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TENSOR TRIANGULATED CATEGORY STRUCTURES IN THE DERIVED CATEGORY OF A VARIETY WITH BIG (ANTI)CANONICAL BUNDLE

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Let X be a smooth projective variety over $\mathbb{C}$ with big (anti)canonical bundle. It is known that for such X the Balmer spectrum of the tensor triangulated category of perfect complexes $\mathrm{Perf}(X)$, equipped with the derived tensor product $\otimes^{\mathbb{L}}_X$, recovers the space X. We study the possible tensor triangulated category structures one can put on $\mathrm{Perf}(X)$. As an application, we prove a monoidal version of the well-known Bondal–Orlov reconstruction theorem.

1. Introduction

Bondal and Orlov [2001] showed that if X is a smooth projective variety over $\mathbb{C}$ with ample (anti)canonical bundle then its bounded derived category $D^b(X)$ completely recovers the space. More precisely, they showed that:

Theorem 1.1 [Bondal and Orlov 2001, Theorem 2.5]. *Let X be an irreducible smooth projective variety with ample (anti)canonical bundle. If $D^b(X) \simeq D^b(Y)$ for some other smooth algebraic variety Y, then $X \cong Y$.*

This theorem came in contrast with the discovery by Mukai [1987] that for an abelian variety A, there exists an equivalence $D^b(A) \simeq D^b(\hat{A})$, as triangulated categories, between the bounded derived category of A and the bounded derived category of its dual $\hat{A}$.

This observation sparked the study of what are now called Fourier–Mukai partners of a given variety X, that is, those varieties which are triangulated equivalent to the bounded derived category of X.

Bondal and Orlov's reconstruction pointed out that a (birational) geometric condition on the variety can introduce some control on these derived equivalences, and with this in mind, Kawamata generalized this theorem for varieties with big (anti)canonical bundle, clarifying from a geometric point of view the role of this condition on the possible equivalence of derived categories. Namely he showed:

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Theorem 1.2 [Kawamata 2002, Theorem 1.4]. *Let X, Y be smooth projective varieties such that there is an equivalence*

$$\mathcal{F} : D^b(X) \xrightarrow{\simeq} D^b(Y),$$

as triangulated categories. Then:

(1) $\dim X = \dim Y$.

(2) *If the canonical divisor K_X is nef, so is K_Y and there is an equality in the numerical Kodaira dimensions $\nu(X)$ and $\nu(Y)$.*

(3) *If X is of general type, then X and Y are birational, and furthermore, there is a smooth projective variety $p : Z \to X$, $q : Z \to Y$ such that $p^*K_X \simeq q^*K_Y$.*

This theorem should be understood as a strong indication of a relationship between the birational geometry of a variety and its derived category.

On the other hand, Balmer showed [Balmer 2002; Bondal and Orlov 2001] that when equipped with the derived tensor product $\otimes^{\mathbb{L}}_X$, the derived category of perfect complexes $\mathrm{Perf}(X)$ of any coherent scheme X can recover the space X by what is now known as the Balmer spectrum $\mathrm{Spc}(\mathrm{Perf}(X), \otimes^{\mathbb{L}}_X)$. The Balmer spectrum can be constructed for a general tensor triangulated category, that is, a triangulated category equipped with a compatible monoidal structure, and produce a locally ringed space.

The existence of nonisomorphic Fourier–Mukai partners Y for a smooth variety X implies, using the Balmer spectrum construction, that the bounded derived category $D^b(X)$ can be equipped with at least as many tensor triangulated category structures as nonisomorphic Fourier–Mukai partners, up to monoidal equivalence.

In other words, if $\mathrm{FM}(X)$ is the set of isomorphism classes of Fourier–Mukai partners of X and $\mathrm{TTS}(X)$ is the set of equivalence classes of tensor triangulated category structures on the bounded derived category $D^b(X)$, there exists an injection

$$\mathrm{FM}(X) \to \mathrm{TTS}(X), \quad Y \mapsto (\otimes^{\mathbb{L}}_Y, \mathcal{O}_Y),$$

where the pair $(\otimes^{\mathbb{L}}_Y, \mathcal{O}_Y)$ denotes the tensor triangulated category structure given by the derived tensor product $\otimes^{\mathbb{L}}_Y$ with unit $\mathcal{O}_Y$.

Our main interest in this work is the study of this function, its surjectivity and the properties that one can deduce about possible tensor triangulated category structures outside of the image of this injection, all under the condition that the (anti)canonical bundle of X is big.

In Section 2 we give a brief general overview of the results we will need about general derived categories of quasicoherent sheaves on a smooth projective variety, together with a reminder of the Balmer spectrum construction through Thomason's classification theorem.

In Section 3, given a tensor triangulated category structure $(D^b(X), \boxtimes, \mathbb{1})$ with unit $\mathbb{1}$ on a bounded derived category $D^b(X)$, we introduce the notion of almost

spanning class with respect to a thick subcategory I (Definition 3.9) and we show (Theorem 3.10) that if X is a smooth projective variety of general type then there exists a proper tensor ideal I_{X^*} of $(D^b(X), \otimes^{\mathbb{L}}_X, \mathcal{O}_X)$ such that the set of tensor powers of ω_X forms an almost spanning sequence with respect to this ideal I_{X^*}. This result is meant to highlight the more general behavior of almost spanning classes through the use of Thomason's classification theorem and properties of the Balmer spectrum. We see that this collection of objects can be used to prove the following result:

Lemma 1.3 (Lemma 3.12). *Suppose X is a smooth projective variety of general type. If $\boxtimes$ is a tensor triangulated structure on $D^b(X)$ with unit $\mathcal{O}_X$, and U is a $\boxtimes$-invertible object such that $U \boxtimes I_{X^*} \subseteq I_{X^*}$, then there is a natural equivalence between the functors induced by $U \boxtimes \_$ and $U \otimes^{\mathbb{L}}_X \_$ in $D^b(X)/I_{X^*}$.*

When the $\otimes^{\mathbb{L}}_X$-tensor ideal I_{X^*} is also a $\boxtimes$-tensor ideal for a tensor triangulated category structure as described in the previous lemma, then we obtain that the Picard group of $\boxtimes$-invertible objects is a subgroup of the Picard group of $\otimes^{\mathbb{L}}_X$-invertible objects (Corollary 3.15). This hypothesis holds true in particular when the (anti)canonical bundle of X is ample.

With this observation, our main corollary is the following monoidal version of the Bondal–Orlov reconstruction theorem:

Corollary 1.4 (Corollary 3.18). *Let X be a smooth projective variety with ample (anti)canonical bundle. If $\omega_X[n]$ is an invertible object for a tensor triangulated structure $\boxtimes$ on $D^b(X)$ with unit $\mathcal{O}_X$, then $\boxtimes$ and $\otimes^{\mathbb{L}}_X$ coincide on objects.*

2. Derived categories and the Balmer reconstruction

Throughout the rest of this work we will be working exclusively with smooth projective varieties over $\mathbb{C}$. We will omit the mention of the base field. We recommend [Huybrechts 2006] as a good reference for the material concerning derived categories in this section.

The goal of this section is to introduce the basic results and notions we will be using for our results. Let us start by recalling that if X is a smooth projective variety then there exists an equivalence as triangulated categories between the derived category $\mathrm{Perf}(X)$ of perfect complexes on X and the bounded derived category $D^b(X)$. As a consequence of this, whenever we work with such a variety we will at times make no distinction between these two categories.

One important feature of these categories is the existence of Serre functors.

Definition 2.1. Let $\mathcal{T}$ be a triangulated category. An autoequivalence $S : \mathcal{T} \to \mathcal{T}$ satisfying $\mathrm{Hom}(A, B) \cong \mathrm{Hom}(B, S(A))^*$ for all objects $A, B \in \mathcal{T}$ is called a Serre functor.

Example 2.2. If the triangulated category is a derived category of a smooth projective scheme of dimension n, we have Grothendieck–Verdier duality, which implies that for every pair of objects $M, N \in D^b(X)$, $\mathrm{Hom}(M, N) = \mathrm{Hom}(N, M \otimes \omega_X[n])^*$, where ω_X is the canonical bundle of X.

This notion was first defined by Bondal and Kapranov [1989]. The following two properties of the Serre functor are essential to our work:

Lemma 2.3 [Bondal and Orlov 2001, Proposition 1.3]. *Let $\mathscr{T}$ be a triangulated category with Serre functor S, and let $\psi : \mathscr{T} \to \mathscr{T}$ be any autoequivalence. Then $\psi \circ S \cong S \circ \psi$.*

Proposition 2.4 [Bondal and Kapranov 1989, Proposition 3.4]. *Let $\mathscr{T}$ be a triangulated category and let S be a Serre functor on $\mathscr{T}$. Then S is unique up to graded isomorphism.*

This latter proposition implies that whenever the Serre functor exists, it is intrinsic to the given category. In our case of interest, because one can write this functor using the derived tensor product $\otimes^{\mathbb{L}}_X$, we have now some possible control on the monoidal structure $\otimes^{\mathbb{L}}_X$ directly from the category without knowledge of X.

Another crucial notion we will use is that of spanning classes.

Definition 2.5. A collection of objects $\{X_i\} \subseteq \mathscr{T}$ of a triangulated category is called a spanning class if the following hold:

(1) If $\mathrm{Hom}(X_i, D[j]) = 0$ for all i and j, then $D \simeq 0$.

(2) If $\mathrm{Hom}(D[j], X_i) = 0$ for all i and j, then $D \simeq 0$.

However, whenever the Serre functor exists in the triangulated category we see that only one of the conditions is necessary and the other will be automatically satisfied by use of the Serre functor isomorphism.

A general way to produce spanning classes in derived categories of abelian categories is from ample sequences:

Definition 2.6. Let $\mathcal{A}$ be an abelian category. A collection of objects $\{L_i\} \subset \mathcal{A}$ is called an ample sequence if for $i \ll 0$ and all $A \in \mathcal{A}$,

(1) $\mathrm{Hom}(L_i, A) \otimes_k L_i \to A$ is surjective,

(2) $\mathrm{Hom}(A, L_i) = 0$, and

(3) $\mathrm{Ext}^j(L_i, A) = 0$ for $j \neq 0$.

As the name suggests, an important example of such sequences comes from collections of tensor powers of ample line bundles. The relation between the two notions of spanning class and ample sequence was shown by Orlov [1997]:

Lemma 2.7. *Let $\mathcal{A}$ be an abelian category of finite homological dimension, and let $\{L_i\}$ be an ample sequence. Then the collection $\{L_i\}$, seen as objects of $\mathcal{D}(\mathcal{A})$, form a spanning class.*

The next example shows how we should exploit the existence of ample sequences.

Example 2.8. Let X be a smooth projective variety with ample canonical bundle. Then the set $\{\omega_X^{\otimes_X}\}$ forms an ample sequence, so by the previous lemma it forms a spanning class in the derived category $D^b(X)$.

As a consequence, we see that any bounded complex $\mathscr{F}$ of coherent sheaves can be resolved by tensor powers of the canonical bundle ω_X. In other words, there exists an exact sequence

$$0 \to \bigoplus_{j_0} (\omega_X^{\otimes i_0}) \to \cdots \to \bigoplus_{j_k} (\omega_X^{\otimes i_k}) \to \mathscr{F} \to 0.$$

Remark 2.9. In general, for a triangulated category $\mathscr{T}$ with a spanning class $\Omega \subset \mathscr{T}$, if $\phi : \mathscr{T} \to \mathscr{T}$ is an autoequivalence then the set $\phi(\Omega)$ is a spanning class.

This implies that in the example above one can resolve any bounded complex $\mathscr{F}$ by tensor powers of sheaves of the form $\omega_X(i)[j]$ for a fixed $i, j \in \mathbb{Z}$.

Tensor triangulated geometry. When dealing with derived categories of coherent sheaves on a variety one can equip this category with a monoidal structure given by the derived tensor product. One can axiomatize this sort of structure in what is known as a tensor triangulated category.

In this subsection we recall Balmer's spectrum construction, which inputs a tensor triangulated category and outputs a locally ringed space, which as we will see recovers a variety whenever we work with the derived category of perfect complexes on said variety.

Definition 2.10. An essentially small tensor triangulated category (TTC) $\mathscr{T}$ is a triangulated category together with the following data:

(1) A closed symmetric monoidal structure given by a functor $\otimes : \mathscr{T} \times \mathscr{T} \to \mathscr{T}$ that is additive and exact (with respect to the k-linear structure) on both entries.

(2) The internal Hom functor $\underline{\hom} : \mathscr{T} \times \mathscr{T} \to \mathscr{T}$ sending triangles to triangles (up to a sign).

(3) Coherent natural isomorphisms for each n and m in $\mathbb{Z}$, $r : x \otimes (y[n]) \cong (x \otimes y)[n]$ and $l : (x[n]) \otimes y \cong (x \otimes y)[n]$, compatible with the symmetry, associative and unit coherence morphisms from the symmetric monoidal category structure (see for example [Dell'Ambrogio 2016, Section 2.1.1] for the explicit diagrams).

We will refer to a TTC by the triple $(\mathscr{T}, \otimes, \mathbb{1}_{\mathscr{T}})$, where $\otimes$ refers to the monoidal structure and $\mathbb{1}_{\mathscr{T}}$ to the unit object. Often if there is no confusion or the unit plays no role we will omit it and write $(\mathscr{T}, \otimes)$ instead.

At times when we deal with a fixed underlying triangulated category $\mathscr{T}$ we will write $\otimes$ or $(\otimes, \mathbb{1})$ to refer to a tensor triangulated category structure on $\mathscr{T}$.

Let us remark however that the functor $\otimes$ and unit $\mathbb{1}_{\mathscr{T}}$ do not completely determine a tensor triangulated category since the compatibility conditions in the symmetric monoidal category structure can in principle change while maintaining the functor $\otimes$ and unit $\mathbb{1}$. As we will explain in the following, this does not represent a problem for our purposes.

We proceed with a number of definitions.

Definition 2.11. Let $\mathscr{T}$ be a triangulated category, and $\mathscr{I} \subseteq \mathscr{T}$ a full triangulated subcategory. We say that $\mathscr{I}$ is thick if it is closed under direct summands, so that if $A \oplus B \in \mathscr{I}$ then $A, B \in \mathscr{I}$.

Definition 2.12. Let $(\mathscr{T}, \otimes)$ be a TTC. We will say that a thick subcategory $\mathscr{I} \subset \mathscr{T}$ is a $\otimes$-ideal if for every $A \in \mathscr{T}$ we have $A \otimes \mathscr{I} \subset \mathscr{I}$

Definition 2.13. Let $(\mathscr{T}, \otimes)$ be a tensor triangulated category. Let $\mathscr{I}$ be a $\otimes$-ideal. We will say that $\mathscr{I}$ is prime if when $A, B \in \mathscr{T}$ with $A \otimes B \in \mathscr{I}$, it follows that $A \in \mathscr{I}$ or $B \in \mathscr{I}$.

As in affine algebraic geometry we can define the spectrum of a tensor triangulated category.

Definition 2.14. Let $(\mathscr{T}, \otimes, \mathbb{1})$ be an essentially small tensor triangulated category. The set of all prime $\otimes$-ideals will be denoted by $\mathrm{Spc}(\mathscr{T}, \otimes, \mathbb{1})$ (alternatively $\mathrm{Spc}(\mathscr{T})$, $\mathrm{Spc}(\otimes, \mathbb{1})$ or $\mathrm{Spc}(\otimes)$, depending on which information is clear from context).

Importantly, whenever the triangulated category $\mathscr{T}$ is nonzero we have that $\mathrm{Spec}(\mathscr{T}, \otimes) \neq \varnothing$ for any tensor triangulated category structure $\otimes$ we can put on $\mathscr{T}$ (see [Balmer 2005, Proposition 2.3]).

On this set we will put a topological structure.

Definition 2.15. Let $(\mathscr{T}, \otimes, \mathbb{1})$ be a TTC. The support of an object $A \in \mathscr{T}$, denoted by $\mathrm{supp}(A)$, is the set $\{\mathfrak{p} \in \mathrm{Spc}(\mathscr{T}) \mid A \notin \mathfrak{p}\}$.

Lemma 2.16 [Balmer 2005, Lemma 2.6]. *Let $S \subset \mathscr{T}$ be a family of objects. The sets of the form $\mathcal{Z}(S) := \bigcap_{A \in S} \mathrm{supp}(A)$ form a basis for a topology on* $\mathrm{Spc}(\mathscr{T})$.

An important result regarding this topology is the following, which restricts the kind of spaces we should be expecting from the construction.

Theorem 2.17 [Balmer 2005, Propositions 2.15, 2.18]. *For any TTC $(\mathscr{T}, \otimes, \mathbb{1})$, the space* $\mathrm{Spc}(\mathscr{T})$ *is a spectral space in the sense of Hochster, meaning it is sober and has a basis of quasicompact open subsets.*

Now that the topology on $\mathrm{Spc}(\mathscr{T})$ has been chosen, the next step is to equip this space with sheaf of rings which will act as the structure sheaf.

To a subset $Y \subset \mathrm{Spc}(\mathscr{T})$ we can assign a thick $\otimes$-ideal denoted by $\mathscr{I}_Y$ and defined as the subcategory supported on Y, meaning $\mathscr{I}_Y := \{A \in \mathscr{T} \mid \mathrm{supp}(A) \subset Y\}$.

Finally, with Y as above, we denote by $\mathbb{1}_{T_Y}$ the image of the unit $\mathbb{1}$ of $\mathscr{T}$ under the localization functor $\pi : \mathscr{T} \to \mathscr{T}/\mathscr{I}_Y$.

Definition 2.18. Let $\mathscr{T}$ be a nonzero TTC. We define a structure sheaf $\mathcal{O}_{\mathrm{Spc}(\mathscr{T})}$ over $\mathrm{Spc}(\mathscr{T})$ as the sheaffification of the assignment $U \mapsto \mathrm{End}(\mathbb{1}_{T_Z})$, where $Z := \mathrm{Spc}(\mathscr{T}) \setminus U$, for an open subset $U \subset \mathrm{Spc}(\mathscr{T})$.

It is not hard to see the assignment $\mathrm{Spc}(F)$ respects composition of exact monoidal functors, so if $F : \mathscr{T} \to \mathscr{T}'$ is such a functor, we get a morphism of ringed spaces, since for a closed $Z = \mathrm{Spc}(\mathscr{T}) \setminus U$ we have $F(\mathscr{I}_Z) \subset \mathscr{I}_{Z'}$, where $Z' = \mathrm{Spc}(\mathscr{T}') \setminus \mathrm{Spc}(F)^{-1}(U)$, which implies there is a morphism $\mathcal{O}_{\mathscr{T}} \to \mathrm{Spc}(F)_* \mathcal{O}_{\mathcal{L}}$, so $\mathrm{Spc} : \mathbb{TTC} \to \mathrm{RS}$ is a functor, where $\mathbb{TTC}$ and RS denote the categories of essentially small tensor triangulated categories and of ringed spaces, respectively.

Under nice conditions (for example, $\mathscr{T}$ rigid), this can be shown to be a functor $\mathrm{Spc} : \mathbb{TTC} \to \mathrm{LRS}$, where LRS denotes the category of locally ringed spaces.

With this construction in mind we can now describe the anticipated reconstruction theorem as described by Balmer.

Theorem 2.19 [Balmer 2005, Corollary 5.6]. *Let X be a quasicompact and quasi-separated scheme. There is a homeomorphism*

$$f : X \xrightarrow{\cong} \mathrm{Spc}(\mathrm{Perf}(X), \otimes_X^{\mathbb{L}}).$$

This homeomorphism follows from Thomason's classification theorem [1997, Theorem 3.15], which establishes a correspondence between certain subsets of a quasicompact and quasiseparated scheme X and $\otimes_X^{\mathbb{L}}$-ideals of $\mathrm{Perf}(X)$. The following is a general version of this classification for tensor triangulated categories as presented by Balmer [2005, Theorem 4.10]

Theorem 2.20. *Let $(\mathscr{T}, \otimes, \mathbb{1})$ be a TTC. Let $\mathscr{S}$ be those subsets $Y \subset \mathrm{Spc}(\mathscr{T})$ which are unions $Y = \bigcup_{i \in I} Y_i$, where Y_i is a closed subset with quasicompact complement for all $i \in I$. Let $\mathscr{R}$ be the set of radical $\otimes$-ideals of $\mathscr{T}$. Then there is an order-preserving bijection $\mathscr{S} \to \mathscr{R}$ given by the assignment which sends Y to the subcategory $\mathscr{T}_Y := \{A \in \mathscr{T} \mid \mathrm{supp}(A) \subset Y\}$ and with inverse sending a radical $\otimes$-ideal $\mathscr{I}$ to the subset $S_{\mathscr{I}} := \bigcup_{A \in \mathscr{I}} \mathrm{supp}(A)$.*

Here by radical $\otimes$-ideal we mean a $\otimes$-ideal $\mathscr{I}$ such that whenever $A^{\otimes n}$ is in $\mathscr{I}$ then A is in $\mathscr{I}$.

In practice every $\otimes$-ideal is automatically a radical $\otimes$-ideal and it certainly depends on the monoidal structure one can put on the triangulated category $\mathscr{T}$. As pointed out by Balmer [2005] this condition is satisfied as soon as the tensor triangulated category is rigid, meaning that every object is dualizable.

When X is a variety the classification theorem can be specialized to a very simple form, as pointed out by Rouquier [2003].

Theorem 2.21. *Let X be a variety. There is a correspondence between the set of closed subsets of X and $\otimes^{\mathbb{L}}_X$-ideals of finite type, that is, those ideals generated by a single object.*

Using the homeomorphism from Theorem 2.19 and the construction of the structure sheaf on $\mathrm{Spc}(\mathrm{Perf}(X), \otimes^{\mathbb{L}}_X)$ from Definition 2.18 we only need the following theorem to complete the reconstruction theorem of Balmer.

Theorem 2.22 [Balmer 2002]. *Let X be scheme with underlying noetherian topological space. There is an isomorphism $\mathcal{O}_X \cong \mathcal{O}_{\mathrm{Spc}(\mathrm{Perf}(X), \mathcal{O}_X)}$.*

The next proposition should inform us how localizations behave under taking Spc.

Proposition 2.23 [Balmer 2005, Proposition 3.11]. *Let $\mathscr{I} \subset \mathscr{T}$ be a thick $\otimes$-ideal. Then the localization functor $\pi : \mathscr{T} \to \mathscr{T}/\mathscr{I}$ is an exact monoidal functor and induces an homeomorphism* $\mathrm{Spc}(\mathscr{T}/\mathscr{I}) \cong \{\mathfrak{p} \in \mathrm{Spc}(\mathscr{T}) \mid \mathscr{I} \subset \mathfrak{p}\}$.

In particular, when this is combined with the classification theorem in the form of Theorem 2.21, we see that open subvarieties U of a variety X are isomorphic to $\mathrm{Spc}(\mathrm{Perf}(X)/\mathscr{I}_Z)$, where Z is the complement of U in X.

Remark 2.24. So far we have been dealing with tensor triangulated categories as described in Definition 2.10, meaning we require there to be a closed symmetric monoidal category structure on $\mathscr{T}$. However, under closer inspection, one sees that nowhere in the classification theorem nor in Balmer's construction does one need the full monoidal structure.

In fact so far we really only need the data of a functor $\otimes : \mathscr{T} \times \mathscr{T} \to \mathscr{T}$, covariant and exact in each variable, together with a unit object $\mathbb{1}$ and isomorphisms corresponding to the symmetric, associative and unit conditions. In other words, if $(\mathscr{T}, \otimes, \mathbb{1})$ and $(\mathscr{T}, \boxtimes, \mathbb{1}')$ are two tensor triangulated categories with underlying triangulated category $\mathscr{T}$ such that $\otimes \simeq \boxtimes$ for every pair of objects in $\mathscr{T}$, and $\mathbb{1} \simeq \mathbb{1}'$, then the Balmer spectra satisfy $\mathrm{Spc}(\mathscr{T}, \otimes, \mathbb{1}) \cong \mathrm{Spc}(\mathscr{T}, \boxtimes, \mathbb{1}')$ as locally ringed spaces. The associators, unitors and braidings of the monoidal categories have no influence in the resulting space.

It is this that justifies our notation $(\mathscr{T}, \otimes, \mathbb{1})$, as we have mentioned before. In the following we shall keep referring to tensor triangulated categories although our results apply for slightly more general but more awkward structures.

3. Tensor triangulated categories and Picard groups

While the Bondal–Orlov reconstruction (Theorem 1.1) tells us that one can directly recover a smooth projective variety X with ample (anti)canonical bundle from the derived category $D^b(X) \simeq \mathrm{Perf}(X)$, there are plenty of smooth projective

varieties which have nonisomorphic Fourier–Mukai partners, varieties Y such that $D^b(X) \simeq D^b(Y)$, which implies that on a given derived category $D^b(X)$ there might be many nonequivalent tensor triangulated category structures.

However, even in the case where our variety X has ample (anti)canonical bundle, as in the hypothesis of the Bondal–Orlov reconstruction theorem, it is not immediate that there is only one possible tensor triangulated category structure. It is, in principle, possible that there might be one such structure $(D^b(X), \boxtimes, \mathbb{1})$ such that $D^b(\mathrm{Spc}(\boxtimes, \mathbb{1})) \not\simeq D^b(X)$ and so Bondal–Orlov does not apply.

In some sense our motivating question is whether Balmer's reconstruction implies Bondal–Orlov. In this section we will be looking into this and related ideas by exploring the possible tensor triangulated categories one can equip on $D^b(X)$ under the slightly more general hypothesis of X having a big (anti)canonical bundle.

We start by mentioning a result by Liu and Sierra [2013] that shows, in particular, that there are smooth projective varieties X with ample anticanonical bundle — thus under the hypothesis of Bondal–Orlov — for which the derived category $D^b(X)$ admits a tensor triangulated category structure $(\boxtimes, \mathbb{1})$ such that $\mathrm{Spc}(\boxtimes, \mathbb{1}) \ncong X$.

Recall that there are varieties X that are known to have derived categories equivalent to the derived category of representations on a quiver (possibly with relations). For example, in the presence of a full strong exceptional collection $\{E_i\}$, we have that $D^b(X)$ is equivalent to $D^b\big(\text{mod-End}\big(\bigoplus E_i\big)\big)$, the derived category of finitely generated modules over the algebra $\mathrm{End}\big(\bigoplus E_i\big)$. This latter algebra on the other hand is equivalent to the path algebra of a quiver, and so we obtain an equivalence between the derived category of X and the derived category of finite-dimensional representations of a quiver $Q = (P_n, E_{ij})$.

The important point here is that this derived category of representations of a quiver comes with a tensor triangulated category structure induced by the tensor product of representations. Recall that for two such representations (V_i, p_{ik}) and (W_j, q_{js}) the tensor product is given entrywise: $(V_i, p_{ik}) \otimes_{\mathrm{rep}} (W_j, q_{js}) := (V_i \otimes W_j, p_{ik} \otimes q_{js})$.

We write $\mathrm{rep}\, Q$ for the abelian k-linear category of finite-dimensional quiver representations on a quiver Q. Let us denote by $(D^b(\mathrm{rep}\, Q), \otimes^{\mathbb{L}}_{\mathrm{rep}}, \mathbb{1}_{\mathrm{rep}})$ the resulting tensor triangulated category structure on $D^b(\mathrm{rep}\, Q)$, where $\otimes^{\mathbb{L}}_{\mathrm{rep}}$ is the derived tensor product coming from $\otimes_{\mathrm{rep}}$, and $\mathbb{1}_{\mathrm{rep}} := (k_i, Id_{ij})$ is the representation given by putting k on every vertex and the identity morphism in each edge of the quiver.

Liu and Sierra [2013, Definition 1.2.5] consider quivers with relations satisfying a compatibility condition with the tensor product and say that in this case the quiver has tensor relations.

Theorem 3.1 [Liu and Sierra 2013, Theorem 2.1.5.1]. *Let Q be a finite ordered quiver with tensor relations. Then* $\mathrm{Spc}((D^b(\mathrm{rep}\, Q), \otimes^{\mathbb{L}}_{\mathrm{rep}}))$ *is the discrete space* $\{P_n\}$.

They also describe completely the structure sheaf in this case.

Theorem 3.2 [Liu and Sierra 2013, Theorem 2.2.4.1]. *Let Q be a finite ordered quiver with tensor relations. Then $\mathscr{O}_Q := \mathscr{O}_{\mathrm{Spc}(\otimes^{\mathbb{L}}_{\mathrm{rep}})}$ is the constant sheaf of algebras k, so that for any open $W \subset \mathrm{Spc}(\otimes^{\mathbb{L}}_{\mathrm{rep}})$ we have $\mathscr{O}_Q(W) = k^{\oplus W}$.*

In particular, for $X = \mathbb{P}^n$, we have by a well-known result of Beilinson [1978] that $D^b(X)$ is equivalent to the category of representations of a quiver with $n+1$ vertices. Thus the derived category $D^b(X)$ has a tensor triangulated category structure $(\otimes^{\mathbb{L}}_{\mathrm{rep}}, \mathbb{1}_{\mathrm{rep}})$ such that $\mathrm{Spc}((\otimes^{\mathbb{L}}_{\mathrm{rep}}, \mathbb{1}_{\mathrm{rep}})) \ncong X = \mathbb{P}^n$. As $\mathbb{P}^n$ is a smooth projective variety with ample anticanonical bundle, this previous result implies that the study of tensor triangulated category structures on $D^b(X)$ is not trivial even in the cases falling under the hypothesis of the Bondal–Orlov reconstruction theorem and might shed some light on the internal structure of the derived category in itself.

In general the behavior of the dynamics of the Balmer spectrum and taking derived categories can be complex. Since the Balmer spectrum is a locally ringed space, it has an abelian category of sheaves of modules which admits a tensor product, and we can derive this category as usual, but the category of sheaves of modules is in general much more complicated than a category of coherent or even quasicoherent sheaves.

Having said that, let us put ourselves in the slightly more general situation of derived categories of varieties of general type. Recall a variety is of general type if its canonical bundle is big. In particular, varieties with ample canonical bundle are of general type.

One alternative characterization of bigness for a variety is the following:

Theorem 3.3 [Lazarsfeld 2004, Example 2.2.9]. *A smooth projective variety is of general type if and only if, for any sheaf $\mathscr{F} \in \mathrm{Coh}(X)$, there exists an integer i_0 depending on $\mathscr{F}$ such that the sheaf $\mathscr{F} \otimes_X \omega_X^i$ is generically globally generated for $i \gg i_0$.*

As a consequence of the Kodaira lemma (see [Lazarsfeld 2004, Proposition 2.2.6]), we have the corollary:

Corollary 3.4. *Let X be a smooth projective variety of general type. Then there exists an open subvariety X^* such that for any $\mathscr{F} \in \mathrm{Coh}(X)$, there exists a positive integer i_0 for which, for any $i \gg i_0$, the sheaf $\mathscr{F}\,|_{X^*} \otimes_X \omega^i_{X^*}$ on X^* is globally generated.*

Let us explain the previous corollary and the nature of the open subvariety X^*. We recall some basic definitions.

Definition 3.5. Let X be a projective variety, and let $\mathscr{L}$ be a line bundle on X. The augmented base locus is the Zariski closed set

$$B_+(\mathscr{L}) := \bigcap_{m \in \mathbb{N}} B(m\mathscr{L} - A),$$

where A is any ample line bundle, and for any line bundle $\mathscr{L}'$ the set $B(\mathscr{L}')$ is defined as the intersection of the base loci of multiples of the line bundle, that is,

$$B(\mathscr{L}') := \bigcap_{m\in\mathbb{N}} Bs(m\mathscr{L}').$$

In [Boucksom et al. 2014] the following theorem characterizing the complement of the augmented base locus is proven:

Theorem 3.6. *Let $\mathscr{L}$ be a big line bundle on a normal projective variety X over an algebraically closed field. Then the complement $X \setminus B_+(\mathscr{L})$ of the augmented base locus is the largest Zariski open subset $U \subseteq X \setminus B(\mathscr{L})$ such that for all large and divisible $m(\mathscr{L}) \in \mathbb{Z}$ the restriction of the morphism*

$$\phi_m : X \setminus B(\mathscr{L}) \dashrightarrow \mathbb{P}H^0(X, m\mathscr{L})$$

to U is an isomorphism onto its image.

Two important observations follow immediately from the definition, the fact that the augmented base locus is independent of the choice of ample line bundle, and Kodaira's decomposition of big line bundles.

Remark 3.7. (1) $B_+(\mathscr{L}) = \varnothing$ if and only if $\mathscr{L}$ is ample.

(2) $B_+(\mathscr{L}) \neq X$ if and only if $\mathscr{L}$ is big.

From the remarks above and Thomason's classification theorem, Theorem 2.20, we know that since there exists a correspondence between closed subsets of the Balmer spectrum and radical tensor ideals in the tensor triangulated category, there exists a radical tensor ideal corresponding to the augmented base locus $B_+(\mathscr{L})$ for any given line bundle $\mathscr{L}$. In particular, the open subvariety X^* from Corollary 3.4 is the complement of the augmented base locus, $X \setminus B_+(\omega_X)$, and corresponds to a $\otimes^{\mathbb{L}}_X$-ideal generated by a single object (using Theorem 2.21) whose homological support gives back the closed subset $B_+(\omega_X)$.

Remark 3.8. Denote by I_{X^*} the $\otimes^{\mathbb{L}}_X$-ideal corresponding to the open subvariety X^*. By Remark 3.7, this ideal must be a proper $\otimes^{\mathbb{L}}_X$-ideal of $D^b(X)$ and is the ideal 0 precisely when the (anti)canonical bundle is ample.

We would like to understand the effect of the positivity of the canonical bundle (in this case the fact that the variety is of general type) on the tensor triangulated structure of the category. We know from Proposition 2.4 that the Serre functor in a triangulated category is unique up to degree whenever it exists and so it is intrinsic to the category. In our concrete case we know furthermore that the Serre functor is isomorphic to $\_ \otimes^{\mathbb{L}}_X \omega_X[n]$, where $n \in \mathbb{N}$ is the dimension of the variety and ω_X is the dualizing sheaf of X.

Let us start with a definition mimicking that of spanning class:

Definition 3.9. Let $(\mathscr{T}, \otimes)$ be a tensor triangulated category, let $\mathscr{I} \subseteq \mathscr{T}$ be a thick subcategory, and let us denote by $\pi : \mathscr{T} \to \mathscr{T}/\mathscr{I}$ the localization functor. We say that a collection of objects $\Omega \subset \mathscr{T}$ is an almost spanning class with respect to $\mathscr{I}$ if the following two conditions hold:

(1) If $X \in \mathscr{T}/\mathscr{I}$ is such that $\mathrm{Hom}_{\mathscr{T}/\mathscr{I}}(\pi(B), X[j]) = 0$ for all $B \in \Omega$ and $j \in \mathbb{Z}$, then $X \cong 0$.

(2) If $X \in \mathscr{T}/\mathscr{I}$ is such that $\mathrm{Hom}_{\mathscr{T}/I}(X[j], \pi(B)) = 0$ for all $B \in \Omega$ and $j \in \mathbb{Z}$, then $X \cong 0$.

It is immediate to see that the previous definition is equivalent to asking that the collection Ω maps through π to a spanning class on the quotient $\mathscr{T}/\mathscr{I}$. When the thick subcategory in question is the 0 subcategory then the definition reduces to that of a spanning class as in Definition 2.5.

Additionally when the triangulated category $\mathscr{T}/\mathscr{I}$ has a Serre functor, only one of the conditions in the definition is necessary as the Serre duality implies the other automatically.

We would like to generalize Lemma 2.7 but for a big canonical bundle instead of an ample one and see that a big bundle induces an almost spanning class in the derived category with respect to a $\otimes^{\mathbb{L}}_X$-ideal $\mathscr{I}$.

Theorem 3.10. *Let X be a smooth projective variety of general type. Then the collection of tensor powers $(\omega_X^{\otimes i})_{i\in\mathbb{Z}}$ forms an almost spanning class with respect to the tensor ideal I_{X^*} in the tensor triangulated category $(D^b(X), \otimes^{\mathbb{L}}_X)$.*

Proof. We need to show that $\pi(\{\omega_X^{\otimes i}\})$ forms a spanning class in the quotient $D^b(X)/I_{X^*}$. As I_{X^*} is the ideal corresponding to the open smooth subvariety X^*, from Corollary 3.4 we know that there is an isomorphism $\mathrm{Spc}(D^b(X)/I_{X^*}) \cong X^*$. Since ω_X restricted to X^* is ample by the characterization of Theorem 3.6, we get that $\{\omega_X^{\otimes^{\mathbb{L}} i}|_{X^*}\}$ forms a spanning class, by Lemma 2.7, of the derived category of X^* which coincides with the quotient category $D^b(X)/I_{X^*}$ by Proposition 2.23. □

The main key in our arguments is the fact that one can construct, as in the ample case, a resolution for any bounded complex of coherent sheaves on X^* in terms of tensor powers of the canonical bundle ω_{X^*} of X^*, with the advantage that one is able to have a concrete description of the derived category of this space in terms of a quotient of the derived category of the larger variety X.

Explicitly, for any bounded complex A of coherent sheaves over X^*, there is a resolution

$$\cdots \to 0 \to \bigoplus_{j_0} (\omega_{X^*}^{\otimes i_0}) \to \cdots \to \bigoplus_{j_k} (\omega_{X^*}^{\otimes i_k}) \to A \to 0.$$

Another thing to notice is that in the example given above for the nonequivalent tensor triangulated category structures on $D^b(\mathbb{P}^n)$, one immediate issue with the two

given such structures was that the units were nonisomorphic. So we should proceed to work with tensor triangulated categories with a fixed unit isomorphic to $\mathscr{O}_X$.

Definition 3.11. Let $(\mathscr{T}, \otimes, \mathbb{1})$ be a TTC. An object $X \in \mathscr{T}$ is $\otimes$-invertible if there exists $X^{-1} \in \mathscr{T}$ such that $X \otimes X^{-1} \cong \mathbb{1}$. We will denote by $\mathrm{Pic}(D^b(X), \otimes)$ the group of isomorphism classes of $\otimes$-invertible objects.

We will make use of the following lemma:

Lemma 3.12. *Suppose X is a smooth projective variety of general type of dimension n. If $\boxtimes$ is a tensor triangulated structure on $D^b(X)$ with unit $\mathscr{O}_X$, and U is a $\boxtimes$-invertible object such that $U \boxtimes I_{X^*} \subseteq I_{X^*}$, then there is a natural equivalence between the functors induced by $U \boxtimes \_$ and $U \otimes_X^{\mathbb{L}} \_$ in $D^b(X)/I_{X^*}$.*

Proof. By our previous discussion we know that any bounded complex can be resolved in $D^b(X)/I_{X^*}$ by a resolution

$$\cdots \to 0 \to \bigoplus_{j_0} (\omega_{X^*}^{\otimes i_0}) \to \cdots \to \bigoplus_{j_k} (\omega_{X^*}^{\otimes i_k}) \to A \to 0.$$

Because the Serre functor in $D^b(X^*)$ is given by $\_ \otimes_X^{\mathbb{L}} \omega_{X^*}[n']$, where n' is the dimension of X^*, and we know any exact equivalence must commute with it, if we let $U\widehat{\boxtimes}$ and $U\widehat{\otimes_X^{\mathbb{L}}}$ denote the endofunctors of $D^b(X)/I_{X^*}$ induced respectively by $U\boxtimes$ and $U\otimes^{\mathbb{L}}$, then, since $U\widehat{\boxtimes}$ is an autoequivalence,

$$(U \,\widehat{\boxtimes}\, \hat{A}) \,\widehat{\otimes_X^{\mathbb{L}}}\, \omega_{X^*}[n'] \cong U \,\widehat{\boxtimes}\, (\hat{A} \,\widehat{\otimes_X^{\mathbb{L}}}\, \omega_{X^*}[n']).$$

As $\mathscr{O}_X$ is a unit for both $\otimes_X$ and $\boxtimes$, and after shifting by $[-n']$, we deduce

$$U \,\widehat{\otimes_X^{\mathbb{L}}}\, \omega_{X^*} \cong U \,\widehat{\boxtimes}\, \omega_{X^*}.$$

From this, the exactness of $\otimes^{\mathbb{L}}$ and $\boxtimes$, and the resolutions in terms of $\omega_{X^*}^i$, we obtain the isomorphisms

$$U \,\widehat{\otimes^{\mathbb{L}}}\, A \cong U \,\widehat{\boxtimes}\, A. \qquad \square$$

Remark 3.13. Let us point out the slight abuse of notation of the functor $U\widehat{\otimes^{\mathbb{L}}}$. This functor would formally be denoted by $\widehat{U}\otimes^{\mathbb{L}}_{D^b(X)/I_{X^*}}$, as it is induced by the object $\widehat{U}$ in the tensor triangulated category $(D^b(X)/I_{X^*}, \otimes^{\mathbb{L}}_{D^b(X)/I_{X^*}})$, but as the only tensor ideal we are taking a quotient by in this section is I_{X^*}, we believe our notation is lighter without losing sight of which functors they represent.

We have the following corollary:

Corollary 3.14. *Let X be a variety of general type, and let $\boxtimes$ be a tensor triangulated category structure on $D^b(X)$ with unit $\mathscr{O}_X$. Then, for any $\boxtimes$-invertible object U such that $U \boxtimes I_{X^*} \subseteq I_{X^*}$, the equivalence $U\widehat{\boxtimes} : D^b(X)/I_{X^*} \to D^b(X)/I_{X^*}$ induced by $U \boxtimes \_$ is equivalent to an equivalence given by objects in the group $\mathrm{Pic}(D^b(X)/I_{X^*}, \widehat{\otimes^{\mathbb{L}}})$ of invertible $\widehat{\otimes^{\mathbb{L}}}$-objects.*

Proof. From Lemma 3.12 we have that if U^{-1} is such that $U \boxtimes U^{-1} \cong \mathscr{O}_X$ then in the quotient $D^b(X)/I_{X^*}$,

$$U \widehat{\otimes^{\mathbb{L}}} \widehat{U^{-1}} \cong U \widehat{\boxtimes} \widehat{U^{-1}} \cong \mathscr{O}_{X^*}.$$

As $(D^b(X)/I_{X^*}, \widehat{\otimes^{\mathbb{L}}})$ is a tensor triangulated category, we have that $\widehat{U} \in D^b(X)/I_{X^*}$ is a $\widehat{\otimes^{\mathbb{L}}}$-invertible object. □

In Lemma 3.12 and Corollary 3.14, the ideal I_{X^*} might not be a $\boxtimes$-tensor ideal and thus the quotient $D^b(X)/I_{X^*}$ does not necessarily carry a tensor triangulated category structure induced by $\boxtimes$. However, our result guarantees that after passing to the quotient, the equivalences induced by the functors $U \boxtimes \_$ are equivalent to equivalences given by invertible objects in $(D^b(X)/I_{X^*}, \otimes^{\mathbb{L}}_X)$ induced by the same object, under the condition that I_{X^*} is stable by $U\boxtimes$.

In particular, we have:

Corollary 3.15. *Let X be a variety of general type and $\boxtimes$ a tensor triangulated structure on $D^b(X)$ with unit $\mathscr{O}_X$. If I_{X^*} is a $\boxtimes$-ideal then the Picard group* $\mathrm{Pic}(D^b(X)/I_{X^*}, \widehat{\boxtimes})$ *is a subgroup of the Picard group* $\mathrm{Pic}(D^b(X)/I_{X^*}, \widehat{\otimes^{\mathbb{L}}_X})$.

Proof. The proof is as in the previous two. If U is in $\mathrm{Pic}(D^b(X)/I_{X^*}, \widehat{\boxtimes})$ then it induces an autoequivalence of $D^b(X)/I_{X^*}$ and so it commutes with the Serre functor on $D^b(X^*) \simeq D^b(X)/I_{X^*}$. By writing a resolution for any bounded complex A in terms of direct sums of derived tensor powers of ω_{X^*} we can use the same argument as in the proof of Lemma 3.12 and we arrive at the isomorphisms

$$U \widehat{\otimes^{\mathbb{L}}} A \cong U \widehat{\boxtimes} A.$$ □

Remark 3.16. In the results above we have chosen to work with varieties of general type, but the same argument applies to varieties with big anticanonical bundle.

The case when our variety has an ample (anti)canonical bundle allows us to relate the Picard group of the full derived category to that of any other tensor triangulated category structure on it.

The following result follows from the previous argument.

Corollary 3.17. *Let X be a variety with ample (anti)canonical bundle. Then if $\boxtimes$ is a tensor triangulated category structure on $D^b(X)$ with unit $\mathscr{O}_X$, the Picard group* $\mathrm{Pic}(D^b(X), \boxtimes)$ *is isomorphic to a subgroup of* $\mathrm{Pic}(D^b(X), \otimes_X)$.

Proof. We just need to notice that in this case the $\otimes_X$-ideal from Corollary 3.4 is the 0 ideal and thus we can resolve any object $A \in D^b(X)$ by a sequence of powers of the Serre functor. By the same reasoning as above we see that

$$U \otimes^{\mathbb{L}} A \cong U \boxtimes A.$$ □

Although Bondal and Orlov had already classified the group of autoequivalences of a derived category of a variety with ample (anti)canonical bundle, we are working without the condition of an equivalence between the derived category of the Balmer spectrum of $\boxtimes$ and the derived category $D^b(X)$, and as such it is not immediate from their result that the Picard group of $\boxtimes$ must involve invertible sheaves over X.

In other words, since $\mathrm{Spc}(\boxtimes)$ is not necessarily isomorphic to X, understanding the autoequivalences of $D^b(X)$ alone does not give us an immediate relationship to the Picard group of $\boxtimes$.

We should think of the following corollary as a monoidal version of the Bondal–Orlov reconstruction theorem:

Corollary 3.18. *Let X be as above. If $\omega_X[n]$ is an invertible object for a tensor triangulated structure $\boxtimes$ on $D^b(X)$ with unit $\mathscr{O}_X$ then $\boxtimes$ and $\otimes^{\mathbb{L}}_X$ coincide on objects.*

Proof. As ω_X is $\boxtimes$-invertible, Corollary 3.17 tells us that for any $A \in D^b(X)$,

$$\omega_X \otimes^{\mathbb{L}}_X A \cong \omega_X \boxtimes A.$$

But we can resolve any other bounded complex B in terms of derived powers of the canonical sheaf, so by the exactness of $\boxtimes$, we have

$$B \otimes^{\mathbb{L}}_X A \cong B \boxtimes A. \qquad \square$$

The nature of this result comes precisely from the fact that the tensor triangulated category structure $(\boxtimes, \mathbb{1})$ does not necessarily come from a derived equivalence $D^b(X) \simeq D^b(Y)$, and although the extra assumption on the unit is required, the result is in this direction slightly more general than the original theorem.

Corollary 3.19. *Let X be a variety with ample (anti)canonical bundle. If $(\mathscr{O}_X, \boxtimes)$ is a tensor triangulated structure on $D^b(X)$ such that $\mathrm{Pic}(\boxtimes) \cong \mathrm{Pic}(D^b(X))$ via the assignment $U \mapsto U$ then $\boxtimes$ coincides with $\otimes^{\mathbb{L}}_X$ on objects.*

Proof. In this case if this morphism is an isomorphism, then ω_X is $\boxtimes$-invertible and the result follows from the previous corollary. $\square$

In fact if we are under the same hypothesis for X then as soon as we are able to show that the generators of $\mathrm{Pic}(D^b(X), \otimes^{\mathbb{L}})$ are $\boxtimes$-invertible then by the previous corollary there must be an equivalence between $\boxtimes$ and $\otimes^{\mathbb{L}}_X$.

Example 3.20. Let $X = \mathbb{P}^n$ be the projective space. In this case we know that $\mathrm{Pic}(D^b(X)) = \mathbb{Z} \oplus \mathbb{Z}$, corresponding to the line bundles plus their shifts. The result above then says that whenever there is a tensor triangulated structure $\boxtimes$ on $D^b(X)$ with unit $\mathscr{O}_X$ then the Picard group of this tensor structure must necessarily be a subgroup of $\mathbb{Z} \oplus \mathbb{Z}$.

If $\omega_X = \mathscr{O}_X(-n-1)$ is $\boxtimes$-invertible then we get that $\boxtimes$ coincides with $\otimes^{\mathbb{L}}_X$. Similarly if $\mathscr{O}_X(-1)$ is $\boxtimes$-invertible.

One natural question to ask when working with Picard groups of tensor triangulated category structures is what the relationship with line bundles on the associated space is. Balmer and Favi [2007, Proposition 4.4] proved the following result:

Proposition 3.21. *Let X be a scheme and consider* $\mathrm{Perf}(X)$, *its derived category of perfect complexes. Then there is a split short exact sequence of abelian groups*

$$0 \to \mathrm{Pic}(X) \to \mathrm{Pic}(\mathrm{Perf}(X), \otimes^{\mathbb{L}}_X) \to C(X;\mathbb{Z}) \to 0,$$

where $C(X;\mathbb{Z})$ stands for the group of locally constant functions from X to $\mathbb{Z}$.

Again, under the hypothesis of X having an ample (anti)canonical bundle, by using Proposition 3.21 we see that for a TTC $(\boxtimes, \mathscr{O}_X)$, if $\mathrm{Spc}(\boxtimes)$ is a scheme then the Picard group of $\mathrm{Spc}(\boxtimes)$ must be a subgroup of the Picard group of $\mathrm{Pic}(D^b(X), \boxtimes) \leq \mathrm{Pic}(D^b(X), \otimes^{\mathbb{L}}_X)$. So a line bundle in $\mathrm{Spc}(\boxtimes)$ has to be $\otimes^{\mathbb{L}}_X$-invertible.

Remark 3.22. From the original proof of the Bondal–Orlov reconstruction theorem, we know that it is actually possible to fully characterize line bundles up to a shift from purely categorical properties. Given the importance of the Picard group of the variety, we can ask whether it is possible to reconstruct the derived tensor product in $D^b(X)$ without having to pass through a reconstruction theorem.

Antieau [2013] sketched a construction in which by considering invertible objects (in the sense of Bondal and Orlov) one can define a collection of tensor products $\otimes^{\mathbb{L}}_U$ by exploiting the resolution by derived tensor powers of ω_X.

The idea is to pick an invertible object U, shown in [Bondal and Orlov 2001] to be isomorphic to a shift of a line bundle in X, and then by use of the resolution we only need to define the products $\omega_X^{\otimes^{\mathbb{L}}_X i}[ni] \otimes^{\mathbb{L}}_U A^\bullet$ for any object $A^\bullet$. As the Serre functor $S \simeq \_ \otimes^{\mathbb{L}}_X \omega_X[n]$ comes from the categorical structure alone then we can set these products to be simply $S^i(A^\bullet)$.

These tensor products $\otimes^{\mathbb{L}}_U$ have U as unit, and they all have X as Balmer spectrum.

In general for a triangulated category $\mathscr{T}$ we have an action by $\mathrm{Aut}(\mathscr{T})$ on the collection $\mathrm{TTS}(\mathscr{T})$.

If $(\otimes, \mathbb{1}) \in \mathrm{TTS}(\mathscr{T})$ and $\phi \in \mathrm{Aut}(\mathscr{T})$ we have a tensor structure defined by

$$X \otimes_\phi Y := \phi^{-1}\big(\phi(X) \otimes \phi(Y)\big),$$

with unit given by $\phi^{-1}(\mathbb{1})$.

We have now justified enough the following definition:

Definition 3.23. Let $\mathscr{T}$ be a triangulated category. Denote by $\mathrm{TTS}(\mathscr{T})$ the collection of equivalence classes of tensor triangulated category structures on $\mathscr{T}$, where we consider two tensor triangulated category structures to be equivalent if there is a monoidal equivalence between the two of them.

To keep some control and avoid counting structures coming from autoequivalences as we saw, we should at least fix the unit object.

Definition 3.24. Let $\mathscr{T}$ be a triangulated category and $U \in \mathscr{T}$ an object. Then the set $\mathrm{TTS}_U(\mathscr{T})$ is the set of equivalence classes of tensor triangulated structures on $\mathscr{T}$, where U is the unit.

This is the set we are mainly interested in classifying.

Let us finish by discussing the original Bondal–Orlov reconstruction theorem in terms of the results we have shown so far.

Theorem 3.25. *Let X be a variety with ample (anti)canonical divisor, and let $\boxtimes$ be a tensor triangulated structure on $D^b(X)$ with unit $\mathscr{O}_X$. If $\mathrm{Spc}(\boxtimes)$ is a smooth projective space with ample (anti)canonical bundle and there is an equivalence $D^b(X) \simeq D^b(\mathrm{Spc}(\boxtimes))$, then $X \cong \mathrm{Spc}(\boxtimes)$.*

Proof. The only thing to note here is that since $\mathrm{Spc}(\boxtimes)$ has ample (anti)canonical bundle, ω_X has to be $\boxtimes$-invertible. Indeed, we recall that one can pick the equivalence $D^b(X) \simeq D^b(\mathrm{Spc}(\boxtimes))$ to send ω_X to $\omega_{\mathrm{Spc}(\boxtimes)}$ and then the assertion follows by applying Corollary 3.17 to $\mathrm{Spc}(\boxtimes)$ so that we obtain that $\mathrm{Pic}(D^b(X), \otimes^{\mathbb{L}}_X)$ is isomorphic via the assignment $\mathscr{L} \mapsto \mathscr{L}$ to a subgroup of $\mathrm{Pic}(D^b(X), \boxtimes)$. Since ω_X is $\boxtimes$-invertible, by Corollary 3.18 we obtain our result. □

Remark 3.26. We need to explain our choice of hypotheses. First, the assumption that $\mathrm{Spc}(\boxtimes)$ is a smooth projective variety is necessary just as in the original Bondal–Orlov theorem formulation. We have added a couple more assumptions, however. We suppose that the (anti)canonical bundle of $\mathrm{Spc}(\boxtimes)$ is also ample to highlight the use of the monoidal structures in the theorem. This hypothesis is however not necessary as it can be directly deduced from the derived equivalence between the two spaces, just as in the original proof of Bondal and Orlov. Alternatively, we can formulate the theorem as follows:

Theorem 3.27. *Let X be a variety with ample (anti)canonical divisor, and let $\boxtimes$ be a tensor triangulated structure on $D^b(X)$ with unit $\mathscr{O}_X$. Suppose that $\mathrm{Spc}(\boxtimes)$ is a smooth projective space and that we have an equivalence $D^b(X) \simeq D^b(\mathrm{Spc}(\boxtimes))$. Then $X \cong \mathrm{Spc}(\boxtimes)$.*

Of more importance is perhaps the choice of unit, as we have seen that there are tensor triangulated category structures on the derived category of such a variety which will produce very different spaces under the Balmer reconstruction. This choice of unit allows us to keep some control in the classification of structures producing the same space.

A natural next step for future work would be to deal with the possible sort of objects which can be units for such a structure.

Remark 3.28. There is some nuance in the way in which Bondal–Orlov follows from our results as we make use of some important technical results from the original proof. We expect however that the discussion in this work has provided

enough of a justification and motivation for looking at this problem in terms of monoidal structures.

We can close our discussion with the following theorem:

Theorem 3.29. *Let X be a smooth projective variety with ample (anti)canonical bundle. If $\boxtimes$ is a tensor triangulated category structure on $D^b(X)$ such that $\mathscr{O}_X$ is its unit and* $\mathrm{Spc}(\boxtimes)$ *is isomorphic to X, then $\boxtimes$ and $\otimes^{\mathbb{L}}_X$ coincide on objects.*

This however does not fully classify $\mathrm{TTS}_{\mathscr{O}_X}(D^b(X))$, since we require Balmer's spectrum to be a Fourier–Mukai partner, but there is no reason to expect in general a relationship between the derived category of the Balmer spectrum and the original triangulated category.

The lack of morphisms between a space X and the Balmer spectrum $\mathrm{Spc}(\boxtimes)$ for some tensor triangulated structure, and thus of functors between the derived categories of these two spaces, is one of the obstacles to being able to understand the possible structures $\boxtimes$.

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