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Given closed possibly nonorientable surfaces M, N, we prove that if a map $f : M \to N$ has geometric degree $d > 0$, then $\chi(M) \le d \cdot \chi(N)$. We give all necessary comments on the definition and properties of geometric degree, which can be defined for any map. Our proof is based on the factorization theorem of Edmonds, a simple natural proof of which is also presented.

1. Introduction

Through this paper, we set M, N to be closed connected surfaces, possibly nonorientable.

Given a map $f : M \to N$, we define its *geometric degree* $\operatorname{Deg} f$ as a minimal cardinality of the preimage of a regular value among all smooth maps in the homotopy class of f. Note that if M and N are orientable, then this number coincides with the usual notion of degree; we discuss features of the definition and prove some of its properties in Section 2.2.

Our goal is to prove the following fact:

Theorem 1. *Let $f : M \to N$ be a map of geometric degree $d > 0$. Then $\chi(M) \le d \cdot \chi(N)$.*

This fact is well known; apparently it was first proved by Kneser [1930] in the case of orientable surfaces. It looks very similar to the assertions "*all maps $S^2 \to N$ are nullhomotopic*" and "*if $\chi(M) > \chi(N)$, then $d = 0$*". These statements can be easily proved using universal covering or the intersection form in cohomology. However, similar elementary approaches to Theorem 1 are not known to the author.

The possible ways to prove Theorem 1 are rather a bit more technical. For orientable surfaces, one can use the Milnor–Wood inequality [1971, Theorem 1.1] or the Gromov norm [1999, §5.35]. In this paper, we present the most elementary proof including the factorization theorem of Edmonds. First, let us recall the notation.

Suppose we have a 2-submanifold with boundary $K \subset M$, such that for every component $K_i \subset K$ its boundary ∂K_i is connected. Collapsing each K_i to a single

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point (see, e.g., [Whitehead 1978, p. 23]) we obtain a smooth manifold Q. The factorization map $p : M \to Q$ is called *a pinch map*.

A smooth map of closed surfaces $q : Q \to N$ is called *a branched covering* if it is a local diffeomorphism outside of a finite subset $P \subset M$ and near every $x_0 \in P$ one can write q as $z \mapsto z^k$, $k > 1$, in some local complex coordinates. The number k is called *the index of the branch point* x_0.

Theorem 2. *Every map $f : M \to N$ is either homotopic to the composition of a pinch map $p : M \to Q$ with a branched covering $q : Q \to N$ for some closed surface Q, or homotopic to a map whose image is a graph embedded into N.*

This theorem is due to Edmonds [1979]. He proved the theorem for surfaces with boundary as well, considering maps f whose restriction to the boundary is a $(\operatorname{Deg} f)$-sheeted covering. The proof of Edmonds was corrected and improved by Skora [1987].

We present another proof of Theorem 2 which is simpler and more natural in some ways. Namely, it does not use induction and constructs the factorization in one step. The idea is to take a triangulation of N and consider a map h homotopic to f which is transversal to $\operatorname{sk}^1(N)$ and has a minimal number of edges of $h^{-1}(\operatorname{sk}^1(N))$, and then to deform h over each triangle. This approach was inspired by Lurie's proof of the Dehn–Nielsen theorem [2009, Lecture 38]; see also [Farb and Margalit 2012, §8.3.1].

Unlike [Edmonds 1979] and [Skora 1987], in our proof we do not control the degree of a branched covering. Also, our approach does not deal with surfaces with boundary, the corresponding generalization is possible but it would require some additional work. Finally, we do not use the theory of absolute degree — see Section 2.2 for some remarks on this subject — so we tried to make our reasoning completely self-contained.

2. Preliminaries

2.1. ***Conventions and notation on surfaces and transversality.*** We use the term *surface* for a 2-manifold, and *closed* with respect to a manifold means that it is compact without boundary. All manifolds will be assumed to be infinitely smooth, as well as maps. The maps we construct sometimes will be not smooth and should be smoothed if needed, but this inaccuracy will not cause difficulties; see, e.g., [Hirsch 1976, Chapter 8].

Every open subset of a surface $U \subset M$ that we take is supposed to be "sufficiently nice" — namely, it should be an interior of a compact 2-submanifold with boundary. We refer to this boundary as ∂U (the notation $\partial \overline{U}$ here would be formally correct, but it is more cumbersome).

When we *cut* a closed surface M *along a closed curve* $C \subset M$, we assume to obtain as a result a compact surface with boundary M', and each point of C will double in M'.

Given surfaces M, N, we say that a map $f : M \to N$ is *transversal* to a stratified subset $\mathcal{S} \subset N$, if f is transversal to its every stratum. Namely, every vertex $y \in \mathcal{S}$ must be a regular value of f and for every edge $C \subset \mathcal{S}$ and any $x \in f^{-1}(C)$ there must be a vector $v \in T_x M$ such that $df(v) \notin T_{f(x)}C$. By the implicit function theorem, $f^{-1}(\mathcal{S}) \subset M$ is a stratified subset. If $\mathcal{S}$ is closed, then the set of maps transversal to $\mathcal{S}$ is a dense and open set in $C^\infty(M, N)$. See, for example, [Goresky and MacPherson 1988, Part 1, §1] for details.

We say that a loop $\varphi : S^1 \to S^1$ has *index* $i \geq 0$, if $[\varphi] = \pm i \in \pi_1(S^1)$.

2.2. *Geometric degree.* Recall, we define a *degree* of a map $f : M \to N$ as a minimal $d \in \mathbb{Z}_{\geq 0}$ such that there is a smooth map $h : M \to N$ homotopic to f and there is a regular value $y \in N$ such that $|h^{-1}(y)| = d$. It is known as *geometric degree*, but in later sections, we will simply call it a *degree* and denote it as $\operatorname{Deg} f$.

The degree theory began with the work of Hopf [1928; 1930] and was developed by Olum [1953] and Epstein [1966]. The most important properties of geometric degree in dimension 2 were proved by Kneser [1928; 1930].

Here we state and sketch the proofs of a few properties of degree. We consider not famous properties, but only those that will be used in Section 4 in order to prove Theorem 1. For a more detailed review, see, e.g., [Brown and Schirmer 2001] or [Sklyarenko 2008] in addition to [Epstein 1966] and [Olum 1953]. As usual, we suppose M, N to be closed connected surfaces, but one can similarly formulate and prove corresponding statements for any pair of closed manifolds of the same dimension.

2.2.1. *Degree of a branched covering.* Suppose a map $f : M \to N$ is *orientation-true*. This means that it takes orientation-preserving/reversing loops in M to orientation-preserving/reversing loops in N, respectively. Clearly, this is equivalent to the equality $f^*(w_1(N)) = w_1(M)$ for Stiefel–Whitney classes, or to the fact $f^*(\mathbb{Z}_N) \simeq \mathbb{Z}_M$, where $\mathbb{Z}_M$ denotes the orientation local system of M with fiber $\mathbb{Z}$. Then f induces a homomorphism $H^2(N; \mathbb{Z}_N) \to H^2(M; \mathbb{Z}_M)$. Here both groups are isomorphic to $\mathbb{Z}$, so the homomorphism $\mathbb{Z} \to \mathbb{Z}$ is a multiplication by an integer called the *cohomological degree* of f, denoted by $\deg f$.

Note that for non-orientation-true maps one can similarly define a cohomological degree as a residue (mod 2), but not as an integer because of $H^2(M; f^*(\mathbb{Z}_N)) \simeq \mathbb{Z}_2$. For more details on local systems, see, for example, [Spanier 1993] or [Whitehead 1978, Chapter VI].

Proposition 3. *If the map* $f : M \to N$ *is orientation-true*, *then* $\operatorname{Deg} f \geq \deg f$.

Proof. Homotope f so that a regular value $y \in N$ has $\operatorname{Deg} f$ preimages. Choose a local orientation of M and N. That allows us to define *the sign* for every preimage $x_i \in f^{-1}(y)$, so that the sum equals $\deg f$. (Formally, here we use local cohomology; see, e.g., [Hirsch 1976, Chapter 5]). □

In fact, the opposite inequation also holds, so we have $\operatorname{Deg} f = \deg f$ for an orientation-true f. The idea of a proof is to cancel a pair of preimages of y which have different signs (see, e.g., [Epstein 1966, p. 380] for dimension > 2), but for surfaces, this strategy is quite a bit more complicated (see, e.g., [Melikhov 2004, Lemma 2]). Another strategy is to apply Theorem 2 (our proof uses Proposition 3, but not the equality $\deg f = \operatorname{Deg} f$ for an orientation-true f).

Corollary 4. *For a k-sheeted branched covering $f : M \to N$ we have* $\operatorname{Deg} f = k$.

Proof. Clearly, f is orientation-true and $\deg f = k$, so by Proposition 3 we have $\operatorname{Deg} f \geq k$. On the other hand, the regular values of f have k preimages, so $\operatorname{Deg} f \leq k$. □

Corollary 5. *If $p : M \to Q$ is an orientation-true pinch map and $q : Q \to N$ is a k-sheeted branched covering, then we have* $\operatorname{Deg}(q \circ p) = k$.

Proof. The reasoning is the same: use the functoriality of cohomology and Proposition 3. □

2.2.2. *Degree of the composition of maps.*

Proposition 6. *Take any map $f : M \to N$. Let $g : N' \to N$ be a k-sheeted covering such that N' is connected and f lifts to N', i.e., there is $f' : M \to N'$ such that $f = g \circ f'$. Then* $\operatorname{Deg} f = k \cdot \operatorname{Deg} f'$.

Proof. Homotope f so that the regular value $y \in N$ has $\operatorname{Deg} f$ preimages. This homotopy can be lifted to a homotopy of f'. So we obtain that every $y' \in g^{-1}(y)$ is a regular value of f', and this immediately implies that $\operatorname{Deg} f \geq k \cdot \operatorname{Deg} f'$.

The opposite inequality is true since for any finite subset $P \subset N'$ we can homotope f' so that each point of P becomes a regular value with $\operatorname{Deg} f'$ preimages. □

In fact, the last argument shows that $\operatorname{Deg}(f_2 \circ f_1) \leq \operatorname{Deg} f_1 \cdot \operatorname{Deg} f_2$ for any maps f_1, f_2. The opposite inequality in Proposition 6 is nontrivial (and does not hold if g is a branched covering) because of the following pathology.

Remark 7. If $f_1 : S^2 \to \mathbb{RP}^2$ is the universal covering, then $\operatorname{Deg} f_1 = 2$ by Corollary 4. If $f_2 : \mathbb{RP}^2 \to S^2$ is a map collapsing a projective line $l \subset \mathbb{RP}^2$ to a point, then $\operatorname{Deg} f_2 = 1$ since it is nonnullhomotopic (the homomorphism $f_2^* H^2(S^2; \mathbb{Z}_2) \to H^2(\mathbb{RP}^2; \mathbb{Z}_2)$ is nonzero).

However, the composition $f_2 \circ f_1$ is nullhomotopic: it takes both hemispheres of the domain S^2 surjectively to the range S^2, but with different orientations. So $\operatorname{Deg}(f_2 \circ f_1) = 0$.

Remark 8. If $f_2 : \mathbb{RP}^2 \to S^2$ is as above, we have $\operatorname{Deg} f_2 = 1$. If $f_3 : S^2 \to S^2$ is any map of degree 7 as an element of $\pi_2(S^2)$, then $\operatorname{Deg} f_3 = 7$ according to Corollary 4. However, $\operatorname{Deg}(f_3 \circ f_2) = 1$, and $f_3 \circ f_2 \sim f_2$.

Indeed, as one can see, there are exactly two homotopy classes of maps $\mathbb{RP}^2 \to S^2$, since the obstruction for such maps to be homotopic lies in $H^2(\mathbb{RP}^2; \pi_2(S^2)) \simeq \mathbb{Z}_2$ (see, e.g., [Whitehead 1978, Chapter VI, §6]). One also can directly construct a homotopy of $f_3 \circ f_2$ to f_2. So we have another example when $\operatorname{Deg}(f_3 \circ f_2) \neq \operatorname{Deg} f_3 \cdot \operatorname{Deg} f_2$. In particular, we note that here f_2 is homotopic to a pinch map, and as f_3 we can take a branched covering.

3. The factorization theorem

3.1. *A map with a minimal graph.*

Proof of Theorem 2. Take some triangulation of N and denote its 1-skeleton by $\mathcal{T} \subset N$. Consider maps $h : M \to N$ homotopic to f which are transversal to $\mathcal{T}$. Then $h^{-1}(\mathcal{T})$ is an embedded graph in M, call it Γ, possibly with *isolated circles* whose images do not cover the vertices of $\mathcal{T}$.

We take h such that Γ has a minimal number of edges $E(\Gamma)$. Each isolated circle is counted as one edge. Then we observe that Γ has the following three properties which we will prove in Section 3.2.

Claim 9. For every edge of Γ the images of its endpoints do not coincide.

For every component $A \subset M \setminus \Gamma$ its image $h(A)$ is contained in a certain triangle $B \subset N$ and $h(\partial A) \subset \partial B$. By Claim 9, for every component $\alpha \subset \partial A$ we may assume that either α is an isolated circle, or α is mapped to ∂B monotonously with index $i_\alpha \neq 0$. In the last case, we call α *essential.*

Claim 10. Either Γ has no isolated circles, or Γ is a union of such circles and has no vertices.

In the case when Γ is a union of isolated circles, the image $h(M)$ is contained in $N \setminus \operatorname{sk}^0(\mathcal{T})$. Note that $N \setminus \operatorname{sk}^0(\mathcal{T})$ can be deformation retracted onto the dual graph of $\mathcal{T}$. This proves the theorem in that case.

Further we will assume that Γ has no isolated circles. Take a triangle $B \subset N$ and a component $A \subset h^{-1}(B)$. Orient ∂B ant ∂A so that $h|_{\partial A}$ preserves the orientation.

Claim 11. The boundary ∂A consists of one component with index 1, or A is orientable and all the components of ∂A have the same orientation with respect to A.

If A is nonorientable, then $h|_A$ is homotopic to a pinch map so that the homotopy is stationary in a neighborhood of ∂A. Indeed, in that case by Claim 11 the boundary ∂A is a circle with index 1. Since h takes ∂A to ∂B monotonously, the restriction of h to a small neighborhood $W \supset \partial A$ is injective. Then $h|_A$ is homotopic

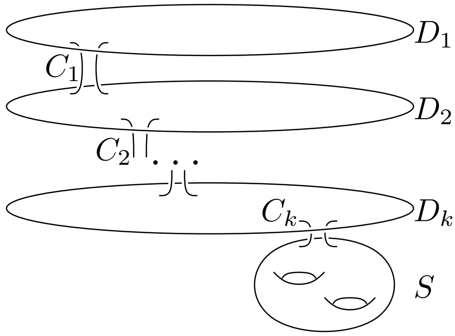

Figure 1. Presentation of A as the connected sum.

to the composition of the collapse of $A \setminus W$ to a point and a homeomorphism $A/(A \setminus W) \to B$ because of contractibility of B.

Otherwise, orient A. We can present A as a connected sum of a certain number of disks $D_1, \dots, D_k$, and a closed surface S (which is possibly a sphere). We may assume that they are joined by cylinders $C_1, \dots, C_k$ in that order; see Figure 1.

Homotope h so that each C_j is mapped to a single point as well as S. Then on every D_j we can homotope h to a branched covering with one critical point of index $i_{\partial D_j}$ or to a diffeomorphism if $i_{\partial D_j} = 1$. This is possible because the restriction $h|_{\partial D_j} : \partial D_j \to \partial B$ is a $i_{\partial D_j}$-sheeted covering. The homotopy is assumed to be stationary in a neighborhood of ∂A and the resulting h is assumed to preserve the orientation. Finally, homotope h near $C_1, \dots, C_{k-1}$ to a branched covering with two branched points of index 2 (see, for instance, [Gabai and Kazez 1987, Figure 2.1]). The subsurface $C_k \cup S \subset A$ remains pinched.

Repeat this for all components $A \subset M \setminus \Gamma$, and the proof of Theorem 2 is complete. $\square$

3.2. *Properties of the minimal graph.*

Proof of Claim 9. Recall that h takes vertices of Γ to vertices of $\mathcal{T}$ and takes interiors of edges of Γ to interiors of edges of $\mathcal{T}$. By transversality of h to the vertices of $\mathcal{T}$, the half-edges of any vertex $v \in \Gamma$ are in bijective correspondence with the half-edges of $h(v) \in \mathcal{T}$. Therefore, since $\mathcal{T}$ has no loops, Γ has no loops either.

Take an edge $e \subset \Gamma$ with endpoints v and w. Suppose $h(v) = h(w)$. Let $e' \subset \mathcal{T}$ be a (closed) edge that contains $h(e)$. Take a small tubular neighborhood $U \supset e'$ and a tubular neighborhood $V \supset e$ such that $h(V) \subset U$ and $h(\partial V) \subset \partial U$.

Since $h(v) = h(w)$, the image $h(V)$ does not cover the whole of e'. Then we can homotope h on V and squeeze $h(V)$ outside U, so that $h(V) \subset \partial U$. When we

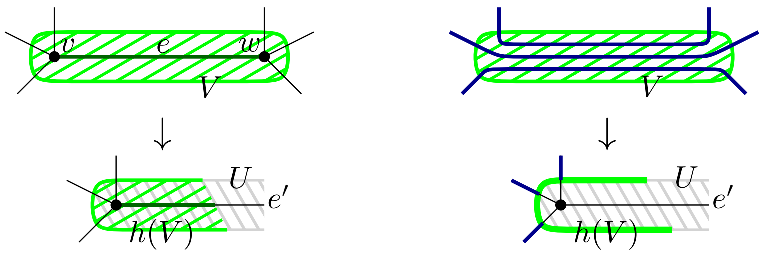

Figure 2. Collapsing of an edge with the same images of the endpoints.

redefine Γ as the preimage of $\mathcal{T}$ under new h, we will see that the edge e disappears and other edges incident on v and w are modified as in Figure 2.

Thus, $E(\Gamma)$ decreased at least by 1, which contradicts the minimality. $\square$

Proof of Claim 10. Suppose Γ has isolated circles as well as vertices. Then find a component $A \subset M \setminus \Gamma$ whose boundary includes at least one isolated circle α_0 and at least one essential component α_1 (to do this, one can consider the dual graph of Γ, mark blue the edges dual to isolated circles, mark red the edges dual to essential curves, and then find a vertex incident to edges with different colors).

Take a triangle $B \subset N$ such that $h(A) \subset B$. Take points $x_0 \in \alpha_0$ and $x_1 \in \alpha_1$ such that $h(x_0) = h(x_1) = y \in \partial B$. Take a non-self-intersecting path γ in A from x_0 to x_1. Then $h \circ \gamma$ is a loop inside the triangle B with basepoint $y \in \partial B$.

Similarly to the proof of Claim 9, we can homotope h to compress $h(\gamma)$ to y and then to squeeze $h(\gamma)$ outside B, so that the homotopy is stationary outside a small neighborhood of γ.

As a result, when we redefine Γ as the preimage of $\mathcal{T}$ under new h, the edges α_0 and α_1 will join together in one curve; see Figure 3. Thus $E(\Gamma)$ decreased by 1, which contradicts the minimality. $\square$

Proof of Claim 11. Suppose the hypothesis of the claim is violated. Then one can find two points $x_1, x_2 \in \partial A$ such that $h(x_1) = h(x_2) = y \in N$ and a non-self-intersecting curve γ in A from x_1 to x_2 which admits a coorientation agreed with the orientation of ∂A at x_1 and x_2. (Indeed, if A is nonorientable, we can take any

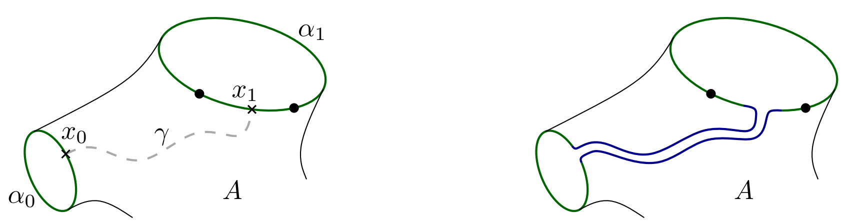

Figure 3. Join of an isolated circle with an essential curve.

x_1, x_2 with the same image and then choose γ properly. And if A is orientable, we take x_1 and x_2 on the components of ∂A with different orientations with respect to A and take any γ.)

Then we can homotope h in a small neighborhood of γ similarly to the proof of Claim 10. Suppose x_1 belongs to the oriented edge v_1w_1 of Γ, and x_2 belongs to the oriented edge v_2w_2. Then after the homotopy Γ modifies as in Figure 3 so that these edges will be replaced by the edges v_1v_2 and w_1w_2.

Note that the homotopy will not change $E(\Gamma)$. But also note that $h(v_1) = h(v_2)$ and $h(w_1) = h(w_2)$, which contradicts Claim 9. □

Remark 12. Clearly, if f is the composition of a pinch map with a branched covering, then $\operatorname{Deg} f \neq 0$. In the case $\operatorname{Deg} f = 0$ one can strengthen Theorem 2 as follows: *f is homotopic to the composition of a retraction of M to a graph $\Gamma' \subset M$ with a projection $\Gamma' \to N$*. Note that as Γ' we can take the dual graph of Γ above, but it may be not isomorphic to the dual graph of $\mathcal{T}$.

4. Estimation of the degree

Note that in Theorem 2 the pinched subsurface of M may be assumed to be connected (or empty). Also, in order to prove Theorem 1, we note the following remark.

Proposition 13. *The resulting map in our proof of Theorem 2 cannot both pinch a nonorientable subsurface of M and have a branch point.*

Note that this assertion refers to the decomposition obtained just in our proof of Theorem 2. Of course, one can compose a pinch of a crosscap with a branched covering (e.g., as in Remark 8), but this is not our case.

Proof. Suppose that the obtained pinch map $p : M \to Q$ collapses a Möbius band $L \subset M$ (and, possibly, some other subsurface). Note that we can "move" L across M. Namely, we can replace L by a point and for any component $A \subset M \setminus \Gamma$ glue a Möbius band instead of any point inside A defining p on it as a collapse to a single point.

For such an A we take a component of $M \setminus \Gamma$ with a disconnected boundary or a component whose boundary has index > 1. Otherwise, if there is no such component, the resulting map will have no branched points, as we can see from the final part of the proof of Theorem 2, and our proposition holds.

Then, after the "moving" of L into A, the statement of Claim 11 does not hold for the obtained map. But the moving of L does not change Γ, which contradicts the minimality of $E(\Gamma)$. □

Proposition 14. *For the factorization $M \xrightarrow{p} Q \xrightarrow{q} N$ from our proof of Theorem 2, if q is a d-fold branched covering, then* $\operatorname{Deg}(q \circ p) = d$.

This assertion is nontrivial in view of Remark 8.

Proof. Note that a pinch map p cannot be contractible since the homomorphism $p^* : H^2(Q; \mathbb{Z}_2) \to H^2(M; \mathbb{Z}_2)$ is nonzero. Therefore $\operatorname{Deg} p = 1$.

Suppose p is not orientation-true. Then by Proposition 13 we have that q is a covering (without branch points). Then $\operatorname{Deg}(q \circ p) = d$ by Proposition 6.

Suppose p is orientation-true. Then $\operatorname{Deg}(q \circ p) = d$ by Corollary 5. □

Proof of Theorem 1. We are given a map $f : M \to N$ of geometric degree $d > 0$. Applying Theorem 2, we obtain a factorization $M \xrightarrow{p} Q \xrightarrow{q} N$. Denote the set of critical values of q by $B \subset N$ (it is finite, possibly empty). By Proposition 14 the number of sheets of q equals d. Then

$$\chi(Q \setminus q^{-1}(B)) = d \cdot \chi(N \setminus B),$$

and therefore $\chi(Q) \leq d \cdot \chi(N)$ (this is similar to the reasoning in the Riemann–Hurwitz formula). To complete our proof, note that $\chi(M) \leq \chi(Q)$. □

Appendix: Stable maps and apparent contours

Now let us show one unexpected application of the factorization theorem.

A map of surfaces $f : M \to N$ is called *generic* if its critical points Σ_f are folds and cusps. A generic map is called *stable* if the set of its critical values $f(\Sigma_f)$ has only transversal self-crossings, called *nodes*. For more details, see, for example [Arnold et al. 2012, Part 1] or [Yamamoto 2017] (see also [Ryabichev 2020]). Denote the number of nodes by $n(f)$.

Note that stable maps form a dense open subset in $C^\infty(M, N)$. Pinches and branched coverings, which we have considered above, are not stable, but one can deform them to stable maps by an arbitrary small homotopy.

Theorem 15. *Every map of closed surfaces $f : M \to N$ is homotopic to a stable map f' such that $n(f') = 0$, i.e., the set of critical values $f'(\Sigma_{f'})$ is a collection of nonintersecting non-self-intersecting curves (possibly with cusps) in N.*

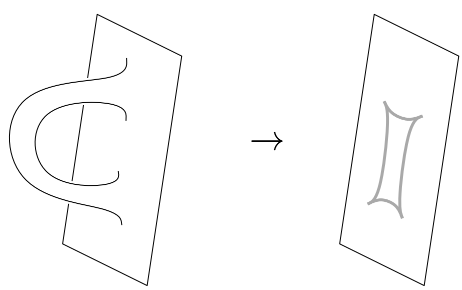

Figure 4. Collapsing of a handle adds a fold curve with 4 cusps.

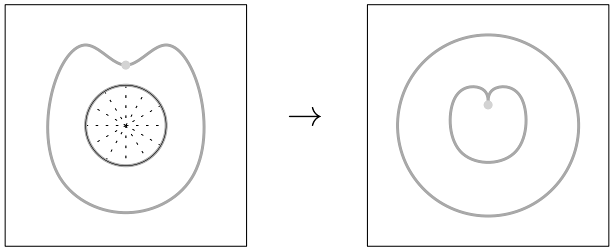

Figure 5. Collapsing of a Möbius band via adding two fold curves with 1 cusp.

This fact connects the study of apparent contours of T. Yamamoto with the theory of branched coverings; see, e.g., [Gabai and Kazez 1987]. Note that if $N = \mathbb{R}^2$, then Theorem 15 can be proved without the factorization theorem, and in the case $N = S^2$, Theorem 15 was proved in [Yamamoto 2010, Theorem 1.4].

Proof. Applying Theorem 2 to the map f, we obtain a factorization $M \xrightarrow{p} Q \xrightarrow{q} N$.

The branched covering q can be homotopied so that each branch point of index i will turn into a fold circle with $i+2$ cusps. See, for example, [Arnold et al. 2012, Part 1, §1.8] for more details.

A collapsing of an orientable handle can be turned into a fold curve with 4 cusps, just as a projection of a plane with a handle in $\mathbb{R}^3$ to the plane; see Figure 4 (see also the discussion in [Yamamoto 2017, §3.2]).

Finally, a collapsing of a Möbius band is homotopic into a map with a fold curve along the Möbius band and with a fold curve with one cusp around it; see Figure 5 (on the left, the opposite points on the inner circle should be identified, we show them connected by dotted lines; see also [Yamamoto 2010, Figures 11 and 12]).

The described homotopies are local, the images of these curves are small and we may assume that they do not cross each other. □

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