and such that $\{\text{coi}(W_x, \iota_x, U_x)\}_{x \in X} \cup \{\text{coi}(W_0, \iota_0, U_0)\}$ is coherent, by Lemma 3.17. Assuming that the elements of $\{\text{coi}(W_i, \iota_j, U_j)\}_{j \leq m}$ have already been chosen such that $\|U_j\| < \frac{1}{2}\|U_{j-1}\|$, each ι_j has nonempty domain and also that the union of collections $\{\text{coi}(W_x, \iota_x, U_x)\}_{x \in X} \cup \{\text{coi}(W_i, \iota_j, U_j)\}_{j \leq m}$ is coherent, we use Lemmas 3.17 and 3.18 to select $U_{m+1} \in \text{Red}_{\kappa_1}$ and $\text{coi } \iota_{m+1}$ from W_{m+1} to U_{m+1} so that ι_{m+1} has nonempty domain, $\|U_{m+1}\| < \frac{1}{2}\|U_m\|$ and $\{\text{coi}(W_x, \iota_x, U_x)\}_{x \in X} \cup \{\text{coi}(W_i, \iota_j, U_j)\}_{j \leq m+1}$ is coherent.

By Lemma 3.12, the collection $\{\text{coi}(W_x, \iota_x, U_x)\}_{x \in X} \cup \{\text{coi}(W_i, \iota_j, U_j)\}_{j \in \omega}$ is coherent. For each $j \in \omega$ we will construct a word $V_j \in \text{Red}_{\kappa_1}$ with $1 \leq |\text{p}^*(V_j)| \leq 2$. Select $\alpha_j < \kappa_1$ such that the word U_j does not end with an α_j -pure subword. This is possible since $\kappa_1 \geq 2$ and U_j can end in at most one pure subword (and might possibly not end in a pure subword). By Lemma 3.19 we select an α_j -pure word $V'_j \in \text{Red}_{\kappa_1} \setminus \text{Pfine}(\{U_x\}_{x \in X} \cup \{U_i\}_{i \in \omega})$ which uses only positive letters such that $\|V'_j\| = \|U_j\|$ and $\overline{V'_j}$ has maximum and minimum elements and $V'_j(\max(\overline{V'_j})) = a_{\alpha_j, d(U_j)+1} = V'_j(\min(\overline{V'_j}))$. If U_{j+1} begins with an α_j -pure subword, then select $\alpha''_j \in \kappa_1 \setminus \{\alpha_j\}$, and again by Lemma 3.19, select $V''_j \in \text{Red}_{\kappa_1} \setminus \text{Pfine}(\{U_x\}_{x \in X} \cup \{U_i\}_{i \in \omega})$ which uses only positive letters such that $\|V''_j\| = \|U_j\|$ and $\overline{V''_j}$ has maximum and minimum elements and $V''_j(\max(\overline{V''_j})) = a_{\alpha''_j, d(U_j)-1} = V''_j(\min(\overline{V''_j}))$ and V''_j is α''_j -pure. If U_{j+1} does not begin with an α_j -pure subword then let $V''_j = E$. Let $V_j = V'_j V''_j$.

We know for each $n \in \omega$ that U_n, V'_n and V''_n are each reduced. By how V'_n was selected, we know that $U_n V'_n$ is reduced since any cancellation would need to pair letters in V'_n with those in U_n , and U_n does not end in an α_j -pure word. Similarly, $U_n V'_n V''_n \equiv U_n V_n$ is reduced.

As $\|U_n V_n\| \leq \frac{1}{2^n}$ we know the expression $\prod_{n \in \omega} U_n V_n \equiv U_0 V_0 U_1 V_1 \cdots$ is a word. By construction each of the words $\prod_{n=0}^m U_n V_n$ is reduced, and therefore the word $U \equiv \prod_{n \in \omega} U_n V_n$ is reduced. We note as well that by how V'_n and V''_n were chosen we can write $\text{p}^*(U) \equiv \prod_{n \in \omega} \text{p}^*(U_n) \text{p}^*(V_n)$, and $1 \leq |\text{p}^*(V_n)| \leq 2$. Let ι be the function $\iota = \bigcup_{j \in \omega} \iota_j$. By Lemma 3.6 (iii) the domain of ι is close in $\text{p}^*(W)$ and the range of ι is close in U , and thus we may write $\text{coi}(W, \iota, U)$. We will show that $\{\text{coi}(W_x, \iota_x, U_x)\}_{x \in X} \cup \{\text{coi}(W_i, \iota_j, U_j)\}_{j \in \omega} \cup \{\text{coi}(W, \iota, U)\}$ is coherent, from which it will immediately follow that $\{\text{coi}(W_x, \iota_x, U_x)\}_{x \in X} \cup \{\text{coi}(W, \iota, U)\}$ is coherent.

Suppose that $y \in X \cup \omega$, $\Lambda_0 \subseteq \text{p}^*(W)$ and $\Lambda_1 \subseteq \text{p}^*(W_y)$ are intervals and $i \in \{-1, 1\}$ are such that $W \upharpoonright_p \Lambda_0 \equiv (W_y \upharpoonright_p \Lambda_1)^i$. If the set $\{n \in \omega \mid I_n \cap \Lambda_0 \neq \emptyset\}$ is infinite, then by the fact that Λ_0 is an interval there exist $m \in \omega$ and intervals $I'_m, I''_m \subseteq I_m$, with I'_m possibly empty, such that $I_m \equiv I'_m I''_m$ and $\Lambda_0 \equiv I''_m \prod_{n=m+1}^\infty I_n$. Certainly $(W_y \upharpoonright_p \Lambda_1)^i \in \text{Pfine}(\{W_x\}_{x \in X} \cup \{W_n\}_{n \in \omega})$, and since $W_n \in \text{Pfine}(\{W_x\}_{x \in X})$ for each n

we have in fact that $\text{Pfine}(\{W_x\}_{x \in X} \cup \{W_n\}_{n \in \omega}) = \text{Pfine}(\{W_x\}_{x \in X})$. Therefore we have $W \downarrow_p \Lambda_0 \equiv (W_y \downarrow_p \Lambda_1)^i \in \text{Pfine}(\{W_x\}_{x \in X})$. But also $(\prod_{n=0}^{m-1} W_n)W \downarrow_p I'_m \in \text{Pfine}(\{W_x\}_{x \in X})$. Thus $W \equiv ((\prod_{n=0}^{m-1} W_n)W \downarrow_p I'_m)(W \downarrow_p \Lambda_0) \in \text{Pfine}(\{W_x\}_{x \in X})$, contrary to the assumptions of our lemma.

Thus we suppose that $y \in X \cup \omega$, $\Lambda_0 \subseteq p^*(W)$ and $\Lambda_1 \subseteq p^*(W_y)$ are intervals and $i \in \{-1, 1\}$ are such that $W \downarrow_p \Lambda_0 \equiv (W_y \downarrow_p \Lambda_1)^i$ and know from this that the set $K = \{n \in \omega \mid I_n \cap \Lambda_0 \neq \emptyset\}$ is finite. If $K = \emptyset$ then $\Lambda_0 = \emptyset = \Lambda_1$ and $[[U \downarrow_p \alpha(\Lambda_0, \iota)]] = [[E]] = [[(U_y \downarrow_p \alpha(\Lambda_1, \iota_y))^i]]$. If K has cardinality 1 then we let $K = \{m\}$ and we can write $I_m \equiv I'_m \Lambda_0 I''_m$ where either or both of I'_m and I''_m may be empty. Since $\{\text{coi}(W_x, \iota_x, U_x)\}_{x \in X} \cup \{\text{coi}(W_j, \iota_j, U_j)\}_{j \in \omega}$ is coherent, we have

$$[[U \downarrow_p \alpha(\Lambda_0, \iota)]] = [[U_m \downarrow_p \alpha(\Lambda, \iota_m)]] = [[(U_y \downarrow_p \alpha(\Lambda_1, \iota_y))^i]].$$

If K is of cardinality at least 2 then we let m_a and m_b be respectively the minimal and maximal elements and write $I_{m_a} \equiv I'_{m_a} I''_{m_a}$, $I_{m_b} \equiv I'_{m_b} I''_{m_b}$ (where either or both of I'_{m_a} and I''_{m_b} may be empty) and $\Lambda_0 \equiv I''_{m_a} I_{m_a+1} \cdots I_{m_b-1} I'_{m_b}$. As $W \downarrow_p \Lambda_0 \equiv (W_y \downarrow_p \Lambda_1)^i$, there exist subintervals $J_0, \dots, J_{m_b-m_a}$ of Λ_1 such that $W \downarrow_p I_j \equiv (W_y \downarrow_p J_j - m_a)^i$ for $m_a < j < m_b$ and $W \downarrow_p I''_{m_a} \equiv (W_y \downarrow_p J_0)^i$ and $W \downarrow_p I'_{m_b} \equiv (W_y \downarrow_p J_{m_b-m_a})^i$. Since $\{\text{coi}(W_x, \iota_x, U_x)\}_{x \in X} \cup \{\text{coi}(W_j, \iota_j, U_j)\}_{j \in \omega}$ is coherent, we have

$$\begin{aligned} [[U \downarrow_p \alpha(\Lambda_0, \iota)]] &= [[U_{m_a} \downarrow_p \alpha(I''_{m_a}, \iota_{m_a})]][[U_{m_a+1} \downarrow_p \alpha(I_{m_a+1}, \iota_{m_a+1})]] \\ &\quad \cdots [[U_{m_b-1} \downarrow_p \alpha(I_{m_b-1}, \iota_{m_b-1})]][[U \downarrow_p \alpha(I'_{m_b}, \iota_{m_b})]] \\ &= \prod_{j \in \{0, \dots, m_b-m_a\}} [[(U_y \downarrow_p \alpha(J_j, \iota_y))^i]] \\ &= [[(U_y \downarrow_p \alpha(\Lambda_1, \iota_y))^i]]. \end{aligned}$$

Suppose now that $\Lambda_0, \Lambda_1 \subseteq p^*(W)$ are intervals and $i \in \{-1, 1\}$ are such that $W \downarrow_p \Lambda_0 \equiv (W \downarrow_p \Lambda_1)^i$. Let $K_0 = \{n \in \omega \mid I_n \cap \Lambda_0 \neq \emptyset\}$ and $K_1 = \{n \in \omega \mid I_n \cap \Lambda_1 \neq \emptyset\}$.

Case 1. K_0 is infinite. In this case, if K_1 is finite then $W \downarrow_p \Lambda_0 \in \text{Pfine}(\{W_x\}_{x \in X})$, and we have already seen that this implies $W \in \text{Pfine}(\{W_x\}_{x \in X})$ since K_0 is infinite, and this is a contradiction. Thus K_1 must be infinite in this case. If $i = -1$ then $W \downarrow_p \Lambda_0 \equiv (W \downarrow_p \Lambda_1)^{-1}$. As Λ_0 and Λ_1 are terminal intervals in $p^*(W)$, let without loss of generality $\Lambda_0 \subseteq \Lambda_1$ and set $Q \equiv W \downarrow_p \Lambda_0$. Then $W \downarrow_p \Lambda_1 \equiv Q^{-1} \equiv PQ$ for nonempty Q and some possibly empty P . Then $Q \equiv Q^{-1}P^{-1} \equiv PQP^{-1}$, so $P \equiv E$, forcing $\Lambda_0 = \Lambda_1$. Then $W \equiv AQ \equiv AQ^{-1}$ for a possibly empty A . Write $Q \equiv BCB^{-1}$ for nonempty cyclically reduced C . Then WW^{-1} has nonempty reduced representative $ABCCB^{-1}A^{-1}$, contradiction.

Therefore $i = 1$ and $W \downarrow_p \Lambda_0 \equiv W \downarrow_p \Lambda_1$, and both Λ_0 and Λ_1 are infinite terminal intervals in $p^*(W)$. If without loss of generality Λ_1 is a proper subinterval of Λ_0 , then since $W \downarrow_p \Lambda_0 \equiv W \downarrow_p \Lambda_1$ we can select a proper terminal subinterval $\Lambda_2 \subseteq \Lambda_1$ such that $W \downarrow_p \Lambda_1 \equiv W \downarrow_p \Lambda_2$, and inductively we select proper terminal subintervals

$\Lambda_{i+1} \subseteq \Lambda_i$ with $W \downarrow_p \Lambda_i \equiv W \downarrow_p \Lambda_{i+1}$. Thus, letting $\lambda \in \Lambda_0 \setminus \Lambda_1$ we see that the nonempty $W \downarrow_p \{\lambda\}$ occurs infinitely often as a subword of W , so that W is not a word, a contradiction. Thus $\Lambda_0 = \Lambda_1$ and $[[U \downarrow_p \alpha(\Lambda_0, \iota)]] = [[(U \downarrow_p \alpha(\Lambda_1, \iota)^i)]]$.

Case 2. K_0 is finite. In this case we know that K_1 is also finite (by applying the argument in Case 1, since $W \downarrow_p \Lambda_1 \equiv (W \downarrow_p \Lambda_0)^i$). Thus $W \downarrow_p \Lambda_0 \in \text{Pfine}(\{W_n\}_{n \in \omega})$. If $K_0 = \emptyset$ then so also $K_1 = \emptyset = \Lambda_0 = \Lambda_1$ and it is easy to see that $[[U \downarrow_p \alpha(\Lambda_0, \iota)]] = [[E]] = [[(U \downarrow_p \alpha(\Lambda_1, \iota)^i)]]$. In case $K_0 \neq \emptyset$, from the correspondence $W \downarrow_p \Lambda_0 \equiv (W \downarrow_p \Lambda_1)^i$ we decompose $\Lambda_0 \equiv \Theta_0 \Theta_1 \cdots \Theta_m$ and $\Lambda_1 \equiv \Theta'_0 \Theta'_1 \cdots \Theta'_m$ so that $W \downarrow_p \Theta_j \equiv (W \downarrow_p \Theta'_{f(j)})^i$ where

$$f(j) = \begin{cases} j & \text{if } i = 1, \\ m - j & \text{if } i = -1, \end{cases}$$

and each Θ_j is a subinterval of one of $I_{\min(K_0)}, \dots, I_{\max(K_0)}$ and each Θ'_j is a subinterval of one of $I_{\min(K_1)}, \dots, I_{\max(K_1)}$. Let $f_0: \{0, \dots, m\} \rightarrow \{\min(K_0), \dots, \max(K_0)\}$ be the nondecreasing surjective function given by $\Theta_j \subseteq I_{f_0(j)}$, and also let $f_1: \{0, \dots, m\} \rightarrow \{\min(K_1), \dots, \max(K_1)\}$ be given by $\Theta'_j \subseteq I_{f_1(j)}$. We have

$$\begin{aligned} [[U \downarrow_p \alpha(\Lambda_0, \iota)]] &= \prod_{j=0}^m [[U_{f_0(j)} \downarrow_p \alpha(\Theta_j, \iota_{f_0(j)})]] \\ &= \prod_{j=0}^m [[(U_{f_1(f(j))} \downarrow_p \alpha(\Theta_{f(j)}, \iota_{f_1(f(j))})^i)]] \\ &= [[(U \downarrow_p \alpha(\Lambda_1, \iota)^i)]] \end{aligned}$$

where the first and third equalities hold by performing a deletion of finitely many pure words in Red_{κ_1} (Lemma 3.13) and the second equality holds by the coherence of the collection $\{\text{coi}(W_n, \iota_n, U_n)\}_{n \in \omega}$. This completes case 2 and this part of the argument.

Suppose $y \in X \cup \omega$, $\Lambda_0 \subseteq p^*(U)$ and $\Lambda_1 \subseteq p^*(U_y)$ are intervals and $i \in \{-1, 1\}$ are such that $U \downarrow_p \Lambda_0 \equiv (U_y \downarrow_p \Lambda_1)^i$. Recalling that $U \equiv \prod_{n \in \omega} (U_n V_n)$ and none of the nonempty p-chunks of V_n are in $\text{Pfine}(\{U_x\}_{x \in X} \cup \{U_n\}_{n \in \omega})$ we see that $\Lambda_0 \subseteq p^*(U_n)$ for some $n \in \omega$. From the coherence of $\{\text{coi}(W_n, \iota_n, U_n)\}_{n \in \omega} \cup \{\text{coi}(W_x, \iota_x, U_x)\}_{x \in X}$ it is easy to see that $[[W \downarrow_p \alpha(\Lambda_0, \iota^{-1})]] = [[(W_y \downarrow_p \alpha(\Lambda_1, \iota_y^{-1}))^i]]$.

Finally suppose intervals $\Lambda_0, \Lambda_1 \subseteq p^*(U)$ and $i \in \{-1, 1\}$ are such that $U \downarrow_p \Lambda_0 \equiv (U \downarrow_p \Lambda_1)^i$. Recall that $U \equiv \prod_{n \in \omega} U_n V_n$ with

$$p^*(U) \equiv \prod_{n \in \omega} p^*(U_n) p^*(V_n)$$

and for all $n \in \omega$ we have $\|U_n\| = \|V_n\| \geq 2\|U_{n+1}\|$ and V_n uses only positive letters, satisfies $1 \leq |p^*(V_n)| \leq 2$ and every nonempty p-chunk of V_n is not an element of $\text{Pfine}(\{U_x\}_{x \in X} \cup \{U_n\}_{n \in \omega})$.

If there exists $\lambda \in \Lambda_0$ and $n \in \omega$ such that $\lambda \in p^*(V_n)$ then $i = 1$ since every pure p -chunk of U which is not in $\text{Pfine}(\{U_x\}_{x \in X} \cup \{U_n\}_{n \in \omega})$ is a p -chunk in some V_m and therefore has positive letters only. Furthermore the order isomorphism $h : \Lambda_0 \rightarrow \Lambda_1$ induced by the word equivalence $U \upharpoonright_p \Lambda_0 \equiv U \upharpoonright_p \Lambda_1$ must have $h(\lambda) = \lambda$, for if $U \upharpoonright_p \{\lambda\}$ is, say, α -pure, then $U \upharpoonright_p \{\lambda\}$ is the unique α -pure p -chunk of U which has value $\|U \upharpoonright_p \{\lambda\}\|$ under the function $\|\cdot\|$. But this implies that h is the identity function since if, say, $\lambda' < \lambda$ and $h(\lambda') < \lambda'$, then $\lambda' < h^{-1}(\lambda') < h^{-2}(\lambda') < \dots < \lambda$ and so the word $U \upharpoonright_p \Lambda_0$ has infinitely many disjoint occurrences of subwords equivalent to $U \upharpoonright_p \{\lambda'\}$, which contradicts the fact that U is a word. Thus $\Lambda_0 = \Lambda_1$ and obviously $[[W \upharpoonright_p \alpha(\Lambda_0, \iota^{-1})]] = [[W \upharpoonright_p \alpha(\Lambda_1, \iota^{-1})]]$.

On the other hand if $\Lambda_0 \cap p^*(V_n) = \emptyset$ for all $n \in \omega$ then $\Lambda_0 \subseteq p^*(U_m)$ for some $m \in \omega$. Thus $U \upharpoonright_p \Lambda_0 \in \text{Pfine}(\{U_x\}_{x \in X} \cup \{U_n\}_{n \in \omega})$, so $\Lambda_1 \cap p^*(V_n) = \emptyset$ for all $n \in \omega$ as well. Thus $\Lambda_1 \subseteq p^*(U_{m'})$ for some $m' \in \omega$. Then

$$\begin{aligned} [[W \upharpoonright_p \alpha(\Lambda_0, \iota^{-1})]] &= [[W_m \upharpoonright_p \alpha(\Lambda_0, \iota_m^{-1})]] \\ &= [[(W_{m'} \upharpoonright_p \alpha(\Lambda_1, \iota_{m'}^{-1}))^i]] \\ &= [[(W \upharpoonright_p \alpha(\Lambda_1, \iota^{-1}))^i]] \end{aligned}$$

since $U_m \upharpoonright_p \Lambda_0 \equiv U_{m'} \upharpoonright_p \Lambda_1$ and $\{\text{coi}(W_n, \iota_n, U_n)\}_{n \in \omega}$ is coherent. □

3F. \mathbb{Q} -type concatenations. In this subsection we will devote our attention to proving the following:

Proposition 3.21. *Suppose that κ_0 and κ_1 are cardinal numbers greater than or equal to 2. Suppose that $\{\text{coi}(W_x, \iota_x, U_x)\}_{x \in X}$ is coherent, that $p^*(W) \equiv \prod_{q \in \mathbb{Q}} I_q$ with each $I_q \neq \emptyset$, $W \upharpoonright_p I_q \in \text{Pfine}(\{W_x\}_{x \in X})$ for each $q \in \mathbb{Q}$, and $W \upharpoonright_p \bigcup \Lambda \notin \text{Pfine}(\{W_x\}_{x \in X})$ for each interval $\Lambda \subseteq \mathbb{Q}$ with more than one point. Suppose also that $|X| < 2^{\aleph_0}$. Then there exists $U \in \text{Red}_{\kappa_1}$ and a coi ι from W to U such that $\{\text{coi}(W_x, \iota_x, U_x)\}_{x \in X} \cup \{\text{coi}(W, \iota, U)\}$ is coherent.*

Proof. Let $\{W_n\}_{n \in \omega}$ be a list such that for each $q \in \mathbb{Q}$ we have some $n \in \omega$ for which either $W \upharpoonright_p I_q \equiv W_n$ or $W \upharpoonright_p I_q \equiv W_n^{-1}$, and $n \neq n'$ implies $W_n \not\equiv W_{n'} \not\equiv W_n^{-1}$. Notice that indeed such a list must be infinite, for otherwise there is some $q' \in \mathbb{Q}$ such that $\{q \in \mathbb{Q} \mid W \upharpoonright_p I_q \equiv W \upharpoonright_p I_{q'}\}$ is infinite, which contradicts the fact that W is a word. By assumption, $\{W_n\}_{n \in \omega} \subseteq \text{Pfine}(\{W_x\}_{x \in X})$. Select $P_0 \in \text{Red}_{\kappa_1}$ and a coi ι_0 from W_0 to P_0 with nonempty domain such that $\{\text{coi}(W_x, \iota_x, U_x)\}_{x \in X} \cup \{\text{coi}(W_0, \iota_0, P_0)\}$ is coherent by Lemma 3.17. Assuming we have chosen P_n and ι_n we select $P_{n+1} \in \text{Red}_{\kappa_1}$ and a coi ι_{n+1} from W_{n+1} to P_{n+1} such that $\|P_{n+1}\| \leq \frac{1}{2} \|P_n\|$, the domain of ι_{n+1} is nonempty, and $\{\text{coi}(W_x, \iota_x, U_x)\}_{x \in X} \cup \{\text{coi}(W_j, \iota_j, P_j)\}_{j=0}^{n+1}$ is coherent by Lemmas 3.17 and 3.18. The collection $\{\text{coi}(W_x, \iota_x, U_x)\}_{x \in X} \cup \{\text{coi}(W_n, \iota_n, P_n)\}_{n \in \omega}$ is coherent by Lemma 3.12.

For each $m \in \omega$ select ordinals $\alpha_{m,b}, \alpha_{m,c} < \kappa_1$ such that P_m does not begin with an initial subword which is $\alpha_{m,b}$ -pure and P_m does not end with a terminal

subword which is $\alpha_{m,c}$ -pure. By Lemma 3.19 we select an $\alpha_{m,b}$ -pure word $V_{m,b}$ which uses only positive letters such that $\|V_{m,b}\| = \|P_m\|$, and $V_{m,b}(\max(\overline{V_{m,b}})) = a_{\alpha_{m,b},d(P_m)+1} = V_{m,b}(\min(\overline{V_{m,b}}))$ and $V_{m,b} \notin \text{Pfine}(\{U_{x \in X}\}_{x \in X} \cup \{P_n\}_{n \in \omega})$. Similarly select an $\alpha_{m,c}$ -pure word $V_{m,c}$ which uses only positive letters such that $\|V_{m,c}\| = \|P_m\|$, and $V_{m,c}(\max(\overline{V_{m,c}})) = a_{\alpha_{m,c},d(P_m)+1} = V_{m,c}(\min(\overline{V_{m,c}}))$ and $V_{m,c} \notin \text{Pfine}(\{U_{x \in X}\}_{x \in X} \cup \{P_n\}_{n \in \omega})$.

Define functions $f_0 : \mathbb{Q} \rightarrow \omega$ and $f_1 : \mathbb{Q} \rightarrow \{\pm 1\}$ by $W \upharpoonright_p I_q \equiv W_{f_0(q)}^{f_1(q)}$. For each $m \in \omega$ the preimage $f_0^{-1}(m)$ is nonempty (by how the list $\{W_n\}_{n \in \omega}$ was chosen) and finite (since W is a word). For each $q \in \mathbb{Q}$ let $U_q \equiv (V_{f_0(q),b} P_{f_0(q)} V_{f_0(q),c})^{f_1(q)}$ and $U \equiv \prod_{q \in \mathbb{Q}} U_q$. Notice that this is a word since for each real number $\epsilon > 0$ the set $\{q \in \mathbb{Q} \mid \|U_q\| \geq \epsilon\}$ is finite. It is easy to see that each U_q is reduced and that moreover $p^*(P_{f_0(q)}^{f_1(q)})$ is a subinterval of $p^*(U_q)$ and $|p^*(U_q) \setminus p^*(P_{f_0(q)}^{f_1(q)})| = 2$.

Lemma 3.22. *U is reduced.*

Proof. For each $n \in \omega$ we let $J_n = \{q \in \mathbb{Q} \mid \|U_q\| = \frac{1}{n+1}\}$. We see that each J_n is finite since U is a word. For any cancellation \mathcal{S} on U we define $L_n(\mathcal{S})$ to be the set of those $q \in J_n$ for which there exists $i \in \overline{U_q}$ which occurs in some ordered pair in \mathcal{S} . Define $L'_n(\mathcal{S}) \subseteq L_n(\mathcal{S})$ to be the set of all $q \in L_n(\mathcal{S})$ for which there exists a unique $q' \in L_n(\mathcal{S})$ such that \mathcal{S} pairs each element in $\overline{U_q}$ with an element in $\overline{U_{q'}}$ and each element in $\overline{U_{q'}}$ with an element in $\overline{U_q}$. Our strategy will be to assume for contradiction that a nonempty cancellation over U exists and then to inductively modify the cancellation into a cancellation which witnesses a cancellation over W , contradicting the reducedness of W .

Suppose that \mathcal{S}_0 is a nonempty cancellation over U and let n_0 be minimal such that $L_{n_0}(\mathcal{S}) \neq \emptyset$. If $L_{n_0}(\mathcal{S}_0) = L'_{n_0}(\mathcal{S}_0)$ then we write $\mathcal{S}_1 = \mathcal{S}_0$ and move on to the next step of our induction. If $L_{n_0}(\mathcal{S}_0) \neq L'_{n_0}(\mathcal{S}_0)$ then we write $L_{n_0}(\mathcal{S}_0) \setminus L'_{n_0}(\mathcal{S}_0) = \{q_0, \dots, q_k\}$ with $q_r < q_{r+1}$ under the ordering on \mathbb{Q} . Define a relation E on $L_{n_0}(\mathcal{S}_0) \setminus L'_{n_0}(\mathcal{S}_0)$ by writing $E(q_{r_0}, q_{r_1})$, where $q_{r_0}, q_{r_1} \in L_{n_0}(\mathcal{S}_0) \setminus L'_{n_0}(\mathcal{S}_0)$, if there exist $i_0 \in \overline{U_{q_{r_0}}}$ and $i_1 \in \overline{U_{q_{r_1}}}$ such that $\langle i_0, i_1 \rangle \in \mathcal{S}_0$. Since each U_q is reduced we see that $E(q_r, q_r)$ is false for all $0 \leq r \leq k$. Also, $E(q_{r_0}, q_{r_1})$ implies that $q_{r_0} < q_{r_1}$ since $\langle i_0, i_1 \rangle \in \mathcal{S}_0$ implies $i_0 < i_1$ in \overline{U} . By how each U_q is defined, we see that $U_q(\min(\overline{U_q})) = U_q(\max(\overline{U_q})) \in \{a_{\alpha_{n_0}, n_0}^{\pm 1}\}$ for each $q \in L_{n_0}(\mathcal{S}_0)$. For $q' \in \bigcup_{n > n_0} L_n(\mathcal{S}_0)$ we have $\|U_{q'}\| < 1/(n_0 + 1)$. Since U_q is reduced for each $q \in L_{n_0}(\mathcal{S}_0)$, we see that for each $q \in L_{n_0}(\mathcal{S}_0)$ at least one of $\max(\overline{U_q})$ or $\min(\overline{U_q})$ must appear in some element of \mathcal{S}_0 . Moreover, by how $L'_n(\mathcal{S}_0)$ is defined, for each $q \in L_{n_0}(\mathcal{S}_0) \setminus L'_{n_0}(\mathcal{S}_0)$ at least one of $\max(\overline{U_q})$ or $\min(\overline{U_q})$ must appear in \mathcal{S}_0 and be paired with some element in $\overline{U_{q'}}$ for some $q' \in L_{n_0}(\mathcal{S}_0) \setminus (L'_{n_0}(\mathcal{S}_0) \cup \{q\})$.

Thus we see that each $q \in L_{n_0}(\mathcal{S}_0) \setminus L'_{n_0}(\mathcal{S}_0)$ must appear as a first or second coordinate in the relation E . Notice as well that if $E(q_{r_0}, q_{r_1})$ and $E(q_{r_2}, q_{r_3})$ where $q_{r_0} < q_{r_2} \leq q_{r_1}$ then $q_{r_0} < q_{r_2} < q_{r_3} \leq q_{r_1}$ by property (4) of cancellations (see

Definition 2.3). Similarly if $E(q_{r_0}, q_{r_1})$ and $E(q_{r_2}, q_{r_3})$ hold and $q_{r_0} \leq q_{r_3} < q_{r_1}$ then we have $q_{r_0} \leq q_{r_2} < q_{r_3} < q_{r_1}$. Since the set $L_{n_0}(\mathcal{S}_0) \setminus L'_{n_1}(\mathcal{S}_0)$ is finite, we therefore have some $0 \leq r < k$ such that $E(q_r, q_{r+1})$. Again, since U_{q_r} and $U_{q_{r+1}}$ are each reduced we must have $\langle \max(\overline{U_{q_r}}, \min(\overline{U_{q_{r+1}}}) \rangle \in \mathcal{S}_0$. Thus $U_{q_r} \equiv (U_{q_{r+1}})^{-1}$ and we let $f : \overline{U_{q_r}} \rightarrow \overline{U_{q_{r+1}}}$ be an order reversing bijection with $U_{q_{r+1}}(f(i)) = (U_{q_r}(i))^{-1}$ witnessing this equivalence.

We let $\mathcal{S}_0^{(1)}$ be given by

$$\begin{aligned} \mathcal{S}_0^{(1)} = \{ \langle i_0, i_1 \rangle \in \mathcal{S}_0 \mid i_0, i_1 \notin \overline{U_{q_r}} \cup \overline{U_{q_{r+1}}} \} \\ \cup \{ \langle i_0, f(i_0) \rangle \mid i_0 \in \overline{U_{q_r}} \} \\ \cup \{ \langle i_0, i_1 \rangle \in \overline{U} \times \overline{U} \mid (\exists i_2 \in \overline{U_{q_r}}) \langle i_0, i_2 \rangle, \langle f(i_2), i_1 \rangle \in \mathcal{S}_0 \} \\ \cup \{ \langle i_0, i_1 \rangle \in \overline{U} \times \overline{U} \mid (\exists i_2 \in \overline{U_{q_r}}) \langle i_1, i_2 \rangle, \langle i_0, f(i_2) \rangle \in \mathcal{S}_0 \} \\ \cup \{ \langle i_0, i_1 \rangle \in \overline{U} \times \overline{U} \mid (\exists i_2 \in \overline{U_{q_r}}) \langle i_2, i_1 \rangle, \langle f(i_2), i_0 \rangle \in \mathcal{S}_0 \}. \end{aligned}$$

It is straightforward to see that $\mathcal{S}_0^{(1)}$ is a cancellation and $L_n(\mathcal{S}_0^{(1)}) \subseteq L_n(\mathcal{S}_0)$ for all $n \in \omega$. But also $L'_{n_0}(\mathcal{S}_0^{(1)}) = L'_{n_0}(\mathcal{S}_0) \sqcup \{q_r, q_{r+1}\}$. Iterating the argument to produce $\mathcal{S}_0^{(2)}, \mathcal{S}_0^{(3)}$, etc. so as to make $L'_{n_0}(\mathcal{S}_0^{(j+1)})$ strictly include $L'_{n_0}(\mathcal{S}_0^{(j)})$ and have $L_{n_0}(\mathcal{S}_0^{(j+1)}) \subseteq L_{n_0}(\mathcal{S}_0^{(j)})$, we see, since $L_{n_0}(\mathcal{S}_0)$ is finite, that eventually $L'_{n_0}(\mathcal{S}_0^{(j)}) = L_{n_0}(\mathcal{S}_0^{(j)})$. Set $\mathcal{S}_1 = \mathcal{S}_0^{(j)}$ for sufficiently large j .

Notice that \mathcal{S}_1 does not pair any element of $\overline{U_q}$ with $\overline{U_{q'}}$ when $q \in L_{n_0}(\mathcal{S}_1)$ and $q' \notin L_{n_0}(\mathcal{S}_1)$. Letting $n_1 \in \omega$ be minimal such that $n_1 > n_0$ and $L_{n_1}(\mathcal{S}_1) \neq \emptyset$ (an $n > n_0$ with $L_n(\mathcal{S}_1) \neq \emptyset$ must exist since \mathbb{Q} is order dense), we may thus repeat the arguments as before to create \mathcal{S}_2 such that $L_{n_1}(\mathcal{S}_2) = L'_{n_1}(\mathcal{S}_2)$ and also \mathcal{S}_2 agrees with \mathcal{S}_1 on $L_{n_0}(\mathcal{S}_1) = L_{n_0}(\mathcal{S}_2)$. Select $n_2 > n_1$ which is minimal such that $L_{n_2}(\mathcal{S}_2) \neq \emptyset$, produce \mathcal{S}_3 , and continue this process inductively. Let \mathcal{S}_∞ equal $\{ \langle i_0, i_1 \rangle \mid (\exists p \in \omega) i_0, i_1 \in \bigcup_{q \in L_{n_p}} U_q \text{ and } \langle i_0, i_1 \rangle \in \mathcal{S}_{p+1} \}$ and we have that \mathcal{S}_∞ is a cancellation such that $L_n(\mathcal{S}_\infty) = L'_n(\mathcal{S}_\infty)$ for all $n \in \omega$ and $\mathcal{S}_\infty \neq \emptyset$.

But now let $\mathcal{S}' = \{ \langle q_0, q_1 \rangle \mid \exists (i_0 \in \overline{U_{q_0}}, i_1 \in \overline{U_{q_1}}) \langle i_0, i_1 \rangle \in \mathcal{S}_\infty \}$ and notice that \mathcal{S}' is a pairing of a subset of elements in \mathbb{Q} that satisfies the comparable properties (1)–(4) of Definition 2.3, and $\langle q_0, q_1 \rangle \in \mathcal{S}'$ implies that $U_{q_0} \equiv (U_{q_1})^{-1}$. Then $W_{q_0} \equiv (W_{q_1})^{-1}$ for $\langle q_0, q_1 \rangle \in \mathcal{S}'$ and it is easy to use \mathcal{S}' to define a nonempty cancellation \mathcal{S} on W , and we have a contradiction. \square

Now that we know that U is reduced, it is easy to see that

$$p^*(U) \equiv \prod_{q \in \mathbb{Q}} p^*(U_q) \equiv \prod_{q \in \mathbb{Q}} (p^*(V_{f_0(q),b}) p^*(P_{f_0(q)}) p^*(V_{f_0(q),c}))^{f_1(q)}.$$

Using the collection $\{\text{coi}(W_n, \iota_n, P_n)\}_{n \in \omega}$ we define the $\text{coi } \iota$ from W to U in the natural way. Namely, let T_q denote the subword $W \upharpoonright_p I_q$, and recall that $W_{f_0(q)}^{f_1(q)} \equiv T_q$ and $U_q \equiv (V_{f_0(q)} P_{f_0(q)} V_{f_0(q)})^{f_1(q)}$. Let $g : p^*(P_{f_0(q)}^{f_1(q)}) \rightarrow p^*(U_q)$ denote the order embedding given by this last equivalence, and ι_q be the function whose domain

$\text{dom}(\iota_q)$ is the image of $\text{dom}(\iota_{f_0(q)})$ under the order isomorphism $f : \mathfrak{p}^*(W_{f_0(q)}^{f_1(q)}) \rightarrow \mathfrak{p}^*(W_q)$, whose image lies in $\mathfrak{p}^*(U_q)$ and such that $\iota_q(i) = g \circ \iota_{f_0(q)} \circ f^{-1}(i)$.

Notice that ι_q is an order isomorphism between its domain and image since $\iota_{f_0(q)}$ is order preserving and exactly one of the following holds:

- f is an order isomorphism between $\mathfrak{p}^*(T_q)$ and $\mathfrak{p}^*(W_{f_0(q)})$ and g is an order embedding from $\mathfrak{p}^*(P_{f_0(q)})$ to $\mathfrak{p}^*(U_q)$.
- f gives an order reversing bijection between $\mathfrak{p}^*(T_q)$ and $\mathfrak{p}^*(W_{f_0(q)})$ and g gives an order reversing embedding from $\mathfrak{p}^*(P_{f_0(q)})$ to $\mathfrak{p}^*(U_q)$.

Since $\text{Close}(\text{dom}(\iota_n), \mathfrak{p}^*(W_n))$, the relation $\text{Close}(\text{dom}(\iota_q), \mathfrak{p}^*(T_q))$ is easily seen to hold. Also, since $|\mathfrak{p}^*(V_{f_0(q),b})| = 1 = |\mathfrak{p}^*(V_{f_0(q),c})|$, we easily see that $\text{Close}(\text{im}(\iota_q), \mathfrak{p}^*(U_q))$. Let ι be the order isomorphism given by $\iota = \bigcup_{q \in \mathbb{Q}} \iota_q$. By Lemma 3.6 (iii) we have $\text{Close}(\text{dom}(\iota), \mathfrak{p}^*(W))$ and $\text{Close}(\text{im}(\iota), \mathfrak{p}^*(U_q))$, so ι is a coi from W to U . We check the coherence of

$$\{\text{coi}(W_x, \iota_x, U_x)\}_{x \in X} \cup \{\text{coi}(W_n, \iota_n, P_n)\}_{n \in \omega} \cup \{\text{coi}(W, \iota, U)\},$$

which will imply the coherence of $\{\text{coi}(W_x, \iota_x, U_x)\}_{x \in X} \cup \{\text{coi}(W, \iota, U)\}$.

Suppose that $x_0 \in X \cup \omega$, $\Lambda_0 \subseteq \mathfrak{p}^*(W)$ and $\Lambda_1 \subseteq \mathfrak{p}^*(W_{x_0})$ are intervals, and $i \in \{-1, 1\}$ are such that $W \upharpoonright_{\mathfrak{p}} \Lambda_0 \equiv (W_{x_0} \upharpoonright_{\mathfrak{p}} \Lambda_1)^i$. Notice that Λ_0 must be a subinterval of some $\mathfrak{p}^*(T_q)$ since \mathbb{Q} is order dense, $W \upharpoonright_{\mathfrak{p}} \Lambda \notin \text{Pfine}(\{W_x\}_{x \in X})$ for each interval $\Lambda \subseteq \mathbb{Q}$ with more than one point and $(W_{x_0} \upharpoonright_{\mathfrak{p}} \Lambda_1)^i \in \text{Pfine}(\{W_x\}_{x \in X} \cup \{W_n\}_{n \in \omega}) = \text{Pfine}(\{W_x\}_{x \in X})$. But letting $f : \mathfrak{p}^*(W_{f_0(q)}^{f_1(q)}) \rightarrow \mathfrak{p}^*(T_q)$ be the natural order isomorphism and $\Lambda'_0 \subseteq \mathfrak{p}^*(W_{f_0(q)}^{f_1(q)})$ be the interval given by $f^{-1}(\Lambda_0)$, it is easy to see that

$$[[U \upharpoonright_{\mathfrak{p}} \alpha(\Lambda_0, \iota)]] = [[(P_{f_0(q)} \upharpoonright_{\mathfrak{p}} \alpha(\Lambda'_0, \iota_{f_0(q)}))^{f_1(q)}]] = [[(U_{x_0} \upharpoonright_{\mathfrak{p}} \alpha(\Lambda_1, \iota_{x_0}))^i]]$$

by how the function ι_q was defined (for the first equality) and the coherence of $\{\text{coi}(W_x, \iota_x, U_x)\}_{x \in X} \cup \{\text{coi}(W_n, \iota_n, P_n)\}_{n \in \omega}$ (for the second equality).

Next, suppose that $\Lambda_0, \Lambda_1 \subseteq \mathfrak{p}^*(W)$ are intervals and $i \in \{-1, 1\}$ are such that $W \upharpoonright_{\mathfrak{p}} \Lambda_0 \equiv (W \upharpoonright_{\mathfrak{p}} \Lambda_1)^i$. Let $J_0 = \{q \in \mathbb{Q} \mid \mathfrak{p}^*(T_q) \cap \Lambda_0 \neq \emptyset\}$ and $J_1 = \{q \in \mathbb{Q} \mid \mathfrak{p}^*(T_q) \cap \Lambda_1 \neq \emptyset\}$. Clearly each of J_0 and J_1 are intervals in \mathbb{Q} . If, say, J_0 is empty or a singleton then $W \upharpoonright_{\mathfrak{p}} \Lambda_0 \in \text{Pfine}(\{W_x\}_{x \in X})$, and so J_1 is not infinite (since we are assuming $W \upharpoonright_{\mathfrak{p}} \Lambda \notin \text{Pfine}(\{W_x\}_{x \in X})$ for each interval $\Lambda \subseteq \mathbb{Q}$ with more than one point). Similarly if J_1 is empty or a singleton then J_0 is finite (hence a singleton or empty). In case J_0 is finite we can argue as before, using the coherence of the collection $\{\text{coi}(W_n, \iota_n, P_n)\}_{n \in \omega}$ to obtain $[[U \upharpoonright_{\mathfrak{p}} \alpha(\Lambda_0, \iota)]] = [[(U \upharpoonright_{\mathfrak{p}} \alpha(\Lambda_1, \iota))^i]]$.

Suppose now that J_0 (and therefore also J_1) is infinite. Since J_0 is order dense and $W \upharpoonright_{\mathfrak{p}} \Lambda \notin \text{Pfine}(\{W_x\}_{x \in X})$ for each interval $\Lambda \subseteq \mathbb{Q}$ with more than one point, we notice that J_0 has a minimum if and only if the word $W \upharpoonright_{\mathfrak{p}} \Lambda_0$ has a nonempty initial subword which is an element of $\text{Pfine}(\{W_x\}_{x \in X})$. Also, if J_0 has minimum q

then $W \downarrow_p (p^*(W_q) \cap \Lambda_0)$ is the maximal initial subword of $W \downarrow_p \Lambda_0$ which is an element in $\text{Pfine}(\{W_x\}_{x \in X})$. Similarly J_0 has a maximum if and only if the word $W \downarrow_p \Lambda_0$ has a nonempty terminal subword which is an element of $\text{Pfine}(\{W_x\}_{x \in X})$, and if J_0 has maximum q then $W \downarrow_p (p^*(W_q) \cap \Lambda_0)$ is the maximal terminal subword of $W \downarrow_p \Lambda_0$ which is an element in $\text{Pfine}(\{W_x\}_{x \in X})$. Let $J'_0 \subseteq J_0$ be the subinterval which consists of J_0 minus any maximum or minimum that J_0 might have. By similar reasoning, we see that for each $q \in J'_0$ the subword T_q is a maximal subword of $W \downarrow_p \Lambda_0$ which is an element of $\text{Pfine}(\{W_x\}_{x \in X})$.

The comparable claims hold for J_1 ; for example J_1 has a minimum if and only if the word $W \downarrow_p \Lambda_1$ has a nonempty initial subword which is an element of $\text{Pfine}(\{W_x\}_{x \in X})$, and if $q \in J_1$ is minimal then $W \downarrow_p (p^*(T_q) \cap \Lambda_1)$ is the maximal initial subword of $W \downarrow_p \Lambda_1$ which is an element in $\text{Pfine}(\{W_x\}_{x \in X})$. Define the interval $J'_1 \subseteq J_1$ similarly. As $W \downarrow_p \Lambda_0 \equiv (W \downarrow_p \Lambda_1)^i$, we see that if $i = 1$:

- J_0 has a minimum if and only if J_1 has one.
- J_0 has a maximum if and only if J_1 has one.
- If $q_0 = \min(J_0)$ and $q_1 = \min(J_1)$, $W \downarrow_p (\Lambda_0 \cap p^*(T_{q_0})) \equiv W \downarrow_p (\Lambda_1 \cap p^*(T_{q_1}))$.
- If $q_0 = \max(J_0)$ and $q_1 = \max(J_1)$, $W \downarrow_p (\Lambda_0 \cap p^*(T_{q_0})) \equiv W \downarrow_p (\Lambda_1 \cap p^*(T_{q_1}))$.
- There is an order isomorphism $h : J'_0 \rightarrow J'_1$ such that $W_{h(q)} \equiv W_g$.

Now if $i = -1$:

- J_0 has a minimum if and only if J_1 has a maximum.
- J_0 has a maximum if and only if J_1 has a minimum.
- If $q_0 = \min(J_0)$ and $q_1 = \max(J_1)$, $W \downarrow_p (\Lambda_0 \cap p^*(T_{q_0})) \equiv (W \downarrow_p (\Lambda_1 \cap p^*(T_{q_1})))^{-1}$.
- If $q_0 = \max(J_0)$ and $q_1 = \min(J_1)$, $W \downarrow_p (\Lambda_0 \cap p^*(T_{q_0})) \equiv (W \downarrow_p (\Lambda_1 \cap p^*(T_{q_1})))^{-1}$.
- There is an order reversing bijection $h : J'_0 \rightarrow J'_1$ such that $T_{h(q)} \equiv (T_q)^{-1}$.

From this and how the ι_q were defined it is clear that

$$U \downarrow_p \alpha \left(\bigcup_{q \in J'_0} p^*(T_q), \iota \right) \equiv \left(U \downarrow_p \alpha \left(\bigcup_{q \in J'_1} p^*(T_q), \iota \right) \right)^i.$$

Now suppose, for example, $i = -1$ and J_0 has maximum and minimum. Let $K \equiv U \downarrow_p \alpha (\bigcup_{q \in J'_0} p^*(T_q), \iota)$. By Lemma 3.13 we have that $[[U \downarrow_p \alpha (\Lambda_0, \iota)]]$ is equal to

$$[[U \downarrow_p \alpha (\Lambda_0 \cap p^*(T_{\min(J_0)}), \iota)]] [[K]] [[U \downarrow_p \alpha (\Lambda_0 \cap p^*(T_{\max(J_0)}), \iota)]]$$

and that $[[U \downarrow_p \alpha (\Lambda_1, \iota)^{-1}]]$ is equal to

$$[[U \downarrow_p \alpha (\Lambda_1 \cap p^*(T_{\max(J_1)}), \iota)^{-1}]] [[K]] [[U \downarrow_p \alpha (\Lambda_1 \cap p^*(T_{\min(J_1)}), \iota)^{-1}]].$$

Since the coi collection $\{\text{coi}(W_n, \iota_n, P_n)\}_{n \in \omega}$ is coherent we also have the equalities

$$[[U \downarrow_p \alpha(\Lambda_0 \cap p^*(T_{\min(J_0)}), \iota)] = [[(U \downarrow_p \alpha(\Lambda_1 \cap p^*(T_{\max(J_1)}), \iota))^{-1}]$$

and

$$[[U \downarrow_p \alpha(\Lambda_0 \cap p^*(T_{\max(J_0)}), \iota)] = [[(U \downarrow_p \alpha(\Lambda_1 \cap p^*(T_{\min(J_1)}), \iota))^{-1}].$$

Thus $[[U \downarrow_p \alpha(\Lambda_0, \iota)] = [[(U \downarrow_p \alpha(\Lambda_1, \iota))^{-1}]$ by direct substitution. All other possibilities can be similarly argued.

Suppose that $x_0 \in X$ and $\Lambda_0 \subseteq p^*(U)$, $\Lambda_1 \subseteq p^*(U_{x_0})$ are intervals and $i \in \{-1, 1\}$ are such that $U \downarrow_p \Lambda_0 \equiv (U_{x_0} \downarrow_p \Lambda_1)^i$. As $(U_{x_0} \downarrow_p \Lambda_1)^i \in \text{Pfine}(\{U_x\}_{x \in X} \cup \{P_n\}_{n \in \omega})$, and $V_{m,b}, V_{m,c} \notin \text{Pfine}(\{U_x\}_{x \in X} \cup \{P_n\}_{n \in \omega})$ for all $m \in \omega$ we see that Λ_0 must be a subinterval of some $p^*(U_q)$, and more particularly a subinterval of $p^*(P_{f_0(q)}^{f_1(q)})$. By how ι_q was defined, and since $\{\text{coi}(W_x, \iota_x, U_x)\}_{x \in X} \cup \{\text{coi}(W_n, \iota_n, P_n)\}_{n \in \omega}$ is coherent it follows that

$$[[W \downarrow_p \alpha(\Lambda_0, \iota^{-1})] = [[(W_{x_0} \downarrow_p \alpha(\Lambda_1, \iota_{x_0}^{-1}))^i].$$

If $n_0 \in \omega$ and $\Lambda_0 \subseteq p^*(U)$, $\Lambda_1 \subseteq p^*(P_{n_0})$ are intervals and $i \in \{-1, 1\}$ are such that $U \downarrow_p \Lambda_0 \equiv (P_{n_0} \downarrow_p \Lambda_1)^i$ then the same argument shows that

$$[[W \downarrow_p \alpha(\Lambda_0, \iota^{-1})] = [[(W_{n_0} \downarrow_p \alpha(\Lambda_1, \iota_{n_0}^{-1}))^i].$$

Finally, suppose that intervals $\Lambda_0, \Lambda_1 \subseteq p^*(U)$ and $i \in \{-1, 1\}$ are such that $U \downarrow_p \Lambda_0 \equiv (U \downarrow_p \Lambda_1)^i$. As before we define

$$J_0 = \{q \in \mathbb{Q} \mid p^*(U_q) \cap \Lambda_0 \neq \emptyset\}, \quad J_1 = \{q \in \mathbb{Q} \mid p^*(U_q) \cap \Lambda_1 \neq \emptyset\}.$$

Once again, the cases where J_0 , hence also J_1 , is empty or a singleton are treated the same. We therefore assume that both J_0 and J_1 are infinite. One sees that J_0 has a minimum if and only if $U \downarrow_p \Lambda_0$ has a nonempty initial subword which is a pure p-chunk (i.e., a word $V_{m,b}^{\pm 1}$ or $V_{m,c}^{\pm 1}$ for some $m \in \omega$) or which is in $\text{Pfine}(\{U_x\}_{x \in X} \cup \{P_n\}_{n \in \omega})$, and not both since the words $V_{m,b}$ and $V_{m,c}$ were not in $\text{Pfine}(\{U_x\}_{x \in X} \cup \{P_n\}_{n \in \omega})$. In either case, J_0 has a minimum if and only if there is an element $\lambda \in \Lambda_0$ for which $U \downarrow_p \{\lambda\} \notin \text{Pfine}(\{U_x\}_{x \in X} \cup \{P_n\}_{n \in \omega})$ which is minimal. Similar such statements for maxima and J_1 apply. Thus we see that when $i = 1$, J_0 has a minimum if and only if J_1 has one, and J_0 has a maximum if and only if J_1 has one. When $i = -1$ the comparable dual statements hold. Let J'_0 be the set J_0 minus any maximal or minimal element and define J'_1 analogously. For each $q \in J'_0$ (or $q \in J'_1$) we have that $U_{f_0(q)}^{f_1(q)}$ is a maximal subword of U which is in $\text{Pfine}(\{U_x\}_{x \in X} \cup \{P_n\}_{n \in \omega})$, and each of $V_{f_0(q),b}^{f_1(q)}$ and $V_{f_0(q),c}^{f_1(q)}$ is a maximal p-chunk of U all of whose nonempty p-chunks are not in $\text{Pfine}(\{U_x\}_{x \in X} \cup \{P_n\}_{n \in \omega})$.

In particular, $U \downarrow_p \bigcup_{q \in J'_0} p^*(U_q)$ is the word obtained from U by removing an initial pure p-chunk (if it exists) and then removing an initial nonempty p-chunk

which is an element of $\text{Pfine}(\{U_x\}_{x \in X} \cup \{P_n\}_{n \in \omega})$ (if it exists) and then removing an initial pure p -chunk (if step two applies) and doing the similar three-step process to the terminal part of the word U . Hence it is clear that $U \downarrow_p \bigcup_{q \in J'_0} p^*(U_q) \equiv (U \downarrow_p \bigcup_{q \in J'_1} p^*(U_q))^i$. Moreover this word equality will pair maximal intervals $\Lambda \subseteq \bigcup_{q \in J'_0} p^*(U_q)$ for which $U \downarrow_p \Lambda \in \text{Pfine}(\{U_x\}_{x \in X} \cup \{P_n\}_{n \in \omega})$ with such intervals in $\bigcup_{q \in J'_1} p^*(U_q)$, and for such a Λ we'll have $U \downarrow_p \Lambda \equiv P_n^{\pm 1}$ for some $n \in \omega$. As $P_n \neq P_{n'} \neq P_n^{-1}$ when $n \neq n'$ we have a bijection $h : J'_0 \rightarrow J'_1$ which is an order isomorphism in case $i = 1$, or an order reversal in case $i = -1$, such that $U_{h(q)} \equiv (U_q)^i$ once again. Thus we get

$$W \downarrow_p \alpha \left(\bigcup_{q \in J'_0} p^*(U_q), \iota^{-1} \right) \equiv \left(W \downarrow_p \alpha \left(\bigcup_{q \in J'_1} p^*(U_q), \iota^{-1} \right) \right)^i.$$

Thus for example, if $i = -1$ and J_0 has maximum and minimum then we let $K \equiv W \downarrow_p \alpha(\bigcup_{q \in J'_0} p^*(U_q), \iota^{-1})$. Then $[[W \downarrow_p \alpha(\Lambda_0, \iota^{-1})]]$ is equal to the product

$$[[W \downarrow_p \alpha(\Lambda_0 \cap p^*(U_{\min(J_0)}), \iota^{-1})]][[K]][[W \downarrow_p \alpha(\Lambda_0 \cap p^*(U_{\max(J_0)}), \iota^{-1})]]$$

by Lemma 3.13. By the same reasoning we have that $[[W \downarrow_p \alpha(\Lambda_1, \iota^{-1})]]^{-1}$ is equal to

$$[[W \downarrow_p \alpha(\Lambda_1 \cap p^*(U_{\max(J_1)}), \iota^{-1})]]^{-1} [[K]][[W \downarrow_p \alpha(\Lambda_1 \cap p^*(U_{\min(J_1)}), \iota^{-1})]]^{-1}.$$

By coherence we get that

$$[[W \downarrow_p \alpha(\Lambda_0 \cap p^*(U_{\min(J_0)}), \iota^{-1})]] = [[W \downarrow_p \alpha(\Lambda_1 \cap p^*(U_{\max(J_1)}), \iota^{-1})]]^{-1},$$

and similarly

$$[[W \downarrow_p \alpha(\Lambda_0 \cap p^*(U_{\max(J_0)}), \iota^{-1})]] = [[W \downarrow_p \alpha(\Lambda_1 \cap p^*(U_{\min(J_1)}), \iota^{-1})]]^{-1},$$

and so the equality $[[W \downarrow_p \alpha(\Lambda_0, \iota^{-1})]] = [[W \downarrow_p \alpha(\Lambda_1, \iota^{-1})]]^{-1}$ is immediate. \square

3G. Arbitrary extensions. In this subsection we will prove the following proposition and then complete the proof of Theorem A as well as prove Theorem B.

Proposition 3.23. *Suppose that κ_0 and κ_1 are cardinal numbers greater than or equal to 2. Suppose that $\{\text{coi}(W_x, \iota_x, U_x)\}_{x \in X}$ is coherent and that $|X| < 2^{\aleph_0}$. Then given $W \in \text{Red}_{\kappa_0}$ there exists $U \in \text{Red}_{\kappa_1}$ and a $\text{coi } \iota$ from W to U such that $\{\text{coi}(W_x, \iota_x, U_x)\}_{x \in X} \cup \{\text{coi}(W, \iota, U)\}$ is coherent.*

Proof. Assume the hypotheses. If W is the empty word E then we let $U \equiv E$ and ι be the empty function. This clearly satisfies the conclusion of the proposition. Thus we may now assume that W is not E and so $p^*(W)$ is nonempty. For each $\lambda \in p^*(W)$ we let ι_λ be the empty function, so ι_λ is a coi from $W \downarrow_p \{\lambda\}$ to E . It is quite trivial to see that $\mathcal{T}_0 = \{\text{coi}(W_x, \iota_x, U_x)\}_{x \in X} \cup \{\text{coi}(W \downarrow_p \{\lambda\}, \iota_\lambda, E)\}_{\lambda \in p^*(W)}$

is coherent. Let $<$ be a well-order on the set $p^*(W)$ and if \mathcal{T} is a collection of cois then we let $h(\mathcal{T})$ denote the set of first words listed in the ordered triples (for example $h(\mathcal{T}_0) = \{W_x\}_{x \in X} \cup \{W \upharpoonright_p \{\lambda\}\}_{\lambda \in p^*(W)}$).

Step 1. Define a function f_0 from an initial subset of the set \aleph_1 of countable ordinals to $p^*(W)$, as well as a function f_1 with the same domain as f_0 and with codomain the set of two letters $\{L, R\}$ and f_2 a function with the same domain as f_0 and with codomain the set of intervals in $p^*(W)$. We shall also extend the coi collection. If each $\lambda \in p^*(W)$ is contained in a maximal interval $I \subseteq p^*(W)$ such that $W \upharpoonright_p I \in h(\mathcal{T}_\zeta)$ then we cease our construction of step 1 and proceed to step 2. If it is not the case that each $\lambda \in p^*(W)$ is contained in a maximal interval $I \subseteq p^*(W)$ such that $W \upharpoonright_p I \in h(\mathcal{T}_\zeta)$ then we select a minimal such λ under the well-ordering $<$ and let $f_0(\zeta) = \lambda$. Note that it is possible that each singleton $\{\lambda\}$ is already maximal such that $W \upharpoonright_p \{\lambda\} \in h(\mathcal{T}_0)$. At least one of two possibilities holds:

Case i. If there is a sequence $\{I_m\}_{m \in \omega}$ such that $\lambda = \min(I_m)$ and I_m is strictly included in I_{m+1} for all $m \in \omega$ with $W \upharpoonright_p I_m \in \text{Pfine}(h(\mathcal{T}_\zeta))$ but $W \upharpoonright_p \bigcup_{m \in \omega} I_m \notin \text{Pfine}(h(\mathcal{T}_\zeta))$, then we let $f_1(\zeta) = L$ (for Left endpoint) and $f_2(\zeta) = \bigcup_{m \in \omega} I_m$. By Proposition 3.20 we select $U_\zeta \in \text{Red}_{\kappa_1}$ and a coi ι_ζ from $W \upharpoonright_p f_2(\zeta)$ to U_ζ such that $\mathcal{T}_{\zeta+1} = \mathcal{T}_\zeta \cup \{\text{coi}(W \upharpoonright_p f_2(\zeta), \iota_\zeta, U_\zeta)\}$ is coherent.

Case ii. If such a sequence as in case i does not exist then there exists a sequence $\{I_m\}_{m \in \omega}$ such that $\lambda = \max(I_m)$ and I_m is strictly included in I_{m+1} for all $m \in \omega$ with $W \upharpoonright_p I_m \in \text{Pfine}(h(\mathcal{T}_\zeta))$, but $W \upharpoonright_p \bigcup_{m \in \omega} I_m \notin \text{Pfine}(h(\mathcal{T}_\zeta))$. In this case we let $f_1(\zeta) = R$ (for Right endpoint) and $f_2(\zeta) = \bigcup_{m \in \omega} I_m$. By Proposition 3.20 applied to the word W^{-1} we select $U_\zeta \in \text{Red}_{\kappa_1}$ and a coi ι_ζ from $W \upharpoonright_p f_2(\zeta)$ to U_ζ such that $\mathcal{T}_{\zeta+1} = \mathcal{T}_\zeta \cup \{\text{coi}(W \upharpoonright_p f_2(\zeta), \iota_\zeta, U_\zeta)\}$ is coherent.

Iterating this recursion and letting $\mathcal{T}_\zeta = \bigcup_{\xi_0 < \zeta} \mathcal{T}_{\xi_0}$ when ζ is a limit ordinal, we define the functions f_0, f_1, f_2 over an increasingly large initial segment of \aleph_1 . We claim, however, that this recursion must terminate at some stage, and thus move us into step 2. If, otherwise, the recursion does not terminate, then the functions f_0, f_1, f_2 are defined on all of \aleph_1 . Since the codomains, $p^*(W)$ and $\{L, R\}$, of f_0 and f_1 are countable, there exists some $\lambda \in p^*(W)$ and, say, $R \in \{L, R\}$, and uncountable $J \subseteq \aleph_1$ such that $f_0(J) = \{\lambda\}$ and $f_1(J) = \{R\}$. Suppose that $\zeta_0, \zeta_1 \in J$ are such that $\zeta_0 < \zeta_1$. Then by construction, at step ζ_1 we see that $f_2(\zeta_1)$ is an interval in $p^*(W)$ with right endpoint λ which is larger than any interval I in $p^*(W)$ with $\lambda = \max(I)$ and $W \upharpoonright_p I \in \text{Pfine}(h(\mathcal{T}_{\zeta_1}))$. As $W \upharpoonright_p f_2(\zeta_0) \in \text{Pfine}(\mathcal{T}_{\zeta_0+1}) \subseteq \text{Pfine}(\mathcal{T}_{\zeta_1})$ we get that $f_2(\zeta_0)$ is strictly included into $f_2(\zeta_1)$. But as J is well ordered under the restriction of the order on \aleph_1 we let $s(\zeta)$ denote the successor of $\zeta \in J$ in J and select $\lambda_\zeta \in f_2(s(\zeta)) \setminus f_2(\zeta)$, giving us an injection from the uncountable set J to the countable set $p^*(W)$, contradiction.

Step 2. From step 1 we obtain a coherent collection \mathcal{T}_ζ of cois, with $|\mathcal{T}_\zeta| < 2^{\aleph_0}$,

and each $\lambda \in p^*(W)$ includes into a maximal interval $I_\lambda \subseteq p^*(W)$ with respect to the property that $W \upharpoonright_p I_\lambda \in \text{Pfine}(h(\mathcal{T}_\zeta))$. Note that it is possible that $I_\lambda = \{\lambda\}$ for each $\lambda \in p^*(W)$. The collection Λ of all such maximal intervals has a natural induced ordering and is necessarily order dense, for if there existed distinct I_λ and $I_{\lambda'}$ between which there are no elements in Λ then the word $W \upharpoonright_p I_\lambda \cup I_{\lambda'}$ would be in $\text{Pfine}(h(\mathcal{T}_\zeta))$, contradicting maximality. As W is not the empty word we know that $\Lambda \neq \emptyset$. If Λ is a singleton then $\Lambda = \{p^*(W)\}$, so $W \in \text{Pfine}(\mathcal{T}_\zeta)$, so by Lemma 3.17 select $U \in \text{Red}_{\kappa_1}$ and ι such that $\mathcal{T}_\zeta \cup \{\text{coi}(W, \iota, U)\}$ is coherent.

If Λ is not a singleton let Λ' be the interval in Λ which excludes $\min(\Lambda)$ and $\max(\Lambda)$ if either or both exist. If Λ' is not empty then it is order isomorphic to \mathbb{Q} , and in either case by Proposition 3.21 we may add, if necessary, a single coi triple to \mathcal{T}_ζ to obtain a coherent collection \mathcal{T}'_ζ such that $W \upharpoonright_p (\bigcup \Lambda') \in \text{Pfine}(h(\mathcal{T}'_\zeta))$. Next, since $W \upharpoonright_p \min(\Lambda), W \upharpoonright_p \max(\Lambda) \in \text{Pfine}(h(\mathcal{T}'_\zeta))$ if either of $\min(\Lambda)$ or $\max(\Lambda)$ exists, we have that $W \in \text{Pfine}(h(\mathcal{T}'_\zeta))$ as W is the concatenation of one or two or three words in $\text{Pfine}(h(\mathcal{T}'_\zeta))$. By Lemma 3.17 we select $U \in \text{Red}_{\kappa_1}$ and a coi ι such that $\mathcal{T}'_\zeta \cup \{\text{coi}(W, \iota, U)\}$ is coherent. Then $\{\text{coi}(W_x, \iota_x, U_x)\}_{x \in X} \cup \{\text{coi}(W, \iota, U)\}$ is coherent and our proposition is proved. \square

Proof of Theorem A. Let κ be a cardinal such that $2 \leq \kappa \leq 2^{\aleph_0}$. It is easy to see from Theorem 2.11 that $|\text{Red}_2| = |\text{Red}_\kappa| = 2^{\aleph_0}$. Thus we let $<$ well-order Red_2 in such a way that each element has fewer than 2^{\aleph_0} predecessors. Similarly let $<'$ well-order Red_κ in such a way that each element has fewer than 2^{\aleph_0} predecessors. We inductively define a coherent collection $\{\text{coi}(W_\zeta, \iota_\zeta, U_\zeta)\}_{\zeta < 2^{\aleph_0}}$ of coi triples from Red_2 to Red_κ .

Recall that each ordinal ζ may be written uniquely as an ordinal sum $\zeta = \beta + m$ where β is either 0 or a limit ordinal and $m \in \omega$, and so ζ can be considered even or odd depending on the parity of m . Select a word $W_0 \in \text{Red}_2$ minimal under $<$ and by Proposition 3.23 select $U_0 \in \text{Red}_\kappa$ and a coi ι_0 such that $\{\text{coi}(W_0, \iota_0, U_0)\}$ is coherent. Suppose that we have defined coherent $\{\text{coi}(W_\zeta, \iota_\zeta, U_\zeta)\}_{\zeta < \mu}$ for all $\mu < \nu < 2^{\aleph_0}$. By Lemma 3.12 we know $\{\text{coi}(W_\zeta, \iota_\zeta, U_\zeta)\}_{\zeta < \nu}$ is coherent. If ν is even then by Lemma 3.19 we select a word $W_\nu \notin \text{Pfine}(\{W_\zeta\}_{\zeta < \nu})$ which is minimal under $<$ and by Proposition 3.23 select $U_\nu \in \text{Red}_\kappa$ and a coi ι_ν such that $\{\text{coi}(W_\zeta, \iota_\zeta, U_\zeta)\}_{\zeta < \nu+1}$ is coherent (using $\kappa_0 = 2$ and $\kappa_1 = \kappa$). Similarly if ν is odd then by Lemma 3.19 we select a word $U_\nu \notin \text{Pfine}(\{U_\zeta\}_{\zeta < \nu})$ which is minimal under $<'$ and by Proposition 3.23 select $W_\nu \in \text{Red}_\kappa$ and a coi ι_ν such that $\{\text{coi}(W_\zeta, \iota_\zeta, U_\zeta)\}_{\zeta < \nu+1}$ is coherent (using $\kappa_0 = \kappa$ and $\kappa_1 = 2$).

Notice that $\text{Pfine}(\{W_\zeta\}_{\zeta < 2^{\aleph_0}}) = \text{Red}_2$ and $\text{Pfine}(\{U_\zeta\}_{\zeta < 2^{\aleph_0}}) = \text{Red}_\kappa$. Thus by Proposition 3.16 we have an isomorphism $\Phi : \mathcal{C}_2 \rightarrow \mathcal{C}_\kappa$. \square

We will derive Theorem B as a consequence of Theorem A. Instead of defining the notions of elementary equivalence and elementary subsumption, we will trust

the reader to know these concepts or to look them up. We will rely on the following classical result.

Lemma 3.24. *Suppose \mathcal{U}_0 is a submodel of \mathcal{U}_1 such that for every $a_0, \dots, a_{n-1} \in U_0$ and $a \in U_1$ there exists an automorphism $\phi : \mathcal{U}_1 \rightarrow \mathcal{U}_1$ such that $\phi(a_i) = a_i$ for all $i < n$ and $\phi(a) \in U_0$. Then \mathcal{U}_0 is an elementary submodel of \mathcal{U}_1 .*

Proof of Theorem B. Certainly if $\gamma = \kappa$ or if $2 \leq \gamma \leq \kappa \leq 2^{\aleph_0}$ then we have $\mathcal{C}_\gamma \simeq \mathcal{C}_\kappa$ (using Theorem A in the second case) and the isomorphism is an elementary embedding. We may therefore assume that $2^{\aleph_0} \leq \gamma < \kappa$, for the result will follow for $2 \leq \gamma < 2^{\aleph_0} < \kappa$ as well by the fact that $\mathcal{C}_\gamma \simeq \mathcal{C}_{2^{\aleph_0}}$ in this case.

The map $\psi_{\gamma,\kappa} : \mathcal{C}_\gamma \rightarrow \mathcal{C}_\kappa$ given by $[[W]] \mapsto [[W]]$ is easily seen to be an injection and we consider \mathcal{C}_γ as the substructure of \mathcal{C}_κ consisting of those $[[W]]$ which have a representative utilizing only letters with first coordinate less than γ . Any bijection $f : \kappa \rightarrow \kappa$ induces a bijection $F_f : \mathcal{A}_\kappa \rightarrow \mathcal{A}_\kappa$ given by $a_{\alpha,n}^{\pm 1} \mapsto a_{f(\alpha),n}^{\pm 1}$ which induces a bijection $\mathcal{F}_f : \mathcal{W}_\kappa \rightarrow \mathcal{W}_\kappa$ given by $W \mapsto \prod_{i \in \bar{W}} F_f(W(i))$. This \mathcal{F}_f induces an automorphism $\bar{\theta}_f : \text{Red}_\kappa \rightarrow \text{Red}_\kappa$ given by $W \mapsto \mathcal{F}_f(W)$ which descends to an automorphism $\bar{\theta}_f : \mathcal{C}_\kappa \rightarrow \mathcal{C}_\kappa$.

Lemma 3.25. *Suppose $\gamma \leq \kappa$ with γ uncountable. If $X \subseteq \mathcal{C}_\gamma$ and $Y \subseteq \mathcal{C}_\kappa$ with $|X|, |Y| < \gamma$ there exists a bijection $f : \kappa \rightarrow \kappa$ such that $\bar{\theta}_f(x) = x$ for all $x \in X$ and $\bar{\theta}_f(Y) \subseteq \mathcal{C}_\gamma$.*

Proof. Assume the hypotheses. For each $x \in X$ fix a representative $W_x \in x$ such that $\text{proj}_0(W) \subseteq \gamma$. For each $y \in Y$ fix a representative W_y . Since each set $\text{proj}_0(W_x)$ is at most countable, the set $\bigcup_{x \in X} \text{proj}_0(W_x)$ is of cardinality at most $\aleph_0 \cdot |X|$. Similarly the set $\bigcup_{y \in Y} \text{proj}_0(W_y)$ is of cardinality at most $\aleph_0 \cdot |Y|$.

Since γ is uncountable, $\bigcup_{x \in X} \text{proj}_0(W_x) \subseteq \gamma$ is of cardinality less than γ and $\bigcup_{y \in Y} \text{proj}_0(W_y) \subseteq \kappa$ is also of cardinality less than γ , we can easily select a bijection $f : \kappa \rightarrow \kappa$ which fixes the elements in $\bigcup_{x \in X} \text{proj}_0(W_x)$ and such that $f(\bigcup_{y \in Y} \text{proj}_0(W_y)) \subseteq \gamma$. The automorphism $\bar{\theta}_f$ satisfies the desired properties. \square

The proof of Theorem B is now complete by appealing to Lemma 3.24. \square

Note that the map $f \mapsto \bar{\theta}_f$ gives a homomorphic injection from the full symmetric group on the set κ , S_κ , to the automorphism group $\text{Aut}(\pi_1(\mathbb{G}S_\kappa))$. Since $\pi_1(\mathbb{G}S_2) \simeq \pi_1(\mathbb{G}S_{2^{\aleph_0}})$ we immediately get the following, which is not obvious a priori:

Corollary 3.26. *The group $\text{Aut}(\pi_1(\mathbb{G}S_2))$ includes a subgroup isomorphic to the full symmetric group $S_{2^{\aleph_0}}$ on a set of size continuum.*

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
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